

MIXING OF THE AVERAGING PROCESS AND ITS DISCRETE DUAL ON FINITE-DIMENSIONAL GEOMETRIES

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We analyze the L^1 -mixing of a generalization of the averaging process introduced by Aldous (2011). The process takes place on a growing sequence of graphs which we assume to be finite-dimensional, in the sense that the random walk on those geometries satisfies a family of Nash inequalities. As a byproduct of our analysis, we provide a complete picture of the total variation mixing of a discrete dual of the averaging process, which we call binomial splitting process. A single particle of this process is essentially the random walk on the underlying graph. When several particles evolve together, they interact by synchronizing their jumps when placed on neighboring sites. We show that, given k the number of particles and n the (growing) size of the underlying graph, the system exhibits cutoff in total variation if $k \rightarrow \infty$ and $k = O(n^2)$. Finally, we exploit the duality between the two processes to show that the binomial splitting process satisfies a version of Aldous' spectral gap identity, namely, the relaxation time of the process is independent of the number of particles.

1. Introduction. Introduced in a series of lectures and expository articles by Aldous ([1, 2, 4]), the averaging process is a Markovian model of mass redistribution among nearest-neighbor sites of a graph. Informally, the averaging process may be described as follows: after initially assigning some real values to each site, at exponentially-distributed times neighboring sites are selected and, then, split *equally* among themselves their total mass.

Originally proposed as a basic mathematical model for social dynamics, the averaging process naturally fits into the growing class of *opinion exchange models* (see, e.g., the recent survey [44]). In this context sites and edges represent agents together with their connections, while the sites' values measure their opinions. Such stochastic models are employed with the scope of quantitatively studying, for example, the conditions and timescales leading to consensus as well as the role of the underlying graph topology in this.

The averaging process is also closely related to a large number of models of mass redistribution from statistical physics, economics and computer science, see, for example, [14], Section 1.1. Moreover, this and other Markovian models with a continuous state space serve as constituent examples to extend the geometric theory of discrete Markov chains (see, e.g., [37, 40, 50]) to the continuous setting. Among the recent works in this direction, we mention [6, 10, 11, 14, 46, 54, 55] as those being concerned with the study of convergence rates and spectral gap's identities for continuous mass redistribution models.

Despite the several analogies with statistical mechanics models, the averaging process shares the distinguishing feature of reaching *equilibrium* at a *single absorbing state* with most examples in opinion exchange dynamics. This singularity, together with the continuous nature of the state space and the heterogeneity of the underlying social network, is what makes these Markovian models mathematically interesting and challenging.

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In this paper we enhance the analysis of the L^1 -transportation metric mixing of an inhomogeneous “unfair” generalization of the averaging process. The setting is that of large undirected graphs satisfying finite-dimensional Nash inequalities; the latter encompasses, for example, the segment, the circle, the discrete d -dimensional torus and all discrete approximations of “nice” Euclidean domains (see Section 2.3 below for further details and examples). We analyze the distance-to-equilibrium at different scales. In particular, we show that the process gradually mixes on the scale $\Theta(1)$, while it exhibits an abrupt behavior when zooming in on finer scales. Furthermore, the bounds that we obtain allow us to carry out a complete analysis of the mixing of a discrete analogue of the averaging process, which we refer to as binomial splitting process. This model is an interacting particle system with a conservation law, and a significant part of this paper is devoted to the study of its spectral gap and total variation (TV) cutoff.

The last decade has registered important breakthroughs on the understanding of the sharp mixing behavior for conservative particle systems. Among the most influential works, we mention those by Caputo et al. [12] on the Aldous’ spectral gap conjecture, and by Lacoïn [34] on the TV cutoff on one-dimensional domains, both for the Symmetric Simple Exclusion process (SSEP). More recently, versions of Aldous’ spectral gap identity have been shown for other models, for example, the Zero Range process (ZRP) [27], Beta and Gibbs samplers on the segment [10, 11], Wright–Fisher and Fleming–Viot processes and multi-allelic Moran models [17, 26, 49, 53]. Similarly, we refer, among others, to [28, 29, 32, 33, 35, 42, 51, 52] for recent developments on the cutoff for SSEP, its asymmetric variants and ZRP.

The binomial splitting process shares some features with some symmetric particle systems mentioned above. Among these, the presence of dualities and intertwining play a prominent role in our work. By means of dual descriptions, on the one hand we derive sharp upper bounds for the averaging process from properties of a-few-particle binomial splitting; on the other hand, we also deduce results on the many-particle binomial splitting through the analysis of the averaging dynamics.

2. Models and main results. We devote this section to the rigorous description of the Markovian models and to the corresponding main results. The underlying common geometry for the two models is represented by a weighted, connected and undirected graph $G = (V, E, (c_{xy})_{xy \in E})$ and site-weights $\pi = (\pi(x))_{x \in V}$, a *nondegenerate* probability measure on V , that is, $\pi(x) > 0$ for all $x \in V$. Moreover, for all $p \in [1, \infty]$, we let $L^p(V, \pi)$ denote the Banach space of functions $\psi : V \rightarrow \mathbb{R}$ endowed with the norm $\|\cdot\|_p$ ($\|\psi\|_p^p := \sum_{x \in V} \pi(x) |\psi(x)|^p$ for $p \in [1, \infty)$ and $\|\psi\|_\infty := \max_{x \in V} |\psi(x)|$ for $p = \infty$). For the case $p = 2$, we use the shorthand notation \mathcal{H} for $L^2(V, \pi)$ and $\langle \cdot, \cdot \rangle$ for the corresponding inner product.

2.1. Binomial splitting process. The binomial splitting process is a natural discrete analogue of the averaging process, in which pairs of sites split particles rather than mass, according to a binomial distribution rather than deterministically. More precisely, for each $k \in \mathbb{N}$, the binomial splitting process with k particles, $\text{Bin}(k)$, is the continuous-time Markov process $(\xi_t)_{t \geq 0}$ on the finite configuration space

$$(2.1) \quad \Omega_k := \left\{ \xi \in \mathbb{N}_0^V \mid \sum_{x \in V} \xi(x) = k \right\},$$

whose evolution is described by the infinitesimal generator

$$(2.2) \quad \mathcal{L}^{\text{Bin}(k)} f = \sum_{xy \in E} c_{xy} (\mathcal{P}_{xy}^{\text{Bin}(k)} - \mathbb{1}) f, \quad f : \Omega_k \rightarrow \mathbb{R}.$$

In the above formula,

$$(2.3) \quad \mathcal{P}_{xy}^{\text{Bin}(k)} f(\xi) := \mathbb{E}_{\Xi_\xi^{xy}} [f],$$

where the Ω_k -valued r.v. Ξ_ξ^{xy} is defined as

$$(2.4) \quad \Xi_\xi^{xy}(z) := \begin{cases} Y & \text{if } z = x, \\ \xi(x) + \xi(y) - Y & \text{if } z = y, \\ \xi(z) & \text{otherwise} \end{cases}$$

with

$$(2.5) \quad Y \sim \text{binomial}\left(\xi(x) + \xi(y), \frac{\pi(x)}{\pi(x) + \pi(y)}\right).$$

Here and in what follows, if not stated otherwise, \mathbb{E}_ν denotes the expectation with respect to the probability law ν ; if $\nu = \text{Law}(X)$, for some r.v. X , we abbreviate $\mathbb{E}_{\text{Law}(X)}$ by \mathbb{E}_X . In equation (2.4), notice the symmetry of the r.v. Ξ_ξ^{xy} with respect to the sites $x, y \in V$.

As a straightforward detailed balance computation shows, the unique invariant and reversible measure for the process $(\xi_t)_{t \geq 0}$ is multinomial with parameters (k, π) , which is denoted by

$$\mu_{k,\pi}(\xi) = k! \prod_{x \in V} \frac{\pi(x)^{\xi(x)}}{\xi(x)!},$$

and the binomial splitting dynamics acts as a local instantaneous thermalization at pairs of sites with rates $(c_{xy})_{x,y \in E}$. It is worth to observe that, while the binomial redistribution rule corresponds to an i.i.d. sampling of the particles' new locations, interaction enters only when forcing them to jump simultaneously. A quantitative analysis of the effects of such a weak dependence lies at the core of our work.

Notice that if the system consists of a single particle, it can be equivalently described through its position $(X_t)_{t \geq 0}$ on V , its reversible measure coinciding with π ; we call $\mathbb{P}^{\text{Bin}(1)}$ the law of such a Markov chain.

2.2. Aldous' spectral gap identity. The first result we present is a spectral gap identity for the binomial splitting process. In words, we show that the k -particle system's spectral gap coincides with the spectral gap of the single-particle system on any graph.

While the literature on spectral gap estimates is too vast to be all mentioned here, the understanding of the exact correspondence between many- and single-particle systems' spectral gaps is limited, besides the trivial case of independent particles, to a few more examples [9–12, 17, 27]. The next theorem adds a further model to the previous list.

Let us first introduce some terminology. Due to reversibility of the multinomial (k, π) , denoted below by $\mu_{k,\pi}$, with respect to the $\text{Bin}(k)$ dynamics, the Rayleigh quotient representation for the spectral gap holds, that is, the smallest positive eigenvalue of $-\mathcal{L}^{\text{Bin}(k)}$ can be equivalently defined by

$$(2.6) \quad \text{gap}_k := \inf_{f \neq 0: \mathbb{E}_{\mu_{k,\pi}}[f]=0} \frac{\mathbb{E}_{\mu_{k,\pi}}[f(-\mathcal{L}^{\text{Bin}(k)} f)]}{\mathbb{E}_{\mu_{k,\pi}}[f^2]}.$$

THEOREM 2.1. *For all graphs $G = (V, E, (c_{xy})_{x,y \in E})$, nondegenerate site-weights $(\pi(x))_{x \in V}$ and $k \in \mathbb{N}$,*

$$(2.7) \quad \text{gap}_k = \text{gap}_1 =: \text{gap}.$$

The proof of this result is deferred to Section 4 below.

2.3. *Asymptotic framework and examples.* In contrast with the spectral gap result in Theorem 2.1, which holds for every fixed (weighted) graph G and nondegenerate site-weights π , all results presented in the forthcoming subsections are framed—in analogy with most of the literature on Markov chains’ mixing times—in an asymptotic setting. More in detail, we will consider a growing sequence of weighted graphs $G_n = (V_n, E_n, (c_{xy})_{xy \in E_n})$ with corresponding site weights $(\pi(x))_{x \in V_n}$. The size of the vertex set $|V_n| = n$ will play the role of the diverging parameter and all the asymptotic notation will refer to the limit $n \rightarrow \infty$; moreover, the dependence on n will be usually omitted. In what follows we use the usual Landau asymptotic notation: given two nonnegative sequences $(f_n)_{n \in \mathbb{N}}, (g_n)_{n \in \mathbb{N}}$, we write

$$\begin{aligned}
 f_n = O(g_n) &\iff \limsup_{n \rightarrow \infty} \frac{f_n}{g_n} < \infty, \\
 f_n = o(g_n) &\iff \limsup_{n \rightarrow \infty} \frac{f_n}{g_n} = 0, \\
 f_n = \Omega(g_n) &\iff \liminf_{n \rightarrow \infty} \frac{f_n}{g_n} > 0, \\
 f_n = \omega(g_n) &\iff \liminf_{n \rightarrow \infty} \frac{f_n}{g_n} = \infty, \\
 f_n = \Theta(g_n) &\iff 0 < \liminf_{n \rightarrow \infty} \frac{f_n}{g_n} \leq \limsup_{n \rightarrow \infty} \frac{f_n}{g_n} < \infty.
 \end{aligned}$$

The main assumption that we require is that the growing graphs are, roughly speaking, finite-dimensional. Nash inequalities will be the analytical tool which encodes the finite-dimensionality of our geometries. One of the strengths of such integral inequalities is that they imply the ultracontractivity of the random walk’s Markov semigroup or, in other words, pointwise heat kernel “on-diagonal” upper bounds. This property, combined with tensorization and comparison results, will turn out to be extremely useful also in our setting of many interacting particles.

Originally developed in the context of parabolic PDEs by Nash ([45], see also [24]), this class of functional inequalities has been first exploited in the context of finite-state Markov chains in [23]. In the reversible setting, [13] established the equivalence between Nash inequalities and ultracontractivity (see also [18]). Several other conditions are known to imply Nash inequalities, for instance: isoperimetry (see, e.g., [50], Section 3.3.2); moderate growth conditions in combination with local Poincaré inequalities ([23], Theorem 5.2); upper and lower Gaussian-like heat kernel bounds or parabolic Harnack inequalities ([7], Section 5.2). Moreover, as most of these conditions, Nash inequalities transfer from infinite graphs to sequences of uniformly roughly isometric finite graphs, see [20], Corollary 2.10 and Proposition 3.1.

Let us now formally present our assumption concerning Nash inequality for the single-particle system, that is, $\text{Bin}(1)$ with law $\mathbb{P}^{\text{Bin}(1)}$ as defined in the end of Section 2.1. For all $n \in \mathbb{N}$, we say that $\text{Bin}(1)$ satisfies a *Nash inequality with (positive) constants* $d = d(n)$, $t_{\text{Nash}} = t_{\text{Nash}}(n)$ and $T = T(n)$ if

$$(2.8) \quad \|\psi\|_2^{2(1+\frac{2}{d})} \leq t_{\text{Nash}}(\mathcal{E}_{\text{Bin}(1)}(\psi) + T^{-1}\|\psi\|_2^2)\|\psi\|_1^{\frac{4}{d}}, \quad \psi \in \mathcal{H},$$

holds, where $\mathcal{E}_{\text{Bin}(1)}$ denotes the Dirichlet form of the single-particle system, namely

$$(2.9) \quad \mathcal{E}_{\text{Bin}(1)}(\psi) := \sum_{xy \in E} c_{xy} \frac{\pi(x)\pi(y)}{\pi(x) + \pi(y)} (\psi(x) - \psi(y))^2.$$

For every fixed $n \in \mathbb{N}$, a Nash inequality for *some* positive constants (d, t_{Nash}, T) always holds; hence, in an asymptotic setting the point lies in finding a sequence of constants $(d, t_{\text{Nash}}, T)_{n \in \mathbb{N}}$ with a suitable asymptotic behavior as, for instance, in the forthcoming Assumption 1. Furthermore, as mentioned in the paragraph above, the integral inequality in equation (2.8) can be translated into a pointwise upper bound for the heat kernel of the single-particle process. More precisely, calling

$$(2.10) \quad h_t^x(y) := \frac{\mathbb{P}^{\text{Bin}(1)}(X_t = y \mid X_0 = x)}{\pi(y)}, \quad x, y \in V, t \geq 0,$$

equation (2.8) implies that (see [50], Theorem 2.3.4)

$$(2.11) \quad \max_{x, y \in V} h_t^x(y) \leq e^{\left(\frac{dt_{\text{Nash}}}{2t}\right)^{\frac{d}{2}}}, \quad t \leq T.$$

Notice that, for the purpose of deriving equation (2.11) above, it suffices to check Nash inequality in equation (2.8) for all $\psi \in \mathcal{P}_\pi \subseteq \mathcal{H}$, the subset of probability densities with respect to π . Moreover, as the above inequality shows, in an asymptotic framework Nash inequalities as in equation (2.8) are most useful when d is small, $T = \Theta(t_{\text{Nash}})$ and t_{Nash} is at most of the same order of the *relaxation time*, that is,

$$(2.12) \quad t_{\text{rel}} := \text{gap}^{-1}.$$

In this case, t_{Nash} plays the role of burn-in time around which all L^p -norms of the Markov chain’s probability density become uniformly bounded and Bin(1) mixes gradually at times $\Theta(t_{\text{rel}})$. This discussion motivates the following assumption on the underlying geometries satisfying a “good” family of Nash inequalities.

ASSUMPTION 1 (Finite-dimensional geometries). For all $n \in \mathbb{N}$, we assume that Nash inequality for Bin(1) holds with positive constants $d = d(n)$, $T = T(n)$ and $t_{\text{Nash}} = t_{\text{Nash}}(n)$ satisfying

$$(2.13) \quad d = O(1), \quad t_{\text{Nash}} = O(t_{\text{rel}}), \quad T = \Theta(t_{\text{Nash}}).$$

Notice that Assumption 1 implies that there exist c_{dim} and $c_{\text{ratio}} \in (0, \infty)$ such that

$$(2.14) \quad \sup_{n \in \mathbb{N}} d \leq c_{\text{dim}} \quad \text{and} \quad \sup_{n \in \mathbb{N}} \frac{t_{\text{Nash}}}{t_{\text{rel}}} \leq c_{\text{ratio}}.$$

Moreover, the assumption of nondegeneracy of the site-weights used in the proof of Theorem 2.1 is replaced in the current asymptotic setting by the following uniform nondegeneracy assumption.

ASSUMPTION 2 (Uniformly elliptic site-weights). We assume that the sequence of site-weights satisfies

$$(2.15) \quad \sup_{n \in \mathbb{N}} \frac{\max_{x \in V} \pi(x)}{\min_{y \in V} \pi(y)} \leq c_{\text{ell}}$$

for some $c_{\text{ell}} \in [1, \infty)$.

REMARK 2.2. If Assumption 1 holds for some uniformly elliptic probability distribution π on V , then, just by simple comparison of norms and Dirichlet forms in equation (2.8), it holds for any other uniformly elliptic π' , with $t'_{\text{Nash}} = \Theta(t_{\text{Nash}})$, $t'_{\text{rel}} = \Theta(t_{\text{rel}})$, etc.

