Model for baryon production in spin-dependent string fragmentation

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We introduce spin-$1/2$ baryons in the string + $^3P_0$ model of hadronization, previously restricted to the production of pseudoscalar and vector mesons. Baryons are modeled as quark–diquark bound states, and baryon production is described by the tunneling of diquark–antidiquark pairs at string breaking points. Diquarks can be scalar or pseudovector, the latter being produced in the relative $^5D_0$ state. Introducing the quark–baryon–diquark coupling, the relevant splitting amplitudes for the emission of baryons are constructed and used to explore analytically the model predictions. We find a Collins effect for baryon production in the fragmentations of transversely polarized quarks or diquarks as well as a baryon spontaneous $\mathbf{p}_{\rm T}$-dependent transverse polarization. The model is presented in a form suitable for implementation in a Monte Carlo event generator. The hadronic decays of polarized hyperons are also included.

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I. INTRODUCTION

Quark spin effects in hadronization, the process of conversion of quarks and gluons in hadrons, have been shown to be non-negligible in different processes such as proton–proton scattering, semi-inclusive deep inelastic scattering (SIDIS), and e^+e^- annihilation to hadrons. They are commonly described in quantum chromodynamics (QCD) by the fragmentation functions (FFs), which encode the underlying nonperturbative and poorly known physics mechanisms.

An interesting phenomenon is the fragmentation $q^\uparrow \to h + X$ of a transversely polarized quark $q^\uparrow$ in the hadron h, known as the Collins effect [1]. It is described by the FF

$$D_{q^\uparrow\to h+X}(z,\mathbf{p}_{\rm T}) = D^h_{1q}(z,\mathbf{p}^2_{\rm T})[1 + a_{C,h}\mathbf{S}_{q,{\rm T}}\cdot\hat{\boldsymbol{\nu}}], \quad (1)$$

where D^h_{1q} is the spin-averaged FF depending on the fraction z of the forward lightcone momentum of q taken by h and on the transverse momentum $\mathbf{p}_{\rm T}$ of h with respect to the quark momentum $\mathbf{k} = |\mathbf{k}|\hat{\boldsymbol{k}}$, and $\hat{\boldsymbol{\nu}} = \hat{\boldsymbol{k}}\times\hat{\mathbf{p}}_{\rm T}$. If the quark transverse polarization $\mathbf{S}_{q,{\rm T}}$ is different from zero, the mixed product in the second term in Eq. (1) produces a modulation in the distribution of the azimuthal angle of the hadron. The amplitude of the modulation is given by the Collins analyzing power [1] $a_{C,h}(z,p_{\rm T})$; the *Collins FF* is $H^{\perp h}_{1q}(z,p_{\rm T}) = a_{C,h}(z,p_{\rm T})D^h_{1q}(z,p_{\rm T})$. The Collins effect has been measured to be nonvanishing in semi-inclusive deep inelastic scattering (SIDIS) with a transversely polarized proton target [2,3] and in e^+e^- annihilation to hadrons [4–8].

If pairs of hadrons h_1h_2 are measured among the products of the fragmentation of the same quark, i.e., in the inclusive process $q^\uparrow \to h_1h_2 + X$, a modulation in the relative momentum of the pair, known as the dihadron production asymmetry, shows up [9,10]. The effect is described by the *interference FF* and has been measured to be nonvanishing in the same conditions as the Collins effect [11–13].

The Collins FF and the interference FF are particularly important, as they provide access to the partonic transverse spin structure of the nucleons and have been extracted by different groups using SIDIS and e^+e^- data (for a review see, *e.g.*, Ref. [14]).

Spin effects show up also in the fragmentation of unpolarized quarks. An example is the *spontaneous* transverse polarization of Λ and $\bar{\Lambda}$ hyperons, i.e. the polarization along the vector perpendicular to the hyperon production plane. The effect was measured to be nonvanishing for the first time in the 70s in high-energy hadronic collisions [15]. More recently, it was measured by the HERMES experiment in deep inelastic scattering by the reaction $l + N \to Y + X$, with $Y = \Lambda, \bar{\Lambda}$ [16], and in e^+e^- annihilation by the BELLE experiment via the reactions $e^+e^- \to Y + X$ and $e^+e^- \to Y + \pi(K) + X$, where the hyperon and

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the meson π or K are produced in opposite hemispheres in the center of mass system of the event [17].

After the first measurements of the spontaneous polarization of hyperons, different models were put forward to explain the observed effects (for a review see, e.g., Ref. [18]). A description of the Λ polarization was given by the Lund group [19] in the frame of the Lund Model (LM) of string fragmentation [20]. The polarization was attributed to the correlation between the total spin and the relative momentum of a $q\bar{q}$ pair produced at a string breaking in the $^3P_0$ state, i.e. with orbital angular momentum $L = 1$ and total spin $S = 1$ such that $\mathbf{J} = \mathbf{L} + \mathbf{S}$ vanishes.

The current QCD description of the spontaneous polarization of a spin 1/2-baryon $B^\uparrow$ produced in the fragmentation $q \to \mathrm{B}^\uparrow + X$ is given in terms of the *polarizing FF* [21,22]. The complete FF takes the form

$$D_{q\to B^\uparrow+X}(z, \mathbf{p}_\mathrm{T}) = \frac{D^B_{1q}(z, \mathbf{p}^2_\mathrm{T})}{2}[1 + a_B \check{\mathbf{S}}_B \cdot \hat{\boldsymbol{\nu}}], \qquad (2)$$

where $\check{\mathbf{S}}_B$ is a chosen direction for the polarization vector of B. The quantity a_B is related to the polarizing FF $D^{\perp B}_{1T,q}(z, p_\mathrm{T}) = a_B(z, p_\mathrm{T}) D^B_{1q}(z, p_\mathrm{T})$, and it gives the magnitude of the polarization of B. The predicted polarization of B is directed along the vector $\hat{\boldsymbol{k}} \times \hat{\mathbf{p}}$, which is perpendicular to the production plane of B. An extraction of the polarizing FFs for Λ and $\bar{\Lambda}$ has been performed for proton–proton data in Ref. [22] and more recently for e^+e^- data from BELLE in Refs. [23–25].

A different approach to tackling the dependence of hadronization on quark spin consists in modeling the underlying physics and implementing the model in a Monte Carlo event generator (MCEG). This is the approach that we have followed by developing the string + $^3P_0$ model of polarized hadronization [26–28]. The string + $^3P_0$ model is a recursive quantum mechanical model that accounts for the systematic propagation of the spin information along the fragmentation chain. It is inspired by the multiperipheral model [29], and implements the string fragmentation dynamics of the LM and the assumption that $q\bar{q}$ pairs at string breakings are produced in the $^3P_0$ relative state. This is achieved by introducing (i) a spin-dependent 2×2 quark propagator proportional to the matrix $\Delta_q = \mu_q + \sigma_z \boldsymbol{\sigma} \cdot \mathbf{k}_\mathrm{T}$, where μ_q is a free complex parameter referred to as the complex mass and $\mathbf{k}_\mathrm{T}$ is the transverse momentum of the fragmenting quark with respect to the string axis, and (ii) a 2×2 vertex matrix Γ_h. The model was implemented in the Pythia 8 MCEG [30] via the StringSpinner package [31] and was shown to reproduce the Collins asymmetries and the dihadron production asymmetries in SIDIS [32] and the Collins asymmetries in e^+e^- [33,34].

The string + $^3P_0$ model is however restricted to the production of pseudoscalar (PS) mesons and vector mesons (VMs) in the final states but lacks the production of baryons. The introduction of the production and decay of polarized spin-1/2 baryons in the string + $^3P_0$ model of Ref. [28] is the goal of this work.[1] We describe baryon production by allowing diquark–antidiquark pairs $(qq) - (\overline{qq})$ to be produced at the string breakings, as suggested in the spinless LM in Ref. [35] and implemented in the Pythia MCEG. We distinguish between scalar $(qq)_0$ and pseudovector (PV) $(qq)_1$ diquarks. In the latter case, the diquark–antidiquark pair should be created in the $^1S_0$ or in the $^5D_0$ state, i.e., with the vacuum quantum numbers, by analogy with the $^3P_0$ state of the $q\bar{q}$ pair creation. In this paper, we introduce a *diquark propagator* Δ_{qq} in spin space based on the dominance of the $^5D_0$ state. We also introduce the spin matrices Γ_B for the quark–baryon–diquark vertices. The Δ_{qq} and Γ_B matrices are the basic objects of our model; from them we build *splitting matrix-functions* $T = F \times \Delta_{qq} \times \Gamma_B$, where F is a function of relevant momenta, baryon species, and diquark species. The T matrices describe the elementary splittings $q \to B + (\overline{qq})$ and $(\overline{qq}) \to \bar{B} + q'$ in spin and momentum space. They allow us to study qualitatively the model predictions as well as to implement the model in a MCEG. The MCEG implementation of the model will be presented in a separate work.

The paper is organized as follows. The splitting matrices of the extended string + $^3P_0$ model with meson and baryon production are constructed in Sec. II. In Sec. III, we study the spin-dependent probability distribution of the produced baryons, in particular the Collins effect for baryon production. The predicted spin states of the produced baryons, among these the spontaneous polarization, are described in Sec. IV. The subsequent decays of the polarized unstable hyperons are described in Sec. V. Section VI describes the rules for propagation of the spin information along the fragmentation chain. We give conclusions in Sec. VII.

II. EXTENDED STRING + $^3P_0$ MODEL WITH MESON AND BARYON PRODUCTION

We start by considering the hadronization

$$q_A \bar{q}_B \to h_1 h_2 \ldots, \qquad (3)$$

of the initial quark–antiquark pair $q_A\bar{q}_B$ into the final state hadrons $h_1, h_2, \ldots$. The hadronization is viewed as the breaking of the string stretched between q_A and $\bar{q}_B$, which occurs by the recursive iteration of the elementary splittings $q \to h + q'$, where q is the fragmenting quark, h is the emitted hadron, and q' is the leftover quark.[2] In the string + $^3P_0$ model

[1] The heavier spin-3/2 baryon states are less frequently produced in hadronization and their inclusion in the model is not compelling. It is left for future work.

[2] Similar to the previous versions of the string + $^3P_0$ model, we do not consider interferences between amplitudes corresponding to the same final state but with permuted ranks of the hadrons.

of Ref. [28], the hadron $h = q\bar{q}'$ was restricted to be either a PS meson or a VM.

To generalize the string $+\,^3P_0$ model by including the production of polarized baryons, which are modeled as quark–diquark bound states, it is necessary to consider the more general splitting

$$Q \to h + Q' = \begin{cases} q \to M + q' \\ q \to B + (\overline{qq}) \\ (\overline{qq}) \to \bar{B} + q' \end{cases}, \tag{4}$$

where the fragmenting particle Q can be a quark ($Q = q$) or an antidiquark $Q = (\overline{qq})$, the hadron h can be a PS meson or a VM ($h = M$) or a baryon ($h = B$) or an antibaryon ($h = \bar{B}$), and the leftover particle Q' can be a quark ($Q' = q'$) or an antidiquark $[Q' = (\overline{qq})]$.

Following the implementation of the LM in Pythia [30], we also introduce the relative probability P_{qq}/P_q for the successive string breaking to occur via the tunneling of a $(qq) - (\overline{qq})$ pair rather than the tunneling of a $q'\bar{q}'$ pair, and the relative probability $P_{(qq)_1}/P_{(qq)_0}$ for the diquark to be PV rather than scalar (apart from the enhancement by a factor of 3 due to the number of states of spin-1 diquarks). These parameters are used to probabilistically chose between the splittings $q \to h + q'$ and $q \to \mathrm{B} + (\overline{qq})$, which are treated separately. Note also that baryons and antibaryons are produced in pairs by the successive splittings $q \to \mathrm{B} + (\overline{qq})$, $(\overline{qq}) \to \bar{B} + q'$. In this work, we neglect the interference between the amplitudes for the two successive splittings to occur via the propagation of a scalar diquark and a PV diquark.

The description of the elementary splitting $Q \to h + Q'$ in momentum and spin space is achieved by the means of a splitting matrix $T_{Q',h,Q}$. We build such matrix using the LM of string fragmentation with the addition of spin matrices Δ (quark or diquark propagators) and Γ (coupling matrices). The general expression for $T_{Q',h,Q}$ is introduced in Sec. II B, while the expressions for Γ and Δ are given in Secs. II C and II D, respectively.

A. Kinematics of the hadron emission

We study the hadronization of the $q_A\bar{q}_B$ pair in the center of mass system of the pair, which is referred to as the string frame. In this frame we introduce the axes $\hat{\mathbf{z}} = \mathbf{k}_A/|\mathbf{k}_A|$, $\hat{\mathbf{y}} = \hat{\mathbf{u}} \times \hat{\mathbf{z}}/|\hat{\mathbf{u}} \times \hat{\mathbf{z}}|$ and $\hat{\mathbf{x}} = \hat{\mathbf{y}} \times \hat{\mathbf{z}}$, where $\mathbf{k}_A$ is the momentum of q_A and $\hat{\mathbf{u}}$ is a reference vector, e.g., the electron beam direction in e^+e^- annihilation. The string axis is along the $\hat{\mathbf{z}}$ axis and it defines the longitudinal direction.

For each splitting, we indicate by k, p, and $k' = k - p$ the four momenta of Q, h, and Q', respectively. The four-momentum of h is parameterized in terms of the longitudinal splitting variable $Z = p^+/k^+$, the transverse momentum (with respect to the string axis) $\mathbf{p}_T = (p_x, p_y)$, and the transverse energy $\varepsilon_h = \sqrt{p^+p^-} = \sqrt{M_h^2 + \mathbf{p}_T^2}$, M_h being the mass of h. The lightcone components for a generic four-vector v are defined as $v^\pm = v^0 \pm v^z$. We have $\mathbf{p}_T = \mathbf{k}_T - \mathbf{k}'_T$, $\mathbf{k}_T$ and $\mathbf{k}'_T$ being the transverse momenta of Q and Q', respectively.

B. The splitting matrix T

To describe the elementary splitting $Q \to h + Q'$ in momentum and spin-space, we generalize the splitting matrix for the splitting $q \to h + q'$ introduced in the string $+\,^3P_0$ model of Ref. [28]. We write the splitting matrix as

$$\begin{aligned} T_{Q',h,Q}(M_h, Z, \mathbf{p}_T, \mathbf{k}_T) = {} & C_{Q',h,Q} D_h(M_h) f^{1/2}_{Q',h,Q}(Z, \mathbf{p}_T, \mathbf{k}_T) \\ & \times N^{-1/2}_{a_{Q'}a_Q}(\varepsilon_h^2) f_T(\mathbf{k}'^2_T) \\ & \times \Delta_{Q'}(\mathbf{k}'_T)\Gamma_h \hat{u}_Q^{-1/2}(\mathbf{k}_T), \end{aligned} \tag{5}$$

where $\Delta_{Q'}$ is the propagator of Q', Γ_h is a matrix connecting the spin state of Q with that of the compound system (Q', h), and h denotes the emitted hadron. The latter matrix has horizontal dimension $2S_Q + 1$ and vertical dimension $(2S_{Q'} + 1) \times (2S_h + 1)$, where S denotes the spin quantum number.

The coefficient $C_{Q',h,Q}$ describes the splitting in flavor-space and is based on the wave function of h in isospin space. The function $|D_h(M_h)|^2$ gives the invariant mass distribution of h. For a fixed hadron mass it is a delta function centered on the nominal squared mass. For a resonance it is a relativistic Breit-Wigner function with mass and width fixed to their nominal values. The Z- and $\mathbf{p}_T$-dependent part of the splitting matrix is the square-root of the spinless splitting function of the LM [36], here indicated by $f_{Q',h,Q}$. The latter is given by

$$\begin{aligned} f_{Q',h,Q}(Z, \mathbf{p}_T, \mathbf{k}_T) = {} & \left(\frac{1-Z}{Z}\right)^{a_{Q'}} \left(\frac{Z}{\varepsilon_h^2}\right)^{a_Q} \\ & \times \exp(-b_L \varepsilon_h^2/Z), \end{aligned} \tag{6}$$

and it describes the distribution of the longitudinal momentum of h in the splitting $Q \to h + Q'$. It depends on the free parameters $a_{Q'}$, a_Q, and b_L. Note that the factor $(1/\varepsilon_h^2)^{a_Q}$ does not explicitly appear in Ref. [36]. Here, we follow the convention in Ref. [37]. This factor does not change the shape of the distribution of the variable Z in Eq. (6).

