

# CYCLIC AND ABELIAN COVERINGS OF REAL VARIETIES

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*Dedicated to Slava (Viatcheslav) Kharlamov on the occasion of his 71-st birthday.*

Abstract. We describe the birational and the biregular theory of cyclic and Abelian coverings between real varieties.

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## INTRODUCTION

While real solutions of polynomial equations were ever since investigated (for instance Newton classified all the possible equations and shapes of real plane cubic curves) a breakthrough came alongside of the impetuous development of complex function theory.

Harnack, Klein and Weichold, just to name a few, [Har76], [Klein82], [Klein76], [Klein92], [Wei83] used the ideas of Riemann in order to study real equations and their real solutions.

In this way the main branch of real algebraic geometry was born, the one which focuses on the pair of sets given by the complex solutions and the real solutions, with complex conjugation  $\sigma$  acting on them.

The abstract formal outcome of this approach is the definition of a real manifold as a pair  $(X, \sigma)$  where  $X$  is a complex manifold and  $\sigma : X \rightarrow X$

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is an antiholomorphic involution (involution means that  $\sigma^2$  is the identity). The substantial outcome was the use of topological methods and of methods from complex manifolds theory for real algebraic geometry.

A strong boost towards the development of real algebraic geometry came from the Hilbert problems 16 and 17, posed in Paris in the year 1900.

Especially problem 16 oriented the research towards the extrinsic geometry: real plane curves were the object of intensive investigations, and, later on, also surfaces of low degree in  $\mathbb{P}^3$  ([Kharl76, Kharl84]).

The Russian school was very influenced by Hilbert's problem 16, with an even more prominent role of topology in the study of this extrinsic geometry of real varieties (see [DegKha00] for an excellent survey): but soon emerged the role of intrinsic geometry.

These methods had been pioneered by Comessatti, who studied the topology of rational surfaces [Come12],[Come14], and later on the real structures on Abelian varieties [Come24-5], complementing previous results of Klein and others on real algebraic curves (see [CilPed96] for an excellent survey).

An important endeavour was the extension of the Castelnuovo-Enriques classification of surfaces to the case of real algebraic surfaces, with initial contributions in [Manin66], [Manin67], [Iskovskih67], [Iskovskih70], [Iskovskih79], [Nikulin79], [Sil89], and then achieved through a long series of papers among which [D-I-K00], [CatFre03], [Fre04], [D-I-K08] (we refer to [Mang17], [Mang20] as an excellent text and for more references of many other works by Mangolte and others).

This said, the contribution of the present paper concerns the intrinsic real algebraic geometry and deals with Abelian coverings between real algebraic varieties  $(X, \sigma)$  and  $(X', \sigma')$  (i.e., Galois coverings such that the Galois group of  $X \rightarrow X'$  is an Abelian group  $G$  normalized by  $\sigma$ ). The complex case was initiated by Comessatti [Come30] who described the cyclic case and, after many intermediate works by several authors, the biregular theory of such coverings, in the case where  $X'$  is smooth, was established in [Par91].

The present paper is divided into two parts: the first one is devoted to the birational theory, i.e., the description of the corresponding field extensions  $\mathbb{R}(X') \subset \mathbb{R}(X)$ , the second is the biregular theory for  $X'$  factorial, where the covering is described according to the scheme of Pardini's paper through building data, which are divisors on  $X'$  and line bundles on  $X'$  satisfying some compatibility relations, and where we have to add the reality constraints.

We should warn the reader that the birational theory, which is well-known and almost trivial in the complex case (where the cyclic case is the one where the field extension  $\mathbb{C}(X') \subset \mathbb{C}(X)$  is obtained by taking the  $n$ -th root  $z$  of a function  $f \in \mathbb{C}(X)$ , hence it is described by the simple formula  $z^n = f$ ), is by no means easy in the real case.

This is the reason why the title distinguishes between the cyclic and the Abelian case: while we can describe the cyclic case as the fibre product (field compositum) of four basic fields extensions, this becomes considerably more complicated in the Abelian case. The underlying reason is a simple arithmetical fact: the group  $G'$  generated by the Galois group  $G$  and by the antiholomorphic involution  $\sigma$  is a semidirect product  $G' = G \rtimes (\mathbb{Z}/2)$ ,

classified by an automorphism  $M : G \rightarrow G$  such that  $M^2 = 1$  (1 is herethe identity of  $G$ ). We dedicate Section 2 to the analysis of the algebraic structure of the group  $G'$ .

In the cyclic case  $M \in (\mathbb{Z}/h)^*$  can be described through a splitting of the cyclic group  $G$  as a direct sum such that, on each summand  $(\mathbb{Z}/h)$ ,  $M$  acts either as multiplication by  $\pm 1$  or, possibly, if  $h$  is divisible by 8, by  $\pm 1 + \frac{h}{2}$ . In the non cyclic case there is no such similar splitting and already the description of such pairs  $(G, M)$  is slightly more complicated (see Lemma 2.5).

Section 3 is dedicated to the description of the corresponding fields extensions.

The easiest case where the field extension is given by  $z^n = f \in \mathbb{R}(X)$  is the one where the group  $G'$  is isomorphic to the dihedral group  $D_n$  (and the field extension  $\mathbb{R}(X) \subset \mathbb{R}(z)$  is not Galois). The other cases become progressively more complicated, and the main result of Section 3, and of the first part, is Theorem 3.2, describing the four basic cases. Since the statement of the theorem requires a complicated notation, we do not follow the usual order of exposition: instead, while preparing the necessary definitions, we provide at the same time the proof.

Section 4 is devoted to the description of the biregular theory of real Abelian coverings in terms of branch divisors and invertible character sheaves, and the results are spelled out in particular in Theorem 4.1 (recalling the cyclic case in the complex setting), in Theorem 4.2, stating the result in the real cyclic case, while for the general Abelian case, to avoid too much repetition, we just indicate how to get the result from Theorem 4.2 mutatis mutandis.

Passing to the biregular theory of more general Galois coverings, with non Abelian Galois group  $G$ , we should recall the real analogue of the complex Riemann existence theorem (in the general version given by Grauert and Remmert [G-R58]).

It is based on the notion of the real fundamental group of  $(X, \sigma)$ , denoted  $\pi^{\mathbb{R}}(X, \sigma)$ , which, in the case where  $X$  has a real point  $x_0 \in X(\mathbb{R})$ , is the semidirect product  $\pi_1(X, x_0) \rtimes (\mathbb{Z}/2)$ , where conjugation by  $(\mathbb{Z}/2)$  is given by  $\sigma_*$ ; else, it is defined as the fundamental group of the Klein variety  $K(X) := X/\sigma$  (see [CatFre03]).

In both cases we have an exact sequence:

$$1 \rightarrow \pi_1(X) \rightarrow \pi_1^{\mathbb{R}}(X) \rightarrow \mathbb{Z}/2 \rightarrow 1,$$

which splits in the first case.

Putting together definitions and results of [G-R58] and of [CatFre03] we obtain the following theorem, whose general statement might be new.

**Theorem 0.1. (Real Riemann existence theorem)** *A real Galois covering with Galois group  $G$  between normal real varieties  $(X, \sigma)$  and  $(X', \sigma')$  is determined by a  $\sigma'$ -invariant Zariski closed subset  $B$  of  $X'$  and a surjective homomorphism*

$$\Psi : \pi_1^{\mathbb{R}}(X' \setminus B, \sigma') \rightarrow G',$$

where  $G' = G \rtimes (\mathbb{Z}/2)$  is determined by an automorphism  $M \in \text{Aut}(G)$  of order 2 (i.e.  $M^2 = 1$ ), such that the following conditions are satisfied:

- i) the action of  $G'$  on  $X$ , which is determined by  $\Psi$ , is such that its restriction to  $\mathbb{Z}/2 < G'$  coincides with the action associated to the real structure  $\sigma$  on  $X$ ;
- ii)  $\Psi(\pi_1(X' \setminus B)) = G$ .

In the case where  $X'$  is smooth, there is a minimal  $B$  which is a divisor. The analysis of its components and of the local monodromies in the Abelian case leads to the biregular theory of such coverings, to which section 4 is devoted. In general, the Riemann existence theorem works fine with complex curves, where the fundamental groups  $\pi_1(X \setminus B)$  are well known.

In a sequel to this paper we shall treat the case of real curves, which goes back to the Klein theory (revived and extended for instance in [Al-Gr71], and many other papers by Seppälä and other authors, for instance [Sep90]): here many arguments, except the final theorems, work for any finite group  $G$ . We shall discuss such coverings in terms of certain numerical and topological data, extending results of [Cat12] in the complex case (this was partly done in [Cat02] and [Cat08]), and with a view to the study of the corresponding moduli spaces of real curves with Abelian symmetry group  $G$  and with a fixed set of invariants ( $M$ , local monodromies, ...).

## 1. THE BASIC SET-UP

In this paper, we consider the following situation:

- 1)  $(X, \sigma)$  is a real projective variety (this means that  $X$  is a complex projective variety given together with an antiholomorphic self map  $\sigma : X \rightarrow X$  which is an involution, i.e.  $\sigma^2$  is the identity);
- 2)  $G$  is a finite subgroup of the group of complex automorphisms of  $X$ , and  $G$  is normalized by  $\sigma$ . If we consider the quotient  $X' := X/G$ , since  $\sigma G = G\sigma$ , we see that  $\sigma$  induces an antiholomorphic involution  $\sigma'$  on  $X'$ , defined by

$$\sigma'(Gx) := \sigma(Gx) = G\sigma x.$$

Hence, in particular,  $(X', \sigma')$  is also a real projective variety.

We can of course consider the more general situation where  $X$  is a real space (replace the condition that  $X$  is a projective variety by the condition that  $X$  is a complex space).

**Definition 1.1.** Saying that  $f : (X, \sigma) \rightarrow (X', \sigma')$  is real  $G$ -Galois covering means that  $G \subset \text{Aut}(X)$  as in 2) above,  $X' \cong X/G$  via  $f$ , and  $f$  is real, that is,  $f \circ \sigma = \sigma' \circ f$ .

In this situation  $\sigma$  normalizes  $G$ , and we have a semidirect product

$$G' := G \rtimes \mathbb{Z}/2, \quad \mathbb{Z}/2 \cong \{1, \sigma\}.$$

If the finite group  $G$  is Abelian, respectively cyclic, we shall say that  $X \rightarrow X'$  is an Abelian, respectively cyclic, covering of real varieties, with group  $G$ . For a projective variety  $X \subset \mathbb{P}^N(\mathbb{C})$  defined by polynomial equations with real coefficients,  $\sigma$  is induced by complex conjugation

$$\sigma(x_0, \dots, x_N) := (\overline{x_0}, \dots, \overline{x_N}).$$

It is important to observe that any real projective variety admits such an embedding, see for instance [Mang17], Théorème 2.6.44.

