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State estimation using a network of distributed observers with unknown inputs[☆]

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ABSTRACT

State estimation for a class of linear time-invariant systems with distributed output measurements (distributed sensors) and unknown inputs is addressed in this paper. The objective is to design a network of observers such that the state vector of the entire system can be estimated, while each observer has access to only local output measurements that may not be sufficient on their own to reconstruct the entire system's state. Existing results in the literature on distributed state estimation assume either that the system does not have inputs, or that all the system's inputs are globally known to all the observers. Accordingly, we address this gap by proposing a distributed observer capable of estimating the overall system's state in the presence of inputs, while each observer only has limited local information on inputs and outputs. We provide a design method that guarantees convergence of the estimation errors to zero under joint detectability conditions. This design suits undirected communication graphs that may have switching topologies and also applies to strongly connected directed graphs. We also give existence conditions that are consistent with existing results on unknown input observers. Finally, simulation results verify the effectiveness of the proposed estimation scheme for various scenarios.

1. Introduction

The increasing ubiquity of embedded systems has empowered sensing equipment with communication and computation capabilities that allow complex algorithms to be deployed on sensors themselves. This is especially beneficial for larger systems comprising many different components, whose state space has a significant size or is spread over a large area. Systems of this kind encompass smart buildings with many networked sensing points (Casado-Vara et al., 2020) or water and power networks (Bartos et al., 2018; Fadel et al., 2015), where measurements are taken over a vast area. In both cases, the centralized computation may result in additional complexity and coordination, hence running distributed algorithms is an effective design choice. Therefore, this paper addresses the state estimation problem for a linear time-invariant (LTI) dynamical system with N sensor nodes. More generally, the *distributed estimation problem* is to design a group of N observers co-located with the sensors such that each observer computes an estimate of the state vector of the entire system, while only having access to some local measurements. In general, these local measurements may not be sufficient to estimate the state, and to overcome this limitation, each observer shares its own estimate with neighboring observers over a communication network.

Many classical algorithms for state estimation, such as the Luenberger observer and the Kalman filter, have been extended in the literature in several ways for distributed state estimation. For instance, the works of Olfati-Saber (2007) and Kamgarpour and Tomlin (2008) extend the classical Kalman filter to distributed systems. In Wang and Morse (2018), a general linear structure of a distributed observer is given, and no assumptions are made on the detectability of the system with respect to the individual node. In Kim et al. (2016) and Han et al. (2019), Luenberger-like observers are designed for distributed state estimation. Such ideas also have been used for more complex scenarios such as resilient distributed state estimation (Mitra et al., 2019; Mitra & Sundaram, 2019), nonlinear distributed estimation (He et al., 2020; Yang et al., 2020), distributed estimation

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in the presence of switching topologies (Ugrinovskii, 2013; Xu et al., 2020), H_∞-based distributed estimation (Shen et al., 2010), distributed moving horizon estimation (Farina et al., 2010), etc.

A limitation that existing works on distributed estimation have in common is the assumption that the global system is *autonomous* (i.e., there are no external inputs) or that the input information is available globally for all nodes. However, in practice, when a system is distributed and is driven by some inputs, it may not be possible for each node to access all control signals. Instead, each node may merely have access to its own part of the system's input, which is available locally. In this case, the existing distributed estimation schemes in the literature may not be effective.

In particular, the problem of distributed state estimation is still open when unknown inputs at some nodes are considered. In the centralized case, unknown input observers (UIOs) (Bhattacharyya, 1978; Chen et al., 1996; Commault et al., 2001; Darouach et al., 1994; Guan & Saif, 1991; Hou & Muller, 1992; Pertew et al., 2005; Wang et al., 1975; Watanabe & Himmelblau, 1982) have historically been an effective tool for dealing with systems with completely unknown exogenous disturbances, where these can also model coupling between subsystems (Hou & Müller, 1994). Little work exists where these observers are instead extended to distributed cases, such as in Chakrabarty et al. (2016). In that work, however, the objective is for each system to estimate their own local state regardless of the neighboring interconnections. In this paper, we aim to leverage the geometric decoupling capabilities offered by the UIO design and devise a distributed version of the algorithm that is capable of estimating the global system state *in the presence of locally unknown inputs*. Therefore, compared to the existing literature in the area of distributed estimation, the main contributions of this paper are listed below:

- The nodes of the distributed observer do not have access to the entire input vector, but rather only subsets of it are assumed to be available at each node.
- The nodes exchange with their neighbors the local estimates of the whole state vector of the system, such that under certain conditions, the estimate of each node converges to the state vector of the system via a suitably designed consensus strategy.
- Under certain detectability conditions, the feasibility of the design of the proposed distributed estimation scheme is guaranteed.

More precisely, we propose a distributed unknown input observer (DUIO) for an LTI system with unknown disturbances, where only the information of local outputs and local inputs is accessible at each node. We provide rigorous (necessary and sufficient) conditions for the existence of such DUIO, depending on a rank condition and an appropriately defined detectability criterion. We also show that any feasible solution of a certain linear matrix inequality (LMI) guarantees asymptotic convergence of the observers' estimates to the real state of the system. Therefore, such LMI condition can be constructively applied to compute the gains of the DUIO, given that the existence conditions are satisfied. Furthermore, when the aforementioned detectability criterion is satisfied, the feasibility of the LMI is always guaranteed. To the best of the authors' knowledge, it is the first time that a distributed consensus-based unknown-input observer architecture is being proposed.

The paper is organized as follows. In Section 2, some notations and basic information on graph theory are provided. The problem is formulated in Section 3. The distributed state estimation scheme in the presence of unknown inputs at each node is proposed in Section 4. Simulation results are provided in Section 5 and concluding remarks and future work are stated in Section 6.

2. Preliminaries

Notation and some concepts and definitions of graph theory are presented in this section.

2.1. Notation

By considering $\mathbb{C}$ as the set of complex numbers, let $\mathbb{C}^- = \{s \in \mathbb{C} : \mathfrak{Re}\ s < 0\}$ and $\mathbb{C}^+ = \{s \in \mathbb{C} : \mathfrak{Re}\ s \geq 0\}$. I_n stands for the $n \times n$ identity matrix. $\mathbf{0}_{n\times m}$ is an $n \times m$ all-zeros matrix. $\mathbf{1}_n$ is an $n \times 1$ all-ones vector. $|\cdot|$ stands for the standard 2-norm. $\otimes$ stands for the Kronecker product. For a matrix $A \in \mathbb{R}^{n\times m}$, $A^\dagger$ represents the pseudo inverse of A such that if A is full row rank, $A^\dagger = A^\top(AA^\top)^{-1}$ and if A is full column rank, $A^\dagger = (A^\top A)^{-1}A^\top$. We say $M \succ 0$, if M is a symmetric positive definite matrix. $\lambda_2(\cdot)$ is the second smallest eigenvalue of a real symmetric matrix. diag$(M_1, M_2, \ldots, M_n)$ represents a block diagonal matrix composed of the matrices $M_1, M_2, \ldots, M_n$. Similarly, diag$_{i\in\mathcal{I}}(M_i)$ is a short-hand notation when the matrices are indexed by a set $\mathcal{I}$. A 'nontrivial' (sub)space V is such that dim$(V) > 0$. Moreover, if $R, S \subseteq X$, we define the subspace $R + S \subseteq X$ as $R + S = \{r+s : r \in R \text{ and } s \in S\}$, and we define the subspace $R \cap S \subseteq X$ as $R \cap S = \{x : x \in R \text{ and } x \in S\}$. Accordingly, the symbol $\oplus$ indicates that the subspaces being added are independent. $V^\perp$ denotes the orthogonal complement of the subspace V. We indicate that two vector spaces V and W are isomorphic by $V \simeq W$. $\alpha_A(s)$ is the minimal polynomial of A, which is factored as follows:

$$\alpha_A(s) = \alpha_A^+(s)\alpha_A^-(s),$$

where the roots of α_A^+ belong to $\mathbb{C}^+$, and the roots of α_A^- belong to $\mathbb{C}^-$ (Wonham, 1985, Chap. 3.6). $UO(C, A)$ denotes the unobservable subspace of the pair (C, A) and is defined by

$$UO(C, A) = \bigcap_{k=1}^{n} \text{Ker } CA^{k-1}.$$

Moreover, $UD(C, A)$ denotes the undetectable subspace of the pair (C, A) and is defined by

$$UD(C, A) = \left(\bigcap_{k=1}^{n} \text{Ker } CA^{k-1}\right) \bigcap \text{ Ker } \alpha_A^+(A).$$

2.2. Graph theory

Communication among the observers is described by an unweighted graph $\mathcal{G} = (\mathbf{N}, \mathcal{E}, \mathcal{A})$ where $\mathbf{N} = \{1, 2, \ldots, N\}$ is the set of nodes (denoting N observers with local measurement), $\mathcal{E} \subseteq \mathbf{N} \times \mathbf{N}$ is the set of edges (denoting communication links), and $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$ denotes the adjacency matrix. We say that $a_{ij} = 1$ if Node j is a neighbor of Node i, that is Node j sends information to Node i, and $a_{ij} = 0$ otherwise (in the case of undirected graphs, $a_{ij} = a_{ji}$). An undirected graph is *connected* if there exists a path of edges connecting each pair of its nodes. Moreover, a directed graph is *strongly connected* if there exists a path in each direction connecting each pair of its nodes.

