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The diffeomorphism type of small hyperplane arrangements is combinatorially determined

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Abstract: It is known that there exist hyperplane arrangements with the same underlying matroid that admit non-homotopy equivalent complement manifolds. Here we show that, in any rank, complex central hyperplane arrangements with up to 7 hyperplanes and the same underlying matroid are isotopic. In particular, the diffeomorphism type of the complement manifold and the Milnor fiber and fibration of these arrangements are combinatorially determined, that is, they depend only on the underlying matroid. To prove this, we associate to every such matroid a topological space, that we call the *reduced realization space*; its connectedness, shown by means of symbolic computation, implies the desired result.

Keywords: Matroids, hyperplane arrangements, realization spaces.

1 Introduction

The central problem in hyperplane arrangement theory is to determine whether the topology or the homotopy type of the complement manifold of an arrangement is described by the combinatorial properties of the arrangement itself. This theory was first developed in [2] with motivations from the study of configuration spaces.

One of the seminal works on the homotopy theory of complex hyperplane arrangements is the computation of the integer cohomology algebra structure of the complement manifold of an arrangement by Orlik and Solomon [14]. Motivated by work of Arnol'd, they exploited techniques of Brieskorn [3] to provide a presentation of this cohomology algebra, in terms of generators and relations, that depends only on the underlying matroid of the arrangement.

This result of [14] has generated many new conjectures and problems, asking which homotopy invariants of the complement manifold of an arrangement are combinatorially determined. A cornerstone in this direction is the isotopy theorem proved by Randell in [17]. It states that the diffeomorphism type of the complement manifold does not change through an isotopy, that is a smooth one-parameter family of arrangements with constant underlying matroid. Afterwards, in [18] Randell proved similar results for more sophisticated invariants such as the Milnor fiber and fibration of an arrangement (see Definition 2.3).

Randell's isotopy theorem can actually be reformulated in terms of matroid realization spaces, which are related to the well-studied matroid stratification of the Grassmannian. In their celebrated paper [9], Gel'fand, Goresky, MacPherson and Serganova studied this stratification and described some of its equivalent reformulations. In particular, Randell's results give rise to the problem of describing the connected components of the matroid strata of the Grassmannian.

On the other hand, Rybnikov [20] found an example of arrangements with the same underlying matroid but non-isomorphic fundamental groups of the corresponding complement manifolds. However, in many remarkable cases the topology of the complement manifold can be recovered simply by the combinatorial data.

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Thus, one important problem is to characterize wider families of arrangements for which Randell's isotopy theorem holds. Several results in this direction appeared in the literature. In particular, Jiang and Yau [10], Nazir and Yoshinaga [13] and Amram, Teicher and Ye [1] focused on some specific classes of line arrangements in the complex projective plane. One-parameter families of isotopic arrangements have also been studied in [22], [23] and [24]. However, the techniques developed in these works seem hardly generalizable to higher dimensions.

To every matroid M we can associate the set of hyperplane arrangements having M as underlying matroid. This set has a natural topological structure as a subset of a space of matrices, and it is called the *realization space* of M. Here, building on previous results of Delucchi and the first-named author, see [6], we associate to a matroid another topological space, called its *reduced realization space* (Definition 3.2). As the name suggests, the latter is a subset of the realization space, and it is obtained by considering hyperplane arrangements of a given shape. Such a shape is determined by what we call the *normal frame* of a matrix (Definition 3.1). Exploiting some ideas from [4] and [19] we study this space, and finally we describe (in Proposition 3.1) how the connectedness of the reduced realization space is related to the one of the "classical" realization space. Moreover, we show by means of symbolic computation and elementary algebraic geometry arguments that for any matroid with ground set of at most 7 elements the associated reduced realization space is either empty or connected.

Thus, by the results of [17] and [18] we can conclude that the diffeomorphism type of the complement manifold and the Milnor fiber and fibrations of complex central hyperplane arrangements with up to 7 hyperplanes are combinatorially determined, that is, they depend only on the underlying matroid of these arrangements.

Overview. Section 2 contains some basic definitions on matroids and complex hyperplane arrangements. In Section 3, we introduce the normal frame of a matrix and the reduced realization space of a matroid, and we deduce some of their properties. Section 4 is devoted to applications in the study of the isotopy type of arrangements with up to 7 hyperplanes. For readability's sake we postpone some of the technical computations to Appendix A.

2 Matroids and arrangements

In this section we provide a quick review of some basic definitions and results about matroids and arrangements. We refer to the book [16] for a detailed treatment of matroid theory and we point to [15] for a general theory of arrangements and to [8] for a survey of their homotopy theory.

2.1 Matroids

A *matroid* M is a pair $(E, \mathfrak{I})$, where E is a finite *ground set* and $\mathfrak{I} \subseteq 2^E$ is a family of subsets of E satisfying the following three conditions:

(I1) $\emptyset \in \mathfrak{I}$;

(I2) If $I \in \mathfrak{I}$ and $J \subseteq I$, then $J \in \mathfrak{I}$;

(I3) If I and J are in $\mathfrak{I}$ and $|I| < |J|$, then there exists an element $e \in J \setminus I$ such that $I \cup \{e\} \in \mathfrak{I}$.

The elements of $\mathfrak{I}$ are called the *independent sets* of M. Maximal independent sets (with respect to inclusion) are called *bases*, and the set of bases of M is denoted by $\mathfrak{B}$. By definition, the *rank* of a subset $S \subseteq E$ is

$$\mathrm{rk}(S) = \max\{|S \cap B| \mid B \in \mathfrak{B}\},$$

and the *rank* of the matroid M is the rank of the ground set E.

A rank d matroid M with ground set $E = \{1, \dots, m\}$ is called *realizable* over $\mathbb{C}$ if there exists a matrix $A \in M_{d,m}(\mathbb{C})$ of d rows and m columns with complex coefficients such that

$$\{J \subseteq E \mid \{A^j\}_{j\in J} \text{ is linearly independent over } \mathbb{C}\}$$

is the family of independent sets of M, where A^j denotes the j-th column of A. We say that A *realizes* M over $\mathbb{C}$.