More generally, if  $X$  is a complex variety, one says that  $\sigma$  is antiholomorphic if it is differentiable and locally induced by an antiholomorphic map of complex manifolds, i.e., such that the derivative  $D\sigma$  of  $\sigma$  satisfies

$$J \circ D\sigma = -D\sigma \circ J,$$

where  $J$  is the complex structure on the (Zariski) tangent space of  $X$  ( $J^2 = -1$ ).

The group  $G$  acts on the function field  $\mathbb{C}(X)$ , and the antiholomorphic map  $\sigma$  induces also an involution  $\tau : \mathbb{C}(X) \rightarrow \mathbb{C}(X)$ , which normalizes the action of  $G$  on  $\mathbb{C}(X)$ , given by  $g(f) := f \circ g^{-1}$ .

It is easy to understand  $\tau$  in the prototype case where  $X$  is a projective variety  $X \subset \mathbb{P}^N(\mathbb{C})$  defined by polynomial equations with real coefficients:  $\tau$  is induced by the map that acts on homogeneous polynomials

$P(x_0, \dots, x_N) = \sum_I a_I x^I$  just by conjugating their coefficients.

i.e.,  $\tau(P) := \sum_I \overline{a_I} x^I$ ; and the field of rational functions which are  $\tau$ -invariant is just the field  $\mathbb{R}(X)$ , such that  $\mathbb{C}(X) = \mathbb{R}(X) \otimes_{\mathbb{R}} \mathbb{C}$ .

In this representation as a tensor product,  $\tau$  is induced by complex conjugation on  $\mathbb{C}$ .

In the general case, since

$$\overline{P(\sigma(x))} = \sum_I \overline{a_I x^I} = \sum_I \overline{a_I} x^I,$$

one defines

$$\tau(f) := \overline{f \circ \sigma}.$$

We shall work, at least in the first part of the paper, with the group  $G'$  generated by  $G$  and  $\tau$  acting on the function field  $\mathbb{C}(X)$ . The same calculation can be performed in the general case working with line bundles instead of function fields.

In the case where the group  $G$  is cyclic,  $G \cong \mathbb{Z}/n$ , since the field  $\mathbb{C}$  contains all roots of unity, the field extension  $\mathbb{C}(X) \supset \mathbb{C}(X')$  has a simple description, as

$$\mathbb{C}(X) = \mathbb{C}(X')[z]/(z^n - f).$$

But in this representation we do not see the action of  $\tau$ . Indeed, in the next section, we show that, through the fibre product of two distinct coverings of  $X$ , we can reduce to four basic cases:

- (1)  $G'$  is the dihedral group  $D_n$ : this is the standard totally real covering case where

$$\mathbb{R}(X) = \mathbb{R}(X')[z]/(z^n - f).$$

- (2)  $G'$  is a direct product  $(\mathbb{Z}/n) \times (\mathbb{Z}/2)$ : this is a case which resembles the one of the complex dihedral coverings (see [CP17]), so we call it the dihedral-like case.

It is important to observe that this is exactly the case where the field extension  $\mathbb{R}(X) \supset \mathbb{R}(X')$  is Galois (with group  $G'$ ): since the subgroup  $\mathbb{Z}/2$  is a normal subgroup of  $G'$  if and only if  $G' \cong G \times \mathbb{Z}/2$ .

- (3)  $n$  is divisible by 8 and  $G'$  is the semidirect product such that

$$\tau g \tau = g^{1+n/2} :$$

this will be called the twisted case, and it can be reduced to the dihedral-like case.

- (4)  $n$  is divisible by 8 and  $G'$  is the semidirect product such that

$$\tau g \tau = g^{-1+n/2} :$$

this will be called the esoteric covering <sup>1</sup>.

## 2. A NUMERICAL LEMMA

In the first part of this section we consider the following situation: we are given a finite group  $G'$ , which is a semidirect product

$$G' := (\mathbb{Z}/n) \rtimes (\mathbb{Z}/2) =: G \rtimes (\mathbb{Z}/2), \quad G := \langle g \mid g^n = 1 \rangle, \quad (\mathbb{Z}/2) := \langle \tau \mid \tau^2 = 1 \rangle.$$

The semidirect product is then classified by an element  $m \in (\mathbb{Z}/n)^*$ , such that  $m^2 = 1$  and such that  $\tau g \tau = g^m$ .

The next lemma shows that, up to letting  $G$  be a direct product of two cyclic groups (of coprime order), we can reduce to four basic situations:  $m = \pm 1$ , or, when  $n$  is divisible by 8,  $m = \pm 1 + n/2$ .

**Lemma 2.1.** *Let  $m \in (\mathbb{Z}/n)^*$  be such that  $m^2 = 1$ . Then we can write  $n = n_1 \cdot n_2 \cdot 2^k$ , where:*

- (1)  $k = 0$  or  $k \geq 3$ ,
- (2) the factors are pairwise relatively prime (and possibly equal to 1),
- (3) using the Chinese remainder theorem to identify

$$\mathbb{Z}/n \cong \mathbb{Z}/n_1 \times \mathbb{Z}/n_2 \times \mathbb{Z}/2^k,$$

$m$  corresponds to  $(1, -1, \pm 1 + 2^{k-1})$ .

In particular, we can write  $n = N_1 N_2$  with  $N_1, N_2$  relatively prime (and possibly equal to 1), and either

- (4)  $m \in \mathbb{Z}/N_1 \times \mathbb{Z}/N_2$  equals  $(1, -1)$ , or
- (5)  $N_2$  is divisible by 8 and  $m \in \mathbb{Z}/N_1 \times \mathbb{Z}/N_2$  equals  $(1, -1 + N_2/2)$ , or
- (6)  $N_2$  is divisible by 8 and  $m \in \mathbb{Z}/N_1 \times \mathbb{Z}/N_2$  equals  $(-1, 1 + N_2/2)$ .

*Proof.* Let us first consider the case where  $n$  is a prime power and show that (the case  $n = 2$  being trivial):

- (i) if  $n = p^e$ , where  $p$  is an odd prime, then  $m = \pm 1$ ;
- (ii) if  $n = 2^e$  and  $e \geq 2$ , then either  $m = \pm 1$  or, in case  $e \geq 3$ , we can also have  $m = \pm 1 + n/2$ .

Indeed in both cases, (i) and (ii),  $m \equiv \pm 1 \pmod{p}$ , where  $\pm 1$  is the case (i), let  $p^h$  the biggest power of  $p$  that divides  $m - \pm 1$ . Then,  $1 = m^2 = (\pm 1 + ap^h)^2 = 1 + 2\pm ap^h + a^2 p^{2h}$ , hence  $h = e$  since  $p > 2$ .

In case (ii), it follows that  $h \geq e - 1$ , hence either  $h = e$  and  $m = \pm 1$ , or  $h = e - 1$  and  $m = \pm 1 + 2^{e-1}$ .

<sup>1</sup>It appears esoteric only because we wish to find explicit algebraic formulae for the field extension, instead of using the real version, Theorem 0.1, of Riemann's existence theorem of [G-R58]

In the general case where  $n > 1$  is any natural number, it suffices to use the primary decomposition of  $\mathbb{Z}/n$  and put together all the primary summands where  $m$  equals respectively  $1, -1, \pm 1 + 2^{k-1}$ . Here  $k = e$ , if  $n = n_1 \cdot n_2 \cdot 2^e$  with  $e \geq 3$  and the summand of  $m$  belonging to  $\mathbb{Z}/2^e$  is equal to  $1 + 2^{e-1}$  (notice that, if  $e = 2, 1 + 2 = 1$ ). This shows (1), (2) and (3).

Finally, to prove (4), (5) and (6) observe that, if the third summand is nontrivial and  $m = (1, -1, 1 + 2^{k-1})$ , then, setting  $N_2 := n_1 \cdot 2^k$ , we have that  $1 + n_1 \cdot 2^{k-1}$  is congruent to 1 modulo  $n_1$  and congruent to  $1 + 2^{k-1}$  modulo  $2^k$ . The other case where  $m = (1, 1, -1 + 2^{k-1})$ , that is,  $m = 1 + n_2 2^{k-1}$ , is entirely similar.

Q

The previous lemma says therefore that there are seven possibilities: three contemplated in (4) (namely  $N_1 = 1$ , or  $N_2 = 1$ , or  $N_1 \neq 1 \wedge N_2$ ) and respectively two for both (5) and (6) (namely,  $N_1 = 1$ , or  $N_1 \neq 1$ , since here we must have  $N_2 = 1$ ). Of these four are ‘pure’ (unmixed cases), and three are direct products of two pure cases.

The way the lemma shall be applied is through the following well-known proposition; when stating our results we shall sometimes take this condition for granted (it distinguishes between irreducible and reducible covers).

*Proposition 2.2. Let  $K$  be a field of characteristic zero and containing all roots of 1.*

*Then  $L := K[z]/(z^n - f)$  is a field if and only if there is no divisor  $h$  of  $n$ , with  $h \geq 2$ , and no element  $a$  of  $K$  such that  $a^h = f$  (equivalently, the same condition with  $h$  a prime number).*

*Assume now that  $L := K[z]/(z^n - f)$  and  $L' := K[x]/(x^m - \varphi)$  are fields, and that  $n, m$  are relatively prime. Then  $L \otimes_K L'$  is a field extension of  $K$ , Galois with Galois group cyclic of order  $nm$ .*

*Proof.* For the first assertion, assume that such an element  $a$  exists and set  $n = hk$ : then  $z^n - f = z^{hk} - a^h$ , which is divisible by  $z^k - a$ .

Conversely, assume that  $F := z^n - f$  is not irreducible in  $K[z]$ . We have an action of the group  $G$  of  $n$ -th roots of 1 on  $K[z]$ ,  $z \mapsto \zeta z$ . Since  $F$  is  $G$ -invariant,  $G$  permutes the irreducible factors of  $F$ . If  $F$  has a linear factor  $z - a$ , we have that  $a^n = f$ . Otherwise, the orbit of a factor  $P$  of minimal degree has cardinality  $< n$ , hence we have a factor  $P$  stabilized by a subgroup of  $G$ , generated by some power  $\zeta^h$ , where  $h \geq 2$  divides  $n$ , and equal to the group of  $k$ -th roots of 1 ( $n = kh$ ).

Therefore this factor is of the form

$$P(z) = \sum_j b_j z^{kj},$$

and, setting  $w := z^k$ , we get that  $Q(w) := \sum_j b_j w^j$  divides  $w^h - f$ .