The Laplacian matrix associated with the graph $\mathcal{G}$ is a matrix $\mathcal{L} = [\ell_{ij}] \in \mathbb{R}^{N\times N}$ defined as

$$\ell_{ij} = \begin{cases} \sum_{j=1, i\neq j}^{N} a_{ij} & i = j \\ -a_{ij} & i \neq j, \end{cases}$$

which always has a zero eigenvalue, and if $\mathcal{G}$ is connected or strongly connected, all the other eigenvalues are on the open right half-plane. If the graph is undirected, $\mathcal{L}$ is also symmetric and all its eigenvalues are real (Ren et al., 2007). In such case, the second

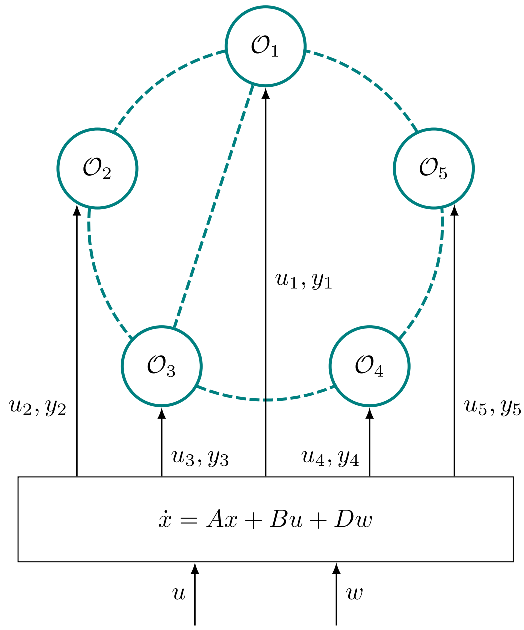

Fig. 1. A distributed observer consisting of five nodes: each local observer $\mathcal{O}_i$ has available local inputs and measurements u_i and y_i. Furthermore, neighboring estimates are exchanged over an undirected communication network (dashed line).

smallest eigenvalue $\lambda_2(\mathcal{L})$ denotes the *algebraic connectivity* of the graph (Olfati-Saber & Murray, 2004).

The graph $\mathcal{G}$ is called balanced, if for all $i \in \mathbf{N}$, $\sum_{i=1}^{N} a_{ij} = \sum_{j=1}^{N} a_{ji}$. If $\mathcal{G}$ is connected or strongly connected and balanced, the right and left eigenvectors associated with the zero eigenvalue are $\mathbf{1}_N/\sqrt{N}$ (Ren et al., 2007). Moreover, the Laplacian matrix associated with any balanced graph is positive semidefinite (Olfati-Saber & Murray, 2004).

3. Problem statement

Consider the dynamical system described as

$$\dot{x} = Ax + Bu + Dw, \tag{1}$$

where $x \in \mathbb{R}^n$ represents the state vector, $u \in \mathbb{R}^m$ is the control input, $w \in \mathbb{R}^q$ is an unknown external disturbance, $A \in \mathbb{R}^{n\times n}$ is the state matrix, $B \in \mathbb{R}^{n\times m}$ denotes the input matrix, and $D \in \mathbb{R}^{n\times q}$ is the disturbance matrix gain. We assume that the system outputs are measured via a distributed measurement system comprising a group of sensors distributed over N nodes, namely

$$y_i = C_i x, \tag{2}$$

with $C_i \in \mathbb{R}^{p_i\times n}$.

In order to further distinguish the locally available signals, we partition the system's inputs into a component u_i, which is local to and assumed to be known at Node i, and a component $\acute{u}_i$, which is unknown and can instead be assimilated to an exogenous disturbance. In symbols, we then have

$$Bu = B_i u_i + \acute{B}_i \acute{u}_i,$$

where $u_i \in \mathbb{R}^{r_i}$, $B_i \in \mathbb{R}^{n\times r_i}$, $\acute{u}_i \in \mathbb{R}^{d_i}$, and $\acute{B}_i \in \mathbb{R}^{n\times d_i}$, with $r_i + d_i = m$. Then, as w is also unknown, we define

$$\acute{w}_i = \begin{bmatrix} \acute{u}_i^\top & w^\top \end{bmatrix}^\top, \bar{B}_i = \begin{bmatrix} \acute{B}_i & D \end{bmatrix},$$

where $\acute{w}_i$ is the locally unknown inputs and $\bar{B}_i$ is the known gain of the unknown terms.

Assumption 1. The matrix $\bar{B}_i$ is full column rank for all $i \in \mathbf{N}$.

Remark 1. Note that Assumption 1 does not cause any loss of generality and is typically made in the literature of estimation with unknown disturbances (Chen et al., 1996). In fact, it is always possible (by means of singular value decomposition, for instance) to decompose $\bar{B}_i$ in a product $\bar{B}_i = \bar{B}_i'\bar{B}_i''$, where $\bar{B}_i'$ is full column rank and $\bar{w}_i' = \bar{B}_i''\acute{w}_i$ constitutes the new unknown input.

As the objective is to reconstruct the state vector x, we consider a distributed observer $\mathcal{O} = \{\mathcal{O}_i\}_{i\in\mathbf{N}}$ comprising N local nodes (or observers) $\mathcal{O}_i$ located at each sensor node, where each observer has access to just its local outputs y_i and local inputs u_i. Furthermore, the local observers exchange their state estimates through a communication network.

Assumption 2. The network communication graph is connected.

To provide a visual example of the proposed architecture, in Fig. 1, an undirected network of distributed observers with five nodes is shown, where the local information of each node includes the local output measurement vector y_i and the local known control input vector u_i.

We can finally characterize the distributed estimation problem. Let $\hat{x}_i$ denote the estimate of x produced by the local observer $\mathcal{O}_i$, then we define the estimation error as

$$e_i = x - \hat{x}_i. \tag{3}$$

A DUIO is hence defined as follows.

Definition 1. The set of observers $\{\mathcal{O}_i\}_{i\in\mathbf{N}}$ is a DUIO for system (1) if for all $i \in \mathbf{N}$,

$$\lim_{t\to+\infty} |e_i(t)| = 0,$$

for all locally unknown inputs $\acute{w}_i$.

That is to say, a distributed observer is a DUIO if the local estimation error terms are decoupled from the disturbances and the input components that are not locally available.

4. Distributed unknown input observer design

In this section, assuming that the communication graph is undirected and fixed, the proposed DUIO design is first presented. Then, the results are extended to scenarios when the undirected communication graph is switching over time, or the communication graph is directed.

4.1. Fundamental results for undirected networks

The basic principle to design an unknown input observer is to derive some algebraic conditions that decouple the observer's error from the unknown disturbances and inputs (see e.g., Chen et al. (1996)). Along this pattern, we propose the following full-order local observer $\mathcal{O}_i$, $i \in \mathbf{N}$:

$$\begin{aligned} \dot{z}_i &= N_i z_i + M_i B_i u_i + L_i y_i + \chi P_i^{-1} \sum_{j=1}^{N} a_{ij}(\hat{x}_j - \hat{x}_i), \\ \hat{x}_i &= z_i + H_i y_i, \end{aligned} \tag{4}$$

where $z_i \in \mathbb{R}^n$ is the state vector of the observer $\mathcal{O}_i$, the matrices N_i, M_i, L_i, P_i, and H_i are defined in the following in order to guarantee the convergence to zero of the error e_i, and χ is a real-valued design parameter.

It can be shown (see Appendix A) that the estimation error of observer (4) with respect to (1) is given by

$$\begin{aligned}\dot{e}_i &= [(I_n - H_iC_i)A - K_iC_i]e_i \\ &+ (I_n - H_iC_i - M_i)B_iu_i + (I_n - H_iC_i)\bar{B}_i\acute{w}_i \\ &+ [(I_n - H_iC_i)A - K_iC_i - N_i]z_i \\ &+ [K_i + ((I_n - H_iC_i)A - K_iC_i)H_i - L_i]y_i \\ &+ \chi P_i^{-1}\sum_{j=1}^{N} a_{ij}(e_j - e_i).\end{aligned} \tag{5}$$

Now, owing to (5), we set the following constraints on the matrices N_i, M_i, L_i, P_i, and H_i, followed by analysis that establishes solvability conditions:

$$(I_n - H_iC_i)\bar{B}_i = \mathbf{0}_{n\times 1}, \tag{6a}$$

$$M_i = I_n - H_iC_i, \tag{6b}$$

$$N_i = M_iA - K_iC_i, \tag{6c}$$

$$L_i = K_i + N_iH_i. \tag{6d}$$

Lemma 1. *(Chen et al., 1996) Eq. (6a) is solvable if and only if*

$$\mathrm{rank}(C_i\bar{B}_i) = \mathrm{rank}(\bar{B}_i),$$

and the general solution is given by

$$H_i = U_i + Y_iV_i, \tag{7}$$

where $Y_i \in \mathbb{R}^{n\times p_i}$ is an arbitrary matrix, and U_i and V_i are defined as follows:

$$U_i = \bar{B}_i(C_i\bar{B}_i)^{\dagger},\ \ V_i = I_{p_i} - C_i\bar{B}_i(C_i\bar{B}_i)^{\dagger}.$$

Remark 2. While U_i is a special solution to the decoupling Eq. (6a), it may be beneficial to also consider Y_i as a design parameter that provides additional degrees of freedom without affecting the decoupling property. This enlarged solution space is useful as it may provide better (e.g., with lower gains) solutions to optimization problems such as the one of Theorem 1, or achieving secondary objectives like noise attenuation (Mondal et al., 2010).