Definition 2.1 (Realization space). For a rank d matroid M with ground set $E = \{1, \dots, m\}$, the *realization space* of M over $\mathbb{C}$ is the set $\mathcal{R}_{\mathbb{C}}(M)$ of all matrices $A \in M_{d,m}(\mathbb{C})$ that realize M over $\mathbb{C}$.

We endow $\mathcal{R}_{\mathbb{C}}(M)$ with the subspace topology of $M_{d,m}(\mathbb{C})$. If $\mathcal{R}_{\mathbb{C}}(M)$ is empty, i.e. if there are no matrices that realize M over $\mathbb{C}$, we say that M is *non-realizable* over $\mathbb{C}$.

2.2 Arrangements

Any finite collection $\mathcal{A} = \{H_1, \dots, H_m\}$ of affine subspaces in $\mathbb{C}^d$ is called an *arrangement*. Its *complement manifold* $M(\mathcal{A})$ is the complement of the union of the H_i in $\mathbb{C}^d$. The arrangement is *central* if every H_i contains the origin. For an arrangement $\mathcal{A} = \{H_1, \dots, H_m\}$ in $\mathbb{C}^d$ we assign a *rank* to each subset $S \subseteq \{1, \dots, m\}$ by

$$\mathrm{rk}_{\mathcal{A}}(S) = \mathrm{codim} \textstyle\bigcap_{i\in S} H_i$$

(where we define the empty set to have codimension $d + 1$). We say that the arrangements $\mathcal{A} = \{H_1, \dots, H_m\}$ and $\mathcal{B} = \{K_1, \dots, K_m\}$ have the same *combinatorial type* if the functions $\mathrm{rk}_{\mathcal{A}}$ and $\mathrm{rk}_{\mathcal{B}}$ coincide.

Given an open interval $(a, b) \subseteq \mathbb{R}$, a *smooth one-parameter* family of arrangements is a collection $\{\mathcal{A}_t\}_{t\in(a,b)}$ of arrangements $\mathcal{A}_t = \{H_1(t), \dots, H_m(t)\}$ in $\mathbb{C}^d$ such that there exist smooth functions from (a, b) to $\mathbb{C}$ for the coefficients of the defining equations of the subspaces $H_i(t)$. With a slight abuse of notation we write $\mathcal{A}_t$ for $\{\mathcal{A}_t\}_{t\in(a,b)}$, omitting the parameter interval (a, b).

Definition 2.2 (Isotopic arrangements). A smooth one-parameter family of arrangements $\mathcal{A}_t$ is an *isotopy* if for any t_1 and t_2 the arrangements $\mathcal{A}_{t_1}$ and $\mathcal{A}_{t_2}$ have the same combinatorial type. In this case we say that $\mathcal{A}_{t_1}$ and $\mathcal{A}_{t_2}$ are *isotopic*.

The following theorem, sometimes referred to as the "isotopy theorem", was proved by Randell [17]. This is one of the pillars on which our work is based, allowing us to focus on isotopic arrangements.

Theorem 2.1 ([17]). *If $\mathcal{A}_{t_1}$ and $\mathcal{A}_{t_2}$ are isotopic arrangements, then the complement manifolds $M(\mathcal{A}_{t_1})$ and $M(\mathcal{A}_{t_2})$ are diffeomorphic.*

A *hyperplane arrangement* is an arrangement of codimension 1 subspaces. Again, a hyperplane arrangement is *central* if each of its subspaces is linear. For a central hyperplane arrangement $\mathcal{A} = \{H_1, \dots, H_m\}$ in $\mathbb{C}^d$ pick linear forms α_i in the dual space $(\mathbb{C}^d)^*$ with $H_i = \ker \alpha_i$. The *underlying matroid* of $\mathcal{A}$ is by definition the matroid $M_{\mathcal{A}}$ with ground set $E_{\mathcal{A}} = \{1, \dots, m\}$ and $\mathfrak{I}_{\mathcal{A}} = \{S \subseteq E \mid \{\alpha_i\}_{i\in S} \text{ is linearly independent over } \mathbb{C}\}$ as independent sets. Clearly, the matroid $M_{\mathcal{A}}$ does not depend on the choice of the linear forms α_i. The *rank* of $\mathcal{A}$ is by definition the rank of $M_{\mathcal{A}}$ and we say that $\mathcal{A}$ is *essential* if its rank is maximal.

Note that a smooth one-parameter family $\mathcal{A}_t$ of central hyperplane arrangements is an isotopy if and only if $M_{\mathcal{A}_{t_1}} = M_{\mathcal{A}_{t_2}}$ for any t_1 and t_2.

Definition 2.3 (Milnor fiber and fibration). Given linear forms $\alpha_i \in (\mathbb{C}^d)^*$ with $H_i = \ker \alpha_i$, the polynomial $Q_{\mathcal{A}} = \prod_{i=1}^m \alpha_i$ is homogeneous of degree m and can be considered as a map

$$Q_{\mathcal{A}} : M(\mathcal{A}) \longrightarrow \mathbb{C}^*$$

that is the projection of a fiber bundle called the *Milnor fibration* of the arrangement; see [12]. The *Milnor fiber* is then the fiber $F_{\mathcal{A}} = Q_{\mathcal{A}}^{-1}(1)$.

The following theorem proved by Randell in [18] states that the Milnor fiber and fibration are also invariants for isotopic arrangements.

Theorem 2.2 ([18]). *Let $\mathcal{A}_t$ be a smooth one-parameter family of central hyperplane arrangements. If $\mathcal{A}_t$ is an isotopy, then for any t_1 and t_2 the Milnor fibrations $Q_{\mathcal{A}_{t_1}}$ and $Q_{\mathcal{A}_{t_2}}$ are isomorphic fiber bundles.*

3 Reduced realization spaces

Throughout this section we suppose that, given a rank d matroid M with ground set $E = \{1, \dots, m\}$, the set $\{1, \dots, d\}$ is a basis of M. We can always assume this after relabelling the elements of the ground set.