Now,  $Q$  must be linear, otherwise it would have a nontrivial stabilizer in the group of  $h$ -th roots of 1, contradicting that the stabilizer of  $P$  has order exactly  $k$ . Since  $Q$  is linear, it is of the form  $w - a$ , hence  $a^h = f$ .

For the second assertion, observe that

$$L \otimes_K L' = K[x, z]/(z^n - f, x^m - \varphi)$$

admits an action by the group  $\mathbb{Z}/n\mathbb{Z}/m = \mathbb{Z}/nm$  (since  $n, m$  are relatively prime) which, given  $\zeta$  a primitive  $n$ -th root of 1, and  $\varrho$  a primitive  $m$ -th root of 1, sends  $z \mapsto \zeta^i z$ ,  $x \mapsto \varrho^j x$ . Hence  $xz$  is an eigenvector for the standard generator of  $\mathbb{Z}/nm$ , in particular it generates the  $K$ -algebra  $L \otimes L'$ , satisfying indeed the relation  $(zx)^{nm} = f^m \varphi^n$ .

By the first assertion, if  $L \otimes L'$  is not a field, there exists  $a \in K$  such that  $a^h = f^m \varphi^n$ , where  $h \geq 2$  divides  $nm$ .

Without loss of generality we may assume that  $h$  is prime, and that it divides  $m$ , so that  $m = hk$ .

Since  $a^h = f^{hk} \varphi^n$ , it follows that, setting  $b := a \cdot f^{-k}$ ,  $b^h = \varphi^n$ .

However, since  $n, m$  are relatively prime, there exist integers  $r, s$  such that  $1 = hr + ns$ .

We finally derive:

$$\varphi = \varphi^{hr+ns} = (\varphi^r b^s)^h =: c^h$$

and since  $h|m$ , we have contradicted that  $L'$  is a field, because of the first assertion of this proposition. Q

**Remark 2.3.** In the case where  $K = C(X)$  is the function field of a factorial variety  $X$ , the condition of existence of  $a$  with  $f = a^h$  can be verified once we write  $f = \frac{\xi}{\zeta}$  as the quotient of two relatively prime sections of a line

bundle  $L$ , and we then take the unique factorization of  $f$ :

$$f = \frac{\prod_i s_i^{n_i}}{\prod_j t_j^{m_j}},$$

where the  $s_i, t_j$  are prime.

$h$  should divide the greatest common divisor  $GCD(n_i, m_j)$  of all the exponents  $n_i, m_j$ . We shall often omit to specify this condition each time.

More generally, we can consider the situation where  $G$  is a finite Abelian group and we have a semidirect product

$$G' = G \rtimes (\mathbb{Z}/2),$$

where the semidirect product is determined by an automorphism

$$M \in \text{Aut}(G), \text{ such that } M^2 = 1 := \text{Id}_G.$$

Using the primary decomposition of  $G$ ,  $G = \bigoplus_p G_p$ , where  $G_p$  is the  $p$ -primary component, it suffices to describe  $M_p : G_p \rightarrow G_p$ .

$G_p$  is isomorphic to a direct sum

$$G_p = \bigoplus_r (\mathbb{Z}/(p^r))^{n_r}.$$

**Lemma 2.4.** *Let  $G_p = \bigoplus_r (\mathbb{Z}/(p^r))^{n_r}$  be a finite Abelian group of exponent a power of  $p$ , where  $p$  is an odd prime, and  $M_p : G_p \rightarrow G_p$  with  $M^2 = 1$ ; then we have a splitting  $G_p = G^+ \oplus G^-$  into  $+1$  and  $-1$  eigenspaces.*

*Proof.* As usual,  $G^+ = \left\{ \frac{1}{2}(x + M_p x) \right\}$ ,  $G^- = \left\{ \frac{1}{2}(x - M_p x) \right\}$ . Q

The following Lemma applies in particular to the case of a finite Abelian group of exponent  $2^s$ ,  $G_2 = \bigoplus_r (\mathbb{Z}/(2^r))^{n_r}$ , endowed with an automorphism  $M_2 : G_2 \rightarrow G_2$  with  $M_2^2 = 1$ :



Lemma 2.5. Let  $G$  be a finite Abelian group endowed with an automorphism  $M : G \rightarrow G$  such that  $M^2 = 1$ .

Let

$$G^+ := \text{Ker}(M - 1), G^- := \text{Ker}(M + 1)$$

be the respective eigenspaces for the eigenvalues  $+1, -1$ , and define  $U := G^+ \cap G^-, W = G^+ + G^-$ .

Then

(1)  $U = G^+_{[2]} = G^-_{[2]}$ , where  $G^\pm$  denotes the subgroup of 2-torsion elements

of  $G^\pm$ , and we have a canonical isomorphism  $W \cong (G^+ \oplus G^-)/U$ ;

(2)  $V := G/W$  is a vector space over  $\mathbb{Z}/2$ , with an induced action by  $M$ , which is the identity;

(3) for  $x \in G$ ,  $M(x) = x + F(x)$ , where  $F : G \rightarrow G^-$  vanishes on  $G^+$ , and equals  $2x$  for  $x \in G^-$ .

(4) For fixed  $G, G^+, G^-$ , we have a set of possible homomorphisms  $M$  with  $M^2 = 1$  and such that  $G^+$  is contained in the  $(+1)$ -eigenspace for  $M$ , and  $G^-$  is contained in the  $(-1)$ -eigenspace for  $M$ , parametrized by the set  $\text{Hom}(V, U)$ .

(5) Take now an Abelian group  $G$  containing subgroups  $W, G^-, G^+$  such that

(I)  $W = G^- + G^+$ ,

(II)  $W \cong (G^- \oplus G^+)/U$ , where  $U = G^+_{[2]} = G^-_{[2]}$

(III)  $G/W \cong V$  is a vector space over  $\mathbb{Z}/2$ .

Let  $F_W : W \rightarrow G^-$  be defined as in (3).

Then there exists an extension  $F : G \rightarrow G^-$  of  $F_W$ , and the set of automorphisms of  $G$  with  $M^2 = 1$  such that  $G^+$  equals the  $(+1)$ -eigenspace for  $M$ ,  $G^-$  equals the  $(-1)$ -eigenspace for  $M$ , is in bijection with all those extensions  $F' = F + \varphi$ ,  $\varphi \in \text{Hom}(V, U)$ , satisfying the further properties:  $G^+ = \text{Ker}(F')$ ,  $G^- = \text{Ker}(F' + 2)$ .

*Proof.* Let us first consider the baby case where  $G$  is a vector space over  $\mathbb{Z}/2$ .

Since  $M : G \rightarrow G$  satisfies  $M^2 = 1$ , we get a Jordan normal form for  $G$  with eigenvalues  $1$  and blocks of length  $\leq 2$ , that is, there is a direct sum  $G = G^+ \oplus G'$  such that  $M(v^+, v') = (v^+ + f(v'), v')$ , with  $f : G' \rightarrow G^+$  linear and injective. Observing that  $G' \cong G/G^+$ , we have established that we have an injective homomorphism  $f : G/G^+ \rightarrow G^+$  such, that for  $y \in G$ , letting  $p_1 : G \rightarrow G/G^+$  be the natural surjection, then  $M(y) = y + f(p_1(y))$ .

In general, since  $M : G \rightarrow G$  satisfies  $M^2 = 1$ , we get two filtrations by submodules

$$0 \subset \text{Im}(M - 1) \subset \text{Ker}(M + 1) =: G^- \subset G = \text{Ker}(M^2 - 1),$$

$$0 \subset \text{Im}(M + 1) \subset \text{Ker}(M - 1) =: G^+ \subset G = \text{Ker}(M^2 - 1),$$

which are invariant since  $M$  acts as  $-1$  on  $G^-$ , and as  $+1$  on  $G^+$ .

Observe then that  $U := G^+ \cap G^- = \{y \mid 2y = 0, My = y\} \subset G_{[2]}$ , hence (1) follows immediately.

Moreover,  $2y = (M+1)y - (M-1)y$  shows that  $G/W$  has exponent 2, hence it is a vector space  $V$  over  $\mathbb{Z}/2$ . As we noticed,  $\text{Im}(M - 1) \subset G^- \subset W$ , hence  $(M - 1) : G/W \rightarrow G/W$  is  $= 0$ , hence (2) holds.

Write  $M(x) = x + F(x)$ .

Here  $F = M - 1$  vanishes on  $G^+$ , and equals  $-2x$  for  $x \in G^-$ , hence (3) is established and  $F|_W$  is canonically determined.

Now,  $F : G \rightarrow G^-$  is an extension of  $F|_W$ .

Observe that  $G$  is an extension:

$$0 \rightarrow W \rightarrow G \rightarrow V \rightarrow 0.$$

Hence we have an exact sequence

$$0 \rightarrow \text{Hom}(V, G^-) \rightarrow \text{Hom}(G, G^-) \rightarrow \text{Hom}(W, G^-) \rightarrow \text{Ext}^1(V, G^-).$$

Since  $V$  has exponent 2, and  $F|_W$  is determined, the choice of an extension  $F$  of  $F|_W$  is determined up to a homomorphism  $\varphi \in \text{Hom}(V, U)$ .

$\varphi$  can be taken arbitrarily, since  $M$  acts as the identity on  $V$ , hence also  $(M + \varphi)^2 = 1$ : hence (4) holds.

Take now an Abelian group  $G$  containing subgroups  $W, G^-, G^+$  as in (5) such that  $W \cong (G^- \oplus G^+)/U$ , where  $U = G_{[2]}^+ = G_{[2]}^-$ .

Take  $F|_W$  as above: the question is whether there exists  $F : G \rightarrow G^-$  extending  $F|_W$ . If the answer is positive, then we define  $M := F + 1$ , and clearly  $M^2 = 1$ . It suffices to see that  $F|_W$  is sent to 0 in  $\text{Ext}^1(V, G^-)$ , because then there exists  $F \in \text{Hom}(G, G^-)$  extending  $F|_W$ .

Now, the coboundary map of the exact sequence is given via the natural pairing

$$(\text{Ext}^1(V, W) \times \text{Hom}(W, G^-)) \rightarrow \text{Ext}^1(V, G^-)$$

by pairing with the extension class of  $0 \rightarrow W \rightarrow G \rightarrow V \rightarrow 0$  (an element of  $\text{Ext}^1(V, W)$ ).

