Dynamic Withdrawals and Stochastic Mortality in GLWB Variable Annuities

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Abstract. In this paper we propose a discrete time model, based on dynamic programming, to price GLWB variable annuities under the dynamic approach within a stochastic mortality framework. Our set-up is very general and only requires the Markovian property for the mortality intensity and the asset price processes. We also show the validity of the bang-bang condition for the set of discrete withdrawal strategies of the model. This result allows to drastically reduce the computational time needed to search the optimal withdrawal in the backward recursive step of our dynamic algorithm and provides, as a by-product, an interesting contract decomposition.

Keywords: GLWB · Dynamic withdrawals · Bang-bang condition · Stochastic mortality

1 Introduction

Variable annuities (VAs) are very flexible life insurance investment products that package living and death benefits endowed with a number of possible guarantees in respect of financial or biometric risks. A rider that can be included in a VA contract in order to provide a post-retirement income is the Guaranteed Lifelong Withdrawal Benefit (GLWB), that offers a lifelong withdrawal guarantee. There has been a number of papers dealing with pricing of the VA products. Most of them are focused on pricing VA guarantees under the *static* policyholder behaviour (see e.g., [1]), meaning that the policyholder always withdraws exactly the guaranteed amount, and never surrenders the contract. Some studies include pricing under the *dynamic* approach, when the policyholder optimally decides the amount to withdraw at each withdrawal date depending on the information available at that date (see, e.g., [2]). According to whether withdrawals are assumed to occur continuously or discretely, the optimal withdrawal problem under the dynamic approach is generally solved using, respectively, stochastic control and dynamic programming [3]. In this paper we propose a discrete time model, based on dynamic programming, to price VAs with GLWB under the dynamic approach within a stochastic mortality framework. Our set-up is very general and only requires the Markovian property for the mortality intensity and the asset price processes. Another contribution of our paper is the verification of the bang-bang condition for the set of discrete withdrawal strategies of the GLWB model. This means that the set of the optimal withdrawals consists of three choices only: zero withdrawal, withdrawal at the contractual amount, complete surrender. This result, proven in our discrete time framework, is particularly remarkable as in the insurance literature either the existence of optimal bang-bang controls is assumed or it requires suitable conditions (see e.g., [4]). The bang-bang condition, beyond drastically reducing the computational time needed to search the optimal withdrawal in the backward recursive step of our dynamic algorithm, allows to clearly separate the various contract components.

The remainder of this paper is organized as follows. In Sect. 2 we describe the structure of the VA contract. In Sect. 3 we introduce our valuation framework and define the optimal withdrawal problem. In Sect. 4 we first define the dynamic programming equations that allow to solve the problem, then we introduce the bang-bang condition and outline the proof of its validity, and after we present the contract decomposition. Finally, Sect. 5 concludes the paper.

2 The Contract Structure

In this section we describe the GLWB rider in our variable annuity contract. At time 0 (contract inception), the policyholder, aged x, pays a single premium P which is entirely invested in a well-diversified and non-dividend paying mutual fund of her own choice. We denote by S_t the market price at time t of each unit of this fund, that drives the return on the investment portfolio built up with the policyholder's payment. The value at time t of such portfolio, that is called 'personal account', is denoted by W_t. The GLWB rider gives the policyholder the right to make periodical withdrawals from her account at some specified dates for the whole life, even if the account value is reduced to zero. The cost of the guarantee is financed by periodical proportional deductions from the personal account value, while the guaranteed withdrawal amount is calculated as a fixed proportion g of the 'benefit base', denoted by A_t, which is initially set equal to the single premium. In addition, the benefit base can be adjusted upward via the 'roll-up' feature, that applies when no withdrawal is made on a specified withdrawal date. Both the complete surrender of the policy and the policyholder's death are events that cause the closure of the contract. The value that remains in the personal account when the policyholder dies is paid to the beneficiary as a death benefit. In particular, from now on we assume that: (i) withdrawals are allowed on a predetermined set of equidistant dates and we take the distance between two consecutive dates as unit of measurement of time; (ii) the death benefit is paid to the beneficiary on the next upcoming withdrawal date. Let τ denote the time of death of the policyholder, so that withdrawals are allowed only at times $i = 1, 2, \ldots$, provided that $\tau > i$. The guaranteed amount that can be withdrawn at time i is equal to gA_i, and the return on the reference fund over the interval $[i-1, i]$ is $R_i = (S_i/S_{i-1}) - 1,\ i = 1, 2, \ldots$. We denote by y_i the actual withdrawal made by the policyholder at time i and, under our dynamic approach, we assume that the set of possible withdrawals at this time is given by the interval $[0, \max\{gA_i, W_i\}]$. If the policyholder does not withdraw anything at time i, the benefit base is proportionally increased according to the roll-up rate, that we denote by b_i (with $0 < b_i < 1$), while, if the withdrawal exceeds gA_i, it is proportionally reduced according to the so called 'pro-rata' adjustment rule. Then the benefit base evolves as follows:

$$A_{i+1} = f_{i+1}^A(W_i, A_i, y_i) = \begin{cases} A_i(1+b_i) & \text{if } \; y_i = 0, \\ A_i & \text{if } \; 0 < y_i \leq gA_i, \\ A_i \dfrac{W_i - y_i}{W_i - gA_i} & \text{if } \; gA_i < y_i \leq W_i \end{cases}, \; i = 1, 2, \ldots, \quad (1)$$

with $A_1 = P$. Moreover, in case of withdrawals exceeding the guaranteed amount, there is also a proportional penalization on the surplus according to a penalty rate, that we denote by k_i (such that $0 < k_i < 1$). Therefore, the net amount (cash-flow) received by the policyholder at time i is given by

$$B_i^{(s)} = f_i^{(s)}(y_i, A_i) = \quad y_i - k_i \max\{y_i - gA_i, 0\}, \quad i = 1, 2, \ldots \; . \quad (2)$$

The policy account value evolves according to the following equation:

$$W_{i+1} = f_{i+1}^W(W_i, R_{i+1}, y_i) = \max\{W_i - y_i, 0\}(1 + R_{i+1})(1 - \varphi), \;\; i = 0, 1, \ldots, \quad (3)$$

where φ (such that $0 < \varphi < 1$) is the insurance fee rate, $W_0 = P$ and $y_0 = 0$. Note that 0 is an absorbent barrier for W because, once it becomes null, it remains so for ever. The contract, however, continues while $A_t > 0$ (and the insured is still alive). Finally, in case of death in the time interval $(i-1, i]$, the death benefit, paid at time i, is

$$B_i^{(d)} = W_i, \quad i - 1 < \tau \leq i \;, \quad i = 1, 2, \ldots \; . \quad (4)$$

In case of surrender at time i, i.e., when $y_i = W_i > gA_i$, the contract is automatically closed because (1) and (3) imply $A_t = W_t = 0$ for all $t > i$, hence no further withdrawals are admitted, nor a death benefit will be paid.

3 The Valuation Framework

In this section we introduce our valuation framework and define the optimal withdrawal problem. Consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, Q)$ supporting all sources of financial and biometric uncertainty, where all random variables and processes are defined. The filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ satisfies the usual conditions of right continuity and completeness, and is such that $\mathcal{F}_0$ is Q-trivial. Q is a risk-neutral probability measure selected by the insurer, for pricing purposes, among the infinitely many equivalent martingale measures existing in incomplete arbitrage-free markets. In this setting, the residual lifetime of the policyholder τ is a stochastic $\mathbb{F}$-stopping time and let $\mu_t := \mu_{x+t}(t)$ be the mortality intensity which determines the probability of death at time t conditional on survival for the policyholder aged x at time 0. Moreover, we suppose there is independence between financial- and biometric-related variables. In this general framework we can consider for μ any (reasonable) Markovian process and denote by

$$p_i(\mu_i) = Q\left(\tau > i+1 \,|\, \tau > i, \mu_i\right) = \mathbb{E}^Q\left[\mathrm{e}^{-\int_i^{i+1}\mu_u\,\mathrm{d}u}\,|\mu_i\right], \quad i = 0, 1, \ldots, \tag{5}$$