Our goal is to introduce a subspace $\mathcal{R}^R_{\mathbb{C}}(M)$ of the realization space $\mathcal{R}_{\mathbb{C}}(M)$ that contains information about the realizability of M over $\mathbb{C}$ and the connectedness of $\mathcal{R}_{\mathbb{C}}(M)$, but it is easier to describe than the full space $\mathcal{R}_{\mathbb{C}}(M)$.

Suppose that $A \in M_{d,m}(\mathbb{C})$ realizes M over $\mathbb{C}$, thus $A \in \mathcal{R}_{\mathbb{C}}(M)$. Since $\{1, \dots, d\}$ is a basis for M, we can perform a change of coordinates in $\mathbb{C}^d$ such that the columns $A^1, \dots, A^d$ of A become the standard basis. The new matrix we obtain realizes M over $\mathbb{C}$ as well. Now we can multiply every row of A by a non-zero scalar without modifying the realizability property. Therefore, for a matrix $A \in M_{d,m}(\mathbb{C})$ realizing M over $\mathbb{C}$ we can try to find an invertible matrix $G \in \mathrm{GL}_d(\mathbb{C})$ of rank d and a complex non-singular diagonal matrix D of rank m such that GAD has as many zeros and ones as possible, and still realizes M over $\mathbb{C}$. Our new space will correspond to the set of these “reduced” matrices. To be more specific, we would like that the new matrix GAD is of the form $(I_d|\tilde{A})$, where I_d is the $d \times d$ identity matrix and $\tilde{A} \in M_{d,m-d}(\mathbb{C})$ is a matrix of d rows and $m - d$ columns with complex coefficients that fulfills the following properties:

- For each column of $\tilde{A}$, the first non-zero entry (from the top to the bottom) equals 1;
- For each row of $\tilde{A}$, the first non-zero entry (from the left to the right) that is not the first non-zero entry (from the top to the bottom) of a column equals 1.

In order to define precisely and to be able to manipulate the object we are going to define, we need a somehow technical notion, the *normal frame* of a matrix. This is a way to encode the “support” of a particular element of the equivalence class of a matrix Q under the left action by $\mathrm{GL}_d(\mathbb{C})$ and the right action by $\mathbb{C}^*$. By “support” we mean that the entries in the normal frame have value 1 for such an element in the equivalence class. Let us consider a matrix $Q \in M_{n,r}(\mathbb{C})$ of n rows and r columns with complex coefficients and let us associate to Q a board $S_0(Q)$ of n rows and r columns with black squares in correspondence to the zero entries of Q and white squares in correspondence to the non-zero entries of Q. We perform the following sequence of operations on the board $S_0(Q)$:

(O1) For each column of $S_0(Q)$ we color blue the first white square from the top to the bottom. We call this board $S_1(Q)$;

(O2) For each row of $S_1(Q)$ we color red the first white square from the left to the right. We call this board $S_2(Q)$;

(O3) We color green each blue or red square of $S_2(Q)$. We call this board $S(Q)$.

Definition 3.1 (Normal frame). The *normal frame* of a matrix $Q \in M_{n,r}(\mathbb{C})$ is

$$\mathcal{P}_Q = \{(i,j) \in \{1, \dots, n\} \times \{1, \dots, r\} \mid \text{the } (i,j)\text{-th square of } S(Q) \text{ is green}\}$$

We are now ready to define the reduced realization space of a matroid.

Definition 3.2 (Reduced realization space). For a rank d matroid M with ground set $E = \{1, \dots, m\}$ and $\{1, \dots, d\}$ as a basis, the *reduced realization space* of M over $\mathbb{C}$ is the set $\mathcal{R}^R_{\mathbb{C}}(M)$ of matrices $A \in M_{d,m}(\mathbb{C})$ that satisfy the following conditions:

(C1) A realizes M over $\mathbb{C}$, that is, A belongs to $\mathcal{R}_{\mathbb{C}}(M)$;

(C2) A is of the form $(I_d|\tilde{A})$, where I_d is the $d \times d$ identity matrix;

(C3) The entries of $\tilde{A}$ with positions in the normal frame $\mathcal{P}_{\tilde{A}}$ equal 1.

We endow $\mathcal{R}^R_{\mathbb{C}}(M)$ with the subspace topology of $M_{d,m}(\mathbb{C})$.

Remark 3.1. For a matrix $A \in M_{d,m}(\mathbb{C})$, Condition 1 is equivalent to

$$\det(A^{j_1}|\dots|A^{j_d}) \neq 0 \text{ if } \{j_1,\dots,j_d\} \in \mathfrak{B} \quad \text{and} \quad \det(A^{j_1}|\dots|A^{j_d}) = 0 \text{ if } \{j_1,\dots,j_d\} \notin \mathfrak{B}, \tag{$*$}$$

where A^j denotes the j-th column of A and $\mathfrak{B}$ is the set of bases of M. For $1 \le i \le d$ and $d+1 \le j \le m$, if we consider the sets $(\{1,\dots,d\} \setminus \{i\}) \cup \{j\}$ of cardinality d, it follows from $(*)$ that, given matrices $\tilde{A}_1$ and $\tilde{A}_2$ in $M_{d,m-d}(\mathbb{C})$ with $(I_d|\tilde{A}_1)$ and $(I_d|\tilde{A}_2)$ in $\mathcal{R}^R_{\mathbb{C}}(M)$, the board $S_0(\tilde{A}_1)$ equals $S_0(\tilde{A}_2)$. Hence all matrices $\tilde{A}$ in $M_{d,m-d}(\mathbb{C})$ with $(I_d|\tilde{A})$ in $\mathcal{R}^R_{\mathbb{C}}(M)$ have the same normal frame. This, together with Condition 2 and $(*)$, implies that $\mathcal{R}^R_{\mathbb{C}}(M)$ can be written as a subset of $M_{d,m}(\mathbb{C})$ satisfying a system of equalities and inequalities of polynomial type.