The function $N_{a_{Q'}a_Q}(\varepsilon_h^2)$, given by

$$N_{a_{Q'},a_Q}(\varepsilon_h^2) = \int_0^1 dZ\, Z^{-1} f_{Q',h,Q}(Z, \mathbf{p}_T, \mathbf{k}_T), \tag{7}$$

plays the role of a normalization factor for the Z-dependent part of the splitting amplitude squared. It depends on the squared transverse energy of h. In the string $+\,^3P_0$ model we assume $a_{q'} = a_q \equiv a$, which leads to $N_{a_{q'},a_q} \equiv N_a(\varepsilon_h^2)$ [28].

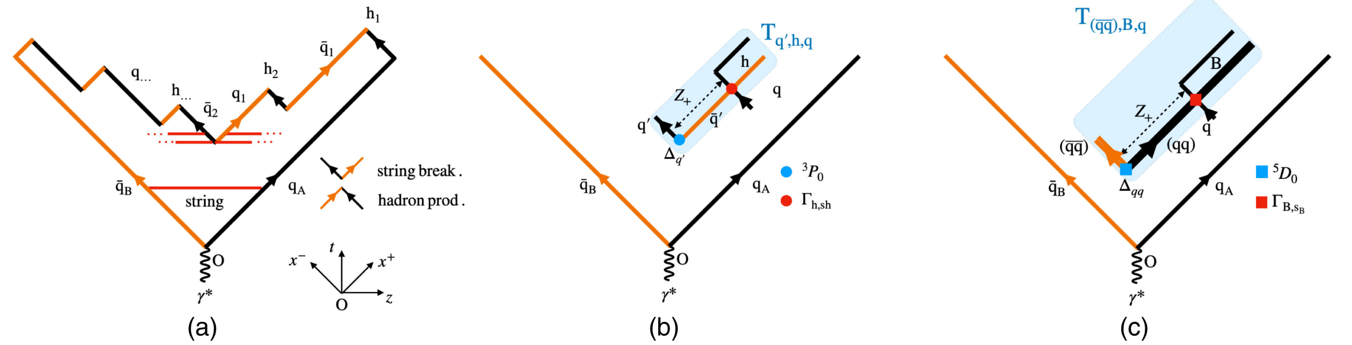

FIG. 1. Spacetime picture of spinless string fragmentation (a), and of string breaking via the tunneling of a quark–antiquark pair (b) or a diquark–antidiquark pair (c). The shaded areas enclose the elements that define the quark-splitting matrices for the emission of a meson (b) and baryon (c).

For diquarks we take $a_{qq} = a_{\overline{qq}} \equiv a_D$ with $a_D \neq a$, as in the implementation of the LM in the Pythia MCEG [30].

The function $f_{\rm T}$ in Eq. (5) provides a damping of the magnitude of the transverse momentum of the q' pair created at the string breaking [see also Fig. 2(a)] and therefore a damping of the transverse momentum of the produced hadron. It is taken to have the exponential form

$$f_{\rm T}(\mathbf{k}_{\rm T}'^2) = \sqrt{b_{\rm T}\pi^{-1}}\exp(-b_{\rm T}\mathbf{k}_{\rm T}'^2/2), \tag{8}$$

with $b_{\rm T}$ being a free parameter. $|f_{\rm T}|^2$ is the generalization of the formula $P_{\rm Tun}(\mathbf{k}_{\rm T}'^2) = \exp[-\pi(m_{q'}^2 + \mathbf{k}_{\rm T}'^2)/\kappa]$ found in Ref. [38] for the tunneling probability of the quark pair, with $\kappa \simeq 0.2\ {\rm GeV}^2$ being the string tension or equivalently the energy per unit length stored in the string. For the tunneling of a $(qq)(\overline{qq})$ pair the same expression holds with $m_{q'} \to m_{qq}$. We thus assume that the function $f_{\rm T}$ describes also the transverse momentum cutoff for the diquarks. Other choices for $f_{\rm T}$ are possible (see Ref. [27]). See also Ref. [31] for the comparison with the spinless LM.

We rewrite the product $\Delta_{Q'}\Gamma_h$ in Eq. (5) in terms of a 2×2 matrix $t_{Q',h,Q}$ connecting only the two spin-1/2 particles. More precisely, for each type of splitting, we define $t_{Q',h,Q}$ by

$$\langle Q', S_{Q'}; h, S_h|\Delta_{Q'}\Gamma_h|Q, S_Q\rangle = \begin{cases} \chi^\dagger(\mathbf{S}_{q'})t_{q',h,q}(\mathbf{k}_{\rm T}', s_h)\chi(\mathbf{S}_q), & Q' = q', h = M \\ \chi^\dagger(\mathbf{S}_B)t_{(\overline{qq}),B,q}(\mathbf{k}_{\rm T}', s_{\overline{qq}})\chi(\mathbf{S}_q), & Q' = \overline{qq}, h = B\,. \\ \chi^\dagger(\mathbf{S}_{q'})t_{q',\bar{B},(\overline{qq})}(\mathbf{k}_{\rm T}', s_{\overline{qq}})\eta(\mathbf{S}_{\bar{B}}), & Q' = q', h = \bar{B} \end{cases} \tag{9}$$

We have introduced the Pauli spinors $\chi(\mathbf{S}_q), \chi(\mathbf{S}_{q'}), \chi(\mathbf{S}_B)$, and $\eta(\mathbf{S}_{\bar{B}}) = \sigma_z\chi(-\mathbf{S}_{\bar{B}})$[3] of, respectively, q, q', B, and $\bar{B}$. $\chi(\mathbf{S})$ indicates the Pauli spinor with polarization vector $\mathbf{S}$. The spin state of h ($\overline{qq}$) is indicated by s_h ($s_{\overline{qq}}$).

[3]This is the analog of the relation $v(k,s) = \gamma_5 u(k,-s)$ involving Dirac spinors.

The general expression for the matrix-function $\hat{u}_Q$ in Eq. (5) is given by [28]

$$\hat{u}_Q(\mathbf{k}_{\rm T}) = \sum_{h,s_h}|C_{Q',h,Q}|^2\int d^2\mathbf{k}_{\rm T}'\int_0^1\frac{dZ}{Z}f_{Q',h,Q}(Z,\mathbf{p}_{\rm T},\mathbf{k}_{\rm T}) \times N^{-1}_{a_{Q'},a_Q}(\varepsilon_h^2)f_{\rm T}^2(\mathbf{k}_{\rm T}^2)t^\dagger_{Q',h,Q}t_{Q',h,Q}, \tag{10}$$

where the summation is taken over the produced hadron h (meson or baryon) and on the spin states of h and Q, except spin-1/2 ones (those are already summed in the product $t^\dagger t$). In the following, for the splittings $q \to B + (\overline{qq})_s$ and $(\overline{qq})_s \to \bar{B} + q'$, we will indicate the corresponding part of $\hat{u}_Q$ by $\hat{U}_q^{(s)}$ and $\hat{u}_{\overline{qq}}^{(s)}$, respectively, with $s = 0, 1$ being the spin of the diquark. An integration over M_h^2 is also understood. It can be shown that $\hat{u}_Q \propto 1_{2\times 2}$ for $Q = q$ and $\hat{u}_Q \propto 1_{3\times 3}$ for $Q = (\overline{qq})_1$ due to the factor $N_{a_{Q'},a_Q}(\varepsilon_h^2)$ in Eq. (7). The proportionality factor is a constant factor, and it can be obtained once the expression for $t_{Q',h,Q}$ is given. The explicit expressions for $t_{Q',h,Q}$ for the splittings in Eq. (4) are as follows.

(a) *Splitting* $q \to M + q'$. The matrix-function $t_{q',h,q}$ for $h = {\rm PS}$, VM can be written as [27,28]

$$t_{q',h,q}(\mathbf{k}_{\rm T}', s_h) = \Delta_{q'}(\mathbf{k}_{\rm T}')\Gamma_{h,s_h}, \tag{11}$$

where the 2×2 matrix $\Delta_{q'}(\mathbf{k}_{\rm T}')$, given in Eq. (20), is the propagator of the quark q' obtained by the $^3P_0$ mechanism, and the 2×2 matrix Γ_{h,s_h}, given in Eq. (15), indicates the coupling of q and q' with the meson h.

(b) *Splitting* $q \to B + \overline{qq}$, *with* $\overline{qq} = (\overline{qq})_1$ or $\overline{qq} = (\overline{qq})_0$. The matrix-function $t_{\overline{qq},B,q}(\mathbf{k}_{\rm T}')$ can be written as

$$t_{\overline{qq},B,q}(\mathbf{k}_{\rm T}', s_{\overline{qq}}) = \begin{cases} \Phi_a^*\Delta_{qq,ab}(\mathbf{k}_{\rm T}')\Gamma_{B,b}, & \overline{qq} = (\overline{qq})_1 \\ \Gamma_{B,0}, & \overline{qq} = (\overline{qq})_0. \end{cases} \tag{12}$$

$\mathbf{\Phi}$, which represents $s_{\overline{qq}}$, is the spin wavefunction of the PV antidiquark whose components in the cartesian basis are Φ_a, $a = x,\ y,\ z$. $\Gamma_{B,b}$, with $b = x,\ y,\ z$, and $\Gamma_{B,0}$ are 2×2 matrices that describe the couplings $q - B - (qq)_1$ and $q - B - (qq)_0$, respectively. The 3×3 matrix $\Delta_{qq}(\mathbf{k}'_{\rm T})$ is the propagator for the PV antidiquark $\overline{qq}$. The explicit expressions for the couplings Γ are given in Eq. (17), while the expressions for the propagator $\Delta_{\overline{qq}}$ are given in Eq. (21).

(c) *Splitting* $Q \to \bar{B} + q'$, *with* $Q = \overline{qq}$. The expression for the matrix $t_{q',\bar{B},Q}$ is

$$t_{q',\bar{B},\overline{qq}}(\mathbf{k}'_{\rm T}, s_{\overline{qq}}) = \begin{cases} \Delta_{q'}(\mathbf{k}'_{\rm T})\sigma_z\Gamma_{B,b}\sigma_z\Phi_b, & \overline{qq} = (\overline{qq})_1 \\ \Delta_{q'}(\mathbf{k}'_{\rm T})\Gamma_{B,0} & \overline{qq} = (\overline{qq})_0 \end{cases}, \tag{13}$$

where the quark propagator $\Delta_{q'}$ and the vertex Γ_B are the same as in Eqs. (11) and (12).

(d) Splittings $\bar{q} \to \bar{B} + (qq)$ and $(qq) \to B + \bar{q}'$. Due to the charge conjugation symmetry, the splitting matrices for the splittings $\bar{q} \to \bar{B} + (qq)$ or $(qq) \to B + \bar{q}'$ can be obtained from Eq. (5) with the substitutions $Q \to \bar{q}$, $h \to \bar{B}$ and $Q' \to (qq)$ or $Q \to (qq)$, $h \to B$ and $Q' \to \bar{q}'$. These splittings can be applied to the description of the target remnant fragmentation in a deep-inelastic scattering event, which will be studied in a separate work.

The splitting matrix for $\bar{q} \to M + \bar{q}'$ can be found in Ref. [33].

C. The coupling matrix Γ

For the splitting $q \to h + q'$, with h a meson, we write according to Eq. (11) [27,28]

$$\langle h, s_h, q', \mathbf{S}_{q'}|\Gamma|q, \mathbf{S}_q\rangle = \chi^\dagger(\mathbf{S}_{q'})\Gamma_{h,s_h}\chi(\mathbf{S}_q), \tag{14}$$

with

$$\Gamma_{h,s_h} = \begin{cases} \sigma_z & \text{if h = PS} \\ G_{\rm L}V^*_{\rm L}1 + G_{\rm T}\mathbf{V}^*_{\rm T}\cdot\boldsymbol{\sigma}_{\rm T}\sigma_z & \text{if h = VM} \end{cases}. \tag{15}$$

$\boldsymbol{\sigma}_{\rm T} = (\sigma_x, \sigma_y)$ is the transverse part of the vector $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ of Pauli matrices and $\mathbf{V} = (V_x, V_y, V_z) = (\mathbf{V}_{\rm T}, V_{\rm L})$ is the spin wave function of the VM in the linearly polarized basis. $\mathbf{V}$ is defined in the VM rest frame obtained from the string frame by the sequence of boosts shown in Ref. [28]. $G_{\rm L}$ and $G_{\rm T}$ are complex coupling constants that apply respectively to longitudinal and transverse (with respect to the string) linear polarizations of the VM.

For the splitting $q \to B + Q'$, we write according to Eq. (12)

$$\langle B, \mathbf{S}_B, Q', s_{Q'}|\Gamma|q, \mathbf{S}_q\rangle = \begin{cases} \chi^\dagger(\mathbf{S}_B)\Gamma_{B,0}\chi(\mathbf{S}_q), & Q' = (\overline{\rm qq})_0 \\ \chi^\dagger(\mathbf{S}_B)\Phi^*_b\Gamma_{B,b}\chi(\mathbf{S}_q) & Q' = (\overline{\rm qq})_1 \end{cases}, \tag{16}$$

with

$$\Gamma_{B,b} = \begin{cases} \boldsymbol{\sigma}_b, & b = x, y, z \\ 1_{2\times 2}, & b = 0 \end{cases}. \tag{17}$$

The coupling in Eq. (17) is obtained by rewriting the covariant amplitude $\bar{u}_B V_{\overline{qq},B,q} u_q$ associated to the splitting $q \to B + \overline{qq}$, where $V_{\overline{qq},B,q}$ is the coupling matrix, in terms of Pauli spinors in the baryon rest frame, see Eq. (16). This frame is reached by the sequence of boosts introduced in Ref. [28]. u_B and u_q are the Dirac spinors of B and q, respectively. In the standard representation, they are

$$u_B = \begin{pmatrix} \chi(\mathbf{S}_B) \\ 0 \end{pmatrix}, \qquad u_q = \frac{1}{\sqrt{2}}\begin{pmatrix} \chi(\mathbf{S}_q) \\ -\sigma_z\chi(\mathbf{S}_q) \end{pmatrix}. \tag{18}$$

Beforehand, the quark spinor u_q has been projected on the subspace with $\alpha_z = -1$, suitable for a quark moving with velocity $v_z = \langle\alpha_z\rangle \simeq -1$ when entering the baryon. This is inspired by the string fragmentation model where the quarks' trajectories are composed of segments with $v_z = dz/dt = \pm 1$ (see Fig. 1).

For $V_{\overline{qq},B,q}$ we have taken $V_{(\overline{qq})_0,B,q} = 1_{4\times 4}$ for a scalar diquark, and $V_{(\overline{qq})_1,B,q} = \gamma_5\gamma^\mu\varepsilon^*_\mu$ for a PV diquark with polarization four-vector ε^μ [39]. Then considering the on-shell relation $\varepsilon^0 = \mathbf{v}\cdot\boldsymbol{\varepsilon}$ and neglecting the transverse velocity $\mathbf{v}_{\rm T}$ we take $\varepsilon^0 = \varepsilon^z$, i.e., $\varepsilon^\mu = (\varepsilon^z, \varepsilon^x, \varepsilon^y, \varepsilon^z)$. Thus the spin-degree of freedom of the antidiquark can be encoded in the 3-vector $\mathbf{\Phi} = (\varepsilon_x, \varepsilon_y, 2\varepsilon_z)$.[4] This is sufficient to implement in the model the required symmetries, shown in Appendix A 1.