We conclude this section with a list of examples of sequences of graphs satisfying Assumption 1 for every uniformly elliptic π as in Assumption 2 (in all the following examples, we omit to remark that $T = \Theta(t_{\text{Nash}})$ holds):

- Lattice discretizations of m -dimensional tori and Euclidean bounded Lipschitz domains $\mathcal{A} \subset \mathbb{R}^m$, $m \geq 1$, where $G = G_n$ is a lattice approximation of \mathcal{A} with nearest-neighbor jumps and uniformly elliptic conductances, see [16]:

$$(2.16) \quad d = m, \quad t_{\text{rel}} = \Theta(n^{2/d}), \quad t_{\text{Nash}} = \Theta(n^{2/d}).$$

As special cases, one recovers the circle and the segment with uniformly elliptic conductances. Notice that, by a comparison argument, the restriction to nearest-neighbor jumps may be relaxed up to include finite-range jumps. Moreover, instances of random walks on finite groups with moderate growth also belong to this same class, see [22, 23].

- Random walks on large finite m -dimensional boxes on the *supercritical percolation cluster*, see [41], equation (6) and Theorem 1.3:

$$(2.17) \quad d = m + o(1), \quad t_{\text{rel}} = \Theta(n^{2/d}), \quad t_{\text{Nash}} = \Theta(n^{2/d}).$$

- The “ n -dog” from [50], Examples 3.3.2 and 3.3.5, namely the lattice discretization of two boxes of \mathbb{R}^2 intersecting only in one corner:

$$(2.18) \quad d = 2, \quad t_{\text{rel}} = \Theta(n \log n), \quad t_{\text{Nash}} = \Theta(n).$$

- Finite fractal graphs with bounded *walk-dimension* $d_w > 2$ and *volume growth exponent* $d_f > 0$:

$$(2.19) \quad d = 2d_f/d_w, \quad t_{\text{rel}} = \Theta(n^{2/d}), \quad t_{\text{Nash}} = \Theta(n^{2/d}).$$

Heat kernel estimates for several examples of this kind have been extensively studied on *infinite* graphs, see, for example, [7, 30, 31]; *finite*-graph examples, including Sierpiński gasket and carpet graphs, are thoroughly discussed in [20].

It is rather straightforward to construct examples of geometries for which our Assumption 1 does *not* hold. For instance, any sequence constructed from the list above with diverging d (e.g., m -dimensional Euclidean boxes with $m = m_n \rightarrow \infty$) belongs to this class. Another example is that of the (homogeneous) complete graphs with n vertices as in Theorem 2.7 below; as a simple computation shows, $t_{\text{rel}} = \frac{2}{n} \rightarrow 0$, while, for all possible choices of $d = O(1)$ and $T = \Theta(t_{\text{Nash}})$, $t_{\text{Nash}} \rightarrow \infty$.

2.4. *Mixing of the binomial splitting.* This section is devoted to the presentation of the results concerning the TV mixing of the binomial splitting process. For all $k \in \mathbb{N}$, we let $(\mathcal{S}_t^{\text{Bin}(k)})_{t \geq 0}$ denote the Markov semigroup associated to the generator $\mathcal{L}^{\text{Bin}(k)}$ in equation (2.2) and, for every initial distribution ν on Ω_k , we adopt the matrix notation $\nu \mathcal{S}_t^{\text{Bin}(k)}$ to refer to the corresponding distribution at time $t \geq 0$.

Recall the definition of TV distance to equilibrium at time $t \geq 0$ when starting from some probability distribution ν over Ω_k

$$(2.20) \quad \mathbf{d}_k^{(\nu)}(t) := \|\nu \mathcal{S}_t^{\text{Bin}(k)} - \mu_{k,\pi}\|_{\text{TV}} = \sup_{A \subset \Omega_k} |\nu \mathcal{S}_t^{\text{Bin}(k)}(A) - \mu_{k,\pi}(A)|,$$

where $\mu_{k,\pi}$ is the multinomial distribution of parameters (k, π) .

In Theorem 2.3 below, we show that, in the asymptotic setting of Section 2.3, the worst-case TV distance exhibits the so-called *cutoff phenomenon*. Discovered in the 80’s by Aldous and Diaconis, [3], the expression *cutoff* refers, in the context of Markov chains, to an abrupt convergence to equilibrium measured with a given distance. In recent years, several systems of interacting particles have been shown to exhibit cutoff in TV distance, see, for example, [8, 10, 11, 28, 34, 36, 39] and references therein.

In analogy with this literature, we consider the *worst-case* mixing, namely, we will take the supremum of the quantity in equation (2.20) over the set of initial distributions which, in

turn, is equivalent to take the maximum over the set Ω_k . For this reason, letting δ_ξ denote the Dirac' distribution at $\xi \in \Omega_k$, we define

$$(2.21) \quad \mathbf{d}_k(t) := \sup_{\xi \in \Omega_k} \mathbf{d}_k^{(\delta_\xi)}(t).$$

We will show that, when $k \rightarrow \infty$, cutoff for the worst-case TV distance in equation (2.21) occurs around the time

$$(2.22) \quad t_{\text{mix}} := \frac{t_{\text{rel}}}{2} \log(k),$$

thus, rightfully referred to as *the* mixing time. Moreover, the quantity in equation (2.21) is bounded away from 0 and 1 in a window of size $\Theta(t_{\text{rel}})$ around t_{mix} . In order to quantify the latter statement, for every $C > 0$ we will consider the quantity

$$(2.23) \quad t_{\mathbf{w}}(C) := Ct_{\text{rel}},$$

which will play the role of the so-called *cutoff window* in the forthcoming Theorem 2.3. In addition, we further define

$$(2.24) \quad t^\pm(C) := t_{\text{mix}} \pm t_{\mathbf{w}}(C).$$

THEOREM 2.3 (Cutoff). *Consider a sequence of graphs and site-weights such that Assumptions 1 and 2 hold. Let us further assume that the total number of particles $k = k_n$ is such that $k \rightarrow \infty$ and there exists some $c_{\text{vol}} \in [0, \infty)$ for which*

$$(2.25) \quad \limsup_{n \rightarrow \infty} \frac{k}{n^2} \leq c_{\text{vol}}.$$

Then, for all $\delta \in (0, 1)$ there exists some $C = C(\delta) > 0$ such that

$$(2.26) \quad \limsup_{n \rightarrow \infty} \mathbf{d}_k(t^+(C)) \leq \delta, \quad \liminf_{n \rightarrow \infty} \mathbf{d}_k(t^-(C)) \geq 1 - \delta$$

hold.

The assumption $k \rightarrow \infty$ is necessary for the validity of Theorem 2.3. Indeed, when $k = 1$, the so-called *product condition* (see, e.g., [37], Proposition 18.4) together with equation (2.11) impose that in order for the cutoff to occur we need $t_{\text{Nash}} \gg t_{\text{rel}}$, which contrasts our Assumption 1. The next proposition shows that, in our setting, the absence of cutoff holds for any sequence $k = O(1)$.

PROPOSITION 2.4. *In the same setting of Theorem 2.3, if $k = k_n = O(1)$, then, there exist $a, b > 0$ independent of n such that*

$$(2.27) \quad e^{-\frac{t}{t_{\text{rel}}}} \leq \mathbf{d}_k(t) \leq ae^{-\frac{t}{t_{\text{rel}}}}, \quad t \geq bt_{\text{rel}}.$$

REMARK 2.5 (High-density regime). In the regime in which $k \rightarrow \infty$ and $k = O(n^2)$ the cutoff time t_{mix} in equation (2.22) coincides with the cutoff time for other recently studied symmetric interacting systems (mostly in 1D), e.g., [10, 11, 34]. While the assumption $k \rightarrow \infty$ is necessary for the validity of Theorem 2.3 (cf. Proposition 2.4), this is not clear for the requirement $k = O(n^2)$. In fact, our techniques break down when dropping that assumption. Hence, determining the emergence of the cutoff phenomenon for the $\text{Bin}(k)$ in the *high-density regime*, that is, when $k = \omega(n^2)$, remains an open problem. Nonetheless, as we will show in Proposition 6.8 below, in this regime a timescale $\Theta(t_{\text{rel}} \log(k))$ is still sufficient for the system to be well mixed.

The conclusions in Theorem 2.3 and Proposition 2.4—whose proofs are postponed to Section 6 below—are drawn from mixing results for the averaging process, presented in the next two sections.

2.5. *Averaging process.* We start by a rigorous definition of the “unfair” averaging process in which sites redistribute their mass proportionally to prescribed site-weights. Since the averaging dynamics conserves the total mass and it is invariant under dilation, with no loss of generality we will assume that the initial configuration is some $\eta \in \Delta$, where Δ is the set of probability distributions over V . More precisely, given a nondegenerate $\pi \in \Delta$, the averaging process is the Markov process $(\eta_t)_{t \geq 0}$ with state space Δ and infinitesimal generator

$$(2.28) \quad \mathcal{L}^{\text{Avg}} f := \sum_{xy \in E} c_{xy} (\mathcal{P}_{xy}^{\text{Avg}} - \mathbb{1}) f, \quad f : \Delta \rightarrow \mathbb{R},$$

where

$$(2.29) \quad \mathcal{P}_{xy}^{\text{Avg}} f(\eta) := f(\eta^{xy}), \quad \eta^{xy}(z) := \begin{cases} \frac{\pi(x)}{\pi(x) + \pi(y)} (\eta(x) + \eta(y)) & \text{if } z = x, \\ \frac{\pi(y)}{\pi(x) + \pi(y)} (\eta(x) + \eta(y)) & \text{if } z = y, \\ \eta(z) & \text{otherwise.} \end{cases}$$

Let $\mathcal{C}(\Delta)$ be the Banach space of continuous functions on the compact metric space $(\Delta, \|\cdot\|_2)$ endowed with the supremum norm. Then, it is easy to check that the operator \mathcal{L}^{Avg} is a bounded linear operator on $\mathcal{C}(\Delta)$ and that generates a Feller Markov contraction semigroup $(\mathcal{S}_t^{\text{Avg}})_{t \geq 0}$ on the same space. All throughout, $\mathbb{P}_\nu^{\text{Avg}}$ and $\mathbb{E}_\nu^{\text{Avg}}$ denote the law and corresponding expectation of the averaging process distributed at time $t = 0$ according to ν ; when ν is a Dirac at η , we simply write $\mathbb{P}_\eta^{\text{Avg}}$ and $\mathbb{E}_\eta^{\text{Avg}}$. An analogous notation will be adopted for the k -particle binomial splitting.

2.6. *Mixing of the averaging.* As it can be read off the definition in equation (2.28), $\pi \in \Delta$ is the unique absorbing point of the averaging dynamics. Moreover, although the sequence of local thermalizations is random, the deterministic nature of the mass redistributions prevents any other point $\eta \in \Delta$ to be visited more than once, breaking down any reversibility. In this context, quantifying the convergence to stationarity entails a sensible choice of distance to equilibrium. Indeed, at any finite time there is a positive probability that only a fraction of edges has been updated. Thus, the distribution of the averaging process is singular with respect to the unique stationary measure, the Dirac δ_π . Hence, convergence to stationarity cannot occur in, for example, TV distance, that is, $\|\text{Law}(\eta_t) - \delta_\pi\|_{\text{TV}} \not\rightarrow 0$ as $t \rightarrow \infty$. Therefore, as in other recent works on the averaging process and related mass redistribution models (see, e.g., [4, 6, 14]), we will employ Wasserstein-type of distances. More precisely, we will adopt L^p -transportation metrics, namely, for all $p \in [1, 2]$,

$$(2.30) \quad \mathbb{E}_\eta^{\text{Avg}} \left[\left\| \frac{\eta_t}{\pi} - 1 \right\|_p \right], \quad \eta \in \Delta, t \geq 0.$$

These metrics set the ground for a quantitative comparison between the averaging and its “noiseless” counterpart. Indeed, as it will become clear with the statement on the duality relations in Proposition 3.3 below, the averaging process $(\eta_t)_{t \geq 0} \subseteq \Delta$ decomposes into two components: a deterministic part $(\pi_t)_{t \geq 0} \subseteq \Delta$ corresponding to the law of the single-particle binomial splitting system, and a “noise” part $(\mathcal{X}_t)_{t \geq 0} \subseteq \mathbb{R}^V$, such that

$$(2.31) \quad \eta_t = \pi_t + \mathcal{X}_t, \quad t \geq 0,$$

with $\mathcal{X}_0 = 0$ and $\sum_{x \in V} \mathcal{X}_t(x) = 0$ a.s., $\mathbb{E}[\mathcal{X}_t] = 0$ for all $t \geq 0$, as well as $\mathcal{X}_t \rightarrow 0$ in law as $t \rightarrow \infty$.

In view of these considerations, it comes natural to compare rates of convergence for the averaging in L^p -transportation distance and for Bin(1) in L^p -distance. This comparison boils

down to analyze the effect of the noise $(\mathcal{X}_t)_{t \geq 0}$ at various scales, leading to possible mismatches in the behaviors of the two processes. A first instance of this phenomenon is shown in the next proposition, which generalizes a result taken from [4] to the inhomogeneous context, that is, when π is not uniform.

PROPOSITION 2.6 (Cf. [4], Proposition 2). *For all graphs $G = (V, E, (c_{xy})_{xy \in E})$ and nondegenerate site-weights $(\pi(x))_{x \in V}$,*

$$(2.32) \quad \mathbb{E}_\eta^{\text{Avg}} \left[\left\| \frac{\eta_t}{\pi} - 1 \right\|_2^2 \right] \leq e^{-\frac{t}{t_{\text{rel}}}} \left\| \frac{\eta}{\pi} - 1 \right\|_2^2, \quad \eta \in \Delta, t \geq 0.$$

The above proposition is key to the proof of Theorem 2.1 and, for completeness, its proof is reported in Section 4 below. In words, Proposition 2.6 above shows that the contraction rate for the averaging’s L^2 -transportation metrics is, in general, off by a factor 2 from the standard L^2 -contraction rate prescribed by Poincaré inequality for Bin(1). In fact, [14], Corollary 2.2, proves that the lack of this pre-factor is exact in the specific context of the homogeneous complete graph.

The result in Proposition 2.6 required no hypothesis other than the nondegeneracy of $\pi \in \Delta$. When turning to the analysis of the asymptotic behaviors, different assumptions on the underlying geometry may lead to dissimilar outcomes. To the best of our knowledge, [14] is the only work so far establishing sharp results on the mixing of the averaging process in an asymptotic setting. As some of their findings directly relate to our results, we schematically report them below, referring the interested reader to [14] for further details.

THEOREM 2.7 ([14], Theorems 1.1 and 1.2). *Consider a sequence of growing complete graphs with n vertices, unitary conductances and uniform site-weights. Then, calling*

$$(2.33) \quad t_{\text{CDSZ}} := \frac{1}{\log(2)} \frac{\log(n)}{n}, \quad t_{\text{CDSZ}, \mathbf{w}}(C) := C \frac{\sqrt{\log(n)}}{n}$$

and

$$(2.34) \quad t_{\text{CDSZ}}^\pm(C) := t_{\text{CDSZ}} \pm t_{\text{CDSZ}, \mathbf{w}}(C),$$

for all $\delta \in (0, 1)$ there exists some $C = C(\delta) > 0$ such that

$$(2.35) \quad \begin{aligned} \limsup_{n \rightarrow \infty} \sup_{\eta \in \Delta} \mathbb{E}_\eta^{\text{Avg}} \left[\left\| \frac{\eta_{t_{\text{CDSZ}}^+(C)}}{\pi} - 1 \right\|_1 \right] &\leq \delta, \\ \liminf_{n \rightarrow \infty} \sup_{\eta \in \Delta} \mathbb{E}_\eta^{\text{Avg}} \left[\left\| \frac{\eta_{t_{\text{CDSZ}}^-(C)}}{\pi} - 1 \right\|_1 \right] &\geq 2 - \delta. \end{aligned}$$

The above theorem allows a comparison between the mixing behaviors of the averaging and of the Bin(1) on the complete graph. Indeed, on the one hand, the L^1 -Wasserstein distance to equilibrium for the averaging sharply drops to zero around times $(\frac{1}{\log(2)} + o(1)) \frac{\log(n)}{n}$; on the other hand, as a simple computation shows, the law of the single Bin-particle reaches equilibrium in TV distance on a strictly shorter timescale, namely $t_{\text{mix}} = \Theta(t_{\text{rel}}) = \Theta(\frac{1}{n})$. In other words, mixing of these two processes differ both qualitatively (i.e., abrupt vs. gradual) and quantitatively (i.e., on different timescales) on the “infinite-dimensional” example of growing complete graphs.

A natural question is whether this disagreement occurs also on finite-dimensional geometries, namely those for which our Assumption 1 holds and, thus, the Bin(1) mixes on a timescale $\Theta(t_{\text{rel}})$ without cutoff. As our next result shows, in such a framework, the mixing behaviors of the averaging and of the Bin(1) match, both occurring gradually at times $\Theta(t_{\text{rel}})$.

PROPOSITION 2.8 (No cutoff). *Consider a sequence of graphs and site-weights such that Assumption 1 holds. Then, there exist $a, b > 0$ independent of n such that*

$$(2.36) \quad e^{-\frac{t}{t_{\text{rel}}}} \leq \mathbb{E}_\eta^{\text{Avg}} \left[\left\| \frac{\eta_t}{\pi} - 1 \right\|_p \right] \leq a e^{-\frac{t}{t_{\text{rel}}}}, \quad t \geq b t_{\text{rel}},$$

for all $p \in [1, 2]$.