The following proposition clarifies how the spaces $\mathcal{R}_{\mathbb{C}}(M)$ and $\mathcal{R}^R_{\mathbb{C}}(M)$ are related. In particular, it shows that the connectedness of $\mathcal{R}^R_{\mathbb{C}}(M)$ implies the one of $\mathcal{R}_{\mathbb{C}}(M)$. This fact is a direct consequence of the connectedness of the complex linear group and of the complex torus.

Proposition 3.1. *For a rank d matroid M with ground set $E = \{1,\dots,m\}$ and $\{1,\dots,d\}$ as a basis, let $A \in \mathcal{R}_{\mathbb{C}}(M)$ be a matrix that realizes M over $\mathbb{C}$. Then there exist an invertible matrix $G \in \mathrm{GL}_d(\mathbb{C})$ of rank d and a complex non-singular diagonal matrix D of rank m such that $GAD \in \mathcal{R}^R_{\mathbb{C}}(M)$. In particular, the following properties hold:*

(P1) *$\mathcal{R}_{\mathbb{C}}(M) \neq \emptyset$ if and only if $\mathcal{R}^R_{\mathbb{C}}(M) \neq \emptyset$;*
(P2) *If $\mathcal{R}^R_{\mathbb{C}}(M)$ is connected, so is $\mathcal{R}_{\mathbb{C}}(M)$.*

To show this result we need two technical lemmas.

Lemma 3.1. *For a matrix $Q \in M_{n,r}(\mathbb{C})$ with at least one non-zero entry, let $S(Q)$ be the board associated to Q. Then, there exists a line (row or column) of $S(Q)$ that contains exactly one green square and such that the board obtained from $S(Q)$ by deleting this line coincides with the one obtained from $S_0(Q)$ by deleting such a line and then performing the steps* 1, 2 *and* 3.

Proof. Without loss of generality we can assume that each line (row or column) of $S_0(Q)$ contains at least one white square. Otherwise, it suffices to delete that black line and study the problem for a smaller board. Set

$$v = \max\{i \in \{1,\dots,n\} \mid \text{the } i\text{-th row of } S_1(Q) \text{ contains a blue square}\}$$

and note that under the assumption that each line of $S_0(Q)$ contains at least one white square, this number v is well defined. We distinguish two cases:

- If $1 \le v < n$, then the statement follows by considering the $(v+1)$-th row.
- If $v = n$, then it suffices to consider the first column for which this maximum is attained. □

Lemma 3.2. *Given a matrix $Q \in M_{n,r}(\mathbb{C})$ there exist complex non-singular diagonal matrices D_1 of rank n and D_2 of rank r such that the entries of $D_1 Q D_2$ with positions belonging to the normal frame $\mathcal{P}_Q$ of Q equal* 1.

Proof. Our proof exploits the same ideas as [4, Proposition 2.7]. We proceed by induction on the cardinality of the normal frame $\mathcal{P}_Q$ of Q. If $|\mathcal{P}_Q| = 0$, then there is nothing to prove, since then all entries of Q are zero. Now we assume our statement to be true for all matrices with normal frame of cardinality strictly less than k, and we consider a matrix Q with normal frame $\mathcal{P}_Q$ of k elements. By Lemma 3.1 there exists a line (row or column) of the board $S(Q)$ that contains exactly one green square and such that the board obtained from $S(Q)$ by deleting this line coincides with the one obtained from $S_0(Q)$ and then performing the steps 1, 2 and 3. Note that our proof will be essentially the same if we suppose that this line is a column. Hence we assume that the line is the i-th row of $S(Q)$. Let (i,j) be the position of the unique green square placed in it. In particular, the entry q_{ij} of Q is non-zero: otherwise, by definition of the steps 1, 2 and 3 there would be a black square in the position (i,j) of the board $S(Q)$. We denote by $\tilde{Q} \in M_{n-1,r}(\mathbb{C})$ the matrix obtained from Q by deleting its i-th row. With the second part of Lemma 3.1 we deduce that the normal frame $\mathcal{P}_{\tilde{Q}}$ of $\tilde{Q}$ has $k-1$ elements. Thus by the inductive hypothesis there exist complex non-singular diagonal matrices $\tilde{D}_1$ of rank $n-1$ and $\tilde{D}_2$ of rank r such that the entries of $\tilde{D}_1 \tilde{Q} \tilde{D}_2$ with positions belonging to the normal frame $\mathcal{P}_{\tilde{Q}}$ of $\tilde{Q}$ equal 1. For $i \in \{1,2\}$

we denote by $\tilde{D}_i(j)$ the j-th diagonal element of the matrix $\tilde{D}_i$. So finally, if we define

$$D_1 = \operatorname{diag}\big(\tilde{D}_1(1), \dots, \tilde{D}_1(i-1), (\tilde{D}_2(j)q_{ij})^{-1}, \tilde{D}_1(i), \dots, \tilde{D}_1(n-1)\big)$$

and set $D_2 = \tilde{D}_2$, one can easily check that all entries of D_1QD_2 with positions belonging to the normal frame $\mathcal{P}_Q$ of Q are equal to 1. □

Proof of Proposition 3.1. Let $A \in \mathcal{R}_{\mathbb{C}}(M)$ be a matrix that realizes M over $\mathbb{C}$. Since $\{1, \dots, d\}$ is a basis of M, there exists an invertible matrix $B \in \mathrm{GL}_d(\mathbb{C})$ of rank d such that $BA = (I_d|Q)$, where $Q \in M_{d,m-d}(\mathbb{C})$. By Lemma 3.2 there exist complex non-singular diagonal matrices D_1 of rank d and D_2 of rank $m-d$ such that the entries of D_1QD_2 with positions belonging to the normal frame $\mathcal{P}_Q$ of Q are equal to 1. Now, for $i \in \{1, 2\}$ denote by $D_i(j)$ the j-th diagonal element of the matrix D_i, and set

$$D = \operatorname{diag}\big(D_1(1)^{-1}, \dots, D_1(d)^{-1}, D_2(1), \dots, D_2(m-d)\big)$$

and $G = D_1B$. With elementary linear algebra arguments it is not hard to see that the matrix GAD realizes the matroid M over $\mathbb{C}$ as well. Hence Condition (C1) in Definition 3.2 is satisfied. By construction the matrix GAD is of the form $(I_d|D_1QD_2)$, so also Conditions (C2) and (C3) in Definition 3.2 are fulfilled.

