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BLOCK TRIANGULAR PRECONDITIONERS FOR DOUBLE SADDLE POINT LINEAR SYSTEMS ARISING IN THE MIXED FORM OF POROELASTICITY EQUATIONS

LUCA BERGAMASCHI*, MASSIMILIANO FERRONATO* AND ÁNGELES MARTÍNEZ†

Abstract. In this paper, we study a class of inexact block triangular preconditioners for double saddle-point symmetric linear systems arising from the mixed finite element and mixed hybrid finite element discretization of Biot’s poroelasticity equations. We extend the theoretical results presented in [2] for general double saddle-point problems with nonzero diagonal blocks, by developing a novel spectral analysis of the preconditioned matrix. This shows that the complex eigenvalues lie in a circle of center $(1, 0)$ and radius smaller than 1. In contrast, the real eigenvalues are described in terms of the roots of a third-degree polynomial with real coefficients. Numerical results are reported to show the quality of the theoretical bounds, which generally turn out to be quite tight, and illustrate the efficiency of the proposed preconditioners in the acceleration of GMRES, especially in comparison with similar block diagonal preconditioning strategies along with the MINRES iteration.

AMS classification: 65F10, 65F50, 56F08.

Keywords: Double saddle point problems. Preconditioning. Krylov subspace methods. Biot’s poroelasticity equations.

1. Introduction. Given positive integer dimensions n, m and l , we define the $(n+m+l) \times (n+m+l)$ double saddle-point linear system [5]

$$(1.1) \quad \mathcal{A}w \equiv \begin{bmatrix} A & B^T & 0 \\ B & -D & C^T \\ 0 & C & E \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} f \\ g \\ h \end{bmatrix} \equiv b,$$

where $A \in \mathbb{R}^{n \times n}$ and $E \in \mathbb{R}^{l \times l}$ are symmetric positive definite (SPD), $B \in \mathbb{R}^{m \times n}$ and $C \in \mathbb{R}^{l \times m}$ have full rank, and $D \in \mathbb{R}^{m \times m}$ is symmetric positive semi-definite (SPSD). Although the assumptions on B and C are not necessary to guarantee the regularity of $\mathcal{A}$, they are useful for our spectral analysis and are always satisfied in the applications under consideration. A detailed analysis of the necessary and sufficient conditions for the regularity of $\mathcal{A}$ can be found in [4]. Finally, $f \in \mathbb{R}^n$, $g \in \mathbb{R}^m$ and $h \in \mathbb{R}^l$ are given vectors. Such linear systems arise in several scientific applications, including constrained least squares problems [62], constrained quadratic programming [35], magma-mantle dynamics [55], to mention a few; see, e.g. [16, 19]. Similar block structures arise in liquid crystal director modeling or the coupled Stokes-Darcy problem, and the preconditioning of such linear systems has been considered in [3, 5, 6, 18, 20]. Symmetric positive definite block preconditioners for (1.1) have been deeply studied in [10, 13].

The double saddle-point structure (1.1) also arises in the mixed form of Biot’s poroelasticity equations. Biot’s model, dating back to [11] for its original derivation, couples Darcy’s flow of a single-phase fluid with the quasi-static mechanical deformation of an elastic, fully saturated porous medium. In its mixed form, Darcy’s law is explicitly considered as a governing equation along with the fluid mass balance, giving rise to the so-called three-field formulation [24, 36, 38, 42, 47, 51, 52], where the solid phase displacements, the pore fluid pressure, and Darcy’s velocity are the independent unknown quantities. Biot’s poroelasticity model is key in several relevant applications, ranging from geosciences, such as groundwater hydrology, geothermal energy extraction, and geological carbon and hydrogen storage problems [28, 48, 50, 54, 58, 61], to biomedicine [21, 33, 45]. The use of a three-field mixed formulation allows for obtaining locally mass-conservative velocity fields [14]. It proves to be particularly robust to strong jumps in the distribution of the material parameters, especially in the permeability tensor, which makes it particularly interesting in many applications. For this reason, the three-field formulation has attracted increasing attention in recent years.

One of the most relevant computational issues posed by the discrete three-field Biot model is the solution to a sequence of double saddle-point linear systems in the form (1.1). In this problem, the use of common triplets of discretization spaces produces $n > \max\{m, l\}$ and often, but not exclusively, $m < l$. Moreover, the diagonal block D is usually singular or numerically close to being so, E is an SPD mass

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matrix, and B and C have full rank. Most of the applications of the three-field formulation are focused on iterative solution schemes with sequential-implicit approaches, e.g. [12, 22, 34, 37], where the poromechanical equilibrium and Darcy's flow subproblems are solved independently, iterating until convergence. Although attractive for their simplicity and ease of implementation, sequential-implicit approaches have a two-fold limitation: (i) suitable splitting strategies, such as the so-called fixed-stress algorithm [43, 44], should be employed to ensure unconditional convergence, and (ii) convergence of these methods is always linear. In recent years, an increasing interest has been shown in the development of specific classes of preconditioners to allow for the efficient monolithic solution of the overall 3×3 block double saddle-point system by Krylov subspace methods. Spectrally equivalent block diagonal preconditioners were first advanced [1, 46, 59], then a family of block triangular and block strategies, combining physically based with fully algebraic Schur complement approximations, was proposed [15, 17, 26, 30, 31]. A review of the block preconditioning methods proposed for the fully implicit monolithic solution of the three-field formulation of Biot's model is available in [25].

This work analyzes the spectral properties of a class of block triangular and block diagonal preconditioners for the solution of the general problem (1.1). In particular, we derive new bounds for the eigenvalues of the preconditioned matrix by extending the results provided in [2, 9, 13], where the simpler case with $D \equiv 0$ and $E \equiv 0$ was addressed. The theoretical results obtained for the general problem (1.1) are validated on the class of block triangular preconditioners developed in [27, 32] for the solution of the discrete stabilized mixed and mixed hybrid finite element formulation of Biot's poroelasticity equations. These block triangular preconditioners are based on both algebraic and physics-based approximations of the Schur complements and generally prove quite efficient in various applications. The theoretical bounds are validated on some test cases, while the overall performance of the inexact block triangular preconditioners in the GMRES acceleration [57] is also compared to that of the related block diagonal variants, which can exploit the cheaper MINRES iteration [49]. Finally, the efficiency of the proposed inexact preconditioners is tested in a large-scale 3D realistic problem.

2. Eigenvalue analysis of the inexact triangular preconditioner. Let us consider the prototype double saddle-point linear system (1.1). We define the first- and second-level Schur complements

$$S = D + BA^{-1}B^T, \quad X = E + CS^{-1}C^T,$$

and denote as $\hat{A}, \hat{S}$ and $\hat{X}$ some related symmetric positive definite approximations of A, S , and X , respectively. In this section, we aim to analyze the eigenvalue distribution of the preconditioned matrix $\mathcal{AP}^{-1}$, where

$$(2.1) \quad \mathcal{P} = \begin{bmatrix} \hat{A} & B^T & 0 \\ 0 & -\hat{S} & C^T \\ 0 & 0 & \hat{X} \end{bmatrix}.$$

The spectral properties of $\mathcal{AP}^{-1}$ will be expressed as a function of the eigenvalues of $\hat{A}^{-1}A, \hat{S}^{-1}\tilde{S}$ and $\hat{X}^{-1}\tilde{X}$, where

$$\tilde{S} = D + B\hat{A}^{-1}B^T, \quad \tilde{X} = E + C\hat{S}^{-1}C^T.$$

Of course, the closer are $\hat{A}$ and $\hat{S}$ to A and S , respectively, the closer are $\tilde{S}$ and $\tilde{X}$ to S and X .

Let

$$(2.2) \quad \mathcal{P}_D = \begin{bmatrix} \hat{A} & 0 & 0 \\ 0 & \hat{S} & 0 \\ 0 & 0 & \hat{X} \end{bmatrix}.$$

Then, finding the eigenvalues of $\mathcal{AP}^{-1}$ is equivalent to solving

$$\mathcal{P}_D^{-\frac{1}{2}} \mathcal{AP}_D^{-\frac{1}{2}} w = \lambda \mathcal{P}_D^{-\frac{1}{2}} \mathcal{P} \mathcal{P}_D^{-\frac{1}{2}} w,$$

or

$$(2.3) \quad \begin{bmatrix} \overline{A} & R^T & 0 \\ R & -\overline{D} & K^T \\ 0 & K & \overline{E} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \lambda \begin{bmatrix} I & R^T & 0 \\ 0 & -I & K^T \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

where $\bar{A} = \hat{A}^{-\frac{1}{2}} A \hat{A}^{-\frac{1}{2}}$, $\bar{D} = \hat{S}^{-\frac{1}{2}} D \hat{S}^{-\frac{1}{2}}$, $\bar{E} = \hat{X}^{-\frac{1}{2}} E \hat{X}^{-\frac{1}{2}}$, $R = \hat{S}^{-\frac{1}{2}} B \hat{A}^{-\frac{1}{2}}$ and $K = \hat{X}^{-\frac{1}{2}} C \hat{S}^{-\frac{1}{2}}$. Notice that

$$(2.4) \quad RR^T = \hat{S}^{-\frac{1}{2}} (\tilde{S} - D) \hat{S}^{-\frac{1}{2}} = \underbrace{\hat{S}^{-\frac{1}{2}} \tilde{S} \hat{S}^{-\frac{1}{2}}}_{\bar{S}} - \bar{D} = \bar{S} - \bar{D},$$

$$(2.5) \quad KK^T = \hat{X}^{-\frac{1}{2}} (\tilde{X} - E) \hat{X}^{-\frac{1}{2}} = \underbrace{\hat{X}^{-\frac{1}{2}} \tilde{X} \hat{X}^{-\frac{1}{2}}}_{\bar{X}} - \bar{E} = \bar{X} - \bar{E}.$$

We define the Rayleigh quotient for a given square matrix H and a nonzero vector w as

$$q(H, w) = \frac{w^* H w}{w^* w}.$$

The following indicators will be used:

$$\begin{aligned} 0 < \gamma_{\min}^A &\equiv \lambda_{\min}(\hat{A}^{-1} A), & \gamma_{\max}^A &\equiv \lambda_{\max}(\hat{A}^{-1} A), & \gamma_A(w) &= q(\bar{A}, w) \in [\gamma_{\min}^A, \gamma_{\max}^A] \equiv I_A, \\ 0 < \gamma_{\min}^S &\equiv \lambda_{\min}(\hat{S}^{-1} \tilde{S}), & \gamma_{\max}^S &\equiv \lambda_{\max}(\hat{S}^{-1} \tilde{S}), & \gamma_S(w) &= q(\bar{S}, w) \in [\gamma_{\min}^S, \gamma_{\max}^S] \equiv I_S, \\ 0 < \gamma_{\min}^X &\equiv \lambda_{\min}(\hat{X}^{-1} \tilde{X}), & \gamma_{\max}^X &\equiv \lambda_{\max}(\hat{X}^{-1} \tilde{X}), & \gamma_X(w) &= q(\bar{X}, w) \in [\gamma_{\min}^X, \gamma_{\max}^X] \equiv I_X, \\ 0 \leq \gamma_{\min}^D &\equiv \lambda_{\min}(\hat{S}^{-1} D), & \gamma_{\max}^D &\equiv \lambda_{\max}(\hat{S}^{-1} D), & \gamma_D(w) &= q(\bar{D}, w) \in [\gamma_{\min}^D, \gamma_{\max}^D] \equiv I_D, \\ 0 < \gamma_{\min}^E &\equiv \lambda_{\min}(\hat{X}^{-1} E), & \gamma_{\max}^E &\equiv \lambda_{\max}(\hat{X}^{-1} E), & \gamma_E(w) &= q(\bar{E}, w) \in [\gamma_{\min}^E, \gamma_{\max}^E] \equiv I_E, \\ 0 < \gamma_{\min}^R &\equiv \lambda_{\min}(RR^T), & \gamma_{\max}^R &\equiv \lambda_{\max}(RR^T), & \gamma_R(w) &= q(RR^T, w) \in [\gamma_{\min}^R, \gamma_{\max}^R] \equiv I_R, \\ 0 \leq \gamma_{\min}^K &\equiv \lambda_{\min}(KK^T), & \gamma_{\max}^K &\equiv \lambda_{\max}(KK^T), & \gamma_K(w) &= q(KK^T, w) \in [\gamma_{\min}^K, \gamma_{\max}^K] \equiv I_K. \end{aligned}$$

Notice that zero can be a lower bound only for $\hat{S}^{-1} D$, because D can be singular, and KK^T , which is rank-deficient if $m < l$. From (2.4)-(2.5) and the definitions above, it follows immediately

$$\gamma_S(w) = \gamma_R(w) + \gamma_D(w), \quad \gamma_X(w) = \gamma_K(w) + \gamma_E(w).$$

In the sequel, we will remove the argument w whenever an indicator γ_* will be used to simplify the notation. Finally, we assume that $1 \in [\gamma_{\min}^A, \gamma_{\max}^A]$. This assumption, which is very commonly satisfied in practice, will simplify some of the bounds that follow.