Analogously, we obtain the coupling for the splitting $\overline{qq} \to \bar{B} + q'$, with Q being an antidiquark $(\overline{qq})$. In this case we write

$$\langle\bar{B}, q'|\Gamma|Q\rangle = \begin{cases} \chi^\dagger(\mathbf{S}_{q'})\Gamma_{B,0}\eta(\mathbf{S}_{\bar{B}}), & Q = (\overline{qq})_0 \\ \chi^\dagger(\mathbf{S}_{q'})\Phi_b\sigma_z\Gamma_{B,b}\sigma_z\eta(\mathbf{S}_{\bar{B}}) & Q = (\overline{qq})_1 \end{cases}, \tag{19}$$

where the matrix Γ_B is the same as in Eq. (17).

[4] We checked that the alternative coupling $V_{(\overline{qq})_1,B,q} = \gamma_5(\gamma^\mu + p^\mu/M_B)\varepsilon^*_\mu$ introduced in Ref. [40] and suitable for a nonrelativistic diquark in the baryon rest frame gives the same coupling in Eq. (17) with $\Phi_z = \varepsilon^z$. We also obtained the same result using the wave functions of the spin-1/2 baryons in spin ⊗ isospin space in the nonrelativistic quark model.

D. The propagator Δ

The quark propagator in Eq. (11) is [28]

$$\Delta(\mathbf{k}'_{\mathrm{T}}) = \mu_q + \sigma_z \boldsymbol{\sigma} \cdot \mathbf{k}'_{\mathrm{T}}, \tag{20}$$

which parameterizes the $^3P_0$ wave function of the $q'\bar{q}'$ pair produced at the string breaking [see, e.g., Fig. 2(a)]. It is given in terms of the complex parameter μ_q. This parameter replaces $k'_z = -\mathrm{i}\sqrt{m^2_{q'} + \mathbf{k}'^2_{\mathrm{T}}}$ of the quark in the middle of the tunneling trajectory. μ_q can in principle depend on the flavor of q' and on $\mathbf{k}'^2_{\mathrm{T}}$, but we take it to be flavor-independent and constant. The imaginary part $\mathrm{Im}(\mu_q)$ is responsible for transverse spin effects, e.g. the Collins effect and the dihadron production asymmetry, while $\mathrm{Im}(\mu_q^2) = 2\mathrm{Re}(\mu_q)\,\mathrm{Im}(\mu_q)$ is responsible for longitudinal spin effects, e.g. the jet-handedness [26].

For a scalar diquark, we take the propagator to be $\Delta_{(qq)_0} = 1$, forgetting a possible phase factor.

For a PV diquark, we assume that string breakings can occur by tunneling of diquark–antidiquark pair $(qq)(\overline{qq})$ from the string medium, as shown in Fig. 2(b), with vacuum quantum numbers $J^{PC} = 0^{++}$. $J = 0$ implies $L = S$, where L is the relative orbital angular momentum of the pair and $\mathbf{S} = \mathbf{S}(qq) + \mathbf{S}(\overline{qq})$ is the total spin. The parity of the $(qq) - (\overline{qq})$ pair is $P = (-1)^L$ and its charge conjugation $C = (-)^{L+S}$. Therefore, $L = S = 0$ ($^1S_0$ state) or $L = S = 2$ ($^5D_0$ state). As a matter of fact, the string medium is not isotropic; there is a chromoelectric field that can produce a kind of Stark effect, *i.e.*, admixture of $J = 2, 4, \ldots$ states. We ignore such an effect.

To construct the PV diquark propagator in Eq. (12), we start from a pure $^5D_0$ state of the diquark pair and, using the fact that in the middle of the tunneling trajectory we have $E^2_{qq} = m^2_{qq} + \mathbf{k}'^2 = 0$, we arrive at a propagator of the form (see Appendix A)

$$\Delta_{qq}(\mathbf{k}'_{\mathrm{T}}) = \begin{pmatrix} k'^2_x + m^2_{qq}/3 & k'_x k'_y & \mu_{qq} k'_x \\ k'_x k'_y & k'^2_y + m^2_{qq}/3 & \mu_{qq} k'_y \\ \mu_{qq} k'_x & \mu_{qq} k'_y & \mu^2_{qq} + m^2_{qq}/3 \end{pmatrix}, \tag{21}$$

where we have replaced first $\mathbf{k}'^2$ by $-m^2_{qq}$, then the remaining k'_z by a phenomenological complex parameter μ_{qq}. The propagator Δ_{qq} is thus a 3×3 matrix that depends on the transverse momentum $\mathbf{k}'_{\mathrm{T}}$ of the diquarks at the exit of the tunneling and on the parameter μ_{qq}. The latter is the analogue of the complex parameter μ_q introduced for the tunneling of quarks via the $^3P_0$ mechanism [see Eq. (20)]. The real and imaginary parts of μ_{qq} are free parameters of the model whose value must be fixed by comparison with data.

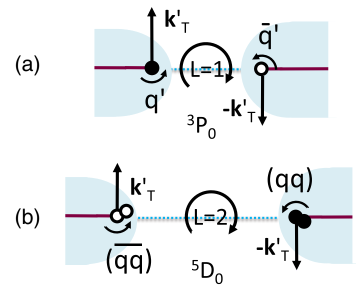

FIG. 2. Quark–antiquark pair tunneling in the $^3P_0$ state (a), and diquark–antidiquark pair tunneling in the $^5D_0$ state (b). Straight arrows indicate transverse momenta, while the curved arrows indicate either the quark/diquark spin or orbital angular momentum. The dotted line indicates the tunneling length.

III. PROBABILITY DISTRIBUTION FOR THE PRODUCED HADRON AND THE COLLINS EFFECT

A. The splitting functions of the model

The splitting matrix in Eq. (5) is related to the splitting probability by

$$\begin{aligned}\frac{dP_{Q\to h+Q'}}{dM_h dZ Z^{-1} d^2\mathbf{p}_{\mathrm{T}}} &= |\langle Q', S_{Q'}; h, S_h | T_{Q',h,Q} | Q, S_Q\rangle|^2 \\ &\equiv F_{Q',h,Q}(Z, \mathbf{p}_{\mathrm{T}}, \mathbf{k}_{\mathrm{T}}, \check{\rho}(Q'), \check{\rho}(h), \rho(Q)),\end{aligned} \tag{22}$$

where we have indicated by $|Q, S_Q\rangle$, $|h, S_h\rangle$, and $|Q', S_{Q'}\rangle$ the spin states of Q, h, and Q', respectively. The function $F_{Q',h,Q}(Z, \mathbf{p}_{\mathrm{T}}, \mathbf{k}_{\mathrm{T}}, \check{\rho}(Q'), \check{\rho}(h), \rho(Q))$ is the triple-polarized splitting function[5] and it describes the energy-momentum sharing between the emitted hadron h and the leftover particle Q', when all involved particles are polarized. With $\rho(X)$, we indicate the spin density matrix of a given particle $X = q, \overline{qq}$, e.g., $\rho(q) = (1 + \boldsymbol{\sigma}\cdot\mathbf{S}_q)/2$ for a quark q with polarization vector $\mathbf{S}_q$. The symbol $\check{\rho}$ indicates the acceptance spin density matrix; for the definition and use of such matrix see, e.g., Ref. [41]. $\check{\rho}(h)$ implements the information on the directions of the decay products of h or on their accepted angular domains. $\check{\rho}(Q')$ implements the spin information coming "backwards in time" from hadron emissions following Q'. The latter information is usually neglected in the recursive string + $^3P_0$ model, namely we take $\check{\rho}(Q') = 1$. This expression is used, at least temporarily, as long as one has no information about its future splittings. For $\check{\rho}(h) = 1$, the resulting function $F_{Q',h,Q}(Z, \mathbf{p}_{\mathrm{T}}, \mathbf{k}_{\mathrm{T}}, \rho(Q))$ is referred to as the (*polarized*)

[5]The splitting function F must be distinguished from the fragmentation function D in Eq. (1). The latter is inclusive with respect to the rank of the produced hadrons. Whereas the splitting function for the initial quark takes into account only the hadron with rank 1, i.e., the first emitted hadron in the recursive process.

splitting function. It gives the probability distribution for emitting h from a polarized Q.

(a) *Case* $q \to M + q'$. Inserting Eq. (11) in Eq. (5), and using Eq. (22) with $h = \mathrm{PS}$, VM yields the triple-polarized splitting function

$$\begin{aligned} &F_{q',h,q}(Z, \mathbf{p}_\mathrm{T}, \mathbf{k}_\mathrm{T}, \check{\rho}(q'), \check{\rho}(h), \rho(q)) \\ &= |C_{q',h,q}|^2 |D_h(M_h)|^2 \\ &\quad \times f_{q',h,q}(Z, \mathbf{p}_\mathrm{T}, \mathbf{k}_\mathrm{T}) N_a^{-1}(\varepsilon_h^2) f_\mathrm{T}^2(\mathbf{k}_\mathrm{T}'^2) \\ &\quad \times \hat{u}_q^{-1} \mathrm{Tr}_{q'}[\Delta_{q'} \Gamma_{h,a} \rho(q) \Gamma_{h,b}^\dagger \Delta_{q'}^\dagger \check{\rho}(q')] \check{\rho}_{ba}(h), \end{aligned} \tag{23}$$

The trace is taken over the spin indices of q', and we have indicated with $\check{\rho}(q')$ the acceptance density matrix of q'. For a PS meson emission $\check{\rho}_{ba}(h)$ and the indices a, b are removed. For VM emission we write $\Gamma_h = \Gamma_{h,a} V_a^*$, with $a = x, y, z$ [see Eq. (15)].

(b) *Case* $q \to B + (\overline{qq})$. The triple-polarized splitting function for baryon production via the splitting $q \to B + \overline{qq}$ can be evaluated inserting Eqs. (5) and (12) for $h = \mathrm{B}$ in Eq. (22). We obtain

$$\begin{aligned} &F_{\overline{qq},B,q}(Z, \mathbf{p}_\mathrm{T}, \mathbf{k}_\mathrm{T}, \check{\rho}(\overline{qq}), \check{\rho}(B), \rho(q)) \\ &= |D_B(m_B^2)|^2 |C_{\overline{qq},B,q}|^2 \\ &\quad \times f_{\overline{qq},B,q}(Z, \mathbf{p}_\mathrm{T}, \mathbf{k}_\mathrm{T}) N_{a_D,a}^{-1}(\varepsilon_B^2) f_\mathrm{T}^2(\mathbf{k}_\mathrm{T}'^2) \\ &\quad \times \hat{U}_q^{-1} \Delta_{qq;ab}(\mathbf{k}_\mathrm{T}') \mathrm{Tr}_B[\Gamma_{B,b} \rho(q) \Gamma_{B,b'}^\dagger \check{\rho}(B)] \Delta_{qq;b'a'}^\dagger(\mathbf{k}_\mathrm{T}') \\ &\quad \times \check{\rho}_{a'a}(\overline{qq}), \end{aligned} \tag{24}$$

where $\check{\rho}(B)$ is the acceptance matrix of B and $\check{\rho}(\overline{qq})$ is the acceptance matrix of $(\overline{qq})$ (it is a 3×3 matrix for a PV diquark and 1 for a scalar diquark). The trace is taken over the Pauli spin indices of B.

(c) *Case* $(\overline{qq}) \to +\bar{B} + q'$. For the antidiquark splitting $(\overline{qq}) \to \bar{B} + q'$, the triple-polarized splitting function is obtained similarly from Eq. (22). We obtain

$$\begin{aligned} &F_{q',\bar{B},(\overline{qq})}(Z, \mathbf{p}_\mathrm{T}, \mathbf{k}_\mathrm{T}, \check{\rho}(q'), \check{\rho}(\bar{B}), \rho(\overline{qq})) \\ &= |D_{\bar{B}}(M_{\bar{B}})|^2 |C_{q',\bar{B},(\overline{qq})}|^2 \\ &\quad \times f_{q',\bar{B},(\overline{qq})}(Z, \mathbf{p}_\mathrm{T}, \mathbf{k}_\mathrm{T}) N_{a,a_D}^{-1}(\varepsilon_{\bar{B}}^2) f_\mathrm{T}^2(\mathbf{k}_\mathrm{T}'^2) \\ &\quad \times \hat{u}_{(\overline{qq})}^{-1} \mathrm{Tr}_{q'}[\Delta_{q'}(\mathbf{k}_\mathrm{T}') \sigma_z \Gamma_{B,a} \sigma_z \check{\rho}(\bar{B}) \sigma_z \Gamma_{B,b}^\dagger \sigma_z \Delta_{q'}^\dagger(\mathbf{k}_\mathrm{T}') \\ &\quad \times \check{\rho}(q')] \rho_{ab}(\overline{qq}), \end{aligned} \tag{25}$$

where the trace is taken over the leftover quark spin indices. $\rho_{ab}(\overline{qq})$ is the spin density matrix of the fragmenting PV antidiquark. $\check{\rho}(\bar{B}) = \sigma_z \rho(-\check{\mathbf{S}}_{\bar{B}}) \sigma_z$ is the acceptance matrix of $\bar{B}$, with $\check{\mathbf{S}}_{\bar{B}}$ the corresponding acceptance polarization vector. For a scalar diquark, we remove the σ_z matrices, $\rho_{ab}(\overline{qq})$, and take $a = b = 0$.

B. Explicit splitting function for meson production

The splitting function associated to the quark splitting $q \to h + q'$, where h is a PS meson or a VM, has been extensively studied in Ref. [28]. We recall it in this section to simplify the comparisons with the new splittings $q \to B + (\overline{qq})$ and $(qq) \to B + \bar{q}'$.

As a preliminary step, the matrix $\hat{u}_q$ in Eq. (10) with $Q' = q'$ can be calculated by inserting Eqs. (20)–(15) in Eq. (11), and by using the obtained expression of $t_{q',h,q}$ in Eq. (10). We decompose the result as [28]

$$\begin{aligned} \hat{u}_q &= \sum_h \hat{u}_{q,h} 1_{2\times 2}, \\ \hat{u}_{q,h} &= |C_{q',h,q}|^2 (|\mu_q|^2 + \langle \mathrm{k}_\mathrm{T}^2 \rangle_{f_\mathrm{T}}) \\ &\quad \times \begin{cases} 1 & \text{if h = PS} \\ f_\mathrm{VM} & \text{if h = VM} \end{cases}, \end{aligned} \tag{26}$$

where for a generic function $H(\mathbf{k}_\mathrm{T}^2)$ we have defined $\langle H(\mathrm{k}_\mathrm{T}^2) \rangle_{f_\mathrm{T}} = \int d^2\mathbf{k}_\mathrm{T} H(\mathrm{k}_\mathrm{T}^2) \mathrm{f}_\mathrm{T}^2(\mathrm{k}_\mathrm{T}^2)$. The constant $f_\mathrm{VM}/(1 + f_\mathrm{VM})$, with $f_\mathrm{VM} = |2|G_\mathrm{T}|^2 + |G_\mathrm{L}|^2$, is the probability for the $q\bar{q}'$ pair to be a VM rather than a PS meson; it is one of the free parameters of the LM implemented in Pythia [30].