Observe also that  $\text{Ext}^1(V, W)$ , since  $V$  is a vector space over  $\mathbb{Z}/2$ , equals

$$V \otimes_{\mathbb{Z}/2} \text{Ext}^1(\mathbb{Z}/2, W) = V \otimes_{\mathbb{Z}/2} (W/2W),$$

similarly  $\text{Ext}^1(V, G^-)$  equals

$$V \otimes_{\mathbb{Z}/2} \text{Ext}^1(\mathbb{Z}/2, G^-) = V \otimes_{\mathbb{Z}/2} (G^-/2G^-),$$

Now, we have that  $F|_W$  sends  $W \rightarrow 2G^-$ , hence  $F|_W$  maps to 0 and there exists  $F \in \text{Hom}(G, G^-)$  extending  $F|_W$ .

Finally,  $M(x) = x \Leftrightarrow F(x) = 0$ , and  $M(x) = -x \Leftrightarrow F(x) = -2x$ .

This said, we have that  $G^+$  is just contained in the (+1)-eigenspace, and similarly  $G^-$  is just contained in the (-1)-eigenspace. Equality holds if  $G^+ = \text{Ker}(F)$ ,  $G^- = \text{Ker}(F + 2)$  Q

**Remark 2.6.** (i) The case where  $G$  is of exponent 2 (a vector space over  $\mathbb{Z}/2$ ), and  $U = W = G^+$  shows that the 2-torsion subgroup of  $G$  can be strictly larger than  $U$ .

(ii) The same example shows that, fixed  $G^+$ , the (+1)-eigenspace for  $M$  contains  $G^+$  and it is equal to it if and only if  $f : V \rightarrow U$  is injective. If we change  $f$  with  $f + \varphi$ , and the latter is not injective, then (+1)-Eigenspace becomes larger, it is equal to the inverse image of  $\text{Ker}(f + \varphi) \subset V$  under the projection  $\rho : G \rightarrow V$ . Likewise also  $U$  becomes larger.

(iii) An easy issue of Lemma 2.5 is the one where  $G$  is any group, and  $M = \pm 1 + \phi$ , where  $\phi : G \rightarrow G_{[2]}$  is a homomorphism vanishing on  $G_{[2]}$ . Let  $\varrho \in \{+1, -1\}$ , so that  $M(x) = \varrho x + \phi(x)$ . Then

$$M^2(x) = \varrho(\varrho x + \phi(x)) + \phi(\varrho x + \phi(x)) = x + \phi(\phi(x)) = x.$$

In the case  $\varrho = 1$ ,  $G^+ = \text{Ker}(\phi) \supset G_{[2]}$ ,  $G^- = \text{Ker}(\phi - 2)$ , and

$$U = \text{Ker}(\phi) \cap \text{Ker}(2) = G_{[2]}.$$

### 3. THE FUNCTION FIELD EXTENSIONS IN THE FOUR BASIC CASES

We begin with a very simple observation.

**Remark 3.1.** Let  $\mathbb{R}(X')$  be a finitely generated extension field of  $\mathbb{R}$  which does not contain  $-1$ , let  $\Phi \in \mathbb{R}(X')$  be not a square, and consider the field extension  $\mathbb{R}(X^0)$  given by  $y^2 = \Phi$ . Then there are two real structures on  $\mathbb{R}(X^0)$ : for one of them (the one we shall choose)  $\tau(y) = y$ , for the other  $\tau'(y) = -y$  ( $\tau' = \iota \circ \tau$ , where  $\iota$  is the Galois involution  $y \mapsto -y$ ).

3.1. The case where  $G'$  is a dihedral group. We have here  $\tau g \tau = g^{-1}$ . Let us assume that the cyclic field extension is generated by  $w$ , such that  $w^n = f$ ,  $f \in \mathbb{C}(X')$ .

If  $g$  is a generator of  $G$ , we have

$$g(w) = \zeta w,$$

where  $\zeta$  is a primitive  $n$ -th root of 1.

Hence

$$g(\tau(w)) = \tau(g^{-1}(w)) = \tau(\zeta^{-1}w) = \zeta\tau(w),$$

so that  $\tau(w)$  is also an eigenvector for  $g$  with eigenvalue  $\zeta$  (in particular there exists  $A \in \mathbb{C}(X')$  with  $\tau(w) = Aw$ ).

Let  $V_\zeta$  be the eigenspace for the eigenvalue  $\zeta$ : then any nonzero element  $z$  inside  $V_\zeta$  generates the extension  $\mathbb{C}(X') \subset \mathbb{C}(X)$ .

If there is now a  $w \in V_\zeta$  such that  $z := \tau(w) + w \neq 0$ , then  $z \in \mathbb{R}(X)$  generates the extension and  $z^n = f \in \mathbb{R}(X')$ , as we want to show.

Otherwise,  $\tau$  would act as  $-1$  on  $V_\zeta$ . This assumption leads to a contradiction, since  $V_\zeta$  is a complex vector space, and we would have  $\tau(\lambda w) = \lambda(-w) \neq -\lambda w$  if  $\lambda$  is not a real number.

3.2. The case where  $G'$  is a direct product  $G' = G \times \langle \tau \rangle$ . We have here  $\tau g \tau = g$ .

A first observation is that  $g$  preserves the two eigenspaces for  $\tau$ , therefore  $g(\mathbb{R}(X)) = \mathbb{R}(X)$ , and  $g(-\mathbb{R}(X)) = -\mathbb{R}(X)$ .

Assume that the field extension is generated by  $z$ , such that  $z^n = f$ ,  $f \in \mathbb{C}(X')$ .

If  $g$  is a generator of  $G$ , we have  $g(z) = \zeta z$ , where  $\zeta$  is a primitive  $n$ -th root of 1.

Hence

$$g(\tau(z)) = \tau(g(z)) = \tau(\zeta z) = \zeta^{-1}\tau(z),$$

so that  $\tau$  exchanges the two eigenspaces  $V_\zeta$  and  $V_{\zeta^{-1}}$ .

Hence we have

$$\varphi := z \cdot \tau(z) \in R(X').$$

Moreover, if  $z = a + \sqrt{-1}b$ ,  $a, b \in R(X)$ , then  $\varphi = a^2 + b^2$ , a function which on the real locus takes real positive values.

If we now set  $w := \tau(z)$ , we have  $w^n = \tau(f)$  and, setting

$$F := f + \tau(f) \Rightarrow F \in R(X'),$$

we have

$$(*) \quad zw = \varphi, \quad z^n + w^n = F.$$

It follows that  $f, \tau(f)$  are roots of the quadratic equation

$$x^2 - Fx + \varphi^n = 0.$$

Let  $\psi \in R(X')$  be twice the imaginary part of  $f$ : then

$$f = \frac{1}{2}(F + \sqrt{F^2 - 4\varphi^n}), \quad \tau(f) = \frac{1}{2}(F - \sqrt{F^2 - 4\varphi^n})$$

and the discriminant  $\Delta$  is minus a square in  $R(X')$ , since

$$(**) \quad \Delta = F^2 - 4\varphi^n = -\psi^2.$$

Conversely, if there are  $F, \varphi, \psi \in R(X')$  satisfying  $(*)$ , then the equations  $(*)$  define a cyclic extension of  $C(X')$  of order  $n$ : since we can eliminate  $w = \varphi/z$ , and for the resulting equation

$$z^{2n} - Fz^n + \varphi^n = 0 \Leftrightarrow (z^n - \frac{1}{2}(F + \sqrt{F^2 - 4\varphi^n}))(z^n - \frac{1}{2}(F - \sqrt{F^2 - 4\varphi^n})) = 0$$

we can take the root  $z$  of  $z^n = f := \frac{1}{2}(F + \sqrt{F^2 - 4\varphi^n})$  or the root  $w$  of  $w^n =$

$$\tau(f) = \frac{1}{2}(F - \sqrt{F^2 - 4\varphi^n}).$$

If  $f := \frac{1}{2}(F + \sqrt{F^2 - 4\varphi^n})$  satisfies the property that there do not exist any divisor of  $n, h > 1$ , and a rational function  $\phi = \alpha + \sqrt{-1}\beta$  with  $(\alpha + \sqrt{-1}\beta)^h = f$ , then Proposition 2.2 applies and the cyclic extension is a field (cf. remark 2.3).

Exchanging  $\psi$  with  $-\psi$  has the effect of replacing  $z$  with  $w$ .

We can then extend  $\tau$  to  $C(X)$  by setting  $w := \tau(z) \Rightarrow z = \tau(w)$ .

Finally, concerning the existence of such functions  $F, \varphi, \psi$  we simply use the rationality of the variety defined by  $(**)$ .

Namely, let  $n = 2m + 1$  for  $n$  odd, else let  $n = 2m + 2$ . Setting

$$F = \varphi^m \hat{F}, \quad \psi = \varphi^m \hat{\psi},$$

we reduce to the respective equations

$$\varphi = \frac{1}{4}(\hat{F}^2 + \hat{\psi}^2), \quad 4\varphi^2 = \hat{F}^2 + \hat{\psi}^2.$$

In the first case, where  $n = 2m + 1$ , we see that the solutions correspond to the choice of two arbitrary functions  $\hat{F}, \hat{\psi} \in R(X')$  such that  $(\hat{F}^2 + \hat{\psi}^2)^m (\hat{F} + \sqrt{-1}\hat{\psi})$  is not of the form  $\phi^h$  for any divisor  $h$  of  $n = 2m + 1$ . Since we have  $4^m (F + \sqrt{-1}\psi) = (\hat{F}^2 + \hat{\psi}^2)^m (\hat{F} + \sqrt{-1}\hat{\psi}) = (\hat{F} + \sqrt{-1}\hat{\psi})^{m+1} (\hat{F} - \sqrt{-1}\hat{\psi})^m$ ,

we conclude that, if  $F + \sqrt{-1}\psi = \phi^h$ , then

$$4^m \phi^h = (\hat{F} + \sqrt{-1}\hat{\psi})^{m+1} (\hat{F} - \sqrt{-1}\hat{\psi})^m.$$

We can for instance show that  $h = 1$  in the following situation: if  $X'$  is factorial, any rational function  $f$  can be written uniquely as  $f = \frac{s}{t}$ , where  $s, t$  are relatively prime sections of a line bundle  $L$ . We shall say that

- $f$  is irreducible if either  $s$  or  $t$  is prime,
- $f$  is strongly irreducible if both  $s$  and  $t$  are prime,
- $F = \frac{A}{B}$  is not associated to  $f = \frac{\hat{F}}{t}$  if  $A$  is relatively prime to  $s$  and  $B$  is relatively prime to  $t$ .