Exploiting (2.3) yields:

$$(2.6) \quad \bar{A}x - \lambda x = (\lambda - 1)R^T y,$$

$$(2.7) \quad Rx - \bar{D}y - (\lambda - 1)K^T z = -\lambda y,$$

$$(2.8) \quad Ky + \bar{E}z = \lambda z.$$

Before characterizing both truly complex and real eigenvalues, we briefly analyze some particular cases where it is easy to show that $\lambda \in \mathbb{R}$. Since $\hat{A}, \hat{S}$ and $\hat{X}$ are SPD, and B has full row rank, then also $\bar{A}$ is SPD and R has full row rank. Independently of the values of $m, l < n$, we can have:

$$\boxed{x \neq 0 \wedge y = 0} \quad \lambda \in [\gamma_{\min}^A, \gamma_{\max}^A] \cap [\gamma_{\min}^E, \gamma_{\max}^E] \in \mathbb{R} \text{ since it is either } \lambda = \gamma_A(x) \text{ or } \lambda = \gamma_E(z), \text{ if } z \neq 0;$$

If $m < n$ (strictly), we can also have

$$\boxed{x \neq 0 \wedge Rx = 0} \quad \lambda \in [\gamma_{\min}^A, \gamma_{\max}^A] \in \mathbb{R} \text{ because premultiplying (2.6) by } x^T \text{ gives } \lambda = \gamma_A(x).$$

To handle the remaining particular cases, we distinguish between the two situations ($m \geq l$ and $m < l$). Recall that C has rank equal to $\min\{m, l\}$ and if $m \geq l$ ($m < l$), then $K(K^T)$ is full row rank.

Case $m \geq l$.

$$\boxed{x = 0} \quad \lambda = 1 \in \mathbb{R}, \text{ because it must be } y \neq 0, \text{ otherwise this also would imply } z = 0;$$

$$\boxed{x \neq 0, y \neq 0, Ky = 0, z \neq 0} \quad \lambda \in [\gamma_{\min}^E, \gamma_{\max}^E] \in \mathbb{R} \text{ since } \lambda = \gamma_E(z).$$

Case $m < l$.

$\boxed{x=0}$ $\lambda \in \{1\} \cup [\gamma_{\min}^E, \gamma_{\max}^E] \in \mathbb{R}$, because it is either $\lambda = 1$ from (2.6), or $K^T z = 0$ with $z \neq 0$, which implies $\lambda = \gamma_E(z)$;
 $\boxed{x \neq 0, K^T z = 0}$ $\lambda \in [\gamma_{\min}^E, \gamma_{\max}^E] \in \mathbb{R}$ since it must be $z \neq 0$ and $\lambda = \gamma_E(z)$.

In all other situations, λ is generally complex. In such cases, the real and imaginary parts satisfy the bounds that follow.

2.1. Complex eigenvalues. We first characterize the eigenvalues with a nonzero imaginary part.

THEOREM 1. *Let $\lambda = a + ib$ be a complex eigenvalue of $\mathcal{AP}^{-1}$, with a the real part, b the imaginary part, i the imaginary unit, and $(x; y; z)$ the corresponding eigenvector. Then, if $0 \leq \gamma_{\min}^D < 1$ any complex eigenvalue satisfies*

$$(2.9) \quad |\lambda - 1| \leq \sqrt{1 - \rho_{\min}}, \quad \text{where} \quad \rho_{\min} = \frac{\gamma_{\min}^A \|x\|^2 + \gamma_{\min}^D \|y\|^2 + \gamma_{\min}^E \|z\|^2}{\|y\|^2 + \gamma_{\min}^E \|z\|^2},$$

otherwise, all eigenvalues are real.

Proof. With no loss of generality, we assume that $(x; y; z)$ is such that $\|x\|^2 + \|y\|^2 + \|z\|^2 = 1$. By multiplying (2.6) by x^* on the left, the transposed conjugate of (2.7) by y on the right, and (2.8) by z^* on the left, we have:

$$(2.10) \quad x^* \bar{A} x - \lambda \|x\|^2 = (\lambda - 1) x^* R^T y,$$

$$(2.11) \quad x^* R^T y - y^* \bar{D} y - (\bar{\lambda} - 1) z^* K y = -\bar{\lambda} \|y\|^2,$$

$$(2.12) \quad z^* K y + z^* \bar{E} z = \lambda \|z\|^2.$$

Eliminating $x^* R^T y$ from (2.11) and (2.10) yields

$$x^* \bar{A} x - \lambda \|x\|^2 = (\lambda - 1) (y^* \bar{D} y + (\bar{\lambda} - 1) z^* K y - \bar{\lambda} \|y\|^2),$$

which, recalling from (2.12) that $z^* K y = -z^* \bar{E} z + \lambda \|z\|^2$, after some algebra provides

$$(2.13) \quad x^* \bar{A} x - \lambda \|x\|^2 = (y^* \bar{D} y - \bar{\lambda} \|y\|^2)(\lambda - 1) + |\lambda - 1|^2 (\lambda \|z\|^2 - z^* E z),$$

i.e.,

$$(2.14) \quad (\gamma_A - \lambda) \|x\|^2 = (\gamma_D - \bar{\lambda})(\lambda - 1) \|y\|^2 + |\lambda - 1|^2 (\lambda - \gamma_E) \|z\|^2.$$

Let us distinguish between the real and imaginary parts of equation (2.14):

$$(2.15) \quad (\gamma_A - a) \|x\|^2 = (\gamma_D a - \gamma_D - |\lambda|^2 + a) \|y\|^2 + |\lambda - 1|^2 (a - \gamma_E) \|z\|^2, \quad (\text{real part}),$$

$$(2.16) \quad -b \|x\|^2 = b \gamma_D \|y\|^2 - b \|y\|^2 + |\lambda - 1|^2 b \|z\|^2, \quad (\text{imaginary part}).$$

If λ is complex, then $b \neq 0$ and from (2.16) we obtain

$$(2.17) \quad |\lambda - 1|^2 = \frac{\|y\|^2 (1 - \gamma_D) - \|x\|^2}{\|z\|^2},$$

Equation (2.17) requires $0 \leq \gamma_{\min}^D < 1$, otherwise no complex eigenvalue occurs. Defining the quantity

$$(2.18) \quad \rho = 1 - |\lambda - 1|^2 = 2a - |\lambda|^2,$$

and substituting (2.17) in (2.15) we have

$$\begin{aligned} 0 &= (\gamma_A - a) \|x\|^2 - (\gamma_D a - \gamma_D - |\lambda|^2 + a) \|y\|^2 - |\lambda - 1|^2 (a - \gamma_E) \|z\|^2 \\ &= (\gamma_A - a) \|x\|^2 - (\gamma_D a - \gamma_D - |\lambda|^2 + a) \|y\|^2 - (a - \gamma_E) (\|y\|^2 (1 - \gamma_D) - \|x\|^2) \\ &= (\gamma_A - \gamma_E) \|x\|^2 - (\gamma_D a - \gamma_D - |\lambda|^2 + a + a - \gamma_E - a \gamma_D + \gamma_E \gamma_D) \|y\|^2 \\ &= (\gamma_A - \gamma_E) \|x\|^2 + (\gamma_D + \gamma_E - \gamma_E \gamma_D - \rho) \|y\|^2, \end{aligned}$$

from which

$$\begin{aligned}\rho &= \gamma_D + \gamma_E - \gamma_E \gamma_D + (\gamma_A - \gamma_E) \frac{\|x\|^2}{\|y\|^2} \\ &= \frac{\gamma_D \|y\|^2 + \gamma_E \|y\|^2 - \gamma_E \gamma_D \|y\|^2 + (\gamma_A - \gamma_E) \|x\|^2}{\|y\|^2} \\ &= \frac{\gamma_E (\|y\|^2 (1 - \gamma_D) - \|x\|^2) + \gamma_A \|x\|^2 + \gamma_D \|y\|^2}{\|y\|^2}.\end{aligned}$$

Then, by using again (2.17), we get

$$(2.19) \quad \rho = \frac{\gamma_E \|z\|^2 |\lambda - 1|^2 + \gamma_A \|x\|^2 + \gamma_D \|y\|^2}{\|y\|^2},$$

which implies $\rho > 0$. Moreover, by using the definition of $\rho = 1 - |\lambda - 1|^2$ we have

$$(2.20) \quad \rho = \frac{\gamma_A \|x\|^2 + \gamma_D \|y\|^2 + \gamma_E \|z\|^2}{\|y\|^2 + \gamma_E \|z\|^2} \geq \frac{\gamma_{\min}^A \|x\|^2 + \gamma_{\min}^D \|y\|^2 + \gamma_E \|z\|^2}{\|y\|^2 + \gamma_E \|z\|^2}.$$

We now show that the right-hand side of (2.20) is an increasing function in the variable γ_E . We have:

$$\begin{aligned}h(t) &\equiv \frac{\gamma_{\min}^A \|x\|^2 + \gamma_{\min}^D \|y\|^2 + t \|z\|^2}{\|y\|^2 + t \|z\|^2} \\ &= 1 + \frac{\overbrace{\|x\|^2 + (\gamma_{\min}^D - 1) \|y\|^2 + (\gamma_{\min}^A - 1) \|x\|^2}^q}{\|y\|^2 + t \|z\|^2} \equiv 1 + \frac{q}{\|y\|^2 + t \|z\|^2}.\end{aligned}$$

Recalling equation (2.17) and $\gamma_A^{\min} < 1$, we can conclude that $q < 0$, hence $h(t)$ is an increasing function for $t \in [\gamma_{\min}^E, \gamma_{\max}^E]$. By setting:

$$\rho_{\min} = \min_{t \in I_E} h(t) \equiv \frac{\gamma_{\min}^A \|x\|^2 + \gamma_{\min}^D \|y\|^2 + \gamma_{\min}^E \|z\|^2}{\|y\|^2 + \gamma_{\min}^E \|z\|^2} < 1,$$

from equation (2.20) we have $1 - |\lambda - 1|^2 \geq \rho_{\min}$, which finally establishes the bound (2.9). $\square$

REMARK 1. In essence, Theorem 1 states that the complex eigenvalues, if any, of the preconditioned matrix $\mathcal{AP}^{-1}$ lie in a circle centered in 1 and with radius $\sqrt{1 - \rho_{\min}}$, being $0 < \rho_{\min} < 1$. This limits also the real part a of any complex λ , such that $a \in [1 - \sqrt{1 - \rho_{\min}}, 1 + \sqrt{1 - \rho_{\min}}]$. Moreover, by using equation (2.18), it can also be written as

$$a = \frac{|\lambda|^2 - |\lambda - 1|^2 + 1}{2} = \frac{|\lambda|^2 + \rho}{2} \geq \frac{\rho}{2} \geq \frac{\rho_{\min}}{2},$$

which implies that the real part of any complex eigenvalue cannot be arbitrarily small.