Lemma 1 provides a geometric condition that allows (6a) in particular to be satisfied. Therefore, by obtaining H_i from (6a), and by setting M_i, N_i, and L_i respectively as (6b), (6c), and (6d), (5) can be simplified as follows:

$$\dot{e}_i = N_ie_i + \chi P_i^{-1}\sum_{j=1}^{N} a_{ij}(e_j - e_i). \tag{8}$$

It should be noted that by virtue of (6) and (7), K_i and Y_i are the design parameters (to be characterized in Theorem 1). Furthermore, under the condition (6a), $\bar{B}_i$ will be in the null space of $I_n - H_iC_i$, such that the term $(I_n - H_iC_i)\bar{B}_i\acute{w}_i$ appearing in (5) does not enter (8). Therefore, any arbitrary w will not have any effect on the estimation errors.

Before introducing the main results on the design and existence of the DUIO, we investigate the detectability properties of the system. For convenience, we first introduce the following definition.

Definition 2 (*Extensive Joint Detectability*). Let

$$\mathcal{A}_i = (I_n - U_iC_i)A. \tag{9}$$

System (1) is extensively jointly detectable from Node i if

$$\bigcap_{i=1}^{N} \mathcal{UD}(C_i, \mathcal{A}_i) = 0. \tag{10}$$

By virtue of the definition of $\mathcal{A}_i$ in (9) and by recalling (7), we define

$$\bar{\mathcal{A}}_i = (I_n - H_iC_i)A = \mathcal{A}_i - Y_iV_iC_iA, \tag{11}$$

so that we can express $N_i = \bar{\mathcal{A}}_i - K_iC_i$. With this, the convergence of the estimation errors in terms of the detectability properties of the pair $(C_i, \bar{\mathcal{A}}_i)$ will be investigated. Accordingly, we introduce a similarity transformation matrix $T_i \in \mathbb{R}^{n\times n}, i \in \mathbf{N}$, as $T_i = \begin{bmatrix}T_{id} & T_{iu}\end{bmatrix}$ in which $T_{iu} \in \mathbb{R}^{n\times \upsilon_i}$ is an orthonormal basis of the undetectable subspace of $\left(C_i, \bar{\mathcal{A}}_i\right)$, where υ_i is the dimension of the undetectable subspace of the pair $(C_i, \bar{\mathcal{A}}_i)$, and $T_{id} \in \mathbb{R}^{n\times(n-\upsilon_i)}$ is an orthonormal basis such that Im T_{id} is orthogonal to Im T_{iu} (Kim et al., 2016). Note that by defining $\mathcal{X} \simeq \mathbb{R}^n$ as the n-dimensional state space of the system, we have $\mathcal{X}$ = Im $T_{id} \oplus$ Im T_{iu}. In the following lemmas, we investigate such detectability properties using a geometric approach. We first prove that the detectability of the pairs $(C_i, \bar{\mathcal{A}}_i)$ and $(C_i, \mathcal{A}_i)$ are equivalent, and then we provide a condition for which all the estimation errors can be steered to zero.

Lemma 2. *The undetectable subspace of the pairs $(C_i, \bar{\mathcal{A}}_i)$ and $(C_i, \mathcal{A}_i)$ are identical for all $Y_i \in \mathbb{R}^{n\times p_i}$.*

Proof. By considering (11), for some $F_i \in \mathbb{R}^{n\times p_i}$, one gets

$$\begin{aligned}\bar{\mathcal{A}}_i + F_iC_i &= \mathcal{A}_i - Y_iV_iC_i\mathcal{A}_i + F_iC_i \\ &= \mathcal{A}_i + \begin{bmatrix}F_i & -Y_iV_i\end{bmatrix}\begin{bmatrix}C_i \\ C_i\mathcal{A}_i\end{bmatrix}.\end{aligned} \tag{12}$$

From (12), it follows that

$$\mathcal{UD}\left(C_i, \bar{\mathcal{A}}_i\right) = \mathcal{UO}\left(\begin{bmatrix}C_i \\ C_i\mathcal{A}_i\end{bmatrix}, \mathcal{A}_i\right) \bigcap \text{ Ker } \alpha^{+}_{\mathcal{A}_i}(\mathcal{A}_i). \tag{13}$$

Meanwhile,

$$\mathcal{UD}\ (C_i, \mathcal{A}_i) = \mathcal{UO}(C_i, \mathcal{A}_i) \bigcap \text{ Ker } \alpha^{+}_{\mathcal{A}_i}(\mathcal{A}_i). \tag{14}$$

By comparing (13) and (14), to show that $\mathcal{UD}\left(C_i, \bar{\mathcal{A}}_i\right) = \mathcal{UD}\ (C_i, \mathcal{A}_i)$, we can show that the unobservable subspaces of $\left(\begin{bmatrix}C_i \\ C_i\mathcal{A}_i\end{bmatrix}, \mathcal{A}_i\right)$ and $(C_i, \mathcal{A}_i)$ are identical. It can be said that

$$\mathcal{UO}\left(\begin{bmatrix}C_i \\ C_i\mathcal{A}_i\end{bmatrix}, \mathcal{A}_i\right) = \bigcap_{k=1}^{n} \text{Ker}\begin{bmatrix}C_i \\ C_i\mathcal{A}_i\end{bmatrix}\mathcal{A}_i^{k-1}. \tag{15}$$

One can observe that

$$\text{Ker}\begin{bmatrix}C_i \\ C_i\mathcal{A}_i\end{bmatrix}\mathcal{A}_i^{k-1} = \text{Ker } C_i\mathcal{A}_i^{k-1} \bigcap \text{ Ker } C_i\mathcal{A}_i^{k},$$

which implies that

$$\bigcap_{k=1}^{n}\text{Ker}\begin{bmatrix}C_i \\ C_i\mathcal{A}_i\end{bmatrix}\mathcal{A}_i^{k-1} = \bigcap_{k=1}^{n+1}\text{Ker } C_i\mathcal{A}_i^{k-1}. \tag{16}$$

Moreover, the unobservable subspace of the pair $(C_i, \mathcal{A}_i)$ is

$$\mathcal{UO}\ (C_i, \mathcal{A}_i) = \bigcap_{k=1}^{n}\text{Ker } C_i\mathcal{A}_i^{k-1}. \tag{17}$$

Now, from (15), (16), and (17), it follows that

$$\mathcal{UO}\ (C_i, \mathcal{A}_i) = \mathcal{UO}\left(\begin{bmatrix}C_i \\ C_i\mathcal{A}_i\end{bmatrix}, \mathcal{A}_i\right),$$

and thus from (13) and (14), we have

$$UD\left(C_i, A_i\right) = UD\left(C_i, \bar{A}_i\right),$$

which completes the proof. ■

Lemma 3. *Let system (1) be extensively jointly detectable. Then, by letting*

$$T_d = \begin{bmatrix} T_{1d} & T_{2d} & \dots & T_{Nd} \end{bmatrix},$$

we have

$$\mathrm{Im}\ T_d = X.$$

Proof. The proof follows standard geometric arguments; however, it is provided in Appendix B for completeness. ■

The presented lemmas let us investigate the stability of the estimation errors, with the hypotheses that a solution to (6) exists. In this case, we leverage standard Lyapunov arguments to obtain an LMI condition that guarantees the stability of the (collective) error e, defined as the stacked vector of local observers' errors as follows:

$$e = \begin{bmatrix} e_1^\top & e_2^\top & \dots & e_N^\top \end{bmatrix}^\top. \tag{18}$$

The stability of the proposed distributed estimation scheme is studied in the following theorem.

Theorem 1 (*Stability*)**.** *Consider the DUIO described in (4) under Assumption 2 and the conditions (6) and (7). Moreover, let*

$$\begin{aligned} \Lambda_i = \ & A^\top (I_n - C_i^\top U_i^\top)P_i + P_i(I_n - U_iC_i)A \\ & - A^\top C_i^\top V_i^\top \bar{Y}_i^\top - \bar{Y}_i V_i C_i A \\ & - C_i^\top \bar{K}_i^\top - \bar{K}_i C_i, \end{aligned} \tag{19}$$

in which the matrices $P_i \succ 0$, $\bar{Y}_i$, and $\bar{K}_i$ are a feasible solution of the following LMI:

$$\sum_{i=1}^{N} \Lambda_i \prec 0. \tag{20}$$

Under these conditions, by considering the DUIO parameters Y_i and K_i as $Y_i = P_i^{-1}\bar{Y}_i$, and $K_i = P_i^{-1}\bar{K}_i$, the estimation error e (18) converges to zero if the gain χ satisfies

$$\chi > \frac{\left| \Lambda + \Lambda_P^\top \left(\sum_{i=1}^{N} \Lambda_i \right)^{-1} \Lambda_P \right|}{2\lambda_2(\mathcal{L})}, \tag{21}$$

where

$$\begin{aligned} \Lambda &= \mathrm{diag}(\Lambda_1, \Lambda_2, \dots, \Lambda_N), \\ \Lambda_P &= \begin{bmatrix} \Lambda_1 & \Lambda_2 & \dots & \Lambda_N \end{bmatrix}. \end{aligned} \tag{22}$$

Proof. We show that along (8), the estimation errors converge to zero. Accordingly, we consider the following Lyapunov candidate of the estimation errors:

$$V = \sum_{i=1}^{N} e_i^\top P_i e_i, \tag{23}$$

which is a positive definite function of the estimation errors. The time derivative of V along (8) can be stated as follows:

$$\dot{V} = e^\top \operatorname*{diag}_{i\in\mathbf{N}}(N_i^\top P_i + P_iN_i)e - 2\chi e^\top(\mathcal{L}\otimes I_n)e. \tag{24}$$