Now, if  $(\hat{F} + \sqrt{-1}\psi) = \frac{s}{t}$ ,  $(\hat{F} - \sqrt{-1}\psi) = \frac{A}{B}$  are irreducible and not associated, then write  $\phi = \frac{a}{b}$ . We obtain that

$$a^h = s^{m+1}A^m, b^h = t^{m+1}B^m.$$

If  $s$  is prime, then the first equality implies that  $h$  divides  $m + 1$ , and if  $t$  is prime, then the second equality also implies  $h|(m + 1)$ . Similarly, if  $A$  is prime, then  $h|n$ , and the same conclusion holds if  $B$  is prime. Hence  $h$  divides  $m, m + 1$ , hence  $h = 1$ .

In the second case, where  $n = 2m + 2$ , we simply have to parametrize the quadric: setting

$$\varphi = 1 + \lambda\tilde{\varphi}, \hat{F} = 2, \hat{\psi} = \lambda\tilde{\psi}$$

we get

$$8\tilde{\varphi} + 4\lambda\tilde{\varphi}^2 = \lambda\tilde{\psi}^2 \Leftrightarrow \lambda = \frac{8\tilde{\varphi}}{\tilde{\psi}^2 - 4\tilde{\varphi}^2},$$

hence the solutions correspond to the choice of two arbitrary functions  $\tilde{\varphi}, \tilde{\psi}$  satisfying a similar condition to the above one for the case where  $n$  is odd. Indeed in this case, up to constants,

$$F + \sqrt{-1}\psi = (1 + \lambda\tilde{\varphi})^m (2 + \sqrt{-1}\lambda\tilde{\psi}).$$

A similar argument shows that, if  $1 + \lambda\tilde{\varphi}, 2 + \sqrt{-1}\lambda\tilde{\psi}$  are irreducible and not associated, then  $h = 1$ .

3.3. The case where  $\tau g \tau = g^{1+n/2}$  (and  $n$  is divisible by 8). The first basic observation here is that  $G'$  contains the index 2 subgroup  $G'^{\text{ev}} := (2\mathbb{Z}/n\mathbb{Z}) \ltimes (\mathbb{Z}/2) =: G^{\text{ev}} (\mathbb{Z}/2)$ , generated by  $g^2$  and by  $\tau$ .

Hence we have a sequence of field extensions

$$C(X') \subset C(X^0) \subset C(X),$$

where  $X^0 := X/G^{\text{ev}}$ . The extension  $C(X^0) \subset C(X)$  is dihedral-like, since  $\tau g^2 \tau = g^{2(1+n/2)} = g^2$ , thus it has the description given in the previous subsection.

For convenience set  $n = 2N$ , and recall that  $N$  is even. Let

$$C(X) = C(X')[z]/(z^N - P), g(z) = \zeta z.$$

Then the extension  $C(X^0) \subset C(X)$  is given as above by ( $w := \tau(z)$ )

$$(*) zw = \varphi, z^N + w^N = F = f + \tau(f), f^2 = P,$$

$$(**) F^2 - 4\varphi^N = -\psi^2.$$

From  $g\tau = \tau g^{1+n/2}$  follows that

$$g(w) = g\tau(z) = \tau g^{1+n/2}(z) = \tau(-\zeta z) = -\zeta^{-1}w,$$

hence  $g(\varphi) = g(zw) = -zw = -\varphi$ , and  $\varphi^2$  is  $G'$  invariant, so that

$$\varphi^2 = \Phi \in R(X).$$

Moreover  $g(F) = -z^N - w^N = -F$ ; hence  $\varphi$  and  $F$  generate the degree two extension  $R(X') \subset R(X^0)$  and there exists therefore  $A \in R(X')$  such that  $F = A\varphi$ . We can rewrite now condition (\*\*) as:

$$A^2 - 4\Phi^{N/2-1} = -(\psi)^2/\Phi.$$

Observe that, since  $g(z^N) = -z^N$ ,  $\psi \in R(X^0)$  satisfies  $g(\psi) = -\psi$ , and in particular  $\psi^2 = \Psi \in R(X')$ .

Moreover, since  $\varphi, \psi$  generate the quadratic extension  $R(X') \subset R(X^0)$  there exists therefore  $B \in R(X')$  such that  $\psi = B\varphi$ .

Hence we can rewrite (\*\*) as

$$(***) A^2 - 4\Phi^{N/2-1} = -B^2.$$

Conversely, given (\*\*\*), we define  $\psi, \varphi$  by  $\psi^2 = \Psi$ , respectively  $\varphi^2 = \Phi$ , set  $F := A\varphi$  and observe that these generate a quadratic extension which we denote  $R(X') \subset R(X^0)$ .

Then we use (\*) to get a cyclic extension  $R(X^0) \subset R(X)$ .

Since  $g(F) = -F$ ,  $g(\varphi) = -\varphi$ , we can extend  $g$  to  $R(X)$  setting  $g(z) := \zeta z$ ,  $g(w) := \zeta^{-1}w$ , and  $\tau$  can be extended setting  $\tau(z) := w$ .

The datum of such functions  $A, \Phi, B$  satisfying (\*\*\*) is shown, exactly as in the previous subsection, to be equivalent to the datum of two arbitrary functions in  $R(X')$ .

3.4. The case where  $\tau g \tau = g^{-1+n/2}$  (and  $n$  is divisible by 8). Again a basic observation here is that  $G'$  contains the index 2 subgroup  $G'^{\text{ev}} := (2\mathbb{Z}/n\mathbb{Z}) \rtimes (\mathbb{Z}/2)$ , generated by  $g^2$  and by  $\tau$ .

Hence we have a sequence of field extensions

$$C(X') \subset C(X^0) \subset C(X),$$

where  $X^0 := X/G'^{\text{ev}}$ . The extension  $C(X^0) \subset C(X)$  is of standard type, since  $\tau g^2 \tau = g^{2(-1+n/2)} = g^{-2}$  so it has the description given in the first subsection (it is generated by a real element which is an eigenvector for  $g^2$  with eigenvalue  $\zeta^2$ ).

Set again  $n = 2N$ , where  $N$  is even.

Pick as usual a generator  $z$  of the extension such that

$$z^n = z^{2N} = P \in C(X'), \quad g(z) = \zeta z.$$

We have  $g^2 \tau(z) = \tau g^{-2}(z) = \zeta^2 \tau(z)$ , and we set

$$W := z + \tau(z) \Rightarrow W \in R(X).$$

If  $W = 0$ , then  $z = -\tau(z)$  and it follows that

$$-\zeta z = g(-z) = g\tau(z) = \tau g^{-1+n/2}(z) = -\zeta \tau(z) = \zeta z,$$

a contradiction.

Hence  $0 \neq W \in R(X)$  and it generates the cyclic extension  $R(X^0) \subset R(X)$ , so that we may write  $W^N = f \in R(X^0)$ .

We can write  $2z = W + u$ , where  $u := z - \tau(z) \in \sqrt{-1}R(X)$ .

The action of  $g$  is as follows:  $g\tau(z) = \tau g^{-1+n/2}(z) = -\zeta\tau(z)$ , hence

$$g(W) = \zeta z - \zeta\tau(z) = \zeta u,$$

$$g(u) = g(z - \tau(z)) = \zeta(z + \tau(z)) = \zeta W.$$

Since  $W^N = f \in R(X^0)$ ,  $g(f) = g(W)^N = -u^N \in R(X^0)$ , hence

$$u^N = h := -g(f).$$

We have  $g(f) = -h, g(h) = -f$  hence

$$y := f + h \Rightarrow y \in R(X^0), g(y) = -y \Rightarrow y^2 = \Phi \in R(X).$$

Likewise  $g(fh) = fh \Rightarrow fh = \Psi \in R(X)$ .

Since every element  $f$  in  $R(X^0)$  can be written in the form  $f = \alpha y + \beta$ ,  $\alpha, \beta \in R(X)$ , the equations  $f + h = y, g(f) = -h, fh = \Psi$ , imply that

$$f = \frac{1}{2}y + \beta, h = \frac{1}{2}y - \beta, \Psi = \frac{1}{4}\Phi - \beta^2.$$

Up to now we have described the field extension as a cyclic extension  $W^N = f = \frac{1}{2}y + \beta$  of a quadratic extension  $y^2 = \Phi$ , and clearly  $g(y) = -y$ .

In this situation, the fact that the global extension is cyclic means that  $g$  extends to the larger field; indeed, we know that  $g(W) = \zeta u$ , where  $u^N = h = \frac{1}{2}y - \beta$ , hence the root  $u$  must be in the field extension; and since  $W, u$  are both in the  $\zeta^2$ -eigenspace for  $g^2$ , this condition amounts to:

$$(a) \exists C, D \in C(X) \text{ such that } u = W(C + yD).$$

Recall that  $\tau(W) = W, \tau(u) = -u$ : hence

$$\tau(C + yD) = -C - yD \Rightarrow C, D \in \sqrt{-1}R(X).$$

Furthermore,

$$g(W) = \zeta u, g(u) = \zeta W \Rightarrow W = ug(C + yD) \Rightarrow$$

$$(b) C^2 - \Phi D^2 = 1.$$

Condition (b) means that  $(C, D, 1)$  is a point of the conic  $X^2 - \Phi Y^2 - Z^2 = 0$ . Since the real conic has already the rational point  $(1, 0, 1)$ , we get a parametrization setting  $C = 1 + D\Theta$ ,  $\Theta \in C(X)$ , and therefore, since  $(1 + D\Theta)^2 - \Phi D^2 = 1$ , we obtain

$$D = \frac{2\Theta}{\Theta^2 - \Phi}, C = \frac{\Phi + \Theta^2}{\Theta^2 - \Phi}.$$

Multiplying numerator and denominator by  $(\tau(\Theta)^2 - \Phi)$ , our condition is that

$$2\Theta(\tau(\Theta)^2 - \Phi), (\Phi + \Theta^2)(\tau(\Theta)^2 - \Phi)$$

are both imaginary.

An easy solution is given by taking  $\Phi = \Theta \cdot \tau(\Theta)$ .

This is the only one, because, writing  $\Theta = A + \sqrt{-1}B$ , the first condition is that  $(A + \sqrt{-1}B)((A - \sqrt{-1}B)^2 - \Phi)$  is imaginary, equivalently,

$$A(A^2 + B^2 - \Phi) = 0.$$

The solution  $A = 0$  must be discarded since then  $C$  is real, a contradiction. Remains exactly the solution

$$A^2 + B^2 = \Phi \Leftrightarrow \Phi = \Theta \cdot \tau(\Theta),$$

and then

$$(Theta) : \Phi = \Theta \cdot \tau(\Theta), \quad D = -\frac{2}{\Theta - \tau(\Theta)}, \quad C = -\frac{\tau(\Theta) + \Theta}{\Theta - \tau(\Theta)}.$$

The condition that  $u$  is a root of  $u^N = h = \frac{1}{2}y - \beta$  is then expressed, using (a), as

$$(a') \quad \frac{1}{2}y - \beta = \left(\frac{1}{2}y + \beta\right)(C + yD)^N,$$

equivalently it can be expressed by requiring that the ratio

$$\frac{\frac{1}{2}y - \beta}{\frac{1}{2}y + \beta} = \frac{1}{\Psi} \left(\frac{1}{2}y - \beta\right)^2$$

is the  $N$ -th power of  $(C + yD)$ .