REMARK 2. A more practical bound on the radius of the circle containing complex eigenvalues can be obtained if we assume $\gamma_{\min}^D > 0$. Considering the function $h(t)$, we have

$$\rho_{\min} = \frac{\gamma_{\min}^A \|x\|^2 + \gamma_{\min}^D \|y\|^2 + \gamma_{\min}^E \|z\|^2}{\|y\|^2 + \gamma_{\min}^E \|z\|^2} \equiv h(\gamma_{\min}^E) \geq h(0) = \frac{\gamma_{\min}^A \|x\|^2 + \gamma_{\min}^D \|y\|^2}{\|y\|^2} \geq \gamma_{\min}^D.$$

2.2. Real eigenvalues. In the following, we aim to characterize the real eigenvalues of the preconditioned matrix not contained in $[\gamma_{\min}^A, \gamma_{\max}^A]$ or $[\gamma_{\min}^E, \gamma_{\max}^E]$. To this end, we first introduce three technical lemmas that will be useful in the analysis that follows.

LEMMA 1. [7, Lemma 1] Let $\lambda \notin [\gamma_{\min}^A, \gamma_{\max}^A]$. Then, for an arbitrary vector $s \in \mathbb{R}^n$, $s \neq 0$, there exists a vector $w \in \mathbb{R}^n$, $w \neq 0$, such that

$$\frac{s^T (\bar{A} - \lambda I)^{-1} s}{s^T s} = \left(\frac{w^T \bar{A} w}{w^T w} - \lambda \right)^{-1} = (\gamma_A - \lambda)^{-1}.$$

LEMMA 2. Let $p(x)$ be the polynomial

$$p(x) = x^3 - a_2x^2 + a_1x - a_0, \quad a_0, a_1, a_2 > 0,$$

and $\alpha = \min \left\{ a_2, \frac{a_0}{a_1} \right\}$, $\beta = \max \left\{ a_2, \frac{a_0}{a_1} \right\}$. Then, $p(x) < 0 \forall x \in (0, \alpha)$ and $p(x) > 0 \forall x > \beta$.

Proof. The statement of the lemma comes from observing that $p(x)$ is the sum of the term $x^3 - a_2x^2$, which is negative in $(0, a_2)$ and positive for $x > a_2$, and of the term $a_1x - a_0$, which monotonically increases with x and changes sign for $x = \frac{a_0}{a_1}$. $\square$

LEMMA 3. Let $\zeta \in \mathbb{R}$ be either $0 < \zeta < \min \left\{ \gamma_{\min}^A, \frac{\gamma_{\min}^R}{\gamma_{\max}^A + \gamma_{\min}^R + \gamma_{\max}^D} \right\}$ or $\zeta \geq \gamma_{\max}^A + \gamma_{\max}^S$. Then, all the eigenvalues of the symmetric matrix $Z \in \mathbb{R}^{m \times m}$

$$(2.21) \quad Z(\zeta) = (1 - \zeta)R(\zeta I - \bar{A})^{-1}R^T - \bar{D} + \zeta I,$$

are either positive or negative.

Proof. Let $w \in \mathbb{R}^m$ be an arbitrary nonzero vector. The Rayleigh quotient $q(w, \zeta)$ associated to the matrix $Z(\zeta)$ in (2.21) reads:

$$(2.22) \quad q(w, \zeta) = \frac{w^T Z w}{w^T w} = (1 - \zeta) \frac{w^T R(\zeta I - \bar{A})^{-1} R^T w}{w^T w} - \frac{w^T \bar{D} w}{w^T w} + \zeta.$$

By setting $z = R^T w$ and recalling that $\gamma_S = \gamma_R + \gamma_D$, we have:

$$(2.23) \quad \begin{aligned} q(w, \zeta) &= (1 - \zeta) \frac{z^T (\zeta I - \bar{A})^{-1} z}{z^T z} \frac{w^T R R^T w}{w^T w} - \frac{w^T \bar{D} w}{w^T w} + \zeta \\ &= (1 - \zeta) \frac{z^T (\zeta I - \bar{A})^{-1} z}{z^T z} (\gamma_S - \gamma_D) - \gamma_D + \zeta, \end{aligned}$$

which yields by applying Lemma 1:

$$(2.24) \quad q(w, \zeta) = \frac{1 - \zeta}{\zeta - \gamma_A} \gamma_S - \gamma_D \frac{1 - \gamma_A}{\zeta - \gamma_A} + \zeta$$

Now, $q(w, \zeta) = 0$ from equation (2.24) implies

$$(2.25) \quad h(\zeta) = \zeta^2 - (\gamma_A + \gamma_S)\zeta + \gamma_S + \gamma_D(\gamma_A - 1) = 0.$$

The roots h_- and h_+ of (2.25) are

$$(2.26) \quad h_- = \frac{\gamma_A + \gamma_S - \sqrt{(\gamma_A + \gamma_S)^2 - 4(\gamma_R + \gamma_D\gamma_A)}}{2} = \frac{2(\gamma_R + \gamma_D\gamma_A)}{\gamma_A + \gamma_S + \sqrt{(\gamma_A + \gamma_S)^2 - 4(\gamma_R + \gamma_D\gamma_A)}},$$

$$(2.27) \quad h_+ = \frac{\gamma_A + \gamma_S + \sqrt{(\gamma_A + \gamma_S)^2 - 4(\gamma_R + \gamma_D\gamma_A)}}{2}.$$

Since $\gamma_A + \gamma_S + \sqrt{(\gamma_A + \gamma_S)^2 - 4(\gamma_R + \gamma_D\gamma_A)} < 2(\gamma_A + \gamma_S)$, we have the following bounds for (2.26) and (2.27):

$$(2.28) \quad h_- \geq \frac{\gamma_R + \gamma_D\gamma_A}{\gamma_A + \gamma_S} \geq \frac{\gamma_R}{\gamma_A + \gamma_R + \gamma_D} \geq \frac{\gamma_{\min}^R}{\gamma_{\max}^A + \gamma_{\min}^R + \gamma_{\max}^D},$$

$$(2.29) \quad h_+ \leq \gamma_A + \gamma_S \leq \gamma_{\max}^A + \gamma_{\max}^S.$$

Hence, the Rayleigh quotient associated to $Z(\zeta)$ cannot be zero under the hypotheses on ζ . $\square$

We are now able to bound the real eigenvalues of the preconditioned matrix $\mathcal{AP}^{-1}$.

THEOREM 2. Let λ be a real eigenvalue of $\mathcal{AP}^{-1}$ with

$$(2.30) \quad \lambda \notin \left[\min \left\{ \gamma_{\min}^A, \frac{\gamma_{\min}^R}{\gamma_{\max}^A + \gamma_{\min}^R + \gamma_{\max}^D} \right\}, \gamma_{\max}^A + \gamma_{\max}^S \right].$$

Then, λ satisfies

$$(2.31) \quad \lambda^3 - (\gamma_A + \gamma_S + \gamma_X)\lambda^2 + (\gamma_A\gamma_X + \gamma_K + \gamma_E\gamma_S + \gamma_D\gamma_A + \gamma_R)\lambda - (\gamma_A\gamma_K + \gamma_E\gamma_A\gamma_D + \gamma_E\gamma_R) = 0,$$

and the following bound holds:

$$(2.32) \quad \min \left\{ \gamma_{\min}^E, \gamma_{\min}^A, \frac{\gamma_{\min}^R}{\gamma_{\max}^A + \gamma_{\min}^R + \gamma_{\max}^D} \right\} \leq \lambda \leq \gamma_{\max}^A + \gamma_{\max}^S + \gamma_{\max}^X.$$

Proof. The equation (2.6) can be written as

$$(2.33) \quad x = (1 - \lambda)(\lambda I - \bar{A})^{-1}R^T y.$$

Introducing x of (2.33) into (2.7), we have

$$(2.34) \quad Z(\lambda)y = (\lambda - 1)K^T z,$$

where $Z(\lambda) = (1 - \lambda)R(\lambda I - \bar{A})^{-1}R^T - \bar{D} + \lambda I$. The hypothesis (2.30) on λ allows us to use Lemma 3, which guarantees the definiteness of the matrix function $Z(\lambda)$. Hence, obtaining $y = (\lambda - 1)Z(\lambda)^{-1}K^T z$ from (2.34) and substituting it in (2.8) yields

$$(2.35) \quad [K(\lambda - 1)Z(\lambda)^{-1}K^T + \bar{E} - \lambda I] z = 0.$$

Premultiplying by z^T on the left and dividing by $z^T z$ yields

$$(2.36) \quad \frac{z^T K(\lambda - 1)Z(\lambda)^{-1}K^T z}{z^T z} + \gamma_E - \lambda = 0.$$

By setting $w = K^T z$, we have

$$(2.37) \quad (\lambda - 1) \frac{w^T Z(\lambda)^{-1} w}{w^T w} \frac{z^T K K^T z}{z^T z} + \gamma_E - \lambda = 0,$$

which becomes:

$$(2.38) \quad (\lambda - 1) \frac{u^T u}{u^T Z(\lambda) u} \frac{z^T K K^T z}{z^T z} + \gamma_E - \lambda = 0,$$

by setting $u = Z(\lambda)^{-1/2} w$. Using now (2.24) in Lemma 3, we get

$$(2.39) \quad \frac{\lambda - 1}{\frac{1 - \lambda}{\lambda - \gamma_A} \gamma_S - \gamma_D \frac{1 - \gamma_A}{\lambda - \gamma_A} + \lambda} \gamma_K + \gamma_E - \lambda = 0,$$

and after simple algebra, we are left with:

$$(2.40) \quad [\lambda^2 - (1 + \gamma_A)\lambda + \gamma_A] \gamma_K + (\gamma_E - \lambda) [\lambda^2 - (\gamma_A + \gamma_S)\lambda + \gamma_S + \gamma_D(\gamma_A - 1)] = 0,$$

which can be rewritten as:

$$(2.41) \quad q(\lambda) = s(\lambda)\gamma_K + (\gamma_E - \lambda)h(\lambda) = 0,$$

where $h(\lambda)$ is the same polynomial as in (2.25). We now observe that $q(0) = \gamma_A\gamma_K + \gamma_E(\gamma_R + \gamma_D\gamma_A) > 0$. Then, denoting by:

$$\lambda_-^{UB} = \frac{\gamma_{\min}^R}{\gamma_{\max}^A + \gamma_{\min}^R + \gamma_{\max}^D} < 1,$$

we notice that $s(\lambda)$, $h(\lambda)$ and $(\gamma_E - \lambda)$ are all positive for any $\lambda \in [0, \min\{\gamma_{\min}^E, \gamma_{\min}^A, \lambda_-^{UB}\}]$, thus proving the left inequality in (2.32).