Pushing further the analogy between averaging dynamics and the “noiseless” Bin(1), we ask whether cutoff occurs on a “finer scale”, that is, when measuring the L^p -transportation metrics on scales of the order of $k^{-1/2}$, for some diverging $k = k_n \rightarrow \infty$. In the forthcoming theorem we prove that, in our asymptotic framework, the averaging process abruptly mixes just like the single particle on finer scales as long as $k = O(n^2)$. An extension similar to that highlighted for the binomial splitting in Remark 2.5 holds for the averaging in the regime in which $k = \omega(n^2)$; see Proposition 5.6 below for further details.

THEOREM 2.9 (Cutoff on a finer scale). *Consider a sequence of graphs and site-weights such that Assumptions 1 and 2 hold. Fix also a sequence $k = k_n$ such that $k \rightarrow \infty$ and there exists some $c_{\text{vol}} \in [0, \infty)$ for which equation (2.25) holds. Then, for all $\delta \in (0, 1)$ there exists some $C = C(\delta) > 0$ such that*

$$(2.37) \quad \limsup_{n \rightarrow \infty} \sqrt{k} \sup_{\eta \in \Delta} \mathbb{E}_\eta^{\text{Avg}} \left[\left\| \frac{\eta_{t^\pm}}{\pi} - 1 \right\|_p \right] \leq \delta, \quad \liminf_{n \rightarrow \infty} \sqrt{k} \sup_{\eta \in \Delta} \mathbb{E}_\eta^{\text{Avg}} \left[\left\| \frac{\eta_{t^\pm}}{\pi} - 1 \right\|_p \right] \geq \frac{1}{\delta},$$

for all $p \in [1, 2]$, where $t^\pm(C)$ are defined as in equation (2.24).

As it will be shown in Section 6 below, the results in Proposition 2.8 and Theorem 2.9 will be instrumental for the proofs of the upper bounds in Proposition 2.4 and Theorem 2.3, respectively.

2.7. *Organization of the paper.* The rest of the paper is organized as follows. Section 3 is devoted to the introduction of several dualities and intertwining relations involving the processes under analysis. In Section 4 we prove the spectral gap identity presented in Section 2.2. The proofs of the mixing results for the averaging are presented in Section 5, while those for the binomial splitting in Section 6.

3. Intertwining and duality relations. In this section, we present the intertwining and (self-)duality relations involving the averaging and the binomial splitting which will be used all throughout. For a general account on these two probabilistic tools in the context of Markov processes and interacting particle systems, the interested reader may refer, for example, to [25, 38, 43] and references therein.

PROPOSITION 3.1 (Multinomial intertwining). *For all $k \in \mathbb{N}$ and functions $f : \Omega_k \rightarrow \mathbb{R}$, we have*

$$(3.1) \quad \mathcal{S}_t^{\text{Avg}} \Lambda_k f = \Lambda_k \mathcal{S}_t^{\text{Bin}(k)} f, \quad t \geq 0$$

where, for all $\eta \in \Delta$,

$$(3.2) \quad \Lambda_k f(\eta) := \mathbb{E}_{\mu_{k,\eta}}[f],$$

and $\mu_{k,\eta}$ stands for multinomial(k, η).

PROOF. Recall Eqs. (2.3) and (2.29). We prove

$$(3.3) \quad \mathcal{P}_{xy}^{\text{Avg}} \Lambda_k f(\eta) = \Lambda_k \mathcal{P}_{xy}^{\text{Bin}(k)} f(\eta), \quad \eta \in \Delta,$$

for all $xy \in E$ and $f : \Omega_k \rightarrow \mathbb{R}$; this yields the analogue of equation (3.1) for the corresponding generators, from which equation (3.1) follows due to the boundedness of the generators involved. (For notational convenience, in this proof we will write, for example, ξ_x for $\xi(x)$.) From now on the proof follows by an elementary direct computation. Starting with the term on the right-hand side in equation (3.3), we rewrite it as

$$(3.4) \quad \begin{aligned} \Lambda_k \mathcal{P}_{xy}^{\text{Bin}(k)} f(\eta) &= \sum_{\xi \in \Omega_k} \mu_{k,\eta}(\xi) \mathcal{P}_{xy}^{\text{Bin}(k)} f(\xi) \\ &= \sum_{\ell=0}^k \sum_{\xi \in \Omega_k} \mathbb{P}(\{\text{Multinomial}(k, \eta) = \xi\} \cap \{\xi_x + \xi_y = \ell\}) \mathcal{P}_{xy}^{\text{Bin}(k)} f(\xi). \end{aligned}$$

Notice that, for all $\ell = 0, 1, \dots, k$, we have

$$\begin{aligned} &\sum_{\xi \in \Omega_k} \mathbb{P}(\{\text{Multinomial}(k, \eta) = \xi\} \cap \{\xi_x + \xi_y = \ell\}) \mathcal{P}_{xy}^{\text{Bin}(k)} f(\xi) \\ &= \sum_{\xi \in \Omega_k} k! \left(\prod_{z \neq x,y} \frac{\eta_z^{\xi_z}}{\xi_z!} \frac{\eta_x^{\xi_x}}{\xi_x!} \frac{\eta_y^{\xi_y}}{\xi_y!} \sum_{\xi' \in \Omega_k} \ell! \left(\mathbf{1}_{\{\xi'_x + \xi'_y = \ell\}} \prod_{z \neq x,y} \mathbf{1}_{\{\xi'_z = \xi_z\}} \right) \right. \\ &\quad \times \frac{\left(\frac{\pi_x}{\pi_x + \pi_y}\right)^{\xi'_x}}{\xi'_x!} \frac{\left(\frac{\pi_y}{\pi_x + \pi_y}\right)^{\xi'_y}}{\xi'_y!} f(\xi') \\ &= \sum_{\xi' \in \Omega_k} k! \left(\prod_{z \neq x,y} \frac{\eta_z^{\xi'_z}}{\xi'_z!} \frac{\left(\frac{\pi_x}{\pi_x + \pi_y}\right)^{\xi'_x}}{\xi'_x!} \frac{\left(\frac{\pi_y}{\pi_x + \pi_y}\right)^{\xi'_y}}{\xi'_y!} \mathbf{1}_{\{\xi'_x + \xi'_y = \ell\}} (\eta_x + \eta_y)^\ell \right) f(\xi') \\ &= \sum_{\xi' \in \Omega_k} k! \left(\prod_{z \neq x,y} \frac{\eta_z^{\xi'_z}}{\xi'_z!} \right) \frac{\left(\frac{\pi_x}{\pi_x + \pi_y}(\eta_x + \eta_y)\right)^{\xi'_x}}{\xi'_x!} \frac{\left(\frac{\pi_y}{\pi_x + \pi_y}(\eta_x + \eta_y)\right)^{\xi'_y}}{\xi'_y!} \mathbf{1}_{\{\xi'_x + \xi'_y = \ell\}} f(\xi'). \end{aligned}$$

Plugging this expression back into equation (3.4), we obtain

$$\begin{aligned} \Lambda_k \mathcal{P}_{xy}^{\text{Bin}(k)} f(\eta) &= \sum_{\ell=0}^k \sum_{\xi \in \Omega_k} \mathbb{P}(\{\text{multinomial}(k, \eta) = \xi\} \cap \{\xi_x + \xi_y = \ell\}) \mathcal{P}_{xy}^{\text{Bin}(k)} f(\xi) \\ &= \sum_{\xi' \in \Omega_k} \mu_{k,\eta^{xy}}(\xi') f(\xi'), \end{aligned}$$

which coincides with

$$\mathcal{P}_{xy}^{\text{Avg}} \Lambda_k f(\eta) = \Lambda_k f(\eta^{xy}) = \mathbb{E}_{\mu_{k,\eta^{xy}}} [f],$$

where η^{xy} and $\mathcal{P}_{xy}^{\text{Avg}}$ are defined as in equation (2.29). Hence, the equality in equation (3.3) follows. \square

Before presenting the duality relations, it turns out to be convenient to introduce a labeled version of the binomial splitting described in Section 2.1. For this reason, for all $k \in \mathbb{N}$, we define the *labeled* binomial splitting process with k particles as the irreducible Markov chain $(\mathbf{X}_t)_{t \geq 0}$ on V^k whose dynamics is described as follows: start with k labeled particles at positions $\mathbf{x} = (x_1, \dots, x_k) \in V^k$; as soon as the Poisson clock of rate c_{xy} rings, the particles

whose position is either x or $y \in V$, independently of each other, place themselves in $x \in V$ with probability $\frac{\pi(x)}{\pi(x)+\pi(y)}$, in $y \in V$ otherwise. Clearly, this Markov process coincides with the process $(\xi_t)_{t \geq 0}$ described in Section 2.1 when the labels of the particles are ignored. More precisely, the corresponding infinitesimal generator

$$(3.5) \quad L^{\text{Bin}(k)} = \sum_{xy \in E} c_{xy} L_{xy}^{\text{Bin}(k)},$$

which is self-adjoint in the k -fold tensor space $\mathcal{H}^{\otimes k} := \mathcal{H} \otimes \dots \otimes \mathcal{H}$, where $\mathcal{H} = L_2(V, \pi)$, maps symmetric functions into symmetric functions, since the particle dynamics does not depend on the particles labels. Rigorously,

$$(3.6) \quad L^{\text{Bin}(k)} \text{Sym}_k = \text{Sym}_k L^{\text{Bin}(k)},$$

where $\text{Sym}_k : \mathcal{H}^{\otimes k} \rightarrow \mathcal{H}^{\otimes k}$ denotes the orthogonal projector

$$(3.7) \quad \text{Sym}_k \psi(x_1, \dots, x_k) := \frac{1}{k!} \sum_{\sigma \in \Sigma_k} \psi(x_{\sigma(1)}, \dots, x_{\sigma(k)}), \quad \text{Sym}_k^2 = \text{Sym}_k$$

and Σ_k the symmetric group on k symbols. We let $(S_t^{\text{Bin}(k)})_{t \geq 0}$ denote the semigroup associated to the labeled k -particle binomial splitting. Moreover, for notational convenience, we will adopt the following shorthand:

$$\pi(\mathbf{x}) = \pi^{\otimes k}(\mathbf{x}) := \pi(x_1) \cdots \pi(x_k), \quad \mathbf{x} = (x_1, \dots, x_k) \in V^k.$$

The next two results are not new and variants of them may be found scattered in the literature in several places for related models, using different techniques, from probably to Lie algebra. For more details about these techniques, we refer the interested reader to, for example, [4, 25, 47, 48]; below, we provide sketches of their proofs for the reader’s convenience.

PROPOSITION 3.2 (Self-duality for the binomial splitting). *Let, for all $k, \ell \in \mathbb{N}$, $\mathbf{x} = (x_1, \dots, x_k) \in V^k$ and $\xi \in \Omega_\ell$,*

$$(3.8) \quad [\xi]_{\mathbf{x}} := \xi(x_1) (\xi(x_2) - \mathbf{1}_{x_2=x_1}) \cdots \left(\xi(x_k) - \sum_{i=1}^{k-1} \mathbf{1}_{x_k=x_i} \right)$$

denote the \mathbf{x} -falling factorial of ξ . Then, for all $\mathbf{x} \in V^k$, $\xi \in \Omega_\ell$ and $t \geq 0$, we have

$$(3.9) \quad \mathbb{E}_\xi^{\text{Bin}(\ell)} \left[\frac{[\xi_t]_{\mathbf{x}}}{\pi(\mathbf{x})} \right] = S_t^{\text{Bin}(k)} \left(\frac{[\xi]}{\pi(\cdot)} \right) (\mathbf{x}).$$

SKETCH OF PROOF. As mentioned above, there are several approaches one might follow to prove this assertion. One option is to proceed by the following two-step argument: first, the self-duality relation between two systems of independent $\text{Bin}(1)$ particles, see, for example, [19], Proposition 2.9.4; second, recovering the binomial splitting processes by “instantaneous thermalization”, see [25], Section 6.3. \square

PROPOSITION 3.3 (Duality between averaging and binomial splitting). *Let, for all $\mathbf{x} \in V^k$ and $\eta \in \Delta$,*

$$(3.10) \quad D(\mathbf{x}, \eta) := \prod_{i=1}^k D(x_i, \eta) := \prod_{i=1}^k \frac{\eta(x_i)}{\pi(x_i)}$$

and

$$(3.11) \quad \bar{D}(\mathbf{x}, \eta) := \prod_{i=1}^k \bar{D}(x_i, \eta) := \prod_{i=1}^k \left(\frac{\eta(x_i)}{\pi(x_i)} - 1 \right).$$

Then, the following duality relations hold: for all $\mathbf{x} \in V^k$, $\eta \in \Delta$, and $t \geq 0$,

$$(3.12) \quad \mathbb{E}_\eta^{\text{Avg}}[D(\mathbf{x}, \eta_t)] = S_t^{\text{Bin}(k)} D(\cdot, \eta)(\mathbf{x})$$

and

$$(3.13) \quad \mathbb{E}_\eta^{\text{Avg}}[\bar{D}(\mathbf{x}, \eta_t)] = S_t^{\text{Bin}(k)} \bar{D}(\cdot, \eta)(\mathbf{x}).$$

REMARK 3.4 (Moment vs. orthogonal duality functions). Functions of the joint system satisfying relations as in Eqs. (3.9), (3.12) and (3.13) are usually referred to as duality functions; more specifically, functions as in equation (3.10) are also known as “moment” or “classical” duality functions, while those as in equation (3.11) take the name of “orthogonal” duality functions, due to their relation with orthogonal polynomials (see, e.g., [48]).

SKETCH OF PROOF. Concerning equation (3.12), the equality follows by Poissonizing the multinomial intertwining in Proposition 3.1 and acting with this new intertwining on the self-duality functions of Proposition 3.2 as explained in [48], Section 5.2. On the other hand, the duality relation in equation (3.13) has been proved for the nonthermalized model in, for example, [48], Section 5.4; since instantaneous thermalization preserves the duality relations, this concludes the proof. \square

4. Proof of the spectral gap identity. This section is completely devoted to the proof of Theorem 2.1. The main ingredient is the following elementary lemma, showing that the duality relation in equation (3.13) allows to map eigenfunctions of the labeled k -particle system to candidate eigenfunctions of the averaging.

LEMMA 4.1. For $k \geq 1$, let $\psi \in \mathcal{H}^{\otimes k}$ be an eigenfunction for $-L^{\text{Bin}(k)}$ associated to the eigenvalue $\lambda \geq 0$, that is,

$$(4.1) \quad L^{\text{Bin}(k)} \psi = -\lambda \psi.$$

Then, $f_\psi \in \mathcal{C}(\Delta)$ defined as

$$(4.2) \quad f_\psi(\eta) := \sum_{\mathbf{x} \in V^k} \pi(\mathbf{x}) \psi(\mathbf{x}) \bar{D}(\mathbf{x}, \eta),$$

solves

$$(4.3) \quad \mathcal{L}^{\text{Avg}} f_\psi = -\lambda f_\psi.$$

In other words, Lemma 4.1 shows that, for every eigenfunction ψ of $-L^{\text{Bin}(k)}$, either f_ψ is identically zero or f_ψ is an eigenfunction for $-\mathcal{L}^{\text{Avg}}$ with the same eigenvalue.

PROOF. By definition, the duality relation equation (3.13) and self-adjointness of $L^{\text{Bin}(k)}$ in $\mathcal{H}^{\otimes k}$, we have

$$(4.4) \quad \begin{aligned} \mathcal{L}^{\text{Avg}} f_\psi(\eta) &= \sum_{\mathbf{x} \in V^k} \pi(\mathbf{x}) \psi(\mathbf{x}) \mathcal{L}^{\text{Avg}} \bar{D}(\mathbf{x}, \cdot)(\eta) \\ &= \sum_{\mathbf{x} \in V^k} \pi(\mathbf{x}) \psi(\mathbf{x}) L^{\text{Bin}(k)} \bar{D}(\cdot, \eta)(\mathbf{x}) \\ &= \sum_{\mathbf{x} \in V^k} \pi(\mathbf{x}) L^{\text{Bin}(k)} \psi(\mathbf{x}) \bar{D}(\mathbf{x}, \eta) = -\lambda f_\psi(\eta). \end{aligned} \quad \square$$

The forthcoming Lemma 4.2 will determine which eigenfunctions ψ give rise to actual eigenfunctions f_ψ . To derive such results we introduce two classes of operators, which are referred to as (particle) *creation* and *annihilation* operators.