Observe that we wrote condition (a') as a condition in  $R(X^0)$ , but this is indeed equivalent to two conditions in  $R(X)$ , since we require that

$$(C+yD)^N = \sum_j \binom{N}{2j} C^{N-2j} D^{2j} \Phi^{j+y} \binom{N}{2j+1} C^{N-2j-1} D^{2j+1} \Phi^j =: M + M'y$$

equals

$$\frac{1}{\Psi} \left(\frac{1}{2}y - \beta\right)^2 = \frac{1}{\Psi} \left(\frac{1}{4}\Phi + \beta^2\right) - y \frac{1}{\Psi}\beta.$$

These equations can be rewritten as two equations for  $\beta$ , namely

$$M\Psi = M\left(\frac{1}{4}\Phi - \beta^2\right) = \frac{1}{4}\Phi + \beta^2, \quad M\Psi = M\left(\frac{1}{4}\Phi - \beta^2\right) = -\beta,$$

hence, using the first equation to eliminate  $\beta^2$  in the second one,

$$(a'') \quad \beta^2(1 + M) = \frac{1}{4}\Phi(M - 1),$$

$$(a''') \quad -\beta = \frac{1}{2}\Phi \frac{M'}{M + 1}.$$

Then  $\beta$  is determined by the equation (a'''), and  $\beta$  is real since,  $C, D$  being imaginary,  $M, M'$  are real; and we want then  $\beta$  to yield a solution of the equation (a''). This amounts to:

$$\left(\frac{1}{2}\Phi \frac{M'}{M + 1}\right)^2(1 + M) = \frac{1}{4}\Phi(M - 1) \Leftrightarrow \Phi(M')^2 = (M^2 - 1).$$

However, this last condition  $1 = (M^2 - \Phi(M')^2) = M^2 - y^2(M')^2$  is automatically true, since

$$M^2 - y^2(M')^2 = (M + M'y)(M - M'y) = (C + yD)^N (C - yD)^N = (C^2 - \Phi D^2)^N = 1.$$

The final conclusion is that also  $\beta$  is determined by  $\Theta \in C(X)$ .

Conversely, it is now easy to see that we get a real cyclic covering with group  $G = \mathbb{Z}/n = \mathbb{Z}/2N$ , and exhibiting the esoteric case  $\tau g \tau = g^{-1+n/2}$  provided that:

- we are given  $\Theta \in C(X)$ , such that
- $\Phi := \Theta \cdot \tau(\Theta)$  is not a square, and
- we are given  $C, D \in {}^{-1}R(X)$  such that condition (b)  $C^2 - \Phi D^2 = 1$  holds (hence  $C, D$  are determined by  $\Theta$  as in formula (Theta)).



In fact, letting  $y$  be defined by  $y^2 = \Phi$  and choosing (see remark 3.1) the real structure on  $C(X^0)$  such that  $\tau(y) = y$ , and letting  $\beta$  be determined by  $\Theta$  according to (a'''), so that condition (a') holds, the covering with group  $G = \mathbb{Z}/n = \mathbb{Z}/2N$  is defined by the extension

$$W^N = f := \frac{1}{2}y + \beta.$$

Defining then  $\tau(W) := W$ ,  $\tau(u) := -u$  we obtain a real cyclic covering exhibiting the esoteric case  $\tau g \tau = g^{-1+n/2}$ .

We shall summarize the above discussion in the following theorem 3.2:

**Theorem 3.2.** *Let  $R(X), R(X')$ , be two real function fields, that is, two finitely generated field extensions of  $\mathbb{R}$  such that  $C(X) := R(X) \otimes_{\mathbb{R}} \mathbb{C}$  and  $C(X')$  are fields with an antilinear automorphism ( $\tau, \tau'$  respectively) induced by complex conjugation on  $\mathbb{C}$ .*

*Then a real cyclic covering  $R(X') \subset R(X)$ , that is, an extension inducing a Galois extension  $C(X') \subset C(X)$  with Galois group  $G = \langle g \cong \mathbb{Z}/n$ , and such that  $\tau$  normalizes  $G$  (then conjugation is given via  $\tau g \tau = g^m, m^2 = 1 \in \mathbb{Z}$ ), is a fibre product of two such real cyclic coverings belonging to the following four basic types:*

- (1)  $m = -1$  if and only if we are in the Standard (totally real) case: there exist  $f \in R(X'), z \in R(X)$  such that

$$z^n = f, g(z) = \zeta z, \langle \zeta \rangle = \mu_n = \{\eta \mid \eta^n = 1\}.$$

- (2)  $m = +1$  if and only if we are in the Dihedral-like case: there exist

$$F, \varphi, \psi \in R(X'), \text{ such that } (**) 4\varphi^n = \psi^2 + F^2,$$

such that the extension is given as

$$C(X) = C(X')(z), \text{ and, for } w := \tau(z), (*) zw = \varphi, z^n + w^n = F.$$

The choice of such  $F, \varphi, \psi \in R(X')$  satisfying (\*\*) is equivalent to the choice of two arbitrary functions  $\tilde{\varphi}, \tilde{\psi} \in R(X')$ .

- (3)  $n$  is divisible by 8 and, defining  $N := \frac{n}{2}$ ,  $m = 1 + N$  if and only if we are in the Twisted case: there exist

$$A, B, \Phi \in R(X'), \text{ such that } (***) A^2 + B^2 = 4\Phi^{N/2-1}$$

and such that  $\varphi^2 = \Phi$  defines a real quadratic field extension  $R(X) \subset R(X^0)$ .

Then, setting  $F := A\varphi, \psi := B\varphi$ , the field extension  $C(X)$  is generated by  $z$  such that ( $w := \tau(z)$ )

$$zw = \varphi, z^N + w^N = F, g(z) = \zeta z, g(w) = -\zeta^{-1}w.$$

The choice of such functions  $A, B, \Phi \in R(X')$  satisfying the above conditions is equivalent to the choice of two arbitrary functions in  $R(X')$ .

- (4)  $n$  is divisible by 8 and, defining  $N := \frac{n}{2}$ ,  $m = -1 + N$  if and only if we are in the Esoteric case: there exist

$$\Phi, \beta \in R(X'), C, D \in i R(X') \text{ such that } (b) C^2 - \Phi D^2 = 1,$$

and moreover, if  $y^2 = \Phi$  defines a quadratic field extension  $R(X') \subset R(X^0) \ni y$ , we have:

$$(a) \quad \sqrt{2}y - \beta = (\sqrt{2}y + \beta)(C + yD) \quad .$$

Then, setting  $f := \frac{1}{2}y + \beta$ , the real extension  $R(X^0) \subset R(X)$  is generated by  $W$ , where

$$W^N = f, \quad \tau(W) = W, \quad g(W) = \zeta u, \quad u := W(C + yD), \quad \tau(u) = -u.$$

The choice of such functions satisfying the above conditions is equivalent to the choice of  $\Theta = A + \sqrt{-1}B \in C(X')$  such that  $\Phi := A^2 + B^2$  is not a square.

#### 4. BIREGULAR STRUCTURE OF THE COVERINGS

The main purpose of this section is to use the description of Abelian coverings through line bundles (invertible sheaves) and effective (branch) divisors  $D_g$ . If  $D$  is an effective divisor, then  $\sigma(D)$  is also an effective divisor. To see this, consider that, if  $(X, \sigma)$  is a real variety, then  $\tau$  acts on  $C(X)$ , in particular it acts on the group of Cartier divisors  $H^0(X, C(X)^*/O_X^*)$ . If a collection  $(U_\alpha, f_\alpha)$ , (here  $f_\alpha \in C(X)^*$ ) defines a Cartier divisor  $D$ , then the conjugate Cartier divisor is given by the collection  $(\sigma(U_\alpha), \tau(f_\alpha))$ . An invertible sheaf  $L = O_X(D)$  has the property that its space of sections  $H^0(X, L)$  is the space

$$\{\varphi \in C(X) \mid \varphi f_\alpha =: \varphi_\alpha \in O_X(U_\alpha)\},$$

hence

$$\tau : H^0(X, L) \rightarrow H^0(X, \tau(L)) = H^0(X, O_X(\sigma D)).$$

Given a cyclic covering of real varieties  $X \rightarrow X'$ , we can replace  $X'$  by a smooth real model  $Y$ , and replace  $X$  by the normalization of the fibre product of the cyclic covering with the resolution  $Y \rightarrow X'$ , and obtain that (as in [Cat12], whose notation we shall now adopt)

- (1)  $Y = X/G, G \cong \mathbb{Z}/n$
- (2)  $X$  is normal, real,
- (3)  $Y$  is real smooth, with  $\sigma'$  induced by  $\sigma$ ;
- (4)  $f : X \rightarrow Y$  is finite and flat.

More generally, we can relax assumption (3) and assume that  $X, Y$  are normal real projective varieties, and  $Y$  is moreover factorial.

By flatness we have a decomposition

$$f_*(O_X) = O_Y \oplus \bigoplus_{\chi \in G^* \setminus \{0\}} O_Y(-L_\chi),$$

with a notation that we now explain.

$C(X)$  is a cyclic Galois extension of  $C(Y)$ , and we denote now by

$$G \cong \mu_n := \{\zeta \in C \mid \zeta^n = 1\}$$

its Galois group, by  $G^*$  the group of characters:  $G^* := \text{Hom}(G, C^*)$ , and we observe that  $G^* \cong \mathbb{Z}/n$ , where the isomorphism  $G^* \xrightarrow{\cong} \mathbb{Z}/n$  associates

to  $\chi \in G^*$  the residue class  $j \in \mathbb{Z}/n$  such that  $\chi(\zeta) = \zeta^j$ , for all  $\zeta \in \mu_n$ .

As in [Cat12] we observe that for each character  $\chi$  of order  $n$  the extension is given by

$$C(X) = C(Y)(w), w^n = F(y) \in C(Y),$$

where  $w$  is a  $\chi$ -eigenvector.

Since  $Y$  is factorial,  $F$  admits a unique prime factorization as a fraction of pairwise relatively prime sections of line bundles, and we can write

$$w^n = \frac{\prod_i \sigma_i^{n_i}}{\prod_j \tau_j^{m_j}},$$

where the sections  $\sigma_i, \tau_j$  are prime.