Based on the conditions on M_i and N_i in (6b) and (6c), it follows that

$$\begin{aligned} N_i^\top P_i + P_iN_i = & \Big((I_n - H_iC_i)A - K_iC_i\Big)^\top P_i \\ & + P_i\Big((I_n - H_iC_i)A - K_iC_i\Big). \end{aligned} \tag{25}$$

According to (25) and the definition of H_i in (7), one gets

$$\begin{aligned} & N_i^\top P_i + P_iN_i = \\ & A^\top P_i + P_iA - A^\top C_i^\top U_i^\top P_i - P_iU_iC_iA \\ & - A^\top C_i^\top V_i^\top Y_i^\top P_i - P_iY_iV_iC_iA - C_i^\top K_i^\top P_i - P_iK_iC_i, \end{aligned}$$

which by considering (19) with $\bar{Y}_i = P_iY_i$ and $\bar{K}_i = P_iK_i$, can be rewritten as

$$N_i^\top P_i + P_iN_i = \Lambda_i. \tag{26}$$

Now, from (26), (24) can be restated as follows:

$$\dot{V} = e^\top \Lambda e - 2\chi e^\top(\mathcal{L}\otimes I_n)e, \tag{27}$$

where Λ is defined in (22). To analyze (27), we decompose the error space into two subspaces. By defining the error space as $E \simeq \mathbb{R}^{Nn}$, one of these subspaces is denoted by $E_c \subseteq E$ ($\dim(E_c) = n$) which is the kernel of $\mathcal{L}\otimes I_n$ and has the form of $\mathbf{1}_N \otimes \omega$, $\omega \in \mathbb{R}^n$. Accordingly, the other subspace is the orthogonal complement subspace of E_c which is denoted by $E_r \subseteq E$ ($\dim(E_c) = Nn - n$) such that $E_c \oplus E_r = E$. Thus, by considering $e_c \in E_c$ and $e_r \in E_r$, (27) yields

$$\dot{V} = e_c^\top \Lambda e_c + 2e_r^\top \Lambda e_c + e_r^\top(\Lambda - 2\chi(\mathcal{L}\otimes I_n))e_r,$$

which since $e_c = \mathbf{1}_N \otimes \omega$ can be restated as follows:

$$\begin{aligned} \dot{V} = \ & \omega^\top \Big(\sum_{i=1}^{N} \Lambda_i \Big)\omega + 2e_r^\top \Lambda_P^\top \omega \\ & + e_r^\top(\Lambda - 2\chi(\mathcal{L}\otimes I_n))e_r. \end{aligned} \tag{28}$$

Moreover, as e_r is orthogonal to the kernel of $\mathcal{L}\otimes I_n$, one gets (Olfati-Saber & Murray, 2004)

$$-e_r^\top(\mathcal{L}\otimes I_n)e_r \le -\lambda_2(\mathcal{L})e_r^\top e_r. \tag{29}$$

Since the graph is connected from Assumption 2, $\lambda_2(\mathcal{L}) \in \mathbb{R}_{>0}$. Hence, by considering (28) and (29), we have

$$\dot{V} \le -\begin{bmatrix} \omega \\ e_r \end{bmatrix}^\top \begin{bmatrix} -\sum_{i=1}^{N} \Lambda_i & -\Lambda_P \\ -\Lambda_P^\top & 2\chi\lambda_2(\mathcal{L})I_{Nn} - \Lambda \end{bmatrix} \begin{bmatrix} \omega \\ e_r \end{bmatrix}. \tag{30}$$

From the inequality (21), one gets:

$$2\chi\lambda_2(\mathcal{L})I_{Nn} - \Lambda - \Lambda_P^\top \left(\sum_{i=1}^{N} \Lambda_i \right)^{-1} \Lambda_P \succ 0. \tag{31}$$

Finally, according to (31) and by invoking the Schur complement (Boyd et al., 1994), the negative definiteness of $\dot{V}$ in (30) can be concluded. Thus, V asymptotically converges to zero, which implies that the estimation error e (and therefore all its components $e_i, \forall i \in \mathbf{N}$) converges to zero. ■

Remark 3. From (27), it follows that

$$\dot{V} = -e^\top \left(2\chi(\mathcal{L}\otimes I_n) - \Lambda\right) e,$$

where according to the proof of Theorem 1, $2\chi(\mathcal{L}\otimes I_n) - \Lambda \succ 0$. Now, by considering (23), one gets

$$\dot{V} \le -\mu V,$$

where

$$\mu = \frac{\lambda_{\min}\big(2\chi(\mathcal{L}\otimes I_n) - \Lambda\big)}{\max_{i\in\mathbf{N}}\big(\lambda_{\max}(P_i)\big)}. \tag{32}$$

From the comparison theorem for scalar ordinary differential equations (Khalil & Grizzle, 2002, Chap. 3), one obtains $0 \le V \le \upsilon$ for all $t \ge 0$, where υ is given by

$$\upsilon(t) = e^{-\mu t}\upsilon(0),$$

namely v converges to zero with time constant $1/\mu$, where since $0 \le V \le v$, $1/\mu$ implies an upper bound for the time constant of the convergence of V to zero as well.

Theorem 2 (*Feasibility*). *If system (1) is extensively jointly detectable, then the LMI (20) is always feasible for some P_i, $\bar{Y}_i$, and $\bar{K}_i$.*

Proof. From (9), (11), and (19), Λ_i can be written as

$$\Lambda_i = (\bar{A}_i - K_i C_i)^\top P_i + P_i(\bar{A}_i - K_i C_i). \tag{33}$$

By considering the similarity transformation matrix $T_i \in \mathbb{R}^{n\times n}$, one can observe that Kim et al. (2016)

$$T_i^\top \bar{A}_i T_i = \begin{bmatrix} \bar{A}_{id} & \mathbf{0}_{(n-v_i)\times v_i} \\ \bar{A}_{ir} & \bar{A}_{iu} \end{bmatrix}, \quad C_i T_i = \begin{bmatrix} C_{id} & \mathbf{0}_{p_i\times v_i} \end{bmatrix}, \tag{34}$$

where the pair $(C_{id}, \bar{A}_{id})$ is detectable. Based on the aforementioned formulation, without loss of generality, let the observer gains $K_i \in \mathbb{R}^{n\times p_i}$ and $P_i \in \mathbb{R}^{n\times n}$ be as follows:

$$K_i = T_i \begin{bmatrix} K_{id} \\ \mathbf{0}_{v_i\times p_i} \end{bmatrix}, \quad P_i = T_i \begin{bmatrix} P_{id} & \mathbf{0}_{(n-v_i)\times v_i} \\ \mathbf{0}_{v_i\times(n-v_i)} & P_{iu} \end{bmatrix} T_i^\top, \tag{35}$$

where $K_{id} \in \mathbb{R}^{(n-v_i)\times p_i}$, $P_{id} \in \mathbb{R}^{(n-v_i)\times(n-v_i)} \succ 0$, and $P_{iu} \in \mathbb{R}^{v_i\times v_i} \succ 0$. From the definition of Λ_i in (33), the definition of K_i and P_i in (35), and the decomposition performed in (34), we have

$$\Lambda_i = T_i \begin{bmatrix} \Lambda_{id} & \Lambda_{ir}^\top \\ \Lambda_{ir} & \Lambda_{iu} \end{bmatrix} T_i^\top, \tag{36}$$

where $\Lambda_{id} \in \mathbb{R}^{(n-v_i)\times(n-v_i)}$, $\Lambda_{ir} \in \mathbb{R}^{v_i\times(n-v_i)}$, and $\Lambda_{iu} \in \mathbb{R}^{v_i\times v_i}$ are as follows:

$$\Lambda_{id} = \Gamma_{id}^\top P_{id} + P_{id}\Gamma_{id},$$
$$\Lambda_{ir} = P_{iu}\bar{A}_{ir},$$
$$\Lambda_{iu} = \bar{A}_{iu}^\top P_{iu} + P_{iu}\bar{A}_{iu},$$

in which $\Gamma_{id} = \bar{A}_{id} - K_{id}C_{id}$. Since $T_i = \begin{bmatrix} T_{id} & T_{iu} \end{bmatrix}$, one gets:

$$T_i \begin{bmatrix} \Lambda_{id} & \Lambda_{ir}^\top \\ \Lambda_{ir} & \Lambda_{iu} \end{bmatrix} T_i^\top = T_{id}\Lambda_{id}T_{id}^\top + + T_{iu}\Lambda_{ir}T_{id}^\top + T_{id}\Lambda_{ir}^\top T_{iu}^\top + T_{iu}\Lambda_{iu}T_{iu}^\top. \tag{37}$$

Now, from (37), and by defining

$$T_d = \begin{bmatrix} T_{1d} & T_{2d} & \dots & T_{Nd} \end{bmatrix},$$
$$\Lambda_d = \mathrm{diag}(\Lambda_{1d}, \Lambda_{2d}, \dots, \Lambda_{Nd}),$$

it follows that

$$\sum_{i=1}^{N}\left(T_i \begin{bmatrix} \Lambda_{id} & \Lambda_{ir}^\top \\ \Lambda_{ir} & \Lambda_{iu} \end{bmatrix} T_i^\top\right) = T_d\Lambda_d T_d^\top + \sum_{i=1}^{N}\left(T_{iu}\Lambda_{ir}T_{id}^\top + T_{id}\Lambda_{ir}^\top T_{iu}^\top + T_{iu}\Lambda_{iu}T_{iu}^\top\right). \tag{38}$$

