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DOTTORANDO
ANKUR ANKUR

COORDINATORE
PROF. FRANCESCO LONGO

Francesco Longo

SUPERVISORE DI TESI
PROF. LORENZO DI PIETRO

Lorenzo Di Pietro

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"In three words I can sum up everything I've learned about life: it goes on."

Robert Frost

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Dedicated to my Parents,
For their endless love and support . . .

Chapter 1

Introduction

Quantum field Theories (QFTs) have stood the test of time and so far has been one of the most beautiful and successful theories in physics. The story of atom has started centuries ago and we have made immense progress in understanding the building block of matter and the universe around us especially in the last century. This has been mainly possible because of triumph of Quantum field theories. Despite being one of the most accepted and celebrated formalism, there is still a lot that is not well understood in the formalism of standard Quantum Field Theory. For example, we still do not have proper non-perturbative formulation of QFT and we still can not properly apply it to strongly coupled system. One goal of present day theoretical physics is to complete the understanding of QFTs in the regime of strong coupling and many attempts has been done to achieve this goal.

The topic of this thesis is an approach to the study of QFT beyond perturbation theory using as a tool the geometry of Anti-de Sitter (AdS) space. This has two main goals:

- To bridge two celebrated approaches to strongly coupled QFT, both based on symmetry and self-consistency, namely the "S-matrix bootstrap" and the "Conformal bootstrap", which apply respectively to massive and conformal quantum field theories. More generally, the AdS geometry allows us to import the techniques and progresses of Conformal Field Theories (CFTs) in the realm of massive quantum field theory;
- To find new computational tools for QFT in cosmological backgrounds, in particular the de Sitter (dS) spacetime characterized by accelerated expansion, which can be related to AdS via an analytic continuation.

Let us now illustrate in more details these points, starting from an explanation of the bootstrap philosophy.

Summarized in a sentence, the bootstrap program is to know the unknowns not by explicitly calculating them but rather using some consistency conditions. There are some consistency conditions that a QFT should follow and the idea is to impose enough conditions so that we can determine the observables. This approach is especially useful in cases where the perturbative calculations are too tedious, or even more importantly in cases where non-perturbative effects are important.

One of the most basic observables in QFTs are the scattering amplitudes, that determine the outcome of the collision between the particle-like excitations of the fields. When applied to scattering amplitudes, this philosophy leads to the so-called "S-matrix boostrap". Examples of such conditions that apply to scattering amplitudes are crossing symmetry [1] and unitarity [2]. Global symmetries that the theory might have can also be used as the consistency condition.

In this context, there has been some progress in 2 to 2 scattering, but things get complicated really fast when considering 2 to n for $n > 2$, i.e. multipoint scattering amplitudes. In $d = 2$, for a class of theories known as integrable QFTs there is no 2 to n scattering for $n > 2$ (see e.g. the discussion in [3, 4] and references therein), though n to n will still remain relevant and one can factor it into product of 2 to 2 scattering amplitudes. In this case the bootstrap program can be carried out successfully [5]. For $d \geq 3$ instead one always has m to n scattering amplitudes for $m \neq n$, i.e. particle production cannot be avoided [6]. This is one of the reason why it is difficult to bootstrap amplitudes in higher dimensions.

In its early incarnation, the goal of the S-matrix bootstrap was to fix completely the scattering amplitude [7]. More recently there has been a revival of this idea with a more humble and realistic approach [8–10] (see also the review [11] and references therein). In the modern take, the goal is essentially to narrow down the space of possible scattering amplitudes, restricting the possible parameter space that is consistent with the constraints, also with the aid of numerical techniques.

With this new approach, in $1 + 1$ dimensions it was found that for a fixed mass spectrum one can bound the couplings from above [9]. This was done by applying crossing symmetry, analyticity and non-linear unitarity. In the case of theories with $O(N)$ symmetry (with the assumtpion that no bound states are present) it was found that the boundary of allowed space contains vertices that correspond to known theories, e.g. some integrable theories [12, 13].

What about higher dimensions? Obviously we cannot expect the same level of progress in the determination of S-matrices in higher dimensions but there has been development in this case as well, due to advancement of numerical techniques.In [10] it is argued that the main difficulty that arises in higher dimensions is that one needs to expand amplitude in terms of partial waves in order to impose unitarity (in lower dimension, unitarity can be applied directly at level of S-matrix), while crossing symmetry and analyticity are more transparent in terms of Mandelstam variables. Thus one needs to use both descriptions to impose the full set of constraints, which is very inconvenient. In the modern approach, numerical techniques are used to impose unitarity.

As mentioned above, apart from scattering amplitudes, another context to which the bootstrap philosophy has been applied very successfully in last few decades is that of Conformal Field Theories (CFT) in the so-called Conformal bootstrap.

CFTs are those QFTs which have additional symmetries known as dilation (which can be thought of as a scaling transformation) and "special conformal transformations", which together with the Poincaré group encompass the conformal group $SO(1, d+1)$ in d (Euclidean) spacetime dimensions. These theories are important because any QFT is scale-invariant in the limit of small (UV) or large (IR) distances. It is not necessary that all scale-invariant theories are also invariant under the full conformal group, but this is typically the case, especially if one insists on the constraint of unitarity [14–16]. Therefore CFTs can be thought of as “landmarks” in the space of QFTs, as any UV-complete QFT can be seen as a renormalization group (RG) flow from a CFT_{UV} to a CFT_{IR} [17]. The importance of CFTs also stems from the fact that they can applied to describe second order phase transitions both in statistical mechanics [18] and in condensed matter physics [19], perhaps the most famous examples being the Ising CFT that describes the critical point of water, and the $O(N)$ CFT that describes the second order magnetization transition in ferromagnets. Except for the trivial examples of free massless fields, CFTs are typically strongly coupled which makes them difficult to study with standard techniques. A perturbative approach can be devised in some cases using the idea of dimensional continuation, leading to the so-called ϵ expansion [20]: for example, if one wants to study the IR fixed point of the ϕ^4 theory in $3d$, instead of computing Feynman diagrams in $3d$, one can compute them in $d = 4 - \epsilon$ in which the fixed point is weakly coupled. However to recover the physical result one eventually needs to perform an uncontrolled extrapolation to the limit $\epsilon \to 1$.

The boostrap philosophy can be applied to the basic observables of CFTs, correlation functions of local operators. Like the S-matrix bootstrap, this is an old idea dating back to ref.s [21, 22], which observed that conformally invariant correlation functions can be reduced to the 2-point and 3-point correlation functions by recursive use of the convergent Operator Product Expansion (OPE) that these theories enjoy, and moreover, analogously to scattering amplitudes, they are constrained by crossing symmetry. Together with the symmetry under the conformal group, and unitarity, crossing symmetry provides a very stringent set of constraints that one can hope to solve, with some further input required to specify the theory, such as additional global symmetries, or the scaling dimensions of light operators. However in practice it is very hard to solve exactly this infinite set of constraints, and in the early days this was accomplished with spectacular success only for some $d = 2$ CFTs [23].

The modern revival of the conformal bootstrap started with [24] that devised a numerical method to implement the constraints to reduce the allowed parameter space (this development in fact inspired also the revival in the S-matrix bootstrap that we mentioned above). Since this paper, various models have been studied using conformal bootstrap, most notably the $3d$ Ising model [25] and $O(N)$ models [26], and also powerful analytical approaches have been developed (for a list of references, see the reviews [27–29]).

Having briefly reviewed these two bootstrap approaches, the S-matrix and the Conformal bootstrap, let us now explain in what sense they can be bridged by studying quantum field theory in AdS. To do so, we need to first explain some basic aspects of QFT in the AdS background.

The geometry of (Euclidean) AdS_D has several features that make it an ideal background to study QFT [30]. It introduces a dimensionful parameter, the radius, which acts as an IR cutoff and can be used to probe the theory at different scales. Differently from other possible IR cutoffs, it preserves a large symmetry, namely the isometry group $SO(1, D)$. Moreover, it admits asymptotic observables, the correlators on the conformal boundary, on which the symmetry acts as the conformal group. This relation between a theory in AdS and a CFT on the boundary has first risen to prominence in the context of the AdS/CFT correspondence [31] (see for instance the reviews [32, 33] for a list of references). In that context the theory living in AdS is a gravitational theory, such as string theory or supergravity, and it can be equivalently described in terms of the boundary CFT, giving a realization of the holographic principle. It is imporant to contrast with the case of a rigid QFT (i.e. without dynamical gravity) in AdS, that will be our concern in this thesis. In the latter case the boundary correlators represent only a subset of the possible observables of the theory, similarly to scattering amplitudes for a QFT in flat space.

The boundary correlators for a QFT in AdS obey all the axioms of a $d = D - 1$ dimensional conformal field theory (CFT), with the only exception of the existence of the stress-tensor operator. They are related to the S-matrix in the flat space limit [8, 34–50]. Therefore the QFT in AdS has boundary conformal correlators, to which the conformal bootstrap can be applied, that are continuosly connected to the scattering amplitudes in flat space, to which the S-matrix bootstrap applies, thus relating directly the two approaches. For example, in [8] the conformal bootstrap was applied and upon taking flat space limit, it exactly matches with the result obtained in flat space with S-matrix bootstrap technique [9]. In other words, the study of QFT in AdS provides us with an additional tool, alternative to the S-matrix bootstrap, to the study of observables in massive quantum field theory: the conformal bootstrap applied to the boundary correlators. This is valuable especially in view of the fact that many properties of correlators in conformal field theories are much better understood compared to the properties of scattering amplitudes [49].

There are other desirable properties of AdS as a background: (i) while offering an IR regulator, in the sense that all correlation functions, even for massless fields, decay fast at large distances, AdS space also has an infinite volume, and thus it allows for sharp phase transitions and symmetry breaking phases. Those would not be visible if the IR behavior would be regulated using a finite volume; (ii) by placing a QFT in AdS, we have a new lens to study the RG flow in the bulk, via the observables of the boundary CFT [51–58]; (iii) in the special case in which the bulk theory is conformal, the AdS background is an efficient tool to study conformal defects [59–64] or boundary conditions [65–67].

Part of the work presented in this thesis concerns precisely the question of how the bulk physics of the massive QFT maps to the boundary correlation functions. A good understanding of this dictionary paves the way to import the progress in CFT to massive quantum field theory, by applying the conformal bootstrap to the boundary correlators either with numerical [8–10] or analytic techniques [68–72]. To make progress in this direction, it is precious to consider benchmark examples of theories that can be studied beyond the perturbative regime, and in which detailed informations can be obtained about both the bulk physics and the boundary correlators. The method that we use to study QFT in AdS at finite coupling is the large N expansion, combined with analytical conformal bootstrap methods. We apply this method to gauge theories.

Let us briefly introduce the idea of the large N expansion. The parameter N represents the number of fields. It is sometimes possible to devise a $1/N$ expansion of observables, which is alternative to the usual expansion at small values of the coupling constants, and therefore allows us to retain finite coupling effects. For example, the $O(N)$ model has been studied in flat space using the large N expansion in [73], which allowed to study the spontaneous symmetry breaking phase even at strong coupling.

The combination of large N and analytic bootstrap has already been applied before to the case of the $O(N)$ and Gross-Neveu models in AdS in [74]. In this thesis we first review this study for the case of $O(N)$ model as it serves as a background before considering gauge interactions. In the $O(N)$ model one has N scalar fields with quartic self-interactions. It is found that in AdS there can be spontaneous symmetry breaking, leading to two possible phases: an $O(N)$ symmetry-preserving phase, in which the scalar fields are massive, and an $O(N)$ symmetry-breaking (gapless) with massless Goldstone bosons. It is also found that for a particular value of the bulk mass-squared the theory enjoys conformal symmetry in the bulk, and thus the setup can be related to a conformal boundary condition for the $O(N)$ CFT in flat space. An AdS analogue of a resonance in flat space is observed in the boundary correlators of the gapless case. This study has been done in two steps, by first introducing a Hubbard-Stratonovich field as an auxiliary field and then computing the exact propagator at large N of this auxiliary field in terms of a unknown "bubble" or 1PI function. In the second step, the 4-point boundary correlation function mediated by this exact propagator is computed and one bootstraps the unknown bubble function by demanding the absence of free double-trace operators on the boundary.

After reviewing the $O(N)$ model [74], the next step is to include gauge interactions, Quantum Electrodynamics (QED) seems like the best candidate to start with. In this thesis, we apply the approach of [74] to a strongly coupled gauge theory: QED in AdS [75]. Asymptotically free gauge theories are a clear target to be studied using the AdS background [76], possibly via the bootstrap of the boundary correlators. It is therefore particularly important to understand how various gauge theory phenomena are encoded in the conformal correlators. We perform the first steps in this direction, studying the simple example of scalar QED (sQED) with N_f flavors in the large N_f limit. This theory is asymptotically free and has an interesting structure of phases for $2 < D < 4$ (D can be kept as a continuous parameter at large N_f). In flat space, it has a Coulomb phase and a Higgs phase separated by a second order phase transition, described by an interacting CFT. In the Coulomb phase the massless excitation is the photon, while in the Higgs phase there are Goldstone bosons of the $\mathbb{CP}^{N_f-1}$ model. We place the theory in AdS_D with Dirichlet-type boundary condition for the gauge field. These phases are still present in AdS_D (and both are allowed for an intermediate range of m^2). In both phases we compute the four-point function of the charged operators created by the scalar electrons of the bulk theory, from which the dimensions of the exchanged operators can be extracted, for arbitrary values of the scaled gauge coupling $\alpha = e^2 N_f$.

As an intermediate step, we compute the bubble diagram corresponding to the bulk two-point correlator of a conserved current in the free theory. To this end, we employ and further develop the technique of the spectral representation for two-point functions of a spinning operator [77]. The spectral representation allows us to readily resum the bubble diagrams and obtain the exact propagator of the photon at the leading order at large N_f. The four-point function is then expressed as an exchange diagram with this exact propagator. In the Coulomb phase, the spin 1 exchanged operators are: a conserved current with protected dimension, and the finite-coupling versions of the spin 1 double-trace operators of the matter fields, whose dimensions and OPE coefficient we can follow to finite values of α (there is a caveat for integer dimension $D = 3$ that we discuss below). In the Higgs phase, the external operators are exactly marginal because the corresponding bulk fields are massless Goldstone bosons. A classical analysis in AdS would suggest that the current operator becomes non-conserved and gets an anomalous dimension. At finite coupling instead the only remnant of this non-conserved current is in a specific feature of the spectrum of the spin 1 double-trace operators, which is the AdS analogue of a resonance in flat space. Going to the deep IR with a tuned value of the mass-squared we reach a critical point with bulk conformal symmetry, corresponding to a BCFT in flat space via a Weyl rescaling, and we can extract the scaling dimensions of the spin 1 boundary operators appearing in the boundary OPE of the gauge field.

Something special happens in the Coulomb phase in integer dimension $D = 3$: the double-trace $j^\mu j_\mu$ of the boundary theory is classically marginal, and the corresponding coupling gets a non-trivial β function triggered by the bulk gauge coupling. Therefore, the conformal symmetry of the boundary gets broken. This is true in ordinary perturbation theory [78, 79] and we explain that it remains true working at large N_f, finite coupling. We comment on the interpretation of this phenomenon as an IR divergence that is not cured by the AdS length scale, and offer a novel point of view on the computation of the β function from the spectral representation of the propagator. This IR divergence persists when we tune the mass-squared of the scalar to the critical value, and hinders the possibility to define a conformal boundary condition for the IR CFT of 3d sQED by considering the RG in AdS with Dirichlet conditions for the gauge field. Therefore, when we talk about the boundary CFT in the Coulomb phase or at the bulk conformal point, we always refer to using a non-integer value of D to regulate this divergence.

After scalar QED, one natural extension is fermionic QED. We first review it in flat space, but while mostly the literature has focused on the CFT case, here we will also look at scattering in the massive phase. In AdS space, the work is still ongoing. Fermionic QED shows parity breaking due to appearance of a Chern-Simons terms when the fermions are massive, and because of this the 1PI bubble has an additional structure as compared to the scalar QED.

After dealing with gauge interactions in AdS space, one can analytically continue to dS space. But let us first explain why dS space is relevant.

Our Universe has a positive curvature and it has an accelerated expansion. Similarly, during the phase of cosmic inflation that preceded the hot Big Bang, the Universe underwent an accelerated expansion. Like during the inflationary phase and in the present phase of the Universe, the dS geometry has also exponentially fast accelerated expansion. In particular with regard to inflation, an especially interesting set of observables in dS are the correlators of operators inserted at the late-time (i.e. infinite future) conformal boundary. We refer to these as "cosmological correlators".

It is a tedious task to directly compute cosmological correlators in dS space and thus a cosmological bootstrap approach has been developed to understand cosmological correlators in better ways, see the recent review [80] for a list of references on the subject. Here constraints like unitarity, singularities, and the fact that dS space admits a late-time conformal boundary were used to bootstrap correlators.

One way to study these cosmological correlators is by doing analytic continuation from AdS [81–84].

These techniques has already led to important improvement in our understanding of the late-time correlators in the dS space. For instance it has led to a derivation of positivity and analyticity constraints for the spectral density that captures the conformal partial wave expansion of late-time four-point function [85, 86]. The existence of these constraints opens the door to the possibility of a non-perturbative cosmological bootstrap approach for the late time correlators [86–89]

In this thesis we have reviewed the rotation from dS to AdS (Euclidean) which is done in such a way that one can relate 4-point boundary correlation functions in AdS to the late time boundary 4-point correlation functions in dS [81–84]. We have also presented the result of [90] to get the exact propagator at large N in dS for $O(N)$ model. It is also found that for the case of $O(N)$, there is no spontaneous symmetry breaking [90] unlike the AdS case [74].

A second goal of this thesis is to present steps towards an extension of the study done in [85, 90] to gauge interactions, particularly scalar QED. An important point to note here is that the insertion of charged operator at the late time boundary is not gauge invariant by itself, and one needs to do an appropriate dressing to make the correlator gauge invariant. We do it by using Wilson lines. Similar subtleties are also expected in case of dynamical gravity and hence we hope that understanding scalar QED in AdS can give us some key insights for the problem of dynamical gravity as well.

The rest of the thesis is organised as follows:

In chapter 2 we review some backround material that is used in the rest of the thesis. We introduce the embedding formalism for both the AdS and dS space. We review the spectral representation which allows to map two-point correlators from coordinate space to functions of a spectral parameter ν, much like Fourier transformation in flat space. We also show how to analytically rotate the Lagrangian from dS to AdS for a scalar field theory.

In chapter 3 we review the $O(N)$ model, first in flat space. We then review the computation in the AdS case, where one uses both the bootstrap and large N techniques. We review both the phases found in this theory. We also present results in the literature for the $O(N)$ model in dS space.

In chapter 4 we discuss scalar QED at large N in flat space, studying both phases: the Coulomb phase and the Higgs phase. We also consider the CFT which separates the two said phase.

In chapter 5 we present the results obtained by applying both the large N and analytical bootstrap methods to scalar QED in AdS. We studied the realization of the different phases in AdS, and we also identify AdS analogue of resonance in flat space in the Higgs phase. We also discuss the case with bulk conformal symmetry, and the issue of IR divergences.

In chapter 6 we discuss the relation between vector propagators in AdS and dS space and then we present the rotation of the Lagrangian of scalar QED from dS to AdS. We discuss the strategy to compute the bubble in dS from this rotated lagrangian.

In chapter 7 we discuss fermionic QED in flat space, pointing out the existence of a bound state in the scattering amplitude of the fermions in the massive phase. We then outline a strategy to study this theory in AdS.

In chapter 8 we summarize the thesis and presented the possible future directions.

In appendices we have discussed the analytic structure of the Proca propagator in AdS, the inversion formula for the spectral representation of spin 1 two-point functions, and its flat space limit.

In this thesis, we present the following publication:

- Scalar QED in AdS.
  -Ankur, Dean Carmi, Lorenzo Di Pietro
  J. High Energ. Phys. 2023, 89 (2023)., arXiv:2306.05551

Also the following two projects which are still ongoing have been included in the thesis:

- Fermionic QED in AdS.
  -Ankur, Dean Carmi, Lorenzo Di Pietro

- Scalar QED in dS
  -Ankur, Lorenzo Di Pietro, Shota Komatsu, Veronica Sacchi, Victor Gorbenko

I have also worked on the following publication during the first few months of my PhD, that is not included in this thesis:

- Dynamic noncommutative BTZ black holes
  -Ankur, Sanjib Dey
  Physics Letters B, 818, (2021) p.136391, arXiv:2105.11130

Chapter 2

Preliminaries

2.1 Introduction

In this chapter we will review some of the preliminaries required to understand the upcoming chapters. Here we will give the definition Anti-de Sitter (AdS) and de Sitter (dS) spaces. We will also discuss the embedding formalism, where we will embed these spaces in higher dimensional Minkowski space. The point of using the embedding formalism is that it makes symmetries manifest, and it will be easier to keep track of indices (as we will see in later chapters). We will also review the Spectral Representation and Harmonic functions as they are the key ingredients that helps us to calculate boundary correlation functions. In this chapter we will be mostly reviewing the work presented in the paper [77]. To discuss physics in AdS, we will give a brief review of Conformal Field theories (CFTs) as we will be using some of the techniques borrowed from CFT to understand the physics in AdS and as well as in dS. We will also introduce late-time cosmological correlators, as these are our key observables in dS space.

2.2 AdS and dS space

Both the AdS and dS space are maximally symmetric lorentzian surfaces with constant Riemann scalar curvature. The difference between the two is related to the sign of that constant curvature, if it is positive, it is considered to be dS space, while if its is negative, it is called AdS. This change of sign, changes the physics drastically but nevertheless, some possibe non-trivial analytic continuation from AdS to dS is possible as we will see in later chapters. One thing to notice is that our universe resemble close to dS space as observed curvature of our universe is considered to be positive as well. One can view both the AdS and dS space as hypersurfaces in higher dimensional space, defined as,

$$\eta_{AB} X^A X^B = \pm R^2. \tag{2.1}$$

In the above equation + sign refers to dS space while the − sign refers to AdS space and R^2 is inversely proportional to the Ricci curvature of the space. Here, our convention of the metric is mostly positive i.e.(-,+,+,+...) and X_A denotes the coordinates of bulk point in higher dimensional space, and they are related to the poincaré coordinates $x_\mu = (\eta, y_a)$ or $x_\mu = (z, y_a)$ (we will be using η for dS geometry and z for AdS geometry in this and next chapters and the index a runs from 1 to d) as follows:

$$X^A_{dS} = \left(\frac{R^2 - \eta^2 + |y|^2}{2\eta}, \frac{Ry^a}{\eta}, \frac{R^2 + \eta^2 - |y|^2}{2\eta}\right), \tag{2.2}$$

$$X^A_{\text{AdS}} = \left(\frac{R^2 + z^2 + |y|^2}{2z}, \frac{Ry^a}{z}, \frac{R^2 - z^2 - |y|^2}{2z}\right). \tag{2.3}$$

In this coordinate system, we can write the metric for both AdS and dS space in the following form:

$$ds^2_{dS} = R^2 \frac{-d\eta^2 + dy^a dy_a}{\eta^2}, \tag{2.4}$$

$$ds^2_{AdS} = R^2 \frac{dz^2 + dy^a dy_a}{z^2}. \tag{2.5}$$

The boundary of AdS space can be defined by taking the limit $z \to 0$ and in the same way we can define late time boundary by taking the limit $\eta \to 0$ for dS case. In this case, we will have boundary points P for both AdS and dS space parameterized by,

$$P^A_{dS/AdS} = \left(\frac{R^2 + |y|^2}{2}, Ry^a, \frac{R^2 - |y|^2}{2}\right). \tag{2.6}$$

Which satisfies the relation $P^2 = 0$ and Y^a is a vector in $\mathcal{R}^d$

2.3 Embedding Formalism AdS

In this section we will understand the map between the AdS geometry and the higher dimensional Minkowski space where the interested geometry is embedded in. It must be emphasized that here it is a rigid geometry where we will be placing our Quantum Fields to interact among themselves and we are not doing a theory of Quantum Gravity as in this case there is no dynamical gravity!

In this formalism we have embedded our Euclidean AdS_{d+1} space into higher dimensional space. This higher dimensional space is $\mathcal{M}_{d+2}$ and from this section onwards we have put $R = 1$, in equations from previous section for sake of convenience.

These vector X^A are still parameterized in terms of Poincare coordinates of AdS_{d+1}, i.e. (z,y^a), and $z = 0$ correspond to the boundary of AdS space. Note that in this thesis we will be using capital latin letters to denote the index of the vector in the

$\mathcal{M}_{d+2}$, small latin index to denote the vector in the $\mathcal{R}^d$ and small greek letter to denote the index for AdS_{d+1} unless otherwise stated.

The point of this section is to write tensors in AdS_{d+1} without indices. We will achieve this in two parts, first we will write a corresponding map which relates the tensor $T_{\mu_1\mu_2\ldots\mu_j}$ in AdS_{d+1} to the corresponding tensor $T_{A_1A_2\ldots A_j}$ in $\mathcal{M}_{d+2}$ and once we have that, we will then write tensor $T_{A_1A_2\ldots A_j}$ without indices. The first step can be achieved by the following relation:

$$T_{\mu_1\ldots\mu_J}(x) = \frac{\partial X^{A_1}}{\partial x^{\mu_1}} \cdots \frac{\partial X^{A_J}}{\partial x^{\mu_J}} T_{A_1\ldots A_J}(X). \tag{2.7}$$

With this relation between tensors in AdS space and Minkowski space, we can work with tensors in Minkowski space. But this achieves only half of the goal, as we still have to keep track of the indices of the tensor. Now we will define another map that will take us from tensors to some polynomial functions (and vice versa) and we will do this with the help of polarization vectors W which satisfies the relation $W^2 = 0 = X \cdot W$ as follows:

$$T(X,W) = W^{A_1} \ldots W^{A_J} T_{A_1\ldots A_J}(X). \tag{2.8}$$

One can find the inversion relation that will give us back the tensors from this well defined polynomials using some differential operators K_A as follows:

$$T_{A_1\ldots A_J}(X) = \frac{1}{J!\left(\frac{d-1}{2}\right)_J} K_{A_1} \ldots K_{A_J} T(X,W). \tag{2.9}$$

Note that these K_A can be explicitly written as,

$$\begin{aligned} K_A =& \frac{d-1}{2}\left(\frac{\partial}{\partial W^A} + X_A\left(X \cdot \frac{\partial}{\partial W}\right)\right) + \left(W \cdot \frac{\partial}{\partial W}\right)\frac{\partial}{\partial W^A} + X_A\left(W \cdot \frac{\partial}{\partial W}\right)\left(X \cdot \frac{\partial}{\partial W}\right) \\ &- \frac{1}{2}W_A\left(\frac{\partial^2}{\partial W \cdot \partial W} + \left(X \cdot \frac{\partial}{\partial W}\right)\left(X \cdot \frac{\partial}{\partial W}\right)\right). \end{aligned} \tag{2.10}$$

In above equation we have introduced the pochammer notation i.e., $(a)_J = \Gamma(a + J)/\Gamma(a)$. and K_A satisfy following important properties (see [77] for more details):

- Transverse i.e. $X^A K_A = 0$
- traceless i.e. $K_A K^A = 0$
- symmetric i.e. $K_A K_B = K_B K_A$.

Thus we have the map from the Polynomial function to the tensor in $\mathcal{M}_{d+2}$ and vice versa. Since we have to deal with differential equations, one needs to define differential operator that can act on these polynomial functions. Since, AdS space is curved, we need to find differential operator in d+2 ($\mathcal{M}_{d+2}$) dimensions, corresponding to covariant derivative in AdS_{d+1}. Note that unlike K_A, these differential operators do not commute and are transverse i.e. $X^A \nabla_A = 0$ and are defined as follows:

$$\nabla_A = \frac{\partial}{\partial X^A} + X_A \left(X \cdot \frac{\partial}{\partial X} \right) + W_A \left(X \cdot \frac{\partial}{\partial W} \right). \tag{2.11}$$

2.4 Propagators in AdS

With the differential operator defined as (2.11), one can write the equation that propagators should satisfy. In this thesis we will use the notation that bulk to bulk propagators are denoted as G while bulk to boundary ones are denoted as K and these G propagators satisfy the following equation of motion:

$$\begin{aligned} \left(\nabla_1^2 - \Delta(\Delta - d) + J\right) G_{\Delta,J}\left(X_1, X_2, W_1, W_2\right) &= -\delta\left(X_1, X_2\right)\left(W_{12}\right)^J + \ldots, \\ \nabla_1 \cdot K_1 G_{\Delta,J}\left(X_1, X_2, W_1, W_2\right) &= \ldots. \end{aligned} \tag{2.12}$$

Where Δ are scaling dimensions and they are related to the mass squared terms as $M^2 = \Delta(\Delta - d) - J$ for general spin J, K_1 is the differential operator defined in 2.10 w.r.t. coordinate X_1 and "..." represent local source terms which contributes only contact terms to the propagator. In this embedding formalism, we can now find the bulk to bulk propagators and here we will write propagators in terms of chordal distance u i.e. defined as half of the square of distance between two points ζ:

$$u = \frac{\zeta}{2} = \frac{(X_1 - X_2)^2}{2} = -1 - X_1 \cdot X_2. \tag{2.13}$$

One thing to note is that most of the formulas presented here will be for general spin J, later on we will use $J = 0$ or $J = 1$ depending on the model we are studying. We will write the bulk to bulk propagators for general spin J as sum of different functions (note that while propagators are written in term of polynomial functions, we can use different K_A operators defined in (2.10) to get back indices) as follows:

$$G_{\Delta,J}\left(X_1, X_2, W_1, W_2\right) = \sum_{k=0}^{J} \left(W_{12}\right)^{J-k} \left(\left(W_1 \cdot X_2\right)\left(W_2 \cdot X_1\right)\right)^k g_k(u, \Delta). \tag{2.14}$$

Where, $W_{12} = W_1 \cdot W_2$ and $g_k(u, \Delta)$ are some generic functions of the chordal distance u and the scaling dimension Δ. By using the above formula it is clear that if one tries to find the propagator for spin 0, we will have no polarization vector as there will be no indices for spin 0 propagator. We can write the spin 0 propagator as follows:

$$G_\Delta\left(X_1,X_2\right) = \frac{\Gamma(\Delta)}{2\pi^{\frac{d}{2}}\Gamma\left(\Delta-\frac{d}{2}+1\right)}\frac{1}{\zeta\left(X_1,X_2\right)^{\Delta}}{}_2F_1\left(\Delta,\Delta-\frac{d}{2}+\frac{1}{2},2\Delta-d+1,-\frac{4}{\zeta\left(X_1,X_2\right)}\right). \tag{2.15}$$

Where the $\zeta = 2u$ defined as (2.13) and mass to scaling dimension relation for spin 0 will remain same as in the [77],

$$m^2 = \Delta(\Delta-d). \tag{2.16}$$

There are two possible solutions to the above equation which we denote as Δ_+ and $\Delta_- = d-\Delta_+$, with $\Delta_+ \geq \frac{d}{2}$. The general solution to the equation of motion of the scalar field behaves near the boundary ($u \to \infty$) as a linear combination of the powers $u^{-\Delta_+}$ and $u^{-\Delta_-}$. The boundary conditions that preserve the AdS isometry set to zero either one of these two modes, leaving only the other one. We will refer to the boundary condition that sets to zero the dominant mode, i.e. $u^{-\Delta_-}$, as "Dirichlet condition". The propagator displayed above behaves like $u^{-\Delta}$ near the boundary, and it gives the correct answer for either of the two boundary conditions, depending on whether the parameter Δ is smaller or larger than $\frac{d}{2}$.

For the case with spin 1, as we know that it has two indices and naturally it can be expanded in terms of these two independent polarization vector as follows:

$$G\left(X_1,X_2,W_1,W_2\right) = W_{12}g_0(u,\Delta)+\left(W_1\cdot X_2\right)\left(W_2\cdot X_1\right)g_1(u,\Delta). \tag{2.17}$$

one can write in terms of two bulk coordinates X_1 and X_2 with indices as well by applying K_A and K_B (2.10),

$$\begin{aligned} G_{A,B}(X_1,X_2) = {} & \left(\eta_{AB}+X_{1A}X_{1B}+X_{2A}X_{2B}-(1+u)X_{1A}X_{2B}\right)g_0(u,\Delta) \\ & +\left(X_{1B}-(1+u)X_{2B}\right)\left(X_{2A}-(1+u)X_{1A}\right)g_1(u,\Delta). \end{aligned} \tag{2.18}$$

where,

$$\begin{aligned} g_0(u,\Delta) &= (d-\Delta)F_1(u)-\frac{1+u}{u}F_2(u), \\ g_1(u,\Delta) &= \frac{(1+u)(d-\Delta)}{u(2+u)}F_1(u)-\frac{d+(1+u)^2}{u^2(2+u)}F_2(u), \\ F_1(u,\Delta) &= \mathcal{N}(2u)^{-\Delta}{}_2F_1\left(\Delta,\frac{1-d+2\Delta}{2},1-d+2\Delta,-\frac{2}{u}\right), \\ F_2(u,\Delta) &= \mathcal{N}(2u)^{-\Delta}{}_2F_1\left(\Delta+1,\frac{1-d+2\Delta}{2},1-d+2\Delta,-\frac{2}{u}\right), \\ \mathcal{N} &= \frac{\Gamma(\Delta+1)}{2\pi^{d/2}(d-1-\Delta)(\Delta-1)\Gamma\left(\Delta+1-\frac{d}{2}\right)}. \end{aligned} \tag{2.19}$$

Note that in [77], the scaling dimension of the spin 1 propagator is related to the mass with the relation $M^2 = \Delta(\Delta - d) - 1$ and the lagrangian that we will be using in later chapters for scalar QED is different than used in [77], and for us the mass of propagator m_A is related to the spin 1 scaling dimension as follows:

$$M^2 = m_A^2 - d = \Delta(\Delta - d) - 1. \tag{2.20}$$

We can now use the above relation to find the scaling dimension for massless vector field, $\Delta = 1, d - 1$. In general there are two solutions for Δ in (2.20) namely Δ_+ and Δ_- and choosing one solution depends on the behaviour of the function at boundary i.e. $z = 0$. The one that decays faster at boundary we call it Dirichlet.

Now, we have our bulk to bulk propagators for the case of $O(N)$ model and Scalar QED, both for the massive and the massless counterparts in AdS space. Naturally, the other ingredient that we would need is bulk to boundary propagators and we can define them by taking a proper limit of these bulk to bulk propagators, sending one of the point to the boundary of AdS for general spin J,

$$K_{\Delta,J}(X, P; W, Z) = \sqrt{\mathcal{C}_{\Delta,J}} \frac{((-2P \cdot X)(W \cdot Z) + 2(W \cdot P)(Z \cdot X))^J}{(-2P \cdot X)^{\Delta+J}} \tag{2.21}$$

where the point P lies on the boundary and the polarization vector corresponding to the boundary point is denoted by Z. The normalization constant $\mathcal{C}_{\Delta,J}$ is fixed as

$$\mathcal{C}_{\Delta,J} = \frac{(J + \Delta - 1)\Gamma(\Delta)}{2\pi^{d/2}(\Delta - 1)\Gamma(\Delta + 1 - h)}. \tag{2.22}$$

One can also introduce corresponding D_Z^A operator to recover indices which is boundary analogue of the operator K_A (2.10) introduced earlier as follows:

$$D_Z^A = \left(h - 1 + Z \cdot \frac{\partial}{\partial Z}\right) \frac{\partial}{\partial Z_A} - \frac{1}{2} Z^A \frac{\partial^2}{\partial Z \cdot \partial Z}. \tag{2.23}$$

where we have used the notation $h = \frac{d}{2}$.