Equation (2.41) can be more explicitly rewritten as

$$(2.42) \quad \lambda^3 - a_2\lambda^2 + a_1\lambda - a_0 = 0,$$

where

$$(2.43) \quad a_2 = \gamma_X + \gamma_S + \gamma_A$$

$$(2.44) \quad a_1 = \gamma_A\gamma_X + \gamma_K + \gamma_E\gamma_S + \gamma_D\gamma_A + \gamma_R$$

$$(2.45) \quad a_0 = \gamma_A\gamma_K + \gamma_E\gamma_A\gamma_D + \gamma_E\gamma_R.$$

Applying Lemma 2 to this cubic polynomial we have

$$\alpha = \frac{\gamma_A\gamma_K + \gamma_E\gamma_A\gamma_D + \gamma_E\gamma_R}{\gamma_A\gamma_X + \gamma_K + \gamma_E\gamma_S + \gamma_D\gamma_A + \gamma_R}, \quad \beta = \gamma_A + \gamma_S + \gamma_X.$$

In this case, it is easy to verify that $\alpha < \beta$, from which we have that

$$\lambda \leq \gamma_{\max}^A + \gamma_{\max}^S + \gamma_{\max}^X,$$

which is the right inequality of (2.32). $\square$

3. Eigenvalue bounds for the block diagonal preconditioner. For the sake of comparison, we consider now as a preconditioner for the double saddle-point linear system (1.1) the block diagonal SPD matrix $\mathcal{P}_D$ defined in (2.2). The use of $\mathcal{P}_D$ as a preconditioner in place of $\mathcal{P}$ allows for employing MINRES instead of GMRES. The analysis that follows is in line with the developments in [9, 13], but, differently from those works, we provide more user-friendly eigenvalue bounds. The eigenvalues of the preconditioned matrix $\mathcal{A}\mathcal{P}_D^{-1}$ are the same as those of $\mathcal{P}_D^{-1/2}\mathcal{A}\mathcal{P}_D^{-1/2}$. Recalling equation (2.3), the corresponding eigenproblem reads:

$$(3.1) \quad \overline{A}x + R^T y = \lambda x,$$

$$(3.2) \quad Rx - \overline{D}y + K^T z = \lambda y,$$

$$(3.3) \quad Ky + \overline{E}z = \lambda z.$$

Assuming $\lambda \notin [\gamma_A^{\min}, \gamma_A^{\max}]$, equation (3.1) can be written as

$$(3.4) \quad x = (\lambda I - \overline{A})^{-1} R^T y,$$

which replaced into (3.2) yields

$$(3.5) \quad (-R(\lambda I - \overline{A})^{-1} R^T + \overline{D} + \lambda I) y = K^T z.$$

Setting $W(\lambda) = -R(\lambda I - \overline{A})^{-1} R^T + \overline{D} + \lambda I$, we now investigate for which values of λ 0 is not included in the interval $[\lambda_{\min}(W(\lambda)), \lambda_{\max}(W(\lambda))]$. Applying Lemma 1, the Rayleigh quotient $q(W(\lambda), s)$ reads:

$$(3.6) \quad q(W(\lambda), s) = \frac{s^T W(\lambda) s}{s^T s} = -\frac{s^T R(\lambda I - \overline{A})^{-1} R^T s}{s^T s} + \gamma_D + \lambda = -\frac{\gamma_R}{\lambda - \gamma_A} + \gamma_D + \lambda = \frac{p(\lambda)}{\lambda - \gamma_A},$$

with

$$p(\lambda) = \lambda^2 - (\gamma_A - \gamma_D)\lambda - \gamma_R - \gamma_A\gamma_D.$$

Denote as λ_-, λ_+ the two roots of p :

$$(3.7) \quad \begin{aligned} \lambda_+ &= \frac{\gamma_A - \gamma_D + \sqrt{(\gamma_A + \gamma_D)^2 + 4\gamma_R}}{2}, \\ \lambda_- &= \frac{\gamma_A - \gamma_D - \sqrt{(\gamma_A + \gamma_D)^2 + 4\gamma_R}}{2} \end{aligned}$$

Since $p(\gamma_A + \gamma_R) = \gamma_R(\gamma_R + \gamma_A + \gamma_D) > 0$ and $p(\gamma_A) = -\gamma_R < 0$, we have that $\gamma_A < \lambda_+ < \gamma_A + \gamma_R$. Moreover, as λ_+ is increasing in γ_A, γ_R and decreasing in γ_D , we have the following, possibly tighter, bound:

$$\lambda_+ \leq \frac{\gamma_{\max}^A + \sqrt{(\gamma_{\max}^A)^2 + 4\gamma_{\max}^R}}{2} \leq \gamma_{\max}^A + \sqrt{\gamma_{\max}^R}.$$

Hence, if $\lambda > 0$, $p(\lambda)$ can be zero only if $\lambda \in [\gamma_{\min}^A, \gamma_{\max}^A + \sqrt{\gamma_{\max}^R}]$. The negative root, λ_- , is increasing in γ_A and decreasing in γ_D, γ_R therefore,

$$\lambda_-(\gamma_A, \gamma_R, \gamma_D) \geq \lambda_-(0, \gamma_R, \gamma_D) = \frac{-\gamma_D - \sqrt{\gamma_D^2 + 4\gamma_R}}{2} \geq -\gamma_D - \sqrt{\gamma_R} \geq -\gamma_{\max}^D - \sqrt{\gamma_{\max}^R}.$$

To bound λ_- from above, we bound its modulus from below:

$$|\lambda_-| = \frac{\gamma_R + \gamma_A \gamma_D}{\lambda_+} > \frac{\gamma_R + \gamma_A \gamma_D}{\gamma_A + \gamma_R} = 1 + \frac{\gamma_A(\gamma_D - 1)}{\gamma_A + \gamma_R}.$$

If $\gamma_D \geq 1$ then $|\lambda_-| \geq 1$, otherwise, the last expression can be conveniently rewritten as

$$\frac{\gamma_R + \gamma_A \gamma_D}{\gamma_A + \gamma_R} = \frac{\gamma_S + (\gamma_A - 1)\gamma_D}{\gamma_A + \gamma_S - \gamma_D},$$

which, upon the assumption $\gamma_D < 1$, can be shown to be decreasing in γ_A , so that

$$\begin{aligned} \frac{\gamma_S + (\gamma_A - 1)\gamma_D}{\gamma_A + \gamma_S - \gamma_D} &\geq \frac{\gamma_S + (\gamma_{\max}^A - 1)\gamma_D}{\gamma_{\max}^A + \gamma_S - \gamma_D} \geq (\text{being the previous increasing in both } \gamma_D \text{ and } \gamma_S, \text{ since } \gamma_{\max}^A > 1), \\ &\geq \frac{\gamma_{\min}^S + (\gamma_{\max}^A - 1)\gamma_{\min}^D}{\gamma_{\max}^A + \gamma_{\min}^S - \gamma_{\min}^D} \geq \frac{\gamma_{\min}^S}{\gamma_{\max}^A + \gamma_{\min}^S}. \end{aligned}$$

Summarizing, as this last quantity is less than one, $\lambda_- \leq -\frac{\gamma_{\min}^S}{\gamma_{\max}^A + \gamma_{\min}^S}$. Therefore, if λ is not included in the set:

$$I_2 = \left[-\gamma_{\max}^D - \sqrt{\gamma_{\max}^R}, -\frac{\gamma_{\min}^S}{\gamma_{\max}^A + \gamma_{\min}^S} \right] \cup \left[\gamma_{\min}^A, \gamma_{\max}^A + \sqrt{\gamma_{\max}^R} \right],$$

then $0 \notin [\lambda_{\min}(W(\lambda)), \lambda_{\max}(W(\lambda))]$ and the eigenvalues of $W(\lambda)$ are either all positive or all negative. Under this condition, we can then proceed as in the proof of Theorem 2. Introducing $y = W(\lambda)^{-1}K^T z$ from (3.5) into (3.3) we obtain:

$$KW(\lambda)^{-1}K^T z + \bar{E}z - \lambda z = 0.$$

By pre-multiplying both sides by z^T and dividing by $z^T z$, and recalling the expression of the Rayleigh quotient for $W(\lambda)$ in (3.6), we have:

$$\frac{\lambda - \gamma_A}{p(\lambda)} \gamma_K + \gamma_E - \lambda = 0,$$

from which we can conclude that the eigenvalues λ of $\mathcal{AP}_D^{-1}$ satisfy the third-degree polynomial:

$$(3.8) \quad \pi(\lambda) = \lambda^3 - \underbrace{(\gamma_A + \gamma_E - \gamma_D)}_{\hat{a}_2} \lambda^2 - \underbrace{(\gamma_R + \gamma_K + \gamma_E \gamma_D + \gamma_A \gamma_D - \gamma_E \gamma_A)}_{\hat{a}_1} \lambda + \underbrace{\gamma_A \gamma_K + \gamma_E \gamma_R + \gamma_E \gamma_A \gamma_D}_{\hat{a}_0}.$$

There are one negative (μ_a) and two positive ($\mu_b < \mu_c$) roots of $\pi(\lambda)$, as also proved in [13, Corollary 2.1]. The next theorem provides bounds on the eigenvalues of the preconditioned matrix $\mathcal{AP}_D^{-1}$.

THEOREM 3. *The eigenvalues of the preconditioned matrix $\mathcal{AP}_D^{-1}$ belong to $I_- \cup I_+$, where*

$$\begin{aligned} I_- &= \left[-\gamma_{\max}^D - \sqrt{\gamma_{\max}^R + \gamma_{\max}^K}, -\frac{\gamma_{\min}^S}{\gamma_{\max}^A + \gamma_{\min}^S} \right] \\ I_+ &= \left[\min\{\gamma_{\min}^E, \gamma_{\min}^A\}, \max \left\{ \gamma_{\max}^A + \gamma_{\max}^E + \sqrt{\gamma_{\max}^K + \gamma_{\max}^R}, \sqrt{\gamma_{\max}^R + \gamma_{\max}^K + \gamma_{\max}^D(\gamma_{\max}^E + \gamma_{\max}^A)} \right\} \right] \end{aligned}$$

Proof. The proof is subdivided into 4 sections, corresponding to the limits for I_+ and I_- , respectively.

1. Lower bound for μ_b . Notice that $\pi(\gamma_E) = \gamma_K(\gamma_A - \gamma_E)$, and $\pi(\gamma_A) = -(\gamma_R + \gamma_A\gamma_D)(\gamma_A - \gamma_E)$. Hence, $\pi(\gamma_A)\pi(\gamma_E) < 0$, which means that $\mu_b \in [\gamma_A, \gamma_E]$ or $\mu_b \in [\gamma_E, \gamma_A]$ depending whether γ_A is larger or smaller than γ_E . It follows that

$$0 < \min\{\gamma_{\min}^E, \gamma_{\min}^A\} \leq \mu_b.$$

2. Upper bound for μ_c . The polynomial $\pi(\lambda)$ can be written also as:

$$\pi(\lambda) = \lambda\varphi(\lambda) + \hat{a}_0,$$

where

$$\varphi(\lambda) = \lambda^2 - (\gamma_A + \gamma_E - \gamma_D)\lambda - (\gamma_R + \gamma_K + \gamma_E\gamma_D + \gamma_A\gamma_D - \gamma_E\gamma_A).$$

Since $\hat{a}_0 > 0$, an upper bound for the largest root of φ is also a bound for μ_c . Direct computation shows that

$$\begin{aligned} \varphi(\gamma_A + \gamma_E + \sqrt{\gamma_K + \gamma_R}) &= (\gamma_A + \gamma_E)^2 + 2(\gamma_A + \gamma_E)\sqrt{\gamma_K + \gamma_R} + \gamma_K + \gamma_R \\ &\quad - (\gamma_A + \gamma_E)^2 - \gamma_D(\gamma_A + \gamma_E) - (\gamma_A + \gamma_E + \gamma_D)\sqrt{\gamma_K + \gamma_R} \\ &\quad - (\gamma_R + \gamma_K + \gamma_E\gamma_D + \gamma_A\gamma_D - \gamma_E\gamma_A). \\ &= (\gamma_E + \gamma_A - \gamma_D)\sqrt{\gamma_K + \gamma_R} + \gamma_E\gamma_A, \end{aligned}$$

which is positive for $\gamma_D < \gamma_E + \gamma_A$. If $\gamma_D \geq \gamma_E + \gamma_A$, positivity of φ is guaranteed under the condition:

$$\lambda > \sqrt{\gamma_R + \gamma_K + \gamma_D(\gamma_E + \gamma_A)}.$$

Summarizing, we have

$$\mu_c \leq \max \left\{ \gamma_{\max}^A + \gamma_{\max}^E + \sqrt{\gamma_{\max}^K + \gamma_{\max}^R}, \sqrt{\gamma_{\max}^R + \gamma_{\max}^K + \gamma_{\max}^D(\gamma_{\max}^E + \gamma_{\max}^A)} \right\}.$$