If the system is extensively jointly detectable it follows by Lemma 3 that $\mathrm{rank}(T_d) = n$. Hence, from (36) and (38), we have

$$\sum_{i=1}^{N}\Lambda_i = T_d\left(\Lambda_d + S\right)T_d^\top, \tag{39}$$

where

$$S = T_d^\dagger \sum_{i=1}^{N}\left(T_{iu}\Lambda_{ir}T_{id}^\top + T_{id}\Lambda_{ir}^\top T_{iu}^\top + T_{iu}\Lambda_{iu}T_{iu}^\top\right) T_d^{\dagger\top}.$$

Considering (39), since T_d is row independent, the LMI (20) is feasible if the following inequality has solution:

$$\Lambda_d + S \prec 0. \tag{40}$$

Let us recall that $\Lambda_d = \mathrm{diag}(\Lambda_{1d}, \Lambda_{2d}, \dots, \Lambda_{Nd})$, where $\Lambda_{id} = \Gamma_{id}^\top P_{id} + P_{id}\Gamma_{id}$ and $\Gamma_{id} = \bar{A}_{id} - K_{id}C_{id}$. Because of the detectability of the pair $(C_{id}, \bar{A}_{id})$, there exists K_{id} such that Γ_{id} is Hurwitz. In this condition, according to the Lyapunov stability criterion (Antsaklis & Michel, 2006, Chap. 6), for each $\beta \in \mathbb{R}_{>0}$ there exists $P_{id} \succ 0$ such that $\Lambda_{id} = \Gamma_{id}^\top P_{id} + P_{id}\Gamma_{id} = -\beta I_{n-v_i}$. On the other hand, there exists a large enough β such that (40) has solution, which guarantees the feasibility of the LMI (20). Hence, by selecting $P_{iu} \succ 0$ and Y_i arbitrarily, and according to the definition of K_i and P_i in (35), the LMI (20) always has solutions for P_i, $\bar{Y}_i$, and $\bar{K}_i$. ■

Theorems 1 and 2 give *constructive* sufficient conditions which can be effectively used to compute the design parameters that achieve error convergence to zero. In the next theorem, we provide necessary and sufficient existence conditions for the proposed observer to be a DUIO in the sense of Definition 1.

Theorem 3 (*Existence*). *Under Assumption 2 and the conditions (6), the observer $\mathcal{O} = \{\mathcal{O}_i\}_{i\in\mathbf{N}}$ comprising local observers in the form (4) is a DUIO for the LTI system (1) if and only if the following conditions hold:*

(i) $\mathrm{rank}(C_i\bar{B}_i) = \mathrm{rank}(\bar{B}_i)$, $\forall i \in \mathbf{N}$,
(ii) $\bigcap_{i=1}^{N} \mathcal{UD}(C_i, A_i) = 0$.

Proof. (Sufficiency)–If (i) holds, (6a) is solvable as stated in Lemma 1. If (ii) is true, then by Theorem 2 we conclude that the LMI (20) admits a solution. Therefore, we can also apply Theorem 1 and conclude that such solution renders e asymptotically stable, i.e., $\forall i \in \mathbf{N}$,

$$\lim_{t\to+\infty} |e_i| = 0.$$

Therefore, $\mathcal{O}$ is a DUIO for (1), according to Definition 1.

(Necessity) – Assume now that $\mathcal{O} = \{O_i\}_{i\in\mathbf{N}}$ is a DUIO for (1), i.e., $\forall i \in \mathbf{N}$, $\lim_{t\to+\infty} |e_i(t)| = 0$. This immediately implies that (6a) is solvable, since it is a necessary condition (see Chen et al. (1996)) for any arbitrary disturbance term $\acute{w}$ in (5) to be exactly canceled. Hence, according to Lemma 1, (i) holds. To prove the necessity of (ii), we proceed by contradiction and assume that there exists a nontrivial subspace $\mathcal{S} \subset \mathcal{X}$ such that

$$\bigcap_{i=1}^{N} \mathcal{UD}(C_i, A_i) = \mathcal{S} \neq 0,$$

which, according to (14), indicates that

$$\mathcal{S} = \left(\bigcap_{i=1}^{N} \mathcal{UO}(C_i, A_i)\right) \bigcap \left(\bigcap_{i=1}^{N} \mathrm{Ker}\ \alpha_{A_i}^{+}(A_i)\right). \tag{41}$$

We consider the factorization of $\alpha_{A_i}^{+}(s)$ as

$$\alpha_{A_i}^{+}(s) = \alpha_{A_i,1}^{+}(s)\alpha_{A_i,2}^{+}(s)\cdots\alpha_{A_i,q_i}^{+}(s), \tag{42}$$

for some positive $q_i \le n$, where $\alpha_{A_i,k}^{+}(s)$, $k \in \{1,\dots,q_i\}$, are real irreducible polynomials and pairwise coprime (Wonham, 1985, Chap. 0.11). An analogous factorization also exists for $\alpha_{A_i}^{-}(s)$. By applying the modal decomposition, $\mathcal{X}$ is decomposed into linearly independent subspaces as

$$\mathcal{X} \simeq \mathcal{X}_i^{-} \oplus \mathcal{X}_i^{+} = \mathcal{X}_i^{-} \oplus \mathcal{X}_{i,1}^{+} \oplus \cdots \oplus \mathcal{X}_{i,q_i}^{+}, \tag{43}$$

where $\mathcal{X}_i^{-} = \mathrm{Ker}\ \alpha_{A_i}^{-}(A_i)$ and $\mathcal{X}_{i,k}^{+} = \mathrm{Ker}\ \alpha_{A_i,k}^{+}(A_i)$. Therefore, thanks to the linear independence of the modes, we have

$$\mathrm{Ker}\ \alpha_{A_i}^{+}(A_i) = \mathrm{Ker}\ \alpha_{A_i,1}^{+}(A_i) \oplus \cdots \oplus \mathrm{Ker}\ \alpha_{A_i,q_i}^{+}(A_i), \tag{44}$$

thus, the second term on the right-hand side of (41) expands as follows:

$$\bigcap_{i=1}^{N} \mathrm{Ker}\, \alpha^{+}_{A_i}(A_i) = \bigcap_{i=1}^{N} \mathcal{X}^{+}_{i,1} \oplus \cdots \oplus \mathcal{X}^{+}_{i,q_i}.$$

Since $\mathcal{S} \neq 0$, there exists $\mathcal{X}^{+}_{\cap} \subseteq \mathcal{X}^{+}_{i}, \forall i \in \mathbf{N}$, whose intersection with the unobservable subspaces of the nodes is nontrivial. Namely, $\mathcal{X}^{+}_{\cap} \subseteq \mathcal{S}$ is by construction an A_i-invariant subspace (cf. Wonham (1985, Chap. 0.11)) of an undetectable mode of all the nodes, that is $\forall i \in \mathbf{N}$, there exists $x \in \mathcal{X}$, $x \neq \mathbf{0}_{n\times 1}$, such that

$$\alpha^{+}_{A_i}(A_i)x = \mathbf{0}_{n\times 1}. \tag{45}$$

By Lemma 2, (41)–(45) hold for $\bar{A}_i$ as well, thus we let $v \in \mathcal{X}^{+}_{\cap}$ be one of such common undetectable modes, and since $\mathcal{S} \subseteq \mathrm{Ker}\, C_i$ for all $i \in \mathbf{N}$, it holds that

$$(\bar{A}_i - K_iC_i)v = \bar{A}_i v. \tag{46}$$

By stacking the error components and from (8), we obtain

$$\begin{aligned}\dot{e} &= \left(\operatorname*{diag}_{i\in\mathbf{N}}(N_i) - \chi \operatorname*{diag}_{i\in\mathbf{N}}\left(P_i^{-1}\right)(\mathcal{L}\otimes I_n)\right)e \\ &= (\Phi - \Pi)e,\end{aligned} \tag{47}$$

where the definitions of Φ and Π follow trivially from the equality. Let $\bar{e} = \mathbf{1}_N \otimes v$. Thanks to Assumption 2, $\bar{e} \in \mathrm{Ker}\,\Pi$, that is $\Pi\bar{e} = \mathbf{0}_{Nn\times 1}$. Moreover, for each block of Φ, $\bar{e}$ satisfies (46) with $v \in \mathcal{X}^{+}_{\cap}$. Therefore, by considering (6) we have

$$(\Phi - \Pi)\bar{e} = \operatorname*{diag}_{i\in\mathbf{N}}(\bar{A}_i - K_iC_i)\bar{e} = \operatorname*{diag}_{i\in\mathbf{N}}(\bar{A}_i)\bar{e}.$$

Now, choosing $\bar{e}_0 = \mathbf{1}_N \otimes v$ as the initial condition for (47) produces an error along the direction of the unstable mode v. This contradicts the asymptotic stability hypothesis, and therefore (ii) must be true. ■

It should be noted that we have formulated Theorem 3 in a way to express the similarities of the conditions derived in our approach to the classical existence conditions (Chen et al., 1996, Theorem 1) for the centralized case. We remark as well that (ii) is a necessary and sufficient condition also appearing in Ugrinovskii (2013).

In the following subsection, we show how the proposed DUIO can be extended to more complex scenarios under some conditions, such as graphs with switching topologies and directed networks.