2.5 Spectral Representation

Like in flat space, we have the fourier transform of propagators, in the same way we have analogue in AdS known as the Spectral Representation of propagators. In this representation we map functions from space of coordinates to space in variable ν. The analogy works very well even the variable ν in spectral representation can be thought of as AdS analogue of the variable p in flat space fourier transformation. This representation will be the key to do computations in the later chapters because of two nice properties, one being able to spilt the bulk to bulk propagators into two bulk to boundary propagators integrated over a common boundary point (as we will see in the later part of this section) and other property is that it helps us to write infinite sum of certain diagrams with "Bubbles" (we will see in later chapters) as geometric sum. In this representation we will expand the propagator in terms of basis functions called harmonic functions denoted by $\Omega_{\nu,J}$, where the subscript denotes that these harmonic functions are parameterized by the spectral parameter ν and the spin J. So first let us see some key properties of these functions, like these functions are eigen functions of the laplacian and are divergence free,

$$\begin{aligned} \left(\nabla_1^2 + h^2 + \nu^2 + J\right)\Omega_{\nu,J}\left(X_1, X_2; W_1, W_2\right) = 0 \\ \nabla_1 \cdot K_1 \Omega_{\nu,J}\left(X_1, X_2; W_1, W_2\right) = 0 \end{aligned} \tag{2.24}$$

We can write these harmonic functions as difference of two bulk to bulk propagators for general spin J as follows:

$$\Omega_{\nu,J}\left(X_1, X_2; W_1, W_2\right) = \frac{i\nu}{2\pi}\left(G_{h+i\nu,J}\left(X_1, X_2; W_1, W_2\right) - G_{h-i\nu,J}\left(X_1, X_2; W_1, W_2\right)\right). \tag{2.25}$$

We can rewrite harmonic functions in term of bulk to boundary propagators as well but this is an integral expression,

$$\Omega_{\nu,J}\left(X_1, X_2; W_1, W_2\right) = \frac{\nu^2\sqrt{\mathcal{C}_{h+i\nu,J}\mathcal{C}_{h-i\nu,J}}}{\pi J!(h-1)_J}\int_\partial dP K_{h+i\nu,J}\left(X_1, P; W_1, D_Z\right) K_{h-i\nu,J}\left(X_2, P; W_2, Z\right), \tag{2.26}$$

It is because of this relation that the spectral representation is also known as the Split representation because we can split the harmonic functions in terms of bulk to boundary propagators integrated over a common boundary point. Note that in this integral, we have Z as polarization vector in one propagator while D_Z in other, this is to make sure that in the index notation, these two propagators are contracted over one index. If we were to find expression for bulk to bulk propagators contracted over one index, one can can use K operators instead of D_Z.

One can find convolution of two harmonic functions as follows:

$$\frac{1}{J!\left(\frac{d-1}{2}\right)_J}\int_{AdS} dY \Omega_{\bar{\nu},J}\left(X_1, Y; W_1, K\right)\int d\nu \Omega_{\nu,J}\left(Y, X_2; W, W_2\right) = \Omega_{\bar{\nu},J}\left(X_1, X_2; W_1, W_2\right) \tag{2.27}$$

This is the other property of harmonic functions that will help us to resum certain families of Feynman diagrams in AdS in later chapters.
So far we have seen that we can write these harmonic function in terms of the propagators but our goal is to expand the propagators in terms of harmonic functions and one can do that as follows [77]:

$$G_{\Delta,J}\left(X_1, X_2; W_1, W_2\right) = \sum_{l=0}^{J}\int d\nu a_l(\nu)\left(\left(W_1 \cdot \nabla_1\right)\left(W_2 \cdot \nabla_2\right)\right)^{J-l}\Omega_{\nu,l}\left(X_1, X_2; W_1, W_2\right) \tag{2.28}$$

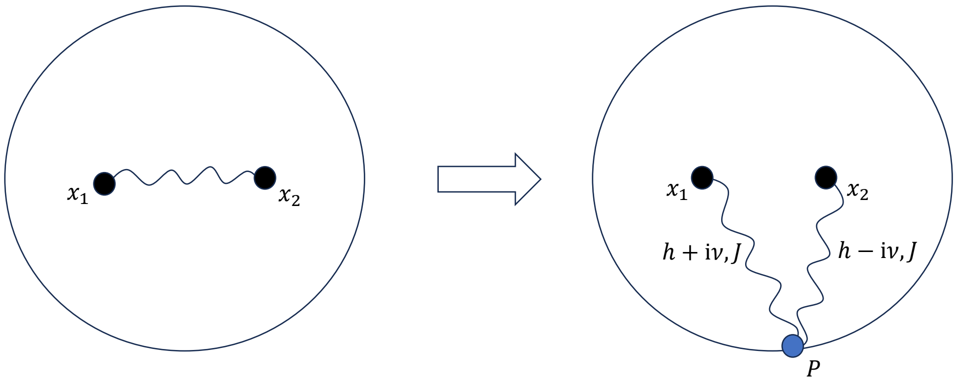


FIGURE 2.1: Splitting the bulk to bulk propagator in terms of bulk to boundary propagator

where $a_l(\nu)$ is generic function of ν. We can use (2.26) in the above expression and it will split the bulk to bulk propagator in terms of two of bulk to boundary propagators as in figure 2.1. Let us see how we can write for spin 0 [74] and spin 1 propagator [77],

$$G_{\Delta=\Delta^+}(X_1,X_2)=\int_{-\infty}^{+\infty}d\nu\frac{1}{\nu^2+\left(\Delta-\frac{d}{2}\right)^2}\Omega_\nu(X_1,X_2) \tag{2.29}$$

The above expression correspond to the bulk to bulk propagator with Dirichlet condition and it is easy to check it is true as we can write the harmonic function using (2.25) and do the integral. To do the integral, one should note that the first term in (2.25) can be only closed in the contour in lower half plane and the second term can be closed in the upper half plane and thus both terms contribute to give full propagator.

Naively one would think this trick would work for any spin J as we can use (2.25) everytime but this is not the case as the expression (2.28) suggest, let us check for spin 1 case that would be relevant to study gauge propagator for Quantum Electrodynamics,

$$\begin{aligned}G_{\Delta,1}(X_1,X_2;W_1,W_2)=&\int\frac{d\nu\Omega_{\nu,1}(X_1,X_2;W_1,W_2)}{\nu^2+(\Delta-h)^2}\\&-\int\frac{d\nu\,(W_1\cdot\nabla_1)(W_2\cdot\nabla_2)\,\Omega_{\nu,0}(X_1,X_2)}{(\Delta-1)(2h-\Delta-1)\,(\nu^2+h^2)}.\end{aligned} \tag{2.30}$$

The first term will give us the bulk to bulk propagator and in addition to this we will have extra contribution from other unphysical poles in this term, which will be cancelled by the second term in the expression (2.30). The details are in (A).

2.6 Review of CFT

We wil briefly review some of the things about conformal field theories (CFTs). They are like any other Quantum Field Theories for most of the part but they have additional symmetries known as "Dilations" or scale transformations $x^\mu \to \lambda x^\mu$ and "Special conformal transformation" which can be viewed as a coordinate transformation $x \to x'(x) = \frac{x^\mu - a^\mu x^2}{1-2(a\cdot x)+a^2x^2}$. These two symmetries extend the Poincaré group to the conformal group with generators (in R^d) given by [17]:

$$\begin{aligned} K_\mu &= 2x_\mu(x^\nu\partial_\nu) - x^2\partial_\mu, \\ D &= x^\mu\partial_\mu, \\ P_\mu &= \partial_\mu, \\ M_{\mu\nu} &= x_\nu\partial_\mu - x_\mu\partial_\nu. \end{aligned} \tag{2.31}$$

These generators satisfy the following relations:

$$\begin{aligned} \left[M_{\mu\nu}, K_\rho\right] &= \delta_{\nu\rho}K_\mu - \delta_{\mu\rho}K_\nu, \\ \left[K_\mu, P_\nu\right] &= 2\delta_{\mu\nu}D - 2M_{\mu\nu}, \\ \left[D, P_\mu\right] &= P_\mu, \\ \left[D, K_\mu\right] &= -K_\mu. \end{aligned} \tag{2.32}$$

This additional symmetry has impact on structure of correlation functions and because of this, it restrict the possible form of the n-point correlation functions. In general we can write n-point correlation functions as sum of $(n-1)$ point correlation functions and this is called Operator Product Expansion or OPE. It is defined as follows:

$$\langle \mathcal{O}_1(x_1)\,\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n)\rangle = \sum_k C_{12k}(x_{12},\partial_2)\,\langle \mathcal{O}_k(x_2)\cdots\mathcal{O}_n(x_n)\rangle \tag{2.33}$$

where, $C_{12k}(x_{12},\partial_2)$ are some differential operators (see [17] for more details). In above expression we are expanding in $|x_1 - x_2|$ and replacing $\mathcal{O}_1$ and $\mathcal{O}_2$ with $\mathcal{O}_k$. Note that $\mathcal{O}$ are primary operators and on the RHS of the equation we have included all the contributions of descendants (i.e. derivatives of primary operators) in the coefficient C_{12k} which are series expansions in the derivative ∂_2, with relative coefficients fixed by the conformal symmetry. In CFTs, three point functions are fixed by conformal symmetry up to some finite constants. Since in this thesis, we are dealing with QED models, one important 3-point function would be of 2 scalar fields and one vector field which would correspond to cubic interaction in QED. In general 3-point functions of two scalar fields and one operator of spin ℓ would be given by

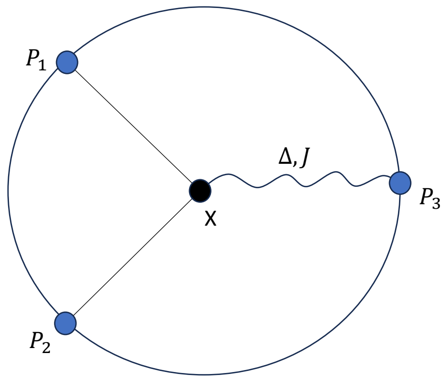


FIGURE 2.2: 3-point witten diagram in AdS space

[17],

$$\langle \phi_1(x_1)\, \phi_2(x_2)\, \mathcal{O}^{\mu_1\cdots\mu_\ell}(x_3) \rangle = \frac{f_{\phi_1\phi_2\mathcal{O}}\left(\left(\frac{x_{13}^{\mu_1}}{x_{13}^2} - \frac{x_{23}^{\mu_1}}{x_{23}^2}\right) \cdots \frac{x_{13}^{\mu_\ell}}{x_{13}^2} - \frac{x_{23}^{\mu_\ell}}{x_{23}^2} - \text{traces} \right)}{x_{12}^{\Delta_1+\Delta_2-\Delta_3+\ell}\, x_{23}^{\Delta_2+\Delta_3-\Delta_1-\ell}\, x_{31}^{\Delta_3+\Delta_1-\Delta_2-\ell}}, \tag{2.34}$$

Here, $\mathcal{O}^{\mu_1\cdots\mu_\ell}$ is spin ℓ operator while phi_1 and ϕ_2 are scalar/ spin 0 operators, $\Delta_1, \Delta_2, \Delta_3$ are scaling dimensions of ϕ_1, ϕ_2 and $\mathcal{O}^{\mu_1\cdots\mu_\ell}$ operator respectively. Note that $f_{\phi_1\phi_2\mathcal{O}}$ is some constant which depends on the theory.

2.7 Three point functions

We need to know the 3-point function in embedding formalism 2.2. As it was mentioned in the Chapter 1 that AdS space has conformal boundary and because of this 3-operators inserted at boundary will result in 3-point CFT function and this would also be fixed by conformal symmetry upto a constant like (2.34). One can consider general cubic term in lagrangian in AdS as follows [77]:

$$g_{\phi_1\phi_2 h} \int_{\text{AdS}} dx \sqrt{g}\, (\phi_2 \nabla_{\mu_1} \cdots \nabla_{\mu_J} \phi_1)\, h^{\mu_1\cdots\mu_J}, \tag{2.35}$$

where $g_{\phi_1\phi_2 h}$ is the coupling constant.

$$\begin{aligned} &\langle \mathcal{O}_{\phi_1}(P_1)\, \mathcal{O}_{\phi_2}(P_2)\, \mathcal{O}_h(P_3, Z) \rangle = \\ &\frac{g_{\phi_1\phi_2 h}}{\sqrt{\mathcal{C}_{\Delta_1}\mathcal{C}_{\Delta_2}\mathcal{C}_{\Delta,J}}} \int_{\text{AdS}} dX K_{\Delta_2,0}(X, P_2)\, \frac{K_{\Delta,J}(X, P_3; K, Z)\, (W\cdot\nabla)^J K_{\Delta_1,0}(X, P_1)}{J! \left(\frac{d-1}{2}\right)_J} = \\ &\frac{g_{\phi_1\phi_2 h}}{\sqrt{\mathcal{C}_{\Delta_1}\mathcal{C}_{\Delta_2}\mathcal{C}_{\Delta,J}}} b(\Delta_1, \Delta_2, \Delta, J)\, \frac{((Z\cdot P_1)\, P_{23} - (Z\cdot P_2)\, P_{13})^J}{P_{12}^{\frac{\Delta_1+\Delta_2-\Delta+J}{2}}\, P_{13}^{\frac{\Delta_1+\Delta-\Delta_2+J}{2}}\, P_{23}^{\frac{\Delta+\Delta_2-\Delta_1+J}{2}}}, \end{aligned} \tag{2.36}$$

In the above expression we have Δ_1,Δ_2 as the scaling dimensions of the scalar fields, Δ for spin J operator and we have used the notation $P_{ij} = -2P_i \cdot P_j$. Also, note that the normalization constants $\mathcal{C}_{\Delta_1}$ and $\mathcal{C}_{\Delta_2}$ for spin 0 are obtained by putting $J = 0$ in the expression (2.22) (note that if spin is not explicitly stated, assume that it is spin 0 in this thesis). This 3 point function in AdS can be fixed up to a constant $(b(\Delta_1, \Delta_2, \Delta, J))$ because of conformal symmetry like we saw in previous section and to find the constant one need to evaluate the AdS integral (see [77] for details). We can clearly see that (2.36) looks very similar to (2.34) and the constant $b(\Delta_1, \Delta_2, \Delta, J)$ can be written in terms of $b(\Delta_1, \Delta_2, \Delta, 0)$ as a recursive relation (see [77]),

$$\begin{aligned} b(\Delta_1, \Delta_2, \Delta, J) &= 2^J \left(\frac{\Delta + \Delta_2 - \Delta_1 - J}{2}\right)_J \frac{\mathcal{C}_{\Delta,J}}{\mathcal{C}_{\Delta,0}} \frac{\mathcal{C}_{\Delta_1}}{\mathcal{C}_{\Delta_1+J}} \frac{(\Delta_1)_J}{(\Delta)_J} b(\Delta_1 + J, \Delta_2, \Delta, 0) = \\ &= \mathcal{C}_{\Delta_1}\mathcal{C}_{\Delta_2}\mathcal{C}_{\Delta,J} \frac{\pi^{\frac{d}{2}}\Gamma\left(\frac{\Delta_1+\Delta_2+\Delta-d+J}{2}\right)\Gamma\left(\frac{\Delta_1+\Delta_2-\Delta+J}{2}\right)\Gamma\left(\frac{\Delta+\Delta_1-\Delta_2+J}{2}\right)\Gamma\left(\frac{\Delta+\Delta_2-\Delta_1+J}{2}\right)}{2^{1-J}\Gamma(\Delta_1)\Gamma(\Delta_2)\Gamma(\Delta+J)} \end{aligned} \tag{2.37}$$

for,

$$b(\Delta_1, \Delta_2, \Delta, 0) = \mathcal{C}_{\Delta_1}\mathcal{C}_{\Delta_2}\mathcal{C}_{\Delta} \frac{\pi^{\frac{d}{2}}\Gamma\left(\frac{\Delta_1+\Delta_2+\Delta-d}{2}\right)\Gamma\left(\frac{\Delta_1+\Delta_2-\Delta}{2}\right)\Gamma\left(\frac{\Delta_1+\Delta-\Delta_2}{2}\right)\Gamma\left(\frac{\Delta+\Delta_2-\Delta_1}{2}\right)}{2\Gamma(\Delta_1)\Gamma(\Delta_2)\Gamma(\Delta)}. \tag{2.38}$$

2.8 Embedding Formalism dS

In this section, we will be writing results of dS corresponding to results of AdS. Note that here we will not be discussing propagators and correlation functions(we will discuss them in later chapters). The main difference that comes in case of dS is from the fact that in dS inner product of two vectors is normalized to 1 and not -1 i.e., $\eta_{AB}X^AX^B = 1$.

In case of dS space, everything will remain same, we will embed this in higher dimensional Minkowski space, and then we want to write propagators and other things in terms of polarization vectors and not indices. One can define similar K operator in dS in the same way we defined in AdS (2.10) as follows:

$$\begin{aligned} K_A^{dS} =& \frac{d-1}{2}\left(\frac{\partial}{\partial W^A} - X_A\left(X\cdot\frac{\partial}{\partial W}\right)\right) + \left(W\cdot\frac{\partial}{\partial W}\right)\frac{\partial}{\partial W^A} - X_A\left(W\cdot\frac{\partial}{\partial W}\right)\left(X\cdot\frac{\partial}{\partial W}\right) \\ &- \frac{1}{2}W_A\left(\frac{\partial^2}{\partial W\cdot\partial W} - \left(X\cdot\frac{\partial}{\partial W}\right)\left(X\cdot\frac{\partial}{\partial W}\right)\right). \end{aligned} \tag{2.39}$$

Note that one can visualize the difference between (2.10) and (2.39), by intuitively connecting X_{AdS} to X_{dS} as $X_{AdS} = iX_{dS}$. In the same way, one we can find the differential operator in dS corresponding to (2.11) as,

$$\nabla_A = \frac{\partial}{\partial X^A} - X_A \left(X \cdot \frac{\partial}{\partial X} \right) - W_A \left(X \cdot \frac{\partial}{\partial W} \right). \tag{2.40}$$

2.9 In-In formalism

When we are dealing with cosmology or physics in dS, one can define correlation function on a fixed time slice and we do it by "in-in" formalism where expectation value of product of fields $A(t)$ at a fixed time slice is defined by (the in-in formalism was applied to cosmology in [91–93], see also reviews in [94, 95]),

$$\langle A(t) \rangle = \left\langle \left(T e^{-i \int_{t_0}^{t} H_{\text{int}}(t') dt'} \right)^{\dagger} A(t) \left(T e^{-i \int_{t_0}^{t} H_{\text{int}}(t'') dt''} \right) \right\rangle. \tag{2.41}$$

Here, the expectation value is defined with respect to the initial state which is generalization of the Bunch-Davis $|\Omega(t_0)\rangle$.

Note that, in this formalism, we have different kind of propagators for internal vertex points depending on where the vertex point is coming from, for example it can come from left (l) (Anti-Time ordered) Hamiltonian or right (r) (Time ordered) Hamiltonian. So, we have 3 different bulk to bulk propagators defined as G_{ll}, G_{rr}, G_{lr}. In next chapter we will discuss the scalar field theory in dS and there these three different propagators are defined as follows:

$$G_\nu^{ll} = W_\nu(s + i\epsilon), \quad G_\nu^{rr} = W_\nu(s - i\epsilon), \quad G_\nu^{lr} = W_\nu\left(s - i\epsilon \,\text{sgn}\left(t_l - t_r\right)\right). \tag{2.42}$$

Note that it is convenient to define them using the $s = X_1 \cdot X_2$ which is related to u as $u = 1 - s$ and the variable ν is related to scaling dimension as $\Delta = \frac{d}{2} \pm i\nu$ and $W_\nu(s)$ is defined as follows:

$$W_\nu(s) = \frac{\Gamma\left(\frac{d}{2} + i\nu\right)\Gamma\left(\frac{d}{2} - i\nu\right)}{(4\pi)^{\frac{d+1}{2}}\Gamma\left(\frac{d+1}{2}\right)} {}_2F_1\left(\frac{d}{2} + i\nu, \frac{d}{2} - i\nu, \frac{d+1}{2}, \frac{1+s}{2}\right). \tag{2.43}$$

We also need to find the propagator that can connect the bulk point to the point at the late time boundary and we can call it as bulk to boundary propagator in this context. Similar to bulk to bulk propagators, we can have two kinds of bulk to boundary propagators depending on the bulk internal vertex point,

$$\begin{aligned} K_\nu^l(s) &= \frac{\frac{1}{4\pi^{\frac{d}{2}+1}}\Gamma\left(\frac{d}{2} - i\nu\right)\Gamma(i\nu)}{(-2s + i\epsilon)^{\frac{d}{2} - i\nu}} \\ K_\nu^r(s) &= \frac{\frac{1}{4\pi^{\frac{d}{2}+1}}\Gamma\left(\frac{d}{2} - i\nu\right)\Gamma(i\nu)}{(-2s - i\epsilon)^{\frac{d}{2} - i\nu}}. \end{aligned} \tag{2.44}$$

2.10 Rotation from AdS to dS

The physics of dS space is very important as dS space resembles very much to our own universe with expanding geometry. Similar to the case of AdS, also dS has a conformal boundary, though in this case it is a spacelike one, namely the "late time boundary" at future infinity (and, if we consider the full global dS, also the analogue boundary at past infinity). So, in this case it makes sense to calculate observables for at fixed time slice and hence we will be using "in-in" formalism explained in previous section. Now the question is, can these late time cosmological correlators be related to conformal boundary correlation function in AdS? To know this, first we need to understand the relation between propagators in AdS and dS and then we can try to answer the said question. As we saw in "in-in" formalism that in dS there are different types of propagators, we need to analytically continue all three of them to AdS space.

First, let us recall few things about AdS and dS space, starting from inner product of the coordinates,

$$\begin{aligned} \eta_{AB}X^A_{AdS}X^B_{AdS} =& -1, \\ \eta_{AB}X^A_{dS}X^B_{dS} =& 1. \end{aligned} \tag{2.45}$$

And the coordinates of AdS and dS space can written as follows:

$$\begin{aligned} X^A_{dS} &= \left(\frac{1-\eta^2+|y|^2}{2\eta}, \frac{y^a}{\eta}, \frac{1+\eta^2-|y|^2}{2\eta} \right), \\ X^A_{\text{AdS}} &= \left(\frac{1+z^2+|y|^2}{2z}, \frac{y^a}{z}, \frac{1-z^2-|y|^2}{2z} \right). \end{aligned} \tag{2.46}$$

Looking at the above parameterization, one reasonable analytic continuation would be $\eta \rightarrow \pm i\eta$ (now after analytically continuing, η would correspond to the z coordinate of EAdS) and since in "in-in" formalism,we have two kinds of internal coordinates depending on if it is coming from left or right hamiltonian, one proposed analytic continuation is [81–84] as follows:

$$\begin{aligned} \eta^l &\rightarrow e^{\frac{i\pi}{2}}\eta^l, \\ \eta^r &\rightarrow e^{-\frac{i\pi}{2}}\eta^r. \end{aligned} \tag{2.47}$$

With above transformation, our distance variable u and s will transform as well. But first recall their definitions,

$$\begin{aligned} u^{AdS} &= \frac{\zeta^{AdS}}{2} = \frac{\left(X_1^{AdS}-X_2^{AdS}\right)^2}{2} = -1 - X_1^{AdS}\cdot X_2^{AdS} = -1 - s^{AdS}, \\ u^{dS} &= \frac{\zeta^{dS}}{2} = \frac{\left(X_1^{dS}-X_2^{dS}\right)^2}{2} = 1 - X_1^{dS}\cdot X_2^{dS} = 1 - s^{dS}. \end{aligned} \tag{2.48}$$

Now let us write scalar propagators for both the AdS and dS space in variable s as follows:

$$G_\nu^{\text{AdS}}\left(s^{\text{AdS}}\right)=\frac{\Gamma\left(\frac{d}{2}+i\nu\right)}{2\pi^{\frac{d}{2}}\Gamma(1+i\nu)\left(-2\left(s^{\text{AdS}}+1\right)\right)^{\frac{d}{2}+i\nu}}{}_2F_1\left(\frac{d}{2}+i\nu,\frac{1}{2}+i\nu,1+2i\nu,\frac{2}{s^{\text{AdS}}+1}\right), \tag{2.49}$$

$$G_\nu^{ll}=W_\nu(s^{dS}+i\epsilon),\quad G_\nu^{rr}=W_\nu(s^{dS}-i\epsilon),\quad G_\nu^{lr}=W_\nu\left(s^{dS}-i\epsilon\,\text{sgn}\,(t_l-t_r)\right). \tag{2.50}$$

where we have used,

$$W_\nu(s^{dS})=\frac{\Gamma\left(\frac{d}{2}\pm i\nu\right)}{(4\pi)^{\frac{d+1}{2}}\Gamma\left(\frac{d+1}{2}\right)}{}_2F_1\left(\frac{d}{2}+i\nu,\frac{d}{2}-i\nu,\frac{d+1}{2},\frac{1+s^{dS}}{2}\right). \tag{2.51}$$

Note, here we have used the notation $\Gamma(a\pm b)=\Gamma(a+b)\Gamma(a-b)$.

In this section, we have defined our propagators in the variable s for both AdS and dS space and hence it makes sense to first see how s transforms under the analytic continuation (2.47). There will be two transformation depending on if the coordinate is coming from left or right Hamiltonian as follows:

$$\begin{aligned} s^{dS}\left(X_1^l,X_2^r\right)&\to s^{\text{AdS}}\left(X_1,X_2\right),\\ s^{dS}\left(X_1^{l(r)},X_2^{l(r)}\right)&\to -s^{\text{AdS}}\left(X_1^{\text{AdS}},X_2^{\text{AdS}}\right). \end{aligned} \tag{2.52}$$

Let us first take the case when one coordinate is coming from left and other one is coming from right hamiltonian i.e. $s\left(X_1^l,X_2^r\right)\to s^{\text{AdS}}\left(X_1,X_2\right)$ and find the relation between the propagator G_ν^{lr} in dS and G_ν^{AdS} in AdS and (we will use hypergeometric identities),

$$G_\nu^{lr}(s^{dS})\to\frac{\Gamma\left(\frac{d}{2}\pm i\nu\right)}{(4\pi)^{\frac{d+1}{2}}\Gamma\left(\frac{d+1}{2}\right)}{}_2F_1\left(\frac{d}{2}+i\nu,\frac{d}{2}-i\nu,\frac{d+1}{2},\frac{1+s^{AdS}}{2}\right). \tag{2.53}$$

Where we have replaced s^{dS} with s^{AdS} and now we can use the following identity:

$$F(a,b,c,z)=(1-z)^{-a}F\left(a,c-b,c,\frac{z}{z-1}\right). \tag{2.54}$$

$G_\nu^{lr}(s^{dS})$ will now become (for convenience, ${}_2F_1$ is just represented by F),

$$G_\nu^{lr}(s^{dS})\to\frac{1}{4\pi^{\frac{d+1}{2}}}\frac{\Gamma\left(\frac{d}{2}\pm i\nu\right)}{\Gamma\left(\frac{d+1}{2}\right)}\left(\frac{1-s^{\text{AdS}}}{2}\right)^{-\left(\frac{d}{2}+i\nu\right)}F\left(\frac{d}{2}+i\nu,\frac{1}{2}+i\nu,\frac{d+1}{2},-\left(\frac{1+s^{\text{AdS}}}{1-s^{\text{AdS}}}\right)\right). \tag{2.55}$$

We further, use the following identities:

$$\begin{aligned}F(a,b,c,z) =&\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}F(a,b,a+b+1-c,1-z)\\&+\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-z)^{(c-a-b)}F(c-a,c-b,1+c-a-b,1-z),\\\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) =&2^{1-2z}\sqrt{\pi}\Gamma(2z).\end{aligned} \tag{2.56}$$

Now the transformation of $G_\nu^{lr}(s^{dS})$ is as follows:

$$\begin{aligned}G_\nu^{lr}(s^{dS}) \rightarrow&\frac{1}{4\pi^{\frac{d+1}{2}}}\frac{\Gamma\left(\frac{d}{2}\pm i\nu\right)}{\Gamma\left(\frac{d+1}{2}\right)\left(\frac{1-s^{\text{AdS}}}{2}\right)^{\left(\frac{d}{2}+i\nu\right)}}\left[\frac{\Gamma\left(\frac{d+1}{2}\right)\Gamma(-2i\nu)}{\Gamma\left(\frac{1}{2}-i\nu\right)\Gamma\left(\frac{d}{2}-i\nu\right)}F\left(\frac{d}{2}+i\nu,\frac{1}{2}+i\nu,2i\nu+1,\frac{2}{1-s^{\text{AdS}}}\right)\right.\\&\left.+\frac{\Gamma\left(\frac{d+1}{2}\right)\Gamma(2i\nu)}{\Gamma\left(\frac{1}{2}+i\nu\right)\Gamma\left(\frac{d}{2}+i\nu\right)\left(\frac{1-s^{\text{AdS}}}{2}\right)^{-2i\nu}}F\left(\frac{d}{2}-i\nu,\frac{1}{2}-i\nu,-2i\nu+1,\frac{2}{1-s^{\text{AdS}}}\right)\right],\\\rightarrow&\frac{1}{4\pi^{\frac{d+1}{2}}}\left[\frac{\Gamma\left(\frac{d}{2}+i\nu\right)\Gamma(-i\nu)}{\sqrt{\pi}2^{1+2(i\nu)}\left(\left(\frac{1-s^{\text{AdS}}}{2}\right)\right)^{\frac{d}{2}+i\nu}}F\left(\frac{d}{2}+i\nu,i\nu+\frac{1}{2},2i\nu+1,\frac{2}{1-s^{\text{AdS}}}\right)+\nu\rightarrow-\nu\right].\end{aligned} \tag{2.57}$$

Finally we can use the identity $F(a,b,c,z)=(1-z)^{-a}F\left(a,c-b,c,\frac{z}{z-1}\right)$ once again and we will have the following relation between AdS and dS propagators:

$$G_\nu^{lr}(s^{dS}) \rightarrow \frac{i\nu}{2\pi}\Gamma(\pm i\nu)\left[G_\nu^{AdS}(s^{AdS})-G_{-\nu}^{AdS}(s^{AdS})\right]. \tag{2.58}$$

Now let us take a look at the other scenario where both the coordinates are coming from the same hamiltonian either right or left i.e. $s\left(X_1^l,X_2^r\right)\rightarrow -s^{AdS}\left(X_1,X_2\right)$. In this case we want to find the similar relation between $G_\nu^{l(r)l(r)}$ and G_ν^{AdS} and we start as follows:

$$W_\nu(s^{dS}) \rightarrow \frac{\Gamma\left(\frac{d}{2}\pm i\nu\right)}{(4\pi)^{\frac{d+1}{2}}\Gamma\left(\frac{d+1}{2}\right)}{}_2F_1\left(\frac{d}{2}+i\nu,\frac{d}{2}-i\nu,\frac{d+1}{2},\frac{1+s^{AdS}}{2}\right) \tag{2.59}$$

Now we can use the following identities to rewrite the transformation:

$$\begin{aligned}F(a,b,c,z) =&\frac{\Gamma(b-a)\Gamma(c)}{\Gamma(b)\Gamma(c-a)(-z)^a}F\left(a,a-c+1,a-b+1,\frac{1}{z}\right)\\&+\frac{\Gamma(a-b)\Gamma(c)}{\Gamma(a)\Gamma(c-b)(-z)^b}F\left(b,b-c+1,b-a+1,\frac{1}{z}\right),\\\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) =& 2^{1-2z}\sqrt{\pi}\Gamma(2z).\end{aligned} \tag{2.60}$$

We can write the transformation as function of ν and $-\nu$ as follows:

$$W_\nu(s^{dS}) \to \frac{1}{4\pi^{\frac{d+1}{2}}}\left[\frac{\Gamma\left(\frac{d}{2}+i\nu\right)\Gamma(-i\nu)}{\sqrt{\pi}2^{1+2(i\nu)}\left(-\left(\frac{1-s^{AdS}}{2}\right)\right)^{\frac{d}{2}+i\nu}}F\left(\frac{d}{2}+i\nu, i\nu+\frac{1}{2}, 2i\nu+1, \frac{2}{1-s^{AdS}}\right)+\nu\to-\nu\right] \tag{2.61}$$

We further use the following identity:

$$F(a,b,c,z) = (1-z)^{-a}F\left(a, c-b, c, \frac{z}{z-1}\right). \tag{2.62}$$

Finally, we can write it as linear sum $G^{AdS}_\nu(s^{AdS})$ and $G^{AdS}_{-\nu}(s^{AdS})$ as follows:

$$W^{dS}_\nu(s^{dS}) \to \frac{i\nu}{2\pi}\Gamma(\pm i\nu)\left[\frac{1}{(-1)^{\left(\frac{d}{2}+i\nu\right)}}G^{AdS}_\nu(s^{AdS}) - \frac{1}{(-1)^{\left(\frac{d}{2}-i\nu\right)}}G^{AdS}_{-\nu}(s^{AdS})\right]. \tag{2.63}$$

Upon further simplification we have the following relations:

$$\begin{aligned} G^{ll}_\nu(s^{dS}) &\to \frac{i\nu}{2\pi}\Gamma(\pm i\nu)\left[e^{i\pi\left(\frac{d}{2}+i\nu\right)}G^{AdS}_\nu(s^{AdS}) - e^{i\pi\left(\frac{d}{2}-i\nu\right)}G^{AdS}_{-\nu}(s^{AdS})\right], \\ G^{rr}_\nu(s^{dS}) &\to \frac{i\nu}{2\pi}\Gamma(\pm i\nu)\left[e^{-i\pi\left(\frac{d}{2}+i\nu\right)}G^{AdS}_\nu(s^{AdS}) - e^{-i\pi\left(\frac{d}{2}-i\nu\right)}G^{AdS}_{-\nu}(s^{AdS})\right]. \end{aligned} \tag{2.64}$$

In the same way one can relate boundary propagators of dS and AdS as well, [81–84]

$$\begin{aligned} K^l_\nu(s) &\to (-\eta_c)^{\frac{d}{2}-i\nu}\, e^{i\frac{\pi}{2}\left(\frac{d}{2}-i\nu\right)}\frac{N_\nu}{N^{\text{AdS}}_{-\nu}}K^{\text{AdS}}_{-\nu}(X,P), \\ K^r_\nu(s) &\to (-\eta_c)^{\frac{d}{2}-i\nu}\, e^{-i\frac{\pi}{2}\left(\frac{d}{2}-i\nu\right)}\frac{N_\nu}{N^{\text{AdS}}_{-\nu}}K^{\text{AdS}}_{-\nu}(X,P), \end{aligned} \tag{2.65}$$

where,

$$\begin{aligned} N^{AdS}_\nu &= \left(\frac{\Gamma\left(\frac{d}{2}+i\nu\right)}{2\pi^{d/2}\Gamma(1+i\nu)}\right)^{\frac{1}{2}}, \\ N_\nu &= \frac{1}{4\pi^{\frac{d}{2}+1}}\Gamma\left(\frac{d}{2}-i\nu\right)\Gamma(i\nu). \end{aligned} \tag{2.66}$$

We have seen how to relate, bulk to boundary and bulk to bulk propagators. To relate the feynman diagrams in dS and AdS, we also need to understand the transformation of the integration measure (for the internal bulk/vertex point) as follows:

$$i\int\frac{d\eta_l}{(-\eta^l)^{d+1}} \to e^{-i\frac{\pi}{2}(d-1)}\int\frac{dz}{z^{d+1}}, \quad -i\int\frac{d\eta_r}{(-\eta^r)^{d+1}} \to e^{i\frac{\pi}{2}(d-1)}\int\frac{dz}{z^{d+1}}. \tag{2.67}$$

Lagrangian in dS

Now we have all the important ingredients to rewrite the lagrangian in AdS that can give us feynman diagrams for dS using the relation of propagators and integration measure we found in previous section. We will consider lagrangian for single field for the moment. As we are doing calculations in "in-in" formalism, we have two fields ϕ^l and ϕ^r, we will further divide each one into two more fields and thus we will have total four scalar fields $\phi_\pm^{l/r}$ depending on $\pm\nu$ part of propagator obtained upon analytic continuation to AdS.