The splitting function can be obtained by inserting Eq. (26) into Eq. (23), and taking $\check{\rho}(q') = 1_{2\times 2}$ and $\check{\rho}(h) = 1_{3\times 3}$ for a VM or $\check{\rho}(h) = 1$ for a PS meson. It reads [28]

$$\begin{aligned} &F_{q',h,q}(M_h, Z, \mathbf{p}_\mathrm{T}, \mathbf{k}_\mathrm{T}, \mathbf{S}_{q,\mathrm{T}}) \\ &= \frac{\hat{u}_{q,h}}{\hat{u}_q} |D_h(M_h)|^2 \left(\frac{1 - Z}{\varepsilon_h^2} \right)^a e^{-b_\mathrm{L} \varepsilon_h^2 / Z} N_a^{-1}(\varepsilon_h^2) \\ &\quad \times f_\mathrm{T}^2(\mathbf{k}_\mathrm{T}'^2) \frac{|\mu_q|^2 + \mathbf{k}_\mathrm{T}'^2}{|\mu_q|^2 + \langle \mathrm{k}_\mathrm{T}^2 \rangle_{f_\mathrm{T}}} \\ &\quad \times \left[1 + c \frac{2\,\mathrm{Im}(\mu_q)}{|\mu_q|^2 + \mathbf{k}_\mathrm{T}'^2} \mathbf{S}_{q,\mathrm{T}} \cdot (\hat{\mathbf{z}} \times \mathbf{k}_\mathrm{T}') \right]. \end{aligned} \tag{27}$$

The first line describes the relative probability of producing the meson h, given by the ratio $\hat{u}_{q,h}/\hat{u}_q$, and the mass distribution of h. The second line gives the distribution of the longitudinal splitting variable Z. The distributions of k_T' and the azimuthal angle of $\mathbf{k}_\mathrm{T}'$ mainly result, respectively, from the third and last lines (they are also affected by the ε_h^2-dependence). Thus, $\mathbf{k}_\mathrm{T}'$ has a mean orientation which is perpendicular to the transverse polarization $\mathbf{S}_{q,\mathrm{T}}$ of the fragmenting quark. The amplitude of this effect is described by the parameter

$$c = \begin{cases} -1, & h = \mathrm{PS} \\ |G_\mathrm{L}|^2 / (2|G_\mathrm{T}|^2 + |G_\mathrm{L}|^2), & h = \mathrm{VM} \end{cases}. \tag{28}$$

The latter c is the fraction of VMs with longitudinal polarization with respect to the string axis.

For the initial fragmenting quark we have $\mathbf{k}_{\rm T}=0$, so we can replace $\mathbf{k}'_{\rm T}$ by $-\mathbf{p}_{\rm T}$ in Eq. (27). Comparing with Eq. (1), we see that the string + $^3P_0$ model predicts a Collins effect of amplitude

$$\hat{a}_{C,h}(p_{\rm T})=-2c\,{\rm Im}(\mu_q)\frac{p_{\rm T}}{|\mu_q|^2+p_{\rm T}^2}, \qquad (29)$$

which, for a produced PS meson ($c=-1$), is of the experimentally observed sign. Note that the Collins effect for VMs is of opposite sign and with reduced magnitude with respect to the PS meson case [28].

C. Explicit splitting function for baryon production

1. Splitting $q\to B+(\overline{qq})_1$

For the production of a baryon B in the splitting $q\to B+(\overline{qq})_1$, where $(\overline{qq})_1$ indicates a PV antidiquark, the matrix $\hat{u}_Q$ of Eqs. (5) and (10) is replaced by the matrix $\hat{U}_q^{(1)}$. The latter can be obtained by first inserting Eqs. (17) and (21) into the expression for $t_{(\overline{qq}),B,q}$ in Eq. (12), and then using the obtained $t_{(\overline{qq}),B,q}$ in Eq. (10). We write the resulting matrix $\hat{U}_q^{(1)}$ as

$$\hat{U}_q^{(1)}=\sum_B \hat{U}_{q,B}^{(1)}1_{2\times 2},$$
$$\hat{U}_{q,{\rm B}}^{(1)}=|C_{(\overline{qq})_1,B,q}|^2\langle H_1^2(\mathbf{k}_{\rm T}^2)+H_2(\mathbf{k}_{\rm T}^2)\rangle_{f_{\rm T}}, \qquad (30)$$

where we have defined the functions

$${\rm H}_1(\mathbf{k}_{\rm T}^2)=|\mu_{qq}|^2+\mathbf{k}_{\rm T}^2,$$
$${\rm H}_2(\mathbf{k}_{\rm T}^2)=\frac{m_{qq}^2}{3}(m_{qq}^2+2\mathbf{k}_{\rm T}^2+2{\rm Re}(\mu_{qq}^2)). \qquad (31)$$

The splitting function for $q\to B+(\overline{qq})_1$ can be obtained by inserting into Eq. (24) the $q-B-(\overline{qq})_1$ coupling in Eq. (17) and the diquark propagator in Eq. (21). Taking $\check{\rho}(B)=1_{2\times 2}$ and $\check{\rho}(\overline{qq})=1_{3\times 3}$, we obtain

$$F_{(\overline{qq})_1,B,q}(M_B,Z,\mathbf{p}_{\rm T},\mathbf{k}_{\rm T},\mathbf{S}_{q,{\rm T}})$$
$$=\frac{\hat{U}_{q,B}^{(1)}}{\hat{U}_q^{(1)}}|D_{\rm B}(M_B)|^2\left(\frac{1-Z}{Z}\right)^{a_D}\left(\frac{Z}{\varepsilon_B^2}\right)^a e^{-b_{\rm L}\varepsilon_B^2/Z}N_{a_D,a}^{-1}(\varepsilon_B^2)$$
$$\times f_{\rm T}^2(\mathbf{k}_{\rm T}'^2)\frac{{\rm H}_1^2(\mathbf{k}_{\rm T}'^2)+{\rm H}_2(\mathbf{k}_{\rm T}'^2)}{\langle {\rm H}_1^2(\mathbf{k}_{\rm T}^2)+{\rm H}_2(\mathbf{k}_{\rm T}^2)\rangle_{f_{\rm T}}}$$
$$\times\left[1+\frac{2\,{\rm Im}(\mu_{qq}){\rm H}_1(\mathbf{k}_{\rm T}'^2)\mathbf{S}_{q,{\rm T}}\cdot(\hat{\mathbf{z}}\times\mathbf{k}'_{\rm T})}{{\rm H}_1^2(\mathbf{k}_{\rm T}'^2)+{\rm H}_2(\mathbf{k}_{\rm T}'^2)}\right]. \qquad (32)$$

The first line of the splitting function describes the splitting in flavor space and the invariant mass distribution of B. The second line gives the longitudinal momentum distribution of B, which depends on the parameter a, a_D, and $b_{\rm L}$. This is at variance with the splitting function for meson production in Eq. (23), which depends only on the parameters a and $b_{\rm L}$.

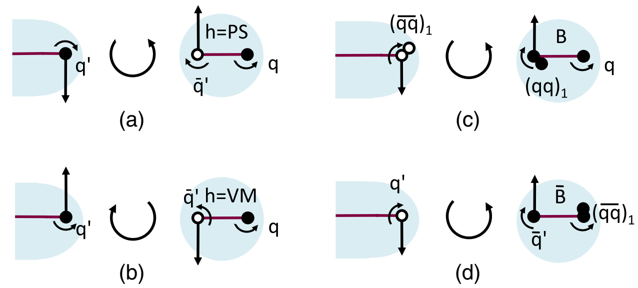

FIG. 3. Fragmentation of a string with a transversely polarized initial quark (a)–(c) or antidiquark (d). String + $^3P_0$ mechanism for PS meson (a), VM (b), and antibaryon (d) production. String + $^5D_0$ mechanism for baryon production (c).

The distribution of the modulus $\mathrm{k}'_{\rm T}$ of the transverse momentum of $(\overline{qq})_1$ is mainly given by the third line of Eq. (32) (it is also affected by the ε_B^2-dependence). It is a fourth-order polynomial in $\mathrm{k}'_{\rm T}$ multiplied by the exponential function $f_{\rm T}^2(\mathbf{k}_{\rm T}'^2)$, and depends also on the diquark mass m_{qq}, which enters the diagonal elements of the PV diquark propagator in Eq. (21). This differs with respect to the meson emission case in Eq. (27), which involves a second-order polynomial in $\mathrm{k}'_{\rm T}$.

The last line of the splitting function in Eq. (32) includes the mixed product $\mathbf{S}_{q,{\rm T}}\cdot(\hat{\mathbf{z}}\times\mathbf{k}'_{\rm T})$ responsible for a Collins effect for the emission of the baryon. For the initial fragmenting quark, we replace $\mathbf{k}'_{\rm T}$ by $-\mathbf{p}_{\rm T}$ in Eq. (32). Comparing with Eq. (1), the predicted Collins effect has the amplitude

$$\hat{a}_{C,B}^{(1)}(p_{\rm T})=-2\,{\rm Im}(\mu_{qq})\frac{p_{\rm T}{\rm H}_1(p_{\rm T}^2)}{{\rm H}_1^2(p_{\rm T}^2)+{\rm H}_2(p_{\rm T}^2)}, \qquad (33)$$

which depends on ${\rm Im}(\mu_{qq})$. Assuming ${\rm Im}(\mu_{qq})<0$, the Collins effect for the baryon production is predicted to have the same sign as that expected from the classical string + $^5D_0$ model and shown in Fig. 3. In addition, it is predicted to have the same sign as the Collins effect for PS meson emission in the splitting $q\to h+q'$.

For a typical diquark mass $m_{qq}\sim 0.5$ GeV, the contribution of the m_{qq}-dependent terms in the denominator in Eq. (33) is expected to be small. In the limit of vanishing diquark mass, Eq. (33) reduces to $\hat{a}_{C,B}^{(1)}\simeq 2\,{\rm Im}(\mu_{qq})\mathrm{k}'_{\rm T}/[|\mu_{qq}|^2+\mathbf{k}_{\rm T}'^2]$. In this limit, the Collins analyzing power for baryon production in the splitting $q\to B+(\overline{qq})_1$ has the same form as the analyzing power for meson production in the splitting $q\to h+q'$ shown in Eq. (29), provided that the substitution $\mu_q\to\mu_{qq}$ is performed. This can be understood in the nonrelativistic quark model of baryons where the diquark plays the role of a color antitriplet, like the antiquark in the wave function of a meson.

2. Splitting $q \to B + (\overline{qq})_0$

For the splitting $q \to B + (\overline{qq})_0$, we start by evaluating the matrix $\hat{U}_q^{(0)}$, which replaces the matrix $\hat{u}_Q$ of Eqs. (5) and (10), by inserting Eq. (12) into Eq. (10) and using the coupling in Eq. (17) for $b = 0$. We obtain

$$\hat{U}_q^{(0)} = \sum_B \hat{u}_{q,B}^{(0)} 1_{2\times 2}, \qquad \hat{U}_{q,\mathrm{B}}^{(0)} = |C_{(\overline{qq})_0,B,q}|^2. \tag{34}$$

The splitting function can be obtained from Eq. (24) by taking $\check{\rho}(B) = 1_{2\times 2}$, removing the diquark propagator and the acceptance matrix $\check{\rho}(\overline{qq})$, and using the coupling for scalar diquarks in Eq. (17). We obtain

$$\begin{aligned} &F_{(\overline{qq})_0,B,q}(M_B, Z, \mathbf{p}_\mathrm{T}, \mathbf{k}_\mathrm{T}, \mathbf{S}_{q,\mathrm{T}}) \\ &= \frac{\hat{U}_{q,B}^{(0)}}{\hat{U}_q^{(0)}} |D_\mathrm{B}(M_B)|^2 \\ &\quad \times \left(\frac{1-Z}{Z}\right)^{a_D} \left(\frac{Z}{\varepsilon_B^2}\right)^a e^{-b_\mathrm{L}\varepsilon_B^2/Z} N_{a_D,a}^{-1}(\varepsilon_B^2) f_\mathrm{T}^2(\mathbf{k}_\mathrm{T}'^2). \end{aligned} \tag{35}$$

As can be seen, for a baryon produced in the splitting $q \to B + (\overline{qq})_0$, the new model predicts a distribution for $\mathbf{k}'_\mathrm{T}$ much simpler than for a baryon produced in the splitting $q \to B + (\overline{qq})_1$ given in Eq. (32). In particular, for the scalar diquark case, the distribution of k'_T is mainly given by $f_\mathrm{T}^2(\mathbf{k}_\mathrm{T}'^2)$ and the distribution of the azimuthal angle of $\mathbf{k}'_\mathrm{T}$ is flat. No Collins effect is thus predicted when the baryon is emitted via the tunneling of scalar diquarks.

a. The effective baryon spectrum. The effective baryon spectrum of the model is obtained by summing the splitting functions in Eqs. (32) and (35), hence,

$$F_{(\overline{qq}),B,q} = \frac{P_{(qq)_1} F_{(\overline{qq})_1,B,q}}{P_{(qq)_0} + P_{(qq)_1}} + \frac{P_{(qq)_0} F_{(\overline{qq})_0,B,q}}{P_{(qq)_0} + P_{(qq)_1}}, \tag{36}$$

where $P_{(qq)_1}/P_{(qq)_0}$ is the suppression of PV diquarks relative to scalar diquarks, introduced in Sec. II. In the summation, each diquark species enters with the weight given by the flavor wave function of B, which is encoded in the coefficients $C_{\overline{qq},B,q}$. In particular, the Collins analyzing power in Eq. (33) is diluted by the scalar diquark contribution.

D. Explicit splitting function for antibaryon production

1. Splitting $(\overline{qq})_1 \to \bar{B} + q'$

The $\hat{u}_{(\overline{qq})}$ matrix for the PV diquark splitting $(\overline{qq})_1 \to \bar{B} + q'$ can be evaluated inserting Eq. (13) into Eq. (10). We obtain

$$\begin{aligned} \hat{u}_{(\overline{qq})_1} &= \sum_{\bar{B}} \hat{u}_{(\overline{qq})_1,\bar{B}} 1_{3\times 3}, \\ \hat{u}_{(\overline{qq})_1,\bar{B}} &= 3|C_{q',\bar{B},(\overline{qq})_1}|^2 \langle |\mu_q|^2 + \mathrm{k}_\mathrm{T}^2 \rangle_{f_\mathrm{T}}. \end{aligned} \tag{37}$$

Inserting Eqs. (37) and (13) into the splitting function (25), and taking $\check{\rho}(\bar{B}) = 1_{2\times 2}$ and $\check{\rho}(q') = 1_{2\times 2}$, we obtain the splitting function

$$\begin{aligned} &F_{q',\bar{B},(\overline{qq})_1}(M_{\bar{B}}, Z, \mathbf{p}_\mathrm{T}, \mathbf{k}_\mathrm{T}, \mathbf{S}_{\overline{qq},\mathrm{T}}) \\ &= \frac{\hat{u}_{(\overline{qq})_1,\bar{B}}}{\hat{u}_{(\overline{qq})_1}} |D_{\bar{B}}(M_{\bar{B}})|^2 \left(\frac{1-Z}{Z}\right)^a \left(\frac{Z}{\varepsilon_{\bar{B}}^2}\right)^{a_D} e^{-b_\mathrm{L}\varepsilon_{\bar{B}}^2/Z} N_{a,a_D}^{-1}(\varepsilon_{\bar{B}}^2) \\ &\quad \times f_\mathrm{T}^2(\mathbf{k}_\mathrm{T}'^2) \frac{|\mu_q|^2 + \mathbf{k}_\mathrm{T}'^2}{\langle |\mu_q|^2 + \mathbf{k}_\mathrm{T}^2 \rangle_{f_\mathrm{T}}} \\ &\quad \times \left[1 - \frac{2\,\mathrm{Im}(\mu_q)}{|\mu_q|^2 + \mathbf{k}_\mathrm{T}'^2} \mathbf{S}_{\overline{qq},\mathrm{T}} \cdot (\hat{\mathbf{z}} \times \mathbf{k}'_\mathrm{T})\right]. \end{aligned} \tag{38}$$

The vector $\mathbf{S}_{\overline{qq},\mathrm{T}}$ is the transverse component of the vector polarization $\mathbf{S}_{\overline{qq}}$ of the antidiquark with respect to the string axis. The components of $\mathbf{S}_{\overline{qq}}$ are obtained from the spin density matrix of $\overline{qq}$ as

$$S_{\overline{qq},c} = \mathrm{i}\varepsilon_{abc}\,\rho_{ab}(\overline{qq}), \tag{39}$$

with $a, b, c = x, y, z$.