4.2. Extension to switching topologies or directed networks

The results presented in Theorem 1 are based on the assumption that the communication graph is undirected and its links are steady and not failing over time. However, by suitably modifying χ, the obtained results can be extended to more general scenarios such as distributed estimation in the presence of switching topologies and distributed estimation in directed networks.

In the presence of switching topologies, let $\mathcal{G}(t)$ describe a communication graph switching over time. Accordingly, the distributed observer proposed in (4) should be modified as follows:

$$\begin{aligned}\dot{z}_i &= N_iz_i + M_iB_iu_i + L_iy_i + \chi P_i^{-1}\sum_{j=1}^{N} a_{ij}(t)(\hat{x}_j - \hat{x}_i), \\ \hat{x}_i &= z_i + H_iy_i,\end{aligned} \tag{48}$$

where $a_{ij}(t) = 1$ if there exists a communication link between Node i and Node j at time t, and it is zero otherwise. Accordingly, we consider an infinite time sequence $t_0, t_1, t_2, \ldots$ starting at $t_0 = 0$, at which $\mathcal{G}(t)$ switches to $\mathcal{G}_k, k = 0, 1, 2, \ldots$, while remaining fixed and connected during the time period $[t_k, t_{k+1})$.

Corollary 1. *Consider the DUIO described in (48) under the conditions (6) and a switching communication graph $\mathcal{G}(t)$, where $\mathcal{G}(t)$ remains connected over time. By letting Λ_i as (19), the estimation error e converges to zero if $Y_i = P_i^{-1}\bar{Y}_i$, and $K_i = P_i^{-1}\bar{K}_i$, where the matrices $P_i \succ 0$, $\bar{Y}_i$, and $\bar{K}_i$ are a feasible solution of the LMI (20), and the gain χ satisfies*

$$\chi > \frac{\left|\Lambda + \Lambda_P^{\top}\left(\sum_{i=1}^{N}\Lambda_i\right)^{-1}\Lambda_P\right|}{2\mathcal{C}(N)}, \tag{49}$$

where Λ and Λ_P are defined in (22), and $\mathcal{C}(N)$ is a lower bound for the algebraic connectivity of graphs with N nodes.

Proof. By considering a common Lyapunov function for the set of switching networks the same as in (23) and following the same steps as in the proof of Theorem 1, for the time period $[t_k, t_{k+1})$ one gets

$$\dot{V} \leq -\begin{bmatrix}\omega\\ e_r\end{bmatrix}^{\top}\begin{bmatrix}-\sum_{i=1}^{N}\Lambda_i & -\Lambda_P\\ -\Lambda_P^{\top} & 2\chi\lambda_2(\mathcal{L}_k)I_{Nn} - \Lambda\end{bmatrix}\begin{bmatrix}\omega\\ e_r\end{bmatrix}, \tag{50}$$

where $\mathcal{L}_k$ is the Laplacian matrix associated with $\mathcal{G}_k$. In this condition, according to (50) and the Schur complement, $\dot{V}$ is negative definite if

$$2\chi\lambda_2(\mathcal{L}_k)I_{Nn} - \Lambda - \Lambda_P^{\top}\left(\sum_{i=1}^{N}\Lambda_i\right)^{-1}\Lambda_P \succ 0,$$

which (49) guarantees this. Therefore, as $\dot{V}$ is negative definite, e converges to zero. ■

It should be noted that for any graph with N nodes there exists a lower bound for the algebraic connectivity of the graph which just depends on N (see Pirani and Sundaram (2016) and Chung (1997)).

Now, let the network communication graph be fixed and directed. When the graph is undirected, the Laplacian matrix associated with the communication graph is semidefinite, and this property has been used in the proof of Theorem 1. However, if the graph is strongly connected, it is possible to modify the proposed DUIO in Theorem 1 such that the obtained results in Theorem 1 are extendable to directed networks as well. In this regard, we first introduce the following lemma.

Lemma 4. *(Li & Duan, 2017) Let $\mathcal{G}$ be a strongly connected directed graph. Then, there exists a unique positive row vector $r = \begin{bmatrix}r_1 & r_2 & \ldots & r_N\end{bmatrix}$ such that $r\mathcal{L} = \mathbf{0}_{1\times N}$ and $r\mathbf{1}_N = N$, and by defining $R := \mathrm{diag}(r_1, \ldots, r_N)$, the symmetric matrix $\hat{\mathcal{L}} := R\mathcal{L} + \mathcal{L}^{\top}R$ is positive semidefinite. Furthermore, $\mathbf{1}_N^{\top}\hat{\mathcal{L}} = \mathbf{0}_{1\times N}$, $\hat{\mathcal{L}}\mathbf{1}_N = \mathbf{0}_{N\times 1}$, and $\lambda_1 = 0$ is an eigenvalue of $\hat{\mathcal{L}}$ while the other eigenvalues of $\hat{\mathcal{L}}$ are positive real.*

Based on Lemma 4, to extend the result of Theorem 1 to strongly connected directed networks, the DUIO (4) can be modified as follows:

$$\begin{aligned}\dot{z}_i &= N_iz_i + M_iB_iu_i + L_iy_i + \chi r_iP_i^{-1}\sum_{j=1}^{N} a_{ij}(\hat{x}_j - \hat{x}_i), \\ \hat{x}_i &= z_i + H_iy_i.\end{aligned} \tag{51}$$

Corollary 2. *Consider the DUIO described in (51) under the conditions (6) and a strongly connected graph. By letting Λ_i as (19), the estimation error e converges to zero if $Y_i = P_i^{-1}\bar{Y}_i$, and $K_i = P_i^{-1}\bar{K}_i$,* *where the matrices $P_i \succ 0$, Y_i, and K_i are a feasible solution of the LMI (20), and the gain χ satisfies*

$$\chi > \frac{\left| \Lambda + \Lambda_P^\top \left(\sum_{i=1}^N \Lambda_i\right)^{-1} \Lambda_P \right|}{2\lambda_2(\hat{\mathcal{L}})}, \tag{52}$$

where Λ and Λ_P are defined in (22).

Proof. Based on the analytical procedure given in Section 4.1, along the DUIO (51), the estimation error takes the following form:

$$\dot{e}_i = N_i e_i + \chi P_i^{-1} \sum_{j=1}^N r_i a_{ij}(e_j - e_i).$$

By considering the same Lyapunov function as in (23) and following the same procedure as in the proof of Theorem 1, one gets

$$\dot{V} = e^\top \Lambda e - 2\chi e^\top (R\mathcal{L} \otimes I_n)\, e$$
$$= e^\top \Lambda e - \chi e^\top \left((R\mathcal{L} + \mathcal{L}^\top R) \otimes I_n\right) e,$$

which after defining $\hat{\mathcal{L}} := R\mathcal{L} + \mathcal{L}^\top R$ can be restated as follows:

$$\dot{V} = e^\top \Lambda e - \chi e^\top \left(\hat{\mathcal{L}} \otimes I_n\right) e. \tag{53}$$

Since $\mathcal{G}$ is strongly connected, by considering Lemma 4, $\hat{\mathcal{L}}$ is symmetric positive semidefinite, $\mathbf{1}_N^\top \hat{\mathcal{L}} = \mathbf{0}_{1\times N}$, $\hat{\mathcal{L}}\mathbf{1}_N = \mathbf{0}_{N\times 1}$, and $\hat{\mathcal{L}}$ has one zero eigenvalue and $N-1$ positive real eigenvalues. By following the same procedure as in the proof of Theorem 1, from (53) one gets

$$\dot{V} \le -\begin{bmatrix}\omega \\ e_r\end{bmatrix}^\top \begin{bmatrix} -\sum_{i=1}^N \Lambda_i & -\Lambda_P \\ -\Lambda_P^\top & 2\chi\lambda_2(\hat{\mathcal{L}}) I_{Nn} - \Lambda \end{bmatrix} \begin{bmatrix}\omega \\ e_r\end{bmatrix}. \tag{54}$$

Based on (54) and the Schur complement, $\dot{V}$ is negative definite if

$$2\chi\lambda_2(\hat{\mathcal{L}}) I_{Nn} - \Lambda - \Lambda_P^\top \left(\sum_{i=1}^N \Lambda_i\right)^{-1} \Lambda_P \succ 0,$$

which is guaranteed by (52). Hence, as $\dot{V}$ is negative definite, e converges to zero. ■

5. Simulation results

The effectiveness of the proposed DUIO is evaluated in this section. We consider a simplified LTI system based on the heat exchange model in multi-zone buildings (Witrant et al., 2009). In this model, a building floor is divided into 9 zones (rooms) by walls with distinct heat exchange rates. Heating, ventilation, and air conditioning (HVAC) is placed only in three zones (Rooms 2, 5, and 6), and one room is affected by an unpredictable temperature disturbance (Room 9). Four observers are placed at different locations at the floor, and it is desired that each observer estimates the temperature of all rooms. Moreover, the input signal of HVAC in each room is assumed to be known to only one node/observer, while the fourth node/observer does not have access to any of the inputs. The schematic diagram in Fig. 2 depicts the described system.