3. Upper bound for μ_a . Since $\pi(\lambda_-) > 0$, we immediately get an upper bound for μ_a given by

$$\mu_a \leq h_- \leq -\frac{\gamma_{\min}^S}{\gamma_{\max}^A + \gamma_{\min}^S}.$$

4. Lower bound for μ_a . Let us write $\pi(\lambda)$ as

$$\pi(\lambda) = \gamma_A q_A(\lambda) + \gamma_E q_E(\lambda) + r(\lambda),$$

where

$$(3.9) \quad q_A(\lambda) = \gamma_K - (\lambda - \gamma_E)(\lambda + \gamma_D),$$

$$(3.10) \quad q_E(\lambda) = -p(\lambda),$$

$$(3.11) \quad r(\lambda) = \lambda(\lambda^2 + \gamma_D\lambda - \gamma_R - \gamma_K).$$

Since μ_A is a root of π , the following relationship must hold:

$$(3.12) \quad \mu_a - \gamma_E = \frac{\gamma_K(\mu_A - \gamma_A)}{p(\mu_a)}.$$

Substituting (3.12) into (3.9) and rearranging yields

$$q_A(\mu_a) = -\frac{\gamma_R\gamma_K}{p(\mu_a)}.$$

Equation (3.12) also tells us that $p(\mu_A) > 0$, because μ_a is negative. As a consequence, both $q_A(\mu_a)$ and $q_E(\mu_a)$ are negative, hence it must be $r(\mu_a) > \pi(\mu_a) = 0$ and therefore the negative root of $r(\lambda)$ is a lower bound for μ_a . The negative root of $r(\lambda)$ satisfies

$$s_- = \frac{-\gamma_D - \sqrt{\gamma_D^2 + 4(\gamma_R + \gamma_K)}}{2} \geq -\gamma_D - \sqrt{\gamma_R + \gamma_K} \geq -\gamma_{\max}^D - \sqrt{\gamma_{\max}^R + \gamma_{\max}^K}.$$

□

4. Mixed form of Biot's poroelasticity equations. The classical formulation of linear poroelasticity couples Darcy's flow of a single-phase fluid with the quasi-static mechanical deformation of an elastic and fully saturated porous medium. In their mixed form, the governing equations consist of a conservation law of linear momentum, a conservation law of fluid mass, and Darcy's law connecting the fluid velocity to the variation of the hydraulic head.

Let $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) be the domain occupied by the porous medium with Γ its Lipschitz-continuous boundary, which is decomposed as $\Gamma = \overline{\Gamma_u} \cup \overline{\Gamma_\sigma} = \overline{\Gamma_p} \cup \overline{\Gamma_q}$, with $\Gamma_u \cap \Gamma_\sigma = \Gamma_p \cap \Gamma_q = \emptyset$. The strong form of the initial boundary value problem (IBVP) consists of finding the displacement $\mathbf{u} : \overline{\Omega} \times]0, t_{\max}[\rightarrow \mathbb{R}^d$, Darcy's velocity $\mathbf{q} : \overline{\Omega} \times]0, t_{\max}[\rightarrow \mathbb{R}^d$, and the excess pore pressure $p : \overline{\Omega} \times]0, t_{\max}[\rightarrow \mathbb{R}$ such that:

$$(4.1a) \quad \nabla \cdot (\mathbb{C}_{dr} : \nabla^s \mathbf{u} - \alpha_b p \mathbf{1}) = \mathbf{0} \quad \text{in } \Omega \times]0, t_{\max}[\quad (\text{equilibrium}),$$

$$(4.1b) \quad \mu \boldsymbol{\kappa}^{-1} \cdot \mathbf{q} + \nabla p = \mathbf{0} \quad \text{in } \Omega \times]0, t_{\max}[\quad (\text{Darcy's law}),$$

$$(4.1c) \quad \alpha_b \nabla \cdot \dot{\mathbf{u}} + S_\epsilon \dot{p} + \nabla \cdot \mathbf{q} = s \quad \text{in } \Omega \times]0, t_{\max}[\quad (\text{continuity}),$$

where $\mathbb{C}_{dr}(\eta, \nu)$ is the rank-four elasticity tensor depending on two independent material parameters, Lamé's constant η and Poisson's ratio ν , ∇^s is the symmetric gradient operator, α_b is the Biot coefficient, and $\mathbf{1}$ is the rank-two identity tensor; μ and $\boldsymbol{\kappa}$ are the fluid viscosity and the rank-two permeability tensor, respectively; S_ϵ is the constrained specific storage coefficient, i.e., the reciprocal of Biot's modulus, and s is the fluid source term. The following set of boundary and initial conditions completes the formulation:

$$(4.2a) \quad \mathbf{u} = \bar{\mathbf{u}} \quad \text{on } \Gamma_u \times]0, t_{\max}[,$$

$$(4.2b) \quad (\mathbb{C}_{dr} : \nabla^s \mathbf{u} - \alpha_b p \mathbf{1}) \cdot \mathbf{n} = \bar{\mathbf{t}} \quad \text{on } \Gamma_\sigma \times]0, t_{\max}[,$$

$$(4.2c) \quad \mathbf{q} \cdot \mathbf{n} = \bar{q} \quad \text{on } \Gamma_q \times]0, t_{\max}[,$$

$$(4.2d) \quad p = \bar{p} \quad \text{on } \Gamma_p \times]0, t_{\max}[,$$

$$(4.2e) \quad \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0 \quad \mathbf{x} \in \overline{\Omega},$$

$$(4.2f) \quad p(\mathbf{x}, 0) = p_0 \quad \mathbf{x} \in \overline{\Omega},$$

where $\bar{\mathbf{u}}$, $\bar{\mathbf{t}}$, $\bar{q}$, and $\bar{p}$ are the prescribed boundary displacements, tractions, Darcy velocity, and excess pore pressure, respectively, whereas p_0 is the initial excess pore pressure. By introducing the spaces:

$$(4.3a) \quad \mathcal{U} = \{\mathbf{u} \in [H^1(\Omega)]^d \mid \mathbf{u}|_{\Gamma_u} = \bar{\mathbf{u}}\}, \quad \mathcal{U}_0 = \{\mathbf{u} \in [H^1(\Omega)]^d \mid \mathbf{u}|_{\Gamma_u} = \mathbf{0}\},$$

$$(4.3b) \quad \mathcal{Q} = \{\mathbf{q} \in [H(\text{div}; \Omega)]^d \mid \mathbf{q} \cdot \mathbf{n}|_{\Gamma_q} = \bar{q}\}, \quad \mathcal{Q}_0 = \{\mathbf{q} \in [H(\text{div}; \Omega)]^d \mid \mathbf{q} \cdot \mathbf{n}|_{\Gamma_q} = 0\},$$

the weak form of the IBVP (4.1) reads: find $\{\mathbf{u}(t), \mathbf{q}(t), p(t)\} \in \mathcal{U} \times \mathcal{Q} \times L^2(\Omega)$ such that $\forall t \in]0, t_{\max}[$:

$$(4.4a) \quad (\nabla^s \boldsymbol{\eta}, \mathbb{C}_{dr} : \nabla^s \mathbf{u})_\Omega - (\text{div } \boldsymbol{\eta}, \alpha_b p)_\Omega = (\boldsymbol{\eta}, \bar{\mathbf{t}})_{\Gamma_\sigma} \quad \forall \boldsymbol{\eta} \in \mathcal{U}_0,$$

$$(4.4b) \quad (\boldsymbol{\phi}, \mu \boldsymbol{\kappa}^{-1} \cdot \mathbf{q})_\Omega - (\text{div } \boldsymbol{\phi}, p)_\Omega = -(\boldsymbol{\phi} \cdot \mathbf{n}, \bar{p})_{\Gamma_p} \quad \forall \boldsymbol{\phi} \in \mathcal{Q}_0,$$

$$(4.4c) \quad (\chi, \alpha_b \text{div } \dot{\mathbf{u}})_\Omega + (\chi, \text{div } \mathbf{q})_\Omega + (\chi, S_\epsilon \dot{p})_\Omega = (\chi, s)_\Omega \quad \forall \chi \in L^2(\Omega).$$

The final fully-discrete form of the continuous problem (4.4) is obtained by introducing a non-overlapping partition $\mathcal{T}_h$ of the spatial domain Ω with the associated finite dimensional counterparts $\mathcal{U}^h$ and $\mathcal{Q}^h$ of the spaces (4.3), and $\mathcal{P}^h$ of $L^2(\Omega)$. Then, discretization in time is performed by a finite difference scheme, e.g., backward Euler, which produces an unconditionally stable first-order solution.

4.1. Mixed Finite Element (MFE) discretization. Let $\{\boldsymbol{\eta}\}$, $\{\boldsymbol{\phi}\}$, and $\{\chi\}$ be the set of basis functions of the discrete spaces $\mathcal{U}^h$, $\mathcal{Q}^h$, and $\mathcal{P}^h$, respectively. For example, a very common choice is based on the triplet $\mathbb{Q}_1 - \mathbb{RT}_0 - \mathbb{P}_0$, with $\mathbb{Q}_1$ the space of bi- or tri-linear piecewise polynomials, according to the size d of the domain Ω , $\mathbb{RT}_0$ the lowest-order Raviart-Thomas space, and $\mathbb{P}_0$ the space of piecewise constant functions. This choice produces a second-order scheme for displacements and a first-order scheme for pressures [24, 51, 52]. The MFE discretization is mass-conservative at the elemental level, robust to jumps in the permeability fields, and provides an explicit representation of Darcy's velocity field, which is of particular interest whenever accuracy in the flux description is essential, e.g., if the Biot model is coupled to an advective transport equation. Denoting by $u_t \in \mathbb{R}^n$, $p_t \in \mathbb{R}^m$, and $q_t \in \mathbb{R}^l$ are the vectors of discrete displacement, pressure, and velocity values at the time t , the fully-discrete form of (4.4) reads:

$$(4.5) \quad \begin{bmatrix} A_{uu} & A_{up} & 0 \\ A_{pu} & A_{pp} & \Delta t A_{pq} \\ 0 & A_{qp} & A_{qq} \end{bmatrix} \begin{bmatrix} u_t \\ p_t \\ q_t \end{bmatrix} = \begin{bmatrix} b_u \\ b_p \\ b_q \end{bmatrix}$$

where Δt is the time discretization step and:

$$\begin{aligned}
(4.6a) \quad & A_{uu} = (\nabla^s \boldsymbol{\eta}, \mathbb{C}_{\text{dr}} : \nabla^s \boldsymbol{\eta})_{\Omega}, & A_{up} &= -(\text{div } \boldsymbol{\eta}, \alpha_b \chi)_{\Omega}, \\
(4.6b) \quad & A_{pu} = (\chi, \alpha_b \text{div } \boldsymbol{\eta})_{\Omega}, & A_{pp} &= (\chi, S_{\epsilon} \chi)_{\Omega}, & A_{pq} &= (\chi, \text{div } \boldsymbol{\phi})_{\Omega}, \\
(4.6c) \quad & & A_{qp} &= -(\text{div } \boldsymbol{\phi}, \chi)_{\Omega}, & A_{qq} &= (\boldsymbol{\phi}, \mu \boldsymbol{\kappa}^{-1} \cdot \boldsymbol{\phi})_{\Omega},
\end{aligned}$$

with:

$$\begin{aligned}
(4.7) \quad & b_u = (\boldsymbol{\eta}, \bar{\mathbf{t}}_t)_{\Gamma_{\sigma}}, \\
(4.8) \quad & b_p = (\chi, \alpha_b \text{div } u_{t-1} + S_{\epsilon} p_{t-1} + \Delta t s_t)_{\Omega}, \\
(4.9) \quad & b_q = -(\boldsymbol{\phi} \cdot \mathbf{n}, \bar{p}_t)_{\Gamma_p}.
\end{aligned}$$

Notice that A_{uu} and A_{qq} are SPD, A_{pp} is SPSD since S_{ϵ} can be 0 in the limiting case of incompressible fluid, $A_{up} = -A_{pu}^T$, and $A_{pq} = -A_{qp}^T$. It is well-known (see, for instance [56]) that the $\mathbb{Q}_1 - \mathbb{RT}_0 - \mathbb{P}_0$ triplet can be unstable close to undrained conditions, i.e., small permeability or time integration step with incompressible fluid. To avoid such an occurrence, a stabilization contribution can be introduced to annihilate the spurious checkerboard modes arising in the pressure solution. For example, here we modify the A_{pp} term by adding an SPSD jump-jump stabilization matrix A_{stab} computed over sets of macro-elements, as proposed in [27, 32].