We can define the matrix D_ϕ as follows ($G^{\text{dS}}_{\phi,ll,+}$ denotes part of the propagator proportional to $G^{\text{AdS}}_{\phi,\nu}$ and same for others):

$$\begin{bmatrix} G^{\text{dS}}_{\phi,ll,+} & G^{\text{dS}}_{\phi,lr,+} \\ G^{\text{dS}}_{\phi,rl,+} & G^{\text{dS}}_{\phi,rr,+} \end{bmatrix} = G^{\text{AdS}}_{\phi,\nu} D_\phi, \tag{2.68}$$

where,

$$D_\phi = \frac{i\nu}{2\pi}\Gamma(\pm i\nu) \begin{bmatrix} e^{i\pi\left(\frac{d}{2}+i\nu\right)} & 1 \\ 1 & e^{-i\pi\left(\frac{d}{2}+i\nu\right)} \end{bmatrix}. \tag{2.69}$$

Similarly, we can define the matrix N_ϕ as follows ($G^{\text{dS}}_{\phi,ll,-}$ denotes the part of the propagator proportional to $G^{\text{AdS}}_{\phi,-\nu}$ and same for the others):

$$\begin{bmatrix} G^{\text{dS}}_{\phi,ll,-} & G^{\text{dS}}_{\phi,lr,-} \\ G^{\text{dS}}_{\phi,rl,-} & G^{\text{dS}}_{\phi,rr,-} \end{bmatrix} = G^{\text{AdS}}_{\phi,-\nu} N_\phi, \tag{2.70}$$

where,

$$N_\phi = -\frac{i\nu}{2\pi}\Gamma(\pm i\nu) \begin{bmatrix} e^{i\pi\left(\frac{d}{2}-i\nu\right)} & 1 \\ 1 & e^{-i\pi\left(\frac{d}{2}-i\nu\right)} \end{bmatrix}. \tag{2.71}$$

Also note that the matrices D_ϕ and N_ϕ are singular and hence non-invertible as one of the eigenvalue happened to be zero. So we can write these fields as follows:

$$\phi_\pm^{l/r} = \boldsymbol{C^{l/r}_{\phi,\pm}} \boldsymbol{\phi}_\pm + C^{l/r}_{\phi,\pm}\phi_\pm. \tag{2.72}$$

In above equation, we are we are picking coefficients $\boldsymbol{C^{l/r}_{\phi,\pm}}, C^{l/r}_{\phi,\pm}$ such that they diagonalize the matrices.

The coefficients $\boldsymbol{C^{l/r}_{\phi,\pm}}, C^{l/r}_{\phi,\pm}$ satisfy the following conditions :

$$\begin{aligned} D_\phi \begin{bmatrix} \boldsymbol{C^l_{\phi,+}} \\ \boldsymbol{C^r_{\phi,+}} \end{bmatrix} &=0, \\ N_{\phi/A} \begin{bmatrix} \boldsymbol{C^l_{\phi,-}} \\ \boldsymbol{C^r_{\phi,-}} \end{bmatrix} &=0, \end{aligned} \tag{2.73}$$

$$
\begin{aligned}
D_\phi \begin{bmatrix} C^l_{\phi,+} \\ C^r_{\phi,+} \end{bmatrix} &= \lambda_{\phi,+} \begin{bmatrix} C^l_{\phi,+} \\ C^r_{\phi,+} \end{bmatrix}, \\
N_\phi \begin{bmatrix} C^l_{\phi,-} \\ C^r_{\phi,-} \end{bmatrix} &= \lambda_{\phi,-} \begin{bmatrix} C^l_{\phi,-} \\ C^r_{\phi,-} \end{bmatrix}.
\end{aligned} \tag{2.74}
$$

Let us look at the eigen equation for D_ϕ,

$$
\begin{aligned}
\frac{i\nu}{2\pi}\Gamma(\pm i\nu)\left[e^{i\pi\left(\frac{d}{2}+i\nu\right)}C^l_{\phi,+} + C^r_{\phi,+}\right] &= \lambda_{\phi,+}C^l_{\phi,+}, \\
\frac{i\nu}{2\pi}\Gamma(\pm i\nu)\left[e^{-i\pi\left(\frac{d}{2}+i\nu\right)}C^r_{\phi,+} + C^l_{\phi,+}\right] &= \lambda_{\phi,+}C^r_{\phi,+}.
\end{aligned} \tag{2.75}
$$

We can consider the solutions to be of the form,

$$
C^l_{\phi,+} = \left(C^r_{\phi,+}\right)^{-1} = e^{-i\frac{\pi}{2}\left(\frac{d}{2}+i\nu\right)}C. \tag{2.76}
$$

Upon solving, the set of equations we have, $C = e^{i\pi\left(\frac{d}{2}+i\nu\right)}$ and hence,

$$
\begin{aligned}
C^l_{\phi,+} &= e^{i\frac{\pi}{2}\left(\frac{d}{2}+i\nu\right)}, \\
C^r_{\phi,+} &= e^{-i\frac{\pi}{2}\left(\frac{d}{2}+i\nu\right)}, \\
\lambda_{\phi,+} &= 2\cos\left(\pi\left(\frac{d}{2}+i\nu\right)\right)\cdot\frac{i\nu}{2\pi}\Gamma(\pm i\nu).
\end{aligned} \tag{2.77}
$$

Similarly, we can solve the following eigen equations for N_ϕ:

$$
\begin{aligned}
-\frac{i\nu}{2\pi}\Gamma(\pm i\nu)\left[e^{i\pi\left(\frac{d}{2}-i\nu\right)}C^l_{\phi,-} + C^r_{\phi,-}\right] &= \lambda_{\phi,-}C^l_{\phi,-}, \\
-\frac{i\nu}{2\pi}\Gamma(\pm i\nu)\left[e^{-i\pi\left(\frac{d}{2}-i\nu\right)}C^r_{\phi,-} + C^l_{\phi,-}\right] &= \lambda_{\phi,-}C^r_{\phi,-}.
\end{aligned} \tag{2.78}
$$

For this case, we can consider solution to be of the following form:

$$
C^l_{\phi,-} = \left(C^r_{\phi,-}\right)^{-1} = e^{-i\frac{\pi}{2}\left(\frac{d}{2}-i\nu\right)}C. \tag{2.79}
$$

Upon solving, the set of equations for N_ϕ we have, $C = e^{i\pi\left(\frac{d}{2}-i\nu\right)}$ and hence,

$$
\begin{aligned}
C^l_{\phi,-} &= e^{i\frac{\pi}{2}\left(\frac{d}{2}-i\nu\right)}, \\
C^r_{\phi,-} &= e^{-i\frac{\pi}{2}\left(\frac{d}{2}-i\nu\right)}, \\
\lambda_{\phi,-} &= -2\cos\left(\pi\left(\frac{d}{2}-i\nu\right)\right)\cdot\frac{i\nu}{2\pi}\Gamma(\pm i\nu).
\end{aligned} \tag{2.80}
$$

Now with this information, we will first write the kinetic term of the lagrangian. Recall that the kinetic term is nothing but the inverse of the propagator. Let us, first consider ϕ_+ term where we have used the notation $C^{\alpha}_{\phi,+} = C^{\alpha}_{+}$ and for $\alpha, \beta = l/r$,

$$\begin{aligned}
\left\langle \phi^{\alpha}_{+}\phi^{\beta}_{+} \right\rangle &= D^{\alpha\beta}_{\phi} G^{AdS}_{\phi,\nu}, \\
\implies C^{\alpha}_{+} C^{\beta}_{+} &= \frac{1}{\langle \phi_+\phi_+ \rangle} D^{\alpha\beta}_{\phi} G^{AdS}_{\phi,\nu}, \\
\implies (C_+ C^T_+) &= \frac{1}{\langle \phi_+\phi_+ \rangle}) D_{\phi} G^{AdS}_{\phi,\nu}, \\
\implies \langle \phi_+\phi_+ \rangle (C^T_+ C_+) C_+ &= D_{\phi} G^{AdS}_{\phi,\nu} C_+, \\
\implies \langle \phi_+\phi_+ \rangle (C^T_+ C_+) &= \lambda_{\phi,+} G^{AdS}_{\phi,\nu}, \\
\implies \langle \phi_+\phi_+ \rangle &= \frac{\lambda_{\phi,+} G^{AdS}_{\phi,\nu}}{(C^T_+ C_+)}.
\end{aligned} \tag{2.81}$$

In above equation $(C^T_+ C_+)$ index α is summed over.

Finally, we can have kinetic term in lagrangian as inverse of this propagator term as follows:

$$\begin{aligned}
L^{Kin}_{\phi,+} &= \frac{(C^T_+ C_+)}{\lambda_{\phi,+}} \left[\frac{1}{2} \left(\partial\phi_+ \partial\phi_+ - m^2 \phi^2_+ \right) \right], \\
&= -\sin(i\pi\nu) \left(\partial\phi_+ \partial\phi_+ - m^2 \phi^2_+ \right), \\
&= -i \sinh(\pi\nu) \left(\partial\phi_+ \partial\phi_+ - m^2 \phi^2_+ \right).
\end{aligned} \tag{2.82}$$

Similarly we have the kinetic term for ϕ_- fields,

$$L^{Kin}_{\phi,-} = i \sinh(\pi\nu) \left(\partial\phi_- \partial\phi_- - m^2 \phi^2_- \right). \tag{2.83}$$

After kinetic term, we can write potential as well [85],

$$\begin{aligned}
V^{AdS}\left(\phi^l_+, \phi^l_-, \phi^r_+, \phi^r_-\right) &= e^{-i\frac{\pi}{2}(d-1)} V\left(\phi^l_+ + \phi^l_-\right) + e^{i\frac{\pi}{2}(d-1)} V\left(\phi^r_+ + \phi^r_-\right), \\
&= e^{-i\frac{\pi}{2}(d-1)} V\left(e^{i\frac{\pi}{2}\left(\frac{d}{2}+i\nu\right)}\phi_+ + e^{i\frac{\pi}{2}\left(\frac{d}{2}-i\nu\right)}\phi_-\right) \\
&+ e^{i\frac{\pi}{2}(d-1)} V\left(e^{-i\frac{\pi}{2}\left(\frac{d}{2}+i\nu\right)}\phi_+ + e^{-i\frac{\pi}{2}\left(\frac{d}{2}-i\nu\right)}\phi_-\right).
\end{aligned} \tag{2.84}$$

Since we have the lagrangian, we can now relate diagrams in dS to diagrams in AdS. We will use similar techniques to study gauge theory in dS, where in addition to scalar fields, we will also have vector field and we will need to find relation between propagators in dS and AdS for vector fields and as well as between feynman diagrams.

Chapter 3

$O(N)$ Model in AdS and dS

3.1 Introduction

$O(N)$ model is a model with N number of scalar fields which are self interacting via a $O(N)$ symmetric quartic coupling. This model is used to describe phase transitions in magnets.

In this chapter we will review what has been done for $O(N)$ model in the literature for flat space, AdS and dS. We will be referring to [73] for the case of flat space, [74] for AdS and [90] for dS. The techniques explored in this chapter will become the basis of the work done in Scalar QED and later chapter of the thesis. One important point to note is that in this section we will be using a combination of large N and bootstrap to calculate observable beyond the standard perturbation theory. The $O(N)$ model is a good starting point to understand interactions without invoking gauge fields, and in next chapters we will see what subtleties arise in case of gauge interactions.

As stated earlier in the introduction of the thesis, AdS has nice properties such as the curvature of AdS acts as IR regulator unlike flat space and we also have infinite volume in AdS space unlike on the sphere and this gives the possibility of phase transition. In this chapter we will see different phases of theory both in flat and AdS space namely symmetry breaking (gapless) and symmetry preserving phase. We will briefly comment on how this changes in dS.

3.2 $O(N)$ Model in flat space

In this model there is $O(N)$ symmetry under $\phi^i \mapsto M^i_j \phi^j$ with $M^T M = \mathbb{I}$ for real scalar fields ϕ^i. First let us write the lagrangian for the $O(N)$ model,

$$\mathcal{L} = \frac{1}{2}\left(\partial\phi^i\right)^2 + \frac{m^2}{2}\left(\phi^i\right)^2 + \frac{\lambda}{2N}\left(\left(\phi^i\right)^2\right)^2 . \tag{3.1}$$

Here, we can see that λ is our quartic coupling, and it is the usual lagrangian of scalar field theory except for the additional index of the field ϕ which denotes the fact that we have N independent scalar fields and the index i runs from 1 to N. Also,

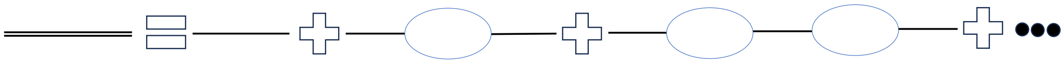


FIGURE 3.1: Resummed σ propagator.
Double lines represent the resummed σ propagator and single black line represents the tree level σ propagator. The loop represent the bubble of scalar fields.

note that summation on the repeated flavour index is implied. We are going to study this theory on flat space $\mathcal{R}^D$ as a reference for later.

In [74] and [73], the effective action is used in the large N limit to study different phases, but in this section we will use different presentation based on resummation of Feynaman diagrams to get the same results. First, we will introduce an auxillary field known as Hubbard Stratonovich field i.e. σ, and rewrite the above lagrangian in terms of this new field, this helps to do computation in large N conveniently. Note that on-shell we have, $\sigma = \frac{\lambda}{\sqrt{N}}\phi^i\phi^i$ and integrating it out we can get previous lagrangian. Consequently, the lagrangian will become (Euclidean signature),

$$\mathcal{L} = \frac{1}{2}\left(\partial\phi^i\right)^2 + \frac{m^2}{2}\left(\phi^i\right)^2 - \frac{1}{2\lambda}\sigma^2 + \frac{1}{\sqrt{N}}\sigma\left(\phi^i\right)^2. \tag{3.2}$$

In the above lagrangian, note that now we have a cubic coupling instead of a quartic coupling. Now we will not be doing expansion in the coupling λ but instead we will be doing expansion in $\frac{1}{N}$ and hence in this way, our results will be valid for any value of λ. The idea here is that we will introduce resummed propagator or exact propagator instead of simply using tree level propagagtor to calculate observables. And we can do this by adding bubbles or 1PI diagrams to the tree level propagator as in the figure 3.1.

Here, we can see that first term is usual tree level propagator while the second term has extra loop and another tree level propagator but we can see that despite of this, even this diagram will contribute at the same order of N, because each extra internal vertex gives a factor of $\frac{1}{\sqrt{N}}$ while the loop gives an additional factor of N as there are N scalar field present in the loop. We can extend this logic to third, fourth and other remaining infinite terms in the summation and hence it will become sum of infinite terms in geometric series. In order to proceed, we need to calculate this loop or bubble function for our case in flat space.

It is clear from the lagrangian (3.2) that the tree level propagator is given by (as there is no other kinetic or quardatic term of σ),

$$\langle\sigma\sigma\rangle = -\lambda. \tag{3.3}$$

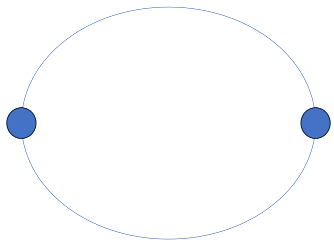


FIGURE 3.2: 1PI diagram or bubble of two scalar fields.

Now let us take a look at the bubble term or the 1PI term in the figue 3.2,

$$2\int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2+m^2}\frac{1}{(k+p)^2+m^2}. \tag{3.4}$$

An explicit calculation with feynman parameter gives:

$$\begin{aligned} B_D\left(p^2,M^2\right) &= \int \frac{d^Dk}{(2\pi)^D}\frac{1}{\left(k^2+M^2\right)\left((k+p)^2+M^2\right)} \\ &= \frac{\Gamma\left(2-\frac{D}{2}\right)\left(M^2\right)^{\frac{D}{2}-2}}{(4\pi)^{D/2}(3-D)}\left(\left(D-6-\frac{4M^2}{p^2}\right){}_2F_1\left(1,2-\frac{D}{2},\frac{1}{2},-\frac{p^2}{4M^2}\right)\right. \\ &\left.+\left(1+\frac{4M^2}{p^2}\right){}_2F_1\left(1,2-\frac{D}{2},-\frac{1}{2},-\frac{p^2}{4M^2}\right)\right). \end{aligned} \tag{3.5}$$

Now we can use tree level propagator and the bubble to find the resummed or exact propagator as in 3.1,

$$\begin{aligned} \langle\sigma\sigma\rangle\,|_{\text{resummed}} &= -\lambda+(-\lambda)2B_D(-\lambda)+\ldots, \\ &= \frac{-\lambda}{1+2\lambda B_D} = -\frac{1}{\frac{1}{\lambda}+2B_D}. \end{aligned} \tag{3.6}$$

This exact propagator is valid as long as there is no VEV of the field ϕ. This phase is called $O(N)$ symmetry preserving phase in which the scalar fields are massive. On the other hand when ϕ fields attain VEV then the phase become gapless as the scalar fields become massless goldstone bosons (shown in the figure 3.3). Let us see what happens in this case, by writing the fields as constant field and fluctuations as below:

$$\begin{aligned} \phi^A &= \pi^A, \\ \phi^N &= (\sqrt{N}\Phi+\rho), \\ \sigma &= \hat{\sigma}+\sqrt{N}\Sigma. \end{aligned} \tag{3.7}$$

Here, note that A runs from 1 to $N-1$ and without the loss of generality we have oriented $\langle\phi^i\rangle$ in the direction of $\mathbf{N}$. Though most of these equations remain same for general dimension D, in gapless phase, we are now focused for $D=3$. We can plug

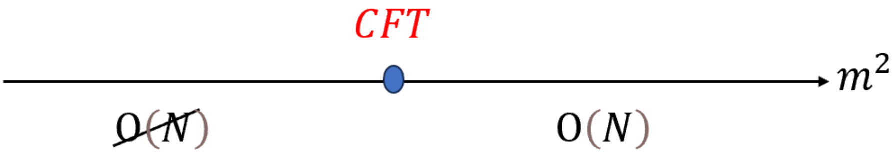


FIGURE 3.3: Different Phase in $O(N)$model

these back to our original lagrangian (3.2) and our modified lagrangian becomes,

$$\mathcal{L} = \frac{1}{\sqrt{N}} \left(\hat{\sigma}\pi^A\pi^A + \sqrt{N}\Sigma\pi^A\pi^A + N\Phi^2\hat{\sigma} + \rho^2\hat{\sigma} + 2\sqrt{N}\Phi\rho\hat{\sigma} + N\sqrt{N}\Sigma\Phi^2 + \sqrt{N}\Sigma\rho^2 + 2N\Phi\rho\Sigma \right)$$
$$\frac{-1}{2\lambda} \left(\hat{\sigma}^2 + 2\sqrt{N}\Sigma\hat{\sigma} + N\Sigma^2 \right) + \frac{m^2}{2} \left(\pi^A\pi^A + N\Phi^2 + \rho^2 + 2\sqrt{N}\Phi\rho \right) + \left(\partial_\mu\rho\right)\left(\partial^\mu\rho\right) + \left(\partial_\mu\pi^A\right)\left(\partial^\mu\pi^A\right). \tag{3.8}$$

Fluctuations in $\hat{\sigma}, \pi, \rho$ should be zero:

$$\langle\hat{\sigma}\rangle = \left\langle \pi^A \right\rangle = \langle\rho\rangle = 0. \tag{3.9}$$

The equation for ρ gives:

$$\begin{aligned} &\left(2\sqrt{N}\Sigma\Phi\rho + m^2\sqrt{N}\Phi\rho\right) = 0, \\ &\Rightarrow \sqrt{N}\Phi(\Sigma + \frac{m^2}{2})\rho = 0, \\ &\Rightarrow \Sigma = \frac{-m^2}{2}. \end{aligned} \tag{3.10}$$

Above, we have assumed Φ is not equal to zero.

Notice that by using the above result, the coefficients $\pi^A\pi^A$ becomes zero which implies mass of these fields are now zero in the broken phase.

For $\hat{\sigma}$, we obtain:

$$\left(\frac{1}{\sqrt{N}}\pi^A\pi^A + \sqrt{N}\Phi - \frac{\sqrt{N}}{\lambda}\Sigma \right)\hat{\sigma} = 0. \tag{3.11}$$

Let us first evaluate the first term of the above expression i.e. $\langle \frac{1}{\sqrt{N}}\pi^A\pi^A \rangle$ as follows:

$$\sqrt{N}\int d^3p\mathcal{P}\left(\pi^A\right), \tag{3.12}$$

where $\mathcal{P}\left(\pi^A\right)$ denotes propagator or π^A fields.

Since π^A have become massless,we have:

$$\sim \sqrt{N}\int \frac{d^3p}{p^2}, \tag{3.13}$$

and the above integral is trivially zero in dimensional regularization. So we have finally,

$$\begin{aligned} &\sqrt{N}\Phi^2 - \frac{\sqrt{N}}{\lambda}\Sigma = 0, \\ &\Phi^2 + \frac{m^2}{\lambda} = 0 \Rightarrow \Phi^2 = -\frac{m^2}{\lambda}. \end{aligned} \tag{3.14}$$

Important terms in $\rho - \hat{\sigma}$ system are as follows:

$$\left(\partial_\mu \rho\right)\left(\partial^\mu \rho\right) + 2\Phi\rho\hat{\sigma} - \frac{\hat{\sigma}^2}{2\lambda} + \frac{1}{\sqrt{N}}\hat{\sigma}\pi^A\pi^A. \tag{3.15}$$

Note that in the above case for the bubble in $\hat{\sigma}$ propagator we have the massless π fields and hence the bubble corresponds to $2B_3(p^2, 0)$.

The $\hat{\sigma} - \rho$ matrix for $d = 3$ is written below (along the diagonal the first entry is $\hat{\sigma}\hat{\sigma}$ and the the last entry is $\rho\rho$ while off-diagonal terms are $\hat{\sigma} - \rho$ and $\rho - \hat{\sigma}$):

$$M = \begin{pmatrix} -\frac{1}{\lambda} - 2B_3\left(p^2, 0\right) & 2\Phi \\ 2\Phi & p^2 \end{pmatrix}. \tag{3.16}$$

The inverse of above matrix is as follows:

$$M^{-1} = \frac{1}{\left[\; p^2\left(-\frac{1}{\lambda} - 2B_3\left(p^2, 0\right)\right) \quad -4\Phi^2 \;\right]} \begin{pmatrix} p^2 & -2\Phi \\ -2\Phi & -\frac{1}{\lambda} - 2B_3\left(p^2, 0\right) \end{pmatrix}. \tag{3.17}$$

From above, the $\hat{\sigma}$ propagator is as follows :

$$\langle\hat{\sigma}\hat{\sigma}\rangle = \frac{p^2}{\left(p^2\left(-\frac{1}{\lambda} - 2B_3\right) - 4\Phi^2\right)}. \tag{3.18}$$

With this new propagator, we can calculate S-matrix Amplitudes for two to two scattering in gapless phase. It is also noted that these two phases are separated by a CFT. Upon analyzing the structure of scattering amplitude in gapless phase, one finds a resonance in the system.

3.3 O(N) model in AdS space

Now we want to implement the same techniques in case of AdS_{d+1} (Euclidean signature) space but there is one problem, the computation to calculate the bubble is not straightforward. The idea that we will use is that AdS space has a conformal boundary that will help us to bootstrap the bubble and hence we do not need to calculate it directly. In this section, we will borrow the techniques like spectral representation and embedding formalism explained in the chapter 2.

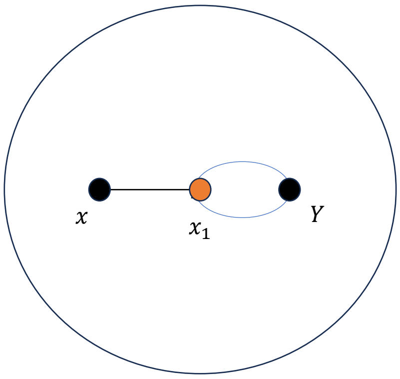


FIGURE 3.4: One bubble and one σ propagator in the bulk

First we write the tree level propagator and the bubble in terms of the spectral representation,

$$\langle\sigma(X)\sigma(Y)\rangle\,|_{\text{tree level}}=\int_{-\infty}^{\infty}d\nu\frac{1}{\lambda^{-1}}\Omega_{\nu}(X,Y), \tag{3.19}$$

$$\textit{scalar bubble in the figure 3.2} = 2\int_{-\infty}^{\infty}d\nu\tilde{B}(\nu)\Omega_{\nu}(X,Y). \tag{3.20}$$

Note that here we do not know the form of $\tilde{B}(\nu)$ and we will bootstrap it later, but first we will write the resummed propagator like in flat space (3.6) but this time using the spectral representation. Note that to do so, we have to use convolution relation of two harmonic functions that we discussed in the previous chapter (2.27) and then we will again be able to write it as a geometric sum.

First let us see, the structure of just one tree level σ propagator and one bubble as in the figure 3.4,

$$\begin{aligned}
&-\int_{AdS}dX_1\int_{-\infty}^{\infty}d\nu\frac{1}{\lambda^{-1}}\Omega_{\nu}(X,X_1)\int_{-\infty}^{\infty}d\bar{\nu}2\tilde{B}(\bar{\nu})\Omega_{\bar{\nu}}(X_1,Y),\\
&-\int_{-\infty}^{\infty}d\nu\frac{1}{\lambda^{-1}}\int_{AdS}dX_1\Omega_{\nu}(X,X_1)\int_{-\infty}^{\infty}d\bar{\nu}2\tilde{B}(\bar{\nu})\Omega_{\bar{\nu}}(X_1,Y),\\
&-\int_{-\infty}^{\infty}d\nu\frac{1}{\lambda^{-1}}\cdot 2\tilde{B}(\nu)\Omega_{\nu}(X,Y).
\end{aligned} \tag{3.21}$$

Similarly we can write the first correction to the tree level propagator which consists of two sigma propagator and one bubble as in the figure 3.1,

$$-\int_{-\infty}^{\infty}d\nu\frac{1}{\lambda^{-1}}\cdot 2\tilde{B}(\nu)\cdot\frac{1}{\lambda^{-1}}\Omega_{\nu}(X,Y). \tag{3.22}$$

Now, in the same way, we can find the structure of 2nd term in the figure 3.1,

$$-\int_{-\infty}^{\infty}d\nu\frac{1}{\lambda^{-1}}\cdot 2\tilde{B}(\nu)\cdot\frac{1}{\lambda^{-1}}\cdot 2\tilde{B}(\nu)\cdot\frac{1}{\lambda^{-1}}\Omega_{\nu}(X,Y). \tag{3.23}$$

We see the pattern of the geometric series and hence the resummed σ propagator at large N is as follows:

$$\langle\sigma(X)\sigma(Y)\rangle = -\int_{-\infty}^{\infty} d\nu \left[\frac{1}{\lambda^{-1}} + \frac{1}{\lambda^{-1}}\cdot 2\tilde{B}(\nu)\cdot\frac{1}{\lambda^{-1}} + \frac{1}{\lambda^{-1}}\cdot 2\tilde{B}(\nu)\cdot\frac{1}{\lambda^{-1}}\cdot 2\tilde{B}(\nu)\cdot\frac{1}{\lambda^{-1}} + \cdots\right]\Omega_\nu(X,Y). \tag{3.24}$$

Performing the sum we obtain:

$$\langle\sigma(X)\sigma(Y)\rangle = -\int_{-\infty}^{\infty} d\nu \frac{1}{\lambda^{-1}+2\tilde{B}(\nu)}\Omega_\nu(X,Y). \tag{3.25}$$

3.3.1 4-point boundary correlation function

We will now be calculating 4 point conformal boundary correlation functions and we will be putting some self consistency conditions on this which will in turn put conditions on the unknown $\tilde{B}(\nu)$ and hence we will be able to bootsrap it (see [74]).

Let us write the four point function up to leading order in $\frac{1}{N}$,

$$\left\langle \phi^i(P_1)\,\phi^j(P_2)\,\phi^k(P_3)\,\phi^l(P_4)\right\rangle = \left[\frac{\delta^{ij}\delta^{kl}}{(P_{12})^{\Delta}(P_{34})^{\Delta}} + \frac{\delta^{ik}\delta^{jl}}{(P_{13})^{\Delta}(P_{24})^{\Delta}} + \frac{\delta^{il}\delta^{jk}}{(P_{14})^{\Delta}(P_{23})^{\Delta}}\right] + \left[\frac{\delta^{ij}\delta^{kl}g_{12|34} + \delta^{ik}\delta^{jl}g_{13|24} + \delta^{il}\delta^{jk}g_{14|23}}{N}\right] + \mathcal{O}(\frac{1}{N^2}). \tag{3.26}$$

Here different g are related to four point boundary correlation function mediated by a σ field. For example $g_{12|34}$ is,

$$\begin{aligned}
g_{12|34} &= 4\int dX_1 dX_2 \left\langle\sigma(X_1)\,\sigma(X_2)\right\rangle \\
&\quad\times K_\Delta(P_1,X_1)\,K_\Delta(P_2,X_1)\,K_\Delta(P_3,X_2)\,K_\Delta(P_4,X_2), \\
&= -4\int_{-\infty}^{\infty} d\nu\frac{1}{\lambda^{-1}+2\tilde{B}(\nu)}\int dX_1 dX_2 \Omega_\nu(X_1,X_2) \\
&\quad\times K_\Delta(P_1,X_1)\,K_\Delta(P_2,X_1)\,K_\Delta(P_3,X_2)\,K_\Delta(P_4,X_2), \\
&= -4\int_{-\infty}^{\infty} d\nu\frac{1}{\lambda^{-1}+2\tilde{B}(\nu)}\frac{\nu^2\sqrt{\mathcal{C}_{\frac{d}{2}+i\nu}\mathcal{C}_{\frac{d}{2}-i\nu}}}{\pi}\int dP_0 \\
&\quad\times\int dX_1 K_\Delta(P_1,X_1)\,K_\Delta(P_2,X_1)\,K_{\frac{d}{2}+i\nu}(P_0,X_1) \\
&\quad\times\int dX_2 K_\Delta(P_3,X_2)\,K_\Delta(P_4,X_2)\,K_{\frac{d}{2}-i\nu}(P_0,X_2).
\end{aligned} \tag{3.27}$$

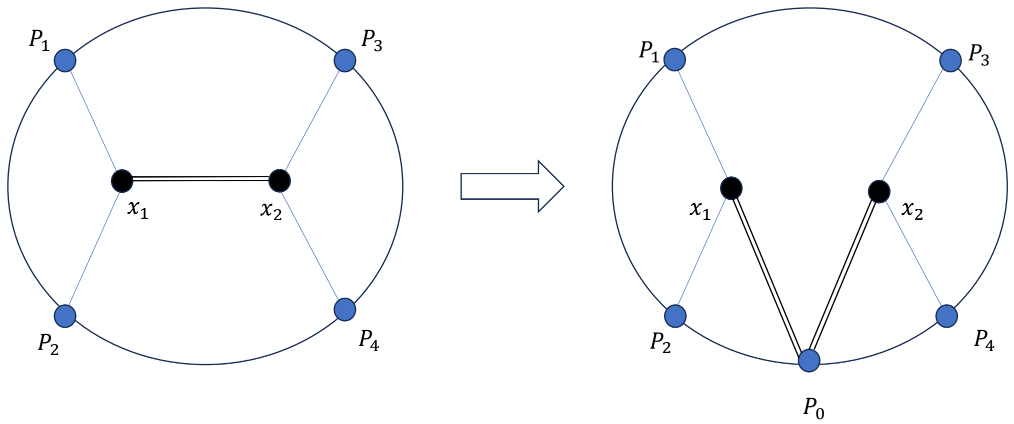


FIGURE 3.5: From 4-point function to two 3-point function
Double line represent the resummed σ propagator. Blue lines represent scalar field.

Note that in above equation K denotes bulk to boundary propagator for spin 0 give by (2.21) for $J = 0$

$$K_\Delta(P, X) = \frac{\sqrt{\mathcal{C}_\Delta}}{(-2P \cdot X)^\Delta}, \tag{3.28}$$

where the point P is at the boundary and point X is in the bulk and the normalization constant is given by:

$$\mathcal{C}_\Delta = \frac{\Gamma(\Delta)}{2\pi^{d/2}\Gamma\left(\Delta - \frac{d}{2} + 1\right)}. \tag{3.29}$$

Also, note that above we have used the relation of harmonic functions and bulk to boundary propagators for spin 0 (see,(2.26)) as follows:

$$\Omega_\nu\left(X_1, X_2\right) = \frac{\nu^2\sqrt{\mathcal{C}_{\frac{d}{2}+i\nu}\mathcal{C}_{\frac{d}{2}-i\nu}}}{\pi} \int dP_0 K_{\frac{d}{2}+i\nu}\left(P_0, X_1\right) K_{\frac{d}{2}-i\nu}\left(P_0, X_2\right). \tag{3.30}$$

We can see that (3.27) which represent 4-point correlation function can be understood as product of two 3-point correlation functions integrated over a common point as shown in the figure (3.5).