We start from the simple observation that the labeled binomial spitting is “consistent”, namely that each subset of k labeled Bin-particles still evolves according to the same Markovian dynamics. More precisely, for all $k \in \mathbb{N}$ the generators $L^{\text{Bin}(k)}$ and $L^{\text{Bin}(k-1)}$ satisfy the following intertwining relations:

$$(4.5) \quad L^{\text{Bin}(k)} \mathfrak{a}_{k,i} = \mathfrak{a}_{k,i} L^{\text{Bin}(k-1)},$$

where, for all $i \in \{1, \dots, k\}$, $\mathfrak{a}_{k,i} : \mathcal{H}^{\otimes(k-1)} \rightarrow \mathcal{H}^{\otimes k}$ is called the annihilation operator of the i th particle for the k -particle system; such an operator is one-to-one and defined by

$$(4.6) \quad (\mathfrak{a}_{k,i})\psi(\mathbf{x}) := \psi(\hat{\mathbf{x}}^i), \quad \hat{\mathbf{x}}^i := (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_k).$$

The adjoint of $\mathfrak{a}_{k,i}$ is then given by $\mathfrak{a}_{k-1,i}^\dagger : \mathcal{H}^{\otimes k} \rightarrow \mathcal{H}^{\otimes(k-1)}$, the so-called creation operator of the i th particle for the k -particle system; such an operator is onto and can be written as

$$(4.7) \quad (\mathfrak{a}_{k-1,i}^\dagger)\psi(\hat{\mathbf{x}}^i) := \sum_{x_i \in V} \pi(x_i)\psi(x_1, \dots, x_i, \dots, x_k), \quad \mathbf{x} := (x_1, \dots, x_k).$$

Notice further that $\text{Im}(\mathfrak{a}_{1,1})$ consists of constant functions, while $\text{Ker}(\mathfrak{a}_{0,1}^\dagger)$ consists of zero mean functions, and that the orthogonal decomposition $\mathcal{H} = \text{Ker}(\mathfrak{a}_{0,1}^\dagger) \oplus_{\perp\pi} \text{Im}(\mathfrak{a}_{1,1})$ holds. The latter decomposition generalizes to the tensor product space $\mathcal{H}^{\otimes k}$ by

$$(4.8) \quad \mathcal{H}^{\otimes k} = \text{Ker}(\mathfrak{a}_{k-1}^\dagger) \oplus_{\perp\pi} \text{Im}(\mathfrak{a}_k),$$

where

$$(4.9) \quad \text{Ker}(\mathfrak{a}_{k-1}^\dagger) := \bigcap_{i=1}^k \text{Ker}(\mathfrak{a}_{k-1,i}^\dagger) = (\text{Ker}(\mathfrak{a}_{0,1}^\dagger))^{\otimes k} \quad \text{and} \quad \text{Im}(\mathfrak{a}_k) := \bigoplus_{i=1}^k \text{Im}(\mathfrak{a}_{k,i}).$$

As an immediate consequence of the definition in equation (3.13), it follows that, for all $\eta \in \Delta$, the function $\bar{D}(\cdot, \eta) : V^k \rightarrow \mathbb{R}$ satisfies, for all $1 \leq i \leq k$,

$$(4.10) \quad (\mathfrak{a}_{k-1,i}^\dagger \bar{D}(\cdot, \eta))(\hat{\mathbf{x}}_i) = \left(\sum_{x_i \in V} \pi(x_i) \bar{D}(x_i, \eta) \right) \bar{D}(\hat{\mathbf{x}}_i, \eta) = 0, \quad \mathbf{x} \in V^k;$$

namely, $\bar{D}(\cdot, \eta) \in \text{Ker}(\mathfrak{a}_{k-1}^\dagger)$.

LEMMA 4.2. Fix $k \in \mathbb{N}$. Given $\psi \in \mathcal{H}^{\otimes k}$, for $f_\psi \in \mathcal{C}(\Delta)$ defined as in equation (4.2), we have

$$(4.11) \quad f_\psi \equiv 0 \iff \text{Sym}_k \psi \in \text{Im}(\mathfrak{a}_k).$$

PROOF. The implication “ \Leftarrow ” is a consequence of the adjointness of $\mathfrak{a}_{k,i}$ and $\mathfrak{a}_{k-1,i}^\dagger$ with equation (4.10). Indeed, if $\text{Sym}_k \psi = \mathfrak{a}_{k,i} \phi$ for some $\phi \in \mathcal{H}^{\otimes(k-1)}$ and $i \in \{1, \dots, k\}$,

$$f_\psi(\eta) = \sum_{\mathbf{x} \in V^k} \pi(\mathbf{x}) \mathfrak{a}_{k,i} \phi(\mathbf{x}) \bar{D}(\mathbf{x}, \eta) = \sum_{\hat{\mathbf{x}}_i \in V^{k-1}} \pi(\hat{\mathbf{x}}_i) \phi(\hat{\mathbf{x}}_i) \mathfrak{a}_{k,i}^\dagger \bar{D}(\hat{\mathbf{x}}_i, \eta) = 0.$$

For the “ \Rightarrow ” part, thanks to the decomposition in equation (4.8), it suffices to show that $f_\psi \neq 0$ for all symmetric and nonzero functions ψ in $\text{Ker}(\mathfrak{a}_{k-1}^\dagger)$, where $f_\psi \in \mathcal{C}(\Delta)$ is defined as in equation (4.2). Thus, fix any such $\psi \in \text{Ker}(\mathfrak{a}_{k-1}^\dagger) \subseteq \mathcal{H}^{\otimes k}$ and consider the corresponding $f_\psi \in \mathcal{C}(\Delta)$. Notice that, by the definition of f_ψ in equation (4.2) and $\bar{D}(\cdot, \pi) = 0$, we have $f_\psi(\pi) = 0$; thus, the conclusion follows if we show that $f_\psi(\eta) \neq 0$ for some $\eta \in \Delta \setminus \{\pi\}$.

By the nondegeneracy of $\pi \in \Delta$, there exists $h > 0$ such that

$$(4.12) \quad \pi + \zeta \in \Delta, \quad \zeta \in \mathcal{T}_h := \left\{ \gamma : V \rightarrow \mathbb{R} \mid \sum_{x \in V} \gamma(x) = 0, \|\gamma\|_\infty \leq h \right\}.$$

Arguing by contradiction, let us suppose that

$$(4.13) \quad f_\psi(\pi + \zeta) = \sum_{\mathbf{x} \in V} \pi(\mathbf{x}) \psi(\mathbf{x}) \prod_{i=1}^k \frac{\zeta(x_i)}{\pi(x_i)} = 0$$

holds for all $\zeta \in \mathcal{T}_h$. Note that in the first identity we only used that, by definition of $\bar{D}(\mathbf{x}, \cdot)$, we have $\bar{D}(\mathbf{x}, \pi + \zeta) = \prod_{i=1}^k \frac{\zeta(x_i)}{\pi(x_i)}$. Then, by homogeneity, equation (4.13) holds for all $\zeta \in \mathcal{T}_\infty$; in other words, ψ is orthogonal to all functions in the linear span of

$$(4.14) \quad \{\varphi^{\otimes k} := \varphi \otimes \dots \otimes \varphi \in \text{Ker}(\mathfrak{a}_{k-1}^\dagger) \mid \varphi \in \text{Ker}(\mathfrak{a}_{0,1}^\dagger)\}.$$

By the polarization identity, the linear span of equation (4.14) is dense in the k -fold symmetric tensor of $\text{Ker}(\mathfrak{a}_{0,1}^\dagger)$, and since $\psi \in \text{Ker}(\mathfrak{a}_{k-1}^\dagger) = (\text{Ker}(\mathfrak{a}_{0,1}^\dagger))^{\otimes k}$ was chosen to be symmetric, ψ must vanish. Hence, the assumption that $f_\psi \equiv 0$ yields $\psi \equiv 0$, a contradiction. \square

4.1. *Proof of Theorem 2.1.* By means of the duality relation in equation (3.13), we constructed in Lemmata 4.1 and 4.2 nontrivial eigenfunctions for $-\mathcal{L}^{\text{Avg}}$ in terms of suitable symmetric eigenfunctions for $-L^{\text{Bin}(k)}$. As we show in Lemma 4.3 below, the main feature of such eigenfunctions for the averaging process is that, as soon as $k \geq 2$, the point $\pi \in \Delta$ is a zero (recall that $f_\psi(\pi) = 0$) of at least the second order with respect to the L^2 -distance (in the sense of equation (4.16)) on the simplex Δ . This property combined with the L^2 -Wasserstein contraction result from Proposition 2.6—for which we provide the proof below—completes the proof of Theorem 2.1.

PROOF OF PROPOSITION 2.6. The variation of $\|\frac{\eta}{\pi} - 1\|_2^2$ after a mass exchange among sites $x, y \in V$ equals

$$(4.15) \quad \begin{aligned} & (\pi(x) + \pi(y)) \left(\frac{\eta(x) + \eta(y)}{\pi(x) + \pi(y)} - 1 \right)^2 - \pi(x) \left(\frac{\eta(x)}{\pi(x)} - 1 \right)^2 - \pi(y) \left(\frac{\eta(y)}{\pi(y)} - 1 \right)^2 \\ &= -\frac{\pi(x)\pi(y)}{\pi(x) + \pi(y)} \left(\frac{\eta(x)}{\pi(x)} - \frac{\eta(y)}{\pi(y)} \right)^2. \end{aligned}$$

Therefore, by definition of spectral gap in equation (2.6) and that of Dirichlet form in equation (2.9), we obtain

$$\mathcal{L}^{\text{Avg}} \left\| \frac{\eta}{\pi} - 1 \right\|_2^2 = -\mathcal{E}_{\text{Bin}(1)} \left(\frac{\eta}{\pi} \right) \leq -\text{gap}_1 \left\| \frac{\eta}{\pi} - 1 \right\|_2^2.$$

An application of Grönwall inequality yields the desired result. \square

LEMMA 4.3. Consider $k \geq 2$, $\psi \in \mathcal{H}^{\otimes k}$ and f_ψ as in equation (4.2). Then,

$$(4.16) \quad C_\psi := \sup_{\eta \in \Delta \setminus \{\pi\}} \frac{|f_\psi(\eta)|}{\|\frac{\eta}{\pi} - 1\|_2^2} \in [0, \infty).$$

PROOF. By Cauchy–Schwarz inequality, we obtain

$$(4.17) \quad |f_\psi(\eta)| = \left| \sum_{\mathbf{x} \in V^k} \pi(\mathbf{x}) \psi(\mathbf{x}) \bar{D}(\mathbf{x}, \eta) \right| \leq \sqrt{\sum_{\mathbf{x} \in V^k} \pi(\mathbf{x}) (\psi(\mathbf{x}))^2} \sqrt{\sum_{\mathbf{x} \in V^k} \pi(\mathbf{x}) (\bar{D}(\mathbf{x}, \eta))^2}.$$

Because of the product structure of both probability measures $\pi = \pi^{\otimes k} : V^k \rightarrow \mathbb{R}$ and duality functions $\bar{D}(\cdot, \eta)$, we have

$$(4.18) \quad \sqrt{\sum_{\mathbf{x} \in V^k} \pi(\mathbf{x})(\bar{D}(\mathbf{x}, \eta))^2} = \left(\sum_{x \in V} \pi(x)(\bar{D}(x, \eta))^2 \right)^{\frac{k}{2}} = \left\| \frac{\eta}{\pi} - 1 \right\|_2^k.$$

The desired conclusion follows because Δ is compact and the L^2 -norm is continuous. \square

In view of the above lemmas, we may conclude by employing a well-known argument due to Chen and Wang ([15]).

PROOF OF THEOREM 2.1. Clearly, we need to consider only the case $k \geq 2$. The inequality

$$(4.19) \quad \text{gap}_k \leq \text{gap}_1$$

follows at once from Eqs. (3.6) and (4.5). Indeed, calling $\xi^x \in \Omega_{k-1}$ the configuration obtained from $\xi \in \Omega_k$ by removing one particle at $x \in V$, the linear operator $J_k : L^2(\Omega_{k-1}, \mu_{k-1, \pi}) \rightarrow L^2(\Omega_k, \mu_{k, \pi})$ defined as

$$(4.20) \quad J_k f(\xi) := \sum_{x \in V} \xi(x) f(\xi^x), \quad \xi \in \Omega_k,$$

is injective; moreover, since its action corresponds to that of a symmetrized annihilation operator on symmetric functions, J_k satisfies

$$(4.21) \quad \mathcal{L}^{\text{Bin}(k)} J_k = J_k \mathcal{L}^{\text{Bin}(k-1)}, \quad k \geq 2,$$

see equation (4.5).

As for the reverse inequality, due to self-adjointness of $-L^{\text{Bin}(k)}$ and the decomposition of $\mathcal{H}^{\otimes k}$ in equation (4.8), it suffices to consider symmetric eigenfunctions in $\text{Ker}(\mathbf{a}_{k-1}^\dagger)$. Let ψ be such an eigenfunction for $-L^{\text{Bin}(k)}$ with corresponding eigenvalue $\lambda > 0$ and $f_\psi \in \mathcal{C}(\Delta)$ defined as in equation (4.2). As already noted in Lemma 4.2, $f_\psi(\pi) = 0$ and $f_\psi \neq 0$; moreover, by Lemma 4.1, $\mathcal{L}^{\text{Avg}} f_\psi = -\lambda f_\psi$. Then, for all $t \geq 0$ and $\eta \in \Delta \setminus \{\pi\}$ such that $f_\psi(\eta) \neq 0$, we have

$$(4.22) \quad \begin{aligned} e^{-\lambda t} |f_\psi(\eta)| &= |\mathbb{E}_\eta^{\text{Avg}}[f_\psi(\eta_t)]| \leq \mathbb{E}_\eta^{\text{Avg}}[|f_\psi(\eta_t)|] \\ &\leq C_\psi \mathbb{E}_\eta^{\text{Avg}} \left[\left\| \frac{\eta_t}{\pi} - 1 \right\|_2^2 \right] \leq C_\psi e^{-\text{gap}_1 t} \left\| \frac{\eta}{\pi} - 1 \right\|_2^2, \end{aligned}$$

where $C_\psi > 0$ is the constant introduced in Lemma 4.3, whereas the last step follows from Proposition 2.6. Since, again by Lemma 4.3, $\frac{|f_\psi(\eta)|}{C_\psi \|\frac{\eta}{\pi} - 1\|_2^2} \in (0, 1]$, we further obtain

$$(4.23) \quad \lambda \geq \text{gap}_1 + \frac{\log\left(\frac{|f_\psi(\eta)|}{C_\psi \|\frac{\eta}{\pi} - 1\|_2^2}\right)}{t}$$

for all $t > 0$. Taking $t \rightarrow \infty$ yields the desired result. \square

5. Proofs from Section 2.6. In this section we prove the mixing results for the averaging stated in Section 2.6. The section is divided in four parts. First in Section 5.1 we prove an easy lower bound for the L^p -Wasserstein distance to equilibrium. We then extract two upper bounds which will be used to estimate the distance from equilibrium at different scales. On the one hand, in Section 5.2, we show that a time $\Theta(t_{\text{rel}})$ suffices to bring the mean L^2 -distance arbitrarily close to zero. On the other hand, in Section 5.3, we use this latter bound

to control the mixing at times $(1 + o(1))\frac{t_{\text{rel}}}{2} \log(k)$, showing that this is sufficient to shrink the L^2 -Wasserstein distance down further to $o(k^{-1/2})$. For this reason, we are going to refer to times of order $\Theta(t_{\text{rel}})$ as “short times” and to times of order $\Theta(t_{\text{rel}} \log(k))$ as “longer times”.

As mentioned in Section 2.6, the latter is shown to hold as long as $k = O(n^2)$. Finally, in Section 5.4 we collect all results of these subsections to prove Proposition 2.8 and Theorem 2.9.

5.1. *Lower bound.* The next lemma follows easily by using the duality relations between the averaging and the one-particle system in Proposition 3.3. In what follows, in analogy with equation (2.10), we define

$$(5.1) \quad h_t^\eta(x) := S_t^{\text{Bin}(1)}\left(\frac{\eta}{\pi}\right)(x) = S_t^{\text{Bin}(1)}D(\cdot, \eta)(x), \quad x \in V, \eta \in \Delta, t \geq 0.$$

LEMMA 5.1. *For all $\eta \in \Delta$, $p \in [1, \infty]$ and $t \geq 0$, we have*

$$(5.2) \quad \mathbb{E}_\eta^{\text{Avg}} \left[\left\| \frac{\eta_t}{\pi} - 1 \right\|_p \right] \geq \|h_t^\eta - 1\|_p.$$

As a consequence,

$$(5.3) \quad \sup_{\eta \in \Delta} \mathbb{E}_\eta^{\text{Avg}} \left[\left\| \frac{\eta_t}{\pi} - 1 \right\|_p \right] \geq \sup_{\eta \in \Delta} \|h_t^\eta - 1\|_p \geq e^{-\frac{t}{t_{\text{rel}}}}.$$

PROOF. Let $q := q(p) \in [1, \infty]$ denote the conjugate exponent of $p \in [1, \infty]$. Then, by the dual formulation of $\|\cdot\|_p$ and Proposition 3.3 with $k = 1$, we have

$$\begin{aligned} \mathbb{E}_\eta^{\text{Avg}} \left[\left\| \frac{\eta_t}{\pi} - 1 \right\|_p \right] &= \mathbb{E}_\eta^{\text{Avg}} \left[\sup_{\|\psi\|_q=1} \sum_{x \in V} \pi(x) \bar{D}(x, \eta_t) \psi(x) \right] \\ &\geq \sup_{\|\psi\|_q=1} \sum_{x \in V} \pi(x) S_t^{\text{Bin}(1)} \bar{D}(\cdot, \eta)(x) \psi(x) = \|h_t^\eta - 1\|_p. \end{aligned}$$

This shows equation (5.2). Then equation (5.3) follows by passing to the supremum in $\eta \in \Delta$, and using the monotonicity of L^p -norms and [37], Lemma 20.11, with $p = 1$. \square

5.2. *Upper bound for short times.* Let us start by noting that, by the duality relation between the averaging and the two-particle system we have, for all $t \geq 0$ and $\eta \in \Delta$,

$$(5.4) \quad \mathbb{E}_\eta^{\text{Avg}} \left[\left\| \frac{\eta_t}{\pi} - 1 \right\|_2^2 \right] = \sum_{x \in V} \pi(x) \mathbb{E}_\eta^{\text{Avg}} \left[\left(\frac{\eta_t(x)}{\pi(x)} \right)^2 - 1 \right]$$

$$(5.5) \quad = \sum_{x \in V} \pi(x) (S_t^{\text{Bin}(2)}D(\cdot, \eta)(x, x) - 1)$$

$$(5.6) \quad = \sum_{x \in V} \pi(x) \left(\sum_{y, z \in V} p_t^{\text{Bin}(2)}((y, z), (x, x)) \frac{\eta(y)\eta(z)}{\pi(x)^2} - 1 \right)$$

$$(5.7) \quad = \sum_{y, z \in V} \eta(y)\eta(z) \sum_{x \in V} \pi(x) \left(\frac{p_t^{\text{Bin}(2)}((y, z), (x, x))}{\pi(x)^2} - 1 \right)$$

$$(5.8) \quad \leq \max_{x, y, z, w \in V} \left| \frac{p_t^{\text{Bin}(2)}((x, y), (z, w))}{\pi(z)\pi(w)} - 1 \right|,$$

where we used the symbol

$$(5.9) \quad p_t^{\text{Bin}(k)}(\mathbf{x}, \mathbf{y}) := \mathbb{P}^{\text{Bin}(k)}(\mathbf{X}_t = \mathbf{y} \mid \mathbf{X}_0 = \mathbf{x}), \quad k \in \mathbb{N}, \mathbf{x}, \mathbf{y} \in V^k, t \geq 0,$$

to refer to the transition probabilities of the labeled particle system. We remark that equation (5.5) follows by equation (3.12), equation (5.6) is just reversibility and to obtain equation (5.8) we used $\eta, \pi \in \Delta$.