It remains to check that properties (P1) and (P2) hold. Property (P1) follows directly from the first part of our statement and the set inclusion $\mathcal{R}^R_{\mathbb{C}}(M) \subseteq \mathcal{R}_{\mathbb{C}}(M)$.

To prove that (P2) is satisfied, we assume that $\mathcal{R}^R_{\mathbb{C}}(M)$ is connected. We show that under this assumption $\mathcal{R}_{\mathbb{C}}(M)$ is actually a path connected space. Since $\mathcal{R}^R_{\mathbb{C}}(M)$ can be expressed as a subset of $M_{d,m}(\mathbb{C})$ satisfying a system of polynomial equalities and inequalities (see Remark 3.1), the connectedness hypothesis of $\mathcal{R}^R_{\mathbb{C}}(M)$ implies that $\mathcal{R}^R_{\mathbb{C}}(M)$ is path connected. Let $A, B \in \mathcal{R}_{\mathbb{C}}(M)$. Using the first part of our statement, let G_1 and G_2 in $\mathrm{GL}_d(\mathbb{C})$ be invertible matrices of rank d and let D_1, D_2 be complex non-singular diagonal matrices of rank m such that G_1AD_1 and G_2BD_2 belong to $\mathcal{R}^R_{\mathbb{C}}(M)$. Since $\mathcal{R}^R_{\mathbb{C}}(M)$ is path connected, we can find a continuous path $\gamma\colon [0,1] \longrightarrow \mathcal{R}^R_{\mathbb{C}}(M)$ such that $\gamma(0) = G_1AD_1$ and $\gamma(1) = G_2BD_2$. Moreover, from the inclusion $\mathcal{R}^R_{\mathbb{C}}(M) \subseteq \mathcal{R}_{\mathbb{C}}(M)$ and the fact that both these spaces are endowed with the subspace topology of $M_{d,m}(\mathbb{C})$, we see that γ is indeed a continuous path with values in $\mathcal{R}_{\mathbb{C}}(M)$. The group $\mathrm{GL}_d(\mathbb{C})$ is path connected, since it is the complement of the complex hypersurface $\{\det(X) = 0\}$ in $M_{d,d}(\mathbb{C})$. Also the space $D_m(\mathbb{C})$ of complex non-singular diagonal matrices of rank m is path connected, since it can be diffeomorphically identified with the complex torus $(\mathbb{C}^*)^m$. Thus there exist continuous paths

$$\sigma_1, \sigma_2\colon [0,1] \longrightarrow \mathrm{GL}_d(\mathbb{C}) \quad \text{and} \quad \tau_1, \tau_2\colon [0,1] \longrightarrow D_m(\mathbb{C})$$

with

$$\begin{aligned} &\sigma_1(0) = I_d, \quad \sigma_1(1) = G_1, \quad \sigma_2(0) = I_d, \quad \sigma_2(1) = G_2, \\ &\tau_1(0) = I_m, \quad \tau_1(1) = D_1, \quad \tau_2(0) = I_m, \quad \tau_2(1) = D_2. \end{aligned}$$

Now, consider $\Gamma_A(t) = \sigma_1(t)A\tau_1(t)$ and $\Gamma_B(t) = \sigma_2(t)B\tau_2(t)$. Again, using elementary linear algebra arguments, we can easily see that for $t \in [0,1]$ the matrices $\Gamma_A(t)$ and $\Gamma_B(t)$ belong to $\mathcal{R}_{\mathbb{C}}(M)$. The joined path

$$\sigma(t) = \begin{cases} \Gamma_A(3t) & \text{if } t \in [0, 1/3] \\ \gamma(3t-1) & \text{if } t \in [1/3, 2/3] \\ \Gamma_B(3-3t) & \text{if } t \in [2/3, 1] \end{cases}$$

is a continuous path $\sigma\colon [0,1] \longrightarrow \mathcal{R}_{\mathbb{C}}(M)$ with $\sigma(0) = A$ and $\sigma(1) = B$. □

4 Applications

In this section we prove that complex central hyperplane arrangements with at most 7 hyperplanes and the same underlying matroid are isotopic, improving the results of [13] and [25] to any rank. The central idea of our proof is to exploit the connectedness of the reduced realization space of the underlying matroid of these arrangements to apply Proposition 3.1.

Theorem 4.1. *Let $\mathcal{A} = \{H_1, \ldots, H_m\}$ and $\mathcal{B} = \{K_1, \ldots, K_m\}$ be central essential hyperplane arrangements in $\mathbb{C}^d$ with the same underlying matroid. If $m \leq 7$, then $\mathcal{A}$ and $\mathcal{B}$ are isotopic arrangements.*

This result implies that the diffeomorphism type of the complement manifold and the Milnor fiber and fibration of these arrangements are uniquely determined by their underlying matroid.

Corollary 4.1. *Let $\mathcal{A} = \{H_1, \ldots, H_m\}$ and $\mathcal{B} = \{K_1, \ldots, K_m\}$ be central essential hyperplane arrangements in $\mathbb{C}^d$ with the same underlying matroid. If $m \leq 7$, then the following properties are fulfilled:*

(1) *The complement manifolds $M(\mathcal{A})$ and $M(\mathcal{B})$ are diffeomorphic.*

(2) *The Milnor fibrations $Q_{\mathcal{A}}$ and $Q_{\mathcal{B}}$ are isomorphic fiber bundles.*

Proof. (1) follows from Theorem 2.1 and (2) is a consequence of Theorem 2.2. □

To prove Theorem 4.1 some preliminary results are required.