Now we will do integral over bulk points X_1 and X_2, which will be given by 3-point conformal boundary correlation functions as follows:

$$\begin{aligned}\int dX K_{\Delta_1}\left(P_1, X\right) K_{\Delta_2}\left(P_2, X\right) K_{\Delta_3}\left(P_3, X\right) = & \frac{\pi^{\frac{d}{2}}}{2}\sqrt{\mathcal{C}_{\Delta_1}\mathcal{C}_{\Delta_2}\mathcal{C}_{\Delta_3}}\,\Gamma\left(-\frac{d}{2} + \frac{\Delta_1+\Delta_2+\Delta_3}{2}\right) \\ & \cdot \frac{\Gamma\left(\frac{\Delta_1+\Delta_2-\Delta_3}{2}\right)\Gamma\left(\frac{\Delta_1-\Delta_2+\Delta_3}{2}\right)\Gamma\left(\frac{-\Delta_1+\Delta_2+\Delta_3}{2}\right)}{\Gamma\left(\Delta_1\right)\Gamma\left(\Delta_2\right)\Gamma\left(\Delta_3\right)} \\ & \cdot \frac{1}{\left(P_{12}\right)^{\frac{\Delta_1+\Delta_2-\Delta_3}{2}}\left(P_{23}\right)^{\frac{\Delta_2+\Delta_3-\Delta_1}{2}}\left(P_{31}\right)^{\frac{\Delta_3+\Delta_1-\Delta_2}{2}}}.\end{aligned} \tag{3.31}$$

We can use (3.31) in (3.27) and the problem will now just reduce to integral over the commmon point P_0 as follows:

$$g_{12|34} = -4\int_{-\infty}^{\infty} d\nu \frac{1}{\lambda^{-1}+2\tilde{B}(\nu)} \frac{\nu^2 \mathcal{C}_{\frac{d}{2}+i\nu}\mathcal{C}_{\frac{d}{2}-i\nu}}{16\pi} \frac{\Gamma^2\left(\Delta-\frac{d}{4}+\frac{i\nu}{2}\right)\Gamma^2\left(\Delta-\frac{d}{4}-\frac{i\nu}{2}\right)\Gamma^2\left(\frac{d}{4}+\frac{i\nu}{2}\right)\Gamma^2\left(\frac{d}{4}-\frac{i\nu}{2}\right)}{\Gamma^2(\Delta)\Gamma^2(\Delta-\frac{d}{2}+1)\Gamma(\frac{d}{2}+i\nu)\Gamma(\frac{d}{2}-i\nu)}$$
$$\times \frac{1}{(P_{12})^{\Delta-\frac{d}{4}-\frac{i\nu}{2}}(P_{34})^{\Delta-\frac{d}{4}+\frac{i\nu}{2}}}\int \frac{dP_0}{(P_{10})^{\frac{d}{4}+\frac{i\nu}{2}}(P_{20})^{\frac{d}{4}+\frac{i\nu}{2}}(P_{30})^{\frac{d}{4}-\frac{i\nu}{2}}(P_{40})^{\frac{d}{4}-\frac{i\nu}{2}}}. \tag{3.32}$$

This integral is well known in literature in terms of conformal block $\mathcal{K}_\Delta$ and here z and $\bar{z}$ are cross ratios defined as $\frac{P_{12}P_{34}}{P_{13}P_{24}} = z\bar{z}$, $\frac{P_{14}P_{23}}{P_{13}P_{24}} = (1-z)(1-\bar{z})$, and the constant $k(\Delta) = \frac{\pi^{\frac{d}{2}}\Gamma(\Delta-\frac{d}{2})\Gamma^2(\frac{d}{2}-\frac{\Delta}{2})}{\Gamma_{d-\Delta}\Gamma^2(\frac{\Delta}{2})}$ and hence,

$$\frac{1}{(P_{12})^{\Delta-\frac{d}{4}-\frac{i\nu}{2}}(P_{34})^{\Delta-\frac{d}{4}+\frac{i\nu}{2}}}\int \frac{dP_0}{(P_{10})^{\frac{d}{4}+\frac{i\nu}{2}}(P_{20})^{\frac{d}{4}+\frac{i\nu}{2}}(P_{30})^{\frac{d}{4}-\frac{i\nu}{2}}(P_{40})^{\frac{d}{4}-\frac{i\nu}{2}}} =$$
$$\frac{1}{(P_{12})^{\Delta}(P_{34})^{\Delta}}\left(k_{\frac{d}{2}-i\nu}\mathcal{K}_{\frac{d}{2}+i\nu}(z,\bar{z}) + k_{\frac{d}{2}+i\nu}\mathcal{K}_{\frac{d}{2}-i\nu}(z,\bar{z})\right). \tag{3.33}$$

Finally, we have our full expression of the σ exchange 4-point boundary Witten diagram in terms of conformal blocks,

$$g_{12|34} = -\frac{1}{(P_{12})^{\Delta}(P_{34})^{\Delta}}\int \frac{d\nu}{2\pi}\frac{1}{\lambda^{-1}+2\tilde{B}(\nu)}$$
$$\cdot\frac{\Gamma^2\left(\Delta-\frac{d}{4}+\frac{i\nu}{2}\right)\Gamma^2\left(\Delta-\frac{d}{4}-\frac{i\nu}{2}\right)\Gamma^4\left(\frac{d}{4}+\frac{i\nu}{2}\right)}{\Gamma^2(\Delta)\Gamma^2(\Delta-\frac{d}{2}+1)\Gamma(\frac{d}{2}+i\nu)\Gamma(i\nu)}\mathcal{K}_{\frac{d}{2}+i\nu}(z,\bar{z}). \tag{3.34}$$

3.3.2 Bootstrapping the Bubble

Now, the next task is to put consistency conditions to bootstrap the unknown $\tilde{B}(\nu)$. We will do that by checking consistency of the 4 point conformal boundary correlation function (3.26) as done in [74]. One can take a projection on the $O(N)$-singlet channel of (3.26) by contracting with $\delta_{ij}\delta_{kl}/N^2$, giving:

$$\frac{\delta_{ij}\delta_{kl}}{N^2}\left\langle \phi^i(P_1)\phi^i(P_2)\phi^k(P_3)\phi^k(P_4)\right\rangle =$$
$$\frac{1}{(P_{12})^{\Delta}(P_{34})^{\Delta}} + \frac{1}{N}\left[\frac{1}{(P_{13})^{\Delta}(P_{24})^{\Delta}} + \frac{1}{(P_{14})^{\Delta}(P_{23})^{\Delta}} + g_{12|34}\right] + \mathcal{O}\left(\frac{1}{N^2}\right). \tag{3.35}$$

Here, we can see that disconnected diagrams from t and u channel contribute at order of $\frac{1}{N}$ along with s channel connected diagram at large N. Note that connected t and u channel in (3.26) after the projection shift to order of $\frac{1}{N^2}$. Let us inspect the OPE expansion of each term in this expression. The first term contains only the identity operator. At $\frac{1}{N}$, there are two kind of terms, the disconnected contribution or free fields (no interaction) that can written as sum of double trace operators [96],

$$\frac{1}{(P_{13})^{\Delta}(P_{24})^{\Delta}}+\frac{1}{(P_{14})^{\Delta}(P_{23})^{\Delta}}=\frac{1}{(P_{12})^{\Delta}(P_{34})^{\Delta}}\sum_{\substack{\ell,n\\ \ell:\text{ even}}}2c_{n,\ell}^{2}\mathcal{K}_{2\Delta+2n+\ell,\ell}(z,\bar{z}),\qquad(3.36)$$

where $\mathcal{K}$ denotes the conformal block and,

$$c_{n,\ell}^{2}=\frac{(-1)^{\ell}\left[\left(\Delta-\frac{d}{2}+1\right)_{n}(\Delta)_{\ell+n}\right]^{2}}{\ell!n!\left(\ell+\frac{d}{2}\right)_{n}(2\Delta+n-d+1)_{n}(2\Delta+2n+\ell-1)_{\ell}\left(2\Delta+n+\ell-\frac{d}{2}\right)_{n}}.\qquad(3.37)$$

The second kind term is $g_{12|34}$, coming from the connected interaction term. The integral expression (3.34) can be simplified by considering the contour integral with closing the contour from below and for that we have to consider poles of this integrand. Now, in (3.34) there are two kinds of poles, one that depend on the coupling parameter λ coming from denominator of the σ propagator i.e. $\lambda^{-1}+2\tilde{B}(\nu)=0$ and the other one is $\frac{d}{2}+i\nu=2\Delta+2n$ (for n a non-negative integer). In the later case note that these poles are coming from the factor in numerator of (3.34) i.e. $\Gamma^{2}\left(\Delta-\frac{d}{4}-\frac{i\nu}{2}\right)$. Since, it is Γ^2, these are double poles and these poles are exactly at the location of scalar double trace primaries for non-interacting theory. Corresponding to these poles, there will be scaling dimension and these scaling dimension can be thought of as representation of energy and we expect that energy states in an interacting theory must be different from non-interacting theory. The only way this would be possible only if contribution from these double poles is cancelled by the first two disconnected / free field terms at the order of $\frac{1}{N}$. But there is a problem now, as mentioned earlier these are double poles while on the side of free field term, these are just simple poles. That means $g_{12|34}$ must have just simple poles overall, and that means that the denominator i.e. $\tilde{B}(\nu)$ must have simple poles at these precise location with corresponding residues that help in cancellation of contribution at these location at large N. This gives us the analytical structure of the unknown function $\tilde{B}(\nu)$. Now we need to ask can there be additional poles in $\tilde{B}(\nu)$?

We know that $\tilde{B}(\nu)$ should be symmetric under the transformation $\nu\to-\nu$ and these will give additional poles corresponding to what we just discussed (double poles) but they will be in the upper half plane so they won't contribute on closing the contour from below. And additionally it is motivated that there will be no other poles in $\tilde{B}(\nu)$ in [74]. The argument is that if there were new poles they will already contribute at one loop correction which will result in new operators in the spectrum and we do not expect this to happen perturbatively, hence there should be no other poles. With the information about poles and expected behaviour at large ν (which will be same as scattering amplitude at large N in flat space), one can uniquely determine the $\tilde{B}(\nu)$ as summation as the following (see [74] for more details):

$$\tilde{B}(\nu) = -\sum_{n=0}^{\infty} \frac{2\Delta + 2n - \frac{d}{2}}{\nu^2 + \left(2\Delta + 2n - \frac{d}{2}\right)^2} \cdot \frac{\left(\frac{d}{2}\right)_n \Gamma(\Delta+n)\Gamma(\Delta+n-\frac{d}{2}+\frac{1}{2})\Gamma(2\Delta+n-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}\Gamma(n+1)\Gamma(\Delta+n+\frac{1}{2})\Gamma(\Delta+n-\frac{d}{2}+1)\Gamma(2\Delta-d+n+1)}. \tag{3.38}$$

This infinite sum can easily be done and the final resulting expression for $\tilde{B}(\nu)$ will be in terms of regularized hypergeometric functions ${}_p\tilde{F}_q[a_1 \dots a_p, b_1 \dots b_q, z]$ given by

$$\begin{aligned}\tilde{B}(\nu) &= \frac{\Gamma(\Delta)\Gamma(\Delta-\frac{d}{2}+\frac{1}{2})\Gamma(2\Delta-\frac{d}{2})}{4(4\pi)^{\frac{d}{2}}} \\ &\times \left(\Gamma\left(\Delta - \frac{d+2i\nu}{4}\right) {}_5\tilde{F}_4\left[\begin{matrix}\left\{\frac{d}{2},\Delta,\Delta-\frac{d}{2}+\frac{1}{2},\Delta-\frac{d+2i\nu}{4},2\Delta-\frac{d}{2}\right\} \\ \left\{\Delta+\frac{1}{2},\Delta-\frac{d}{2}+1,\Delta-\frac{d+2i\nu}{4}+1,2\Delta-d+1\right\}\end{matrix};1\right]\right. \\ &\left.+\Gamma\left(\Delta - \frac{d-2i\nu}{4}\right) {}_5\tilde{F}_4\left[\begin{matrix}\left\{\frac{d}{2},\Delta,\Delta-\frac{d}{2}+\frac{1}{2},\Delta-\frac{d-2i\nu}{4},2\Delta-\frac{d}{2}\right\} \\ \left\{\Delta+\frac{1}{2},\Delta-\frac{d}{2}+1,\Delta-\frac{d-2i\nu}{4}+1,2\Delta-d+1\right\}\end{matrix};1\right]\right).\end{aligned} \tag{3.39}$$

Here, the most important property that we have used is that in AdS, boundary correlation functions manifest conformal symmetry and upon taking singlet projection, disconnected and connected diagrams contribute at the same order which helps us to bootstrap the unknown $\tilde{B}(\nu)$ function.

3.3.3 Different Phases

Like in flat space we have found different phases separated by a CFT, similarly its is found that one has different phases in AdS space as well, and one can do similar things as writing lagrangian in terms of fluctuation as in flat space and higgs mechanism will kick in and now the bubble will consists of massless scalar fields and we can incorporate this information in the bubble found in previous section (3.39) by picking Δ for massless scalar fields i.e. $\Delta = d$. Also, we can write matrix in spectral representation in AdS space corresponding to what we had in flat space in broken phase (3.16) as [74],

$$M(\nu) = \begin{pmatrix} -\frac{1}{\lambda} - 2\tilde{B}(\nu) & 2|\Phi| \\ 2|\Phi| & \nu^2 + \frac{d^2}{4}. \end{pmatrix} \tag{3.40}$$

One can inverse it to find the required propagator of σ field,

$$(M(\nu))^{-1} = \frac{1}{\det M(\nu)} \begin{pmatrix} \nu^2 + \frac{d^2}{4} & -2|\Phi| \\ -2|\Phi| & -\frac{1}{\lambda} - 2\tilde{B}(\nu) \end{pmatrix}. \tag{3.41}$$

Upon analyzing the structure of σ field one finds AdS analogue of flat space resonance in the system. [74]. It has also be noticed that these two separate phases are separated by a CFT [74].

In the upcoming chapter, we will try to apply the same bootstrap technique as in this chapter but to scalar QED and see what results we get.

3.4 O(N) model in dS space

In this section we will present the result of [90], in the context of the $O(N)$ model. The bubble can be written in the spectral representation as follows:

$$B(X,Y) = \int_{-\infty}^{+\infty} d\nu \frac{\nu}{\pi i} \hat{B}(\nu) G_\nu(X,Y) \tag{3.42}$$

Here, $G_\nu(X,Y)$ is the scalar propagator in dS. In [85] this bubble is computed using the rotation to AdS and for the case of dS_3 [90] it is given by

$$\hat{B}(\nu)\big|_{d=2} = \frac{i}{8\pi\nu}\left[\pi - i\coth\left(\pi\nu_\phi\right)\left(\psi\left(-i\nu_\phi + \frac{i\nu}{2} + \frac{1}{2}\right) - \psi\left(i\nu_\phi + \frac{i\nu}{2} + \frac{1}{2}\right)\right)\right]. \tag{3.43}$$

In the above equation, ν_ϕ is related to scaling dimension of scalar propagator in dS space and ψ is the digamma function.

With this bubble, one can calculate the exact propagator of the Hubbard-Stratonovich field at large N as follows (in spectral representation):

$$\langle\sigma(X)\sigma(Y)\rangle = \int_{-\infty}^{+\infty} d\nu \frac{\nu}{\pi i} f_\sigma(\nu) G_\nu(X,Y), \tag{3.44}$$

where,

$$f_\sigma(\nu) = \frac{1}{\frac{1}{\lambda} - 2\hat{B}(\nu)}. \tag{3.45}$$

In [90], by computing the effective potential, it is found that there is no symmetry breaking in dS space unlike what we saw in AdS. As a result only one phase is accessible in dS, namely the gapped symmetry-preserving phase. This suggests that dS behaves effectively as a finite volume from the point of view of symmetry breaking.

Chapter 4

Scalar QED in flat space

4.1 Scalar QED in flat Space

In this section, we will apply large N techniques to scalar QED in flat space [75] and this will help us to calculate observables (in this case, scattering amplitude) beyond standard perturbative techniques which is only valid for weak coupling.

Instead of considering one scalar field for electron, we are considering "N_f" numbers of fields or we can call it "N_f flavors" and the large N_f is still a perturbative technique but instead of doing perturbation in coupling we will be doing perturbation in "$\frac{1}{N_f}$" and in this way results are valid for any finite coupling. This has been done before in case of O(N) model [74] and [73]. Though our computations are valid for general dimensions, in our analysis we will be particularly focused on 3 dimensions of spacetime as in this case we will see different phases of the theory: a Coulomb phase where we see massive scalar fields along with the massless photon field, and a Higgs phase in which the IR excitations are $2N_f - 2$ massless goldstone bosons parametrizing a $\mathbb{CP}^{N_f-1}$ sigma model, while the gauge-field becomes massive. These two phases are separated by a transition point described by a CFT (see the figure 4.1). Note that the continuous symmetries of this theory are a flavor symmetry $SU(N_f)$ rotating the scalar fields, and for integer D a magnetic $U(1)$ $(D-3)$-form symmetry whose conserved current is $\frac{1}{2\pi} \star F$. The lagrangian of our theory is the following (in Euclidean signature):

$$\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} + \left(D_\mu\varphi^a\right)\left(D^\mu\varphi^a\right)^* + m^2\varphi^{a^*}\varphi^a + \frac{\sigma}{\sqrt{N_f}}\left(\varphi^a\varphi^{a^*}\right) - \frac{\sigma^2}{2\lambda} \tag{4.1}$$

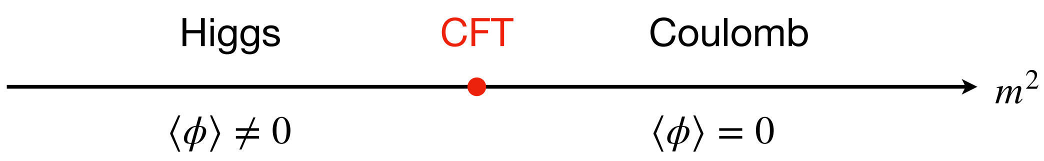


FIGURE 4.1: Different Phase of Scalar QED

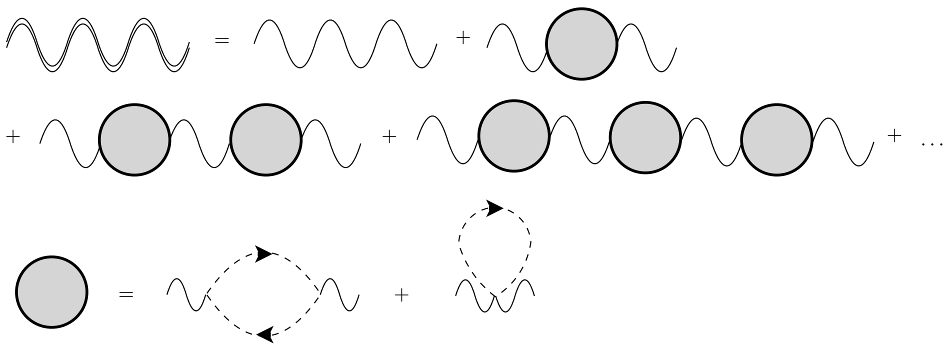


FIGURE 4.2: The double wavy line is the propagator of the photon at leading order $\mathcal{O}(N_f^{-1})$ in the large N_f expansion, and exactly in $\alpha = e^2 N_f$, while the single wavy line is the tree level propagator $\propto e^2$. The grey blob represents the 1-loop 1PI correction to the two-point function that is $\propto N_f$. The dashed line is the propagator of the complex scalars, with the arrow denoting the flow of charge.

The first term in the above equation is usual Maxwell term, in the 2nd term which introduces the interaction between gauge field and scalar field lies the covariant derivative and here we have used the convention $D_\mu = (\partial_\mu + iA_\mu)$, e being the electric charge/coupling and A_μ being the vector field. Note that the index a on the scalar field φ denotes the number of the fields and it runs from 1 to N_f. In the last term, we have introduced the Hubbard-Stratonovich field σ which is an auxillary field [73, 74] i.e. it's equation of motion is not dynamic and is given by $\sigma = \frac{\lambda}{\sqrt{N_f}} \varphi^a \varphi^a$. Integrating σ out one recovers the usual quartic interaction term among scalars with λ as coupling constant.

To obtain the interactions mediated by both the vector field and the Hubbard-Stratonovich field, we need the propagators of both of them in large N_f. These exact propagators in large N_f are a geometric sum of different "bubbles" (1PI diagrams). The exact propagator of the σ field will be same as we saw in previous chapter and for the vector field we will do the resummation as seen in the figure 4.2

Notice that in this exact propagator there will be infinite diagrams to resum and this is the consequence of going beyond standard perturbation theory where we would have just considered the first term in this resummation i.e. the tree level propagator to calculate scattering amplitudes. Also, in this way we keep both the $\alpha = N_f e^2$ and λ finite.

4.2 Bubble in the Flat Space (Coulomb Phase)

In this section we will be explicitly calculating the bubble for the case of scalar QED in general dimensions D. Note that we need the bubble only for the vector field as the bubble for the σ field will remain the same as we saw in last chapter for $O(N)$ model.

As we see in the figure 4.2 we have 2 diagrams to consider while calculating this bubble, one is coming from the cubic interaction and the other is coming from the quartic interaction, combined we can write the required integral as the following in the momentum space by integrating over the internal momenta k,

$$I_{\mu\nu} = N_f \int \frac{d^D k}{(2\pi)^D} \left[\frac{(2k+p)_\mu (2k+p)_\nu}{(k^2+m^2)\left((k+p)^2+m^2\right)} - \frac{2\delta_{\mu\nu}}{k^2+m^2} \right]. \tag{4.2}$$

This integral can be written in terms of two functions denoted by $F_1(p)$ and $F_2(p)$ in momentum space and now we will find the form of these two functions.

$$I_{\mu\nu} \equiv N_f \left[F_1(p) \left(\frac{p_\mu p_\nu - \delta_{\mu\nu}}{p^2} \right) + F_2(p)\delta_{\mu\nu} \right] p^{D-2}. \tag{4.3}$$

First multiply the above two equations with $p_\mu p_\nu$ and then will have,

$$I_1 = I_{\mu\nu} p^\mu p^\nu = N_f \int \frac{d^D k}{(2\pi)^D} \left[\frac{((2k+p)\cdot p)^2}{(k^2+m^2)\left[(k+p)^2+m^2\right]} - \frac{2p^2}{(k^2+m^2)} \right]. \tag{4.4}$$

We can further rewrite this integral using $D1$ and $D2$ which are functions of (k, p, m),

$$\begin{aligned} I_1 &= N_f \int \frac{d^D k}{(2\pi)^D} \left[\frac{D_1^2 + D_2^2 - 2D_1 D_2}{D_1 D_2} - \frac{2p^2}{k^2+m^2} \right], \\ &= N_f \int \frac{d^D k}{(2\pi)^D} \left[\left[\frac{D_1}{D_2} + \frac{D_2}{D_1} - 2 \right] - \frac{2p^2}{k^2+m^2} \right], \end{aligned} \tag{4.5}$$

where $D1 = \left[(k+p)^2 + m^2\right], D2 = \left[k^2+m^2\right]$.

Using the fact that we $\int \frac{D_2}{D_1} = \int \frac{D_1}{D_2}$, we can clearly see that integral I_1 is zero and hence we the function $F_2(p)$ must be zero as I_1 also equates to $I_1 = N_f F_2(p) \cdot p^d$ and we can explicitly see it as follows:

$$\begin{aligned} I_1 &= N_f \int \frac{d^D k}{(2\pi)^D} \left[\frac{2\left(D_1 - D_2\right)}{D_2} - \frac{2p^2}{k^2+m^2} \right], \\ &= N_f \int \frac{d^D k}{(2\pi)^D} \left[\frac{2\left(2k\cdot p + p^2\right)}{k^2+m^2} - \frac{2p^2}{k^2+m^2} \right], \\ &= 0. \end{aligned} \tag{4.6}$$

This is expected behaviour as due to gauge invariance, $I_{\mu\nu}$ must be transverse.

Similarly we can now multiply $I_{\mu\nu}$ by $\delta_{\mu\nu}$ and proceed to calculate the function $F_1(p)$,

$$I_2 = N_f \int \frac{d^D k}{(2k)^D} \left[\frac{(2k+p)^2}{(k^2+m^2)\left((k+p)^2+m^2\right)} - \frac{2D}{k^2+m^2} \right]. \tag{4.7}$$

We can further rewrite the this integral as follows:

$$I_2 = N_f \int \frac{d^D k}{(2k)^D} \left[\frac{(4-2D)}{k^2+m^2} - \frac{(p^2+4m^2)}{(k^2+m^2)\,[(k+p)^2+m^2]} \right] . \tag{4.8}$$

Now the first integral in above expression can be done using dimensional regularization and note that 2nd integral is what has already been computed in [74] for the case of O(N) model. Denoting the scalar bubble by $B^0(p^2, m^2)$ we get,

$$I_2 = N_f \left[\frac{(4-2D)}{(2\pi)^D} (\pi)^{D/2} \left(m^2\right)^{\frac{D}{2}-1} \Gamma\left(1-\frac{D}{2}\right) - \left(p^2+4m^2\right) \cdot B^0(p^2, m^2) \right] . \tag{4.9}$$

When comparing with $I_2 = N_f\,[F_1(1-D)]\,p^{D-2}$, we can find the expression for the function $F_1(p)$. Note that since we have found $F_2(p)$ to be zero, we obtain the following:

$$\begin{aligned} I_{\mu\nu} &= -N_f B^{(1)}(p^2, m^2) \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) , \\ B^{(1)}(p^2, m^2) &= \frac{1}{D-1} \left[\left(p^2+4m^2\right) B^{(0)}(p^2, m^2) - \frac{(4-2D)}{(4\pi)^{\frac{D}{2}}} \left(m^2\right)^{\frac{D}{2}-1} \Gamma\left(1-\frac{D}{2}\right) \right] . \end{aligned} \tag{4.10}$$

We refer to $B^{(1)}(p^2, m^2)$ as the spin 1 bubble function.

With the gauge fixing term $\mathcal{L}_{g.f.} = \frac{1}{2e^2\xi}(\partial_\mu A^\mu)^2$, the tree level gauge propagator can be written as follows:

$$\langle A_\mu(p) A_\nu(-p) \rangle|_{\text{tree}} = \frac{e^2}{p^2} \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) + e^2 \xi \frac{p_\mu p_\nu}{p^4} . \tag{4.11}$$

Now, with the spin 1 bubble we can calculate the exact propagator as follows:

$$\begin{aligned} \langle A_\mu(p) A_\nu(-p) \rangle|_{\frac{1}{N_f}} &= \frac{e^2 \xi p_\mu p_\nu}{p^4} + \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) \left[\frac{e^2}{p^2} + \frac{e^2}{p^2} \cdot \left(-N_f B^{(1)}(p^2, m^2) \right) \cdot \frac{e^2}{p^2} + \ldots + \right] \\ &= \frac{1}{N_f} \left(\frac{\alpha}{p^2 + \alpha B^{(1)}(p^2, M^2)} \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) + \zeta \frac{p_\mu p_\nu}{p^4} \right) \end{aligned} \tag{4.12}$$

Where, $\alpha = e^2 N_f$ is new coupling constant and $\zeta = e^2 N_f \xi$ note that in large N_f limit, α will remain fixed.

4.3 Scattering in the Coulomb phase

The simplest observable to compute is the scattering amplitude of the charged scalars $\phi^{*a}\phi^b \to \phi^{*c}\phi^d$. The contribution from the gauge field is given by the diagrams in figure 4.3 . It is immediate to write down the resulting amplitude using the exact

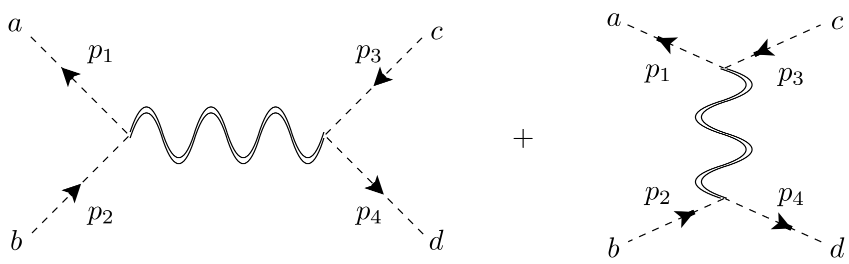


FIGURE 4.3: Diagrams that compute the scattering amplitude at leading order $\mathcal{O}(N_f^{-1})$. The letters denote the $SU(N_f)$ flavor index and the momenta are all ingoing with $p_1 + p_2 + p_3 + p_4 = 0$.

photon propagator

$$\begin{aligned} i\,T_{ab\to cd} &= i\,\frac{1}{N_f}\left(\delta^{ab}\delta^{cd}\,\mathcal{T}(s,t) + \delta^{ac}\delta^{bd}\,\mathcal{T}(t,s)\right)\,, \\ i\mathcal{T}(s,t) &= -\frac{\alpha(s-4M^2+2t)}{s-\alpha B^{(1)}(-s,M^2)}\,. \end{aligned} \tag{4.13}$$

The amplitude is crossing symmetric under simultaneous exchange of the flavor indices b and c and the Mandelstam variables s and t. It also has analytic properties that are expected in the interacting theory, e.g. as a function of complex s for fixed t there is a pole at $s = 0$ due to the photon exchange and a two-particle branch-cut starting at $s = 4M^2$.

It is also interesting to check the unitarity of the amplitude, in particular after projecting to the singlet sector. Note that we can view the amplitude as the matrix element

$$i\,T_{ab\to cd} = \langle c,d|\,i\,T\,|a,b\rangle\,, \tag{4.14}$$

where the asymptotic two-particle states are normalized as

$$\langle a',b'|a,b\rangle = \delta^{aa'}\delta^{bb'}\times(\text{momentum conserving delta's})\,. \tag{4.15}$$

Therefore the unit normalized flavor singlet state is

$$|\mathrm{S}\rangle = \frac{1}{\sqrt{N_f}}\sum_a |a,a\rangle\,, \tag{4.16}$$

and the amplitude in the singlet sector is

$$i\,T_{\mathrm{S}\to\mathrm{S}} = i\,\mathcal{T}(s,t) + \mathcal{O}(N_f^{-1})\,. \tag{4.17}$$

Note that t only appears in the combination $s - 4M^2 + 2t = (s-4M^2)\cos\theta$ in the numerator, where θ is the scattering angle. Therefore the decomposition in partial waves contains only spin $J = 1$. In the normalization of [4] we have

$$f_{J=1}^{S\to S}(s) = -\frac{\pi}{(16\pi)^{\frac{D-1}{2}}\Gamma(\frac{D+1}{2})}\frac{\alpha(s-4M^2)}{s-\alpha B^{(1)}(-s,M^2)}\,. \tag{4.18}$$

Since this projection to the singlet sector is not suppressed by any small parameter, the full non-linear unitarity constraint applies to it, for any α. In fact elastic unitarity is saturated, i.e. for any α and any real $s > 4M^2$ we have

$$2\Im f_1^{S\to S}(s) = 2\frac{(s-4M^2)^{\frac{D-3}{2}}}{\sqrt{s}}|f_1^{S\to S}(s)|^2\,. \tag{4.19}$$

This can be easily checked using the following identity valid for real $s > 4M^2$

$$\operatorname{Im} B^{(0)}(-s,M^2) = -\frac{M^{D-4}}{2^{D+1}\pi^{\frac{D-3}{2}}\Gamma\left(\frac{D-1}{2}\right)}\sqrt{\frac{4M^2}{s}}\left(\frac{s}{4M^2}-1\right)^{\frac{D-3}{2}}\,. \tag{4.20}$$

The fact that at leading order in the $1/N_f$ expansion there is no particle production is a consequence of the fact that we are resumming only one-loop diagrams. Note that there is an additional contribution to the $\phi^{*a}\phi^b \to \phi^{*c}\phi^d$ amplitude from the exchange of the σ field, i.e. from the scalar self-interaction, which however only contributes to the $J = 0$ partial wave and similarly, when projected to the singlet sector, at leading order at large N_f saturates elastic unitarity.

For $D \leq 4$ at subleading order in the $1/N_f$ expansion we expect that IR divergences make the amplitude of the charged particles ill-defined in the Coulomb phase, see e.g. [97] for a recent discussion of IR divergences in sQED with $D = 3$. Therefore one would need to consider some dressing of the asymptotic states, or replace the scattering amplitude with some inclusive observable.

4.4 Higgs Phase

Note that at leading order at large N_f, the Vacuum Expectation Values (VEVs) of the hubbard-stratonovich as well as the scalar fields is unaffected by the presence of gauge fields and thus those results remain same as in the previous chapter but note that there is one small change as there are N_f complex scalar fields or one can say $2N_f$ real fields. So, even in presence of the gauge field, the mass term (now M^2) of scalar field is shifted to $m^2 + \Sigma$. In this phase, while doing the minimization of the effective potential, we do require $M^2 = 0$, i.e. $\Sigma = -m^2$ because of the non-zero VEV of $\phi^{a\,*}\phi^a$. In this case global symmetry $SU(N_f)$ is broken to $SU(N_f - 1)$. Let us expand the fields around the minimum and proceed to find the bubble in this case,

$$\begin{aligned}
\phi^A &= \pi^A \,, \\
\phi^N &= \left(\sqrt{N_f}\Phi + \frac{\rho}{\sqrt{2}}\right) e^{i\frac{\theta}{\sqrt{2N_f}\Phi}} \,, \\
\sigma &= -\sqrt{N_f}\, m^2 + \hat{\sigma} \,.
\end{aligned} \tag{4.21}$$

In the above equation, the values of the index A runs from 1 to $N_f - 1$ and the field fluctuations are denoted by the Goldstone bosons π^A, the radial mode ρ and the Hubbard-Stratonovich field $\hat{\sigma}$. We can expand our lagrangian in terms of these fluctuations as well,

$$\begin{aligned}
\mathcal{L} = & \frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} + N_f\Phi^2 A_\mu A^\mu + (D^\mu \pi^A)(D_\mu \pi^A)^* + \frac{1}{2}\left(\partial_\mu \rho\right)^2 + \frac{1}{2}\left(\partial_\mu \theta\right)^2 \\
& + \sqrt{N_f}\left(\frac{m^2}{\lambda} + \Phi^2\right)\hat{\sigma} - \frac{\hat{\sigma}^2}{2\lambda} + \sqrt{2}\,\Phi\,\hat{\sigma}\,\rho + \frac{\hat{\sigma}}{\sqrt{N_f}}\left(\frac{1}{2}\rho^2 + \pi^A\pi^{A\,*}\right) \\
& + \sqrt{2N_f}\,\Phi\, A_\mu\left(\partial^\mu\theta\right) + \sqrt{2N_f}\Phi\,\rho\, A_\mu A^\mu + 2\rho A_\mu\left(\partial^\mu\theta\right) + \frac{1}{2}\rho^2 A_\mu A^\mu \\
& + \frac{1}{\sqrt{2N_f}\,\Phi}\rho^2 A_\mu\left(\partial^\mu\theta\right) + \frac{1}{\sqrt{2N_f}\,\Phi}\rho\left(\partial_\mu\theta\right)^2 + \frac{1}{4N_f\Phi^2}\,\rho^2\left(\partial_\mu\theta\right)^2 \,.
\end{aligned} \tag{4.22}$$

Note that now the terms containing interaction with the gauge field and that are relevant at leading order are same as the coulomb case (ϕ^a being replaced by π^A) except for the term with $A_\mu \partial^\mu \theta$ and the term that is responsible for the photon to get massive i.e. $m_A^2 = 2e^2 N_f \Phi^2$. We can remove the first term by modifying our gauge fixing term as the following,

$$\mathcal{L}_{g.f.} = \frac{N_f}{2\zeta}\left(\partial_\mu A^\mu + \sqrt{\frac{2}{N_f}}\,\zeta\,\Phi\,\theta\right)^2 \,. \tag{4.23}$$

Note that this new term did not introduce any other relevant interaction containing the gauge field at leading order at large N_f. Hence in summary, in the Higgs case the photon gets massive and scalar fields become goldstone bosons (massless). Using this result in the coulomb phase bubble, we can get the expression for the bubble in the higgs phase where now the loop is of $N_f - 1$ massless scalars,

$$B^{(1)}(p^2, 0) = -\frac{\pi}{(16\pi)^{\frac{D-1}{2}}\,\Gamma\left(\frac{D+1}{2}\right)\sin\left(\frac{\pi D}{2}\right)}(p^2)^{\frac{D-2}{2}} \,. \tag{4.24}$$

The massive tree level propagator is given by,

$$\begin{aligned}\langle A_\mu(p)A_\nu(-p)\rangle|_{\text{mass-tree}} &= \frac{e^2}{(p^2+m_A^2)}\left[\delta_{\mu\nu} - \frac{(1-\xi)p_\mu p_\nu}{m_A^2\xi + p^2}\right],\\ &= \frac{e^2}{(p^2+m_A^2)}\left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) + S_{\mu\nu},\\ S_{\mu\nu} &= \frac{e^2}{(p^2+m_A^2)}\left[\frac{p_\mu p_\nu}{p^2} - \frac{(1-\xi)p_\mu p\nu}{m_A^2\xi + p^2}\right] = \frac{e^2\xi p_\mu p_\nu}{p^2\left(p^2+m_A^2\xi\right)}.\end{aligned} \tag{4.25}$$

And thus the resummed propagator will be,

$$\begin{aligned}\langle A_\mu(p)A_\nu(-p)\rangle|_{\text{exact}} &= S_{\mu\nu}\\ &+ \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\left[\frac{e^2}{p^2+m_A^2} + \frac{e^2}{p^2+m_A^2}\cdot\left(-N_f B^{(1)}(p^2,0)\right)\cdot\frac{e^2}{(p^2+m_A^2)} + \dots\right],\\ &= \frac{e^2\xi k_\mu k_\nu}{k^2\left(k^2+m_A^2\xi\right)} + \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\frac{e^2}{(p^2+m_A^2) + \alpha B^{(1)}(p^2,0)}.\end{aligned} \tag{4.26}$$

The bubble diagram at $M^2 = 0$ reads

$$B^{(1)}(p^2,0) = -\frac{\pi}{(16\pi)^{\frac{D-1}{2}}\Gamma\left(\frac{D+1}{2}\right)\sin\left(\frac{\pi D}{2}\right)}(p^2)^{\frac{D-2}{2}}. \tag{4.27}$$

Neglecting the bubble the massive photon is a stable particle corresponding to the pole at $p^2 = -m_A^2$. This value of p^2 on the negative real axis is precisely on the branch-cut of the power appearing in the bubble function. For $2 < D < 4$ and any α the actual pole of the exact propagator is for complex values of p^2 and not in the first sheet.[1] The massive photon becomes a resonance, as expected given that it can decay to pions.

An observable in the Higgs phase is the scattering amplitude of the pions, that can be computed at leading order at large N_f with the same techniques showed in section 4.3. Like we saw in the Coulomb phase, this amplitude is directly determined by the exact propagator of the photon, and therefore in this case it will contain a spin 1 resonance. Like in the Coulomb phase, the $J = 1$ partial amplitude in the singlet sector saturates elastic unitarity. An important difference with the Coulomb phase is that in the Higgs phase we do not expect any IR divergence, therefore this scattering amplitude remains an interesting observable of the theory also at subleading order in the $1/N_f$ expansion, or at finite N_f.

[1]The equation for the zero of the denominator is $p^2 + C(p^2)^\gamma = -m_A^2$, with $C > 0$ and $0 < \gamma < 1$. On the first sheet $\text{Arg}(p^2) \in (-\pi,\pi)$ and $\text{Arg}(C(p^2)^\gamma) = \gamma\,\text{Arg}(p^2)$. Note that the solution cannot have $\text{Arg}(p^2) = 0$ because in that case both p^2 and $C(p^2)^\gamma$ are positive real numbers. As a result, in the first sheet the arguments of p^2 and $C(p^2)^\gamma$ are either both in $(0,\pi)$ and their sum has a positive imaginary part, or both in $(-\pi,0)$ and their sum has negative imaginary part. In either case their sum cannot equal the real number $-m_A^2$. This proves that the solution is not in the first sheet.

