POLYNOMIAL MULTIPLE VARIANCE IMPULSE RESPONSE MEASUREMENT

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ABSTRACT

The paper discusses a multiple variance methodology for measuring the impulse response for small signals of a mildly nonlinear system, i.e., the first-order kernel of a Volterra model. It is shown with theory that using multiple variance inputs and any linear impulse response measurement method, it is possible to accurately estimate the first-order kernel by applying a polynomial interpolation. The values of the input gains that minimize the influence of noise are also determined. The experimental results, considering an emulated scenario, show how the proposed method can be effectively used to accurately estimate the room impulse response in case of nonlinearities in the measurement system.

***Index Terms*—** Polynomial multiple variance, impulse response, Volterra filter.

1. INTRODUCTION

Different techniques have been proposed for measuring the impulse response of a linear system and have been applied to room impulse response measurement [1, 2]. A common problem in many of these techniques is the artifacts introduced by nonlinearities caused by the measurement system (e.g., originated by the amplifier or the loudspeaker used in the measurement) or present in the system we identify, which may be mildly nonlinear instead of a linear system. To contrast the effect of these nonlinearities, the exponential sweep (ES) technique has been introduced in [3, 4]. Unfortunately, it has been recently shown [5] that the ES techniques are also affected by nonlinearities, even if much less than the other methods. More recently, techniques robust towards the nonlinearities have been developed by directly taking into account the nonlinear nature of the measurement system. In [6, 7] the measurement system has been modelled as an orthogonal nonlinear filter, a Legendre nonlinear (LN) filter in [8, 6] or a Wiener nonlinear (WN) filter in [9, 7], and its first-order kernel has then been identified using a perfect periodic sequence (PPS) using the cross-correlation method. In [10, 11], the measurement system has been modelled as a Volterra filter [12] and its first-order kernel has been identified with an orthogonal periodic sequence (OPS), again computing a cross-correlation. It is interesting to note that in a Volterra filter, the first-order kernel coincides with the impulse response for small signal amplitudes (if we neglect the constant term).

In this paper, the measurement system will be modelled as a Volterra filter, but a different identification technique will be applied for measuring the first-order kernel. In particular, a polynomial multiple variance (PMV) approach will be applied in combination with any classical linear identification technique, like the popular maximal length sequences (MLSs) [13] or ESs [3, 4]. In the multiple variance approach, the same input is applied multiple times with different gains and, thus, different input powers. In the proposed approach, the corresponding outputs are used to identify the impulse response with the adopted linear technique. It will be shown that the first-order kernel of the Volterra filter can be accurately obtained with a polynomial interpolation of the measured impulse responses.

A multiple variance technique has already been proposed in [14] but for a different purpose. Specifically, [14] uses the multiple variances to contrast the locality of the identification of a Wiener model with the Lee-Schetzen method. Using the single input variance, the derived model represents well the behavior of the nonlinear system only for input powers close to the chosen variance. In the approach of [14], to expand the validity of the model, the different kernels are estimated with different gains of the input signal minimizing the mean square error between the output of the system and of the model. In contrast, in this paper the multiple variance approach is used to estimate with high accuracy a single kernel of the nonlinear model, the first-order one, and a polynomial interpolation is used for this purpose, minimizing the mean square deviation (MSD) between the estimated kernel and the real one.

The paper is organized as follows. Section 2 presents the theory of the polynomial multiple variance (PMV) method discussing the proposed approach, how it is affected by noise, and how to optimally choose the gains. Section 3 provides some experimental results emulating a room impulse response measurement in presence of nonlinearities. Section 4 presents the concluding remarks.

Throughout the paper, small boldface letters are used to denote vectors and bold capital letters are used to denote matrices, $E[\cdot]$ indicates mathematical expectation.

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2. POLYNOMIAL MULTIPLE VARIANCE METHOD

This section first briefly reviews the Volterra filters, then discusses the proposed polynomial multiple variance method. The mean square deviation of the estimated coefficients is also computed and is used to derive the optimal value of the multiple input gains.