Lemma 4.1. *For a rank d matroid M with ground set $E = \{1, \ldots, m\}$ let A and B be two matrices that realize M over $\mathbb{C}$ and belong to the same connected component of $\mathcal{R}_{\mathbb{C}}(M)$. Then, there exists $\varepsilon > 0$ and a smooth path $\sigma\colon (-\varepsilon, 1+\varepsilon) \longrightarrow M_{d,m}(\mathbb{C})$ such that $\sigma(0) = A$, $\sigma(1) = B$ and $\sigma(t) \in \mathcal{R}_{\mathbb{C}}(M)$ for $t \in (-\varepsilon, 1+\varepsilon)$.*

Proof. Let $\mathfrak{B}$ be the set of bases of M and write

$$\mathcal{R}_{\mathbb{C}}(M) = \left\{ A \in M_{d,m}(\mathbb{C}) \;\middle|\; \begin{array}{ll} \det(A^{j_1}|\ldots|A^{j_d}) \neq 0 & \text{if } \{j_1, \ldots, j_d\} \in \mathfrak{B} \\ \det(A^{j_1}|\ldots|A^{j_d}) = 0 & \text{if } \{j_1, \ldots, j_d\} \notin \mathfrak{B} \end{array} \right\}$$

where A^j is the j-th column of A. This expresses the space $\mathcal{R}_{\mathbb{C}}(M)$ as a subset of $M_{d,m}(\mathbb{C})$ satisfying a system of equalities and inequalities of polynomial type. As a consequence, each connected component $\mathcal{C}$ of $\mathcal{R}_{\mathbb{C}}(M)$ is actually a piecewise linear path connected space, i.e. the following property holds:

For all $A, B \in \mathcal{C}$ there exist $\varepsilon > 0$ and a piecewise linear path $\gamma\colon (-\varepsilon, 1+\varepsilon) \to \mathcal{C}$ with $\gamma(0) = A$ and $\gamma(1) = B$.

Since the equalities and inequalities that define $\mathcal{R}_{\mathbb{C}}(M)$ are of polynomial type, it is not hard to see that it is possible to reparametrize the path γ by stopping of infinite order at each point where it it is not smooth (using pieces like $t \mapsto e^{-1/t^2}$) in order to find a smooth path $\sigma\colon (-\varepsilon, 1+\varepsilon) \to M_{d,m}(\mathbb{C})$ such that $\sigma(0) = A$, $\sigma(1) = B$ and $\sigma(t) \in \mathcal{C}$ for $t \in (-\varepsilon, 1+\varepsilon)$. □

Lemma 4.2. *Let M be a rank d matroid with ground set $E = \{1, \ldots, m\}$ and $\{1, \ldots, d\}$ as a basis. If M is realizable over $\mathbb{C}$ and $1 \leq d \leq m \leq 7$, then the space $\mathcal{R}^R_{\mathbb{C}}(M)$ is non-empty and connected.*

Proof. Since by hypothesis M is realizable over $\mathbb{C}$, the space $\mathcal{R}_{\mathbb{C}}(M)$ is non-empty, and by Proposition 3.1 we get $\mathcal{R}^R_{\mathbb{C}}(M) \neq \emptyset$. The space $\mathcal{R}^R_{\mathbb{C}}(M)$ can be expressed as a subset of $M_{d,m}(\mathbb{C})$ satisfying a system of polynomial equalities and inequalities (see Remark 3.1). By [21, Chapter 7, Theorem 7.1] to prove connectedness it is enough to show that $\mathcal{R}^R_{\mathbb{C}}(M)$ is irreducible in the Zariski topology. We checked this for all matroids M satisfying the hypothesis by a direct computation with the aid of the computer algebra system Sage [7]; for more details, see Appendix A. □

Proof of Theorem 4.1. It is clear that the only interesting case is when $d \geq 1$ and $m \geq 1$. To prove our statement we have to distinguish the two Cases $1 \leq d \leq m$ and $1 \leq m < d$.

Case $1 \leq d \leq m$. Let M be the underlying matroid of the arrangements $\mathcal{A}$ and $\mathcal{B}$. Relabelling the hyperplanes $\{H_i\}$ of $\mathcal{A}$ and $\{K_i\}$ of $\mathcal{B}$, we can suppose that $\{1, \ldots, d\}$ is a basis of M; note that we can always do this, since $\mathcal{A}$ and $\mathcal{B}$ are essential arrangements. Pick linear forms α_i and β_i such that $H_i = \ker \alpha_i$ and $K_i = \ker \beta_i$. We denote by α_i^j and β_i^j the j-th component of α_i and β_i, respectively. Set $A = (\alpha_i^j)^t$ and $B = (\beta_i^j)^t$. Now consider the space $\mathcal{R}_{\mathbb{C}}(M)$. The matrices A and B belong to $\mathcal{R}_{\mathbb{C}}(M)$. Hence, to prove that $\mathcal{A}$ and $\mathcal{B}$ are isotopic arrangements (compare Definition 2.2) it is enough to show that there exist $\varepsilon > 0$ and a smooth path $\sigma\colon (-\varepsilon, 1+\varepsilon) \to M_{d,m}(\mathbb{C})$ with $\sigma(0) = A$, $\sigma(1) = B$ and $\sigma(t) \in \mathcal{R}_{\mathbb{C}}(M)$ for t in $(-\varepsilon, 1+\varepsilon)$. By Lemma 4.1 it suffices to check that $\mathcal{R}_{\mathbb{C}}(M)$ is connected. To see this, thanks to Proposition 3.1, we can just verify the connectedness of $\mathcal{R}^R_{\mathbb{C}}(M)$. Thus the statement follows from Lemma 4.2.