4.5 CFT at the phase transition

For completeness let us now briefly review the evidence that at large N_f there is a second order transition at $m^2 = m_c^2$, namely at $M^2 = 0$ and $|\Phi|^2 = 0$. The photon propagator with this value of the parameters and at leading order in the large N_f expansion is

$$\langle A_\mu(p)A_\nu(-p)\rangle|_{\frac{1}{N_f}} = \frac{1}{N_f}\left(\frac{\alpha}{p^2+\alpha B^{(1)}(p^2,0)}\left(\delta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right)+\zeta\frac{p_\mu p_\nu}{p^4}\right), \tag{4.28}$$

where $B^{(1)}(p^2,0)$ is the power of momentum in eq. (4.27). In the IR limit $(p^2)^{\frac{D-2}{2}} \ll \alpha$ the propagator approaches the α-independent limit

$$\langle A_\mu(p)A_\nu(-p)\rangle|_{\frac{1}{N_f},\mathrm{IR}} = \frac{1}{N_f}\left(\frac{1}{B^{(1)}(p^2,0)}\left(\delta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right)+\zeta\frac{p_\mu p_\nu}{p^4}\right), \tag{4.29}$$

from which we compute the two-point function of the gauge-invariant field strength operator

$$\begin{aligned}\langle \tfrac{1}{2\pi}F_{\rho\mu}(p)\tfrac{1}{2\pi}F_{\sigma\nu}(-p)\rangle|_{\frac{1}{N_f},\mathrm{IR}} &= \frac{C_F}{N_f}\frac{\delta_{\mu\nu}p_\rho p_\sigma-\delta_{\rho\nu}p_\mu p_\sigma-\delta_{\mu\sigma}p_\rho p_\nu+\delta_{\rho\sigma}p_\mu p_\nu}{(p^2)^{\frac{D-2}{2}}}\,,\\ C_F &\equiv -\frac{(16\pi)^{\frac{D-1}{2}}\Gamma\left(\frac{D+1}{2}\right)\sin\left(\frac{\pi D}{2}\right)}{4\pi^3}\,.\end{aligned} \tag{4.30}$$

Note that $C_F > 0$ for $2 < D < 4$. In position space this correlator is

$$\begin{aligned}\langle \tfrac{1}{2\pi}F_{\rho\mu}(x)\tfrac{1}{2\pi}F_{\sigma\nu}(0)\rangle|_{\frac{1}{N_f},\mathrm{IR}} &= \frac{C_F}{N_f}\frac{16}{(4\pi)^{\frac{D}{2}}\Gamma(\frac{D}{2}-1)}\frac{I_{\mu\nu}I_{\rho\sigma}-I_{\rho\nu}I_{\mu\sigma}}{(x^2)^2}\,,\\ I_{\mu\nu} &\equiv \delta_{\mu\nu}-2\frac{x_\mu x_\nu}{x^2}\,.\end{aligned} \tag{4.31}$$

This takes precisely the form of the correlator for a two-form primary operator of scaling dimension $\Delta_F = 2$ in a D dimensional CFT. Note that the unitarity bound for a two-form is $\Delta \geq \max(2, D-2)$, so in the range $2 < D < 4$ this bound is saturated by F, reflecting the existence of the null operator corresponding to the Bianchi identity $dF = 0$. In integer D the hodge dual $\star F$ gives the conserved current for a $(D-3)$-form symmetry, but only in $D = 3$ this conserved current is compatible with the unitarity of the CFT.

At leading order at large N_f, and restricting to local operators, the CFT is the product of two decoupled sectors, the mean field theory of the field strength operator, and the free CFT of the matter fields, restricted to the singlet sector of the $U(1)$ gauge symmetry and with the conserved current removed $J^\mu = 0$. Corrections to the CFT data can be computed systematically in $1/N_f$ expansion, e.g. using a diagrammatic approach with the exact propagator and the standard interaction vertices involving the matter fields. In addition in $D = 3$ there are also local monopole operators, for which the $1/N_f$ expansion is less straightforward but has also been developed, see e.g. [98–100].

Chapter 5

Scalar QED in AdS

5.1 Introduction

In this chapter, we will go through similar calculation as in previous chapter but in AdS space instead of flat space. Scalar QED is a good example to start studying gauge theories in AdS space, later we will try to extend it to the fermionic QED in the chapter 7. Another advantage of studying interactions in AdS is that we can analytically continue them to dS space (see chapter 6). All of this provides good motivation to focus on theories in AdS space.

In this chapter we will present the work done in [75]. Let us recall the lagrangian scalar QED (in Euclidean signature),

$$\mathcal{L} = \frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + m^2 \phi^a \phi^{a*} + \left(D_\mu \phi^a\right)\left(D^\mu \phi^a\right)^* + \frac{\sigma}{\sqrt{N}}\left(\phi^a \phi^{a*}\right) - \frac{\sigma^2}{2\lambda} \tag{5.1}$$

where, ϕ are complex scalar fields, D_μ is covariant derivative for QED, and $F_{\mu\nu}$ is the field strength and e^2 is gauage coupling and λ is quartic coupling for scalar fields. The possible relevant curvature coupling $R\,\phi^a \phi^{a\,*}$ is absorbed in the definition of the coupling m^2. Note that since we are trying to go beyond standard perturbation theory to get result for any finite coupling especially for strong ones, we will employ large N techniques and thus here, we have N_f number of scalar fields and the index a runs from 1 to N_f and we will be doing perturbation in $\frac{1}{N_f}$ and not in the coupling constant. And like in chapter 3, here also we will have a bubble but this time this bubble is made up of complex scalar fields and not real scalar fields. Again, we will not explicitly calculate it like we did in the flat space in previous chapter 4, instead here we will bootstrap it like we saw in the $O(N)$ model, by using conformal consistency conditions and applying analytical bootstrap methods. When calculating the 4-point boundary correlation function, since it is scalar QED, we will have exchange of vector field and we will be using "resummed propagator" to do so. Also note that in the lagrangian, we still have σ (Hubbard-Startonovich) propagator which was there in $O(N)$ model,so there will also be the four point boundary correlation function mediated by resummed σ propagator and calculations will exactly be same apart from some constant factors as in this case we have complex scalar fields and not the real scalar fields. In this chapter we are focused on the contribution from the resummed vector field at large N. We will stay in arbitrary dimensions $D = d+1$, and thus we are setting up the problem in AdS_{d+1} and we are doing it in the embedding formalism introduced in chapter 2

5.2 Resummed Propagator

Like we saw in the last chapter, where we did Scalar QED in flat space, we need the exact or resummed propagator at large N_f in the figure 4.2. In this chapter we will be using spectral representation introduced in chapter 2, to find the expression for the resummed propagator.

Let us first write the tree level photon propagator in terms of spectral representation,

$$\begin{aligned}\langle A_M(X)A_N(Y)\rangle_{\text{pert. theory}} &\equiv G^{(1)}_{MN}(X,Y)\\ &= \int_{-\infty}^{+\infty} d\nu \frac{e^2}{\nu^2+\left(\frac{d}{2}-1\right)^2}\Omega^{(1)}_{\nu MN}(X,Y)+\nabla^X_M\nabla^Y_N L(u).\end{aligned} \tag{5.2}$$

Let us recall the fact that AdS coordinates obey, $X^2 = Y^2 = -1$ and the harmonic functions $\Omega^{(1)}_{\nu MN}(X,Y)$ are the eigen functions of the laplacian operator and they are transverse as well,

$$\begin{aligned}\Box^X\Omega^{(1)}_{\nu MN}(X,Y) &= \left(\nu^2+\frac{d^2}{4}+1\right)\Omega^{(1)}_{\nu MN}(X,Y),\\ \nabla^X_M\Omega^{(1)}_{\nu\ N}(X,Y) &= 0.\end{aligned} \tag{5.3}$$

ν is a variable which is very similar to momentum p in flat space and the second term in the expression of tree level propagator is the gauge dependent part which is function of half of the chordal distance $u = -1 - X\cdot Y$,

$$L(u) = e^2\xi G^{(0)}_d \star G^{(0)}_d(u) = e^2\xi\int_{-\infty}^{+\infty} d\nu \frac{1}{\left(\nu^2+\frac{d^2}{4}\right)^2}\Omega^{(0)}_\nu(u). \tag{5.4}$$

where, ξ is gauge coupling, $\star$ denotes the convolution, $\Omega^{(0)}_\nu(u)$ and $G^{(0)}_d$ are harmonic function and propagator for massless field with $\Delta = d$ for spin 0 respectively.

We can assume the bubble for spin 1 to take the following form:

$$\langle J_M(X)J_N(Y)\rangle = -N_f\int_{-\infty}^{+\infty} d\nu B^{(1)}(\nu)\Omega^{(1)}_{\nu MN}(X,Y). \tag{5.5}$$

Note, that factor of N_f is there because there are N_f no. of complex scalar fields.Now, let us look at how one photon and one bubble will look and we will be using (2.27) (see the figure 5.1),

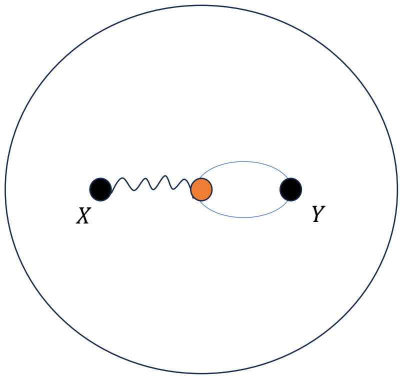


FIGURE 5.1: wavy line is photon propagator and the other object is bubble and they are connected by internal vertex.

$$\int_{-\infty}^{+\infty} d\nu \frac{e^2}{\nu^2 + \left(\frac{d}{2} - 1\right)^2} \cdot (-N_f) B^{(1)}(\nu) \Omega^{(1)}_{\nu MN}(X, Y) + \nabla^X_M \nabla^Y_N L(u). \tag{5.6}$$

Now let us look at two photons connected at either end of bubble which will be the first term in correction to the tree level propagator in the figure 4.2,

$$\int_{-\infty}^{+\infty} d\nu \frac{e^2}{\nu^2 + \left(\frac{d}{2} - 1\right)^2} \cdot (-N_f) B^{(1)}(\nu) \cdot \frac{e^2}{\nu^2 + \left(\frac{d}{2} - 1\right)^2} \Omega^{(1)}_{\nu MN}(X, Y) + \nabla^X_M \nabla^Y_N L(u). \tag{5.7}$$

Similarly, we can write the second term in correction which is made up of 3 photons and 2 bubbles in the figure 4.2,

$$\int_{-\infty}^{+\infty} d\nu \frac{e^2}{\nu^2 + \left(\frac{d}{2} - 1\right)^2} \cdot B^{(1)}(\nu) \cdot \frac{(-N_f) e^2}{\nu^2 + \left(\frac{d}{2} - 1\right)^2} \cdot B^{(1)}(\nu) \cdot \frac{(-N_f) e^2}{\nu^2 + \left(\frac{d}{2} - 1\right)^2} \Omega^{(1)}_{\nu MN}(X, Y) + \nabla^X_M \nabla^Y_N L(u). \tag{5.8}$$

We can continue with the rest of infinite terms in the figure 4.2 and thus, the re-summed propagator will look like,

$$\int_{-\infty}^{+\infty} d\nu \left[\frac{e^2}{\nu^2 + \left(\frac{d}{2}-1\right)^2} + \frac{e^2}{\nu^2 + \left(\frac{d}{2}-1\right)^2} \cdot \frac{(-N_f)e^2 B^{(1)}(\nu)}{\nu^2 + \left(\frac{d}{2}-1\right)^2} \right.$$
$$\left. + \frac{e^2}{\nu^2 + \left(\frac{d}{2}-1\right)^2} \cdot \frac{(-N_f)e^2 B^{(1)}(\nu)}{\nu^2 + \left(\frac{d}{2}-1\right)^2} \cdot \frac{(-N_f)e^2 B^{(1)}(\nu)}{\nu^2 + \left(\frac{d}{2}-1\right)^2} + \ldots \right] \Omega^{(1)}_{\nu MN}(X,Y)$$
$$+ \nabla^X_M \nabla^Y_N L(u). \tag{5.9}$$

Clearly, it is a geometric series and we can sum these infinite terms to get the result as follows:

$$\langle A_M(X) A_N(Y) \rangle_{\text{large } N_f}$$
$$= \frac{1}{N_f} \int_{-\infty}^{+\infty} d\nu \frac{\alpha}{\nu^2 + \left(\frac{d}{2}-1\right)^2 + \alpha B^{(1)}(\nu)} \Omega^{(1)}_{\nu MN}(X,Y) + \nabla^X_M \nabla^Y_N L(u). \tag{5.10}$$

Here,we have introduced the new coupling parameter as $\alpha = e^2 N_f$ which is fixed at the limit of large N_f.

5.3 4-point Function

To calculate boundary 4-point function mediated by the above resummed vector field, we will first write the leading contribution which is given below,

$$\left\langle \phi^a(P_1) \phi^{*b}(P_2) \phi^{*c}(P_3) \phi^d(P_4) \right\rangle \Big|_{O\left(N_f^0\right)}$$
$$= \delta^{ab}\delta^{cd} \frac{1}{(-2P_1 \cdot P_2)^{\Delta} (-2P_3 \cdot P_4)^{\Delta}} + \delta^{ac}\delta^{bd} \frac{1}{(-2P_1 \cdot P_3)^{\Delta} (-2P_2 \cdot P_4)^{\Delta}}. \tag{5.11}$$

Points P_i are the points on boundary and the letter a denotes the index of the field (a runs from 1 to N_f). Also, note that in above equation unlike the $O(N)$ model, there are only 2 channels (with disconnected diagrams), s and t as here the field are complex and not real and as a result in the OPE decomposition we will have double trace operators of any integer spin J, with dimensions $2\Delta + 2n + J$. Scaling dimensions δ of complex scalar fields are related to mass of the complex scalar fields as $m^2 = \Delta(\Delta - d)$. Here if $\Delta > \frac{d}{2} - 1$, there is no conserved spin 1 operator because the boundary theory is non-local.

We will consider next contribution at large N_f which will also contain s and t channel contribution only but this time we have connected diagrams mediated by a vector field whose expression is as follows:

$$\left\langle \phi^a(P_1)\,\phi^{*b}(P_2)\,\phi^{*c}(P_3)\,\phi^d(P_4)\right\rangle\Big|_{O\left(N_f^{-1}\right)} = \delta^{ab}\delta^{cd}g_{12|34} + \delta^{ac}\delta^{bd}g_{13|24}. \tag{5.12}$$

for,

$$\begin{aligned} g_{ij|kl} = \frac{1}{N_f}\int_{-\infty}^{+\infty} d\nu \frac{\alpha}{\nu^2 + \left(\frac{d}{2}-1\right)^2 + \alpha B^{(1)}(\nu)} \\ 4\int_{X,Y} K_\Delta(P_i,X)\, i\nabla_M^X K_\Delta(P_j,X)\, K_\Delta(P_k,Y)\,(-i)\nabla_N^Y K_\Delta(P_l,Y)\,\Omega_\nu^{(1)MN}(X,Y). \end{aligned} \tag{5.13}$$

where, we have used $\int_{X,Y} = \int d^{d+1}X \int d^{d+1}Y$ to denote integral over X and Y bulk points and let us recall that $K_\Delta(P,X)$ represent bulk to boundary propagator for spin 0 whose expression is as follow (as we discussed in chapter 2),

$$\begin{aligned} K_\Delta(P,X) &= \frac{\sqrt{\mathcal{C}_\Delta}}{(-2X\cdot P)^\Delta}, \\ \mathcal{C}_\Delta &\equiv \frac{\Gamma(\Delta)}{2\pi^{\frac{d}{2}}\Gamma\left(\Delta-\frac{d}{2}+1\right)}. \end{aligned} \tag{5.14}$$

We should remember that our theory is gauge invariant and hence in calculating observables such as boundary 4-point correlation functions, only the transverse part of the bulk to bulk photon propagator contributes as the longitudinal part is gauge dependent and hence unphysical. We will now calculate the integral in the $g_{ij|kl}$ as [77]

$$\begin{aligned} &\int_{X,Y} K_\Delta(P_i,X)\, i\nabla_M^X K_\Delta(P_j,X)\, K_\Delta(P_k,Y)\,(-i)\nabla_N^Y K_\Delta(P_l,Y)\,\Omega_\nu^{(1)MN}(X,Y) \\ &= \frac{1}{\left(-2P_i\cdot P_j\right)^\Delta\left(-2P_k\cdot P_l\right)^\Delta}\frac{1}{8\pi^{\frac{d}{2}}\Gamma(\Delta)^2\Gamma\left(1-\frac{d}{2}+\Delta\right)^2}\mathcal{F}^{(1)}_{\frac{d}{2}+i\nu}(u,v), \\ &= \frac{1}{\left(-2P_i\cdot P_j\right)^\Delta\left(-2P_k\cdot P_l\right)^\Delta}\frac{1}{8\pi^{\frac{d}{2}}\Gamma(\Delta)^2\Gamma\left(1-\frac{d}{2}+\Delta\right)^2}\left(C_\nu\mathcal{K}^{(1)}_{\frac{d}{2}+i\nu}(u,v) + (\nu\to-\nu)\right). \end{aligned} \tag{5.15}$$

Here, first we used the split representation for $\Omega_\nu^{(1)}(X,Y)$ and we split it into bulk to boundary spin-1 propagator and then we pair up the three bulk to boundary propagators to give a boundary 3-point correlation function and in this way we have two 3-point boundary correlation function. Note that these two boundary 3-point correlation functions are then integrated over a common point on the boundary. The last integral is done using conformal partial wave $\mathcal{F}^{(1)}_{\frac{d}{2}+i\nu}(u,v)$ (u and v are conformal invariant cross ratios) for spin-1 which was further divided into conformal blocks of spin-1 $\mathcal{K}^{(1)}_{\frac{d}{2}+i\nu}(u,v)$ with and the coefficient C_ν is given by,

$$C_\nu \equiv \frac{\Gamma\left(\frac{d}{4}+\frac{i\nu}{2}+\frac{1}{2}\right)^4 \Gamma\left(-\frac{d}{4}+\Delta \pm \frac{i\nu}{2}+\frac{1}{2}\right)^2}{2\pi\left(\frac{d}{2}+i\nu-1\right)\Gamma(i\nu)\Gamma\left(\frac{d}{2}+i\nu+1\right)}. \tag{5.16}$$

Finally we have 4-point connected boundary correlation function given by,

$$\begin{aligned} g_{ij|kl} = & \frac{1}{N_f}\frac{1}{(-2P_i\cdot P_j)^\Delta(-2P_k\cdot P_l)^\Delta}\frac{1}{2\pi^{\frac{d}{2}}\Gamma(\Delta)^2\Gamma\left(1-\frac{d}{2}+\Delta\right)^2} \\ & \int_{-\infty}^{+\infty} d\nu \frac{\alpha}{\nu^2+\left(\frac{d}{2}-1\right)^2+\alpha B^{(1)}(\nu)}\left(C_\nu \mathcal{K}^{(1)}_{\frac{d}{2}+i\nu}(u,v)+(\nu\to-\nu)\right). \end{aligned} \tag{5.17}$$

In the above expression, one can note that for the case of $\alpha << 1$, we can neglect the bubble contribution and it will become boundary 4-point function in perturbation theory with exchange via tree level photon propagator. In this case, clearly, there is a pole at $\nu = \pm i\left(\frac{d}{2}-1\right)$ and this correspond to having a conserved current operator in the OPE. In case of finite α or strong coupling, there can still be a conserved current operator in OPE if the pole is still at that location and for that, we need the following condition,

$$B^{(1)}\left(\pm i\left(\frac{d}{2}-1\right)\right) = 0. \tag{5.18}$$

5.4 Bootstrapping the Bubble

In the previous two sections, we have used spectral representation to find the resummed propagator and split representation of the Ω_ν to find the expression of the 4-point function in terms of the bubble. Let us recall that this bubble in not known like in the case of flat space in the previous chapter 4, so the only thing left is to find the expression for the bubble. Like we saw in the $O(N)$ model in chapter 3 [74]. Here we will not directly compute the bubble but instead we will bootstrap it using self consistency condition for the boundary 4-point function. First we will take the singlet projection of the full 4-point function by contracting it with the $\frac{1}{N_f^2}\delta^{ab}\delta^{cd}$,

$$\begin{aligned} \frac{1}{N_f^2}\left\langle \phi^a(P_1)\phi^{*a}(P_2)\phi^{*b}(P_3)\phi^b(P_4)\right\rangle = & \frac{1}{(-2P_1\cdot P_2)^\Delta(-2P_3\cdot P_4)^\Delta} \\ & + \frac{1}{N_f}\left(\frac{1}{(-2P_1\cdot P_3)^\Delta(-2P_2\cdot P_4)^\Delta}+g_{12|34}\right)+\mathcal{O}\left(N_f^{-2}\right). \end{aligned} \tag{5.19}$$

Here, we again notice that upon taking singlet projection we have only s channel disconnected diagram at the leading order and t channel disconnected diagram now contribute at the order of $\frac{1}{N_f}$ along with the connected s channel diagram mediated by a photon.Also, the t channel connected diagram shifted to the order of $\frac{1}{N_f^2}$. This fact will help us to bootstrap the unknown bubble. Let us further analyse the expression, the first term is just simply identity operator. The second term contains two contribution, the disconnected or the free field (no interaction) one can be expanded into conformal blocks $\mathcal{K}_{2\Delta+2n+J}$ with coefficients proportional to $c_{n,J}^2$,

$$c_{n,J}^2 = \frac{2^J \left(\Delta - \frac{d}{2} + 1\right)_n^2 (\Delta)_{n+J}^2}{J!n! \left(J + \frac{d}{2}\right)_n (2\Delta + n - d + 1)_n (2\Delta + 2n + J - 1)_J \left(2\Delta + n + J - \frac{d}{2}\right)_n}. \tag{5.20}$$

Now let us look at the second term at order of $\frac{1}{N_f}$, there are two kind of poles in $g_{12|34}$, double poles at $\nu = \nu_n^\pm \equiv \pm i\left(2\Delta + 2n + 1 - \frac{d}{2}\right)$ (for n non-negative integer)coming from $\Gamma\left(-\frac{d}{4} + \Delta \pm \frac{i\nu}{2} + \frac{1}{2}\right)^2$. In ordinary perturbation theory $\alpha \ll 1$, the term involving the function $B^{(1)}(\nu)$ in the denominator can be neglected, and these double poles have the effect of producing an $\mathcal{O}(\alpha)$ anomalous dimension for the spin 1 double-trace operators of dimension $2\Delta + 2n + 1$. The other poles are α dependent coming from the denominator $\nu^2 + \left(\frac{d}{2} - 1\right)^2 + \alpha B^{(1)}(\nu)$ of the exact photon propagator. Now in an interacting theory we expect the scaling dimensions of the spin 1 double trace operators to be different from the non-interacting theory and this would make sense only if the disconnected term at $\frac{1}{N_f}$ is getting cancelled by a negative contribution from $g_{12|34}$. But as we have noticed there are double poles in $g_{12|34}$ while on the side of free field term, there are just simple poles. That means $g_{12|34}$ must have just simple poles overall, and that means that the denominator i.e. $B^{(1)}(\nu)$ must have simple poles at these precise location with corresponding residues that help in cancellation of contribution at these location at large N_f. This gives us the following condition,

$$c_{n,J=1}^2 = 2\pi i \operatorname{Res}\left[2\frac{1}{2\pi^{\frac{d}{2}}\Gamma(\Delta)^2\Gamma\left(1 - \frac{d}{2} + \Delta\right)^2} \frac{\alpha}{\nu^2 + \left(\frac{d}{2} - 1\right)^2 + \alpha B^{(1)}(\nu)} C_\nu\right]\Bigg|_{\nu=\nu_n^-}, \tag{5.21}$$

Near the pole, bubble will behave as,

$$B^{(1)}(\nu) \underset{\nu\sim\nu_n^-}{\sim} \frac{b_n^{(1)}}{i\left(\nu - \nu_n^-\right)} + \dots \tag{5.22}$$

and we can solve the (5.21) to give,

$$b_n^{(1)} = \frac{\Gamma\left(\frac{d}{2}+n+1\right)\Gamma(n+\Delta+1)\Gamma\left(-\frac{d}{2}+n+\Delta+\frac{1}{2}\right)\Gamma\left(-\frac{d}{2}+n+2\Delta+1\right)}{(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}+1\right)\Gamma(n+1)\Gamma\left(n+\Delta+\frac{3}{2}\right)\Gamma\left(-\frac{d}{2}+n+\Delta+1\right)\Gamma(-d+n+2\Delta+1)}. \tag{5.23}$$

In ordinary perturbation theory, i.e. in an expansion in α, the bubble function appears in the numerator as a loop correction to the photon exchange diagram. At any finite order in perturbation theory we cannot have new operators appearing in the spectrum, but rather we can only generate a series of corrections to the OPE data of the GFF theory. As a result, the singularities in (5.22) are the only singularities of $B^{(1)}(\nu)$ in the complex plane.

If in addition the function $B^{(1)}(\nu)$ would decay at infinity in the complex plane, by a simple contour argument the function would be uniquely fixed in terms of the location of the poles and the residues. However in generic dimension $B^{(1)}(\nu)$ does not decay, and this manifests in a divergence of the sum over poles with the prescribed residues. This is due to (bulk) UV divergences in the loop that computes the bubble. The summand (symmetrized under $\nu \to -\nu$) behaves as

$$B_n^{(1)}(\nu) \equiv \frac{2i\nu_n^- b_n^{(1)}}{\nu^2-(\nu_n^-)^2} \underset{n\to\infty}{\sim} -\frac{n^{d-2}}{(4\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2}+1)}(1+\dots)\,, \tag{5.24}$$

where the dots denote a series of $1/n$ corrections, such that the coefficients of the $1/n^{2k}$ and $1/n^{2k+1}$ corrections are (even) polynomials in ν of degree ν^{2k}. In any d we can make the sum convergent by subtracting sufficient terms, say $2m$, in the Taylor expansion of the summand $B_n^{(1)}(\nu)$ around $\nu = 0$, which contains only even powers. After resumming the resulting convergent series, we account for the subtraction by adding a polynomial in ν of degree $2m$ with arbitrary coefficients.[1]

From the structure described above we see that no subtraction is needed only for $d < 1$. In the more interesting range $1 \le d < 3$ we have (in [88], expression for

[1]The same procedure can be derived from the fact that $B^{(1)}(\nu)$ behaves at infinity as

$$B^{(1)}(\nu) \underset{|\nu|\to\infty}{\propto} |\nu|^{d-1}\,. \tag{5.25}$$

This growth implies that the contour argument determines $B^{(1)}(\nu)$ when $d < 1$, while for $1 \le d < 3$ we can only use it to determine $B^{(1)}(\nu) - B^{(1)}(0)$, for $3 \le d < 5$ we can only use it to determine $B^{(1)}(\nu) - B^{(1)}(0) - \frac{1}{2}\nu^2 B^{(1)''}(0)$, and so on. The behavior for $\nu \to \infty$ corresponds to the flat space limit in momentum space with $\nu \sim Lp$, see the appendix C, and therefore can be fixed by computing the behavior of the spin 1 bubble in $\mathbb{R}^{d+1}$. However this limit also requires to scale $\Delta \sim Lm \to \infty$. Since we cannot prove that the leading power in the growth at large ν does not depend on Δ (though a posteriori this will turn out to be true) we prefer to use the argument based on the structure of the series in the main text.

bubble as sum of poles can also be found but it is not regularized.)

$$B^{(1)}(\nu)|_{1\leq d<3} = \sum_{n=0}^{\infty}\left[B_n^{(1)}(\nu) - B_n^{(1)}(0)\right] + a_0\,, \tag{5.26}$$

where the infinite sum is now convergent, but a_0 is a constant that remains undetermined. We have the further constraint (5.18) coming from the condition of gauge invariance, and we can use it to fix a_0. We get

$$B^{(1)}(\nu)|_{1\leq d<3} = \sum_{n=0}^{\infty}\left[B_n^{(1)}(\nu) - B_n^{(1)}(i\left(\tfrac{d}{2}-1\right))\right]\,. \tag{5.27}$$

Note that, if one computes the loop with a choice of regulator, the coefficient a_0 is UV divergent if the regulator does not preserve gauge invariance (e.g. a sharp cutoff). The coupling that reabsorbs the UV divergence in a_0 is in fact the mass of the gauge field. In the range $3 \leq d < 5$ we need to perform one more subtraction to get a convergent sum

$$B^{(1)}(\nu)|_{3\leq d<5} = \sum_{n=0}^{\infty}\left[B_n^{(1)}(\nu) - B_n^{(1)}(0) - \frac{\nu^2}{2}B_n^{(1)''}(0)\right] + a_0 + a_1(\nu^2 + (\tfrac{d}{2}-1)^2)\,. \tag{5.28}$$

and there are two undetermined constants $a_{0,1}$. We can again impose the gauge-invariance condition (5.18) to fix a_0, obtaining

$$\begin{aligned} B^{(1)}(\nu)|_{3\leq d<5} = &\sum_{n=0}^{\infty}\left[B_n^{(1)}(\nu) - B_n^{(1)}(i\left(\tfrac{d}{2}-1\right)) - \frac{\nu^2+(\frac{d}{2}-1)^2}{2}B_n^{(1)''}(0)\right] \\ &+ a_1(\nu^2+(\tfrac{d}{2}-1)^2)\,. \end{aligned} \tag{5.29}$$

However in this case the coefficient a_1 remains undetermined, and in fact computing explicitly the loop with a UV regulator one would find that a_1 is UV divergent, even with a gauge-invariant regulator. The UV divergence in a_1 is reabsorbed in the gauge coupling, which indeed is finite in the range $1 \leq d < 3$ but needs to be renormalized in the range $3 \leq d < 5$. More subtractions are needed if d is further increased.

The final sums simplify when d is even. For $d = 2$ (i.e. AdS_3) we obtain

$$B^{(1)}(\nu)|_{d=2} = \frac{\nu\left[-2(2\Delta-3)\nu + (\nu^2+4(\Delta-1)^2)\left(i\,\psi\left(\Delta-\frac{i\nu}{2}\right) - i\,\psi\left(\Delta+\frac{i\nu}{2}\right)\right)\right]}{16\pi\left(\nu^2+1\right)}\,, \tag{5.30}$$

where $\psi(x)$ denotes the digamma function. For $d = 4$ (i.e. AdS_5) we obtain

$$\begin{aligned} B^{(1)}(\nu)|_{d=4} = &\frac{\nu^2+1}{2048\pi^2\nu\left(\nu^2+4\right)}\left[(\Delta-1)(2\Delta-5)(2\Delta-7)\nu\left(4-3\nu^2\right)\right. \\ &\left.-4\left(\nu^2+(2\Delta-3)^2\right)\left(\nu^2+(2\Delta-5)^2\right)\left(i\,\psi\left(\Delta-\tfrac{i\nu}{2}-\tfrac{1}{2}\right) - i\psi\left(\Delta+\tfrac{i\nu}{2}-\tfrac{1}{2}\right)\right)\right] \\ &+ \tilde{a}_1(\nu^2+1)\,, \end{aligned} \tag{5.31}$$

where the tilde denotes that we reabsorbed a Δ-dependent constant in the undetermined coefficient. As a check of the result, we can compute the flat-space limit by taking $\nu = Lp$, p being the modulus of the momentum in flat space, and $\Delta = Lm$, and sending $L \to \infty$, see the appendix C. We find that the AdS results approach the flat space answer for the bubble computed with a dimreg regulator in eq. (4.10), up to a polynomial in the momentum when $d \geq 3$ which reflects the ambiguity in the choice of the regulator (which we left unspecified in AdS).

We can also write the sum in generic d in terms of a generalized hypergeometric function. In the rest of the paper we focus on the range $1 \leq d < 3$ in which case the expression reads

$$
\begin{aligned}
&B^{(1)}(\nu)\Big|_{1\leq d<3} = \mathcal{B}(\nu) - \mathcal{B}\left(i\left(\frac{d}{2}-1\right)\right),\\
&\mathcal{B}(\nu) \equiv \frac{\pi^{\frac{1-d}{2}}2^{-2\Delta}\nu^2\Gamma(\Delta+1)\Gamma\left(2\Delta-\frac{d}{2}+1\right)}{\left(\Delta-\frac{d}{4}+\frac{1}{2}\right)\Gamma\left(\Delta+\frac{3}{2}\right)\Gamma\left(\Delta-\frac{d}{2}+1\right)^2\left(\nu^2+4\left(\Delta-\frac{d}{4}+\frac{1}{2}\right)^2\right)}\\
&{}_7F_6\left[\begin{matrix}\frac{d}{2}+1,\Delta+1,\Delta-\frac{d}{2}+\frac{1}{2},2\Delta-\frac{d}{2}+1,\Delta-\frac{d}{4}+\frac{1}{2},\Delta-\frac{d}{4}+\frac{i\nu}{2}+\frac{1}{2},\Delta-\frac{d}{4}-\frac{i\nu}{2}+\frac{1}{2}\\ \Delta+\frac{3}{2},\Delta-\frac{d}{2}+1,2\Delta-d+1,\Delta-\frac{d}{4}+\frac{3}{2},\Delta-\frac{d}{4}+\frac{i\nu}{2}+\frac{3}{2},\Delta-\frac{d}{4}-\frac{i\nu}{2}+\frac{3}{2}\end{matrix};1\right].
\end{aligned}
\tag{5.32}
$$

Having fixed the form of the exact propagator of the photon, we now study the physical observables that we can extract from it in the various phases of the theory.

5.5 Coulomb Phase in AdS

For $m^2 \geq m^2_{c,1}$ there is stable minimum of the AdS effective potential at $\phi^a = 0$ and with a non-zero expectation value $\propto \Sigma$ for the Hubbard-Stratonovich field, which gives a physical mass-squared $M^2 = m^2 + 2\Sigma$ to the scalar fluctuations above the Breitenlohner-Freedman (BF) bound [101]. The analysis of the effective potential at leading order at large N_f is not affected by the gauge field and therefore we will not repeat it here but simply refer to the case of the self-interacting scalars [74], from which one can also read off the (scheme-dependent) value of $m^2_{c,1}$. This phase of the gauge theory is the Coulomb phase, in which the photon mediates a long-range force between the scalars. We concentrate in the range $1 \leq d < 3$ in which the theory is strongly-coupled at large distances. We assume Dirichlet boundary conditions for the gauge field.

The observables we will consider are the scaling dimensions of the spin 1 boundary operators that are exchanged in the connected four-point of the charged operators, at the leading order in the $1/N_f$ expansion. Equivalently, these are the operators that appear in the boundary channel expansion of the bulk two-point function of the gauge field. Setting the AdS scale $L = 1$, they depend on two parameters, the

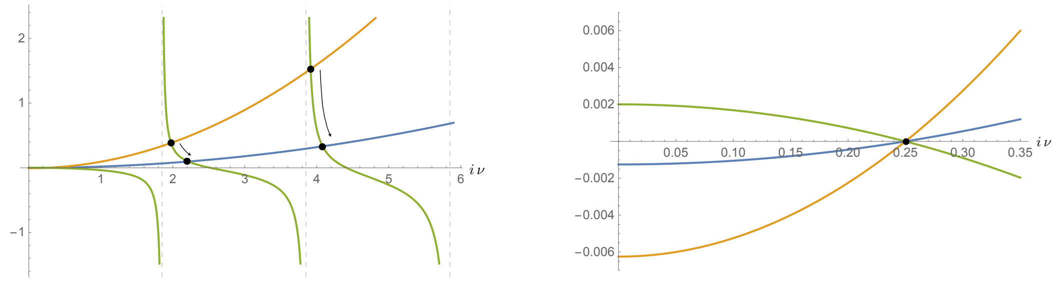


FIGURE 5.2: On the left: Plot of $B^{(1)}(\nu)$ (green line) on the negative imaginary ν axis, together with $-\frac{1}{\alpha}(\nu^2 + (\frac{d}{2}-1)^2)$ for $\alpha = 10$ (orange curve) and $\alpha = 50$ (blue curve) in units of the AdS radius. We have taken $d = 5/2$ and $\Delta = 21/20$ for the external operators. The intersections are highlighted with black dots, the corresponding values of ν give the scaling dimensions of spin 1 operators via $\Delta = \frac{d}{2} + i\nu$. The arrows denote the direction of increasing coupling constant α. The dashed vertical lines correspond to the spin 1 double-trace operators in the free theory. On the right: Zoom of the previous plot near the origin. There all the curves $-\frac{1}{\alpha}\left(\nu^2 + \left(\frac{d}{2}-1\right)^2\right)$ for any α intersect $B^{(1)}(\nu)$ at the point $i\nu = \frac{d}{2} - 1$. The corresponding operator is the conserved current of the global symmetry.

gauge coupling α and the mass-squared M^2, which we will trade with the scaling dimension Δ of the boundary charged operator.