The system model is in the form of (1) where A, B, and D are given in Appendix C. By decomposing u as $u = [u_1\ u_2\ u_3]^\top$, we assume that B_1, B_2, and B_3 are as given in Appendix C. Note that B_4 is not defined since Node 4 is assumed to be without local inputs. Accordingly, we have

$$\bar{B}_1 = \begin{bmatrix} B_2 & B_3 & D \end{bmatrix}, \quad \bar{B}_2 = \begin{bmatrix} B_1 & B_3 & D \end{bmatrix},$$
$$\bar{B}_3 = \begin{bmatrix} B_1 & B_2 & D \end{bmatrix}, \quad \bar{B}_4 = \begin{bmatrix} B_1 & B_2 & B_3 & D \end{bmatrix}.$$

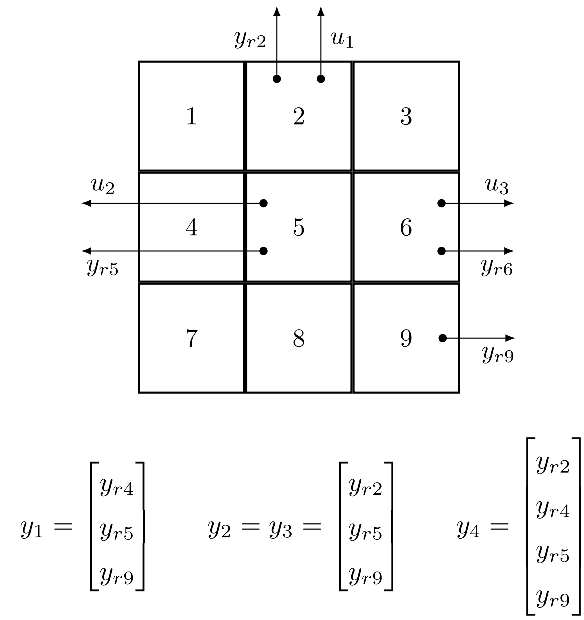

Fig. 2. Diagram of the simulated system. Each square represents a different building zone, the arrows indicate which quantities are known from each subsystem (cf. Fig. 1). Underneath, the measurements y_i available to Node i are denoted based on the measurements y_{rk} of Room k.

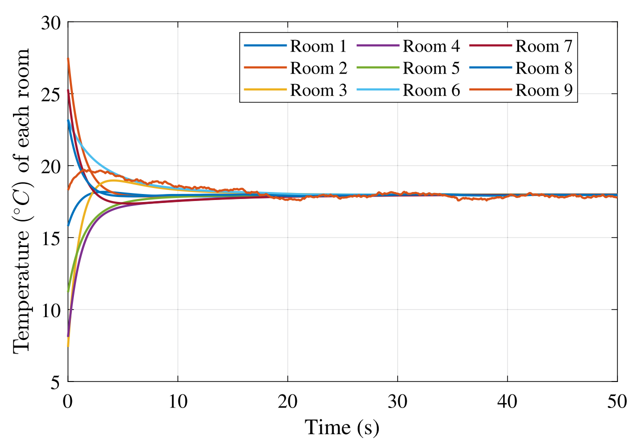

Fig. 3. True temperature of each room for all four scenarios.

Moreover, the output matrices are considered as given in Appendix C. Without loss of generality, the control input is selected as $u = -F(x - x_d)$, where $F \in \mathbb{R}^{3\times 9}$ is provided in Appendix C and x_d is the desired state that indicates the desired temperature in each room, which is 18 °C in this case. Moreover, w is set as the band-limited white noise with noise power set to 2. The initial temperature of each room is chosen arbitrarily between 7 °C and 32 °C, which is given by

$$x_0 = \begin{bmatrix} 15.8 & 27.5 & 7.4 & 8.1 & 11.2 & 23.2 & 25.3 & 23.2 & 18.3 \end{bmatrix}^\top.$$

In this condition, the states of the system are shown in Fig. 3.

The proposed distributed estimation strategies are first evaluated in three scenarios corresponding to communication topologies being fixed and connected, fixed and strongly connected

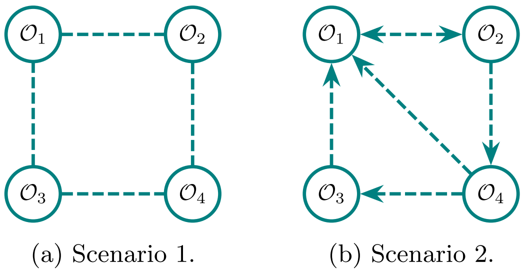

Fig. 4. Network communication topologies in Scenarios 1 and 2.

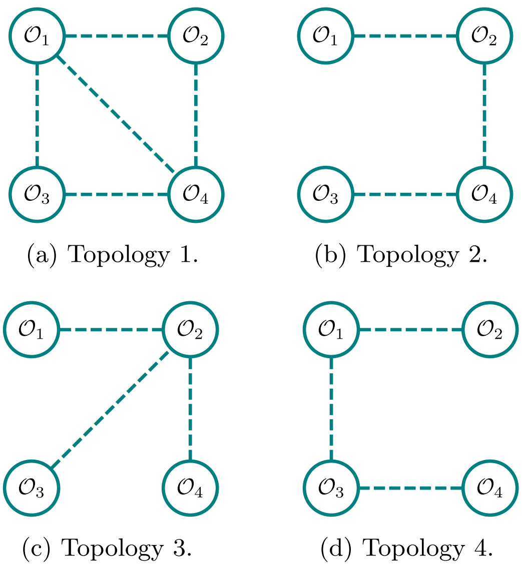

Fig. 5. Switching set of the network topologies in Scenario 3.

directed, and switching connected, as shown in Figs. 4–5, respectively. The last scenario is designed to evaluate the effect of parameter perturbations. Note that the initial conditions of all the observers are set as zero vectors for all four scenarios.

Scenario 1 (*Undirected Graph*)**.** In the first scenario, the nodes are assumed to be connected via the unweighted undirected communication graph depicted in Fig. 4(a) implying that $\lambda_2(\mathcal{L}) = 2$. Distributed state estimation is based on the distributed observers (4) where N_i, M_i, L_i, H_i, and P_i are obtained from the solution of the LMI (20) as provided in the supplementary document (see Appendix C), computed using the CVX toolbox (Boyd & Vandenberghe, 2004). Moreover, following (21), χ is set to 41.31. Under these conditions, the estimation errors $e_i, i \in \mathbf{N}$, at all nodes are shown in Fig. 6. According to the figure, the estimation error of all the states at all the nodes converges to zero asymptotically. From (32), the time constant is calculated as $\mu^{-1} = 3.889$ s. In this regard, the evolution of the Lyapunov function V along with $e^{-\mu t}V(0)$ is depicted in Fig. 7.

Scenario 2 (*Directed Graph*)**.** In the second scenario, the nodes are assumed to be connected via the unbalanced directed communication graph depicted in Fig. 4(b). Distributed state estimation is based on the distributed observers given in (51) where N_i, M_i, L_i, H_i, and P_i still are the same as Scenario 1. According to Lemma 4, $R = \text{diag}(0.5714, 1.714, 0.5714, 1.143)$, and following (52), χ is set to 114.7. Under these conditions, the norm of the collective estimation error e converges to zero asymptotically (since the plot

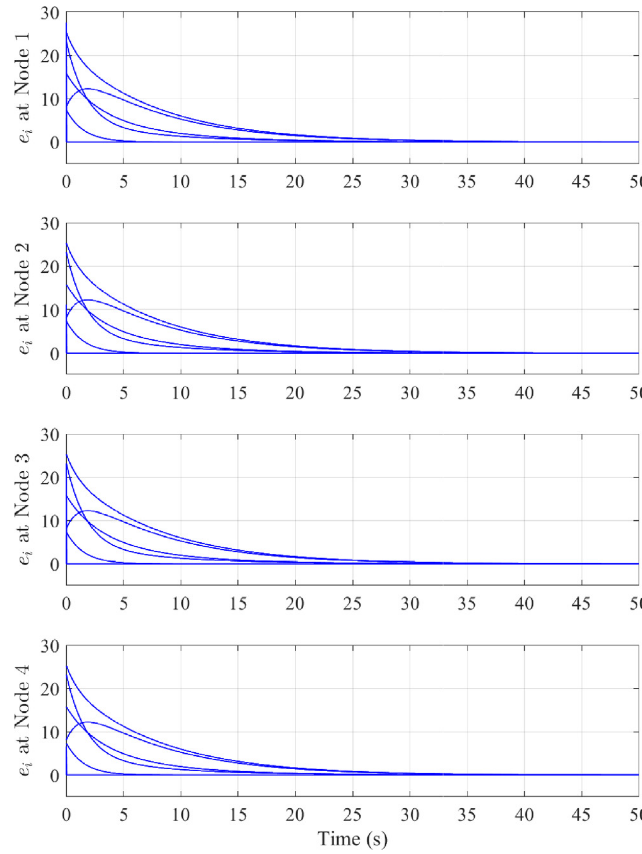

Fig. 6. Estimation error generated by the proposed distributed UIO in the presence of unknown inputs for all the nodes in Scenario 1.

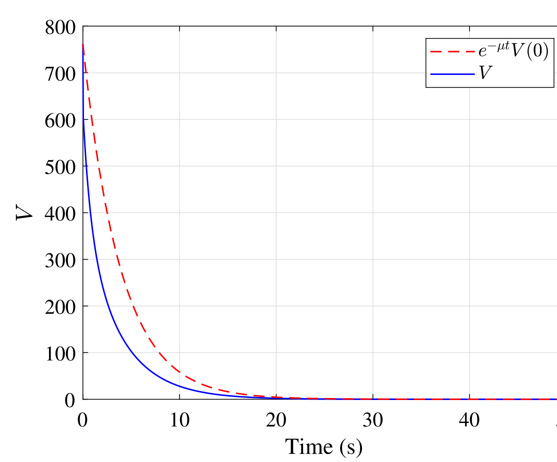

Fig. 7. Lyapunov function V generated by the proposed distributed observer in Scenario 1.

of the results for this scenario is almost the same as Fig. 8, it is omitted to avoid repetition).