Comparing Eqs. (38) and (27) with $c = -1$, it can be seen that the distribution of the transverse momentum $\mathbf{k}'_\mathrm{T}$ of the leftover quark q' in the splitting $(\overline{qq})_1 \to \bar{B} + q'$ is the same as in $q \to \mathrm{PS} + q'$. This is in agreement with the classical string + ${}^5D_0$ mechanism in Fig. 3(d).

2. Splitting $(\overline{qq})_0 \to \bar{B} + q'$

Inserting Eq. (13) into Eq. (25), the splitting function for a scalar antidiquark reads

$$\begin{aligned} &F_{q',\bar{B},(\overline{qq})_0}(M_{\bar{B}}, Z, \mathbf{p}_\mathrm{T}, \mathbf{k}_\mathrm{T}) \\ &= \frac{\hat{u}_{(\overline{qq})_0,\bar{B}}}{\hat{u}_{(\overline{qq})_0}} |D_{\bar{B}}(M_{\bar{B}})|^2 \left(\frac{1-Z}{Z}\right)^a \left(\frac{Z}{\varepsilon_{\bar{B}}^2}\right)^{a_D} e^{-b_\mathrm{L}\varepsilon_{\bar{B}}^2/Z} N_{a,a_D}^{-1}(\varepsilon_{\bar{B}}^2) \\ &\quad \times f_\mathrm{T}^2(\mathbf{k}_\mathrm{T}'^2) \frac{|\mu_q|^2 + \mathbf{k}_\mathrm{T}'^2}{\langle |\mu_q|^2 + \mathbf{k}_\mathrm{T}^2 \rangle_{f_\mathrm{T}}}, \end{aligned} \tag{40}$$

where

$$\begin{aligned} \hat{u}_{(\overline{qq})_0} &= \sum_{\bar{B}} \hat{u}_{(\overline{qq})_0,\bar{B}}, \\ \hat{u}_{(\overline{qq})_0,\bar{B}} &= |C_{q',\bar{B},(\overline{qq})_0}|^2 \langle |\mu_q|^2 + \mathbf{k}_\mathrm{T}^2 \rangle_{f_\mathrm{T}}. \end{aligned} \tag{41}$$

$\hat{u}_{(\overline{qq})_0}$ is obtained by inserting Eq. (13) with $(\overline{qq}) = (\overline{qq})_0$ in Eq. (10).

For $\mathbf{k}_\mathrm{T} = \mathbf{0}$, Eq. (40) results in a flat distribution for the azimuthal angle of $\mathbf{k}'_\mathrm{T}$, as expected by the fact that the fragmenting scalar antidiquark does not carry any spin information. Concerning the modulus $|\mathbf{k}'_\mathrm{T}|$, its distribution is similar to that in $q \to h + q'$ [Eq. (27)], since in both amplitudes enters the quark propagator in Eq. (20).

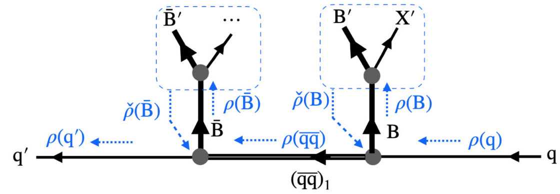

FIG. 4. Production of a baryon B and antibaryon $\bar{B}$ in the fragmentation chain initiated by the quark q. The dotted arrows show the propagation of the spin information by means of spin density matrices and acceptance matrices. The decaying baryons are taken to be hyperons.

IV. BARYON POLARIZATION

A. The spin density matrix of the baryon

The spin density matrix of the baryon B produced in the quark splitting $q \to B + (\overline{qq})_1$, shown in the fragmentation chain in Fig. 4, can be obtained from the last line of the triple-polarized splitting function in Eq. (24), with $\hat{u}_q = \hat{U}_q^{(1)}$ of Eq. (30), by summing over the polarization states of the PV diquark [i.e., $\check{\rho}(\overline{qq}) = 1_{3\times 3}$] and interpreting the trace operation as $\mathrm{Tr}[\rho^{(1)}(B)\check{\rho}(B)]$. The baryon spin density matrix $\rho^{(1)}(B)$ is then given by

$$\rho^{(1)}(B) = \frac{\Delta_{ab}\Gamma^b \rho(q)\Gamma^{b'\dagger}\Delta^\dagger_{b'a}}{\mathrm{Tr}[\ldots]}, \tag{42}$$

where the dots in the denominator represent the same expression as in the numerator, and the trace is taken over the baryon spin indices.

By inserting in Eq. (42) the explicit expression for the diquark propagator in Eq. (21) and coupling in Eq. (17), the spin density matrix $\rho^{(1)}(B)$ of the emitted baryon can be explicitly evaluated. The polarization vector $\mathbf{S}_B^{(1)}$ of the baryon can be evaluated as $\mathbf{S}_B^{(1)} = \mathrm{Tr}[\boldsymbol{\sigma}\rho^{(1)}(B)]$. We use the decomposition $\mathbf{S}_B^{(1)} = (\mathbf{S}_{B,\mathrm{T}}^{(1)}, S_{B,\mathrm{L}}^{(1)})$, with $\mathbf{S}_{B,\mathrm{T}}^{(1)}$ and $S_{B,\mathrm{L}}^{(1)}$ being the transverse and longitudinal components, respectively, with respect to the string axis.

The transverse component of the baryon polarization is

$$\begin{aligned}\mathbf{S}_{B,\mathrm{T}}^{(1)} = \frac{1}{N_q(\mathbf{S}_q)}\Big[&-2\,\mathrm{Im}(\mu_{qq})\mathrm{H}_1(\mathbf{k}'^2_\mathrm{T})\hat{\mathbf{z}}\times\mathbf{k}'_\mathrm{T}\\ &-\mathbf{S}_{q,\mathrm{T}}(\mathrm{H}_1^2(\mathbf{k}'^2_\mathrm{T}) + \mathrm{H}_2(\mathbf{k}'^2_\mathrm{T}))\\ &+2(\mathbf{k}'_\mathrm{T}\cdot\mathbf{S}_{q,\mathrm{T}})\mathbf{k}'_\mathrm{T}\left(\mathrm{H}_1(\mathbf{k}'^2_\mathrm{T}) + \frac{2m^2_{qq}}{3}\right)\\ &+2\mathrm{Re}(\mu_{qq})S_{q,z}\mathbf{k}'_\mathrm{T}\left(\mathrm{H}_1(\mathbf{k}'^2_\mathrm{T}) + \frac{2m^2_{qq}}{3}\right)\Big],\end{aligned} \tag{43}$$

where the normalization function $N_q(\mathbf{S}_q)$ is given by

$$\begin{aligned}N_q(\mathbf{S}_q) = {}&\mathrm{H}_1^2(\mathbf{k}'^2_\mathrm{T}) + \mathrm{H}_2(\mathbf{k}'^2_\mathrm{T})\\ &+2\,\mathrm{Im}(\mu_{qq})\mathrm{H}_1(\mathbf{k}'^2_\mathrm{T})\mathbf{S}_{q,\mathrm{T}}\cdot(\hat{\mathbf{z}}\times\mathbf{k}'_\mathrm{T}).\end{aligned} \tag{44}$$

For a baryon emitted by the initial quark we have $\mathbf{k}'_\mathrm{T} = -\mathbf{p}_\mathrm{T}$, and the first term in Eq. (43) gives a baryon transverse polarization in the direction $\hat{\mathbf{z}}\times\mathbf{p}_\mathrm{T}$. Taking an unpolarized q and comparing with Eq. (2), the spontaneous polarization of B results in

$$\hat{a}_B^{(1)}(p_\mathrm{T}) = 2\,\mathrm{Im}(\mu_{qq})\frac{p_\mathrm{T}\mathrm{H}_1(p_\mathrm{T}^2)}{\mathrm{H}_1^2(p_\mathrm{T}^2) + \mathrm{H}_2(p_\mathrm{T}^2)}. \tag{45}$$

For $\mathrm{Im}(\mu_{qq}) < 0$, the baryon spontaneous is negative, in agreement with the BELLE data on the spontaneous polarization of Λ and $\bar{\Lambda}$ hyperons produced in e^+e^- annihilation. Note that the size of the spontaneous polarization in Eq. (45) depends on $\mathrm{Im}(\mu_{qq})$,[6] which could be determined from a more quantitative comparison with data by the means of a MC event generator, which will be presented in a separate work.

The terms in the second and third lines of Eq. (43) give contributions from the transverse quark polarization. However, their sum is generally not collinear to $\mathbf{S}_{q,\mathrm{T}}$. For instance, if we take $m_{qq} = \mu_{qq} = 0$, then $\mathbf{S}_{q,\mathrm{T}}$ and $\mathbf{S}_{B,\mathrm{T}}^{(1)}$ are symmetrical about $\mathbf{k}'_\mathrm{T}$. A study of the transverse polarization transfer from quarks to Λ and $\bar{\Lambda}$ hyperons was performed by the COMPASS experiment using SIDIS data with a transversely polarized proton target [42].

The last term in Eq. (43) gives instead a conversion of the longitudinal polarization of the quark to the transverse polarization of the baryon.

For the longitudinal polarization ($S_{B,\mathrm{L}} \equiv S_{B,z}$) of the baryon, we find

$$\begin{aligned}S_{B,\mathrm{L}}^{(1)} = \Big\{&-\left(\mathbf{k}'^4_\mathrm{T} - |\mu_{qq}|^4 + \frac{m^2_{qq}}{3}\left(\frac{m^2_{qq}}{3} + 2\mathbf{k}'^2_\mathrm{T}\right)\right)S_{q,z}\\ &+2\mathrm{Re}(\mu_{qq})\mathbf{S}_{q,\mathrm{T}}\cdot\mathbf{k}'_\mathrm{T}\left(\mathrm{H}_1(\mathbf{k}'^2_\mathrm{T}) + \frac{2m^2_{qq}}{3}\right)\Big\}\\ &\times N_q^{-1}(\mathbf{S}_q).\end{aligned} \tag{46}$$

[6]Also, that $\hat{a}_B^{(1)}$ has the same modulus as $\hat{a}_{C,B}^{(1)}$ in Eq. (33) but differs in sign. This can be understood from Fig. 3(c). In this figure, $\mathbf{p}_\mathrm{T}$ is upward and $(qq)_1$ is spinning clockwise. The correlation between $(qq)_1$ and $\mathbf{S}_q$ is negative (q is spinning anticlockwise), leading to a positive $\hat{a}_{C,B}^{(1)}$, whereas the correlation between $(qq)_1$ and $\mathbf{S}_B$ is positive, leading to a negative $\hat{a}_B^{(1)}$.

The first line gives the transfer of longitudinal polarization from the quark to the baryon, while the second line describes the conversion of the quark transverse polarization to the longitudinal polarization of the baryon.

If the leftover diquark in the splitting $q \to B + (\overline{qq})_0$ is a scalar diquark, then, according to Eqs. (24) and (17), the spin density matrix of the baryon is

$$\rho^{(0)}(B) = \rho(q), \tag{47}$$

which leads to the baryon polarization vector

$$\mathbf{S}^{(0)}_{B,\mathrm{T}} = \mathbf{S}_{q,\mathrm{T}}, \qquad S^{(0)}_{B,\mathrm{L}} = S_{q,z}. \tag{48}$$

Hence, the quark polarization is completely transferred to the baryon due to the fact that the scalar diquark does not carry spin information.

Comparing with Eqs. (43)–(46), one can see that the contribution to the transverse spin transfer from the quark to the baryon via a leftover scalar diquark has opposite sign with respect to the case of a leftover PV diquark.

a. Effective spin density matrix of the baryon The effective spin density matrix for the baryon B produced in the splitting $q \to B + (\overline{qq})$ predicted by the model is obtained by the weighted sum of Eqs. (42) and (47). We obtain

$$\rho(B) = \frac{P_{(qq)_1} F_{(\overline{qq})_1,B,q}\rho^{(1)}(B) + P_{(qq)_0} F_{(\overline{qq})_0,B,q}\rho^{(0)}(B)}{P_{(qq)_1} F_{(\overline{qq})_1,B,q} + P_{(qq)_0} F_{(\overline{qq})_0,B,q}}. \tag{49}$$

The polarization vector of B can then be obtained analogously by the substitutions $\rho^{(1)}(B) \to \mathbf{S}^{(1)}_B$ and $\rho^{(0)}(B) \to \mathbf{S}^{(0)}_B$.

B. Spin density matrix of the antibaryon

The spin density matrix of the antibaryon $\bar{B}$ produced in the splitting $(\overline{qq})_1 \to \bar{B} + q'$ can be obtained from the triple-polarized splitting function in Eq. (25) by taking $\check{\rho}(q') = 1_{2\times 2}$ and interpreting the spin-dependent part as $\mathrm{Tr}[\rho^{(1)}(\bar{B})\check{\rho}(\bar{B})]$, with $\rho^{(1)}(\bar{B}) = \sigma_z \rho^{(1)}(-\mathbf{S}_{\bar{B}})\sigma_z$. The antibaryon spin density matrix is thus

$$\rho^{(1)}(\bar{B}) = \frac{\sigma_z \Gamma^\dagger_{B,b} \sigma_z \Delta^\dagger_{\bar{q}'} \Delta_{\bar{q}'} \sigma_z \Gamma_{B,a} \sigma_z \rho_{ab}(qq)}{\mathrm{Tr}[\ldots]}, \tag{50}$$

where the trace operation in the denominator is taken over the baryon spin indices. The resulting transverse and longitudinal components of the $\bar{B}$ polarization can be derived using the quark propagator in Eq. (20) and the coupling in Eq. (17). The transverse component of the polarization vector reads

$$\mathbf{S}^{(1)}_{\bar{B},\mathrm{T}} = \frac{1}{N_{\overline{qq}}(\mathbf{S}_{\overline{qq}})}[(|\mu_q|^2 + \mathbf{k}'^2_{\mathrm{T}})\mathbf{S}_{\overline{qq},\mathrm{T}} - 2\,\mathrm{Im}(\mu_q)\hat{\mathbf{z}} \times \mathbf{k}'_{\mathrm{T}} + 4\,\mathrm{Im}(\mu_q)(\mathrm{Re}\rho_{\mathrm{T}}(\overline{qq})[\hat{\mathbf{z}} \times \mathbf{k}'_{\mathrm{T}}])], \tag{51}$$

where $\rho_{\mathrm{T}}(\overline{qq})$ is the 3×3 matrix with components $\rho_{\mathrm{T},ij}(\overline{qq}) = \rho_{ij}(\overline{qq})$ for $i, j = x,\ y$ and zero elsewhere. The vector $\mathbf{S}_{\mathrm{qq}}$ is defined in Eq. (39). The normalization function is given by

$$N_{\overline{qq}}(\mathbf{S}_{\overline{qq}}) = |\mu_q|^2 + \mathbf{k}'^2_{\mathrm{T}} - 2\,\mathrm{Im}(\mu_q)\mathbf{S}_{\overline{qq},\mathrm{T}} \cdot (\hat{\mathbf{z}} \times \mathbf{k}'_{\mathrm{T}}). \tag{52}$$

The first term in Eq. (51) describes the transfer from diquark transverse polarization to the $\bar{B}$ transverse polarization. The second term is a source of spontaneous polarization for $\bar{B}$ [c.f. with Eq. (2)] and, since in the string + $^3P_0$ model $\mathrm{Im}(\mu_q) > 0$, the polarization has the same sign as that arising in the splitting $q \to B + (\overline{qq})_1$ in Eq. (43). The model thus predicts the same sign for the spontaneous polarization of baryons and antibaryons, as observed for Λ and $\bar{\Lambda}$ hyperons in e^+e^- annihilation at BELLE [17]. The last term describes the transfer of the transverse tensor polarization from the diquark to the antibaryon.