It is easy to verify that the fully-discrete problem (4.5) can be just slightly modified to possess the double saddle-point structure (1.1). In fact, by multiplying by -1 and Δt the second and third row, respectively, we build the system matrix in (1.1) with:

$$(4.10) \quad A = A_{uu}, \quad B = -A_{pu}, \quad C = \Delta t A_{pq}, \quad D = A_{pp} + A_{stab}, \quad E = \Delta t A_{qq}.$$

4.2. Mixed Hybrid Finite Element (MHFE) discretization. The hybrid formulation is derived by replacing the discrete function space $\mathcal{Q}^h$ for Darcy's velocity representation with that obtained by using discontinuous piecewise polynomials defined on every element T of the partition $\mathcal{T}_h$:

$$(4.11) \quad \mathcal{W}^h = \{\mathbf{w}^h \in [L^2(\Omega)]^d \mid \mathbf{w}^h|_T \in [\mathbb{RT}_0(T)], \forall T \in \mathcal{T}_h\}.$$

The continuity of normal fluxes through inter-element edges or faces is enforced by Lagrange multipliers π , which are discretized through piecewise constant functions over each edge or face e . From a physical viewpoint, π represent a pressure value, hence for consistency, it must be $\pi|_{\Gamma_p} = \bar{p}$. Denoting by $\mathcal{B}^h$ the finite-dimensional function space for π and by $\mathcal{B}_0^h$ its counterpart taking homogeneous values along Γ_p , the semi-discrete weak form (4.4) becomes: find $\{\mathbf{u}(t), \mathbf{w}(t), p(t), \pi(t)\} \in \mathcal{U}^h \times \mathcal{W}^h \times \mathcal{P}^h \times \mathcal{B}^h$ such that $\forall t \in]0, t_{\max}[$:

$$\begin{aligned}
(4.12a) \quad & (\nabla^s \boldsymbol{\eta}, \mathbb{C}_{\text{dr}} : \nabla^s \mathbf{u})_{\Omega} - (\text{div } \boldsymbol{\eta}, \alpha_b p)_{\Omega} = (\boldsymbol{\eta}, \bar{\mathbf{t}})_{\Gamma_{\sigma}} & \forall \boldsymbol{\eta} \in \mathcal{U}_0^h, \\
(4.12b) \quad & (\boldsymbol{\varphi}, \mu \boldsymbol{\kappa}^{-1} \cdot \mathbf{w})_{\Omega} - \sum_{T \in \mathcal{T}_h} [(\text{div } \boldsymbol{\varphi}, p)_T - (\boldsymbol{\varphi} \cdot \mathbf{n}_e, \pi)_{\partial T}] = 0 & \forall \boldsymbol{\varphi} \in \mathcal{W}^h, \\
(4.12c) \quad & (\chi, \alpha_b \text{div } \dot{\mathbf{u}})_{\Omega} + \sum_{T \in \mathcal{T}_h} (\chi, \text{div } \mathbf{w})_T + (\chi, S_{\epsilon} \dot{p})_{\Omega} = (\chi, s)_{\Omega} & \forall \chi \in \mathcal{P}^h, \\
(4.12d) \quad & \sum_{T \in \mathcal{T}_h} (\zeta^h, \mathbf{w} \cdot \mathbf{n}_e)_{\partial T} = (\zeta, \bar{q})_{\Gamma_q} & \forall \zeta \in \mathcal{B}_0^h.
\end{aligned}$$

In (4.12), $\{\boldsymbol{\varphi}\}$ and $\{\zeta\}$ are the set of basis functions of the spaces $\mathcal{W}^h$ and $\mathcal{B}^h$, respectively. The MHFE fully discrete weak form is finally obtained by introducing a backward Euler discretization of the time variables. Denoting by $w_t \in \mathbb{R}^{2p}$ and $\pi_t \in \mathbb{R}^l$ the vectors of discrete velocity and Lagrange multiplier values at time t , from (4.12) we obtain:

$$(4.13) \quad \begin{bmatrix} A_{uu} & 0 & A_{up} & 0 \\ 0 & A_{ww} & A_{wp} & A_{w\pi} \\ A_{pu} & \Delta t A_{pw} & A_{pp} & 0 \\ 0 & A_{\pi w} & 0 & 0 \end{bmatrix} \begin{bmatrix} u_t \\ w_t \\ p_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} b_u \\ 0 \\ b_p \\ b_{\pi} \end{bmatrix},$$

TABLE 5.1
Cantilever beam test case: material properties.

Parameter	Value	Unit
Time integration step (Δt)	1×10^{-5}	[s]
Lamé's constant (η)	1.428571×10^5	[Pa]
Poisson's ratio (ν)	0.4	[-]
Biot's coefficient (α_b)	1.0	[-]
Constrained specific storage (S_ε)	0	[Pa]
Isotropic permeability (κ)	1×10^{-7}	[m ²]
Fluid viscosity (μ)	1×10^3	[Pa · s]

where the additional matrices with respect to (4.6) are:

$$(4.14a) \quad A_{ww} = (\varphi, \mu \kappa^{-1} \cdot \varphi)_\Omega, \quad A_{wp} = - \sum_{T \in \mathcal{T}_h} (\operatorname{div} \varphi, \chi)_T, \quad A_{w\pi} = \sum_{T \in \mathcal{T}_h} (\varphi \cdot \mathbf{n}_e, \zeta)_{\partial T}$$

$$(4.14b) \quad A_{pw} = \sum_{T \in \mathcal{T}_h} (\chi, \alpha_b \operatorname{div} \varphi)_T,$$

$$(4.14c) \quad A_{\pi w} = \sum_{T \in \mathcal{T}_h} (\zeta, \varphi \cdot \mathbf{n}_e)_{\partial T},$$

and:

$$(4.15) \quad b_\pi = (\zeta, \bar{q}_t)_{\Gamma_q}.$$

Notice that A_{ww} is SPD and block diagonal, while $A_{wp} = -A_{pw}^T$ and $A_{w\pi} = A_{\pi w}^T$. With the low-order triplet $\mathbb{Q}_1 - \mathbb{RT}_0 - \mathbb{P}_0$ a stabilization is still necessary, so the SPSP contribution A_{stab} is added to A_{pp} as with the MFE discretization. The block system (4.14) can be reduced by static condensation [14] by eliminating the w_t variables:

$$(4.16) \quad \begin{bmatrix} A_{uu} & A_{up} & 0 \\ A_{pu} & A_{pp} + A_{stab} - \Delta t A_{pw} A_{ww}^{-1} A_{wp} & \Delta t A_{pw} A_{ww}^{-1} A_{w\pi} \\ 0 & A_{\pi w} A_{ww}^{-1} A_{wp} & A_{\pi w} A_{ww}^{-1} A_{w\pi} \end{bmatrix} \begin{bmatrix} u_t \\ p_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} b_u \\ b_p \\ -b_\pi \end{bmatrix}.$$

Even though there is a loss of sparsity relative to the 4×4 block form (4.13), the reduction to the 3×3 structure (4.16) is always convenient because the computation of the new blocks involving A_{ww}^{-1} can be efficiently carried out at the elemental level and the resulting sparsity is comparable to that of system (4.5) [14]. The MHFE discretization inherits the mass-conservation and robustness properties of the MFE formulation. In addition, it enriches the pressure representation at the cost of suppressing the explicit computation of Darcy's velocity field. Hence, the MHFE formulation is generally preferred to the MFE one whenever the knowledge of the fluxes is not of specific interest.

As with the MFE discretization, the fully-discrete problem (4.16) can be easily modified to possess the double saddle-point structure (1.1). By multiplying by -1 and Δt the second and third row, respectively, we build the matrix in (1.1) with:

$$(4.17) \quad A = A_{uu}, \quad B = -A_{pu}, \quad C = \Delta t A_{\pi w} A_{ww}^{-1} A_{wp}, \quad D = A_{pp} + A_{stab} - \Delta t A_{pw} A_{ww}^{-1} A_{pw}^T, \quad E = \Delta t A_{\pi w} A_{ww}^{-1} A_{w\pi}.$$

5. Numerical results. A set of numerical experiments, concerning the mixed form of Biot's poroelasticity equations introduced in Section 4, is used to investigate the quality of the bounds on the eigenvalues of the preconditioned matrix and the effectiveness of the proposed triangular preconditioners on a problem with increasing size of the matrices involved. As a reference test case for the experiments, we considered the porous cantilever beam problem originally introduced in [53] and already used in [32]. The domain is the unit square or cube for the 2-D and 3-D case, respectively, with no-flow boundary conditions along all sides, zero displacements along the left edge, and a uniform load applied on top. Material properties are summarized in Table 5.1.

First, the 2-D version of the cantilever beam problem is used to compute the full eigenspectrum of the preconditioned matrix $\mathcal{AP}^{-1}$ with both the MFE and MHFE discretizations of the mixed Biot's

poroelasticity equations. The eigenspectrum is compared against the bounds introduced in Section 2 for a relatively small-sized matrix, thus validating the theoretical outcomes. The bounds on the eigenvalues of the preconditioned matrix $\mathcal{AP}_D^{-1}$ given in Section 3 are also used to compare the expected performance of GMRES and MINRES. Subsequently, the three-dimensional cantilever beam problem, discretized using the MHFE formulation, is employed with progressively larger test matrices to evaluate the efficiency and scalability of the proposed preconditioners.

5.1. Eigenvalue bounds validation. Let us consider a uniform discretization of the unit square with spacing $h = 1/40$, such that the sizes of the matrix blocks in $\mathcal{A}$ are $n = 3362$, $m = 1600$, and $l = 3200$. To check the relative error, we use a randomly manufactured solution with the corresponding right-hand side. Both a right-preconditioned GMRES with $\mathcal{P}$ and a preconditioned MINRES with $\mathcal{P}_D$ are used as a solver. The exit test is on the 2-norm of the residual vector normalized to its initial value, with a tolerance equal to either 10^{-13} (right-preconditioned GMRES) or 10^{-10} (preconditioned MINRES), to obtain comparable relative errors in the solution. The strict values of the tolerance are set to obtain a relative error norm of the order of 10^{-6} . For the numerical experiments, we use a MATLAB implementation run on an Intel Core Ultra 7 165U Notebook at 3.8 GHz with 16 GB RAM.

5.1.1. MFE discretization. The system matrix (1.1) has the blocks defined as in (4.10). To set up the preconditioner, we have to define the approximations $\hat{A}$, $\hat{S}$, and $\hat{X}$.