In order to control the quantity in equation (5.8) we now derive a Nash inequality for the two-particle system from the analogous one for $\text{Bin}(1)$. The main ingredient is the following comparison result between the two-particle system, that is, $\text{Bin}(2)$, and the product chain of two one-particle systems, that is, $\text{Bin}(1) \otimes \text{Bin}(1)$.

LEMMA 5.2 (Comparison of Dirichlet forms). *For all $\psi \in \mathcal{H}^{\otimes 2}$,*

$$(5.10) \quad \frac{1}{2} \mathcal{E}_{\text{Bin}(1) \otimes \text{Bin}(1)}(\psi) \leq \mathcal{E}_{\text{Bin}(2)}(\psi) \leq \mathcal{E}_{\text{Bin}(1) \otimes \text{Bin}(1)}(\psi),$$

where $\mathcal{E}_{\text{Bin}(2)}$ and $\mathcal{E}_{\text{Bin}(1) \otimes \text{Bin}(1)}$ denote the Dirichlet forms on $\mathcal{H}^{\otimes 2}$ of the corresponding processes.

PROOF. Recall that

$$(5.11) \quad L^{\text{Bin}(2)} = \sum_{xy \in E} c_{xy} L_{xy}^{\text{Bin}(2)} \quad \text{and} \quad L^{\text{Bin}(1) \otimes \text{Bin}(1)} = \sum_{xy \in E} c_{xy} (L_{xy}^{\text{Bin}(1)} \oplus L_{xy}^{\text{Bin}(1)}),$$

with $A \oplus B := A \otimes \mathbb{1} + \mathbb{1} \otimes B$ denoting the Kronecker sum of two operators. For all $xy \in E$ and $\psi \in \mathcal{H}^{\otimes 2}$, it is simple to check that

$$(5.12) \quad \begin{aligned} & (L_{xy}^{\text{Bin}(2)} - L_{xy}^{\text{Bin}(1)} \oplus L_{xy}^{\text{Bin}(1)})\psi(z, w) \\ &= \begin{cases} \left(\frac{\pi(y)}{\pi(x) + \pi(y)}\right)^2 (\psi(y, y) + \psi(x, x) - \psi(x, y) - \psi(y, x)) & \text{if } z = w = x, \\ \left(\frac{\pi(x)}{\pi(x) + \pi(y)}\right)^2 (\psi(y, y) + \psi(x, x) - \psi(x, y) - \psi(y, x)) & \text{if } z = w = y, \\ -\left(\frac{\pi(x)}{\pi(x) + \pi(y)}\right)\left(\frac{\pi(y)}{\pi(x) + \pi(y)}\right) (\psi(y, y) + \psi(x, x) - \psi(x, y) - \psi(y, x)) & \text{if } (z, w) = (x, y) \text{ or } (y, x), \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Combining the above two identities, we obtain, for all $\phi, \varphi \in \mathcal{H}^{\otimes 2}$,

$$(5.13) \quad \begin{aligned} & \sum_{z, w \in V} \pi(z)\pi(w)\phi(z, w)(L^{\text{Bin}(2)} - L^{\text{Bin}(1) \otimes \text{Bin}(1)})\varphi(z, w) \\ &= \sum_{xy \in E} c_{xy} \left(\frac{\pi(x)\pi(y)}{\pi(x) + \pi(y)}\right)^2 (\varphi(x, x) + \varphi(y, y) - \varphi(x, y) - \varphi(y, x)) \\ & \quad \times (\phi(x, x) + \phi(y, y) - \phi(x, y) - \phi(y, x)). \end{aligned}$$

Therefore, for all $\psi \in \mathcal{H}^{\otimes 2}$,

$$\begin{aligned}
 \mathcal{F}_{\text{Bin}(2)}(\psi) &:= \sum_{z,w \in V} \pi(z)\pi(w)\psi(z,w)(L^{\text{Bin}(2)} - L^{\text{Bin}(1) \otimes \text{Bin}(1)})\psi(z,w) \\
 (5.14) \quad &= \sum_{xy \in E} c_{xy} \left(\frac{\pi(x)\pi(y)}{\pi(x) + \pi(y)} \right)^2 (\psi(x,x) + \psi(y,y) - \psi(x,y) - \psi(y,x))^2 \\
 &\geq 0.
 \end{aligned}$$

As a consequence of the definition of $\mathcal{F}_{\text{Bin}(2)}$, we get

$$(5.15) \quad \mathcal{E}_{\text{Bin}(2)}(\psi) = \mathcal{E}_{\text{Bin}(1) \otimes \text{Bin}(1)}(\psi) - \mathcal{F}_{\text{Bin}(2)}(\psi),$$

yielding, since $\mathcal{F}_{\text{Bin}(2)}(\psi) \geq 0$, the second inequality in equation (5.10). For what concerns the first inequality in equation (5.10), we show that

$$(5.16) \quad \frac{1}{2} \mathcal{E}_{\text{Bin}(1) \otimes \text{Bin}(1)}(\psi) - \mathcal{F}_{\text{Bin}(2)}(\psi) \geq 0$$

holds for all $\psi \in \mathcal{H}^{\otimes 2}$. For this purpose, let us recall that

$$\begin{aligned}
 &\mathcal{E}_{\text{Bin}(1) \otimes \text{Bin}(1)}(\psi) \\
 &= \sum_{xy \in E} c_{xy} \frac{\pi(x)\pi(y)}{\pi(x) + \pi(y)} \left\{ \sum_{z \in V} \pi(z) ((\psi(x,z) - \psi(y,z))^2 + (\psi(z,x) - \psi(z,y))^2) \right\},
 \end{aligned}$$

therefore, equation (5.16) is equivalent to show that

$$\begin{aligned}
 &\sum_{xy \in E} c_{xy} \frac{\pi(x)\pi(y)}{\pi(x) + \pi(y)} \left\{ \sum_{z \in V} \pi(z) (\psi(x,z) - \psi(y,z))^2 \right. \\
 &\quad \left. - \frac{\pi(x)\pi(y)}{\pi(x) + \pi(y)} ((\psi(x,x) - \psi(y,x)) + (\psi(y,y) - \psi(x,y)))^2 \right\} \\
 &\quad + \sum_{xy \in E} c_{xy} \frac{\pi(x)\pi(y)}{\pi(x) + \pi(y)} \left\{ \sum_{z \in V} \pi(z) (\psi(z,x) - \psi(z,y))^2 \right. \\
 &\quad \left. - \frac{\pi(x)\pi(y)}{\pi(x) + \pi(y)} ((\psi(x,x) - \psi(x,y)) + (\psi(y,y) - \psi(y,x)))^2 \right\}
 \end{aligned}$$

is nonnegative. Now, we claim that this holds because each expression between curly brackets is nonnegative. Indeed, for all $xy \in E$, focusing on the first expression between curly brackets (the second one can be dealt with analogously) and setting

$$(5.17) \quad \begin{aligned}
 u &:= (\psi(x,x) - \psi(y,x)), & p &:= \pi(x), \\
 v &:= (\psi(y,y) - \psi(x,y)), & q &:= \pi(y),
 \end{aligned}$$

we have

$$\begin{aligned}
 &\sum_{z \neq x,y} \pi(z) (\psi(x,z) - \psi(y,z))^2 + \left(pu^2 + qv^2 - \frac{pq}{p+q} (u+v)^2 \right) \\
 &\geq pu^2 + qv^2 - \frac{pq}{p+q} (u+v)^2 \geq 0.
 \end{aligned}$$

This concludes the proof. \square

REMARK 5.3. The first inequality in Lemma 5.2 and the intertwining relation equation (4.5) ensure that not only $\text{gap}_2 = \text{gap}$, as shown in Theorem 2.1, but also that the spectral gap of the *labeled* two-particle system equals gap .

PROPOSITION 5.4. *Under Assumption 1, there exist $C, c > 0$, independent of n , such that*

$$(5.18) \quad \max_{x,y,z,w \in V} \left| \frac{p_t^{\text{Bin}(2)}((x,y),(z,w))}{\pi(z)\pi(w)} - 1 \right| \leq ce^{-\frac{t}{t_{\text{rel}}}}, \quad t \geq Ct_{\text{rel}}.$$

Therefore, as a consequence of the inequalities in (5.4)–(5.8), we have

$$(5.19) \quad \sup_{\eta \in \Delta} \mathbb{E}_\eta^{\text{Avg}} \left[\left\| \frac{\eta_t}{\pi} - 1 \right\|_2^2 \right] \leq ce^{-\frac{t}{t_{\text{rel}}}}, \quad t \geq Ct_{\text{rel}}.$$

PROOF. By [50], Theorem 2.3.4, Assumption 1 implies

$$(5.20) \quad \max_{x,y \in V} \frac{p_t^{\text{Bin}(1)}(x,y)}{\pi(y)} \leq e \left(\frac{dt_{\text{Nash}}}{2t} \right)^{\frac{d}{2}}, \quad t \leq T,$$

and an analogous inequality for the product chain $\text{Bin}(1) \otimes \text{Bin}(1)$: letting $d' := 2d$,

$$(5.21) \quad \max_{x,y,z,w \in V} \frac{p_t^{\text{Bin}(1) \otimes \text{Bin}(1)}((x,y),(z,w))}{\pi(z)\pi(w)} \leq e^2 \left(\frac{d't_{\text{Nash}}}{t} \right)^{\frac{d'}{2}}, \quad t \leq T,$$

where $p_t^{\text{Bin}(1) \otimes \text{Bin}(1)}$ is defined in analogy to equation (5.9). Because of reversibility of $\text{Bin}(1) \otimes \text{Bin}(1)$, the converse to Nash’s argument due to [13] (see also [50], Theorem 2.3.7) ensures that

$$(5.22) \quad \|\psi\|_2^{2(1+\frac{2}{d'})} \leq C't_{\text{Nash}} \left(\mathcal{E}_{\text{Bin}(1) \otimes \text{Bin}(1)}(\psi) + \frac{1}{2T} \|\psi\|_2^2 \right) \|\psi\|_1^{\frac{4}{d'}} \quad \forall \psi \in \mathcal{H}^{\otimes 2},$$

for some constant $C' > 0$ depending only on d . By combining equation (5.22) with the first inequality in Lemma 5.2, we further obtain the following integral version of the Nash inequality for $\text{Bin}(2)$:

$$(5.23) \quad \|\psi\|_2^{2(1+\frac{2}{d'})} \leq 2C't_{\text{Nash}} \left(\mathcal{E}_{\text{Bin}(2)}(\psi) + \frac{1}{4T} \|\psi\|_2^2 \right) \|\psi\|_1^{\frac{4}{d'}} \quad \forall \psi \in \mathcal{H}^{\otimes 2},$$

which, again by [50], Theorem 2.3.4, implies

$$(5.24) \quad \max_{x,y,z,w \in V} \frac{p_t^{\text{Bin}(2)}((x,y),(z,w))}{\pi(z)\pi(w)} \leq e \left(\frac{d'C't_{\text{Nash}}}{t} \right)^{\frac{d'}{2}}, \quad t \leq 4T.$$

By Chapman–Kolmogorov equation, as well as Cauchy–Schwarz and Poincaré inequalities, we obtain, for all $t \geq s \geq 0$,

$$(5.25) \quad \begin{aligned} & \max_{x,y,z,w \in V} \left| \frac{p_t^{\text{Bin}(2)}((x,y),(z,w))}{\pi(z)\pi(w)} - 1 \right| \\ & \leq \max_{x,y \in V} \left\| \frac{p_{t/2}^{\text{Bin}(2)}((x,y), \cdot)}{\pi \otimes \pi} - 1 \right\|_{\mathcal{H}^{\otimes 2}}^2 \\ & \leq \exp \left\{ -\frac{t-s}{t_{\text{rel}}} \right\} \max_{x,y \in V} \left\| \frac{p_{s/2}^{\text{Bin}(2)}((x,y), \cdot)}{\pi \otimes \pi} - 1 \right\|_{\mathcal{H}^{\otimes 2}}^2, \end{aligned}$$

where the last inequality follows by Remark 5.3. Setting $s := 8T = \Theta(t_{\text{Nash}})$ in the last displacement, equation (5.24) yields, for all $t \geq 8T \geq 0$ and some constant $c > 0$,

$$(5.26) \quad \max_{x,y,z,w \in V} \left| \frac{p_t^{\text{Bin}(2)}((x,y),(z,w))}{\pi(z)\pi(w)} - 1 \right| \leq c \exp\left\{-\frac{t}{t_{\text{rel}}}\right\},$$

where we used the fact that $T = O(t_{\text{rel}})$ thanks to Assumption 1. \square

5.3. *Upper bound for longer times.* The next proposition provides the upper bound which will turn out to be central for the proof of Theorem 2.9.

PROPOSITION 5.5. *Under Assumptions 1 and 2, there exists a constant $C' > 0$ such that for all $t > Ct_{\text{rel}}$, with C as in Proposition 5.4, and $\eta \in \Delta$,*

$$(5.27) \quad \mathbb{E}_\eta^{\text{Avg}} \left[\left\| \frac{\eta_t}{\pi} - 1 \right\|_2^2 \right] \leq C' \left(\frac{1}{n} \exp\left\{-\frac{t}{t_{\text{rel}}}\right\} + \exp\left\{-\frac{2t}{t_{\text{rel}}}\right\} \right) \left\| \frac{\eta}{\pi} - 1 \right\|_2^2.$$

PROOF. We start by rewriting the left-hand side of equation (5.27) as

$$(5.28) \quad \begin{aligned} \mathbb{E}_\eta^{\text{Avg}} \left[\left\| \frac{\eta_t}{\pi} - 1 \right\|_2^2 \right] &= \sum_{x \in V} \pi(x) \mathbb{E}_\eta^{\text{Avg}} \left[\left(\frac{\eta_t(x)}{\pi(x)} \right)^2 - 1 \right] \\ &= \sum_{x \in V} \pi(x) \left(\left(\mathbb{E}_\eta^{\text{Avg}} \left[\frac{\eta_t(x)}{\pi(x)} \right] \right)^2 - 1 \right) \\ &\quad + \sum_{x \in V} \pi(x) \left(\mathbb{E}_\eta^{\text{Avg}} \left[\left(\frac{\eta_t(x)}{\pi(x)} \right)^2 \right] - \left(\mathbb{E}_\eta^{\text{Avg}} \left[\frac{\eta_t(x)}{\pi(x)} \right] \right)^2 \right) \\ &= \|h_t^\eta - 1\|_2^2 + \sum_{z \in V} \pi(z) (S_t^{\text{Bin}(2)} - S_t^{\text{Bin}(1) \otimes \text{Bin}(1)}) \left(\frac{\eta}{\pi} \otimes \frac{\eta}{\pi} \right) (z, z), \end{aligned}$$

where we used the duality in equation (3.12) with $k = 1$ and $k = 2$ and the definition in equation (5.1). The rest of the proof is devoted to estimating equation (5.28): while the bound for the first term is straightforward (see equation (5.29)), as for the second term, first we rewrite it in equation (5.30) below, then we split it into two parts in equation (5.37), and conclude bounding these two expressions in Eqs. (5.37) and (5.40).