Scenario 3 (*Switching Undirected Graph*)**.** In the third scenario, the nodes are assumed to be connected under the switching communication topology depicted in Fig. 5, such that the information exchange starts from Topology 1 and switches to the next topology every 0.1 s (after Topology 4 the graph switches back to Topology 1). Distributed state estimation is based on the distributed observers given in (48) where N_i, M_i, L_i, H_i, and P_i are the

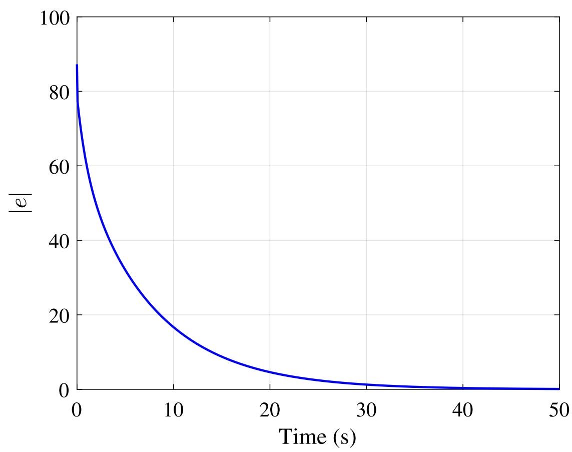

Fig. 8. Norm of the collective estimation error generated by the proposed distributed UIO in the presence of unknown inputs and parameter perturbation in Scenario 4.

same as Scenario 1. Moreover, $\mathcal{C}(4)$ is calculated as 4.167×10^{-2}, and χ is set to 1.978×10^{3} by following (49). Under these conditions, the norm of the collective estimation error e converges to zero asymptotically (since the plot of the results for this scenario is almost the same as Fig. 8, it is omitted to avoid repetition).

Scenario 4 (*With Parameter Perturbation*)**.** In the fourth scenario, the effect of system parameter perturbation is evaluated. The system perturbation is formulated as in Chen et al. (1996) such that

$$Dw = \Delta A x + \Delta B u$$

for some $w \in \mathbb{R}^q$, where ΔA and ΔB are given in Appendix C.

The topology of the nodes and the distributed state estimation scheme are identical to those stated in Scenario 1. Under these conditions, the collective estimation error e converges to zero asymptotically as shown in Fig. 8.

6. Conclusions and future work

Distributed state estimation of a class of LTI systems was addressed, where the system outputs were measured via a network of sensors distributed within N nodes, and the local measurements at each node were not sufficient for local state estimation. We proposed a DUIO consisting of N local observers co-located with the N nodes and connected via a communication network such that the full state vector of the system was estimated by each local observer. The proposed architecture allowed to account for partial measurements, but more notably for inputs that may not be available locally at a node, together with other unknown disturbances. The feasible solution of an LMI provided adequate choices of parameters that guaranteed convergence of the estimation errors, under some joint detectability conditions. Furthermore, we provided necessary and sufficient existence conditions that were in line with existing theorems for the centralized case. Finally, we extended our main result to include more complex scenarios in our study, such as switching network topologies and directed communication links. It should be noted that this study was a primary effort on DUIOs, and many problems such as designing DUIOs in the presence of measurement noise as well as expanding the obtained results to discrete-time domain remain open to be studied as future work.

Appendix A. Derivation of Eq. (5)

The equation to be proved is obtained by expanding the error definition (3) along with the system dynamics (1), the output Eq. (2), and the local observer (4). We start by noting that

$$e_i = x - z_i - H_i y_i = (I_n - H_i C_i)x - z_i. \tag{A.1}$$

Taking the time derivative of (A.1) yields

$$\begin{aligned}\dot{e}_i = \; & (I_n - H_i C_i)(Ax + B_i u_i + \bar{B}_i \acute{w}_i) \\ & - N_i z_i - M_i B_i u_i - L_i y_i - \chi P_i^{-1} \sum_{j=1}^{N} a_{ij}(\hat{x}_j - \hat{x}_i),\end{aligned}$$

which by adding and subtracting the term $(I_n - H_i C_i)A\hat{x}_i$ to the right-hand side and since $\hat{x}_i = z_i + H_i y_i$, can be restated as follows:

$$\begin{aligned}\dot{e}_i = \; & (I_n - H_i C_i)Ae_i + (I_n - H_i C_i - M_i)B_i u_i \\ & + (I_n - H_i C_i)\bar{B}_i \acute{w}_i + (I_n - H_i C_i)A(z_i + H_i y_i) \\ & - N_i z_i - L_i y_i - \chi P_i^{-1} \sum_{j=1}^{N} a_{ij}(\hat{x}_j - \hat{x}_i).\end{aligned} \tag{A.2}$$

According to the definition of e_i, one gets

$$-K_i C_i e_i + K_i C_i (x - \hat{x}_i) = \mathbf{0}_{n\times 1}. \tag{A.3}$$

Since $\hat{x}_i = z_i + H_i y_i$, from (A.3), it follows that

$$-K_i C_i e_i + K_i y_i - K_i C_i z_i - K_i C_i H_i y_i = \mathbf{0}_{n\times 1}. \tag{A.4}$$

We note that

$$\hat{x}_j - \hat{x}_i = x - \hat{x}_i - (x - \hat{x}_j) = e_i - e_j. \tag{A.5}$$

Now, by adding the zero term (A.4) to the right-hand side of (A.2) and by considering (A.5), after grouping similar terms, we finally obtain (5). ■

Appendix B. Proof of Lemma 3

Since T_{iu} is an orthonormal basis of the undetectable subspace of $(C_i, \bar{A}_i)$, we have

$$\operatorname{Im} T_{iu} = \mathcal{UD}(C_i, \bar{A}_i). \tag{B.1}$$

According to the analysis in the proof of Lemma 1 of Yang et al. (2020), one gets

$$\operatorname{Im} T_d = \left(\bigcap_{i=1}^{N} \mathcal{UD}(C_i, \bar{A}_i) \right)^{\perp}. \tag{B.2}$$

From Lemma 2, we have

$$\mathcal{UD}\left(C_i, \bar{A}_i\right) = \mathcal{UD}\left(C_i, A_i\right).$$

Hence from (B.2), it follows that

$$\operatorname{Im} T_d = \left(\bigcap_{i=1}^{N} \mathcal{UD}(C_i, A_i) \right)^{\perp}. \tag{B.3}$$

Finally, by (B.3) and under the hypothesis of extensive joint detectability (10), we have $\operatorname{Im} T_d = 0^{\perp} = \mathcal{X}$ (Wonham, 1985, Chap. 0.12), which completes the proof. ■

Appendix C. Simulation parameters in Section 5

The following system parameters are used for all four scenarios except ΔA and ΔB used only in Scenario 4. The parameters of the proposed observers are provided in the supplementary document (available at the repository [https://github.com/ang-b/yang-distributed-uio]).

$$A = 10^{-2} \times \begin{bmatrix} -30 & 25 & 0 & 5 & 0 & 0 & 0 & 0 & 0 \\ 25 & -65 & 25 & 0 & 15 & 0 & 0 & 0 & 0 \\ 0 & 25 & -65 & 0 & 0 & 40 & 0 & 0 & 0 \\ 5 & 0 & 0 & -60 & 15 & 0 & 40 & 0 & 0 \\ 0 & 15 & 0 & 15 & -75 & 10 & 0 & 35 & 0 \\ 0 & 0 & 40 & 0 & 10 & -70 & 0 & 0 & 20 \\ 0 & 0 & 0 & 40 & 0 & 0 & -50 & 10 & 0 \\ 0 & 0 & 0 & 0 & 35 & 0 & 10 & -60 & 15 \\ 0 & 0 & 0 & 0 & 0 & 20 & 0 & 15 & -35 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0.1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.1 & 0 & 0 & 0 \end{bmatrix}^\top,$$

$$B_1 = \begin{bmatrix} 0 & 0.1 & \mathbf{0}_{1\times 7} \end{bmatrix}^\top \quad B_2 = \begin{bmatrix} \mathbf{0}_{1\times 4} & 0.1 & \mathbf{0}_{1\times 4} \end{bmatrix}^\top,$$

$$B_3 = \begin{bmatrix} \mathbf{0}_{1\times 5} & 0.1 & \mathbf{0}_{1\times 3} \end{bmatrix}^\top, D = \begin{bmatrix} \mathbf{0}_{1\times 8} & 0.1 \end{bmatrix}^\top,$$

$$C_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$C_3 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$C_4 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\Delta A = 10^{-4} \times \begin{bmatrix} & & & & \mathbf{0}_{8\times 9} & & & & \\ \hline 81 & 90 & 12 & 91 & 63 & 9 & 27 & 54 & 95 \end{bmatrix},$$

$$\Delta B = 10^{-3} \times \left[\begin{array}{c|c} \mathbf{0}_{3\times 8} & \begin{matrix} 15 \\ 97 \\ 95 \end{matrix} \end{array} \right]^\top.$$

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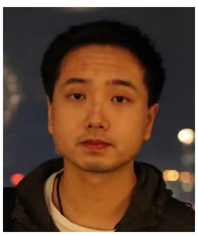

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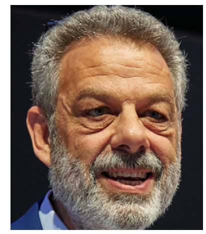

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