Case $1 \le m < d$. This follows from elementary complex linear algebra arguments. As previously, for the hyperplanes $\{H_i\}$ of $\mathcal{A}$ and $\{K_i\}$ of $\mathcal{B}$ we choose linear forms α_i and β_i with $H_i = \ker \alpha_i$ and $K_i = \ker \beta_i$. We denote by α_i^j and β_i^j the j-th component of α_i and β_i, respectively. Set $A = (\alpha_i^j)^t$ and $B = (\beta_i^j)^t$. Now consider the space

$$\mathcal{S}_{d,m}(\mathbb{C}) = \{Q \in M_{d,m}(\mathbb{C}) \mid \operatorname{rk} Q = m\}$$

and note that A and B belong to $\mathcal{S}_{d,m}(\mathbb{C})$ since the arrangements $\mathcal{A}$ and $\mathcal{B}$ are essential. Again, to show that $\mathcal{A}$ and $\mathcal{B}$ are isotopic it suffices to prove that there exist $\varepsilon > 0$ and a smooth path $\sigma\colon (-\varepsilon, 1+\varepsilon) \to M_{d,m}(\mathbb{C})$ with $\sigma(0) = A$, $\sigma(1) = B$ and $\sigma(t) \in \mathcal{S}_{d,m}(\mathbb{C})$ for t in $(-\varepsilon, 1+\varepsilon)$. With the same arguments as for Lemma 4.1, it is enough to verify the connectedness of $\mathcal{S}_{d,m}(\mathbb{C})$. To prove this, we write

$$\mathcal{S}_{d,m}(\mathbb{C}) = \bigcup_{1 \le i_1 < \cdots < i_m \le d} \mathcal{S}_{d,m}^{i_1 \dots i_m}(\mathbb{C})$$

where

$$\mathcal{S}_{d,m}^{i_1 \dots i_m}(\mathbb{C}) = \{Q \in M_{d,m}(\mathbb{C}) \mid \det(Q_{i_1}^t | \dots | Q_{i_m}^t) \neq 0\}$$

and Q_i is the i-th row of Q. Each space $\mathcal{S}_{d,m}^{i_1 \dots i_m}(\mathbb{C})$ is path connected, since it is the complement of the complex hypersurface $\{\det(Q_{i_1}^t | \dots | Q_{i_m}^t) = 0\}$ in $M_{d,m}(\mathbb{C})$. Hence, to conclude our proof it is sufficient to show that

$$\bigcap_{1 \le i_1 < \cdots < i_m \le d} \mathcal{S}_{d,m}^{i_1 \dots i_m}(\mathbb{C}) \neq \emptyset,$$

or equivalently that

$$\prod_{1 \le i_1 < \cdots < i_m \le d} \det(Q_{i_1}^t | \dots | Q_{i_m}^t)$$

is not the zero polynomial. None of the factors $\det(Q_{i_1}^t | \dots | Q_{i_m}^t)$ is the zero polynomial. Thus, the statement follows from the fact that the ring of polynomials in dm variables with complex coefficients is an integral domain. □

Remark 4.1. Note that in the Case $1 \le m < d$ we never used the assumption $m \le 7$, so in this situation the result of Theorem 4.1 holds without any numerical restriction.

A Appendix: Checking connectedness of reduced realization spaces

We show by a direct test that Lemma 4.2 holds. For a rank d matroid M with ground set $E = \{1, \dots, m\}$ and $\{1, \dots, d\}$ as a basis, we consider a matrix $G_{0,M} \in M_{d,m}(\mathbb{C})$ with all entries equal to -1 and perform the following sequence of operations:

(S1) We insert a $d \times d$ identity matrix corresponding to the first d columns of $G_{0,M}$. We call this matrix $G_{1,M}$.

(S2) For $1 \le i \le d$ and $d+1 \le j \le m$ we set the (i,j)-th entry of $G_{1,M}$ equal 0 if $(\{1, \dots, d\} \setminus \{i\}) \cup \{j\}$ is not a basis of M. We call this matrix $G_{2,M}$.

(S3) Let $\tilde{G}_{2,M}$ be the $d \times (m-d)$ matrix such that $G_{2,M} = (I_d | \tilde{G}_{2,M})$. We set the entries of $\tilde{G}_{2,M}$ with positions in the normal frame $\mathcal{P}_{\tilde{G}_{2,M}}$ equal 1. We call this matrix $\tilde{G}_{3,M}$ and we set $G_{3,M} = (I_d | \tilde{G}_{3,M})$.

(S4) We denote by s_M the number of -1 entries of $G_{3,M}$.

(S5) We replace the -1 entries of $G_{3,M}$ with symbolic variables $t_1, \dots, t_{s_M}$ and call this matrix G_M.

Algorithm 1 (`TestIrreducibility`).

Require: `case` $= (d, m)$ a pair from Equation $(**)$.

Ensure: `True` if the reduced realization spaces of all realizable matroids of type `case` are irreducible, `False` otherwise.

Compute the list `subsets` of all subsets of d elements of $\{1, \dots, m\}$ and order it with respect to the reverse lexicographic term order.
for `matroid` in `all_matroids[case]` **do**
  Compute the first basis for `matroid` in the list `subsets` and call it `basis`.
  Set G = `FillMatrix(case, basis)`.
  ▷ Computing the (in)equalities for $X_{\texttt{matroid}}$.
  Substitute the -1 entries of G with symbolic variables.
  Set `equalities = emptylist` and `inequalities = emptylist`.
  for `subset` in `subsets` **do**
    Set `det` to be the $d \times d$ minor corresponding to the submatrix of G whose columns are prescribed by `subset`.
    if `subset` is a basis for `matroid` **then Add** `det` to `inequalities`.
    else Add `det` to `equalities`.
    end if
  end for
  ▷ Checking irreducibility of the zero set determined by only the equalities.
  Set `ideal` to be the ideal generated by `equalities`.
  if the zero set of `ideal` is not geometrically irreducible **then**
    return `False`.
  end if
end for
return `True`.

Algorithm 2 (`FillMatrix`).
Require: `case` = (d, m), a pair from Equation $(**)$; `basis`, a subset of $\{1, \dots, m\}$ of cardinality d.
Ensure: a matrix G with entries belonging to $\{-1, 0, 1\}$ and ensuring Conditions 2 and 3 from Definition 3.2.