The first term in equation (5.28), by Poincaré inequality, can be bounded from above by

$$(5.29) \quad \|h_t^\eta - 1\|_2^2 \leq \exp\left\{-\frac{2t}{t_{\text{rel}}}\right\} \left\| \frac{\eta}{\pi} - 1 \right\|_2^2.$$

As for the second term in equation (5.28), noting that $S_t^{\text{Bin}(1) \otimes \text{Bin}(1)} \left(\frac{\eta}{\pi} \otimes \frac{\eta}{\pi} \right) = h_t^\eta \otimes h_t^\eta$, by the integration by parts formula (see, e.g., [38], Proposition VIII.1.7) we obtain

$$\begin{aligned} \mathcal{N}_t^\eta(\eta) &:= \sum_{z \in V} \pi(z) (S_t^{\text{Bin}(2)} - S_t^{\text{Bin}(1) \otimes \text{Bin}(1)}) \left(\frac{\eta}{\pi} \otimes \frac{\eta}{\pi} \right) (z, z) \\ &= \sum_{z \in V} \pi(z) \int_0^t S_{t-s}^{\text{Bin}(2)} (L^{\text{Bin}(2)} - L^{\text{Bin}(1) \otimes \text{Bin}(1)}) S_s^{\text{Bin}(1) \otimes \text{Bin}(1)} \left(\frac{\eta}{\pi} \otimes \frac{\eta}{\pi} \right) (z, z) \, ds \\ &= \sum_{z \in V} \pi(z) \int_0^t \sum_{v,w \in V} p_{t-s}^{\text{Bin}(2)}((z,z),(v,w)) (L^{\text{Bin}(2)} - L^{\text{Bin}(1) \otimes \text{Bin}(1)}) (h_s^\eta \otimes h_s^\eta)(v,w) \, ds. \end{aligned}$$

Hence, by the explicit computation in equation (5.13) and the duality relation in equation (3.12) with $k = 1, 2$ we have

$$(5.30) \quad \mathcal{N}_t(\eta) = \int_0^t \sum_{xy \in E} c_{xy} \frac{\pi(x)\pi(y)}{\pi(x) + \pi(y)} (h_s^\eta(x) - h_s^\eta(y))^2 \Phi_{t-s}(x, y) \, ds,$$

where

$$(5.31) \quad \begin{aligned} \Phi_{t-s}(x, y) &:= \frac{\pi(x)\pi(y)}{\pi(x) + \pi(y)} \sum_{z \in V} \pi(z) \mathbb{E}_{\delta_z}^{\text{Avg}} [(D(x, \eta_{t-s}) - D(y, \eta_{t-s}))^2] \\ &= \frac{\pi(x)\pi(y)}{\pi(x) + \pi(y)} \sum_{z \in V} \pi(z) (g_{t-s}^{x,x}(z, z) + g_{t-s}^{y,y}(z, z) - g_{t-s}^{x,y}(z, z) - g_{t-s}^{y,x}(z, z)), \end{aligned}$$

with

$$(5.32) \quad g_{t-s}^{x,y}(z, z) := \frac{p_{t-s}^{\text{Bin}(2)}((x, y), (z, z))}{\pi(z)\pi(z)} = S_{t-s}^{\text{Bin}(2)}\left(\frac{\mathbf{1}_{(z,z)}(\cdot, \cdot)}{\pi \otimes \pi}\right)(x, y).$$

Concerning the function $t \mapsto \Phi_t(x, y)$, it is easy to check that

$$(5.33) \quad \Phi_0(x, y) = 1, \quad x, y \in V \text{ such that } xy \in E.$$

Further, Assumption 2 yields

$$(5.34) \quad |\Phi_{t-s}(x, y)| \leq 2c_{\text{cell}}, \quad t - s \geq 0.$$

Indeed, since $\sum_{z \in V} p_{t-s}^{\text{Bin}(2)}((a, b), (z, z)) \in [0, 1]$ for $a, b \in V$,

$$\begin{aligned} |\Phi_{t-s}(x, y)| &\leq \frac{\pi(x)\pi(y)}{\pi(x) + \pi(y)} 2 \max \left\{ \sum_{z \in V} \frac{p_{t-s}^{\text{Bin}(2)}((x, x), (z, z))}{\pi(z)}, \right. \\ &\quad \left. \sum_{z \in V} \frac{p_{t-s}^{\text{Bin}(2)}((y, y), (z, z))}{\pi(z)}, \sum_{z \in V} \frac{p_{t-s}^{\text{Bin}(2)}((x, y), (z, z))}{\pi(z)} \right\}, \end{aligned}$$

and equation (5.34) follows at once estimating the above maximum by $(\min_{z \in V} \pi(z))^{-1}$, $\frac{\pi(x)}{\pi(x) + \pi(y)} \leq 1$, and Assumption 2.

We now split, for some $r \in (0, t)$ to be fixed later, the integral in equation (5.30) as

$$(5.35) \quad \begin{aligned} \mathcal{N}_t(\eta) &= \int_0^r \sum_{xy \in E} c_{xy} \frac{\pi(x)\pi(y)}{\pi(x) + \pi(y)} (h_s^\eta(x) - h_s^\eta(y))^2 \Phi_{t-s}(x, y) \, ds \\ &\quad + \int_r^t \sum_{xy \in E} c_{xy} \frac{\pi(x)\pi(y)}{\pi(x) + \pi(y)} (h_s^\eta(x) - h_s^\eta(y))^2 \Phi_{t-s}(x, y) \, ds. \end{aligned}$$

The second term on the right-hand side in equation (5.35), thanks to equation (5.34), is bounded by

$$(5.36) \quad \begin{aligned} &\int_r^t \sum_{xy \in E} c_{xy} \frac{\pi(x)\pi(y)}{\pi(x) + \pi(y)} (h_s^\eta(x) - h_s^\eta(y))^2 \Phi_{t-s}(x, y) \, ds \\ &\leq c_{\text{cell}} \int_r^t 2\mathcal{E}_{\text{Bin}(1)}(h_s^\eta) \, ds \leq c_{\text{cell}} \|h_r^\eta - 1\|_2^2. \end{aligned}$$

Hence, by an application of Poincaré inequality we obtain

$$\begin{aligned}
 (5.37) \quad & \int_r^t \sum_{xy \in E} c_{xy} \frac{\pi(x)\pi(y)}{\pi(x) + \pi(y)} (h_s^\eta(x) - h_s^\eta(y))^2 \Phi_{t-s}(x, y) \, ds \\
 & \leq c_{\text{ell}} \exp\left\{-\frac{2r}{t_{\text{rel}}}\right\} \left\| \frac{\eta}{\pi} - 1 \right\|_2^2.
 \end{aligned}$$

Recalling that $t > Ct_{\text{rel}}$, we can choose $r = t - Ct_{\text{rel}}$ so that, by Proposition 5.4, for all $s \in [0, r]$,

$$(5.38) \quad |\Phi_{t-s}(x, y)| \leq \frac{4c_{\text{ell}}c}{n} \exp\left\{-\frac{t-s}{t_{\text{rel}}}\right\},$$

where the constant c is the same as in Proposition 5.4. Therefore, the first term on the right-hand side in equation (5.35) can be bounded by

$$\begin{aligned}
 (5.39) \quad & \int_0^r \sum_{xy \in E} c_{xy} \frac{\pi(x)\pi(y)}{\pi(x) + \pi(y)} (h_s^\eta(x) - h_s^\eta(y))^2 \Phi_{t-s}(x, y) \, ds \\
 & \leq \frac{2c_{\text{ell}}c}{n} \int_0^r 2\mathcal{E}_{\text{Bin}(1)}(h_s^\eta) e^{-\frac{t-s}{t_{\text{rel}}}} \, ds.
 \end{aligned}$$

By an integration by parts and Poincaré inequality, we obtain

$$\begin{aligned}
 & \int_0^r 2\mathcal{E}_{\text{Bin}(1)}(h_s^\eta) e^{-\frac{t-s}{t_{\text{rel}}}} \, ds \\
 & = e^{-\frac{t}{t_{\text{rel}}}} \left\| \frac{\eta}{\pi} - 1 \right\|_2^2 - e^{-\frac{t-r}{t_{\text{rel}}}} \|h_r^\eta - 1\|_2^2 + \frac{1}{t_{\text{rel}}} \int_0^r e^{-\frac{t-s}{t_{\text{rel}}}} \|h_s^\eta - 1\|_2^2 \, ds \\
 & \leq e^{-\frac{t}{t_{\text{rel}}}} \left\| \frac{\eta}{\pi} - 1 \right\|_2^2 + e^{-\frac{t}{t_{\text{rel}}}} \left\| \frac{\eta}{\pi} - 1 \right\|_2^2 \int_0^r \frac{1}{t_{\text{rel}}} e^{-\frac{s}{t_{\text{rel}}}} \, ds \\
 & \leq 2e^{-\frac{t}{t_{\text{rel}}}} \left\| \frac{\eta}{\pi} - 1 \right\|_2^2,
 \end{aligned}$$

so that equation (5.39) is bounded above by

$$\begin{aligned}
 (5.40) \quad & \int_0^r \sum_{xy \in E} c_{xy} \frac{\pi(x)\pi(y)}{\pi(x) + \pi(y)} (h_s^\eta(x) - h_s^\eta(y))^2 \Phi_{t-s}(x, y) \, ds \\
 & \leq \frac{4c_{\text{ell}}c}{n} e^{-\frac{t}{t_{\text{rel}}}} \left\| \frac{\eta}{\pi} - 1 \right\|_2^2.
 \end{aligned}$$

Collecting Eqs. (5.37) and (5.40) and recalling that $r = t - Ct_{\text{rel}}$ we conclude that

$$(5.41) \quad \mathcal{N}_t(\eta) \leq c_{\text{ell}} \left(\frac{4c}{n} \exp\left\{-\frac{t}{t_{\text{rel}}}\right\} + e^{2C} \exp\left\{-\frac{2t}{t_{\text{rel}}}\right\} \right) \left\| \frac{\eta}{\pi} - 1 \right\|_2^2.$$

By combining Eqs. (5.29) and (5.41) we finally obtain

$$(5.42) \quad \mathbb{E}_\eta^{\text{Avg}} \left[\left\| \frac{\eta_t}{\pi} - 1 \right\|_2^2 \right] \leq c_{\text{ell}} \left(\frac{4c}{n} \exp\left\{-\frac{t}{t_{\text{rel}}}\right\} + (1 + e^{2C}) \exp\left\{-\frac{2t}{t_{\text{rel}}}\right\} \right) \left\| \frac{\eta}{\pi} - 1 \right\|_2^2,$$

and defining properly the constant C' in the statement, we obtain the desired result. \square

5.4. *Proofs of Proposition 2.8 and Theorem 2.9.*

PROOF OF PROPOSITION 2.8. The lower bound in equation (2.36) follows immediately by Lemma 5.1. Concerning the upper bound, recall the estimate in equation (5.19) in Proposition 5.4. Then, by setting $t = bt_{\text{rel}}$ for large enough b and applying Jensen’s inequality, we conclude the proof of equation (2.36) for $p = 2$. The corresponding result for $p \in [1, 2)$ follows again by Jensen’s inequality. \square

PROOF OF THEOREM 2.9. As in the proof of Proposition 2.8, the lower bound is an immediate consequence of Lemma 5.1. For the upper bound we will exploit Proposition 5.5. Take $t = t^+(C)$ as in equation (2.24) for some sufficiently large C to be fixed later. Fixing now $s = \frac{C}{2}t_{\text{rel}}$, $k \rightarrow \infty$ ensures that $t - s > Ct_{\text{rel}}$ for large enough n ; hence, we can apply Proposition 5.5 in the time window $[s, t]$, namely,

$$(5.43) \quad \mathbb{E}_\eta^{\text{Avg}} \left[\left\| \frac{\eta_t}{\pi} - 1 \right\|_2^2 \right] = \mathbb{E}_\eta^{\text{Avg}} \left[\mathbb{E}_\eta^{\text{Avg}} \left[\left\| \frac{\eta_t}{\pi} - 1 \right\|_2^2 \mid \eta_s \right] \right] \leq C' \left(\frac{1}{n} \exp \left\{ -\frac{t-s}{t_{\text{rel}}} \right\} + \exp \left\{ -\frac{2(t-s)}{t_{\text{rel}}} \right\} \right) \mathbb{E}_\eta^{\text{Avg}} \left[\left\| \frac{\eta_s}{\pi} - 1 \right\|_2^2 \right],$$

where we used the Markov property and the estimate in equation (5.27) for the process starting from η_s and evolving for a time $t - s$. Now, if C is large enough, we can use Proposition 5.4 to bound the expectation in the right-hand side of equation (5.43), that is,

$$(5.44) \quad \mathbb{E}_\eta^{\text{Avg}} \left[\left\| \frac{\eta_t}{\pi} - 1 \right\|_2^2 \right] \leq cC' \left(\frac{1}{n} \exp \left\{ -\frac{t}{t_{\text{rel}}} \right\} + \exp \left\{ -\frac{2t-s}{t_{\text{rel}}} \right\} \right).$$

Multiplying both sides in equation (5.44) by k , and substituting the values of $t = t_{\text{mix}} + Ct_{\text{rel}}$ and $s = \frac{C}{2}t_{\text{rel}}$, we obtain

$$(5.45) \quad k \mathbb{E}_\eta^{\text{Avg}} \left[\left\| \frac{\eta_t}{\pi} - 1 \right\|_2^2 \right] \leq cC' \left(\frac{\sqrt{k}}{n} e^{-C} + e^{-\frac{3}{2}C} \right).$$

The upper bound in equation (2.37) follows for all $p \in [1, 2]$ by applications of Jensen’s inequality. \square

We conclude this section by showing that when the assumption $k = O(n^2)$ in equation (2.25) is dropped, the arguments used so far show that a weaker version of Theorem 2.3 still holds.

PROPOSITION 5.6 (Pre-cutoff). *Consider a sequence of graphs and site-weights such that Assumptions 1 and 2 hold. Then, if $k/n^2 \rightarrow \infty$, for all $\delta \in (0, 1)$,*

$$(5.46) \quad \limsup_{n \rightarrow \infty} \sqrt{k} \sup_{\eta \in \Delta} \mathbb{E}_\eta^{\text{Avg}} \left[\left\| \frac{\eta_{T^+}}{\pi} - 1 \right\|_p \right] \leq \delta, \quad \liminf_{n \rightarrow \infty} \sqrt{k} \sup_{\eta \in \Delta} \mathbb{E}_\eta^{\text{Avg}} \left[\left\| \frac{\eta_{T^-}}{\pi} - 1 \right\|_p \right] \geq \frac{1}{\delta},$$

for all $p \in [1, 2]$, where

$$(5.47) \quad T^+ := a \frac{t_{\text{rel}}}{2} \log(k) + Ct_{\text{rel}}, \quad T^- := \frac{t_{\text{rel}}}{2} \log(k) - Ct_{\text{rel}},$$

$$(5.48) \quad a := 2 \frac{\log(k/n)}{\log(k)} \in [1, 2],$$

and some $C = C(\delta) > 0$.

PROOF. The upper bound (for $p = 2$) follows directly by equation (5.44). The lower bound (for $p = 1$) follows by Lemma 5.1. \square

Notice that, as soon as $k = \Omega(n^{2+\varepsilon})$ for some $\varepsilon > 0$, then $T^+ - T^- = \Theta(t_{\text{rel}} \log(k))$, namely, their first order terms do not coincide. On the other hand, Proposition 5.6 implies a cutoff also for all those $k = n^{2+o(1)}$, but with a larger window having size $\omega(t_{\text{rel}})$.

6. Proofs from Section 2.4. In this section, we provide the proofs of the results from Section 2.4. More specifically, in Sections 6.1–6.3 below we derive some intermediate results needed for such proofs, which are then presented in Section 6.4 below. In particular, in Section 6.1 we provide an upper bound for the TV-distance of the binomial splitting process when the initial distribution of particles is multinomial in terms of the L^2 -distance of the averaging process. The crucial ingredient in this step is the multinomial intertwining in Proposition 3.1. In Section 6.2 we extend this bound to all possible initial distribution by means of “multi-colored” auxiliary processes defined therein. Finally, in Section 6.3 we use Wilson’s method, originally introduced in [56], to show a matching lower bound.

6.1. *Upper bound for multinomial initial distributions.*

LEMMA 6.1. *For all $k \in \mathbb{N}$ and $\eta \in \Delta$, recall the definition of $\mu_{k,\eta}$ as the multinomial distribution with parameters (k, η) . Then,*

$$(6.1) \quad \|\mu_{k,\eta} \mathcal{S}_t^{\text{Bin}(k)} - \mu_{k,\pi}\|_{\text{TV}} \leq \sqrt{ek \mathbb{E}_\eta^{\text{Avg}} \left[\left\| \frac{\eta_t}{\pi} - 1 \right\|_2^2 \right]}, \quad t \geq 0.$$

We defer its proof after the following preliminary result, in which we derive an upper bound on the TV distance between two multinomial distributions with the same sample size.