Writing

$$n_i = N_i + nn'_i, \quad m_j = -M_j + nm'_j$$

with  $0 \leq N_i, M_j \leq n - 1$  and setting

$$z := w \cdot \prod_i \sigma_i^{N_i} \prod_j \tau_j^{M_j},$$

we get a rational section  $z$  of a line bundle on  $Y$  and we have

$$z^n = \prod_i \sigma_i^{N_i} \prod_j \tau_j^{M_j}.$$

Put together the prime factors which appear with the same exponent, obtaining:

$$z^n = \prod_{i=1}^n \delta_i.$$

Here each factor  $\delta_j$  is reduced, but not irreducible, and corresponds to a Cartier divisor that we shall denote  $D_j$ . Calculating the local monodromy around  $D_j$  is easily seen that  $D_j$  is exactly the divisorial part of the branch locus  $D := \sum_j D_j$  where the local monodromy is the  $j$ -th power of the

standard generator  $\gamma := e^{\frac{2\pi\sqrt{-1}}{n}}$  of  $G \cong \mu_n$ .

We write characters additively, in the sense that we view them as  $G^* = \text{Hom}(G, \mathbb{Z}/n)$ .

To  $\chi$  we associate the normal covering

$$Z_\chi := X/\ker(\chi).$$

A similar argument (see for instance [Cat12] page 285, section 1) shows now that we have a linear equivalence

$$(*) \quad nL_\chi \cong \sum_i [\chi(i)]D_i$$

where  $[r]$ , for  $r \in \mathbb{Z}/n$ , is the unique residue class in  $\{0, 1, \dots, n - 1\}$ .

We observe for further use the following formula:

$$(l) \quad [\chi(i)] + [\chi'(i)] = [(\chi + \chi')(i)] + \varrho_{\chi, \chi'}^i n,$$

which defines the numbers  $\varrho_{\chi, \chi'}^i \in \{0, 1\}$ .

The following theorem is in part a special case of the structure theorem for Abelian coverings due to Pardini ([Par91], see also [Come30]).

Theorem 4.1. *i) Given a factorial variety  $Y$ , the datum of a pair  $(X, \gamma)$  where  $X$  is a normal variety and  $\gamma$  is an automorphism of order  $n$  such that,  $G$  being the subgroup generated by  $\gamma$ , one has  $X/G \cong Y$ , is equivalent to the datum of reduced effective divisors  $D_1, \dots, D_{n-1}$  without common components, and of a divisor class  $L$  such that we have the following linear equivalence*

$$(*) \quad nL \equiv \sum_i iD_i$$

and moreover, setting  $h := \text{G.C.D}\{i \mid D_i \neq 0\}$ , either  $(**)$   $h = 1$  or the divisor class

$$(***) \quad L' := \frac{n}{h}L - \sum_i \frac{i}{h}D_i$$

has order precisely  $h$ .

*ii) If  $L$  is the geometric line bundle whose sheaf of regular sections is  $\mathcal{O}_Y(L)$ , then  $X$  is the normalization of the singular covering*

$$X' \subset L, X' := \{(y, z) \mid z^n = \sum_{i=1}^{n-1} \delta_i y^i\}.$$

$\underbrace{\hspace{1.5cm}}_{2\pi\sqrt{-1}}$

And  $\gamma$  acts by  $z \mapsto e^{2\pi\sqrt{-1}/n} z$ .

*iii) The scheme structure of  $X$  is explicitly given as*

$$X := \text{Spec}(\mathcal{O}_Y \oplus \bigoplus_{\chi \in G^* \setminus \{0\}} \mathcal{O}_Y(-L_\chi))$$

where the divisor classes  $L_\chi$  are recursively determined by  $L_1 := L$ , and by  $L_{\chi+\xi} \equiv L_\chi + L_\xi - \sum_i \delta_{\chi, \xi}^i D_i$ .

Finally the ring structure is given by the multiplication maps

$$\mathcal{O}_Y(-L_\chi) \times \mathcal{O}_Y(-L_\xi) \rightarrow \mathcal{O}_Y(-L_{\chi+\xi})$$

determined by the section

$$\sum_i \delta_i^{\chi, \xi} \in \mathcal{H}(\mathcal{O}(-L_\chi - L_\xi + L_{\chi+\xi}))_X$$

At this stage we want to consider the extra data coming from the real structure.

It is convenient to view  $X$  embedded in  $\bigoplus_{\chi \in G^* \setminus \{0\}} L_\chi$  and to write the ring structure via the fundamental equation

$$(***) \quad z_\chi \cdot z_\xi = z_{\chi+\xi} \sum_i \delta_i^{\chi, \xi},$$

where  $z_\chi$  is a fibre variable on the geometric line bundle  $L_\chi$  (it is the natural section on  $L_\chi$  of the pull back of  $\mathcal{O}_Y(-L_\chi)$ ) and where we use the convention that  $z_0 = 1$ .

The action of  $\tau$  on  $C(X)$ , hence also on the subfield  $C(Y)$ , induces, by the description that we recalled above of the building data of the cover, an action of  $\tau$  on the building data  $L_\chi, D_i, z_\chi, \delta_i$ .

For simplicity of calculations, we write from now on the characters as elements of  $Z/n$ .

From the relations  $\tau\gamma\tau = \gamma^m$ , and  $\gamma z_j = \zeta^j z_j$ , we obtain that

$$\gamma(\tau(z_j)) = \tau(\gamma^m(z_j)) = \zeta^{-mj}\tau(z_j),$$

hence  $\tau(z_j)$  is an eigenvector for  $\zeta^{-mj}$ , and  $\tau$  maps

$$L_j \rightarrow L_{-mj}.$$

Observe here that  $j = -mj$  if and only if  $j(m+1) = 0 \in Z/n$ ; in particular, if  $m = -1$ , all the line bundles are real and linearized. The line bundles corresponding to  $\{j | j(m+1) \neq 0 \in Z/n\}$  come in pairs, which are exchanged by  $\tau$ .

We look now at the branch divisors  $D_i$ . It is clear that the antiholomorphic map  $\sigma'$  carries the branch locus to itself, but we show now how the divisors  $D_i$  are permuted.

We know that the local monodromy around  $D_i$  is given by  $\gamma^i$ , and we take local coordinates  $(y, \delta_i)$  ( $y = y_1, \dots, y_{d-1}$ ), where  $d = \dim(Y)$ , at the general point of  $D_i$ . Then, in appropriate local coordinates,  $\sigma'$  is given by  $(y, \delta_i)$ , and since  $\sigma\gamma^i\sigma = \gamma^{mi}$ , and the orientation in the normal bundle is reversed by  $\sigma'$ , it follows that

$$\sigma'(D_i) = D_{-mi},$$

in particular  $\tau(\delta_i) = \delta_{-mi}$  and  $D_i$  is real if and only if  $i = -mi$ .

Applying now  $\tau$  to the fundamental equation (\*\*\*) we obtain, since  $\tau(z_j) = z_{-mj}$ , that the real structure extends to the covering if and only if

$$z_{-mj} \cdot z_{-mh} = z_{-m(j+h)} \prod_k \delta_{-mk}^{\sigma_{j,h}^k},$$

and since

$$z_{-mj} \cdot z_{-mh} = z_{-m(j+h)} \prod_i \delta_i^{i_{-mj,-mh}},$$

this is equivalent to requiring that

$$\prod_i \delta_i^{i_{-mj,-mh}} = \prod_i \delta_i^{\bar{\sigma}_{j,h}^{mi}},$$

a property which clearly holds true by virtue of (I) (just observe that  $-mj(i) = j(-mi), -mh(i) = h(-mi)$ ).

It is now easy to derive the real version of the biregular theory of cyclic coverings.

**Theorem 4.2.** *i) Given a real factorial variety  $(Y, \tau_Y)$ , the datum of a real finite cyclic covering  $(X, \tau_X) \rightarrow (Y, \tau_Y)$ , where  $X$  is a normal variety and the group  $G \cong Z/n$  with  $X/G \cong Y$  is generated by an automorphism  $\gamma$  of order  $n$ , is equivalent to:*

- (1) *the datum of an element  $m \in Z/n$  with  $m^2 = 1$ ,*
- (2) *the datum of reduced effective divisors  $D_1 = \text{div}(\delta_1), \dots, D_{n-1} = \text{div}(\delta_{n-1})$  without common components, and such that*

$$\tau_Y(\delta_i) = \delta_{-mi}, \quad \text{for } i = 1, \dots, n-1,$$

- (3) and the datum of a divisor class  $L$  such that we have the following linear equivalence

$$(*) \quad nL \equiv \sum_i iD_i.$$

The above data should satisfy the following conditions:

- (I) setting  $h := \text{G.C.D.}\{i \mid D_i \neq 0\}$ , either  $(**) \quad h = 1$  or, setting  $n = hd$ , the divisor class

$$(***) \quad L' := \frac{n}{h}L - \sum_i \frac{i}{h}D_i$$

has order precisely  $h$  (this condition guarantees that  $Y$  is an irreducible variety, otherwise it is just a normal scheme);

- (II) defining divisor classes  $L_\chi$  recursively by  $L_0 = \mathcal{O}_Y$ ,  $L_1 := L$ , and by  $L_{\chi+\xi} \equiv L_\chi + L_\xi - \sum_i \varrho_{\chi,\xi}^i D_i$ , there are choices of tautological

sections  $z_j$  on  $L_j$  of the pull back of  $\mathcal{O}_Y(-L_j)$  such that

$$\tau_Y(z_j) = z_{-mj},$$

in particular  $\tau(L_j) \cong L_{-mj}$ .

- ii) In the situation of i)  $X$  is the normalization of the singular covering

$$X' \subset L, X' := \{(y, z) \mid z^n = \prod_{i=1}^m \delta_i^i\},$$

and  $\gamma$  acts by  $z \mapsto e^{\frac{2\pi\sqrt{-1}}{n}} z$ .

- iii) The scheme structure of  $X$  is explicitly given as

$$X := \text{Spec}(\mathcal{O}_Y \oplus \bigoplus_{\chi \in G^* \setminus \{0\}} \mathcal{O}_Y(-L_\chi)),$$

that is,  $X$  is embedded in  $\bigoplus_{\chi \in G^* \setminus \{0\}} L_\chi$  and defined by the fundamental equations

$$(***) \quad z_\chi \cdot z_\xi = z_{\chi+\xi} \prod_i \delta_i^{\varrho_{\chi,\xi}^i}.$$

- iv) The real structure  $\tau_X$  on  $X$  is defined by extending  $\tau_Y$  from  $\mathcal{O}_Y$  to  $\mathcal{O}_X$  via the action of  $\tau_Y$  on the  $z_i, \delta_i$ 's (clearly then  $\tau g \tau = g^m$  for all  $g \in G$ ).