The antibaryon longitudinal polarization resulting from Eq. (50) is

$$S^{(1)}_{\bar{B},\mathrm{L}} = \frac{1}{N_{\overline{qq}}(\mathbf{S}_{\overline{qq}})}[(|\mu_q|^2 + \mathbf{k}'^2_{\mathrm{T}})S_{\overline{qq},\mathrm{L}} + 4\,\mathrm{Im}(\mu_q)\hat{\mathbf{z}} \cdot (\mathrm{Re}\rho(\overline{qq})[\hat{\mathbf{z}} \times \mathbf{k}'_{\mathrm{T}}])], \tag{53}$$

where $S_{\overline{qq},\mathrm{L}} = \mathrm{i}[\rho_{xy}(\overline{qq}) - \rho_{yx}(\overline{qq})]$ is the longitudinal vector polarization of the diquark. The first term describes the transfer of longitudinal polarization from the diquark to the antibaryon while the second term proportional to $\mathrm{Re}(\rho_{zy})k'_x - \mathrm{Re}(\rho_{zx})k'_y$ describes the conversion of the diquark oblique polarization to the antibaryon longitudinal polarization.

Concerning the scalar antidiquark splitting $(\overline{qq})_0 \to \bar{B} + q'$, the corresponding spin density matrix of the produced $\bar{B}$ can be obtained from Eq. (50) by removing $\rho_{ab}(\overline{qq})$ and taking for $\Gamma_{B,0}$ the unit matrix according to Eq. (17). The antibaryon spin density matrix reads

$$\rho^{(0)}(\bar{B}) = \frac{\Delta^\dagger_{\bar{q}'}\Delta_{\bar{q}'}}{\mathrm{Tr}[\ldots]} = \frac{1}{2}\left[1 + \frac{2\,\mathrm{Im}(\mu_q)}{|\mu_q|^2 + \mathbf{k}'^2_{\mathrm{T}}}\boldsymbol{\sigma}_{\mathrm{T}} \cdot (\hat{\mathbf{z}} \times \mathbf{k}'_{\mathrm{T}})\right], \tag{54}$$

which gives for the polarization vector of the antibaryon

$$\mathbf{S}^{(0)}_{\bar{B},\mathrm{T}} = \frac{2\,\mathrm{Im}(\mu_q)}{|\mu_q|^2 + \mathbf{k}'^2_{\mathrm{T}}}\hat{\mathbf{z}} \times \mathbf{k}'_{\mathrm{T}}, \qquad S^{(0)}_{\bar{B},\mathrm{L}} = 0. \tag{55}$$

Antibaryons produced in scalar antidiquark splittings have therefore spontaneous polarization, which originates from the correlation between the spins and transverse momenta of the quark pair produced via the $^3P_0$ mechanism. Our result in Eq. (55), obtained with the quantum-mechanical string + $^3P_0$ model, is reminiscent of the expression for the spontaneous polarization of Λ hyperons produced in unpolarized pp scattering obtained by the Lund group using a semiclassical model of hadronization involving the $^3P_0$ mechanism [19].

V. POLARIZED HYPERON DECAY

The spin density matrix $\rho(B)$ $[\rho(\bar{B})]$ of the produced baryon (antibaryon) is used for the description of the decay of the particle in its rest frame (see Fig. 4). In this work, we consider the two-body decays of hyperons $B = Y$, of the type $Y \to B' + X$, where $B' = p, n, Y'$ is another baryon and $X = \gamma, \pi$ the remaining decay product (see Fig. 5). This applies to the hyperons $Y = \Lambda, \Sigma, \Xi$. We distinguish among three types of decays: the nonleptonic (NL) decays ($\Lambda \to p + \pi$, $\Sigma^+ \to p + \pi^0$, etc.), the electromagnetic (EM) decay $\Sigma^0 \to \Lambda + \gamma$, and the weak radiative (WR) decays ($\Xi^0 \to \Sigma^0 + \gamma$, $\Lambda \to n + \gamma$, etc.). In this section, we focus on the NL decay, the most common among the hyperons, and describe the other decays in Appendix B.

In the string rest frame, where the four-momentum of Y is p, we indicate by p' the four-momentum of B'. The decay is described in the rest frame of Y. In this frame, we indicate the four-momenta of the decay products by capital letters, *i.e.* P' and Q'. We define the rest frame of Y by the combination of boosts introduced in Ref. [28]:

(i) A longitudinal boost $B_{\rm L}^{-1}(p_z/\varepsilon_Y)$ that brings Y at a reference frame with $p_{\rm L} = 0$, but conserving $\mathbf{p}_{\rm T}$, and
(ii) A transverse boost $B_{\rm T}^{-1}(\mathbf{p}_{\rm T}/M_Y)$ that brings Y at rest.

In the rest frame of Y, the baryon B' has four-momentum $P' = (E_{B'}, \mathbf{P}')$, with $\mathbf{P}' \equiv |\mathbf{P}'|\hat{\boldsymbol{P}}'$. The two-body kinematics is solved by $E_{B'} = [M_Y^2 - M_X^2 + M_{B'}^2]/(2M_Y)$ and $|\mathbf{P}'| = \sqrt{E_{B'}^2 - M_{B'}^2}$ [43]. The direction of B' is given by $\hat{\boldsymbol{P}}' = (\cos\phi_{\mathbf{P}'}\sin\theta_{\mathbf{P}'}, \sin\phi_{\mathbf{P}'}\sin\theta_{\mathbf{P}'}, \cos\theta_{\mathbf{P}'})$, where $\phi_{\mathbf{P}'}$ and $\theta_{\mathbf{P}'}$ are the azimuthal and polar angles of B' in the rest frame of Y.

After $\hat{\boldsymbol{P}}'$ has been generated according to the polarized angular distribution $dN_{Y\to B'+X}/d\Omega_{\mathbf{P}'}$ of the decay $Y \to B' + X$, where $d\Omega_{\mathbf{P}'} = d\phi_{\mathbf{P}'} d\cos\theta_{\mathbf{P}'}$, p' is obtained by the sequence of boosts $p' = B_{\rm L}(p_z/\varepsilon_Y)B_{\rm T}(\mathbf{p}_{\rm T}/M_Y)P'$.

A. Nonleptonic decay

To obtain the polarized angular distribution in the rest frame of Y, the amplitude associated to the decay $Y \to B' + X$ is needed. The latter reads

$$\bar{u}_{B'}(p')\hat{\mathcal{M}}_{\rm cov.}^{Y\to B'+X'}u_Y(p) = \chi^\dagger(\mathbf{S}_{B'})\hat{\mathcal{M}}\chi(\mathbf{S}_Y), \quad (56)$$

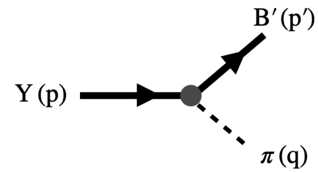

FIG. 5. Diagram for the nonleptonic decay $Y \to B' + \pi$ of a hyperon Y. In parentheses is indicated the four-momentum of each particle.

where $\bar{u}_{B'}$ is the on-shell Dirac spinor of B' and $\hat{\mathcal{M}}_{\rm cov.}$ is the covariant decay matrix element. The righthand side expression is obtained in the rest frame of Y. It is the decay amplitude in Pauli spin space, with $\chi(\mathbf{S}_{B'})$ being the spinor with polarization vector $\mathbf{S}_{B'}$ that describes the spin state of B' in its rest frame, obtained by a direct boost from the rest frame of Y. The reduction of $\hat{\mathcal{M}}_{\rm cov.}$ in Pauli spin space is the 2×2 matrix $\hat{\mathcal{M}}$. It depends on the decay process and determines the angular distribution of the decay products in the rest frame of Y.

For the weak decay $Y \to B' + \pi$ [Fig. 5(b)], we parameterize the covariant matrix element as $\hat{\mathcal{M}}_{\rm cov} \propto [A_s 1_{4\times 4} - B_p\gamma_5]$, with A_s and B_p being complex parameters [43]. The parameter A_s gives a parity violating contribution to the decay amplitude. We obtain for the reduced matrix element

$$\hat{\mathcal{M}}_{\rm NL} \propto A_s + \tilde{B}_p\boldsymbol{\sigma}\cdot\hat{\boldsymbol{P}}', \quad (57)$$

where we have introduced the parameter $\tilde{B}_p = B_p|\mathbf{P}'|/(E_{B'} + M_{B'})$.

The decay distribution in the rest frame of Y for both Y and B' polarized is

$$\frac{dN^{\rm NL}_{Y\to B'+\pi}}{d\Omega_{\mathbf{P}'}} \propto {\rm Tr}_{B'}[\hat{\mathcal{M}}_{\rm NL}\rho(Y)\hat{\mathcal{M}}^\dagger_{\rm NL}\check{\rho}(B')]. \quad (58)$$

The explicit decay distribution can be obtained inserting Eq. (57) into Eq. (58). If the detector of B' does not favor a special polarization $\check{\rho}(B') = 1_{2\times 2}$ and one gets the known angular distribution

$$\frac{dN_{Y\to B'+\pi}}{d\Omega_{\mathbf{P}'}} = \frac{1}{4\pi}(1 + \alpha_Y\mathbf{S}_Y\cdot\hat{\boldsymbol{P}}'). \quad (59)$$

The parameter $\alpha_Y = 2{\rm Re}A_s^*\tilde{B}_p/(|A_s|^2 + |\tilde{B}_p|^2)$ depends on the decaying hyperon. It is measured for different hyperons and can be taken from Ref. [43].

The spin density matrix of B' can be deduced from Eq. (58). We obtain

$$\rho(B') \equiv \frac{1}{2}[1 + \boldsymbol{\sigma}\cdot\mathbf{S}_{B'}] = \frac{\hat{\mathcal{M}}_{\rm NL}\rho(Y)\hat{\mathcal{M}}^\dagger_{\rm NL}}{{\rm Tr}[\ldots]}, \quad (60)$$

where the explicit expression for the polarization vector $\mathbf{S}_{B'}$ can be obtained using the matrix element in Eq. (57) and reads

$$\mathbf{S}_{B'} = (1+\alpha_Y \mathbf{S}_Y\cdot\hat{\boldsymbol{P}}')^{-1}[(\alpha_Y+\mathbf{S}_Y\cdot\hat{\boldsymbol{P}}')\hat{\boldsymbol{P}}' + \beta_Y \mathbf{S}_Y\times\hat{\boldsymbol{P}}' + \gamma_Y\hat{\boldsymbol{P}}'\times(\mathbf{S}_Y\times\hat{\boldsymbol{P}}')]. \tag{61}$$

The parameters $\beta_Y = 2\,\mathrm{Im}(A_s^*\tilde{B}_p)/(|A_s|^2+|\tilde{B}_p|^2)$ and $\gamma_Y = (|A_s|^2-|\tilde{B}_p|^2)/(|A_s|^2+|\tilde{B}_p|^2)$ depend on the decaying hyperon and are related to α_Y by $\alpha_Y^2+\beta_Y^2+\gamma_Y^2=1$. They have been measured for different hyperons and can be found in Ref. [43].

Finally, the acceptance matrix of Y, used in the Collins-Knowles recipe [44,45], reads

$$\check{\rho}(Y) = \hat{\mathcal{M}}_{\mathrm{NL}}^{\dagger}\check{\rho}(B')\hat{\mathcal{M}}_{\mathrm{NL}} \propto 1+\boldsymbol{\sigma}\cdot\check{\mathbf{S}}_Y, \tag{62}$$

which is evaluated at the generated value of $\hat{\boldsymbol{P}}'$. The expression for the acceptance polarization vector $\check{\mathbf{S}}_Y$ can be obtained from Eq. (61) by the replacements $\mathbf{S}_Y\to\check{\mathbf{S}}_{B'}$ and $\beta_Y\to-\beta_Y$. For a not-analyzed baryon B' the vector $\check{\mathbf{S}}_Y$ is nonvanishing due to the parity violation in the NL decay, and it reads $\check{\mathbf{S}}_Y = \alpha_Y\hat{\boldsymbol{P}}'$.

VI. PROPAGATION OF THE SPIN INFORMATION ALONG THE FRAGMENTATION CHAIN

After a baryon B has been emitted in the quark splitting $q\to B+(\overline{qq})$ and B has decayed, it is necessary to continue the propagation of the spin information along the fragmentation chain. The propagation of the spin information is shown in Fig. 4 by the dotted lines ending with an arrow for the chain of splittings $q\to B+(\overline{qq})_1$, $(\overline{qq})_1\to\bar{B}+q'$. Such lines represent either spin density matrices or acceptance matrices.

To propagate the spin information along the string fragmentation chain, it is necessary to evaluate the spin density matrix of the leftover particle after each elementary splitting. For the meson emission case $q\to M+q'$, this has been done in Ref. [28]. Here we study the splittings $q\to B+(\overline{qq})_1$ followed by $(\overline{qq})_{0,1}\to\bar{B}+q'$ involving the production of a baryon $B=p,n,Y$. If the two splittings in Fig. 4 are connected by a scalar antidiquark $(\overline{qq})_0$, the polarization of q is not transferred to q'. Even in this case, the q' is polarized and its spin density matrix is needed to propagate the spin correlations along the chain starting from the production of $\bar{B}$.

A. Spin density matrix of $(\overline{qq})$ in reaction $q\to B+(\overline{qq})$

The spin density matrix $\rho_{aa'}(\overline{qq})$ of the PV antidiquark $(\overline{qq})_1$ in the splitting $q\to B+(\overline{qq})_1$ can be read off from the triple-polarized splitting function in Eq. (24), by interpreting the last line as $\rho_{aa'}(\overline{qq})\check{\rho}_{a'a}(\overline{qq})$. It gives

$$\rho_{aa'}(\overline{qq}) = \frac{\Delta_{ab}\mathrm{Tr}_B[\Gamma_b\rho(q)\Gamma_{b'}^{\dagger}\check{\rho}(B)]\Delta_{b'a'}^{\dagger}}{\mathrm{Tr}[\ldots]}, \tag{63}$$

where the trace appearing in the denominator is taken over the antidiquark spin indices. As can be seen, the spin state of the antidiquark depends on the spin state of the fragmenting quark q via $\rho(q)$ and on the orientation of the decay products in the decay of B via the acceptance matrix $\check{\rho}(B)$ for a hyperon $B=Y$, as expected by the Collins-Knowles recipe [44,45]. For a stable baryon $(B=p,\,n)$ we take $\check{\rho}(B)=1_{2\times2}$.

B. Spin density matrix of q' in reaction $(\overline{qq})\to\bar{B}+q'$

The spin density matrix $\rho(q')$ of the leftover quark q' in the splitting $(\overline{qq})\to\bar{B}+q'$ can be obtained from Eq. (25), by interpreting the last line of the triple-polarized splitting function as $\mathrm{Tr}[\rho(q')\check{\rho}(q')]$. It gives

$$\rho(q') \propto \begin{cases} \Delta_{q'}\check{\rho}(\bar{B})\Delta_{q'}^{\dagger}, & (\overline{qq})_0 \\ \Delta_{q'}\sigma_z\Gamma_{B,a}\sigma_z\rho_{aa'}(\overline{qq})\check{\rho}(\bar{B})\sigma_z\Gamma_{B,a'}^{\dagger}\sigma_z\Delta_{q'}^{\dagger}, & (\overline{qq})_1 \end{cases}. \tag{64}$$

As can be seen, in the splitting of a scalar antidiquark $(\overline{qq})_0$ the quark q' can also be polarized despite the absence of spin information coming from the diquark. In particular, for a nonanalyzed antibaryon $\bar{B}$, the polarization of q' arises from the correlation between spin and transverse momentum at string breaking due to the $^3P_0$ mechanism. For a fragmenting PV antidiquark $(\overline{qq})_1$, the polarization of q' depends on $\mathbf{k}'_{\mathrm{T}}$, on the spin density matrix $\rho(\overline{qq})$ of the antidiquark and on the orientation of the decay products of $\bar{B}$, via $\check{\rho}(\bar{B})$, for a hyperon $\bar{B}=\bar{Y}$ if the decay is analyzed.