LEMMA 6.2. *For all $k \in \mathbb{N}$ and $\eta, \pi \in \Delta$,*

$$(6.2) \quad \|\mu_{k,\eta} - \mu_{k,\pi}\|_{\text{TV}} \leq 1 \wedge \left\| \frac{\mu_{k,\eta}}{\mu_{k,\pi}} - 1 \right\|_{L^2(\Omega_k, \mu_{k,\pi})} \leq \sqrt{ek} \left\| \frac{\eta}{\pi} - 1 \right\|_2.$$

PROOF. Jensen’s inequality yields the first inequality. As for the second one, by simple manipulations with multinomial distributions, we have

$$(6.3) \quad \begin{aligned} \left\| \frac{\mu_{k,\eta}}{\mu_{k,\pi}} - 1 \right\|_{L^2(\Omega_k, \mu_{k,\pi})}^2 &= \sum_{\xi \in \Omega_k} \mu_{k,\pi}(\xi) \left(\left(\frac{\mu_{k,\eta}(\xi)}{\mu_{k,\pi}(\xi)} \right)^2 - 1 \right) \\ &= \left(\sum_{\xi \in \Omega_k} \mu_{k,\pi}(\xi) \prod_{x \in V} \left(\frac{\eta(x)}{\pi(x)} \right)^{2\xi(x)} \right) - 1 \\ &= \left(\sum_{\mathbf{x} \in V^k} \pi(\mathbf{x}) (D(\mathbf{x}, \eta))^2 \right) - 1 \\ &= \left(\sum_{x \in V} \pi(x) (D(x, \eta))^2 \right)^k - 1 \\ &= \left(1 + \left\| \frac{\eta}{\pi} - 1 \right\|_2^2 \right)^k - 1. \end{aligned}$$

(Note that for the third step, we just used that, for any labeled configuration $\mathbf{x} \in V^k$ and its corresponding unlabeled one $\xi \in \Omega_k$, $D(\mathbf{x}, \eta) = \prod_{x \in V} \frac{\eta(x)^{\xi(x)}}{\pi(x)}$ holds for all $\eta \in \Delta$.) The inequality $1 \wedge [(1 + a)^k - 1] \leq eka$, $a \geq 0$, yields the desired result. \square

PROOF OF LEMMA 6.1. By using the intertwining relation in Proposition 3.1, we have

$$(6.4) \quad \begin{aligned} \|\mu_{k,\eta} \mathcal{S}_t^{\text{Bin}(k)} - \mu_{k,\pi}\|_{\text{TV}} &:= \max_{A \subseteq \Omega_k} |\Lambda_k \mathcal{S}_t^{\text{Bin}(k)} \mathbf{1}_A(\eta) - \Lambda_k \mathbf{1}_A(\pi)| \\ &= \max_{A \subseteq \Omega_k} |\mathcal{S}_t^{\text{Avg}} \Lambda_k \mathbf{1}_A(\eta) - \Lambda_k \mathbf{1}_A(\pi)|. \end{aligned}$$

By moving the maximum within the expectation, the right-hand side above is further bounded by

$$(6.5) \quad \|\mu_{k,\eta} \mathcal{S}_t^{\text{Bin}(k)} - \mu_{k,\pi}\|_{\text{TV}} \leq \mathbb{E}_\eta^{\text{Avg}} [\|\mu_{k,\eta_t} - \mu_{k,\pi}\|_{\text{TV}}].$$

Finally, applying Lemma 6.2 and Jensen’s inequality concludes the proof. \square

6.2. *Multicolored processes and upper bound for general initial conditions.* The upper bound in Lemma 6.1 in the previous section was derived only for particle systems initialized according to multinomial distributions; as a particular case, setting $\eta = \delta_x \in \Delta$, Lemma 6.1 holds true for $\mu_{k,\eta} = \mu_{k,\delta_x}$, that is, the Dirac measure on the configuration with all k particles piled at $x \in V$. Of course, the same upper bound carries over to arbitrary convex combinations of multinomial distributions: for all probability measures ν on Δ ,

$$(6.6) \quad \left\| \int_{\Delta} \nu(d\eta) \mu_{k,\eta} \mathcal{S}_t^{\text{Bin}(k)} - \mu_{k,\pi} \right\|_{\text{TV}} \leq \sqrt{ek} \mathbb{E}_\nu^{\text{Avg}} \left[\left\| \frac{\eta_t}{\pi} - 1 \right\|_2 \right], \quad t \geq 0.$$

However, while De Finetti’s theorem ensures that such convex combinations would exhaust the space of probability measures on Ω_k if V were infinite, this is far from being true when V is finite (see, e.g., [5, 21]). In fact, it is simple to check that there exists a constant $\gamma \in (0, 1)$ such that

$$(6.7) \quad \sup_{\mu} \inf_{\nu} \left\| \mu - \int_{\Delta} \nu(d\eta) \mu_{k,\eta} \right\|_{\text{TV}} \geq \gamma$$

holds for all n and $k \in \mathbb{N}$ large enough, where the above supremum runs over all probability measures on Ω_k . Hence, multinomial initial distributions do not suffice to approximate via convex combinations all initial conditions.

In order to bypass this obstacle and derive an upper bound for all initial conditions, we introduce below what we call *multicolored averaging* and *binomial splitting processes*, and prove a corresponding intertwining relation between them.

Let us start by assigning to each site $z \in V$ a different color. For each $z \in V$, consider a probability measure $\eta^{(z)} \in \Delta$ on V equipped with the corresponding color. We now construct the multicolored averaging $(\vec{\eta}_t)_{t \geq 0} = ((\eta_t^{(z)})_{z \in V})_{t \geq 0} \subseteq \Delta^V$ started from $\vec{\eta} = (\eta^{(z)})_{z \in V}$ by means of the following grand coupling: at time $t = 0$, each z -coordinate $\eta_0^{(z)}$ is set equal to $\eta^{(z)} \in \Delta$; then, at independent exponential times of rates $(c_{xy})_{xy \in E}$, let all coordinates $(\eta_t^{(z)})_{t \geq 0}$, $z \in V$, thermalize simultaneously their values at the same edge. In other words, each colored coordinate evolves as the averaging process, with the constraint that all share the same random sequence of edge updates. More precisely, recalling the definition in equation (2.28), the generator of this colored process writes as

$$(6.8) \quad \vec{\mathcal{L}}^{\text{Avg}} f := \sum_{xy \in E} c_{xy} \left(\bigotimes_{z \in V} (\mathcal{P}_{xy}^{\text{Avg}} - \mathbb{1}) \right) f,$$

for all $f \in \mathcal{C}(\Delta^V) = \bigotimes_{z \in V} \mathcal{C}(\Delta)$.

An entirely analogous construction carries over for the multicolored binomial splitting started from the configuration $\xi \in \Omega_k$: at time $t = 0$, for all sites $z \in V$, the $\xi(z)$ particles at $z \in V$ are given the color assigned to $z \in V$; then, following the same edge updates, each colored particle system performs the binomial splitting dynamics. Hence, we write $(\vec{\xi}_t)_{t \geq 0} = ((\xi_t^{(z)})_{z \in V})_{t \geq 0} \subseteq \prod_{z \in V} \Omega_{\xi(z)}$ to indicate the vector of such colored binomial splitting process started from $\xi \in \Omega_k$, with generator given by

$$(6.9) \quad \vec{\mathcal{L}}^{\text{Bin}(\xi)} f := \sum_{xy \in E} c_{xy} \left(\bigotimes_{z \in V} (\mathcal{P}_{xy}^{\text{Bin}(\xi(z))} - \mathbb{1}) \right) f$$

for all $f : \prod_{z \in V} \Omega_{\xi(z)} \rightarrow \mathbb{R}$. Notice that choosing $\xi \in \Omega_k$ with

$$(6.10) \quad w \mapsto \xi(w) := \begin{cases} k & \text{if } w = z, \\ 0 & \text{otherwise} \end{cases}$$

as initial configuration, the projection of $(\vec{\xi}_t)_{t \geq 0}$ on its z -coordinate corresponds to the $\text{Bin}(k)$ system started with all k particles at $z \in V$.

Having introduced such multicolored processes, the following intertwining relation generalizes Proposition 3.1 to such processes.

PROPOSITION 6.3 (Intertwining for the multicolored processes). *For all $k \in \mathbb{N}$, $\xi \in \Omega_k$ and $t \geq 0$,*

$$(6.11) \quad \vec{S}_t^{\text{Avg}} \vec{\Lambda}_\xi f = \vec{\Lambda}_\xi \vec{S}_t^{\text{Bin}(\xi)} f, \quad f : \prod_{z \in V} \Omega_{\xi(z)} \rightarrow \mathbb{R},$$

where

$$(6.12) \quad \vec{\Lambda}_\xi f(\vec{\eta}) := \mathbb{E}_{(\otimes_{z \in V} \mu_{\xi(z), \eta(z)})} [f],$$

and

$$(6.13) \quad \vec{S}_t^{\text{Avg}} := e^{t \vec{\mathcal{L}}^{\text{Avg}}} \quad \text{and} \quad \vec{S}_t^{\text{Bin}(\xi)} := e^{t \vec{\mathcal{L}}^{\text{Bin}(\xi)}}, \quad t \geq 0.$$

PROOF. As in the proof of Proposition 3.1, checking the identity of equation (6.11) for the corresponding bounded generators, by linearity, it suffices to verify that

$$(6.14) \quad \left(\bigotimes_{z \in V} (\mathcal{P}_{xy}^{\text{Avg}} - \mathbb{1}) \right) \vec{\Lambda}_\xi f = \vec{\Lambda}_\xi \left(\bigotimes_{z \in V} (\mathcal{P}_{xy}^{\text{Bin}(\xi(z))} - \mathbb{1}) \right) f$$

holds for all $xy \in E$, $\xi \in \Omega_k$ and $f : \prod_{z \in V} \Omega_{\xi(z)} \rightarrow \mathbb{R}$. The rest of the argument goes on as in the proof of Proposition 3.1 due to the product structure of the operators involved. \square

In view of the above intertwining, we are now ready to extend the upper bound in Lemma 6.1 to all initial conditions.

LEMMA 6.4. *For all $k \in \mathbb{N}$ and $t \geq 0$,*

$$(6.15) \quad \sup_{\mu} \|\mu \mathcal{S}_t^{\text{Bin}(k)} - \mu_{k, \pi}\|_{\text{TV}} \leq \sqrt{ek \sup_{\eta \in \Delta} \mathbb{E}_{\eta}^{\text{Avg}} \left[\left\| \frac{\eta_t}{\pi} - 1 \right\|_2^2 \right]},$$

where the supremum on the left-hand side runs over all probability measures on Ω_k .

PROOF. First observe that, since all measures μ write as convex combinations of Dirac measures $(\delta_\xi)_{\xi \in \Omega_k}$,

$$(6.16) \quad \sup_{\mu} \|\mu \mathcal{S}_t^{\text{Bin}(k)} - \mu_{k,\pi}\|_{\text{TV}} \leq \sup_{\xi \in \Omega_k} \|\delta_\xi \mathcal{S}_t^{\text{Bin}(k)} - \mu_{k,\pi}\|_{\text{TV}}.$$

Then, since $(\xi_t)_{t \geq 0}$ started from $\xi \in \Omega_k$ can be obtained as a projection of the multicolored process $(\vec{\xi}_t)_{t \geq 0}$ also started from $\xi \in \Omega_k$, we have

$$(6.17) \quad \|\delta_\xi \mathcal{S}_t^{\text{Bin}(k)} - \mu_{k,\pi}\|_{\text{TV}} \leq \left\| \left(\bigotimes_{z \in V} \mu_{\xi(z), \delta_z} \right) \vec{\mathcal{S}}_t^{\text{Bin}(\xi)} - \left(\bigotimes_{z \in V} \mu_{\xi(z), \pi} \right) \right\|_{\text{TV}}.$$

Arguing as in the proof of Lemma 6.1, by means of the intertwining relation in Proposition 6.3, it follows that

$$(6.18) \quad \begin{aligned} & \left\| \left(\bigotimes_{z \in V} \mu_{\xi(z), \delta_z} \right) \vec{\mathcal{S}}_t^{\text{Bin}(\xi)} - \left(\bigotimes_{z \in V} \mu_{\xi(z), \pi} \right) \right\|_{\text{TV}} \\ & \leq \sup_{\vec{\eta} \in \Delta^V} \mathbb{E}_{\vec{\eta}}^{\text{Avg}} \left[\left\| \left(\bigotimes_{z \in V} \mu_{\xi(z), \eta_t^{(z)}} \right) - \left(\bigotimes_{z \in V} \mu_{\xi(z), \pi} \right) \right\|_{\text{TV}} \right] \\ & \leq \sup_{\vec{\eta} \in \Delta^V} \mathbb{E}_{\vec{\eta}}^{\text{Avg}} \left[\left(1 \wedge \sum_{z \in V} \left\| \frac{\mu_{\xi(z), \eta_t^{(z)}}}{\mu_{\xi(z), \pi}} - 1 \right\|_{L^2(\Omega_{\xi(z), \mu_{\xi(z), \pi}})} \right)^{\frac{1}{2}} \right], \end{aligned}$$

where in the last inequality we employed [39], Proposition 7, to bound from above the TV distance between two product measures. (Here, $\mathbb{E}_{\vec{\eta}}^{\text{Avg}}$ denotes expectation with respect to the law of the multicolored averaging started from $\vec{\eta} \in \Delta^V$.) By Jensen’s inequality and the observation that each colored marginal of $(\vec{\eta}_t)_{t \geq 0} \subseteq \Delta^V$ evolves like the averaging process, Lemma 6.2 further yields

$$(6.19) \quad \begin{aligned} & \sup_{\vec{\eta} \in \Delta^V} \mathbb{E}_{\vec{\eta}}^{\text{Avg}} \left[\left(1 \wedge \sum_{z \in V} \left\| \frac{\mu_{\xi(z), \eta_t^{(z)}}}{\mu_{\xi(z), \pi}} - 1 \right\|_{L^2(\Omega_{\xi(z), \mu_{\xi(z), \pi}})} \right)^{\frac{1}{2}} \right] \\ & \leq \sup_{\eta \in \Delta} \sqrt{\sum_{z \in V} e^{\xi(z)} \mathbb{E}_{\eta}^{\text{Avg}} \left[\left\| \frac{\eta_t}{\pi} - 1 \right\|_2^2 \right]} = \sup_{\eta \in \Delta} \sqrt{ek \mathbb{E}_{\eta}^{\text{Avg}} \left[\left\| \frac{\eta_t}{\pi} - 1 \right\|_2^2 \right]}. \end{aligned}$$

Combining the inequalities in (6.16)–(6.19) concludes the proof. \square

6.3. *Lower bound.* The lower bound proof is based on Wilson’s method (see, e.g., [37], Theorem 13.28). Roughly speaking, the method amounts in exhibiting a *distinguishing statistics*, F , whose knowledge of means and variances in and out of equilibrium suffices to obtain a lower bound for the TV-distance to equilibrium. Further, as we will now explain, such a distinguishing statistic is built from an eigenfunction of the Markov generator of the single-particle system.

Let $\psi : V \rightarrow \mathbb{R}$ be an eigenfunction of $-L^{\text{Bin}(1)}$ associated to the spectral gap and such that $\|\psi\|_2 = 1$; in particular, we have

$$(6.20) \quad \sum_{x \in V} \pi(x) \psi(x) = 0 \quad \text{and} \quad \sum_{x \in V} \pi(x) (\psi(x))^2 = 1.$$

Wilson’s method will be applied to the observable $F : \Omega_k \rightarrow \mathbb{R}$ defined as

$$(6.21) \quad F(\xi) := \sum_{x \in V} \psi(x) \xi(x).$$

To the purpose of analyzing F , recall that the mean and covariance of a multinomial distribution are given by

$$(6.22) \quad \begin{aligned} \mathbb{E}_{\mu_{k,\eta}}[\xi(x)] &= k\eta(x), \\ \text{Cov}_{\mu_{k,\eta}}(\xi(x), \xi(y)) &= -k\eta(x)\eta(y)\mathbf{1}_{\{x \neq y\}} + k\eta(x)(1 - \eta(x))\mathbf{1}_{\{x=y\}}. \end{aligned}$$

Recall that $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathcal{H} = L^2(V, \pi)$.

LEMMA 6.5 (Mean and variance at equilibrium). *For every sequence $k = k_n$,*

$$(6.23) \quad \mathbb{E}_{\mu_{k,\pi}}[F] = 0 \quad \text{and} \quad \text{Var}_{\mu_{k,\pi}}(F) = k.$$

PROOF. The proof of Lemma 6.5 follows by the definition of F in equation (6.21), as well as the identities in Eqs. (6.20) and (6.22). \square

LEMMA 6.6 (Mean and variance out of equilibrium). *For every sequence $k = k_n$, $\eta \in \Delta$ and $t \geq Ct_{\text{rel}}$, where C is as in Proposition 5.4,*

$$(6.24) \quad \mathbb{E}_{\mu_{k,\eta} S_t^{\text{Bin}(k)}}[F] = ke^{-\frac{t}{t_{\text{rel}}}} \langle \psi, D(\cdot, \eta) \rangle,$$

and

$$(6.25) \quad \text{Var}_{\mu_{k,\eta} S_t^{\text{Bin}(k)}}(F) \leq C' \left[k + \frac{k^2}{n} \left(\exp \left\{ -\frac{2t}{t_{\text{rel}}} \right\} \|D(\cdot, \eta)\|_\infty^2 + \exp \left\{ -\frac{t}{t_{\text{rel}}} \right\} \right) \right],$$

for some absolute constant $C' > 1$.