5.5.1 $1 \leq d < 3\,,\ d \neq 2$: scaling dimensions from weak to strong coupling

We first consider $d \neq 2$ in order to regulate the IR divergence that appears in AdS_3. The spectrum of the spin 1 boundary operators is determined by the poles of the exact propagator (5.10) in the complex ν plane, i.e. by the zeroes of the denominator

$$\frac{1}{\alpha}\left(\nu^2 + \left(\frac{d}{2}-1\right)^2\right) + B^{(1)}(\nu) = 0\,. \tag{5.33}$$

The solutions $\{\nu_n^*\}_{n\geq 0}$ are located on the negative imaginary ν axis, and they correspond to the exchange of an operator with scaling dimension $\Delta_n = \frac{d}{2} + i\nu_n^*$ (since the function is symmetric under $\nu \to -\nu$ we could equivalently look at the positive imaginary axis). We cannot find a closed form expression for these solutions, but we can easily visualize them by plotting the function $B^{(1)}(\nu)$ in eq. (5.32) as a function of ν along the negative imaginary axis, for a given Δ and d. This is showed in figure 5.2. We see that for small values of α the scaling dimensions approach their minimal values $\Delta_n^{(0)} = 2\Delta + 2n + 1$ which are just the values in the free theory. The anomalous dimension increases monotonically as a function of the coupling, with no level crossing. The maximum value of Δ_n is still separated by a gap from $\Delta_{n+1}^{(0)}$ and is reached in the limit $\alpha \to \infty$, corresponding to the zeroes of $B^{(1)}(\nu)$. In addition to this tower of solutions, for any value of α there is also an additional zero at $i\nu^* = \frac{d}{2} - 1$ which corresponds to the conserved current, as follows from the condition of gauge invariance (5.18) that we used to fix the bubble function.

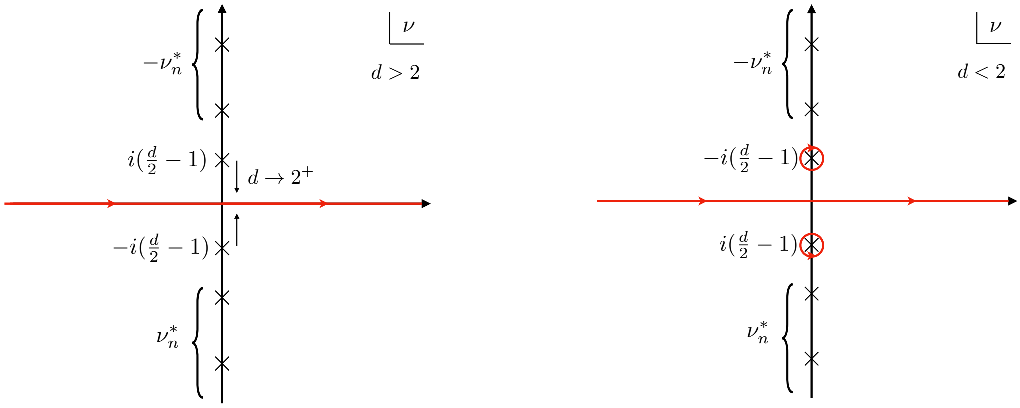


FIGURE 5.3: Poles (crosses) and integration contour (red curve) of the spectral representation of the photon propagator in the complex ν plane. The poles at $\pm\nu_n^*$ gives the finite coupling version of the double-trace operators arising from the matter fields. The poles at $\pm i(\frac{d}{2}-1)$ give the conserved current. For $d < 2$ they cross the contour and we need to add to the contour two circles surrounding them.

Note that as we go from $d > 2$ to $d < 2$ the conserved current poles in the upper- and lower-half ν plain cross the integration contour on the real ν axis, as illustrated in figure 5.3. As a result to ensure continuity in d the contour needs to be changed by adding circles surrounding these two poles, similarly to what is done for scalar AdS propagators with alternate boundary conditions. This is related to the fact that the two boundary modes of the vector field in AdS

$$A_\mu \underset{z\to 0}{\sim} z^{d-2} e^2 j_\mu + a_\mu + \dots , \tag{5.34}$$

exchange dominance as we go from $d > 2$ to $d < 2$. Here we are using Poincaré coordinates (z, x^μ), $\mu = 1, \dots, d$, with boundary at $z = 0$, j_μ denotes the boundary conserved current and a_μ the boundary gauge field. The dots denote subleading contributions from descendants, and also from higher dimensional operators when the gauge field is coupled to matter. The Dirichlet boundary condition sets $a_\mu = 0$.

The contribution from the piece of the contour surrounding the pole naively requires evaluating $\Omega^{(1)}_{\nu\, MN}$ at $\nu = i(\frac{d}{2}-1)$, however the harmonic function itself is singular there, it has a single pole. One should then evaluate the residue at the resulting double pole, but alternatively we observe that the residue of $\Omega^{(1)}_{\nu\, MN}$ is longitudinal, namely

$$\Omega^{(1)}_{\nu\, MN}(X,Y) \underset{\nu\to i(\frac{d}{2}-1)}{\sim} \frac{\nabla^X_M \nabla^Y_N F(u)}{\nu - i(\frac{d}{2}-1)} + \bar{\Omega}^{(1)}_{MN}(X,Y)\,. \tag{5.35}$$

Recalling that the ν integral computes the two-point function of the gauge field, thanks to gauge-invariance we can ignore the longitudinal piece and simply consider the finite term denoted as $\bar{\Omega}^{(1)}_{MN}(X,Y)$.

5.5.2 $d = 2$: IR divergence and breaking of conformal invariance

In the limit $d \to 2$ the poles associated to the conserved current pinch the contour, see figure 5.3, making the propagator of the photon with Dirichlet boundary condition singular in AdS_3. This singularity arises in the spectral representation of the photon propagator from the behaviour of the ν integral in eq. (5.2) around $\nu = 0$

$$\sim \int \frac{d\nu}{\nu^2} \,. \tag{5.36}$$

The analogy between the ν integral and momentum space integrals in flat space suggests the interpretation of this divergence as an IR divergence in the bulk of AdS. On the other hand from the point of view of the boundary conformal theory this manifests like a UV divergence and can be reabsorbed in a running coupling, which leads to a breaking of conformal invariance.

We can understand the relation between this divergence and the running of the coupling as follows: adding the marginal interaction on the boundary

$$\delta S_{\text{boundary}} = \frac{\kappa_0}{2} \int d^d x \, \hat{j}_\mu \hat{j}^\mu \,, \tag{5.37}$$

gives an additional contribution to the propagator of the gauge field, represented by the diagram in fig. 5.4. Using $d \neq 2$ as a regulator, the contribution of this diagram is expected to be proportional to the value of the harmonic function at the pole that is pinching the contour, namely $\bar{\Omega}^{(1)}_{MN}(X,Y)$ up to longitudinal terms, and therefore the pole $\frac{1}{d-2}$ can be absorbed by a renormalization of the coupling $\kappa_0 \propto \frac{\mu^{d-2}}{d-2}$. This in turn gives rise to a β function for κ.

Instead of computing the diagram, a simple way to obtain this β function is by looking at the boundary condition of the gauge field [78, 79] (similar results in the scalar case were discussed earlier in [102, 103]). In $d = 2$ the boundary conserved current appears as the coefficient of a logarithmic mode in the near boundary expansion of the gauge field

$$A_\mu|_{d=2} \underset{z\to 0}{\sim} \log z \, e^2 \, \hat{j}_\mu + a_\mu + \dots \,. \tag{5.38}$$

Using dimreg and comparing (5.38) with (5.34) we see that the $d = 2$ current $\hat{j}_\mu$ is related to d dimensional counterpart as

$$j_\mu = \frac{1}{d-2} \hat{j}_\mu + \mathcal{O}(1) \,. \tag{5.39}$$

The resulting pole in the near-boundary expansion must be reabsorbed by a redefinition of the constant mode

$$a_\mu = -\kappa_0 \, \hat{j}_\mu \,, \quad \kappa_0 = \left(\frac{e^2}{d-2} + \kappa(\mu) \right) \mu^{2-d} \,. \tag{5.40}$$

This mixed boundary condition corresponds to turning on the double-trace coupling

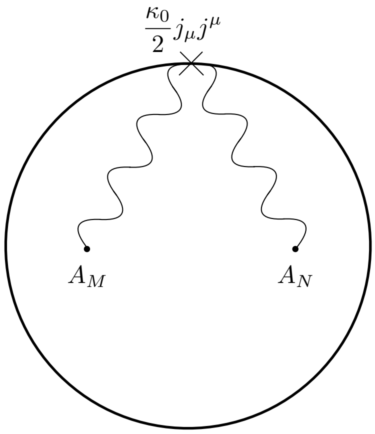


FIGURE 5.4: The correction to the propagator at leading order in the boundary current-current interaction. The wavy lines are bulk-to-boundary propagators, and the boundary point is integrated over.

$\frac{\kappa_0}{2}\,\hat{j}_\mu \hat{j}^\mu$[103] which is classically marginal in $d=2$.[2] Here κ_0 denotes the bare dimreg coupling, $\kappa(\mu)$ the renormalized coupling, and μ is the dimreg scale. We then obtain the leading order β function for the coupling

$$0 = \frac{d\kappa_0}{d\log\mu} \Rightarrow \beta_\kappa = \frac{d\kappa}{d\log\mu} = -e^2 + \mathcal{O}(\kappa^2, e^4, e^2\kappa) \,, \tag{5.41}$$

which depends only on the bulk coupling and not on κ itself at leading order.

We can use the same logic to compute the β function of κ at leading order for small κ and large N_f, at any value of α. The only difference is that the boundary OPE coefficient e^2 of $\hat{j}_\mu$ in the expansion of the gauge field gets now replaced by the coefficient of $\frac{1}{\nu^2}$ in the expansion around $\nu \to 0$ of the full propagator (5.10). In this way we get

$$\beta_\kappa|_{\text{large } N_f} = -\frac{1}{N_f}\frac{\alpha}{1+\frac{\alpha}{8\pi}\left((3-2\Delta)+2(\Delta-1)^2\psi(\Delta)\right)} + \mathcal{O}(\kappa^2, N_f^{-2}, N_f^{-1}\kappa) \,. \tag{5.42}$$

As a result, both in perturbation theory and at large N_f the Dirichlet boundary condition in AdS$_3$ in the Coulomb phase does not preserve the isometry and it does not allow to define a set of boundary conformal correlator. It would be interesting to explore the existence of fixed points for the coupling κ with some appropriate scaling of the coupling with e^2 or $\frac{1}{N_f}$. As the derivation did not use any detail of the matter sector, the existence of this boundary running couplings is a generic phenomenon for 3d gauge theories in AdS, and persists even for the pure gauge theory. An important exception is the case with a Chern-Simons term in the Lagrangian.

[2] To fix the normalization of the current-current coupling, one needs to consider the d-dimensional boundary action in the presence of the source. With the normalization in (5.34) one finds that the coupling between the source and the current is $-\int d^dx\,(d-2)\,a_\mu j^\mu$ which in the limit $d\to 2$ gives $-\int d^dx\,a_\mu \hat{j}^\mu$.

5.6 Higgs Phase in AdS

For $m^2 \leq m^2_{c,2}$ the AdS effective potential has minimum with $\phi^a = \sqrt{N_f}\Phi^a \neq 0$ and with vanishing mass-squared of the scalar fluctuations $M^2 = 0$. This is the Higgs phase, in which the gauge field gets a mass $m_A^2 = 2e^2\Phi^2$, and the $N_f - 1$ massless scalar fluctuations correspond to Goldstone bosons for the spontaneous breaking of the flavor symmetry. We refer again to [74] for the discussion of the effective potential, we simply recall that $m^2_{c,1} < m^2_{c,2}$ and as a result in AdS there is a range of parameters in which the Coulomb and the Higgs phase are both possible.

The Lagrangian for the Higgs phase in AdS is the same as the one in flat space in eq. (4.22). The corresponding large N_f propagator of the gauge field is

$$\begin{aligned}&\langle A_M(X)A_N(Y)\rangle_{\text{large } N_f,\text{ Higgs phase}} \\ &= \frac{1}{N_f}\int_{-\infty}^{+\infty} d\nu \, \frac{\alpha}{\nu^2 + (\Delta_A - \frac{d}{2})^2 + \alpha B^{(1)}(\nu)|_{\Delta=d}} \, \Omega^{(1)}_{\nu\, MN}(X,Y) + \nabla^X_M \nabla^Y_N L(u)\,,\end{aligned} \tag{5.43}$$

where $m_A^2 = (\Delta_A - 1)(\Delta_A - d + 1)$. Goldstone bosons in AdS are associated to the existence of a conformal manifold of boundary theories, on which the bulk global symmetry acts [74]. In this Higgs phase, a $U(1)$ factor of the spontaneously broken symmetry is gauged, and in the boundary conformal theory this means that the marginal operators are charged under the would-be $U(1)$ symmetry, which consequently is explicitly broken by the marginal couplings. The current operator is therefore not protected anymore, and classically it would get a scaling dimension Δ_A above the unitarity bound $\geq d - 1$.

The observables we will consider are the scaling dimensions of the spin 1 boundary operators that are exchanged in the connected four-point of the Goldstone bosons π^A, at the leading order in the $1/N_f$ expansion. Equivalently, these are the operators that appear in the boundary channel expansion of the bulk two-point function of the massive gauge field. Setting the AdS scale $L = 1$, they depend on two parameters, the gauge coupling α and the mass-squared m_A^2, which we will trade with the scaling dimension Δ_A. Unlike the Coulomb phase, having generated a mass for the gauge field there is no IR divergence in this phase, and there is no need to discuss $d = 2$ separately.

5.6.1 Spin 1 resonance in AdS

The spectrum of spin 1 operators is determined by the zeroes of the denominator of the photon propagator

$$\frac{1}{\alpha}\left(\nu^2 + (\Delta_A - \tfrac{d}{2})^2\right) + B^{(1)}(\nu)|_{\Delta=d} = 0\,. \tag{5.44}$$

We show the solutions $\{\nu_n^*\}$ in fig. 5.5. They determine the scaling dimensions of the exchanged operators via $\Delta_n = \frac{d}{2} + i\nu_n^*$. Like in the Coulomb phase, the scaling

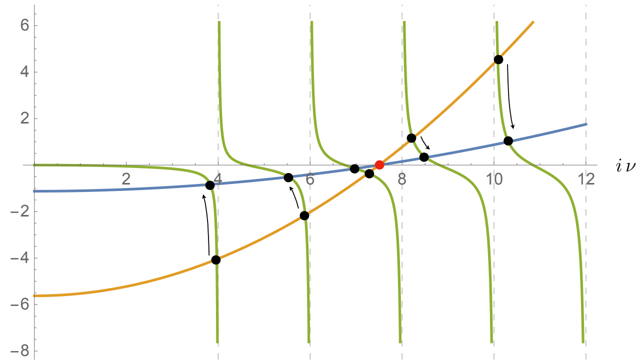


FIGURE 5.5: Plot of $B^{(1)}(\nu)$ (green line) on the negative imaginary ν axis, together with $-\frac{1}{\alpha}(\nu^2+(\Delta_A-\frac{d}{2})^2)$ for $\alpha=10$ (orange curve) and $\alpha=50$ (blue curve) in units of the AdS radius, and $\Delta_A=8.5$. We are in $d=2$, i.e. AdS_3. The intersections are highlighted with black dots, the corresponding values of ν give the scaling dimensions of spin 1 operators via $\Delta=\frac{d}{2}+i\nu$. The arrows denote the direction of increasing coupling constant α. The dashed vertical lines correspond to the spin 1 double-trace operators in the limit $\alpha\to 0$ with Δ_A fixed. The red dot correspond to the dimension Δ_A of the non-conserved spin one operator associated to the massive vector classically. There is no operator with this scaling dimension in the interacting theory.

dimensions approach the ones of the spin 1 double trace operators $\Delta_n^{(0)}=2d+2n+1$ in the limit of small α, and increase monotonically as we increase α, without level crossing.

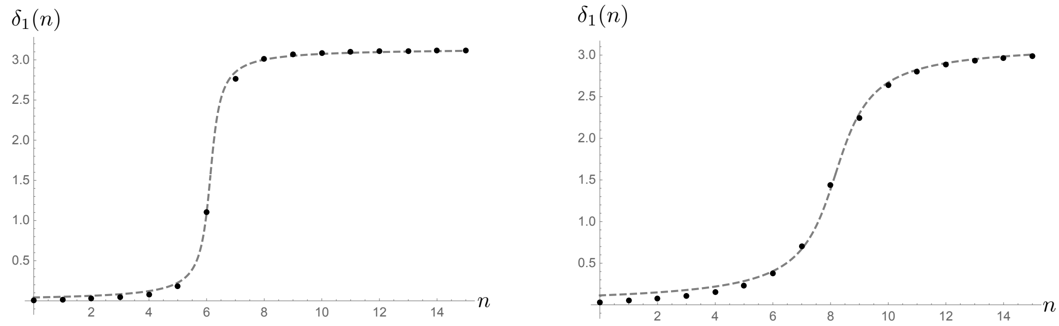
Besides the absence of the conserved current, we see a new feature in the spectrum of the Higgs phase if we compare the anomalous dimensions $\Delta_n-\Delta_n^{(0)}$ in the two regimes $\Delta_n^{(0)}<\Delta_A$ and $>\Delta_A$. Note that contrarily to the classical expectation, in the interacting theory there is no spin 1 operator with dimension Δ_A, due to the resummation of the bubble. As proposed in [8] the quantity

$$\delta_{l=1}(n)=\frac{\pi}{2}\left(\Delta_n^{(0)}-\Delta_n\right)\,, \tag{5.45}$$

is related in the flat space limit to the spin 1 phase shift $\delta_{l=1}(s)$ in the scattering amplitude of the pions (the relation between n and s involves a certain average over a window of the discrete values of n centered around the value such that $\Delta_n^{(0)}\sim\sqrt{s}$, see [8] for the precise formulation of the flat space limit). The feature in $\delta_{l=1}(n)$ displayed in fig. 5.6 when $\Delta_n^{(0)}\sim\Delta_A$ is the AdS avatar of the existence of a resonance in flat space, which is characterized by a similar step behaviour of $\delta_{l=1}(s)$ around $s\sim m_A^2$.

5.7 Conformal Point

Let us now consider the limit in which λ and α are sent to $+\infty$, and the mass-squared of the charged scalars is fine-tuned to have bulk conformal symmetry. Tuning the mass-squared is equivalent to fixing a particular value for the scaling dimension Δ of the boundary charged operator, so the first question to ask is what value of Δ, if



$\Delta_A = 16.5\,,\ \alpha = 10\,,\ x = 6.1\,,\ y = 0.3\,.$

$\Delta_A = 20.5\,,\ \alpha = 50\,,\ x = 8.2\,,\ y = 0.9\,.$

FIGURE 5.6: The black dots are the values of (5.45), related to the anomalous dimension of the n-th double-trace operator, as a function of n at finite coupling in the Higgs phase. The dashed line is the a fit with a Breit-Wigner phase shift, i.e. Arg $\left(\frac{-1}{n-x+iy}\right)$. Both plots are for $d = 2$, i.e. AdS_3 and the value of the parameters Δ_A and α, as well as of the fitted parameters x and y, are indicated under each panel. Note that as expected the resonance broadens as the coupling α increases.

any, gives rise to conformal symmetry in the bulk. Note that, unlike in flat space, the correlation length is always finite in AdS, i.e. the correlation functions always decay exponentially as a function of the large geodesic distance [30], and there is no symmetry enhancement, so it is more subtle to detect the conformal point. Nevertheless, there are some important consequences of the bulk conformal symmetry: up to a Weyl rescaling the theory in AdS becomes equivalent to a conformal boundary condition for the CFT on a half flat-space (at least this is the case if also the boundary condition preserves conformal symmetry, more on this below). As a result there is a convergent bulk OPE expansion for correlation functions, and among the boundary operators there is a displacement operator with protected scaling dimension $D = d + 1$.

In similar setups, a criterion to detect the conformal value of a free parameter Δ from the two-point function $\langle OO \rangle$ of a bulk operator O was proposed in [74]. It uses the fact that the bulk OPE expansion of the two-point function of identical operators contains the identity, and this contribution to the bulk OPE is simply a power law $\zeta^{-\Delta_O}$ of the chordal distance squared ζ that is going to zero. In a massive theory this leading power is generically accompanied by subleading integer shifted "pseudo-descendant" powers $\zeta^{-\Delta_O+k}$, with $k \in \mathbb{N}$, but in a CFT, barring the existence of other primary operators of integer dimension k, these powers must be absent. It was found in examples that there exists a value of Δ setting to zero simultaneously the coefficients of all of these powers, and this determines the conformal value. This criterion can be also implemented in ν space, i.e. from the spectral representation of the two-point function: in this case one requires that the expansion at large ν matches the expansion of the spectral representation of the power-law $\zeta^{-\Delta_O}$, given by [74]

$$\widehat{\zeta^{-\Delta_O}}(\nu) = (4\pi)^{\frac{d+1}{2}} \frac{\Gamma(\frac{d+1}{2} - \Delta_O)}{4^{\Delta_O}\Gamma(\Delta_O)} \frac{\Gamma(-\frac{d}{2} + \Delta_O \pm i\nu)}{\Gamma(\frac{1}{2} \pm i\nu)} . \tag{5.46}$$

Applying this criterion to the bulk two-point function of the operator $\Phi^*\Phi$ (or equivalently, the Hubbard-Stratonovich field σ) in sQED at leading order at large N_f, the resulting value of Δ will be identical to the one found in the $O(N)$ model, simply because at this order this two-point function is not affected by the gauge interactions. Therefore one finds that in $d = 2$ the value is $\Delta = 1$ [74]. More generally for any dimension $1 < d < 3$ we know from the study of the $O(N)$ BCFT at large N in flat space [104–106] that the conformal value is $\Delta = d - 1$. We can then plug this value in the spectral representation of the large N_f two-point function of the bulk gauge field, which in this limit becomes simply

$$\begin{aligned} &\langle A_M(X) A_N(Y)\rangle_{\text{large } N_f\text{, conformal point}} \\ &= \frac{1}{N_f}\int_{-\infty}^{+\infty} d\nu\, \frac{1}{B^{(1)}(\nu)|_{\Delta=d-1}}\, \Omega^{(1)}_{\nu\, MN}(X,Y) + \nabla^X_M \nabla^Y_N L(u)\,, \end{aligned} \tag{5.47}$$

and read-off the spectrum of spin 1 boundary operators appearing in the boundary OPE of the gauge field from the poles of $(B^{(1)}(\nu)|_{\Delta=d-1})^{-1}$ and also their bulk-to-boundary OPE coefficients squared from the residues. In general d we cannot find these values analytically, but for any specific d their numerical values can be extracted from the explicit expression (5.32), and one can check the positivity of the squared OPE coefficients.

At the integer value $d = 2$, i.e. AdS$_3$, the bubble function evaluated at $\Delta = 1$ simplifies to

$$B^{(1)}(\nu)|_{d=2,\Delta=1} = \frac{\nu^3 \coth\left(\frac{\pi\nu}{2}\right)}{16(\nu^2+1)}\,, \tag{5.48}$$

which, besides the double zero at $\nu = 0$, has single zeroes at $\nu_{k,\pm} = \pm i(2k+1)$, with $k \in \mathbb{N}$ and $k \geq 1$, giving boundary operators of dimension $\Delta_k = 2k + 2$. The corresponding residues are

$$2\pi i \text{Res}_{\nu_{k,+}} \left[(B^{(1)}(\nu)|_{d=2,\Delta=1})^{-1} \right] = \frac{256k(k+1)}{(2k+1)^3} \geq 0\,. \tag{5.49}$$

However in this case, as we have explained in section 5.5.2, the double pole at $\nu = 0$ and the associated divergence in the integral representation of the propagator imply the existence of a spin one operator of dimension 1 in the boundary spectrum, whose scalar bilinear is classically marginal and breaks conformal invariance.[3] Note that the operator giving rise to the boundary running coupling in the deep IR $\alpha \to \infty$ is not simply the boundary mode of the gauge field which is visible at weak coupling: recall that the boundary current is related to the bulk gauge field by $A_\mu \underset{z\to 0}{\sim} e^2 j_\mu \log z$,

[3] It makes sense to talk about the bilinear of the operator and to sum up the scaling dimensions because we are working at large N_f.

therefore when we take $e^2 \to \infty$ to reach the IR j_μ is set to 0. Instead it is an operator from the matter sector that mixes with the gauge field at strong coupling, and that is why its properties are controlled by the bubble function.

Because of this, we cannot use Weyl rescaling to flat space and we fail to construct a conformal boundary condition for the IR fixed point of sQED with a Dirichlet boundary condition for the gauge field in AdS_3. Note that this does not exclude the possibility that in flat space we might be able to define a conformal boundary condition for the IR fixed point of 3d gauge theories by starting the RG flow with a Dirichlet condition for the gauge field in the UV, because AdS and flat space with a boundary are not equivalent along the RG flow. It would be interesting to explore this question purely in flat space, for instance using the large N_f expansion as a computational tool.

Chapter 6

Scalar QED in dS

In this chapter we will study scalar QED in dS. As we have already seen in chapter 2 in the case of a scalar field theory, one way to study late-time correlators in dS is by doing a rotation from dS to AdS. In this chapter we will see how to generalize that procedure to gauge theories. Gauge theories in dS are particularly interesting as insertions of charged operators on the late-time boundary are not gauge invariant by themselves, and we have to do an appropriate dressing to make the correlation functions gauge invariant.

6.1 Rotation from dS to AdS

In this chapter we will see what is the relation between AdS and dS space physics for the case of Scalar QED. Note that this work is still in progress.

The goal of this section is to compute lagrangian in AdS (Euclidean signature) that will give us the same theory a in dS space. As computations in dS space are tedious to do, our goal is to relate the observables in dS space to the observables in AdS space.

First, let us find the massless propagator of the gauge field in dS in the embedding space introduce in the chapter 2. The Lagrangian of this theory is

$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}\left(\nabla_\mu A^\mu\right)^2. \tag{6.1}$$

The equation of motion can be calculated as follows:

$$\begin{aligned}
&\left[-\nabla^2 g^{\nu\mu} + \nabla^\mu\nabla^\nu - \frac{1}{\xi}\nabla^\nu\nabla^\mu\right]A_\mu =0\\
\implies &\left[-\nabla^2 g^{\nu\mu} + \left(1-\frac{1}{\xi}\right)\nabla^\nu\nabla^\mu + R^{\rho\nu\sigma\mu}g_{\rho\sigma}\right]A_\mu =0\\
\implies &\left[\left(-\nabla^2 + d\right)g^{\nu\mu} + \left(1-\frac{1}{\xi}\right)\nabla^\nu\nabla^\mu\right]A_\mu =0.
\end{aligned} \tag{6.2}$$

We can now write the embedding space equivalent of the above equation for general gauge parameter ξ as follows:

$$\begin{aligned}&\left(-\nabla_X^2+d+\left(1-\frac{1}{\xi}\right)\frac{1}{\frac{d-1}{2}}\left(W_X\cdot\nabla_X\right)\left(K_X\cdot\nabla_X\right)\right)G_A^{dS}\left(X,Y;W_X,W_Y\right)\\&=\left(W_X\cdot W_Y\right)\delta^{d+1}(X,Y).\end{aligned}\tag{6.3}$$

Note that in the above equation, we have used the ∇ and K operator defined for dS in (2.40) and (2.39) respectively. We can write the propagator G_A^{dS} as,

$$G_A^{dS}\left(X,Y;W_X,W_Y\right)=\left(W_X\cdot W_Y\right)G_0(u)+\left(W_X\cdot Y\right)\left(W_Y\cdot X\right)G_1(u).\tag{6.4}$$

We can solve the differential equation for both the G_0 and G_1 component to find the propagator. We are restricting to $d=3$ as it will model close to our universe and we have noticed simplification in the "Yennie Gauge" i.e. $\xi=3$ [107, 108]. G_0 and G_1 are given by,

$$\begin{aligned}G_0&=\frac{1}{4\pi^2u},\\G_1&=\frac{1}{8\pi^2u^2}.\end{aligned}\tag{6.5}$$

Similarly, the equation of motion in AdS for massless vector field is :

$$\begin{aligned}&\left(-\nabla_X^2-d+\left(1-\frac{1}{\xi}\right)\frac{1}{\frac{d-1}{2}}\left(W_X\cdot\nabla_X\right)\left(K_X\cdot\nabla_X\right)\right)G^{AdS}\left(X,Y;W_X,W_Y\right)\\&=\left(W_X\cdot W_Y\right)\delta^{d+1}(X,Y).\end{aligned}\tag{6.6}$$

One can solve it using similar structure for AdS propagator as in (2.17) for $d=3$ and $\xi=3$ and the solution for Dirichlet boundary condition would be :

$$\begin{aligned}g_0&=\frac{1}{4\pi^2u}-\frac{1}{4\pi^2(u+2)},\\g_1&=\frac{1}{8\pi^2u^2}-\frac{1}{8\pi^2(u+2)^2}.\end{aligned}\tag{6.7}$$

Similarly, one can solve for Neumann boundary condition as the following:

$$\begin{aligned}g_0&=\frac{1}{4\pi^2u}+\frac{1}{4\pi^2(u+2)},\\g_1&=\frac{1}{8\pi^2u^2}+\frac{1}{8\pi^2(u+2)^2}.\end{aligned}\tag{6.8}$$

We want to compute correlation functions at late time boundary and thus we will be using "in-in" formalism here. For that we will be defining different vector propagators in dS like we saw for scalar field theory. We found the following relation with the massless propagators in AdS upon rotation,

$$\begin{aligned} G^{dS}_{A,ll/rr} =& \frac{1}{2}(G^{AdS}_{A,+} + G^{AdS}_{A,-}) \\ G^{dS}_{A,lr/rl} =& -\frac{1}{2}(G^{AdS}_{A,+} - G^{AdS}_{A,-}). \end{aligned} \tag{6.9}$$

In the above equation, $G^{AdS}_{A,+}$ and $G^{AdS}_{A,-}$ denotes massless photon propagator in AdS with the Dirichlet and Neumann boundary condition respectively.

We can define the matrix D_A as follows (where we have used the notation that $G^{\mathrm{dS}}_{A,ll,+}$ denotes the part of the propagator that is proportional to $G^{\mathrm{AdS}}_{A,+}$ and same for the others):

$$\begin{bmatrix} G^{\mathrm{dS}}_{A,ll,+} & G^{\mathrm{dS}}_{A,lr,+} \\ G^{\mathrm{dS}}_{A,rl,+} & G^{\mathrm{dS}}_{A,rr,+} \end{bmatrix} = G^{\mathrm{AdS}}_{A,+} D_A. \tag{6.10}$$

Where the matrix D_A is given by,

$$D_A = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \tag{6.11}$$

In the same way we have defined the matrix N_A as follows (where we have used the notation that $G^{\mathrm{dS}}_{A,ll,-}$ denotes the part of the propagator that is proportional to $G^{\mathrm{AdS}}_{A,-}$ and same for the others):

$$\begin{bmatrix} G^{\mathrm{dS}}_{A,ll,-} & G^{\mathrm{dS}}_{A,lr,-} \\ G^{\mathrm{dS}}_{A,rl,-} & G^{\mathrm{dS}}_{A,rr,-} \end{bmatrix} = G^{\mathrm{AdS}}_{A,-} N_A. \tag{6.12}$$

Where the matrix N_A is given by,

$$N_A = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}. \tag{6.13}$$

Also note that the matrices D and N are singular and hence non-invertible as one of the Eigenvalues happens to be zero. So we can write these fields as follows:

$$A^{l/r}_{\pm} = \boldsymbol{C^{l/r}_{A,\pm} A_{\pm}} + C^{l/r}_{A,\pm} A_{\pm}. \tag{6.14}$$

Here we are picking the coefficients $\boldsymbol{C^{l/r}_{A,\pm}}$ and $C^{l/r}_{A,\pm}$ in a such a way that they diagonalize the matrices. and the condition on these coefficients is as follows:

$$\begin{aligned} D_A \begin{bmatrix} \boldsymbol{C^{l}_{A,+}} \\ \boldsymbol{C^{r}_{A,+}} \end{bmatrix} =& 0, \\ N_A \begin{bmatrix} \boldsymbol{C^{l}_{A,-}} \\ \boldsymbol{C^{r}_{A,-}} \end{bmatrix} =& 0, \end{aligned} \tag{6.15}$$

$$\begin{aligned} D_A \begin{bmatrix} C^l_{A,+} \\ C^r_{A,+} \end{bmatrix} &= \lambda_{A,+} \begin{bmatrix} C^l_{A,+} \\ C^r_{A,+} \end{bmatrix}, \\ N_A \begin{bmatrix} C^l_{A,-} \\ C^r_{A,-} \end{bmatrix} &= \lambda_{A,-} \begin{bmatrix} C^l_{A,-} \\ C^r_{A,-} \end{bmatrix}. \end{aligned} \tag{6.16}$$

Let us look at the eigen equation for D_A,

$$\begin{aligned} \frac{1}{2}\left[C^l_{A,+} - C^r_{A,+}\right] &= \lambda_{A,+} C^l_{A,+}, \\ \frac{1}{2}\left[-C^l_{A,+} + C^r_{A,+}\right] &= \lambda_{A,+} C^r_{A,+}. \end{aligned} \tag{6.17}$$

We can consider solution to be of the following form,

$$C^l_{A,+} = \left(C^r_{A,+}\right)^{-1}. \tag{6.18}$$

Upon solving, the set of equations we have

$$\begin{aligned} C^l_{A,+} &= i, \\ C^r_{A,+} &= -i, \\ \lambda_{A,+} &= 1. \end{aligned} \tag{6.19}$$

Similarly, we can solve the following eigen equation for N_ϕ:

$$\begin{aligned} \frac{1}{2}\left[C^l_{A,-} + C^r_{A,-}\right] &= \lambda_{A,-} C^l_{A,-}, \\ \frac{1}{2}\left[C^l_{A,-} + C^r_{A,-}\right] &= \lambda_{A,-} C^r_{A,-}. \end{aligned} \tag{6.20}$$

For this case, we can consider solution to be of the form,

$$C^l_{A,-} = \left(C^r_{A,-}\right)^{-1}. \tag{6.21}$$

Upon solving, we get the following results:

$$\begin{aligned} C^l_{A,-} &= 1, \\ C^r_{A,-} &= 1, \\ \lambda_{A,-} &= 1. \end{aligned} \tag{6.22}$$

Now with this information, we will first write the kinetic term of the lagrangian. Recall that kinetic term is nothing but the inverse of the propagator. Let us first consider for A_+ where we have used the notation $C^\alpha_{A,+} = C^\alpha_+$ and for $\alpha, \beta = l/r$,

$$\langle A_+ A_+ \rangle = \frac{\lambda_{A,+} G^{AdS}_{A,+}}{(C^T_+ C_+)}. \tag{6.23}$$

In the above equation in $(C_+^T C_+)$, the index α is summed over. Finally, we can have kinetic term in lagrangian as inverse of this propagator term,

$$\begin{aligned} L_{A,+}^{Kin} &= \frac{(C_+^T C_+)}{\lambda_{A,+}} \left[\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2\xi} \left(\nabla_\mu A_+^\mu \right)^2 \right] \\ &= -2 \left[\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2\xi} \left(\nabla_\mu A_+^\mu \right)^2 \right]. \end{aligned} \tag{6.24}$$

Similarly we have the following kinetic term for A_-,

$$\begin{aligned} L_{A,-}^{Kin} &= \frac{(C_-^T C_-)}{\lambda_{A,-}} \left[\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2\xi} \left(\nabla_\mu A_-^\mu \right)^2 \right] \\ &= 2 \left[\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2\xi} \left(\nabla_\mu A_-^\mu \right)^2 \right]. \end{aligned} \tag{6.25}$$

We can now plug the expression of the dS fields in terms of the AdS fields A_+ and A_- inside the dS Lagrangian. In this way we will obtain the AdS lagrangian whose boundary correlators correspond to the dS late-time correlators.