2.1. Volterra filters

Volterra filters are polynomial filters that derive from the double truncation with respect to order and memory of the Volterra series. They can approximate arbitrarily well any causal, discrete-time, time-invariant, continuous nonlinear system, whose input–output relationship can be expressed by a nonlinear function f of the N most recent input samples,

$$y(n) = f[x(n), x(n-1), ..., x(n-N+1)], \tag{1}$$

where the input signal $x(n)$ belongs to a compact in $\mathbb{R}$.
A Volterra filter of order K and memory N has input-output relationship

$$y(n) = \sum_{k=0}^{K} y_k(n), \tag{2}$$

where $y_k(n)$ is a homogeneous polynomial term of order k, i.e., $y_0(n) = h_0 \in \mathbb{R}$,

$$y_1(n) = \sum_{i=0}^{N-1} h_{1,i}x(n-i), \tag{3}$$

and $y_k(n)$ for any $k \in [2, K]$ in triangular form is

$$y_k(n) = \sum_{i_1=0}^{N-1}\sum_{i_2=i_1}^{N-1} ... \sum_{i_k=i_{k-1}}^{N-1} h_{k,i_1,...,i_k} x(n-i_1)\cdot...\cdot x(n-i_k). \tag{4}$$

The proposed method relies on the homogeneity property of the terms $y_k(n)$. If the input $x(n)$ is multiplied by a factor A, then $y_k(n)$ is multiplied by a factor A^k.

2.2. The proposed method

Let us assume we are dealing with a nonlinear system that can be represented as a Volterra filter of order K and memory N,

$$y(n) = H(x)(n) = \sum_{k=0}^{K} y_k(n) + \nu(n), \tag{5}$$

where H is the operator representing the nonlinear system, and $\nu(n)$ is an output additive noise. Our objective is to measure the coefficients of the linear kernel $h_{1,i}$ in (3) for $i = 0, .., N-1$.
Assume a linear funtional $\mathcal{L}_i$ is available such that, for a certain input $x(n)$,

$$\mathcal{L}_i y_1(n) = h_{1,i}, \tag{6}$$

i.e., $\mathcal{L}_i$ perfectly estimates $h_{1,i}$ in case of a linear system and noise absence. Examples of these linear functionals are those obtained with the identifications methods based on MLSs, PPSs, linear or exponential sweeps, or any other method that allows to estimate the impulse response of a linear system from the knowledge of $x(n)$ and $y_1(n)$. Since $\mathcal{L}_i$ is linear,

$$\mathcal{L}_i y(n) = \mathcal{L}_i H(x)(n) = \sum_{k=0}^{K} \mathcal{L}_i y_k(n) + \mathcal{L}_i \nu(n), \tag{7}$$

and in the considered conditions (7) does not equal to $h_{1,i}$ both for the presence of noise and of the nonlinear terms $y_k(n)$, for $k \neq 1$. In order to estimate $h_{1,i}$, we adopt a multiple variance method, and we apply the input $x(n)$ multiple times, multiplied by different factors A_m, for $m = 1, ..., M$. For the homogeneity property of $y_k(n)$, when $x(n)$ is multiplied by a factor A_m,

$$y_{A_m}(n) = H(A_m x)(n) = \sum_{k=0}^{K} A_m^k y_k(n) + \nu_m(n), \tag{8}$$

where $y_k(n)$ is given in (4) and $\nu_m(n)$ is the additive output noise on $y_{A_m}(n)$. Introducing the $(K+1) \times 1$ vector

$$\boldsymbol{h}_i = [\mathcal{L}_i y_0, \mathcal{L}_i y_1(n), ..., \mathcal{L}_i y_K(n)]^T, \tag{9}$$

then according to (8),

$$\mathcal{L}_i y_{A_m}(n) = [1, A_m, A_m^2, ..., A_m^K] \cdot \boldsymbol{h}_i + \mathcal{L}_i \nu_m(n). \tag{10}$$

By defining the $M \times 1$ vectors

$$\boldsymbol{d}_i = [\mathcal{L}_i y_{A_1}(n), \mathcal{L}_i y_{A_2}(n), ..., \mathcal{L}_i y_{A_M}(n)]^T, \tag{11}$$

$$\boldsymbol{\nu}_i = [\mathcal{L}_i \nu_1(n), \mathcal{L}_i \nu_2(n), ..., \mathcal{L}_i \nu_M(n)]^T, \tag{12}$$

the $(K+1) \times M$ Vandermonde matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & \dots & 1 \\ A_1 & A_2 & \dots & A_M \\ A_1^2 & A_2^2 & \dots & A_M^2 \\ \vdots & & & \vdots \\ A_1^K & A_2^K & \dots & A_M^K \end{bmatrix}, \tag{13}$$

writing (10) for $m = 1, ..., M$ in matrix form results in

$$\boldsymbol{d}_i = \boldsymbol{A}^T \boldsymbol{h}_i + \boldsymbol{\nu}_i. \tag{14}$$