Remark 4.3. (a) Since  $Y$  is a complete variety, condition (I) in Theorem 4.2, ensuring connectedness of the covering, may be replaced by  $H^0(\mathcal{O}_Y(-L_\chi)) = 0$  for all  $\chi \neq 0$ .

(b) Condition (II) in Theorem 4.2 holds if it holds for  $j = 1$ , as it follows inductively by the linear equivalences (II).

In order to spell out concretely condition (II) for  $j = 1$ ,  $\tau(L_1) \cong L_{-m}$ , let us set  $\mu = [-m]$ , and use the recursive definition of the  $L_\chi$ 's, for convenience working in the Picard group of  $Y$ .

Then

$$L_{-m} = L_\mu = \mu L_1 - \sum_i (\varrho_{1,1}^i + \varrho_{2,1}^i + \cdots + \varrho_{\mu-1,1}^i) D_i.$$

Now, because of the equation

$$nL_\mu = \mu \sum_i iD_i - \sum_i n(\varrho_{1,1}^i + \varrho_{2,1}^i + \cdots + \varrho_{\mu-1,1}^i)D_i = \sum_i [\mu i]D_i,$$

$$\eta := \tau(L_1) - L_\mu$$

satisfies

$$n\eta = \tau\left(\sum_i iD_i\right) - \sum_i [\mu i]D_i = \sum_i (i - [\mu i])D_i = 0,$$

hence  $\eta$  is an  $n$ -torsion divisor (a torsion divisor of order dividing  $n$ ). If  $\eta = 0$ , one can still alter the choice of  $L_1$  keeping the divisors  $D_i$  fixed, in view of (3), adding another  $n$ -torsion divisor, call it  $\lambda$ .

We can then satisfy condition (II) for  $j = 1$  if we find  $\lambda$  which solves the equation

$$\eta = \tau(\lambda) - \mu\lambda.$$

For  $m = 1, \mu = 1$ , where condition (II) for  $j = 1$  simply means that  $L_1$  should be real, we want to find  $\lambda$  solving:

$$\eta := \tau(L_1) - L_1 = \tau(\lambda) - \lambda.$$

Remark 4.4. A quite analogous theorem holds, mutatis mutandis, for real Abelian coverings, describing real normal (not necessarily connected) schemes.

These are the changes to be done:

- $\mathbb{Z}/n$  is replaced by a finite Abelian group  $G$ .
- in (1),  $m$  is replaced by  $M \in \text{Aut}(G)$  such that  $M^2 = 1$ .
- in (2), we choose divisors  $D_g$  for  $g \in G, g \neq 0$ .
- (3) and (I) disappear, while (II) is replaced by : the character sheaves  $L_\chi$  must satisfy

$$L_{\chi+\xi} \equiv L_\chi + L_\xi - \sum_i \varrho_{\chi,\xi}^i D_i$$

(see [Par91])

- in (II) we replace  $z_j$  by  $z_\chi, L_j$  by  $L_\chi$ , and so on..
- ii) disappears, iii) and iv) are identical.
- If  $Y$  is complete, then connectedness of  $X$  is verified by imposing that the global sections of  $O_X$  are just the constants.

### REFERENCES

[AI-Gr71] NORMAN L. ALLING, NEWCOMB GREENLEAF, *Foundations of the theory of Klein surfaces*. Lecture Notes in Mathematics. 219. Berlin-Heidelberg-New York: Springer-Verlag. VIII,117 p. (1971).

[Cat02] FABRIZIO CATANESE, *Deformation types of real and complex manifolds*. Contemporary trends in algebraic geometry and algebraic topology (Tianjin, 2000), 195–238, Nankai Tracts Math., 5, World Sci. Publ., River Edge, NJ, (2002).

[Cat08] FABRIZIO CATANESE, *Differentiable and deformation type of algebraic surfaces, real and symplectic structures*. Symplectic 4-manifolds and algebraic surfaces, 55–167, Lecture Notes in Math., 1938, Springer, Berlin, (2008).

- [Cat12] FABRIZIO CATANESE: *Irreducibility of the space of cyclic covers of algebraic curves of fixed numerical type and the irreducible components of  $\text{Sing}(M_g)$* . Advances in geometric analysis, 281–306, Adv. Lect. Math. (ALM), 21, Int. Press, Somerville, MA, (2012).
- [CatFre03] FABRIZIO CATANESE, PAOLA FREDIANI *Real hyperelliptic surfaces and the orbifold fundamental group*. J. Inst. Math. Jussieu 2 (2003), no. 2, 163–233
- [CP17] FABRIZIO CATANESE, FABIO PERRONI: *Dihedral Galois covers of algebraic varieties and the simple cases*. J. Geom. Phys. 118 (2017), 67–93.
- [CilPed96] CIRO CILIBERTO, CLAUDIO PEDRINI, *Real abelian varieties and real algebraic curves*. Lectures in real geometry (Madrid, 1994), 167–256, De Gruyter Exp. Math., 23, de Gruyter, Berlin, (1996).
- [Come12] ANNIBALE COMESSATTI, *Fondamenti per la geometria sopra le superficie razionali dal punto di vista reale*. Math. Ann. 73, 1–72 (1912).
- [Come14] ANNIBALE COMESSATTI, *Sulla connessione delle superficie razionali reali*. Annali di Mat. (3) 23, 215–283 (1914).
- [Come24-5] ANNIBALE COMESSATTI, *Sulle varietà abeliane reali, I-II* Annali di Mat. (4) 2, 67–106 (1924), (4) 3, 27–71 (1925).
- [Come30] ANNIBALE COMESSATTI, *Sulle superficie multiple cicliche*. Rendiconti Seminario Padova 1, 1-45 (1930)
- [D-I-K00] ALEXANDER I. DEGTYAREV, ILYA ITENBERG, VIATCHESLAV M. KHARLAMOV, *Real Enriques surfaces*. Lecture Notes in Mathematics, 1746. Springer-Verlag, Berlin, (2000) xvi+259 pp.
- [D-I-K08] ALEXANDER I. DEGTYAREV, ILYA ITENBERG, VIATCHESLAV M. KHARLAMOV, *On deformation types of real elliptic surfaces*. Amer. J. Math. 130 (2008), no.6, 1561–1627.
- [DegKha00] ALEXANDER I. DEGTYAREV, VIATCHESLAV M. KHARLAMOV, *Topological properties of real algebraic varieties: Rokhlin’s way*. Uspekhi Mat. Nauk 55 (2000), no. 4(334), 129–212; translation in Russian Math. Surveys 55 (2000), no. 4, 735–814.
- [Fre04] PAOLA FREDIANI, *Real Kodaira surfaces*. Collect. Math. 55, No. 1, 61–96 (2004).
- [G-R58] HANS GRAUERT, REINHOLD REMMERT, *Komplexe Räume*. Math. Ann. 136, 245–318 (1958).
- [Har76] AXEL HARNACK, *On the Multi holiness of plane algebraic curves. (Über die Vieltheiligkeit der ebenen algebraischen Curven.)* Clebsch Ann. X, 189–199 (1876).
- [Iskovskih67] VASILII A. ISKOVSKIĬ, *Rational surfaces with a pencil of rational curves*. Mat. Sb. (N.S.) 74 (116), 608–638 (1967).
- [Iskovskih70] VASILII A. ISKOVSKIĬ, *Rational surfaces with a sheaf of rational curves and with a positive square of canonical class*. Mat. Sb. (N.S.) 83 (125), 90–119 (1970).
- [Iskovskih79] VASILII A. ISKOVSKIĬ, *Minimal models of rational surfaces over arbitrary fields*. Izv. Akad. Nauk SSSR Ser. Mat. 43 no.1, 19–43 (1979).
- [Khar176] VIATCHESLAV M. KHARLAMOV, *Topological types of nonsingular surfaces of degree 4 in  $\mathbb{RP}^3$* . Funkcional. Anal. i Prilozhen. 10 (1976), no. 4, 55–68.
- [Khar184] VIATCHESLAV M. KHARLAMOV, *On the classification of nonsingular surfaces of degree 4 in  $\mathbb{RP}^3$  with respect to rigid isotopies*. Funktsional. Anal. i Prilozhen. 18 (1984), no. 1, 49–56.
- [Klein76] FELIX KLEIN, *Über eine neue Art von Riemannschen Flächen*, Math. Annalen 10, (1876).
- [Klein82] FELIX KLEIN, *On Riemann’s theory of algebraic functions and their integrals. (Über Riemanns Theorie der algebraischen Functionen und ihrer Integrale.)* Leipzig. Teubner (1882).
- [Klein92] FELIX KLEIN, *Über Realitätsverhältnisse bei der einem beliebigen Geschlechte zugehörigen Normalkurve der  $\phi$* , Math. Annalen 42 (1892).
- [Mang17] FRÉDÉRIC MANGOLTE, *Real algebraic varieties. (Variétés algébriques réelles.)* Cours Spécialisés (Paris) 24. Paris: Société Mathématique de France (SMF) vii, 484 p. (2017).



- [Mang20] FRÉDÉRIC MANGOLTE, *Real algebraic varieties*. Springer Monographs in Mathematics. Cham: Springer xviii, 444 p. (2020).
- [Manin66] YURIĬ I. MANIN, *Rational surfaces over perfect fields*. Inst. Hautes Études Sci. Publ. Math. 30, 55–113 (1966).
- [Manin67] YURIĬ I. MANIN, *Rational surfaces over perfect fields. II*. Mat. Sb. (N.S.) 72 (114), 161–192 (1967).
- [Nikulin79] VYACHESLAV V. NIKULIN, *Integer symmetric bilinear forms and some of their geometric applications*. Izv. Akad. Nauk SSSR Ser. Mat. 43, no. 1, 111–177, 238 (1979).
- [Par91] RITA PARDINI, *Abelian covers of algebraic varieties*. J. Reine Angew. Math. 417 (1991), 191–213.
- [Sep90] MIKA SEPPÄLA, *Real algebraic curves in the moduli space of complex curves*, Comp. Math. 74 (1990), 259–283.
- [Sil89] ROBERT SILHOL, *Real algebraic surfaces*. Lecture Notes in Mathematics, 1392. Springer-Verlag, Berlin, 1989. x+215 pp.
- [Wei83] GUIDO WEICHOLD, *On symmetric Riemann surfaces and the moduli of periodicity of the accompanying abelian normal integrals of the first kind. (Über symmetrische Riemannsche Flächen und die Periodizitätsmoduln der zugehörigen Abelschen Normalintegrale erster Gattung)*. Diss. Leipzig. Schlömilch Z. XXVIII. 321–352 (1883).

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