First let us take a look at cubic interaction,

$$L_{dS}^{Int}\left(A_\mu, \phi_1, \phi_2\right) = eA_\mu \left[\phi_1 \partial_\mu \phi_2 - \phi_2 \partial_\mu \phi_1\right] + \frac{e^2 A_\mu A^\mu}{2} \left[(\phi_1)^2 + (\phi_2)^2 \right]. \tag{6.26}$$

Where, we have chosen to write the complex field ϕ in terms of ϕ_1 and ϕ_2 real scalar fields. The kinetic term, mass term and the self interaction quartic term for these scalar fields in the rotated lagrangian will be same as we see earlier in simple scalar field theory.

After the rotation (2.47) the required interaction lagrangian will be given by by $L^{Int,l} + L^{Int,r}$ where,

$$\begin{aligned} L^{Int,l} &= e^{-\frac{i\pi}{2}(d-1)} L_{dS}^{Int} \left(A_+^l + A_{--}^l, \phi_{1,+}^l + \phi_{1,-}^l, \phi_{2,+}^l + \phi_{2,-}^l \right), \\ L^{Int,r} &= e^{\frac{i\pi}{2}(d-1)} L_{dS}^{Int} \left(A_+^r + A_-^r, \phi_{1,+}^r + \phi_{1,-}^r, \phi_{2,+}^r + \phi_{2,-}^r \right). \end{aligned} \tag{6.27}$$

Note that since we have relation (upon rotation) for scalar fields for general dimension $d+1$ and for massless vector fields only for $d = 3$, we have kept d for scalar fields in further calculations where it is implied that $d = 3$. The reason for doing so is that, we have noticed that the dependence on d vanishes in the rest of the expression as we will see and hence the only d dependence that will be relevant to generalize the expressions will come from the vector fields. First let us take a look at cubic interaction term,

$$\begin{aligned} L_1^{Int,l} =& ee^{-\frac{i\pi}{2}(d-1)} \left(iA_+^\mu + A_-^\mu \right) \Big[e^{i\pi(\frac{d}{2}+i\nu)} \left(\phi_{1,+}\partial_\mu \phi_{2,+} - \phi_{2,+}\partial_\mu \phi_{1,+} \right) \\ &+ e^{i\pi(\frac{d}{2}-i\nu)} \left(\phi_{1,-}\partial_\mu \phi_{2,-} - \phi_{2,-}\partial_\mu \phi_{1,-} \right) \\ &+ e^{i\pi(\frac{d}{2})} \left(\phi_{1,+}\partial_\mu \phi_{2,-} - \phi_{2,+}\partial_\mu \phi_{1,-} + \phi_{1,-}\partial_\mu \phi_{2,+} - \phi_{2,-}\partial_\mu \phi_{1,+} \right) \Big], \end{aligned} \tag{6.28}$$

$$\begin{aligned} L_1^{Int,r} &= e e^{\frac{i\pi}{2}(d-1)} \left(-iA_+^\mu + A_-^\mu\right) \left[e^{-i\pi\left(\frac{d}{2}+i\nu\right)} \left(\phi_{1,+}\partial_\mu\phi_{2,+} - \phi_{2,+}\partial_\mu\phi_{1,+}\right)\right. \\ &\quad e^{-i\pi\left(\frac{d}{2}-i\nu\right)} \left(\phi_{1,-}\partial_\mu\phi_{2,-} - \phi_{2,-}\partial_\mu\phi_{1,-}\right) \\ &\quad \left. +e^{-i\pi\left(\frac{d}{2}\right)} \left(\phi_{1,+}\partial_\mu\phi_{2,-} - \phi_{2,+}\partial_\mu\phi_{1,-} + \phi_{1,-}\partial_\mu\phi_{2,+} - \phi_{2,-}\partial_\mu\phi_{1,+}\right)\right]. \end{aligned} \tag{6.29}$$

Now we can add both terms to give final cubic term as follows:

$$\begin{aligned} & L_1^{Int,l} + L_1^{Int,r} : \\ & (2e)\left(\phi_{1,+}\partial_\mu\phi_{2,+} - \phi_{2,+}\partial_\mu\phi_{1,+}\right) \left[(-A_+^\mu)\sin\left(\pi(i\nu + \frac{1}{2})\right) + (A_-^\mu)\cos\left(\pi(i\nu + \frac{1}{2})\right)\right] \\ & +(2e)\left(\phi_{1,-}\partial_\mu\phi_{2,-} - \phi_{2,-}\partial_\mu\phi_{1,-}\right) \left[(-A_+^\mu)\sin\left(\pi(-i\nu + \frac{1}{2})\right) + (A_-^\mu)\cos\left(\pi(-i\nu + \frac{1}{2})\right)\right] \\ & +(2e)\left(\phi_{1,+}\partial_\mu\phi_{2,-} - \phi_{2,+}\partial_\mu\phi_{1,-} + \phi_{1,-}\partial_\mu\phi_{2,+} - \phi_{2,-}\partial_\mu\phi_{1,+}\right)\left[(-A_+^\mu)\right]. \end{aligned} \tag{6.30}$$

Here note that in the last term, there is mixing of ϕ_+ and ϕ_- fields in the cubic interaction with the Dirichlet field condition and not with the Neumann one. This coupling is not gauge invariant and we expect this to give mass to the Dirichlet vector field because it mixes with ϕ_+ and ϕ_-.

Let us take a look at quartic interaction now,

$$\begin{aligned} L_2^{Int,l} =& \frac{e^2}{2}\left[-(A_+)^2 + (A_-)^2 + 2i(A_+ \cdot A_-)\right] \times \\ & \left[e^{i\pi(\frac{1}{2}+i\nu)}\left(\phi_{1,+}^2 + \phi_{2,+}^2\right) + e^{i\pi(\frac{1}{2}-i\nu)}\left(\phi_{1,-}^2 + \phi_{2,-}^2\right) + 2e^{i\pi(\frac{1}{2})}\left(\phi_{1,+}\phi_{1,-} + \phi_{2,+}\phi_{2,-}\right)\right], \end{aligned} \tag{6.31}$$

$$\begin{aligned} L_2^{Int,r} =& \frac{e^2}{2}\left[-(A_+)^2 + (A_-)^2 - 2i(A_+ \cdot A_-)\right] \times \\ & \left[e^{-i\pi(\frac{1}{2}+i\nu)}\left(\phi_{1,+}^2 + \phi_{2,+}^2\right) + e^{-i\pi(\frac{1}{2}-i\nu)}\left(\phi_{1,-}^2 + \phi_{2,-}^2\right) + 2e^{-i\pi(\frac{1}{2})}\left(\phi_{1,+}\phi_{1,-} + \phi_{2,+}\phi_{2,-}\right)\right]. \end{aligned} \tag{6.32}$$

One can add these two terms to finally give the following quartic interaction term,

$$\begin{aligned} & L_2^{Int,l} + L_2^{Int,r} : \\ & (e^2)\left(\phi_{1,+}^2 + \phi_{2,+}^2\right)\left[\left(-(A_+)^2 + (A_-)^2\right)\left(\cos\pi(i\nu + \frac{1}{2})\right) - 2(A_+ \cdot A_-)\left(\sin\left(\pi(i\nu + \frac{1}{2})\right)\right)\right] \\ & +(e^2)\left(\phi_{1,-}^2 + \phi_{2,-}^2\right)\left[\left(-(A_+)^2 + (A_-)^2\right)\left(\cos\pi(-i\nu + \frac{1}{2})\right) - 2(A_+ \cdot A_-)\left(\sin\left(\pi(-i\nu + \frac{1}{2})\right)\right)\right] \\ & +(e^2)\left(\phi_{1,+}\phi_{1,-} + \phi_{2,+}\phi_{2,-}\right)\left[(-4(A_+ \cdot A_-))\right] \end{aligned} \tag{6.33}$$

6.2 Observables for a gauge theory in dS

Once we have the Lagrangian that will relate the boundary correlators in AdS to the late time boundary correlators in dS, the next goal is to perform the computation in AdS.

Some things that we can compute are: the late-time four point function of the scalar fields at the tree-level, namely the photon exchange diagram; the one-loop corrections to the photon propagator, i.e. the dS spin 1 bubble; combining the previous two, one can compute the late-time correlation function at one-loop and also at finite coupling in the large N_f expansion, upon resumming the bubble. This work is currently in progress.

One important subtlety that arises is that in dS we need to include an appropriate dressing of the charged operator insertions at late times to make the correlator gauge invariant. We do so by including Wilson lines attached to the late-time insertion points. In order to preserve conformal invariance, the Wilson lines are restricted to lie along geodesics. We hope that this can give us an important insight to attack the problem with dynamical gravity in dS, where similar subtleties related to gauge-invariance arise (in that case, under diffeomorphisms). Another problm we plan to investigate regards the nature of the Higgs phase. In the case of the $O(N)$ model in dS space [90], there was no spontaneous symmetry breaking, so it is natural to wonder if can we see a Higgs phase in the case of scalar QED.

Chapter 7

Fermionic QED

In this chapter, we will study fermionic QED at large N_f mostly in flat space for $D = 3$, and we discuss the first step towards the study of this theory on AdS background, similarly to what we did for the scalar QED case. In flat space we will see that there is a possibility of different massive phases separated by a CFT, similarly to the case of scalar QED. Due to appearance of Chern-Simons terms in 3 dimensions when the fermions are massive, parity is broken in the massive phases of fermionic QED.

7.1 Fermionic QED in Flat space

This section is a review of fermionic QED but mostly literature has focused on the CFT and here we will also look at scattering in the massive phase. The lagrangian of the theory (in Euclidean signature) is given by,

$$\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}^a \partial_\mu \gamma^\mu \psi^a + ieA_\mu \bar{\psi}^a \gamma^\mu \psi^a + m\bar{\psi}^a \psi^a. \tag{7.1}$$

Here also, there are N number of fermionic matter fields ψ and the index a runs from 1 to N. Also note that summation over index a is implied. The Dirac adjoint of the fermionic field is denoted by $\bar{\psi}$, γ_μ are the Dirac gamma matrices, A_μ is the vector field and the index μ in the lagrangian refers to the spacetime index. e is the gauge coupling and m is the mass term of the fermionic field.

We have taken γ matrices to coincide with the σ Pauli matrices. Some useful identities:

$$\begin{aligned} \gamma^\nu &= \sigma^\nu \\ \{\gamma^\mu, \gamma^\nu\} &= 2\delta^{\mu\nu} \\ \mathrm{Tr}\{\gamma^\mu \gamma^\nu \gamma^\sigma\} &= i\epsilon^{\mu\nu\sigma}\,\mathrm{Tr}(\mathbb{I}) = 2i\epsilon^{\mu\nu\sigma}. \end{aligned} \tag{7.2}$$

Now, we will take a look at the form of fermionic bubble in flat space 7.1.

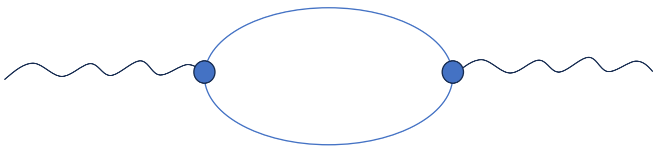


FIGURE 7.1: Fermionic Bubble in flat space coming from cubic interaction term.

$$
\begin{aligned}
&= -\frac{d^3k}{(2\pi)^3}\,\mathrm{Tr}\left[(-ie\gamma^{\mu})\left(\frac{i\,\not k+m}{k^2+m^2}\right)(-ie\gamma^{\nu})\left(\frac{i(\not k+\not p)+m}{(k+p)^2+m^2}\right)\right],\\
&= e^2\int\frac{d^3k}{(2\pi)^3}\,\mathrm{Tr}\left[\frac{\left(ik_\sigma\gamma^{\mu}\gamma^{\sigma}+m\gamma^{\mu}\right)\left(i(k+p)_\delta\gamma^{\nu}\gamma^{\delta}+m\gamma^{\nu}\right)}{(k^2+m^2)((k+p)^2+m^2)}\right],\\
&= e^2\int\frac{d^3k}{(2\pi)^3}\,\mathrm{Tr}\left[\frac{-k_\sigma(k+p)_\delta\gamma^{\mu}\gamma^{\sigma}\gamma^{\nu}\gamma^{\delta}+imk_\sigma\gamma^{\mu}\gamma^{\sigma}\gamma^{\nu}+im(k+p)_\delta\gamma^{\mu}\gamma^{\nu}\gamma^{\delta}+m^2\gamma^{\mu}\gamma^{\nu}}{(k^2+m^2)((k+p)^2+m^2)}\right].
\end{aligned}
\tag{7.3}
$$

One can simplify it further using the trace identities obtaining,

$$
I^{\mu\nu} = 2e^2\int\frac{d^3k}{(2\pi)^3}\frac{\left[\left(k\cdot(k+p)+m^2\right)g^{\mu\nu}-2k^{\mu}k^{\nu}-k^{\mu}p^{\nu}-k^{\nu}p^{\mu}-mp_\sigma\epsilon^{\mu\nu\sigma}\right]}{(k^2+m^2)\left[(k+p)^2+m^2\right]}. \tag{7.4}
$$

Because of the last term in numerator, there will be one extra term as compared to the the scalar bubble we saw in chapter 4 and we can assume the fermionic bubble to take the following form,

$$
I^{\mu\nu} = \left[F_1(p)\left(\frac{p^{\mu}p^{\nu}}{p^2}-\delta^{\mu\nu}\right)+F_2(p)\delta^{\mu\nu}\right]p+F_3(p)p_\lambda\epsilon^{\mu\nu\lambda}. \tag{7.5}
$$

In the above equation note that the last term i.e. $F_3(p)p_\lambda\epsilon^{\mu\nu\lambda}$ breaks the parity and is special for $D=3$. We can now equate (7.4) and (7.5) and contract both sides by $p_\mu p_\nu$ and after simplification it would give us, $F_2(p)=0$ which is expected from the gauge invariance. Contracting with $\delta_{\mu\nu}$ instead gives

$$
-2F_1(p)\cdot p = e^2\left[-\frac{\sqrt{m^2}}{2\pi}-\left(p^2-4m^2\right)B_3^0\right]. \tag{7.6}
$$

Where B_3^0 is scalar bubble in flat space for $D=3$. Thus we can write $F_1(p)$ as,

$$
F_1(p) = \frac{e^2}{2p}\left[\frac{\sqrt{m^2}}{2\pi}+\left(p^2-4m^2\right)\frac{1}{8\pi\sqrt{m^2}}\frac{\arctan\left(\sqrt{p^2/4m^2}\right)}{\sqrt{p^2/4m^2}}\right]. \tag{7.7}
$$

Finally, to get the form of $F_3(p)$, we can contract $I^{\mu\nu}$ with $p^{\lambda}\epsilon_{\lambda\mu\nu}$ and simplify it to give,

$$
F_3(p) = -2e^2mB_3^0. \tag{7.8}
$$

The above bubble is with one loop of fermionic fields. to get the bubble with N_f number of fermionic fields, we can multiply it with N_f. After having our fermionic bubble fixed, we can now get the resummed photon propagator by doing geometric same as we did with the case of scalar QED 4.2. We thus find the resummed propagator as,

$$\langle A_\mu(p)A_\nu(-p)\rangle|_{\frac{1}{N_f}} = \frac{\xi p_\mu p_\nu}{p^4} + \frac{\left(\sqrt{p^2}+(\alpha f)\right)}{\sqrt{p^2}\left((\alpha g)^2+\left(\sqrt{p^2}+(\alpha F)\right)^2\right)}\left(\delta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right) + \frac{(\alpha g)p^\lambda \epsilon_{\lambda\mu\nu}}{p^2\left((\alpha g)^2+\left(\sqrt{p^2}+(\alpha f)\right)^2\right)}. \tag{7.9}$$

In the above equation, we have used the gauge fixing term as $\mathcal{L}_{g.f.} = \frac{1}{2\xi}(\partial_\mu A^\mu)^2$, and the new coupling parameter is given by $\alpha = N_f e^2$, and we refer the function f and g as $F_1(p) = e^2 f$ and $F_3(p) = e^2 g$. Note that this resummed propagator has one additional structure as compared to the scalar QED. Recall that the additional structure contains a function g which is linear in m and thus we can have different propagator for different sign of m and thus different phases as well. These two phases are separated by $m = 0$ and it denotes the CFT point. Studying the amplitude of $\psi\bar{\psi} \to \psi\bar{\psi}$ at leading order in $\frac{1}{N_f}$ one finds a pole in the spin 1 partial wave, meaning that massive vector exchange can be interpreted as the bound state.

7.2 Fermionic QED in AdS

In this section, we will discuss the ongoing work in AdS. The idea is to apply again the bootstrap and large N_f techniques. In this case, when we write the bubble function in the spectral representation, it will contain an additional structure due to the breaking of parity, similarly to what we discussed in flat space. Unlike the scalar QED case, where in the spectral representation the bubble contained only the spin 1 harmonic function $\Omega^{(1)}_{\nu\, MN}(X,Y)$, here we need to add a parity odd transverse structure. We propose that this parity odd structure can be written as $F^{MN}_\nu(X,Y) = \epsilon^{MNRS}X_R\nabla^Y_S\Omega^{(0)}_{\nu\, MN}(X,Y)$. This term is inspired from the flat space structure. As a result, to compute the resummed propagator with this bubble, we need to be able to compute also convolution integrals that involve the new parity odd structure. The first required integrals is given below:

$$\int_{AdS} dY \Omega^{(1)M}_{\bar{\nu},N}(X,Y)F^{NP}_\nu(Y,Z) \propto F^{MP}_\nu(X,Z)\,. \tag{7.10}$$

This proportionality is due to the fact that on the L.H.S. there is a parity odd term (the product of a parity even and a parity odd term) and hence on the R.H.S. we need a parity odd term as well. Another required integral is

$$\int_{AdS} dY F^{M}_{\bar{\nu},N}(X,Y) F^{NP}_{\nu}(Y,Z) \propto \Omega^{(1)MP}_{\nu}(X,Z) \tag{7.11}$$

This proportionality is due to the fact that on the L.H.S. there is parity even term (the product of two parity odd terms) and hence on the R.H.S. we need a parity even term as well.

We are currently figuring out the constants of proportionality which might depend on $\nu, \bar{\nu}$. Once we have the resummed propagator, we can try to bootstrap the bubble by computing the 4-point boundary correlation function with the exact propagator containing the unknown bubble. Then, we can study the realization of the massive phase in AdS, as well as the point with bulk conformal symmetry. Another thing that we hope to find is the AdS analogue of the bound state in flat space.

Chapter 8

Summary and Outlook

In this thesis, we have motivated the study of massive QFTs in AdS and dS space beyond standard perturbation theory. We have used a combination of techniques from two approaches, namely large N and analytical conformal bootstrap, to achieve this goal. We have reviewed the embedding formalism (which makes correlation functions convenient to compute) and also the spectral representation which plays a key role in resumming the Feynman diagrams that dominate at large N. As the thesis is focused on both AdS and dS space, we have explained the rotation from dS to AdS which allows one to relate 4-point late time boundary correlation functions in dS with the 4-point boundary correlation functions in AdS [81–84].

We have then reviewed the $O(N)$ model in both flat and AdS space [85],describing the different phases of the theory, namely the symmetry breaking and the symmetry preserving phase. We have seen how a combination of large N and conformal bootstrap can be applied to get results beyond standard perturbation theory in AdS. We have presented the AdS analogue of a resonance. Upon analytical continuation of this model to dS space, one finds that there is no spontaneous symmetry breaking [90].

After reviewing the $O(N)$ model, we have presented the results obtained in [75] on scalar QED. Here we have also used a combination of large N and analytical bootstrap to get results for any finite coupling. We found two phases of the theory: a Coulomb phase where the scalar fields are massive and the vector field is massless, and a Higgs phase where the scalar fields are massless Goldstone boson and the vector field is massive. In the latter phase, we also noticed AdS analogue of resonance in flat space. We briefly discussed the case with conformal symmetry in the bulk. We also explained the first steps to extend this work to fermionic QED in AdS. The presence of parity breaking in $D = 3$ requires us to add parity-odd structures to the bubble before we attempt to bootstrap it. We also saw that for fermionic QED in flat space, there is a spin 1 bound state and there are two phases separated by a CFT.

In this thesis, we also explained the rotation of the Lagrangian from dS to AdS for scalar QED. The main motivation of this work is to understand gauge invariance in dS: we need to do an appropriate dressing of the charged operators at the late-time boundary to make the correlation functions gauge invariant. An important outcome of the study of the rotation to AdS is that we have noticed that vector field with Dirichlet boundary condition couples with ϕ fields with different boundary conditions. This coupling is not gauge invariant and hence we expect this to give mass to the Dirichlet vector field.

We conclude by mentioning some possible future directions:

- Complete the projects about fermionic QED in AdS and scalar QED in dS.
- Considering purely a Chern-Simons kinetic term for the gauge field, it would be interesting to study the boundary correlation functions for Chern-Simons matter theory in AdS_3, and to try to elucidate the unusual properties under crossing symmetry of the scattering of anyons [109–111] from the point of view of the boundary conformal correlators;
- Scattering amplitudes of charged particles in abelian gauge theories in flat space have IR divergences in $D \leq 4$. A direction for the future is to understand the AdS counterpart of the inclusive observables that give finite results, using the behavior in the flat-space limit as a diagnostic of the IR properties, see [50, 112] for work in this direction. In particular one could compute the 1 loop diagrams in AdS that correspond to the IR divergent amplitude in flat space, and study their behavior in the flat-space limit to look for an appropriate prescription that gives a finite result. The IR divergences can also be studied at large N_f and finite coupling, by computing at next-to-leading order in the $1/N_f$ expansion;
- It would be interesting to try to apply bootstrap techniques to the boundary correlators of gauge theories in AdS. An important problem in this direction is to understand what are the minimal set of assumptions that allow to single out a particular gauge theory. A nice feature of the Dirichlet boundary condition is that the gauge group becomes a global symmetry at the boundary and therefore is visible in the conformal bootstrap. A natural target in this case is the four-point function of the non-abelian currents, see [113] for recent numerical progress on this problem. Even for a fixed gauge group and matter content, in this setup one always finds not just a single conformal theory but rather a continuous family of them parametrized by the dimensionless combination of the gauge coupling and the AdS radius. Therefore important inputs for the bootstrap problem can come from the regime of weak coupling where the data of the conformal theory can be reliably computed in perturbation theory.
- It would be interesting to obtain explicit position-space expressions for loop diagrams with gauge fields. For this, one can use similar techniques to those obtained in [114, 115] for scalar and fermionic diagrams. Presumably there will be various relations between the spinning diagrams and the scalar diagrams.

- After understanding scalar QED in dS, one natural extension would be to do fermionic QED in dS. One can compute the fermionic bubble in this case and get the exact photon propagator for dS at large N_f.
- As was mentioned earlier, understanding the subtlety in gauge theory in dS can help in attacking the problem in dS with dynamical gravity. One natural step forward would be to include dynamical gravity in the case of $O(N)$ and QED models.

Appendix A

Curious case of Proca-Propagator

In this section we will take a closer look at the proca propagator 2.30. As it was mentioned before that second term in the propagator cancels the unphysical terms in the first term of the propagator. Here, we will explicitly show this calculation.

$$G_{\Delta,1}\left(X_1,X_2;W_1,W_2\right)=\int\frac{d\nu\Omega_{\nu,1}\left(X_1,X_2;W_1,W_2\right)}{\nu^2+(\Delta-h)^2} \\ -\int\frac{d\nu\left(W_1\cdot\nabla_1\right)\left(W_2\cdot\nabla_2\right)\Omega_{\nu,0}\left(X_1,X_2\right)}{(\Delta-1)(2h-\Delta-1)\left(\nu^2+h^2\right)}. \tag{A.1}$$

Note that $h=\frac{d}{2}$ we can write $\Omega_{\nu,J}\left(X_1,X_2;W_1,W_2\right)$ as difference of two bulk to bulk propogator.

$$\Omega_{\nu,J}\left(X_1,X_2;W_1,W_2\right)=\frac{i\nu}{2\pi}\left(G_{h+i\nu,J}\left(X_1,X_2;W_1,W_2\right)-G_{h-i\nu,J}\left(X_1,X_2;W_1,W_2\right)\right) \tag{A.2}$$

also we have for spin 1,

$$G_{\Delta,1}\left(X_1,X_2,W_1,W_2\right)=W_{12}g_0(u,\Delta)+\left(W_1\cdot X_2\right)\left(W_2\cdot X_1\right)g_1(u,\Delta) \tag{A.3}$$

here we have used the notation $W_{12}=W_1\cdot W_2$ and ,

$$\left(W_1\cdot\nabla_1\right)\left(W_2\cdot\nabla_2\right)\Omega_{\nu,0}\left(X_1,X_2\right)=\frac{i\nu}{2\pi}\left[-W_{12}\left(G'_{h+i\nu}(u)-G'_{h-i\nu}(u)\right)\right. \\ \left.+\left(W_1\cdot X_2\right)\left(W_2\cdot X_1\right)\left(G''_{h+i\nu}(u)-G''_{h-i\nu}(u)\right)\right] \tag{A.4}$$

In above expression, $G'(u)$ denotes derivative of $G(u)$ with respect to u. Also G_Δ denotes the spin 0 bulk to bulk propagator and we can use above expression in A.1,

$$G_{\Delta',1}\left(X_1,X_2;W_1,W_2\right)=\int\frac{d\nu}{\nu^2+(\Delta'-h)^2}\cdot\frac{i\nu}{2\pi}\left(W_{12}\left(g_0(u,\Delta)-g_0(u,\bar{\Delta})\right)\right. \\ \left.+\left(W_1\cdot X_2\right)\left(W_2\cdot X_1\right)\left(g_1(u,\Delta)-g_1(u,\bar{\Delta})\right)\right) \\ -\int\frac{d\nu\cdot i\nu}{2\pi(\Delta'-1)(2h-\Delta'-1)\left(\nu^2+h^2\right)} \\ \left[-W_{12}\left(G'_\Delta(u)-G'_{\bar{\Delta}}(u)\right)+\left(W_1\cdot X_2\right)\left(W_2\cdot X_1\right)\left(G''_\Delta(u)-G''_{\bar{\Delta}}(u)\right)\right] \tag{A.5}$$

Note that in above expression, $\Delta = h + i\nu$.
Let us consider only W_{12} term and only with Δ.

$$G_{\Delta',1}\left(X_1, X_2; W_1, W_2\right)|_{\Delta,W_{12}} = \int \frac{d\nu \cdot W_{12} \cdot i\nu}{2\pi}\left[\frac{g_0(u,\Delta)}{\nu^2 + (\Delta' - h)^2} + \frac{G'_\Delta(u)}{(\Delta'-1)(2h-\Delta'-1)\left(\nu^2+h^2\right)}\right] \tag{A.6}$$

where,

$$\begin{aligned} g_0(u,\Delta) =&(d-\Delta)\mathcal{N}(2u)^{-\Delta}{}_2F_1\left(\Delta, \frac{1-d+2\Delta}{2}, 1-d+2\Delta, -\frac{2}{u}\right) \\ &- \frac{1+u}{u}\mathcal{N}(2u)^{-\Delta}{}_2F_1\left(\Delta+1, \frac{1-d+2\Delta}{2}, 1-d+2\Delta, -\frac{2}{u}\right) \end{aligned} \tag{A.7}$$

and,

$$\mathcal{N} = \frac{\Gamma(\Delta+1)}{2\pi^{d/2}(d-1-\Delta)(\Delta-1)\Gamma\left(\Delta+1-\frac{d}{2}\right)} \tag{A.8}$$

$$\begin{aligned} G'_\Delta(u) =&A\left(\frac{2}{u^2}\right)\left[\Delta\left(\frac{-2}{u}\right)^{\Delta-1}{}_2F_1\left(\Delta, \frac{1-d+2\Delta}{2}, 1-d+2\Delta, -\frac{2}{u}\right)\right. \\ &\left.+\frac{\Delta}{2}\left(\frac{-2}{u}\right)^{\Delta}{}_2F_1\left(\Delta+1, \frac{3-d+2\Delta}{2}, 2-d+2\Delta, -\frac{2}{u}\right)\right] \end{aligned} \tag{A.9}$$

where,

$$A = \frac{\Gamma(\Delta)(-4)^{-\Delta}}{2\pi^{d/2}\Gamma\left(\Delta+1-\frac{d}{2}\right)} \tag{A.10}$$

looking at the four terms of the integrand proportional to W_{12},
first term:

$$\begin{aligned} =& \frac{i\nu}{2\pi}\frac{\Gamma(\Delta+1)}{\Gamma(\Delta+1-\mathrm{h})}\frac{(2\mathrm{u})^{-\Delta}(2\,\mathrm{h}-\Delta)}{2\pi^{\mathrm{h}}\left(\nu^2+(\Delta'-\mathrm{h})^2\right)(2\,\mathrm{h}-1-\Delta)(\Delta-1)} \\ &{}_2F_1\left(\Delta, \frac{1-2\,\mathrm{h}+2\Delta}{2}, 1-2\,\mathrm{h}+2\Delta, -\frac{2}{\mathrm{u}}\right) \end{aligned} \tag{A.11}$$

2nd term:

$$\begin{aligned} =& \frac{i\nu}{2\pi}\frac{\Gamma(\Delta+1)}{\Gamma(\Delta+1-h)}\frac{-(2\mathrm{u})^{-\Delta}\left(\frac{1+\mathrm{u}}{\mathrm{u}}\right)}{2\pi^{\mathrm{h}}\left(\nu^2+(\Delta'-\mathrm{h})^2\right)(2\,\mathrm{h}-1-\Delta)(\Delta-1)} \\ &{}_2F_1\left(\Delta+1, \frac{1-2\,\mathrm{h}+2\Delta}{2}, 1-2\,\mathrm{h}+2\Delta, -\frac{2}{\mathrm{u}}\right) \end{aligned} \tag{A.12}$$

3rd term:

$$\begin{aligned} =& \frac{i\nu}{2\pi}\frac{\Gamma(\Delta+1)}{\Gamma(\Delta+1-h)}\frac{(2\mathrm{u})^{-\Delta}\left(-\frac{1}{u}\right)}{2\pi^{\mathrm{h}}\left(\nu^2+\mathrm{h}^2\right)(2\,\mathrm{h}-1-\Delta')(\Delta'-1)} \\ &{}_2F_1\left(\Delta, \frac{1-2\,\mathrm{h}+2\Delta}{2}, 1-2\,\mathrm{h}+2\Delta, -\frac{2}{\mathrm{u}}\right) \end{aligned} \tag{A.13}$$

4th term:

$$
\begin{aligned}
&= \frac{i\nu}{2\pi}\frac{\Gamma(\Delta+1)}{\Gamma(\Delta+1-h)}\frac{(2\mathrm{u})^{-\Delta}\left(\frac{1}{u^2}\right)}{2\pi^h\left(\nu^2+h^2\right)(2h-1-\Delta')(\Delta'-1)} \\
&{}_2F_1\left(\Delta+1,\frac{3-2h+2\Delta}{2},2-2h+2\Delta,-\frac{2}{u}\right)
\end{aligned}
\tag{A.14}
$$

Note that, in all of these 4 terms $\frac{{}_2F_1}{\Gamma[\Delta+1-h]}$ have no poles. Poles in each terms are as follow:
First and 2nd term:

$$
\begin{aligned}
\nu &= \pm i\left(\Delta'-\mathrm{h}\right)\,(\text{Expected Poles}) \\
\nu &= i\left(\mathrm{h}+n+1\right) \\
\nu &= i\left(\mathrm{h}-1\right) \\
\nu &= -i\left(\mathrm{h}-1\right).
\end{aligned}
\tag{A.15}
$$

3rd and 4th term:

$$
\begin{aligned}
\nu &= \pm i\mathrm{h} \\
\nu &= i\left(\mathrm{h}+n+1\right)
\end{aligned}
\tag{A.16}
$$

For the integral, closing the contour from below. We noted that residue of terms 1st and 2nd at $\nu = -i\left(\mathrm{h}-1\right)$ were exactly cancelled by residue of 3rd and 4th term at $\nu = -i\mathrm{h}$. Similar cancellation will take place for the shadow part with terms involving $\bar{\Delta}$ and Same thing will happen for terms proportional to $(W_1\cdot X_2)\,(W_2\cdot X_1)$.

Appendix B

Inversion Formula

We have a bulk two-point function in AdS_{d+1} of conserved current operators

$$\langle J(X, W_1) J(Y, W_2) \rangle \equiv F_J(X, Y; W_1, W_2) \,. \tag{B.1}$$

In the spectral representation we rewrite F_J in an expansion in a basis of transverse spin 1 harmonic functions in AdS_{d+1}

$$F_J(X, Y; W_1, W_2) = \int_{-\infty}^{+\infty} d\nu \, \tilde{F}_J(\nu) \, \Omega^{(1)}_{\nu}(X, Y; W_1, W_2) \,. \tag{B.2}$$

We want to invert this formula and find an expression for $\tilde{F}_J(\nu)$ as an integral of $F_J(X, Y; W_1, W_2)$ over spacetime.

We start from the identity

$$\begin{aligned} \int_{AdS_{d+1}} d^{d+1}X \, \Omega^{(1)}_{\nu'}(Z, X; W_1, K) \Omega^{(1)}_{\nu}(X, Y; W, W_2) \\ = \frac{d-1}{2} \frac{\delta(\nu - \nu') + \delta(\nu + \nu')}{2} \Omega^{(1)}_{\nu}(Z, Y; W_1, W_2) \,. \end{aligned} \tag{B.3}$$

We want to evaluate it at $Z = Y$. We will use that

$$\Omega^{(1)}_{\nu}(Z, Y; W_1, W_2) = \frac{i\nu}{2\pi} \left(G_{\frac{d}{2}+i\nu}(Z, Y; W_1, W_2) - G_{\frac{d}{2}-i\nu}(Z, Y; W_1, W_2) \right) \,, \tag{B.4}$$

and

$$G_{\Delta}(Z, Y; W_1, W_2) = W_1 \cdot W_2 \, g^{\Delta}_0(u) + (W_1 \cdot Y)(W_2 \cdot Z) \, g^{\Delta}_1(u) \,, \ \ u \equiv \frac{(Y - Z)^2}{2} \,. \tag{B.5}$$

Therefore

$$\begin{aligned} \Omega^{(1)}_{\nu}(Z, Y; W_1, W_2) = \frac{i\nu}{2\pi} \Bigg[W_1 \cdot W_2 \left(g_0^{\frac{d}{2}+i\nu}(u) - g_0^{\frac{d}{2}-i\nu}(u) \right) \\ + (W_1 \cdot Y)(W_2 \cdot Z) \left(g_1^{\frac{d}{2}+i\nu}(u) - g_1^{\frac{d}{2}-i\nu}(u) \right) \Bigg] \,. \end{aligned} \tag{B.6}$$

Acting with $\frac{(K_1)_A(K_2)_B}{(\frac{d-1}{2})^2}$ we obtain

$$\begin{aligned} &\frac{(K_1)_A(K_2)_B}{(\frac{d-1}{2})^2}\Omega^{(1)}_\nu(Z,Y;W_1,W_2) \\ &= \frac{i\nu}{2\pi}\left[(\eta_{AB}+X_AX_B+Y_AY_B+X\cdot Y\,X_AY_B)\left(g_0^{\frac{d}{2}+i\nu}(u)-g_0^{\frac{d}{2}-i\nu}(u)\right)\right. \\ &\qquad\left.+(X_B+X\cdot Y\,Y_B)(Y_A+X\cdot Y\,X_A)\left(g_1^{\frac{d}{2}+i\nu}(u)-g_1^{\frac{d}{2}-i\nu}(u)\right)\right]. \end{aligned} \tag{B.7}$$

We now contract with η^{AB} and we obtain

$$\begin{aligned} &\frac{(K_1)\cdot(K_2)}{(\frac{d-1}{2})^2}\,\Omega^{(1)}_\nu(Z,Y;W_1,W_2) \\ &= \frac{i\nu}{2\pi}\left[(d+(1+u)^2)\left(g_0^{\frac{d}{2}+i\nu}(u)-g_0^{\frac{d}{2}-i\nu}(u)\right)\right. \\ &\qquad\left.-u(1+u)(2+u)\left(g_1^{\frac{d}{2}+i\nu}(u)-g_1^{\frac{d}{2}-i\nu}(u)\right)\right]. \end{aligned} \tag{B.8}$$

In the limit $Z\to Y$ in which u goes to zero, both terms $g_{0,1}^{\frac{d}{2}+i\nu}(u)-g_{0,1}^{\frac{d}{2}-i\nu}(u)$ have a finite limit. We are left with

$$\frac{(K_1)\cdot(K_2)}{(\frac{d-1}{2})^2}\,\Omega^{(1)}_\nu(Z,Y;W_1,W_2)|_{Z=Y} = \frac{d(d+1)\nu\,(d^2+4\nu^2)\sinh(\pi\nu)\Gamma\left(\frac{d}{2}\pm i\nu-1\right)}{2^{d+4}\pi^{\frac{d+3}{2}}\Gamma\left(\frac{d+3}{2}\right)}. \tag{B.9}$$

As a result we obtain

$$\begin{aligned} &\frac{(K_1)\cdot(K_2)}{(\frac{d-1}{2})^3}\int_{AdS_{d+1}} d^{d+1}X\,\Omega^{(1)}_{\nu'}(Z,X;W_1,K)\Omega^{(1)}_\nu(X,Y;W,W_2)|_{Z=Y} = C^{(1)}_\nu\frac{\delta(\nu-\nu')+\delta(\nu+\nu')}{2}, \\ &C^{(1)}_\nu = \frac{d(d+1)\nu\,(d^2+4\nu^2)\sinh(\pi\nu)\Gamma\left(\frac{d}{2}\pm i\nu-1\right)}{2^{d+4}\pi^{\frac{d+3}{2}}\Gamma\left(\frac{d+3}{2}\right)}. \end{aligned} \tag{B.10}$$

To obtain the inversion formula, we will then simply redefine in eq. (B.2) $W_1\to W$, then multiply both sides by $\Omega^{(1)}_\nu(Z,X;W_1,K)$, where K is the differential operator that frees the index contracted with W, and we then act with $(K_1\cdot K_2)$ and integrate X over AdS_{d+1}. Finally we set $Y=Z$. We then get

$$\frac{(K_1)\cdot(K_2)}{(\frac{d-1}{2})^3}\int_{AdS_{d+1}} d^{d+1}X\,\Omega^{(1)}_\nu(Y,X;W_1,K)F_J(X,Y;W,W_2) = C^{(1)}_\nu\,\tilde{F}_J(\nu), \tag{B.11}$$

Note that the dependence on $W_{1,2}$ on the left-hand side disappears after we act with $K_1\cdot K_2$.