Provided $M \geq K+1$ and $\boldsymbol{A}\boldsymbol{A}^T$ is invertible,

$$\boldsymbol{h}_i = (\boldsymbol{A}\boldsymbol{A}^T)^{-1}\boldsymbol{A}\boldsymbol{d}_i - (\boldsymbol{A}\boldsymbol{A}^T)^{-1}\boldsymbol{A}\boldsymbol{\nu}_i. \tag{15}$$

Neglecting the noise effect, $\boldsymbol{h}_i$ can be estimated with

$$\widehat{\boldsymbol{h}}_i = (\boldsymbol{A}\boldsymbol{A}^T)^{-1}\boldsymbol{A}\boldsymbol{d}_i, \tag{16}$$

and thus $h_{1,i}$ can be evaluated as

$$\widehat{h}_{1,i} = \boldsymbol{e}_2^T \boldsymbol{h}_i = \boldsymbol{e}_2^T (\boldsymbol{A}\boldsymbol{A}^T)^{-1}\boldsymbol{A}\boldsymbol{d}_i, \tag{17}$$

with $\boldsymbol{e}_2$ the second column of the $(K+1) \times (K+1)$ identity matrix. Note that $\widehat{\boldsymbol{h}}_i$ collects the coefficients of a polynomial interpolations of $\boldsymbol{d}_i$ and $\widehat{h}_{1,i}$ is the first-order coefficient.

2.3. The noise effect

If the nonlinear system has an order lower than or equal to K and is not affected by noise, then equation (17) allows to perfectly estimate its first-order kernel. In noise presence the measurement will be affected by an error,

$$\epsilon_i = \widehat{h}_{1,i} - h_{1,i} = \boldsymbol{e}_2^T(\boldsymbol{A}\boldsymbol{A}^T)^{-1}\boldsymbol{A}\boldsymbol{\nu}_i. \qquad (18)$$

The MSD of ϵ_i is

$$\begin{aligned} \text{MSD} &= E\left[(\widehat{h}_{1,i} - h_{1,i})^T(\widehat{h}_{1,i} - h_{1,i})\right] \\ &= \boldsymbol{e}_2^T(\boldsymbol{A}\boldsymbol{A}^T)^{-1}\boldsymbol{A}E[\boldsymbol{\nu}_i\boldsymbol{\nu}_i^T]\boldsymbol{A}^T(\boldsymbol{A}\boldsymbol{A}^T)^{-1}\boldsymbol{e}_2. \qquad (19) \end{aligned}$$

In the hypothesis that the nonlinear terms $\mathcal{L}_i\nu_l(n)$ are uncorrelated and Gaussian distributed with zero mean and variance σ_ν^2, the MSD simplifies to

$$\text{MSD} = \boldsymbol{e}_2^T(\boldsymbol{A}\boldsymbol{A}^T)^{-1}\boldsymbol{e}_2\sigma_\nu^2. \qquad (20)$$

2.4. The optimal gains

From (20), it is clear that the MSD depends on the optimal choice of the gains A_m, which affect the Vandermonde matrix $\boldsymbol{A}$. In the mathematical literature [15] the values of the so-called *nodes*, i.e., of the gains A_m, that optimize the condition number of $\boldsymbol{A}$ have been obtained and are symmetrically distributed (i.e., they appear in pairs $(A_m, -A_m)$). The gains that optimize the conditioning do not minimize the MSD. Anyway, in our experiments the choice of symmetric gains has proved optimal also for minimizing (20).

Considering symmetric gains and fixing the maximum gain $A_1 = 1$, a small optimization program has been written to find the A_m that minimize (20). It was found that many gains result very close to each other, especially for low values, and introducing a small quantization, which does not affect the resulting MSD, they can be set equal. For example, the optimal gains for orders $K = 5$ and 6, and $M = 10$ and an 8-bit quantization are the following:

$$\left[1, \frac{217}{256}, \frac{87}{256}, \frac{87}{256}, \frac{87}{256}\right] \qquad (21)$$

and their symmetric values.