Create a $d \times m$ matrix G, and fill it with -1 entries.
Set `non_basis` to be equal to the set $\{1, \dots, m\} \setminus$ `basis`.
▷ Imposing Condition 2.
Insert in G a $d \times d$ identity matrix in correspondence to the columns of `basis`.
▷ Inserting as many zeroes as possible in G.
for j in `non_basis` **do**
  for $i \in \{1, \dots, d\}$ **do**
    if $\big(\{1, \dots, d\} \setminus \{i\}\big) \cup \{j\}$ is not a basis of `matroid` **then**
      Set $G(i, j) = 0$.
    end if
  end for
end for
▷ Computing the normal frame and imposing Condition 3.
▷ Inserting 1s column by column.
for j in `non_basis` **do**
  Set $r = 1$.
  while $G(r, j) = 0$ **do**
    Increase r by 1.
    if $r = d + 1$ **then Break** the loop.
    end if
  end while
  if $r \le d$ **then Set** $G(r, j) = 1$.
  end if

```
end for
                                                    ▷ Inserting 1s row by row.
for i ∈ {1, . . . , d} do
    Set c = 1.
    while G(i, c) = 0 or G(i, c) = 1 do
        Increase c by 1.
        if c = m + 1 then Break the loop.
        end if
    end while
    if c ≤ m then Set G(i, c) = 1.
    end if
end for
return G.
```

Definition A.1. For a rank d matroid M with ground set $E = \{1, \dots, m\}$ and $\{1, \dots, d\}$ as a basis, the *reduced variety* of M over $\mathbb{C}$ is the quasi-projective variety X_M defined by

$$X_M = \left\{ (z_1, \dots, z_{s_M}) \in \mathbb{A}^{s_M}_{\mathbb{C}} \;\middle|\; \begin{array}{lll} \det(G_M^{j_1}|\dots|G_M^{j_d}) \neq 0 & \text{if} & \{j_1, \dots, j_d\} \in \mathfrak{B} \\ \det(G_M^{j_1}|\dots|G_M^{j_d}) = 0 & \text{if} & \{j_1, \dots, j_d\} \notin \mathfrak{B} \end{array} \right\}$$

where G_M^j is the j-th column of G_M and $\mathfrak{B}$ denotes the set of bases of M.

Remark A.1. The defining equalities and inequalities of X_M have integer coefficients.

Comparing Definition A.1 and Definition 3.2, it is not hard to see that the quasi-projective variety X_M is isomorphic to the space $\mathcal{R}^R_{\mathbb{C}}(M)$ endowed with the Zariski topology (see Remark 3.1 for more details). Taking this into account, from now on we are concerned with the determination of the irreducibility of X_M. Note that if $d = 1$, $d = m$, or $d = m - 1$, then the reduced variety X_M is either empty (in which case $\mathcal{R}^R_{\mathbb{C}}(M) = \emptyset$, so by Proposition 3.1 the matroid M is not realizable over $\mathbb{C}$), or equals a point (thus in particular $\mathcal{R}^R_{\mathbb{C}}(M)$ is irreducible). Hence we are left with the cases when (d, m) belongs to

$$\{(2, 4), (2, 5), (2, 6), (2, 7), (3, 5), (3, 6), (3, 7), (4, 6), (4, 7), (5, 7)\}. \qquad (**)$$

All matroids in these cases are classified, see [11], and the tables describing them are available at the link www-imai.is.s.u-tokyo.ac.jp/~ymatsu/matroid/index.html.

For all matroids M in the cases described by the set $(**)$ we computed the equalities and inequalities defining X_M. Note that if $\hat{X}_M$ is the subset of $\mathbb{A}^{s_M}_{\mathbb{C}}$ defined by

$$\hat{X}_M = \left\{ (z_1, \dots, z_{s_M}) \in \mathbb{A}^{s_M}_{\mathbb{C}} \mid \det(G_M^{j_1}|\dots|G_M^{j_d}) = 0 \text{ if } \{j_1, \dots, j_d\} \notin \mathfrak{B} \right\}$$

where G_M^j is the j-th column of G_M and $\mathfrak{B}$ is the set of bases of M, then by elementary topology arguments the irreducibility of $\hat{X}_M$ implies the one of X_M, if the latter is non-empty. We checked that $\hat{X}_M$ is always irreducible, hence we conclude that X_M is always either empty, or irreducible. There are algorithms that decide whether an algebraic set defined by rational equalities (as $\hat{X}_M$, recall Remark A.1) is irreducible or not, see for example [5]; in our case, via a direct inspection helped by computations with Sage, we noticed that all sets $\hat{X}_M$ fall into one of these families

- Linear varieties
- Rational hypersurfaces
- Quadrics of rank strictly bigger than 2,

or they are cones over such varieties, and so are irreducible by easy algebraic geometry arguments. The Sage code we used to perform the test is available at the link
http://matteogallet.altervista.org/main/papers/hyperplanes2015/hyperplanes.sage

The algorithm `TestIrreducibility` provided in Algorithm 1 describes the pseudocode of the main procedure we implemented, and the algorithm `FillMatrix` presented in Algorithm 2 sketches the pseudocode of the ancillary algorithm we used to build the matrix G_M (compare the definition of the operations 1, 2, 3, 4 and 5).

Remark A.2. Note that the assumption $m \leq 7$ does not play any role in any of the algorithms presented in this Appendix. The only reason to limit ourselves to the case $m \leq 7$ is the fact that for $m > 7$ the total number of matroids becomes significantly bigger, and moreover both the number and the degree of the equalities and inequalities defining X_M increases. Thus the computations for checking whether X_M is irreducible become more and more expensive in terms of memory and time, and moreover the cases when $\hat{X}_M$ does not fall into one of the simple families of varieties reported above become much more frequent. Hence one would need to improve the existing algorithm and to find new families of algebraic varieties that ensure irreducibility in order to attack the cases when $m > 7$, taking also into account the already known cases of matroids for which the variety X_M is reducible.

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