C. Recursive fragmentation chain

The complete fragmentation chain is simulated recursively by applying the splittings $q\to M+q'$, $q\to B+(\overline{qq})$ or $(\overline{qq})\to\bar{B}+q'$ until the energy-momentum in the string falls below some threshold after which the exit condition is called and the fragmentation chain is stopped. The recursive algorithm, including the exit condition, is similar to that provided in Ref. [28] for the production of mesons, and is not reported here.

The main difference is that for a fragmenting quark q, the relative probability P_{qq}/P_q is used to probabilistically decide whether to emit a baryon B by the splitting $q\to B+(\overline{qq})$ or a meson M by the splitting $q\to M+q'$. PV diquarks at string breakings are also further suppressed compared to scalar diquarks. The suppression is regulated by the further parameter $P_{(\overline{qq})_1}/P_{(\overline{qq})_0}$. According to Pythia, it is $P_{qq}/P_q\sim0.1$ and $P_{(\overline{qq})_1}/P_{(\overline{qq})_0}\sim0.03$, hence a string breaking occurs via the tunneling of quark–antiquark pairs roughly ten times more frequently than via the tunneling of diquark–antidiquark pairs, and PV diquarks are seldomly produced [30]. For a given baryon, however, the contribution of PV diquarks can be enhanced by its flavor wavefunction. Strange scalar and PV diquarks are further suppressed with respect to nonstrange diquarks by few additional parameters; see also Ref. [46].

VII. CONCLUSIONS

We have extended the string $+\,^3P_0$ model of spin-dependent string fragmentation by introducing the production and decay of polarized spin $1/2$ baryons. To include baryon production, we assume that a quark of the recursive fragmentation chain can be replaced by an antidiquark (or an antiquark by a diquark) of spin 0 (scalar) or 1 (pseudovector). The 3×3 propagator of a pseudovector diquark is inspired from a tunneling mechanism producing a diquark–antidiquark pair in the $^5D_0$ state while breaking the string, in analogy with the $^3P_0$ mechanism of $q\bar{q}$ pair creation assumed for the quark propagator. Its adopted precise form depends on a complex parameter μ_{qq}, which plays the analog role of the complex mass μ_q of the $^3P_0$ mechanism. Using the pseudovector diquark propagator and introducing the couplings of quarks and diquarks to baryons, we have constructed the *matrix splitting amplitudes* for the description of the elementary quark and diquark splittings to baryons. They constitute the essential ingredients required for the systematic propagation of the spin correlations along the fragmentation chain.

Using these matrices, we show that a Collins effect is predicted for the emission of baryons in the fragmentation of transversely polarized quarks due to the correlation between spin and transverse momentum of the tunneling spin-1 diquarks. For $\mathrm{Im}(\mu_{qq})<0$, the Collins effect for baryon production has the same sign as for the PS meson emission in the string $+\,^3P_0$ model, as also predicted by a classical string $+\,^5D_0$ mechanism.

Furthermore, we show that the model reproduces a spontaneous transverse polarization for the emitted baryon along the vector perpendicular to the production plane of the baryon, as well as the transverse spin transfer from a fragmenting transversely polarized quark to the emitted baryon. For a hyperon, the predicted spontaneous polarization has the same sign as that measured in e^+e^- annihilation to hadrons by the BELLE experiment. This is an encouraging result that motivates studies of more quantitative predictions of the model. This calls for the implementation of the model in the Monte Carlo event generator Pythia, which will be presented in a separate work.

Finally, the model presented in this work can be regarded as the basis for new interesting applications such as the description of the spin effects in the target fragmentation region in deep-inelastic scattering events, as well the simulation of the spin-dependent *fracture functions*. We also note that the model allows for the successive production of $B\bar{B}$ pairs, while configurations such as $BM_1M_2\ldots\bar{B}$, where mesons $M_1M_2\ldots$ are produced between the baryons like in the *pop-corn model* [38,47], are neglected. The production of such final state hadronic configurations can be achieved by either introducing the splittings of (anti) diquarks to mesons and a leftover (anti)diquark, or by endowing the popcorn model with the spin degree of freedom. The exploration of these applications is planned for future work.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: CONSTRUCTION OF THE SPIN-1 DIQUARK PROPAGATOR

1. Theoretical constraints on the propagator

Indicating by $\mathbf{\Phi}$ and $\mathbf{\Phi}'$ the initial and final (with respect to the propagation) three-dimensional spin wavefunctions of the diquark, we require the amplitude $\mathbf{\Phi}'^{\dagger}\Delta(\mathbf{k}_{\mathrm{T}})\mathbf{\Phi}$ to be invariant under

(1) simultaneous rotation of $\mathbf{\Phi}$, $\mathbf{\Phi}'$ and $\mathbf{k}_{\mathrm{T}}$ about the $\hat{\mathbf{z}}$ axis,
(2) symmetry about any plane containing the $\hat{\mathbf{z}}$ axis,
(3) longitudinal boost of k and k' for fixed $\mathbf{\Phi}$ and $\mathbf{\Phi}'$,
(4) quark line reversal (LR) symmetry.

The symmetry (2) implies (1). The LR symmetry in (4) is also called *left-right symmetry* in the Lund Model of string fragmentation [36]. We find that the LR symmetry implies for the propagator the condition

$$\Delta_{\overline{qq}}(-R\mathbf{k}) = R\Delta^{\mathrm{t}}_{\overline{qq}}(\mathbf{k})R, \tag{A1}$$

where $R=\mathrm{diag}(-1,1,-1)$ is the rotation matrix of an angle π about the $\hat{\mathbf{y}}$ axis and on the righthand side the transpose of the propagator appears.

The required invariances (1)–(4) are satisfied by the general form

$$\Delta(\mathbf{k}_{\mathrm{T}}) = g_1(\mathbf{k}_{\mathrm{T}}^2)\Delta_1(\mathbf{k}_{\mathrm{T}}) + g_2(\mathbf{k}_{\mathrm{T}}^2)\Delta_2(\mathbf{k}_{\mathrm{T}}) + g_3(\mathbf{k}_{\mathrm{T}}^2)\Delta_3(\mathbf{k}_{\mathrm{T}}) + g_4(\mathbf{k}_{\mathrm{T}}^2)\Delta_4(\mathbf{k}_{\mathrm{T}}), \tag{A2}$$

with

$$\Delta_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \Delta_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$\Delta_3 = \begin{pmatrix} k_x^2 & k_x k_y & 0 \\ k_y k_x & k_y^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \Delta_4 = \begin{pmatrix} 0 & 0 & k_x \\ 0 & 0 & k_y \\ k_x & k_y & 0 \end{pmatrix}. \tag{A3}$$

The functions $g_i(\mathbf{k}_\mathrm{T}^2)$ are *a priori* unknown, complex, and independent. In the following subsection, we employ a mechanism for the production of a diquark–antidiquark pair at the string breakings that leads to simple expressions for these functions.

2. Diquark pair production from vacuum: The $^5D_0$ mechanism

Let us start with the two possible wave functions of the $(qq)_1 - (\overline{qq})_1$ pair having the $J^{PC} = 0^{++}$ quantum numbers of the vacuum

$$\Psi_{ab}^{^1S_0} \propto \delta_{ab}, \tag{A4}$$

$$\Psi_{ab}^{^5D_0} \propto -\frac{\mathbf{k}^2}{3}\delta_{ab} + \mathbf{k}_a\mathbf{k}_b, \tag{A5}$$

where $\mathbf{k} = \mathbf{k}(\overline{qq}) - \mathbf{k}(qq)$ is the relative momentum of the pair.

The most general $J^{PC} = 0^{++}$ wave function is of the form

$$\Psi_{ab}(\mathbf{k}) = A(\mathbf{k}^2)\Psi_{ab}^{^1S_0} + B(\mathbf{k}^2)\Psi_{ab}^{^5D_0}. \tag{A6}$$

As a temporary choice, let us take for the diquark propagator $\Delta_{qq;ab}(\mathbf{k}) = \Psi_{ab}(\mathbf{k})$, which gives

$$\Delta_{qq}(\mathbf{k}) = B(\mathbf{k}^2) \times \begin{pmatrix} k_x^2 + D(\mathbf{k}^2) & k_x k_y & k_x k_z \\ k_x k_y & k_y^2 + D(\mathbf{k}^2) & k_y k_z \\ k_z k_x & k_z k_y & k_z^2 + D(\mathbf{k}^2) \end{pmatrix}, \tag{A7}$$

with $D(\mathbf{k}^2) = A(\mathbf{k}^2)/B(\mathbf{k}^2) - \mathbf{k}^2/3$. Without loss of generality, we take $B(\mathbf{k}^2) = 1$, hence $D(\mathbf{k}^2) = A(\mathbf{k}^2) - \mathbf{k}^2/3$. This temporary choice of the propagator does not satisfy Eq. (A2), because it depends on k_z. For a definitive choice, we must first replace $A(\mathbf{k}^2)$ and $B(\mathbf{k}^2)$ with $A(\mathbf{k}_\mathrm{T}^2)$ and $B(\mathbf{k}_\mathrm{T}^2)$.

The latter function is *a priori* unknown. A possible choice can be inspired from the semiclassical picture of string breaking in Fig. 2(b). The figure represents the positions on the string and the momenta of (qq) and $(\overline{qq})$ when they exit the tunneling region for a given $\mathbf{k}_\mathrm{T}$. There, $k_z = 0$ and the orbital angular momentum of the pair is $\mathbf{L} = -d\hat{\mathbf{z}} \times \mathbf{k}$, where d is the $(qq) - (\overline{qq})$ distance. Energy conservation in the tunneling yields $d = 2\sqrt{m_{qq}^2 + \mathbf{k}_\mathrm{T}^2}/\kappa$, leading to $\mathbf{L} = -2\sqrt{m_{qq}^2 + \mathbf{k}_\mathrm{T}^2}\hat{\mathbf{z}} \times \mathbf{k}_\mathrm{T}/\kappa$. Assuming a typical diquark mass $m_{qq} \sim 0.5$ GeV [47] and using the value of the string tension $\kappa \simeq 0.2\ \mathrm{GeV}^2$, the Schwinger formula for the tunneling probability of the diquark–antidiquark pair (see Sec. II B) yields an average orbital angular momentum $\langle L \rangle \simeq 1.3$. Considering this large average value of L, we assume the $(qq)_1 - (\overline{qq})_1$ pair to be produced essentially in the $^5D_0$ state. This corresponds to taking $A(\mathbf{k}_\mathrm{T}'^2) = 0$ in Eq. (A7).

In the center of the tunneling region, the longitudinal relative diquark momentum is $k_z = -\mathrm{i}(m_{qq}^2 + \mathbf{k}_\mathrm{T}^2)^{1/2}$. Replacing first $\mathbf{k}^2$ by $-m_{qq}^2$, then the remaining k_z by a phenomenological complex parameter μ_{qq}, our model result for the spin-1 diquark propagator is

$$\Delta_{qq}(\mathbf{k}_\mathrm{T}') = \begin{pmatrix} k_x'^2 + m_{qq}^2/3 & k_x' k_y' & \mu_{qq} k_x' \\ k_x' k_y' & k_y'^2 + m_{qq}^2/3 & \mu_{qq} k_y' \\ \mu_{qq} k_x' & \mu_{qq} k_y' & \mu_{qq}^2 + m_{qq}^2/3 \end{pmatrix}. \tag{A8}$$

Comparing with Eq. (A2), the model gives $g_1 = m_{qq}^2/3$, $g_2 = \mu_{qq}^2 + m_{qq}^2/3$, $g_3 = 1$ and $g_4 = \mu_{qq}$. Equation (A8) is thus a particular case of Eq. (A2).

Note that the parameter μ_{qq} is *a priori* pure imaginary given by $\mu_{qq} = k_z = -\mathrm{i}\sqrt{m_{qq}^2 + \mathbf{k}_\mathrm{T}^2}$ and $\mu_{qq}^2 = -(m_{qq}^2 + \mathbf{k}_\mathrm{T}^2)$ is real. Since our model is, however, phenomenological we take μ_{qq} to be a complex parameter whose real and imaginary parts are to be determined from data. This generalization is similar to the parameter μ_q of the string $+\ ^3P_0$ model [26] and has the advantage of including in the model all possible spin transfers between the quark q and the baryon B in the splitting $q \to B + (\overline{qq})_1$. In fact, as can be seen from Eqs. (51) and (53), a pure imaginary μ_{qq} does not lead to the conversion of transverse (longitudinal) quark polarization to longitudinal (transverse) baryon polarization in $q \to B + (\overline{qq})_1$.

APPENDIX B: RADIATIVE DECAYS OF HYPERONS

In this section, we study the electromagnetic decay $\Sigma^0 \to \Lambda + \gamma$ and the weak radiative decays of polarized hyperons, *i.e.*, decays of the type $Y \to B' + \gamma$ shown by the diagram in Fig. 6. We label the momenta of the particles as in Sec. V.

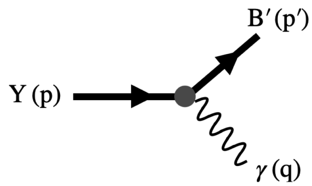

FIG. 6. Diagram for the electromagnetic or radiative weak decay $Y \to B' + \gamma$ of a hyperon Y. In parentheses is indicated the four-momentum of each particle.

1. Electromagnetic decay

The covariant matrix element describing the electromagnetic decay $Y \to B' + \gamma$ can be obtained by writing $\hat{\mathcal{M}}_{\rm cov} = \hat{\mathcal{M}}^{\mu}_{\rm cov.}\Phi^{*}_{\gamma,\mu}$, Φ^{μ}_{γ} being the polarization four-vector of the emitted photon, and expanding $\hat{\mathcal{M}}^{\mu}_{\rm cov.}(p', q)$ in terms of Dirac matrices, $q = p - p'$ being the four-momentum of the photon. Requiring parity invariance and electromagnetic current conservation leads to $\hat{\mathcal{M}}_{\rm cov} \propto -{\rm i}\sigma^{\mu\nu} q_{\nu}\Phi^{*}_{\gamma,\mu}$, where $\sigma^{\mu\nu} = {\rm i}[\gamma^{\mu}, \gamma^{\nu}]/2$. In the rest frame of Y, it gives the reduced matrix element

$$\hat{\mathcal{M}}_{\rm EM} \propto -\boldsymbol{\sigma} \cdot [\hat{\boldsymbol{P}}' \times \boldsymbol{\Phi}^{*}_{\gamma}], \tag{B1}$$

where we have kept only the spin-dependent part of the matrix element. $\boldsymbol{\Phi}_{\gamma}$ is the vector amplitude of γ in the Coulomb gauge.

The matrix element in Eq. (B1) leads to the decay distribution

$$\frac{dN^{\rm EM}_{Y \to B'+\gamma}}{d\Omega_{\mathbf{P}'}} \propto \sum_{\boldsymbol{\Phi}_{\gamma}} {\rm Tr}_{B'}[\hat{\mathcal{M}}_{\rm EM}\rho(Y)\hat{\mathcal{M}}^{\dagger}_{\rm EM}\check{\rho}(B')], \tag{B2}$$

where we have summed over the polarization states of the (not analyzed) γ, and $\check{\rho}(B')$ is the acceptance matrix of B'. Inserting Eq. (B1) into Eq. (B2), the angular distribution of B' is obtained and taking $\check{\rho}(B') = 1_{2\times 2}$. We get $dN^{\rm EM}_{Y\to B'+\gamma}/d\Omega_{\mathbf{P}'} = (4\pi)^{-1}$, hence an isotropic angular distribution in the rest frame of Y.