When M is close to $K+1$, the MSD assumes large values, but by increasing M, the MSD rapidly decreases, as shown in Fig. 1. As a rule of thumb, M should be at least $2K$ to obtain reasonable values of MSD.

It should be noted that for assigned M and gains A_m, the MSD in (20) is an increasing function of K, which means that we should always choose the minimum order that fits the nonlinear systems we are dealing with.

3. EXPERIMENTAL RESULTS

The experimental results emulate a room impulse response measurement in presence of nonlinearities originated by the

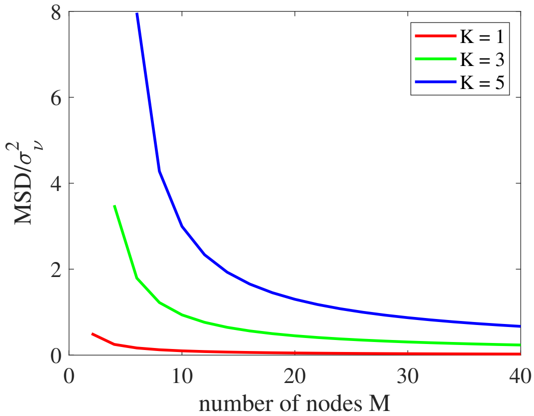

Fig. 1. Optimal value of MSD/σ_ν^2 vs M.

power amplifier or the loudspeaker. The nonlinearities are generated with a real device, a Behringer MIC100 vacuum tube preamplifier. The preamplifier has a potentiometer that allows to set different distortion levels acting on its gain and 13 different settings (setting 0 till 12) have been considered, which correspond to different gains. Fig. 2 shows the second, third, and total harmonic distortions we have at the different settings on a sinusoidal signal having the maximum amplitude considered in the experiments.

Working with a sampling frequency $F_{\text{S}} = 44\,100$ Hz, different test signals have been applied to the preamplifier. The test signals were:

(i) MLSs of period $2^{16} - 1$;
(ii) ESs of length 2^{16};
(iii) a Gaussian input signal for an OPS of period 2^{21}.

The resulting output signals have then been convolved with a previously measured room impulse response having 8192 sample length and a Gaussian noise has been added to have a 40 dB output signal-to-noise (SNR) on the lowest power signals. The preamplifier outputs have been recorded with an SNR that was always greater than 57 dB. The test signals are composed of multiple repetitions of the MLSs and of the ESs, multiplied by the gains in (21) and their opposite values. The ESs have been generated as in [16] and sweeps between 18.90 Hz and 21 203 Hz. The initial and final frequencies have been optimized to obtain a sequence that starts and ends with a zero sample. The OPS input has a much longer period and a lower power than the other sequences and the identification using an OPS of order 3 and maximum diagonal number 5 [11] is very protected against nonlinearities. It will be used as ground truth in the following comparisons.

Figs. 3 and 4 show the identification results obtained at the different settings. The results are given in terms of log spectral distance (LSD) in the band $[100, 18\,000]$ Hz, falling strictly inside the passband. Considering $|\hat{H}_{\text{R}}(k)|$ the measured room magnitude response using an FFT on T samples and $|H_{\text{R}}(k)|$ the reference room magnitude response obtained with the OPS, the LSD is defined in the band

$B = [k_1 \frac{F_S}{T}, k_2 \frac{F_S}{T}]$, with k_1 and $k_2 \in \mathbb{N}$, as follows:

$$\text{LSD} = \sqrt{\frac{1}{k_2 - k_1 + 1} \sum_{k=k_1}^{k_2} \left[10 \log_{10} \frac{|H_R(k)|^2}{|\hat{H}_R(k)|^2} \right]^2}. \quad (22)$$

Fig. 3 shows the results obtained with the MLSs. The thin curves are obtained by identifying the impulse responses with the classical MLS method applied on 10 MLS periods with gains $\pm\frac{87}{256}, \pm\frac{217}{245}, \pm 1$. The bold curves are obtained with the PMV method, identifying first the impulse responses on one MLS period for each gain in (21) and negated values, and then using the formula in (17). Thus, the comparison is performed considering the same number of input samples in the identification. The MLSs are the most affected by nonlinearities and the PMV method provides the best identification results for $K = 3, 4$ and $5, 6$. In the PMV method, the same results are obtained for orders $K = 1$ and 2, 3 and 4, 5 and 6, respectively, thanks to the symmetry of the gains. At settings 0 and 1, i.e, with the lowest distortion, the same result is obtained by the PMV method for all orders K, but for all other settings the PMV method for $K = 3, 4$ and $5, 6$ gives the best outcome, with a very flat curve for all distortions.