Since A is a standard elasticity matrix, an appropriate off-the-shelf approximation $\hat{A}$ can be computed by using an Algebraic MultiGrid (AMG) tool. In this case, we used the Matlab function `hsl_mi20`. Regarding the approximation $\hat{S}$ of the first-level Schur complement matrix $\tilde{S} = D + B\hat{A}^{-1}B^T$, we first observe that it is not possible to compute explicitly $\tilde{S}$ because $\hat{A}^{-1}$ is not available. A simple and common purely algebraic strategy with saddle-point problems relies on replacing $\tilde{S}$ with:

$$(5.1) \quad \tilde{S}^{(1)} = D + B(\text{diag}(A))^{-1}B^T,$$

then setting $\hat{S}$ as the incomplete Cholesky factorization of $\tilde{S}^{(1)}$ with no fill-in. For the second-level Schur complement matrix $\tilde{X} = E + C\hat{S}^{-1}C^T$, we can follow a similar strategy, where $\hat{S}^{-1}$ is approximated with the inverse of $\text{diag}(\tilde{S}^{(1)})$ and then $\hat{X}$ is the incomplete factorization of $\tilde{X}$ with no fill-in. We denote the block triangular preconditioner built with these choices as $\mathcal{P}^{(1)}$, while the corresponding block diagonal option is $\mathcal{P}_D^{(1)}$.

In this case, the intervals I_* for the indicators γ_* read:

$$\begin{aligned} I_A &= [5.01 \times 10^{-5}, \quad 1.035] \\ I_S &= [0.448, \quad 3.515] & I_R &= [7.7 \times 10^{-3}, \quad 3.470] & I_D &= [5.3 \times 10^{-3}, \quad 0.983] \\ I_X &= [0.999, \quad 1.030] & I_K &= [4.3 \times 10^{-5}, \quad 0.035] & I_E &= [0.995, \quad 1.000] \end{aligned}$$

Using Theorems 2 and 3, the indicators above give the eigenvalue bounds for the real eigenvalues reported in Table 5.2. These bounds are compared to the exact spectral intervals of $\mathcal{AP}^{(1)-1}$ and $\mathcal{AP}_D^{(1)-1}$. We observe that the accuracy of the bounds is extremely good; they are sufficiently tight, particularly for the eigenvalues that are close to 0.

The real spectral interval turns out to be quite wide for both preconditioners, especially in consideration of the small size of the problem at hand. The reason for this occurrence is twofold: (i) the interval I_A has a very small left extreme, and (ii) the intervals I_A and I_S do not overlap. While the first issue can be simply addressed by improving the quality of the off-the-shelf AMG approximation of A , the second issue can be fixed by scaling $\tilde{S}$ by a factor $\omega^{-1} > 1$, similarly to the strategy used in [8]. This will proportionally decrease $\gamma_{\max}^S$ and $\gamma_{\max}^R$, thus reducing the real spectral interval for the triangular preconditioner. Hence, we define $\tilde{S}(\omega) = \omega^{-1}\tilde{S}$, and use, for instance, $\omega = 10^{-1}$, with the new preconditioners denoted as $\mathcal{P}^{(1,\omega)}$ and $\mathcal{P}_D^{(1,\omega)}$. The new indicator intervals become:

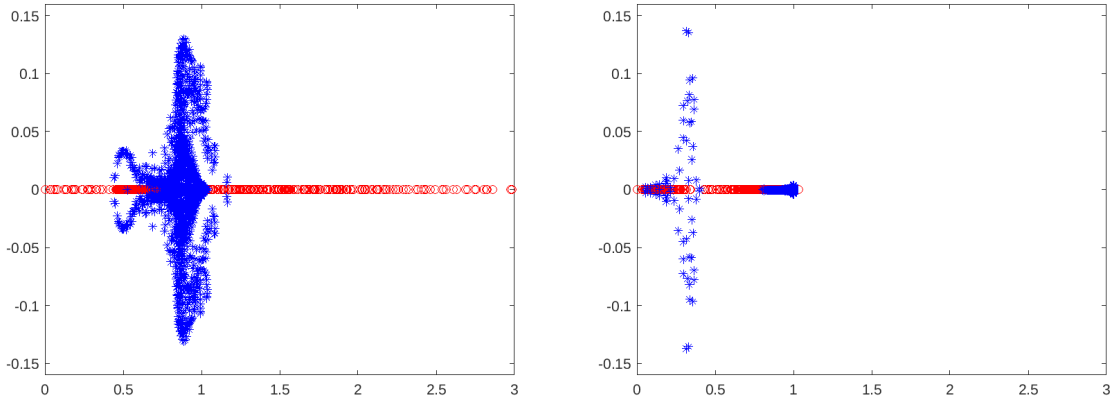
$$\begin{aligned} I_S &= [0.045, \quad 0.352] & I_R &= [7.7 \times 10^{-4}, \quad 0.347] & I_D &= [5.3 \times 10^{-4}, \quad 0.098] \\ I_X &= [0.999, \quad 1.030] & I_K &= [4.3 \times 10^{-6}, \quad 0.004] & I_E &= [0.999, \quad 1.000] \end{aligned}$$

from which we obtain the bounds for the real eigenvalues shown in Table 5.2. It can be seen that the spectral interval for the real positive eigenvalues has been narrowed. In contrast, reducing the indicators

TABLE 5.2

Eigenvalues and bounds of the preconditioned matrix for the block triangular and block diagonal preconditioners.

	Bounds	True eigenvalues
$\mathcal{AP}^{(1)-1}$	$[5.0098\text{e-}05, 5.5536]$	$[5.0113\text{e-}05, 3.4649]$
$\mathcal{AP}_D^{(1)-1}$	$[-2.8548, -0.3021] \cup [5.0098\text{e-}05, 3.9014]$	$[-1.4950, -0.3652] \cup [5.0105\text{e-}05, 2.3850]$
$\mathcal{AP}^{(1,\omega)-1}$	$[5.0098\text{e-}05, 2.3865]$	$[5.0113\text{e-}05, 1.0346]$
$\mathcal{AP}_D^{(1,\omega)-1}$	$[-0.9672, -0.0415] \cup [5.0098\text{e-}05, 2.6262]$	$[-0.2971, -0.0440] \cup [5.0105\text{e-}05, 1.2381]$
$\mathcal{AP}^{(2)-1}$	$[5.0098\text{e-}05, 3.5249]$	$[5.0113\text{e-}05, 1.4321]$
$\mathcal{AP}_D^{(2)-1}$	$[-1.7058, -0.2978] \cup [5.0098\text{e-}05, 3.1750]$	$[-0.9295, -0.3608] \cup [5.0105\text{e-}05, 1.6997]$
$\mathcal{AP}^{-1}$	$[5.0098\text{e-}05, 3.3065]$	$[5.0113\text{e-}05, 1.0346]$
$\mathcal{AP}_D^{-1}$	$[-1.3486, -0.2166] \cup [5.0098\text{e-}05, 3.0332]$	$[-0.8323, -0.2826] \cup [5.0105\text{e-}05, 1.4150]$

FIG. 5.1. MFE discretization: eigenvalue distribution for $\mathcal{AP}^{(1)-1}$ (left) and $\mathcal{AP}^{(1,\omega)-1}$ with $\omega = 10^{-1}$ (right). The red and blue markers denote the real and complex eigenvalues, respectively.

γ_*^S, γ_*^R and γ_*^D slightly widens the region of purely complex eigenvalues, as shown in Figure 5.1. In total, the behavior of the Krylov solvers is positively influenced by the reduced real spectral intervals, as confirmed by the convergence results shown in Table 5.3.

Alternatively to equation (5.1), a second way to approximate the first-level Schur complement relies on replacing $\tilde{S}$ with:

$$(5.2) \quad \tilde{S}^{(2)} = D + S_K,$$

where S_K is the diagonal “fixed-stress” matrix, introduced as a preconditioner in [17, 60], which is a suitable scaling of the lumped mass matrix of the pressure shape functions. The lumped structure of the approximation is suggested by the nature of the differential operators underlying A and B , which represent a discrete div-grad and a discrete divergence operator, respectively, as in the classical Stokes problem [23]. The values of the diagonal entries of S_K are computed as:

$$(5.3) \quad [S_K]_i = \frac{\alpha_b^2 |\Omega_i|}{\bar{K}_i}, \quad i = 1, \dots, l,$$

where $|\Omega_i|$ is the measure of the i -th finite element and $\bar{K}_i$ is an estimate of the associated bulk modulus, is dictated by physical considerations [60]. A more general algebraic strategy for computing the entries (5.3) can be implemented following the ideas sketched in [17]. Let us denote with $\mathcal{N}^{(i)}$ the subset of $\mathcal{N} = \{1, 2, \dots, n\} \subset \mathbb{N}$ containing the indices of the nonzero components of the i -th row $B^{(i)}$ of B , and define $R^{(i)}$ as the restriction operator from $\mathbb{R}^n$ to $\mathbb{R}^{|\mathcal{N}^{(i)}|}$ such that:

$$(5.4) \quad A^{(i,i)} = R^{(i),T} A R^{(i)}$$

TABLE 5.3

Number of iterations n_{it} and CPU time T of right-preconditioned GMRES and preconditioned MINRES with block triangular and block diagonal preconditioners.

	GMRES			MINRES		
	prec.	n_{it}	T [s]	prec.	n_{it}	T [s]
MFE	$\mathcal{P}^{(1)}$	85	0.95	$\mathcal{P}_D^{(1)}$	227	2.12
	$\mathcal{P}^{(1,\omega)}$	62	0.69	$\mathcal{P}_D^{(1,\omega)}$	167	1.60
	$\mathcal{P}^{(2)}$	62	0.72	$\mathcal{P}_D^{(2)}$	142	1.39
MHFE	$\mathcal{P}$	54	0.72	$\mathcal{P}_D$	148	1.51

is the sub-matrix of A made by the entries lying in the rows and columns with indices in $\mathcal{N}^{(i)}$. Since $A^{(i,i)}$ is a diagonal symmetric block of an SPD matrix, it is nonsingular and can be regularly inverted. The i -th diagonal entry of S_K can be algebraically computed as:

$$(5.5) \quad [S_K]_i = r(B^{(i)})A^{(i,i),-1}r(B^{(i),T}), \quad i = 1, \dots, l,$$

where $r(\cdot)$ is the operator restricting a vector to its nonzero entries.

Since S_K is diagonal, the first-level Schur complement approximation $\tilde{S}^{(2)}$ of equation (5.2) is diagonally dominant and $\hat{S}$ can be simply set as $\text{diag}(\tilde{S}^{(2)})$. As a consequence, the second-level Schur complement matrix $\tilde{X} = E + C\hat{S}^{-1}C^T$ can be explicitly computed, with $\hat{X}$ defined again as the incomplete factorization of $\tilde{X}$ with no fill-in. The block triangular and block diagonal preconditioners built with these choices as denoted as $\mathcal{P}^{(2)}$ and $\mathcal{P}_D^{(2)}$, respectively.

Using $\tilde{S}^{(2)}$ yields the following intervals for the indicators γ_* :

$$\begin{aligned} I_A &= [5.01 \times 10^{-5}, \quad 1.035] \\ I_S &= [0.439, \quad 1.489] & I_R &= [3.4 \times 10^{-3}, \quad 1.310] & I_D &= [4.7 \times 10^{-3}, \quad 0.559] \\ I_X &= [0.998, \quad 1.001] & I_K &= [4.0 \times 10^{-6}, \quad 0.005] & I_E &= [0.993, \quad 1.003] \end{aligned}$$

The bounds and the eigenvalues of the preconditioned matrices $\mathcal{AP}^{(2)-1}$ and $\mathcal{AP}_D^{(2)-1}$ are provided in Table 5.2. By comparing these bounds with those for $\mathcal{AP}^{(1,\omega)-1}$, we can predict a similar GMRES convergence, while the more clustered negative eigenvalues suggest an improved MINRES performance with respect to $\mathcal{P}_D^{(1,\omega)}$. Both observations are confirmed in Table 5.3, where the GMRES number of iterations is the same for the two approaches, whereas the fixed-stress approach provides a reduction of the MINRES iterations from 167 to 142. Hence, we can conclude that the use of the fixed-stress approximation for the first-level Schur complement is generally helpful.