We will now use the following decomposition of the two-point function

$$F(X,Y;W_1,W_2) = (W_1\cdot W_2)\,F_0(u) + (W_1\cdot Y)(W_2\cdot X)\,F_1(u)\,. \tag{B.12}$$

Acting with $K_1\cdot\nabla_X$ we find

$$K_1\cdot\nabla_X F(X,Y;W_1,W_2) = \frac{d-1}{2}(W_2\cdot X)\mathrm{Div}\,, \tag{B.13}$$

where

$$\mathrm{Div} = (u+1)F_0'(u) + (d+1)F_0(u) - (d+2)(u+1)F_1(u) + (1-(u+1)^2)F_1'(u)\,. \tag{B.14}$$

This combination needs to vanish, to enforce that the operator is transverse.
Let us plug this decomposition in the formula above. We use

$$\begin{aligned}
&\frac{(K_1)\cdot(K_2)}{(\frac{d-1}{2})^3}\Omega_\nu^{(1)}(Y,X;W_1,K)(W\cdot W_2)\,F_0(u)\\
&= \frac{(K_1)\cdot(K_2)}{(\frac{d-1}{2})^2}\Omega_\nu^{(1)}(Y,X;W_1)_B G^{BD}(X)(W_2)_D\,F_0(u)\\
&= \Omega_\nu^{(1)}(Y,X)_{AB}G^{BD}(X)G^A_D(Y)\,F_0(u)\\
&= \Omega_\nu^{(1)}(Y,X)_{AB}\eta^{AB}\,F_0(u)\,.
\end{aligned} \tag{B.15}$$

where $G^{AB}(Z) = \eta^{AB} + Z^AZ^B$ is the transverse projector in Z, and

$$\begin{aligned}
&\frac{(K_1)\cdot(K_2)}{(\frac{d-1}{2})^3}\Omega_\nu^{(1)}(Y,X;W_1,K)(W\cdot Y)(W_2\cdot X)\,F_1(u)\\
&= \frac{(K_1)\cdot(K_2)}{(\frac{d-1}{2})^2}\Omega_\nu^{(1)}(Y,X;W_1)_B G^{BD}(X)Y_D(W_2\cdot X)\,F_1(u)\\
&= \Omega_\nu^{(1)}(Y,X)_{AB}G^{BD}(X)G^{AC}(Y)Y_DX_C\,F_1(u)\\
&= \Omega_\nu^{(1)}(Y,X)_{AB}X^AY^B\,F_1(u)\,.
\end{aligned} \tag{B.16}$$

We will also use the analogous decomposition for the harmonic function (which is transverse, and in fact one can check that the corresponding Div vanishes)

$$\Omega_\nu^{(1)}(X,Y;W_1,W_2) = (W_1\cdot W_2)\,\omega_0^\nu(u) + (W_1\cdot Y)(W_2\cdot X)\,\omega_1^\nu(u)\,. \tag{B.17}$$

Freeing up the indices, this decomposition becomes

$$\begin{aligned}
\Omega_\nu^{(1)}(Y,X)_{AB} &= (\eta_{AB} + Y_AY_B + X_AX_B + Y\cdot X\,Y_AX_B)\,\omega_0^\nu(u)\\
&\quad + (Y_B + Y\cdot X\,X_B)(X_A + Y\cdot X\,Y_A)\,\omega_1^\nu(u)\,,
\end{aligned} \tag{B.18}$$

and therefore

$$\eta^{AB}\Omega_\nu^{(1)}(Y,X)_{AB} = \left(d+(1+u)^2\right)\omega_0^\nu(u) \\ - u(1+u)(2+u)\,\omega_1^\nu(u)\,, \tag{B.19}$$

and

$$\Omega_\nu^{(1)}(Y,X)_{AB}X^AY^B = \left[(1+u)\left[1-(1+u)^2\right]\right]\omega_0^\nu \\ + \left[1-2(1+u)^2+(1+u)^4\right]\omega_1^\nu \tag{B.20}$$

As a result we obtain the inversion formula,

$$\tilde{F}_J(\nu) \\ = \frac{\mathrm{Vol}\left(S^d\right)}{C_\nu^{(1)}}\int_0^{+\infty} du\sqrt{g(u)}\,\left[\left(d+(1+u)^2\right)F_0(u) - u(1+u)(2+u)\,F_1(u)\right]\omega_0^\nu(u)\,. \\ + \frac{\mathrm{Vol}\left(S^d\right)}{C_\nu^{(1)}}\int_0^{+\infty} du\sqrt{g(u)}\,\left[-u(1+u)(2+u)\,F_0(u) + (1-(1+u)^2)^2\,F_1(u)\right]\omega_1^\nu(u). \tag{B.21}$$

Note that $\sqrt{g(u)} = \frac{1}{2}(u(2+u))^{\frac{d-1}{2}}$ and $\mathrm{Vol}\left(S^d\right) = \frac{2\pi^{\frac{d+1}{2}}}{\Gamma\left(\frac{d+1}{2}\right)}$.
Also the expression for $\omega_1^\nu(u)$ and $\omega_0^\nu(u)$ are given below,

$$\omega_0^\nu(u) = \frac{i\nu}{2\pi}\left(g_0^{\frac{d}{2}+i\nu}(u) - g_0^{\frac{d}{2}-i\nu}(u)\right)\,, \tag{B.22}$$

This can be explicitly written in terms of hypergeometric functions,

$$\omega_0^\nu(u) = \frac{\nu\sinh(\pi\nu)\left(d^2+4\nu^2\right)\Gamma\left(\frac{d}{2}-1+i\nu\right)\Gamma\left(\frac{d}{2}-1-i\nu\right)}{2^{d+4}\pi^{\frac{d+3}{2}}\Gamma\left(\frac{d+3}{2}\right)} \\ \left[(d+1)\,{}_2F_1\left(\frac{d}{2}-i\nu,\frac{d}{2}+i\nu;\frac{d+1}{2};-\frac{u}{2}\right)\right. \\ \left.-(u+1)\,{}_2F_1\left(\frac{d}{2}+1-i\nu,\frac{d}{2}+1+i\nu;\frac{d+3}{2};-\frac{u}{2}\right)\right] \\ \omega_1^\nu(u) = \frac{\nu\sinh(\pi\nu)\left(d^2+4\nu^2\right)\Gamma\left(\frac{d}{2}-1+i\nu\right)\Gamma\left(\frac{d}{2}-1-i\nu\right)}{2^{d+4}\pi^{\frac{d+3}{2}}\Gamma\left(\frac{d+3}{2}\right)u(2+u)} \\ \left[(d+1)(1+u)\,{}_2F_1\left(\frac{d}{2}-i\nu,\frac{d}{2}+i\nu;\frac{d+1}{2};-\frac{u}{2}\right)\right. \\ \left.-\left(d+(1+u)^2\right)\,{}_2F_1\left(\frac{d}{2}+1-i\nu,\frac{d}{2}+1+i\nu;\frac{d+3}{2};-\frac{u}{2}\right)\right]. \tag{B.23}$$

Appendix C

Flat space Limit

Note that the decomposition (B.12) can be written in any coordinates x^μ on AdS_{d+1} as

$$F(x,y)_{\mu\nu} = -\frac{\partial u}{\partial x^\mu \partial y^\nu} F_0(u) + \frac{\partial u}{\partial x^\mu}\frac{\partial u}{\partial y^\nu} F_1(u)\ . \tag{C.1}$$

To take the flat-space limit, we need to restore the radius of AdS L in our formulae. To do that, we replace the parameter u by u/L^2. We then take $L \to \infty$, and identify u with half of the square-distance in flat space $\mathbb{R}^{d+1}$, which in Cartesian coordinates can be written simply as $(x-y)^2/2$. We will also use the notation $r = |x-y|$, so that after the limit $L \to \infty$ we can identify u with $r^2/2$. The flat-space limit is then

$$\begin{aligned} &L^{-2\alpha}\left(-\frac{\partial u}{\partial x^\mu \partial y^\nu} F_0\left(L^{-2}u\right) + L^{-2}\frac{\partial u}{\partial x^\mu}\frac{\partial u}{\partial y^\nu} F_1\left(L^{-2}u\right)\right) \\ &\underset{L\to\infty}{\longrightarrow} \delta_{\mu\nu} f_0(u) - (x-y)_\mu (x-y)_\nu f_1(u), \end{aligned} \tag{C.2}$$

where the lower-case is used to denote the functions in flat space and 2α can be defined as mass dimension of the function $f_0(u)$ in flat space. From this we see that,

$$\begin{aligned} L^{-2\alpha} F_0(L^{-2}u) &\underset{L\to\infty}{\longrightarrow} f_0(u)\ , \\ L^{-2\alpha-2} F_1(L^{-2}u) &\underset{L\to\infty}{\longrightarrow} f_1(u)\ . \end{aligned} \tag{C.3}$$

Equivalently we can write that the expansion of the AdS functions is

$$\begin{aligned} F_0\left(L^{-2}u\right) &\underset{L\to\infty}{=} L^{2\alpha}\left(f_0(u) + \mathcal{O}\left(L^{-2}\right)\right) \\ F_1\left(L^{-2}u\right) &\underset{L\to\infty}{=} L^{2\alpha}L^2\left(f_1(u) + \mathcal{O}\left(L^{-2}\right)\right). \end{aligned} \tag{C.4}$$

The resulting two-point function in flat space is then written as

$$f(x,y)_{\mu\nu} = \delta_{\mu\nu} f_0(u) - (x-y)_\mu (x-y)_\nu f_1(u)\ . \tag{C.5}$$

Given a function of the distance $h(u)$, we can obtain a transverse function by applying to it a transverse projector

$$\begin{aligned}
&(\partial^2\delta_{\mu\nu} - \partial_\mu\partial_\nu)h(u) \\
&= \delta_{\mu\nu}\partial_\rho((x-y)^\rho h'(u)) - \partial_\mu((x-y)_\nu h'(u)) \\
&= \delta_{\mu\nu}(2uh''(u) + d\,h'(u)) - (x-y)_\mu(x-y)_\nu h''(u)\,.
\end{aligned} \tag{C.6}$$

Therefore transversality is equivalent to the existence of a function $h(u)$ such that

$$\begin{aligned}
f_0(u) &= 2uh''(u) + d\,h'(u)\,, \\
f_1(u) &= h''(u)\,.
\end{aligned} \tag{C.7}$$

Taking a derivative of $f_0(u)$ we get

$$f_0'(u) = (2+d)f_1(u) + 2uf_1'(u)\,. \tag{C.8}$$

Indeed this equation coincides with the flat-space limit (C.3) of Div $= 0$.

C.0.1 Inversion formula in flat space

The Fourier transform in flat space is

$$f(x,y)_{\mu\nu} = \int \frac{d^{d+1}p}{(2\pi)^{d+1}}(-p^2\delta_{\mu\nu} + p_\mu p_\nu)\tilde{h}(p^2)e^{-ip(x-y)}\,. \tag{C.9}$$

$\tilde{h}(p^2)$ is a radial Fourier transform of $h(u)$.

Comparing with equation (C.5) we get

$$(-p^2\delta_{\mu\nu} + p_\mu p_\nu)\tilde{h}(p^2) = \delta_{\mu\nu}\tilde{f}_0(p^2) + \frac{\partial^2}{\partial p^\mu \partial p^\nu}\tilde{f}_1(p^2)\,. \tag{C.10}$$

Moreover

$$\frac{\partial^2}{\partial p^\mu \partial p^\nu}\tilde{f}_1(p^2) = \frac{\partial}{\partial p^\mu}(\tilde{f}_1'(p^2)2p_\nu) = 2\delta_{\mu\nu}\tilde{f}_1'(p^2) + 4p_\mu p_\nu \tilde{f}_1''(p^2)\,. \tag{C.11}$$

As a result

$$\tilde{h}(p^2) = -\frac{1}{p^2}(\tilde{f}_0(p^2) + 2\tilde{f}_1'(p^2)) = 4\tilde{f}_1''(p^2)\,. \tag{C.12}$$

$\tilde{f}_{0,1}$ can be written as a single integral over the distance, using the radial Fourier transform

$$\tilde{f}_{0,1}(p^2) = (p^2)^{-\frac{d-1}{4}}(2\pi)^{\frac{d+1}{2}}\int_0^{+\infty} dr\, r^{\frac{d+1}{2}} J_{\frac{d-1}{2}}(\sqrt{p^2}\,r) f_{0,1}(r^2/2)\,, \tag{C.13}$$

where we used the definition $u = \frac{r^2}{2}$ in the argument of the functions in position space. Using the identity

$$\frac{d}{dp^2}\left((p^2)^{-\frac{d-1}{4}} J_{\frac{d-1}{2}}(\sqrt{p^2}\,r)\right) = -\frac{r}{2}(p^2)^{-\frac{d+1}{4}} J_{\frac{d+1}{2}}(\sqrt{p^2}\,r) \tag{C.14}$$

we can then write also $\tilde{h}(p^2)$ as a single integral as follows

$$\begin{aligned}\tilde{h}(p^2) = -(p^2)^{-\frac{d+3}{4}}(2\pi)^{\frac{d+1}{2}}\int_0^{+\infty} dr\, r^{\frac{d+1}{2}}\Big[&J_{\frac{d-1}{2}}(\sqrt{p^2}\,r) f_0(r^2/2) \\ &-\frac{r}{\sqrt{p^2}} J_{\frac{d+1}{2}}(\sqrt{p^2}\,r) f_1(r^2/2)\Big]\,.\end{aligned} \tag{C.15}$$

Another possibility is to use the second equality in (C.12) and use the following identity for the Bessel function

$$\begin{aligned}&\frac{d^2}{d(p^2)^2}\left((p^2)^{-\frac{d-1}{4}} J_{\frac{d-1}{2}}\left(\sqrt{p^2}\,r\right)\right) \\ &\quad = -\frac{1}{4}r^2(p^2)^{-\frac{d+3}{4}}\left[J_{\frac{d-1}{2}}(\sqrt{p^2}\,r) - \frac{d+1}{\sqrt{p^2}\,r} J_{\frac{d+1}{2}}(\sqrt{p^2}\,r)\right]\,.\end{aligned} \tag{C.16}$$

This gives the following alternative integral expression,

$$\begin{aligned}\widetilde{h}(p) = -\left(p^2\right)^{-\frac{d+3}{4}}(2\pi)^{\frac{d+1}{2}}\int_0^{+\infty} dr r^{\frac{d+5}{2}}\Big[&J_{\frac{d-1}{2}}\left(\sqrt{p^2}r\right) \\ &-\frac{d+1}{\sqrt{p^2}r} J_{\frac{d+1}{2}}\left(\sqrt{p^2}r\right)\Big] f_1\left(r^2/2\right)\end{aligned} \tag{C.17}$$

C.0.2 Limits of the kernel function

In (B.21) the kernel of the transform is the special function

$$\omega_0^\nu(u) = \frac{i\nu}{2\pi}\left(g_0^{\frac{d}{2}+i\nu}(u) - g_0^{\frac{d}{2}-i\nu}(u)\right)\,, \tag{C.18}$$

where

$$\begin{aligned}g_0^{\frac{d}{2}+i\nu}(u) = &\frac{\pi^{-\frac{d}{2}}2^{-\frac{d}{2}-i\nu}\Gamma\left(\frac{d}{2}+i\nu+1\right)u^{-\frac{d}{2}-i\nu-1}}{\left((d-2)^2+4\nu^2\right)\Gamma(i\nu+1)}\left(u(d-2i\nu)\,{}_2F_1\left(\frac{d}{2}+i\nu, i\nu+\frac{1}{2}; 2i\nu+1; -\frac{2}{u}\right)\right. \\ &\left.-2(u+1)\,{}_2F_1\left(\frac{d}{2}+i\nu+1, i\nu+\frac{1}{2}; 2i\nu+1; -\frac{2}{u}\right)\right)\,.\end{aligned} \tag{C.19}$$

In this subsection we will compute the expansion of $\omega_0^\nu(u)$ in the flat-space limit in which we replace u with $L^{-2}u = L^{-2}r^2/2$ and at the same time $\nu = Lp$, where p is the modulus of the momentum in flat space. Since $g_0^{\frac{d}{2}+i\nu}(u)$ is a sum of two hypergeometric functions, we will also accordingly split $\omega_0^\nu(u)$ in two terms that we call $\omega_0^\nu(u)|_{\text{1st hyp}}$ and $\omega_0^\nu(u)|_{\text{2nd hyp}}$.

To find the approximation of $\omega_0^\nu(u)|_{\text{1st hyp}}$ we use the following rewriting of the first hypergeometric function, exploiting the integral expression

$$\begin{aligned}
&2^{-\frac{d}{2}-i\nu}\left(\frac{r^2}{2L^2}\right)^{-\frac{d}{2}-i\nu-1}{}_2F_1\left(\frac{d}{2}+i\nu, i\nu+\frac{1}{2}; 2i\nu+1; -\frac{4L^2}{r^2}\right)\\
&= 2^{-\frac{d}{2}-i\nu}\left(\frac{r^2}{2L^2}\right)^{-\frac{d}{2}-i\nu-1}\frac{\Gamma(2i\nu+1)}{\Gamma(i\nu+\frac{1}{2})^2}\int_0^1 dt\, t^{i\nu-\frac{1}{2}}(1-t)^{i\nu-\frac{1}{2}}\left(1+\frac{4L^2 t}{r^2}\right)^{-\frac{d}{2}-i\nu}\\
&\underset{t=\frac{ry}{2L}}{=} 2^{-\frac{d}{2}-i\nu}\left(\frac{r^2}{2L^2}\right)^{-\frac{d}{2}-i\nu-1}\frac{\Gamma(2i\nu+1)}{\Gamma(i\nu+\frac{1}{2})^2}\left(\frac{r}{2L}\right)^{i\nu+\frac{1}{2}}\int_0^{\frac{2L}{r}} dy\, y^{i\nu-\frac{1}{2}}\left(1-\frac{ry}{2L}\right)^{i\nu-\frac{1}{2}}\left(1+\frac{2Ly}{r}\right)^{-\frac{d}{2}-i\nu}\\
&= 2^{-\frac{d}{2}-i\nu}\left(\frac{r^2}{2L^2}\right)^{-\frac{d}{2}-i\nu-1}\frac{\Gamma(2i\nu+1)}{\Gamma(i\nu+\frac{1}{2})^2}\left(\frac{r}{2L}\right)^{\frac{d}{2}+2i\nu+\frac{1}{2}}\int_0^{\frac{2L}{r}} dy\, y^{-\frac{d+1}{2}}\left(1-\frac{ry}{2L}\right)^{i\nu-\frac{1}{2}}\left(1+\frac{r}{2Ly}\right)^{-\frac{d}{2}-i\nu}\\
&\underset{\nu=pL,\, L\to\infty}{\approx} L^{\frac{d}{2}+2}\, 2^{\frac{1-d}{2}} r^{-\frac{d+3}{2}}\frac{e^{i\frac{\pi}{4}}\sqrt{p}}{\sqrt{\pi}}\int_0^\infty dy\, y^{-\frac{d+1}{2}} e^{-i\frac{rp}{2}(y+\frac{1}{y})}+\dots\,.
\end{aligned} \tag{C.20}$$

In the last line we use the asymptotic approximation for the ratio of Gamma functions

$$\frac{\Gamma(2i\nu+1)}{\Gamma(i\nu+\frac{1}{2})^2}\approx\frac{2^{2i\nu}e^{\frac{i\pi}{4}}\sqrt{\nu}}{\sqrt{\pi}} \tag{C.21}$$

The prefactor of the first hypergeometric function is

$$\frac{i\nu}{2\pi}\frac{\pi^{-\frac{d}{2}}\Gamma\left(\frac{d}{2}+i\nu+1\right)}{((d-2)^2+4\nu^2)\,\Gamma(i\nu+1)}L^{-2}u(d-2i\nu)\underset{\nu=pL,\, L\to\infty}{\approx} -L^{\frac{d}{2}-2}\,2^{-3}\pi^{-\frac{d}{2}-1}\,r^2\,i^{\frac{d}{2}+2}\,p^{\frac{d}{2}}+\dots\,. \tag{C.22}$$

Putting things together we obtain

$$\begin{aligned}
\omega_0^\nu(L^{-2}u)|_{\text{1st hyp}} &\underset{\nu=pL,\, L\to\infty}{\approx} L^d\, 2\,\mathrm{Re}\left[-2^{-\frac{d+5}{2}}e^{\frac{i\pi(d+5)}{4}}\pi^{-\frac{d+3}{2}}p^{\frac{d+1}{2}}r^{\frac{1-d}{2}}\int_0^\infty dy\, y^{-\frac{d+1}{2}}e^{-i\frac{rp}{2}(y+\frac{1}{y})}\right]\\
&= L^d\, 2^{-\frac{d+3}{2}}\pi^{-\frac{d+1}{2}}p^{\frac{d+1}{2}}r^{\frac{1-d}{2}}J_{\frac{d-1}{2}}(rp)\,.
\end{aligned} \tag{C.23}$$

The 2nd hypergeometric function in $g_0^{\frac{d}{2}+i\nu}(u)$ gets rewritten as

$$
\begin{aligned}
&2^{-\frac{d}{2}-i\nu}\left(\frac{r^2}{2L^2}\right)^{-\frac{d}{2}-i\nu-1} {}_2F_1\left(\frac{d}{2}+i\nu+1, i\nu+\frac{1}{2}; 2i\nu+1; -\frac{2L^2}{r^2}\right)\\
&= 2^{-\frac{d}{2}-i\nu}\left(\frac{r^2}{2L^2}\right)^{-\frac{d}{2}-i\nu-1} \frac{\Gamma(2i\nu+1)}{\Gamma(i\nu+\frac{1}{2})^2}\int_0^1 dt\, t^{i\nu-\frac{1}{2}}(1-t)^{i\nu-\frac{1}{2}}\left(1+\frac{4L^2t}{r^2}\right)^{-\frac{d}{2}-i\nu-1}\\
&\underset{t=\frac{ry}{2L}}{=} 2^{-\frac{d}{2}-i\nu}\left(\frac{r^2}{2L^2}\right)^{-\frac{d}{2}-i\nu-1} \frac{\Gamma(2i\nu+1)}{\Gamma(i\nu+\frac{1}{2})^2}\left(\frac{r}{2L}\right)^{i\nu+\frac{1}{2}}\int_0^{\frac{2}{r}} dy\, y^{i\nu-\frac{1}{2}}\left(1-\frac{ry}{2L}\right)^{i\nu-\frac{1}{2}}\left(1+\frac{2Ly}{r}\right)^{-\frac{d}{2}-i\nu-1}\\
&= 2^{-\frac{d}{2}-i\nu}\left(\frac{r^2}{2L^2}\right)^{-\frac{d}{2}-i\nu-1} \frac{\Gamma(2i\nu+1)}{\Gamma(i\nu+\frac{1}{2})^2}\left(\frac{r}{2L}\right)^{\frac{d}{2}+2i\nu+\frac{3}{2}}\int_0^{\frac{2L}{r}} dy\, y^{-\frac{d+3}{2}}\left(1-\frac{ry}{2L}\right)^{i\nu-\frac{1}{2}}\left(1+\frac{r}{2Ly}\right)^{-\frac{d}{2}-i\nu-1}\\
&\underset{\nu=pL,\ L\to\infty}{\approx} L^{\frac{d}{2}+1}\, 2^{-\frac{d+1}{2}} r^{-\frac{d+1}{2}} \frac{e^{i\frac{\pi}{4}}\sqrt{p}}{\sqrt{\pi}}\int_0^\infty dy\, y^{-\frac{d+3}{2}} e^{-i\frac{rp}{2}(y+\frac{1}{y})} + \dots\, .
\end{aligned}
\tag{C.24}
$$

The prefactor of the second hypergeometric function is

$$
-\frac{i\nu}{2\pi}2(u+1)\frac{\pi^{-\frac{d}{2}}\Gamma\left(\frac{d}{2}+i\nu+1\right)}{((d-2)^2+4\nu^2)\,\Gamma(i\nu+1)} \underset{\nu=pL,\ L\to\infty}{\approx} -L^{\frac{d}{2}-1}\, 2^{-2}\, \pi^{-\frac{d}{2}-1}\, i^{\frac{d}{2}+1}\, p^{\frac{d}{2}-1}\, .
\tag{C.25}
$$

As a result we get

$$
\begin{aligned}
\omega_0^\nu(u)|_{2^{\text{nd}}\,\text{hyp}} &\underset{\nu=pL,\ L\to\infty}{\approx} L^d\, 2\,\text{Re}\left[-2^{-\frac{d+5}{2}} e^{\frac{i\pi(d+3)}{4}} \pi^{-\frac{d+3}{2}} p^{\frac{d-1}{2}} r^{-\frac{d+1}{2}} \int_0^\infty dy\, y^{-\frac{d+3}{2}} e^{-i\frac{rp}{2}(y+\frac{1}{y})}\right]\\
&= -L^d\, 2^{-\frac{d+3}{2}} \pi^{-\frac{d+1}{2}} p^{\frac{d-1}{2}} r^{-\frac{d+1}{2}} J_{\frac{d+1}{2}}(rp)\, .
\end{aligned}
\tag{C.26}
$$

Putting the first and the second hypergeometric together we obtain

$$
\omega_0^\nu(u) \underset{\nu=pL,\ L\to\infty}{\approx} L^d\, 2^{-\frac{d+3}{2}} \pi^{-\frac{d+1}{2}} p^{\frac{d+1}{2}} r^{\frac{1-d}{2}} \left(J_{\frac{d-1}{2}}(rp) - \frac{1}{rp} J_{\frac{d+1}{2}}(rp)\right)\, .
\tag{C.27}
$$

Now simplifying ω_1^ν

$$
\omega_1^\nu(u) = \frac{i\nu}{2\pi}\left(g_1^{\frac{d}{2}+i\nu}(u) - g_1^{\frac{d}{2}-i\nu}(u)\right)\, ,
\tag{C.28}
$$

where

$$
\begin{aligned}
g_1^{\frac{d}{2}+i\nu}(u) = \frac{\pi^{-\frac{d}{2}} 2^{-\frac{d}{2}-i\nu}\Gamma\left(\frac{d}{2}+i\nu+1\right) u^{-\frac{d}{2}-i\nu-1}}{((d-2)^2+4\nu^2)\,\Gamma(i\nu+1)} &\left(\frac{2(1+u)(d/2-i\nu)}{(2+u)}\, {}_2F_1\left(\frac{d}{2}+i\nu, i\nu+\frac{1}{2}; 2i\nu+1; -\frac{2}{u}\right)\right.\\
&\left. -\frac{2(d+(1+u)^2)}{u(2+u)}\, {}_2F_1\left(\frac{d}{2}+i\nu+1, i\nu+\frac{1}{2}; 2i\nu+1; -\frac{2}{u}\right)\right)\, .
\end{aligned}
\tag{C.29}
$$

Taking the first hypergeometric function:

$$\begin{aligned} &2^{-\frac{d}{2}-i\nu}\left(\frac{r^2}{2L^2}\right)^{-\frac{d}{2}-i\nu-1}{}_2F_1\left(\frac{d}{2}+i\nu,i\nu+\frac{1}{2};2i\nu+1;-\frac{4L^2}{r^2}\right) \\ &\underset{\nu=pL,\ L\to\infty}{\approx} L^{\frac{d}{2}+2}\,2^{\frac{1-d}{2}}r^{-\frac{d+3}{2}}\frac{e^{i\frac{\pi}{4}}\sqrt{p}}{\sqrt{\pi}}\int_0^\infty dy\,y^{-\frac{d+1}{2}}e^{-i\frac{rp}{2}(y+\frac{1}{y})}+\dots\,. \end{aligned} \tag{C.30}$$

Prefactor of first hypergeometric function:

$$\frac{i\nu}{2\pi}\frac{\pi^{-\frac{d}{2}}\Gamma\left(\frac{d}{2}+i\nu+1\right)}{((d-2)^2+4\nu^2)\,\Gamma(i\nu+1)}\frac{2(1+u/L^2)(d/2-i\nu)}{(2+u/L^2)}\underset{\nu=pL,\ L\to\infty}{\approx}-L^{\frac{d}{2}}\,2^{-3}\pi^{-\frac{d}{2}-1}\,i^{\frac{d}{2}+2}\,p^{\frac{d}{2}}+\dots\,. \tag{C.31}$$

now, the first hypergeometric can be simplified to:

$$\begin{aligned} \omega_1^\nu(L^{-2}u)|_{1^{\rm st}\,\rm hyp} &\underset{\nu=pL,\ L\to\infty}{\approx} L^{d+2}\,2\,\mathrm{Re}\left[-2^{-\frac{d+5}{2}}e^{\frac{i\pi(d+5)}{4}}\pi^{-\frac{d+3}{2}}p^{\frac{d+1}{2}}r^{-\frac{d+3}{2}}\int_0^\infty dy\,y^{-\frac{d+1}{2}}e^{-i\frac{rp}{2}(y+\frac{1}{y})}\right] \\ &= L^{d+2}\,2^{-\frac{d+3}{2}}\pi^{-\frac{d+1}{2}}p^{\frac{d+1}{2}}r^{-\frac{d+3}{2}}J_{\frac{d-1}{2}}(rp)\,. \end{aligned} \tag{C.32}$$

Now looking at 2nd hypergeometric fucntion:

$$\begin{aligned} &2^{-\frac{d}{2}-i\nu}\left(\frac{r^2}{2L^2}\right)^{-\frac{d}{2}-i\nu-1}{}_2F_1\left(\frac{d}{2}+i\nu+1,i\nu+\frac{1}{2};2i\nu+1;-\frac{2L^2}{r^2}\right) \\ &\underset{\nu=pL,\ L\to\infty}{\approx} L^{\frac{d}{2}+1}\,2^{-\frac{d+1}{2}}r^{-\frac{d+1}{2}}\frac{e^{i\frac{\pi}{4}}\sqrt{p}}{\sqrt{\pi}}\int_0^\infty dy\,y^{-\frac{d+3}{2}}e^{-i\frac{rp}{2}(y+\frac{1}{y})}+\dots\,. \end{aligned} \tag{C.33}$$

Prefactor of 2nd hypergeometric function:

$$\frac{i\nu}{2\pi}\frac{\pi^{-\frac{d}{2}}\Gamma\left(\frac{d}{2}+i\nu+1\right)}{((d-2)^2+4\nu^2)\,\Gamma(i\nu+1)}\left(-\frac{2(d+(1+u/L^2)^2)}{(u/L^2)(2+u/L^2)}\right)\underset{\nu=pL,\ L\to\infty}{\approx}-\frac{d+1}{r^2}L^{\frac{d}{2}+1}\,2^{-2}\,\pi^{-\frac{d}{2}-1}\,i^{\frac{d}{2}+1}\,p^{\frac{d}{2}-1}\,. \tag{C.34}$$

Now the 2nd hypergeometric can be simplified to:

$$\begin{aligned} \omega_1^\nu(u)|_{2^{\rm nd}\,\rm hyp} &\underset{\nu=pL,\ L\to\infty}{\approx} L^{d+2}\,2\,\mathrm{Re}\left[-2^{-\frac{d+5}{2}}e^{\frac{i\pi(d+3)}{4}}\pi^{-\frac{d+3}{2}}p^{\frac{d-1}{2}}r^{-\frac{d+5}{2}}(d+1)\int_0^\infty dy\,y^{-\frac{d+3}{2}}e^{-i\frac{rp}{2}(y+\frac{1}{y})}\right] \\ &= -L^{d+2}\,2^{-\frac{d+3}{2}}\pi^{-\frac{d+1}{2}}p^{\frac{d-1}{2}}r^{-\frac{d+5}{2}}(d+1)J_{\frac{d+1}{2}}(rp)\,. \end{aligned} \tag{C.35}$$

Putting the first and the second hypergeometric together we obtain

$$\omega_1^{\nu}(u) \underset{\nu=pL,\ L\to\infty}{\approx} L^{d+2}\, 2^{-\frac{d+3}{2}} \pi^{-\frac{d+1}{2}} p^{\frac{d+1}{2}} r^{\frac{1-d}{2}} \frac{1}{r^2} \left(J_{\frac{d-1}{2}}(rp) - \frac{d+1}{rp} J_{\frac{d+1}{2}}(rp) \right) . \quad \text{(C.36)}$$

Also noting that $\sqrt{g(u)}\, du = \frac{1}{2}\, dr\, r^d L^{-d-1}$ in flat space limit, the integral in B.21 will become,

$$L^{-2\alpha+d+1} \widetilde{F}_J(\nu)\Big|_{\nu=pL\ \ L\to\infty} \xrightarrow{} p^2 \widetilde{h}\left(p^2\right) . \quad \text{(C.37)}$$

Appendix D

Important Integrals

Following are the important relations and integrals that might come handy:

$$\int_{\partial} dP \frac{P^{A_1} \dots P^{A_{2l}}}{(-2P \cdot Y)^{2h+2l}} = \frac{\pi^h (2h+2l)_{-h} Y^{A_1} \dots Y^{A_{2l}}}{(-Y^2)^{h+2l}} - \text{ traces} . \tag{D.1}$$

For P to be the point on boundary and Y to be a generic coordinate.

Also we have the following relation:

$$(W_1 \cdot \nabla_1)^J K_{\Delta_2} = \frac{\mathcal{C}_{\Delta_2} (W_1 \cdot \nabla_1)^J}{(-2P_2 \cdot X_1)^{\Delta_2}} = \mathcal{C}_{\Delta_2} (\Delta_2)_J \frac{(2W_1 \cdot P_2)^J}{(-2P_2 \cdot X_1)^{\Delta_2 + J}}, \tag{D.2}$$

$$\left[\nabla^2, (W \cdot \nabla_X)^n\right] = -n (2h - 1 + 2\mathcal{D}_W - n) (W \cdot \nabla_X)^n \tag{D.3}$$

for,$\mathcal{D}_W = W \cdot \partial_W$.

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