the probability of survival up to $i+1$ for the policyholder still alive at age $x+i$ given the mortality intensity's values up to i. Consequently, $q_i(\mu_i) = 1 - p_i(\mu_i)$ is the probability of death before $i+1$ conditional on survival at time i. Concerning the financial uncertainty, we assume the instantaneous interest rate to be deterministic and constant, and denote it by r. The reference price S, instead, can be any Markovian process whose discounted value is a martingale under Q. Consider now a withdrawal strategy $y = (y_i)_{i\in\mathbb{N}^+}$, where y_i denotes the actual withdrawal made at time i (in case of survival). This is a stochastic process, adapted to the filtration $\mathbb{F}$, because at each withdrawal date the policyholder takes her withdrawal decision once she knows the values of all state variables. This strategy is *admissible* if it belongs to the set of admissible withdrawal strategies $Y = (Y_i)_{i\in\mathbb{N}^+}$, where $Y_i = [0,\, \max\{W_i,\, gA_i\}]$. Then we define the initial value of the GLWB variable annuity as the solution of the following optimization problem:

$$V_0 = \sup_{y\in Y}\mathbb{E}^Q\left[\sum_{i=1}^{\infty}\mathrm{e}^{-ri}\left(\mathbf{1}_{\{\tau>i\}}f_i^{(s)}(y_i, A_i) + \mathbf{1}_{\{i-1<\tau\leq i\}}W_i\right)\right], \tag{6}$$

where the account value and the benefit base satisfy (3) and (1) respectively. Hence the policyholder is assumed to maximize the present expected value, under Q, of all the future cash-flows generated by the VA contract.

4 Dynamic Programming

In this section we implement a dynamic programming algorithm for discrete stochastic control problems to solve (6). In particular, as we act in a Markovian framework, for each i we denote by $V_i(W_i, A_i, \mu_i)$ the contract value at time i (before the periodic withdrawal) and by $v_i(W_i, A_i, \mu_i)$ the contract value at the same time when, moreover, the policyholder is then alive. Clearly $V_i(W_i, A_i, \mu_i) = \mathbf{1}_{\{\tau>i\}}v_i(W_i, A_i, \mu_i)$ and $V_0 = V_0(P, P, \mu_0) = v_0(P, P, \mu_0)$.

Since the algorithm proceeds backward, we need a starting point. To this end, we assume that there is an ultimate age for the policyholder beyond which her survival probability is null. We denote by ω this age, that typically is in the range 110-120 years, and let $n = \max\{i \in \mathbb{N} : \omega - x \leq i+1\}$, hence $n < \omega - x \leq n+1$. Then Eq. (5) is valid only for $i < n$, while $p_i(\mu_i) \equiv 0$ for $i \geq n$. Therefore, the optimal problem (6) can be rewritten as

$$V_0 = \sup_{y\in Y}\mathbb{E}^Q\left[\sum_{i=1}^{n}\mathrm{e}^{-ri}\left(\mathbf{1}_{\{\tau>i\}}f_i^{(s)}(y_i, A_i) + \mathbf{1}_{\{i-1<\tau\leq i\}}W_i\right) + \mathrm{e}^{-r(n+1)}\mathbf{1}_{\{\tau>n\}}W_{n+1}\right] \tag{7}$$

We take $n+1$ as starting point of our backward dynamic algorithm, and define the following terminal condition:

$$v_{n+1}(W_i, A_i, \mu_i) \equiv 0. \tag{8}$$

Then we proceed backward and, for $i = n, n-1, \dots, 1$, we define the Bellman recursive equation of the problem as follows:

$$v_i(W_i, A_i, \mu_i) = \sup_{y_i \in Y_i} \Bigg(f_i^{(s)}(y_i, A_i) \; + \; q_i(\mu_i) \max\{W_i - y_i,\, 0\}(1-\varphi) \; + \tag{9}$$

$$\mathbb{E}^Q \left[\mathrm{e}^{-\int_i^{i+1} \mu_u \, \mathrm{d}u} v_{i+1}\left(f_{i+1}^W(W_i, R_{i+1}, y_i), f_{i+1}^A(W_i, A_i, y_i), \mu_{i+1} \right) \mathrm{e}^{-r} | W_i, A_i, \mu_i \right] \Bigg).$$

Finally, the initial contract value is given by

$$v_0(P, P, \mu_0) = q_0(\mu_0) P (1-\varphi) + \mathbb{E}^Q \left[\mathrm{e}^{-\int_0^1 \mu_u \, \mathrm{d}u} v_1(P(1+R_1)(1-\varphi), P, \mu_1) e^{-r} \right]. \tag{10}$$