PROOF. To prove the identity in equation (6.24) we use the self-duality in Proposition 3.2 with $k = 1, 2$, the first identity in equation (6.22), self-adjointness of $S_t^{\text{Bin}(1)}$ and the fact that ψ is an eigenfunction:

$$(6.26) \quad \begin{aligned} \mathbb{E}_{\mu_{k,\eta} S_t^{\text{Bin}(k)}}[F] &= \sum_{x \in V} \pi(x) \psi(x) \mathbb{E}_{\mu_{k,\eta}}^{\text{Bin}(k)} \left[\frac{\xi_t(x)}{\pi(x)} \right] \\ &= \sum_{x \in V} \pi(x) \psi(x) \mathbb{E}_{\mu_{k,\eta}} \left[S_t^{\text{Bin}(1)} \left(\frac{\xi(\cdot)}{\pi(\cdot)} \right) (x) \right] \\ &= k \sum_{x \in V} \pi(x) \psi(x) S_t^{\text{Bin}(1)}(D(\cdot, \eta))(x) \\ &= ke^{-\frac{t}{t_{\text{rel}}}} \sum_{x \in V} \pi(x) \psi(x) D(x, \eta). \end{aligned}$$

For what concerns the proof of equation (6.25), we have

$$\begin{aligned} &\text{Var}_{\mu_{k,\eta} S_t^{\text{Bin}(k)}}(F) \\ &= \sum_{x \in V} \psi(x)^2 \mathbb{E}_{\mu_{k,\eta}}^{\text{Bin}(k)}[\xi_t(x)] \\ &\quad + \sum_{x,y \in V} \psi(x)\psi(y) \{ \mathbb{E}_{\mu_{k,\eta}}^{\text{Bin}(k)}[\xi_t(x)(\xi_t(y) - \mathbf{1}_{x=y})] - \mathbb{E}_{\mu_{k,\eta}}^{\text{Bin}(k)}[\xi_t(x)] \mathbb{E}_{\mu_{k,\eta}}^{\text{Bin}(k)}[\xi_t(y)] \}. \end{aligned}$$

As already noticed in equation (6.26), we have, for all $x \in V$,

$$(6.27) \quad \mathbb{E}_{\mu_{k,\eta}}^{\text{Bin}(k)}[\xi_t(x)] = k\pi(x) S_t^{\text{Bin}(1)}(D(\cdot, \eta))(x);$$

analogously, it follows that, for all $x, y \in V$,

$$(6.28) \quad \mathbb{E}_{\mu_{k,\eta}}^{\text{Bin}(k)}[\xi_t(x)]\mathbb{E}_{\mu_{k,\eta}}^{\text{Bin}(k)}[\xi_t(y)] = k^2\pi(x)\pi(y)S_t^{\text{Bin}(1)\otimes\text{Bin}(1)}(D(\cdot, \eta))(x, y),$$

$$(6.29) \quad \mathbb{E}_{\mu_{k,\eta}}^{\text{Bin}(k)}[\xi_t(x)(\xi_t(y) - \mathbf{1}_{x=y})] = k(k-1)\pi(x)\pi(y)S_t^{\text{Bin}(2)}(D(\cdot, \eta))(x, y).$$

Therefore

$$\begin{aligned} & \text{Var}_{\mu_{k,\eta}S_t^{\text{Bin}(k)}}(F) \\ &= k \sum_{x \in V} \pi(x)\psi(x)^2 S_t^{\text{Bin}(1)}(D(\cdot, \eta))(x) - k \left(\sum_{x \in V} \pi(x)\psi(x) S_t^{\text{Bin}(1)}(D(\cdot, \eta))(x) \right)^2 \\ (6.30) \quad &+ k(k-1) \sum_{x,y \in V} \pi(x)\pi(y)\psi(x)\psi(y) (S_t^{\text{Bin}(2)} - S_t^{\text{Bin}(1)\otimes\text{Bin}(1)})(D(\cdot, \eta))(x, y) \\ &= k \text{Var}_{\eta S_t^{\text{Bin}(1)}}(\psi) \\ &+ k(k-1) \sum_{x,y \in V} \pi(x)\pi(y)\psi(x)\psi(y) (S_t^{\text{Bin}(2)} - S_t^{\text{Bin}(1)\otimes\text{Bin}(1)})(D(\cdot, \eta))(x, y). \end{aligned}$$

By Assumption 1, the choice of $t > Ct_{\text{rel}}$ and $\|\psi\|_2^2 = 1$, recalling equation (2.11), we obtain

$$(6.31) \quad \text{Var}_{\eta S_t^{\text{Bin}(1)}}(\psi) \leq \langle \psi^2, S_t^{\text{Bin}(1)} D(\cdot, \eta) \rangle = \langle \psi^2, h_t^\eta \rangle \leq c,$$

for some constant $c > 1$ independent of n . On the other hand, by the integration by parts formula, we rewrite

$$\sum_{x,y \in V} \pi(x)\pi(y)\psi(x)\psi(y) (S_t^{\text{Bin}(2)} - S_t^{\text{Bin}(1)\otimes\text{Bin}(1)})(D(\cdot, \eta))(x, y)$$

as $(L^{(2)} - L^{(1,1)}) := L^{\text{Bin}(2)} - L^{\text{Bin}(1)\otimes\text{Bin}(1)}$

$$\begin{aligned} & \int_0^t \sum_{x,y \in V} \pi(x)\pi(y) (S_{t-s}^{\text{Bin}(2)} (L^{(2)} - L^{(1,1)}) S_s^{\text{Bin}(1)\otimes\text{Bin}(1)}) (\psi \otimes \psi)(x, y) D((x, y), \eta) ds \\ &= \int_0^t \sum_{x,y \in V} \pi(x)\pi(y) e^{-\frac{2s}{t_{\text{rel}}}} (L^{(2)} - L^{(1,1)}) (\psi \otimes \psi)(x, y) S_{t-s}^{\text{Bin}(2)} D(\cdot, \eta)(x, y) ds, \end{aligned}$$

where the last identity is an immediate consequence of self-adjointness of the Bin(2)-semigroup and the fact that ψ is eigenfunction for Bin(1). By performing a change of variables and using the explicit computation in equation (5.13), we further obtain

$$\begin{aligned} & \sum_{x,y \in V} \pi(x)\pi(y)\psi(x)\psi(y) (S_t^{\text{Bin}(2)} - S_t^{\text{Bin}(1)\otimes\text{Bin}(1)})(D(\cdot, \eta))(x, y) \\ (6.32) \quad &= e^{-\frac{2t}{t_{\text{rel}}}} \sum_{xy \in E} c_{xy} \frac{\pi(x)\pi(y)}{\pi(x) + \pi(y)} (\psi(x) - \psi(y))^2 \int_0^t e^{\frac{2s}{t_{\text{rel}}}} \frac{\pi(x)\pi(y)}{\pi(x) + \pi(y)} \Psi_s(x, y) ds, \end{aligned}$$

where, for notational convenience, we write, for all $s \geq 0$ and $x, y \in V$,

$$\begin{aligned} \Psi_s(x, y) &:= \mathbb{E}_\eta^{\text{Avg}} \left[\left(\frac{\eta_s(x)}{\pi(x)} - \frac{\eta_s(y)}{\pi(y)} \right)^2 \right] \\ &= \sum_{z,w \in V} \eta(z)\eta(w) \left[\frac{p_s^{\text{Bin}(2)}((z, w), (x, x))}{\pi(x)^2} + \frac{p_s^{\text{Bin}(2)}((z, w), (y, y))}{\pi(y)^2} \right. \\ &\quad \left. - 2 \frac{p_s^{\text{Bin}(2)}((z, w), (x, y))}{\pi(x)\pi(y)} \right]. \end{aligned}$$

Let us now provide two upper bounds for $\Psi_s(x, y)$ depending on the value of $s \geq 0$. On the one hand, for all $s \geq 0$ and $x, y \in V$, by $(a - b)^2 \leq 2a^2 + 2b^2$ and the duality in Proposition 3.3 (in equation (3.12)) with $k = 2$, we have

$$\begin{aligned}
 \Psi_s(x, y) &\leq 2\mathbb{E}_\eta^{\text{Avg}} \left[\left(\frac{\eta_s(x)}{\pi(x)} \right)^2 \right] + 2\mathbb{E}_\eta^{\text{Avg}} \left[\left(\frac{\eta_s(y)}{\pi(y)} \right)^2 \right] \\
 &\leq 4 \max_{z \in V} \left(\frac{\eta(z)}{\pi(z)} \right)^2 = 4 \|D(\cdot, \eta)\|_\infty^2.
 \end{aligned}
 \tag{6.33}$$

On the other hand, by Hölder inequality and $\eta \in \Delta$, Proposition 5.4 ensures that

$$\Psi_s(x, y) \leq 4 \max_{x, y, z, w \in V} \left| \frac{p_s^{\text{Bin}(2)}((x, y), (z, w))}{\pi(z)\pi(w)} - 1 \right| \leq 4ce^{-\frac{s}{t_{\text{rel}}}}, \quad s \geq Ct_{\text{rel}}.
 \tag{6.34}$$

As done in the proof of Proposition 5.5, we split the integral in equation (6.32) in two, obtaining

$$\begin{aligned}
 &\int_0^t \exp\left\{ \frac{2s}{t_{\text{rel}}} \right\} \frac{\pi(x)\pi(y)}{\pi(x) + \pi(y)} \Psi_s(x, y) \, ds \\
 &\leq \frac{4c_{\text{cell}}}{n} \|D(\cdot, \eta)\|_\infty^2 \int_0^{Ct_{\text{rel}}} \exp\left\{ \frac{2s}{t_{\text{rel}}} \right\} \, ds + \frac{4cc_{\text{cell}}}{n} \int_{Ct_{\text{rel}}}^t \exp\left\{ \frac{s}{t_{\text{rel}}} \right\} \, ds \\
 &\leq \frac{t_{\text{rel}}}{n} (2c_{\text{cell}}e^{2C} \|D(\cdot, \eta)\|_\infty^2 + 4cc_{\text{cell}}e^{\frac{t}{t_{\text{rel}}}}) \\
 &\leq \tilde{C} \frac{t_{\text{rel}}}{n} (\|D(\cdot, \eta)\|_\infty^2 + e^{\frac{t}{t_{\text{rel}}}}),
 \end{aligned}
 \tag{6.35}$$

where we absorbed all constants in the quantity $\tilde{C} > 0$. Plugging equation (6.35) into equation (6.32) and noticing that by our choice of ψ we have $\mathcal{E}_{\text{Bin}(1)}(\psi) = \frac{1}{t_{\text{rel}}}$, we obtain

$$\begin{aligned}
 &\sum_{x, y \in V} \pi(x)\pi(y)\psi(x)\psi(y) (S_t^{\text{Bin}(2)} - S_t^{\text{Bin}(1) \otimes \text{Bin}(1)})(D(\cdot, \eta))(x, y) \\
 &\leq \frac{\tilde{C}}{n} \left(\exp\left\{ -\frac{2t}{t_{\text{rel}}} \right\} \|D(\cdot, \eta)\|_\infty^2 + \exp\left\{ -\frac{t}{t_{\text{rel}}} \right\} \right).
 \end{aligned}
 \tag{6.36}$$

Collecting the estimates in Eqs. (6.31) and (6.36), and going back to equation (6.30) we finally obtain equation (6.25). \square

The next proposition is a straightforward application of [37], Proposition 7.9, using the estimates in Lemmata 6.5 and 6.6.

PROPOSITION 6.7. *For all $k \in \mathbb{N}$, $\eta \in \Delta$ and $t \geq Ct_{\text{rel}}$,*

$$\|\mu_{k, \eta} \mathcal{S}_t^{\text{Bin}(k)} - \mu_{k, \pi}\|_{\text{TV}} \geq 1 - \frac{8}{a_t(k, \eta)},
 \tag{6.37}$$

where

$$a_t(k, \eta) := \frac{k \langle \psi, D(\cdot, \eta) \rangle^2}{1 + \frac{k}{n} (\|D(\cdot, \eta)\|_\infty^2 + \exp\{\frac{t}{t_{\text{rel}}}\})},
 \tag{6.38}$$

where $\psi : V \rightarrow \mathbb{R}$ is an eigenfunction of $-L^{\text{Bin}(1)}$ associated to the eigenvalue gap such that $\|\psi\|_2 = 1$.

6.4. *Proofs of Proposition 2.4 and Theorem 2.3.*

PROOF OF PROPOSITION 2.4. The upper bound in Proposition 2.4 follows at once from Lemma 6.4 and the upper bound in Proposition 2.8. Concerning the lower bound, by the injectivity of the operator J_k in equation (4.20) and the relation in equation (4.21), a projection argument ensures that

$$(6.39) \quad \|\mu_{k,\eta} \mathcal{S}_t^{\text{Bin}(k)} - \mu_{k,\pi}\|_{\text{TV}} \geq \|\eta \mathcal{S}_t^{\text{Bin}(1)} - \pi\|_{\text{TV}}.$$

Thus, the second inequality in equation (5.3) concludes the proof. \square

PROOF OF THEOREM 2.3. The upper bound in Theorem 2.3 follows by Lemma 6.4 and the upper bound in Theorem 2.9. For what concerns the lower bound, we exploit Proposition 6.7 by showing that $\sup_{\eta \in \Delta} a_{t^-}(k, \eta)$, with $t^- = t_{\text{mix}} - t_w(C)$, is increasing to infinity as C grows. Notice that, with this choice of $t = t^-$, we have

$$(6.40) \quad a_{t^-}(k, \eta) = \frac{k \langle \psi, D(\cdot, \eta) \rangle^2}{1 + \frac{k}{n} (\|D(\cdot, \eta)\|_\infty^2 + e^{-C} \sqrt{k})}.$$

Concerning the numerator in equation (6.40) we have

$$(6.41) \quad \sup_{\eta \in \Delta} |\langle \psi, D(\cdot, \eta) \rangle| \geq \sup_{\substack{\eta \in \Delta \\ \|D(\cdot, \eta)\|_\infty \leq 2}} |\langle \psi, D(\cdot, \eta) \rangle| \geq \sup_{\substack{\|\varphi\|_\infty \leq 2 \\ \|\varphi\|_{1,+} = 1}} \langle \psi_+, \varphi \rangle \vee \sup_{\substack{\|\varphi\|_\infty \leq 2 \\ \|\varphi\|_{1,-} = 1}} \langle \psi_-, \varphi \rangle,$$

where

$$\begin{aligned} V_+ &:= \{x \in V : \psi_+(x) := \max\{\psi(x), 0\} \geq 0\}, \\ V_- &:= \{x \in V : \psi_-(x) := \max\{-\psi(x), 0\} > 0\}, \end{aligned}$$

and

$$(6.42) \quad \|\varphi\|_{1,\pm} := \sum_{x \in V_\pm} \pi(x) |\varphi(x)|.$$

Let us define $\pi(V_\pm) := \sum_{x \in V_\pm} \pi(x)$; then, $\pi(V_-) = 1 - \pi(V_+)$ and, since $\langle \psi, 1 \rangle = 0$ and by Assumption 2, $\pi(V_\pm) \in [\frac{1}{c_{\text{ell}} n}, 1 - \frac{1}{c_{\text{ell}} n}] \subseteq (0, 1)$. Hence, for

$$(6.43) \quad \varphi_\pm(x) := \frac{\mathbf{1}_{V_\pm}(x)}{\pi(V_\pm)}, \quad x \in V,$$

we have, by definition, $\|\varphi_\pm\|_{1,\pm} = 1$; moreover, by the fact that $\pi(V_-) = 1 - \pi(V_+)$, for all $n \in \mathbb{N}$, we have

$$(6.44) \quad 1 \leq \|\varphi_+\|_\infty \wedge \|\varphi_-\|_\infty \leq 2.$$

Hence, for each $n \in \mathbb{N}$, at least one between φ_+ and φ_- satisfies the constraints on the right-hand side in equation (6.41); therefore, since $\langle \psi_+, 1 \rangle = \langle \psi_-, 1 \rangle = \frac{1}{2} \|\psi\|_1$, this yields

$$(6.45) \quad \sup_{\substack{\eta \in \Delta \\ \|D(\cdot, \eta)\|_\infty \leq 2}} |\langle \psi, D(\cdot, \eta) \rangle| \geq \sup_{\substack{\|\varphi\|_\infty \leq 2 \\ \|\varphi\|_{1,+} = 1}} \langle \psi_+, \varphi \rangle \vee \sup_{\substack{\|\varphi\|_\infty \leq 2 \\ \|\varphi\|_{1,-} = 1}} \langle \psi_-, \varphi \rangle \geq \frac{1}{2} \|\psi\|_1,$$

where the first inequality follows from equation (6.41) and the second one follows by the lower bound in equation (6.44). As a consequence, we obtain

$$(6.46) \quad \sup_{\eta \in \Delta} a_{t^-}(k, \eta) \geq \sup_{\|D(\cdot, \eta)\|_\infty \leq 2} \frac{k \langle \psi, D(\cdot, \eta) \rangle^2}{1 + \frac{k}{n} (\|D(\cdot, \eta)\|_\infty^2 + e^{-C} \sqrt{k})} \geq \frac{\frac{1}{4} \|\psi\|_1^2}{\frac{1}{k} + \frac{2}{n} + e^{-C} \frac{\sqrt{k}}{n}}.$$

In order to conclude, we need to show that the eigenfunction ψ of $-L^{\text{Bin}(1)}$ satisfies

$$(6.47) \quad \liminf_{n \rightarrow \infty} \|\psi\|_1 > 0,$$

which, by Hölder inequality and $\|\psi\|_2 = 1$, holds if we prove that

$$(6.48) \quad \limsup_{n \rightarrow \infty} \|\psi\|_\infty < \infty.$$

The above follows by noting that, for all $t \geq 0$ and $x \in V$,

$$(6.49) \quad |\psi(x)| = e^{\frac{t}{t_{\text{rel}}}} |S_t^{\text{Bin}(1)} \psi(x)| = e^{\frac{t}{t_{\text{rel}}}} |\langle h_t^x, \psi \rangle| \leq e^{\frac{t}{t_{\text{rel}}}} \|h_t^x\|_2 \|\psi\|_2,$$

where we used Cauchy–Schwarz inequality. Hence, plugging $t = Ct_{\text{rel}}$ into equation (6.49), using Assumption 1 and recalling that $\|\psi\|_2 = 1$, equation (6.48) holds and the desired claim follows. \square

Similar to what has been done at the end of Section 5, we show how to use our arguments to get some quantitative bounds on the TV mixing of the particle system in the case in which $k = \omega(n^2)$.

PROPOSITION 6.8 (Pre-cutoff for the binomial splitting). *Consider a sequence of graphs and site-weights such that Assumptions 1 and 2 hold. Then, if $k/n^2 \rightarrow \infty$, for all $\delta \in (0, 1)$,*

$$(6.50) \quad \limsup_{n \rightarrow \infty} \mathbf{d}_k(T^+) \leq \delta, \quad \liminf_{n \rightarrow \infty} \mathbf{d}_k(T^-) \geq 1 - \delta,$$

where

$$(6.51) \quad T^+ := a \frac{t_{\text{rel}}}{2} \log(k) + Ct_{\text{rel}}, \quad T^- := b \frac{t_{\text{rel}}}{2} \log(k) - Ct_{\text{rel}},$$

$$(6.52) \quad a := 2 \frac{\log(k/n)}{\log(k)} \in [1, 2], \quad b := 2 \frac{\log(n)}{\log(k)} \in (0, 1],$$

and some $C = C(c_{\text{cell}}, c_{\text{ratio}}, \delta) > 0$.

Let us observe that, compared to Proposition 5.6, the effect of $k = \omega(n^2)$ alters not only the first order of T^+ , but also that of T^- .

PROOF. The lower bound follows directly by Proposition 6.7, while, for the upper bound, it is enough to combine Lemma 6.4 and the estimate in equation (5.44). \square

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