If B' is a hyperon to also be decayed, its spin density matrix is needed and can be read off from Eq. (B2). We obtain

$$\begin{aligned}\rho(B') &= \frac{\sum_{\boldsymbol{\Phi}_{\gamma}} \hat{\mathcal{M}}_{\rm EM}\rho(Y)\hat{\mathcal{M}}^{\dagger}_{\rm EM}}{{\rm Tr}[\ldots]} \\ &= \frac{1}{2}[1 - (\mathbf{S}_Y \cdot \hat{\boldsymbol{P}}')\boldsymbol{\sigma} \cdot \hat{\boldsymbol{P}}'],\end{aligned} \tag{B3}$$

where the second line is obtained by using Eq. (B1). The polarization vector of B' is thus $\mathbf{S}_{B'} = -(\mathbf{S}_Y \cdot \hat{\boldsymbol{P}}')\hat{\boldsymbol{P}}'$, in agreement with helicity conservation in the decay process.

Once the decay of Y has been simulated, according to the Collins-Knowles recipe [44,45], it returns the acceptance matrix (also named decay matrix) $\check{\rho}(Y) = 1 + \boldsymbol{\sigma} \cdot \check{\mathbf{S}}_Y$, where $\check{\mathbf{S}}_Y$ is the acceptance polarization vector of Y. This is necessary to propagate the information about the relative orientation of the decay products of Y back to the emission vertex of Y, and to continue the propagation of the spin correlations along the fragmentation chain. The acceptance matrix can be deduced from Eq. (B2) interpreting the decay distribution as ${\rm Tr}[\rho(Y)\check{\rho}(Y)]$. We obtain the not-normalized acceptance matrix

$$\check{\rho}(Y) = \sum_{\boldsymbol{\Phi}_{\gamma}} \hat{\mathcal{M}}^{\dagger}_{\rm EM}\check{\rho}(B')\hat{\mathcal{M}}_{\rm EM} = 1 - (\check{\mathbf{S}}_{B'} \cdot \hat{\boldsymbol{P}}')\boldsymbol{\sigma} \cdot \hat{\boldsymbol{P}}', \tag{B4}$$

evaluated at the generated value of $\hat{\boldsymbol{P}}'$. The acceptance polarization vector of Y is therefore $\check{\mathbf{S}}_Y = -(\check{\mathbf{S}}_{B'} \cdot \hat{\boldsymbol{P}}')\hat{\boldsymbol{P}}'$, where $\check{\mathbf{S}}_{B'}$ is the acceptance polarization vector stemming from the decay of an unstable B'. If the decay of B' is not analyzed, it is $\check{\mathbf{S}}_Y = \mathbf{0}$.

2. Weak radiative decay

To describe the weak radiative decay $Y \to B' + \gamma$, we take the covariant matrix element $\hat{\mathcal{M}}_{\rm cov} = F_1\Gamma^{\mu}\Phi^{*}_{\gamma,\mu} + iF_2\sigma^{\mu\nu}q_{\mu}\Phi^{*}_{\gamma,\nu} + F_3\gamma^{\mu}\gamma_5\Phi^{*}_{\gamma,\mu}$, where F_1, F_2, and F_3 are form factors [43]. In the rest frame of Y, the covariant matrix element reduces to

$$\hat{\mathcal{M}}_{\rm WR} \propto -iF_M\boldsymbol{\sigma} \cdot (\hat{\boldsymbol{P}}' \times \boldsymbol{\Phi}^{*}_{\gamma}) + G\boldsymbol{\sigma} \cdot \boldsymbol{\Phi}^{*}_{\gamma}, \tag{B5}$$

where the relevant form factors are defined to be $F_M = (M - M')[F_2 - F_1/(M + M')]$ and $G = F_3$ [43], $\hat{\boldsymbol{P}}'$ is the direction of the momentum of the decay baryon B', and $\boldsymbol{\Phi}_{\gamma}$ is the vector amplitude of the photon in the Coulomb gauge.

The angular distribution of B' can be calculated using Eq. (B2) by the substitution $\hat{\mathcal{M}}_{\rm EM} \to \hat{\mathcal{M}}_{\rm WR}$, and by summing over the polarization states of B'. We obtain

$$\frac{dN^{\rm WR}_{Y\to B'+\gamma}}{d\Omega_{\mathbf{P}'}} = \frac{1}{4\pi}(1 + \alpha_{Y,\gamma}\mathbf{S}_B \cdot \hat{\boldsymbol{P}}'), \tag{B6}$$

where the parameter $\alpha_{Y,\gamma} = 2{\rm Re}(F^{*}_{M}G)/[|F_M|^2 + |G|^2]$ depends on the decaying hyperon and can be found in Ref. [43].

Substituting $\hat{\mathcal{M}}_{\rm EM}$ in Eq. (B3) with $\hat{\mathcal{M}}_{\rm WR}$ in Eq. (B5), one can obtain the spin density matrix of B'. We obtain

$$\begin{aligned}\rho(B') &= \frac{\sum_{\boldsymbol{\Phi}_{\gamma}} \hat{\mathcal{M}}_{\rm WR}\rho(Y)\hat{\mathcal{M}}^{\dagger}_{\rm WR}}{{\rm Tr}[\ldots]} \\ &= \frac{1}{2}\left[1 - \boldsymbol{\sigma} \cdot \hat{\boldsymbol{P}}'\frac{\alpha_{Y,\gamma} + \mathbf{S}_Y \cdot \hat{\boldsymbol{P}}'}{1 + \alpha_{Y,\gamma}\mathbf{S}_Y \cdot \hat{\boldsymbol{P}}'}\right].\end{aligned} \tag{B7}$$

The spin density matrix of B' depends thus only on the combination of form factors given by the measured parameter $\alpha_{Y,\gamma}$. Moreover, B' is polarized even if Y is unpolarized, due to parity nonconservation. For $\mathbf{S}_Y = \mathbf{0}$, it is $\mathbf{S}_{B'} = -\alpha_{Y,\gamma}\hat{\boldsymbol{P}}'$.

The acceptance matrix associated with the radiative decay of Y can be obtained as in Eq. (B4) using $\hat{\mathcal{M}}_{\mathrm{WR}}$ in Eq. (B5) instead of $\hat{\mathcal{M}}_{\mathrm{EM}}$. We obtain

$$\check{\rho}(Y) = \frac{1}{2}\left[1 + \boldsymbol{\sigma}\cdot\hat{\boldsymbol{P}}'\,\frac{\alpha_{Y,\gamma} - \check{\mathbf{S}}_{B'}\cdot\hat{\boldsymbol{P}}'}{1 - \alpha_{Y,\gamma}\check{\mathbf{S}}_{B'}\cdot\hat{\boldsymbol{P}}'}\right].$$

If the decay of B' is not analyzed, the acceptance matrix of B is characterized by the polarization vector $\check{\mathbf{S}}_Y = \alpha_{Y,\gamma}\hat{\boldsymbol{P}}'$.

[1] J. C. Collins, Fragmentation of transversely polarized quarks probed in transverse momentum distributions, Nucl. Phys. **B396**, 161 (1993).

[2] A. Airapetian *et al.* (HERMES Collaboration), Azimuthal single- and double-spin asymmetries in semi-inclusive deep-inelastic lepton scattering by transversely polarized protons, J. High Energy Phys. 12 (2020) 010.

[3] C. Adolph *et al.* (COMPASS Collaboration), Collins and Sivers asymmetries in muonproduction of pions and kaons off transversely polarised protons, Phys. Lett. B **744**, 250 (2015).

[4] R. Seidl *et al.* (Belle Collaboration), Measurement of azimuthal asymmetries in inclusive production of hadron pairs in e^+e^- annihilation at $\sqrt{s} = 10.58$-GeV, Phys. Rev. D **78**, 032011 (2008); **86**, 039905(E) (2012).

[5] H. Li *et al.* (Belle Collaboration), Azimuthal asymmetries of back-to-back $\pi^{\pm} - (\pi^0, \eta, \pi^{\pm})$ pairs in e^+e^- annihilation, Phys. Rev. D **100**, 092008 (2019).

[6] J. P. Lees *et al.* (*BABAR* Collaboration), Measurement of Collins asymmetries in inclusive production of charged pion pairs in e^+e^- annihilation at *BABAR*, Phys. Rev. D **90**, 052003 (2014).

[7] J. P. Lees *et al.* (*BABAR* Collaboration), Collins asymmetries in inclusive charged KK and $K\pi$ pairs produced in e^+e^- annihilation, Phys. Rev. D **92**, 111101 (2015).

[8] M. Ablikim *et al.* (BESIII Collaboration), Measurement of azimuthal asymmetries in inclusive charged dipion production in e^+e^- annihilations at $\sqrt{s} = 3.65$ GeV, Phys. Rev. Lett. **116**, 042001 (2016).

[9] J. C. Collins, S. F. Heppelmann, and G. A. Ladinsky, Measuring transversity densities in singly polarized hadron hadron and lepton—hadron collisions, Nucl. Phys. **B420**, 565 (1994).

[10] A. Bianconi, S. Boffi, R. Jakob, and M. Radici, Two hadron interference fragmentation functions. Part I. General framework, Phys. Rev. D **62**, 034008 (2000).

[11] A. Airapetian *et al.* (HERMES Collaboration), Effects of transversity in deep-inelastic scattering by polarized protons, Phys. Lett. B **693**, 11 (2010).

[12] C. Adolph *et al.* (COMPASS Collaboration), A high-statistics measurement of transverse spin effects in dihadron production from muon–proton semi-inclusive deep-inelastic scattering, Phys. Lett. B **736**, 124 (2014).

[13] A. Vossen *et al.* (Belle Collaboration), Observation of transverse polarization asymmetries of charged pion pairs in e^+e^- annihilation near $\sqrt{s} = 10.58$ GeV, Phys. Rev. Lett. **107**, 072004 (2011).

[14] M. Anselmino, A. Mukherjee, and A. Vossen, Transverse spin effects in hard semi-inclusive collisions, Prog. Part. Nucl. Phys. **114**, 103806 (2020).

[15] A. D. Panagiotou, Λ^0 polarization in hadron—nucleon, hadron—nucleus and nucleus-nucleus interactions, Int. J. Mod. Phys. A **05**, 1197 (1990).

[16] A. Airapetian *et al.* (HERMES Collaboration), Transverse polarization of Λ hyperons from quasireal photoproduction on nuclei, Phys. Rev. D **90**, 072007 (2014).

[17] Y. Guan *et al.* (Belle Collaboration), Observation of transverse $\Lambda/\bar{\Lambda}$ hyperon polarization in e^+e^- annihilation at Belle, Phys. Rev. Lett. **122**, 042001 (2019).

[18] J. Felix, On Theoretical studies of Λ^0 polarization, Mod. Phys. Lett. A **14**, 827 (1999).

[19] B. Andersson, G. Gustafson, and G. Ingelman, A semi-classical model for the polarization of inclusively produced Λ^0 particles at high-energies, Phys. Lett. **85B**, 417 (1979).

[20] B. Andersson, G. Gustafson, G. Ingelman, and T. Sjostrand, Parton fragmentation and string dynamics, Phys. Rep. **97**, 31 (1983).

[21] P. J. Mulders and R. D. Tangerman, The complete tree level result up to order 1/Q for polarized deep inelastic leptoproduction, Nucl. Phys. **B461**, 197 (1996); **B484**, 538(E) (1997).

[22] M. Anselmino, D. Boer, U. D'Alesio, and F. Murgia, Λ polarization from unpolarized quark fragmentation, Phys. Rev. D **63**, 054029 (2001).

[23] U. D'Alesio, F. Murgia, and M. Zaccheddu, First extraction of the Λ polarizing fragmentation function from Belle e^+e^- data, Phys. Rev. D **102**, 054001 (2020).

[24] D. Callos, Z.-B. Kang, and J. Terry, Extracting the transverse momentum dependent polarizing fragmentation functions, Phys. Rev. D **102**, 096007 (2020).

[25] U. D'Alesio, L. Gamberg, F. Murgia, and M. Zaccheddu, Transverse Λ polarization in e^+e^- processes within a TMD factorization approach and the polarizing fragmentation function, J. High Energy Phys. 12 (2022) 074.

[26] A. Kerbizi, X. Artru, Z. Belghobsi, F. Bradamante, and A. Martin, Recursive model for the fragmentation of polarized quarks, Phys. Rev. D **97**, 074010 (2018).

[27] A. Kerbizi, X. Artru, Z. Belghobsi, and A. Martin, Simplified recursive $^3P_0$ model for the fragmentation of polarized quarks, Phys. Rev. D **100**, 014003 (2019).

[28] A. Kerbizi, X. Artru, and A. Martin, Production of vector mesons in the String + $^3P_0$ model of polarized quark fragmentation, Phys. Rev. D **104**, 114038 (2021).

[29] D. Amati, A. Stanghellini, and S. Fubini, Theory of high-energy scattering and multiple production, Nuovo Cimento **26**, 896 (1962).

[30] C. Bierlich *et al.*, A comprehensive guide to the physics and usage of Pythia 8.3, SciPost Phys. Codebases 8 (2022).

[31] A. Kerbizi and L. Lönnblad, StringSpinner—adding spin to the Pythia string fragmentation, Comput. Phys. Commun. **272**, 108234 (2022).

[32] A. Kerbizi and L. Lönnblad, Extending StringSpinner to handle vector-meson spin, Comput. Phys. Commun. **292**, 108886 (2023).

[33] A. Kerbizi and X. Artru, String fragmentation of a quark pair with entangled spin states: Application to e^+e^- annihilation, Phys. Rev. D **109**, 054029 (2024).

[34] A. Kerbizi, L. Lönnblad, and A. Martin, Quark spin effects in e^+e^- annihilation: A Monte Carlo event generator study, Phys. Rev. D **110**, 074029 (2024).

[35] B. Andersson, G. Gustafson, and T. Sjostrand, A model for baryon production in quark and gluon jets, Nucl. Phys. **B197**, 45 (1982).

[36] B. Andersson, G. Gustafson, and B. Soderberg, A general model for jet fragmentation, Z. Phys. C **20**, 317 (1983).

[37] X. Artru, Correspondences between the symmetric Lund model and the dual resonance model, Z. Phys. C **26**, 83 (1984).

[38] A. Casher, H. Neuberger, and S. Nussinov, Chromoelectric flux tube model of particle production, Phys. Rev. D **20**, 179 (1979).

[39] A. Bacchetta, F. Conti, and M. Radici, Transverse-momentum distributions in a diquark spectator model, Phys. Rev. D **78**, 074010 (2008).

[40] R. Jakob, P. J. Mulders, and J. Rodrigues, Modeling quark distribution and fragmentation functions, Nucl. Phys. **A626**, 937 (1997).

[41] X. Artru, M. Elchikh, J.-M. Richard, J. Soffer, and O. V. Teryaev, Spin observables and spin structure functions: Inequalities and dynamics, Phys. Rep. **470**, 1 (2009).

[42] M. G. Alexeev *et al.* (COMPASS Collaboration), Probing transversity by measuring Λ polarisation in SIDIS, Phys. Lett. B **824**, 136834 (2022).

[43] R. L. Workman *et al.* (Particle Data Group), Review of particle physics, Prog. Theor. Exp. Phys. **2022**, 083C01 (2022).

[44] J. C. Collins, Spin correlations in Monte Carlo Event generators, Nucl. Phys. **B304**, 794 (1988).

[45] I. G. Knowles, Spin correlations in parton—parton scattering, Nucl. Phys. **B310**, 571 (1988).

[46] B. Assi, C. Bierlich, P. Ilten, T. Menzo, S. Mrenna, M. Szewc, M. K. Wilkinson, A. Youssef, and J. Zupan (MLhad Collaboration), Post-hoc reweighting of hadron production in the Lund string model, SciPost Phys. **19**, 104 (2025).

[47] B. Andersson, G. Gustafson, and T. Sjostrand, Baryon production in jet fragmentation and Υ decay, Phys. Scr. **32**, 574 (1985).