4.1 Bang-Bang Analysis

At each time step $i = n, n-1, \dots 1$, Eq. (9) requires to solve a real-valued optimization problem where the domain of y_i is the whole interval $Y_i = [0,\, \max\{W_i,\, gA_i\}]$. Moreover, this problem must be solved for every possible triplet of state variables (W_i, A_i, μ_i). Then the computational effort could be substantial. A property that drastically reduces this effort is the *bang-bang* condition, which states that the set of the optimal withdrawals consists of three choices only: zero withdrawal, withdrawal at the contractual amount, complete surrender. Such a condition is satisfied for our problem, indeed the optimal solution of (9) is $y_i = 0$, or $y_i = gA_i$, or $y_i = W_i$.

Now we outline the proof, that can be made by backward induction. First of all, through tedious computations it is easy to show that the function to maximise at step n is a continuous linear spline, defined in the closed interval Y_n, with a single knot given by $\min\{W_n,\, gA_n\}$, and that its maximizer belongs to the set $\{W_n,\, gA_n\}$. In addition, the value function at this step takes the form $v_n(W_n, A_n, \mu_n) = C_n(\mu_n) W_n + D_n(\mu_n) g A_n$, where C_n and D_n are two (constant) functions such that $0 \leq C_n(\mu_n) < 1$ and $D_n(\mu_n) > 0$. Then, assuming $v_{i+1}(W_{i+1}, A_{i+1}, \mu_{i+1}) = C_{i+1}(\mu_{i+1}) W_{i+1} + D_{i+1}(\mu_{i+1}) g A_{i+1}$ for $i = n-1, \dots, 1$, with $0 \leq C_{i+1}(\mu_{i+1}) < 1$ and $D_{i+1}(\mu_{i+1}) > 0$ (almost surely), it is easy to show that the function to maximize at step i is a linear spline defined in the closed interval Y_i. This function is discontinuous at 0, where it takes a value strictly greater than its right limit, and continuous in the (only) knot given by $\min\{W_i,\, gA_i\}$. Hence the conclusion is that its maximizer belongs to the set $\{0,\, W_i,\, gA_i\}$ and also at this step the value function takes the form $v_i(W_i, A_i, \mu_i) = C_i(\mu_i) W_i + D_i(\mu_i) g A_i$, with $0 \leq C_i(\mu_i) < 1$ and $D_i(\mu_i) > 0$.

4.2 Contract Decomposition

It is clear that the valuation algorithm aimed at producing the contract value under the dynamic approach can be used to obtain, as simplified cases, also the contract values under alternative policyholder behaviours, namely under the static and the mixed[1] approaches. To obtain the value under the static approach it is sufficient to fix $y_i = gA_i$ for any $i = 1, 2, \ldots, n$, without searching any maximum, while to obtain the value under the mixed approach the search of the maximum must be restricted to the subset $\{gA_i, W_i\}$. To distinguish between these three different values we denote them, respectively, by $V_0^{dynamic}$, V_0^{static} and V_0^{mixed}. Then we can see the dynamic contract as the combination of three components: the *basic GLWB contract*, i.e., the static one, the *surrender option* (with value given by $V_0^{surrender} := V_0^{mixed} - V_0^{static}$), and the *roll-up option* (whose value is $V_0^{rollup} := V_0^{dynamic} - V_0^{mixed}$):

$$V_0^{dynamic} = V_0^{static} + V_0^{surrender} + V_0^{rollup}.$$

5 Conclusion

In this paper we have proposed a discrete time model, based on dynamic programming, to price GLWB variable annuities under the dynamic approach within a stochastic mortality framework. We have verified, by backward induction, the bang-bang condition for the set of discrete withdrawal strategies of the model, and offered an interesting contract decomposition. We have considered a quite general set-up, only requiring the Markovian property for the mortality intensity and the asset price processes. However, to keep the curse of dimensionality of our valuation algorithm manageable, we have assumed constant interest rates. Our next step is the numerical implementation of the model by focussing on a square root process for the mortality intensity and an exponential Lévy process for the asset price. Moreover, the inclusion of stochastic interest rates is a challenging topic for future research.

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[1] That is withdrawal of the guaranteed amount or complete surrender, see [3].