Fig. 4 shows the results for the ESs. The thin curves are obtained by identifying the impulse responses with the deconvolution method applied on 10 sweeps, having the same gain and separated by silence periods, and averaging the resulting impulse responses. The bold curves are obtained with the PMV method, identifying first the impulse responses on 1 sweep, and then using the formula in (17). The ESs are much less affected by the nonlinearities, as can be appreciated with the different vertical scale compared with Fig. 3. In this case, the PMV method with $K = 1, 2$ provides the best results till setting 3, while for all other settings the best results are obtained for $K = 3, 4$. The figure also highlights that the MSD is higher if the order K is larger than the prevailing order of nonlinearities of the system under modelling.

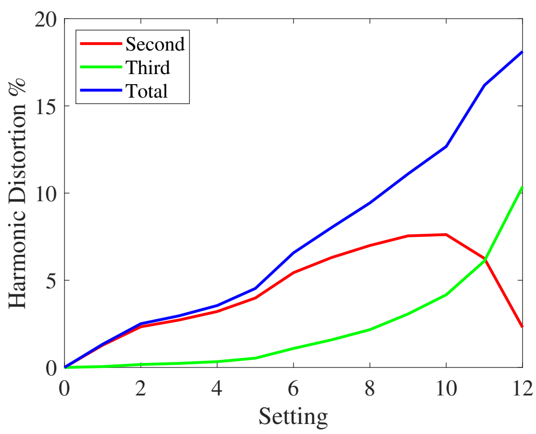

Fig. 2. Second, third, and total harmonic distortion of the MIC100 preamplifier at the different settings.

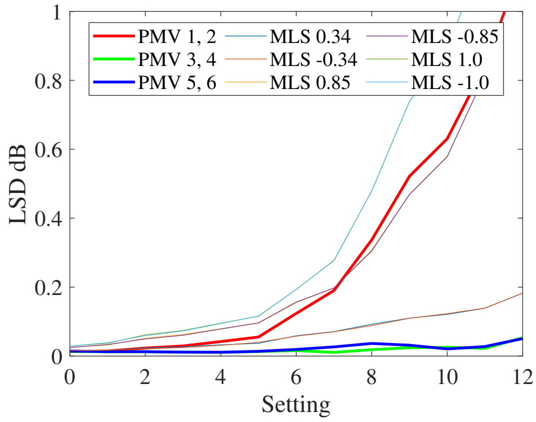

Fig. 3. LSD for MLSs at different gains A_m and for PMV based on MLSs at different orders K.

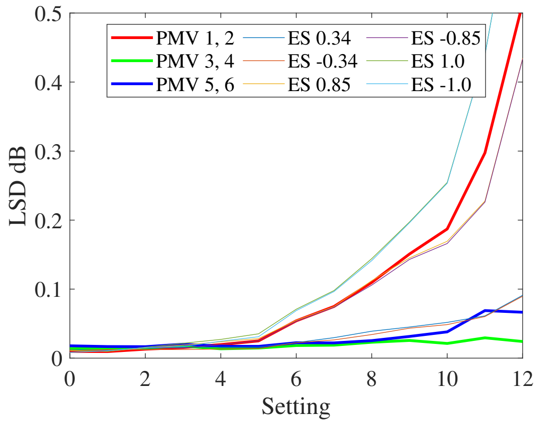

Fig. 4. LSD for ESs and for PMV based on ESs at different orders K.

4. CONCLUSIONS

It has been shown with theory and experiments that using multiple variance inputs and any linear impulse response measurement method, by polynomial interpolation it is possible to accurately estimate the first-order kernel of a Volterra filter, i.e., the impulse response for small signals. The values of the input gains that minimize the influence of noise have also been determined. The experimental results, considering an emulated scenario, have illustrated the accuracy of the proposed approach.

It could be argued that with multiple measurements of sufficiently low amplitude, it is possible to avoid the nonlinearities and contrast at the same time the effect of noise. It can be proved with theory that, for the same number of measurements, the PMV approach provides a lower MSD if the amplitude of the input signal has to be reduced by more than a factor 4 or 5 to obtain negligible nonlinearities. The proof will be included in a paper under preparation.

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