5.1.2. MHFE discretization. The system matrix (1.1) now has the block definition (4.17). To set up the preconditioner, we use as $\hat{A}$ the same `hsl_mi20` AMG operator as before, while, following the outcome of the MFE numerical experiments, we use the diagonal of the “fixed-stress” approximation (5.2) for $\hat{S}$ and the incomplete Cholesky factorization of $\tilde{X}$ with no fill in for $\hat{X}$. The block triangular and block diagonal preconditioners built with these choices are simply denoted as $\mathcal{P}$ and $\mathcal{P}_D$, respectively.

The intervals I_* for the indicators γ_* read:

$$\begin{aligned} I_A &= [5.01 \times 10^{-5}, \quad 1.035] \\ I_S &= [0.286, \quad 1.086] & I_R &= [3.4 \times 10^{-3}, \quad 0.655] & I_D &= [3.1 \times 10^{-3}, \quad 0.536] \\ I_X &= [0.797, \quad 1.186] & I_K &= [0, \quad 0.005] & I_E &= [0.795, \quad 1.186] \end{aligned}$$

from which Theorem 1 and Remark 1 yield the bound $|\lambda - 1| \leq \sqrt{1 - \gamma_{\min}^D} = 0.998$ for the complex eigenvalues. The eigenvalues of $\mathcal{AP}^{-1}$ are shown in Figure 5.2. As for the real eigenvalues, we have the bounds and true spectral intervals reported in Table 5.2. Again, the bounds are very tight, especially for the lower (resp. upper) bounds for the positive (resp. negative) eigenvalues, as also clear from Figure 5.3 where the true spectral intervals are compared with the intervals predicted by the bounds.

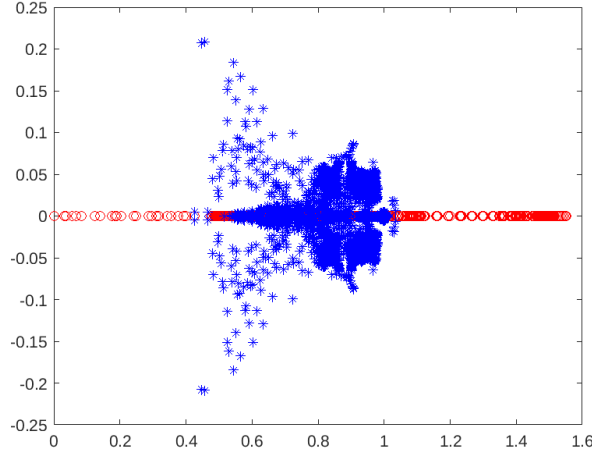


FIG. 5.2. *MHFE discretization: eigenvalue distribution for AP^{-1} . The red and blue markers denote the real and complex eigenvalues, respectively.*

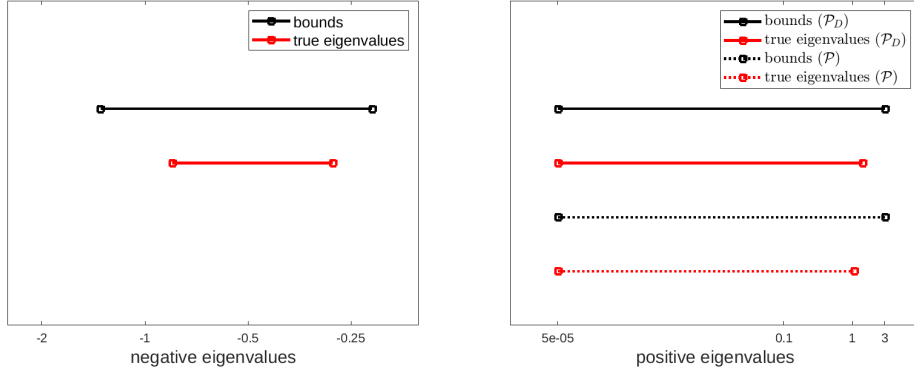


FIG. 5.3. *Bounds (black line) vs true real eigenvalues (red line) for the preconditioners $\mathcal{P}_D$ and $\mathcal{P}$ in the MHFE setting.*

The GMRES and MINRES convergence, shown in Table 5.3, is consistent with the eigenvalue distribution, showing an overall similar algebraic behavior between the MFE and MHFE discretizations.

5.2. Computational efficiency and scalability. We consider a set of uniform hexahedral MHFE discretizations of the unit cube with spacing $h = 0.100, 0.050, 0.025$, which gives rise to the double saddle-point linear systems (1.1) whose size and number of nonzeros are reported in Table 5.4. The linear systems are solved by using either GMRES or MINRES preconditioned with $\mathcal{P}$ or $\mathcal{P}_D$, respectively. We considered a right-hand side corresponding to an imposed solution of all ones, and the zero vector as the initial guess.

In addition to the `hsl_mi20` routine previously mentioned, for $\hat{A}$ we also use the Matlab implementation of the AMG operator belonging to the Chronos software package [39, 41], which is specifically designed to address an elasticity matrix such as A . In fact, the prolongation is constructed by a least square fit of the rigid body modes, i.e., the near-null kernel of the elasticity operator, and further improved via energy minimization [29]. The smoother is based on an adaptive variant of the factorized sparse approximate inverse preconditioner [40], which allows for a more aggressive coarsening than other classical smoothers, such as Jacobi or Chebychev. The approximations $\hat{S}$ and $\hat{X}$ of the Schur complements are as in Subsection 5.1.2.

The convergence results are summarized in Table 5.5. We observe that the AMG preconditioner from the Chronos software package performs better than the HSL package in terms of scalability, which is almost perfect with both GMRES and MINRES. This outcome also shows that the approximations introduced for the first- and second-level Schur complements do not affect the overall scalability of $\mathcal{P}$ and $\mathcal{P}_D$, which depends only on the quality of $\hat{A}$ as a preconditioner for A . The Preconditioned Conjugate Gradient

TABLE 5.4

3D cantilever beam problem on a unit cube: size and number of nonzeros of the test matrices ($N = n + m + l$).

$1/h$	n	m	l	N	nonzeros
10	3993	1000	3300	8293	0.3×10^6
20	27783	8000	25200	60983	2.4×10^6
40	206763	64000	196800	467563	19.2×10^6

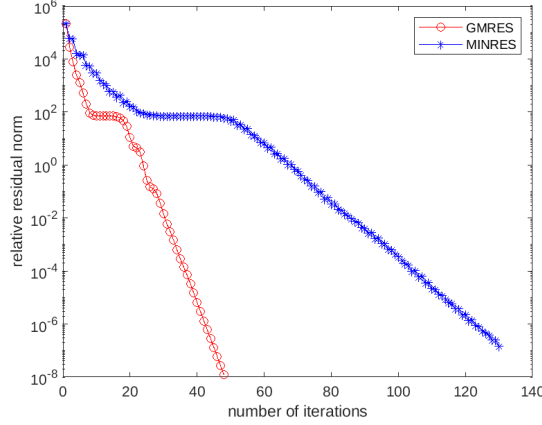


FIG. 5.4. 3D cantilever beam problem on a unit cube: convergence profile of MINRES and GMRES for the finest mesh.

(PCG) method for solving the linear system with the (1,1) block A , accelerated by the Chronos AMG operator, requires 25, 26, and 27 iterations, for $h = 0.100, 0.050$ and 0.025 , respectively. Therefore, no loss of scalability is introduced by the approximations of the two Schur complements proposed in this work. Finally, the results confirm the superior computational efficiency of GMRES over MINRES for our setting, in terms of both convergence rate and CPU time (see also Figure 5.4, where the absolute residual is plotted vs the iteration number) to obtain roughly the same relative error, see the last column of Table 5.5.

6. Conclusions. Double-saddle point linear systems naturally arise in the monolithic solution to the coupled Biot’s poroelasticity equations written in mixed form. Concerning the general 3×3 block structure of the matrix $\mathcal{A}$ in equation (1.1), the block A is an SPD elasticity matrix, D and E are mass matrices where D can be singular, B is a discrete divergence, and C is a scaled Gram matrix. The family of block triangular preconditioners $\mathcal{P}$ introduced in References [27, 32] has been considered for this double saddle-point problem by combining physically-based and purely algebraic Schur complement approximations.

In this work, we have analyzed the general spectral properties of the preconditioned matrix $\mathcal{A}\mathcal{P}^{-1}$ by extending the results already available for the simpler case with $D = E = 0$. We have proved that:

- the complex eigenvalues, if any, lie in a circle with center $(1, 0)$ and radius smaller than 1;
- the real eigenvalues are the roots of a third-degree polynomial with real coefficients and are bounded by quantities that depend on the quality of the inner preconditioners $\hat{A}$, $\hat{S}$, and $\hat{X}$ of the (1,1) block A , and the first- and second-level Schur complements S and X ;
- the eigenvalues obtained by using a block diagonal preconditioner are clustered in two intervals, one on the negative and one on the positive side of the real axis, with the bounds of the positive interval similar to those obtained with the block triangular preconditioner.

The numerical experiments on a set of test problems arising from mixed and mixed hybrid discretizations of Biot’s poroelasticity equations have validated the established bounds, which generally turn out to be very tight. Moreover, some numerical features of the investigated family of preconditioners have been clarified on consideration of their theoretical spectral properties. The results that follow are worth summarizing.

1. The use of GMRES with a block triangular preconditioner is usually more efficient than MINRES with a block diagonal preconditioner, both in terms of convergence rate and computation time, the GMRES overhead in terms of memory requirements being relatively modest, due to the small number of iterations.
2. Complex eigenvalues have a limited influence on the solver convergence. The key factor for the

TABLE 5.5

3D cantilever beam problem on a unit cube: number of iterations n_{it} and CPU time T with different AMG approximations for $\hat{A}$ obtained from either the HSL routine `hsl_mi20` or the Chronos software package.

$1/h$	solver	HSL			Chronos		
		n_{it}	T [s]	rel. err.	n_{it}	T [s]	rel. err.
10	GMRES	27	0.15	1.7×10^{-5}	46	0.10	1.58×10^{-6}
	MINRES	79	1.06	2.4×10^{-5}	111	0.08	1.71×10^{-6}
20	GMRES	38	2.63	1.0×10^{-5}	47	0.61	1.60×10^{-6}
	MINRES	135	7.86	1.1×10^{-5}	122	1.05	1.44×10^{-6}
40	GMRES	71	37.26	1.0×10^{-5}	47	4.48	1.17×10^{-6}
	MINRES	264	127.55	5.5×10^{-6}	129	9.21	1.09×10^{-6}

overall performance is the quality of the approximation of the elasticity matrix, which generally controls the lower bound of the eigenspectrum of the preconditioned matrix.

- Another factor that affects the solver convergence is the overlap between the spectral intervals of $\bar{A}$ and $\bar{S}$. This is automatically ensured by using a “fixed-stress” approximation of $B\hat{A}^{-1}B^T$ in the computation of the first-level Schur complement. In addition, in this way, $\hat{S}$ can be set as a simple diagonal matrix and $\hat{X}$ can be computed exactly.
- The overall scalability of the proposed family of block triangular preconditioners for the mixed form of Biot’s poroelasticity equations depends mainly on the scalability of $\hat{A}$ as a preconditioner of the elasticity matrix A . In this context, the AMG operator provided by the Chronos software package appears to be a valuable tool.

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