Small energy stabilization for 1D nonlinear Klein Gordon equations

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1. Introduction

Let $m > 0$ and $V \in \mathcal{S}(\mathbb{R}, \mathbb{R})$ (Schwartz function) with set of eigenvalues

$$\sigma_{\mathrm{d}}(L_1) = \{\lambda_j^2 \mid j = 1, \cdots, N\} \text{ with } 0 < \lambda_1 < \cdots < \lambda_N < m, \text{ where } L_1 = -\partial_x^2 + V + m^2. \tag{1.1}$$

We assume there exist $C > 0$ and $a_1 > 0$ such that

$$|V^{(l)}(x)| \le C e^{-a_1|x|} \text{ for all } 0 \le l \le N + 1. \tag{1.2}$$

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Let $f \in C^\infty(\mathbb{R},\mathbb{R})$ s.t. $f(0)=f'(0)=0$. We consider the nonlinear Klein-Gordon (NLKG) equation

$$\dot{\mathbf{u}} = \mathbf{J}\left(\mathbf{L}_1\mathbf{u} + \mathbf{f}[\mathbf{u}]\right), \ \mathbf{u} = {}^t(u_1\ u_2) : \mathbb{R}\times\mathbb{R} \to \mathbb{R}^2, \tag{1.3}$$

where

$$\mathbf{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{L}_1 = \begin{pmatrix} L_1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{f}[\mathbf{u}] = \begin{pmatrix} f(u_1) \\ 0 \end{pmatrix}.$$

Denoting by ϕ_j a real valued eigenfunction with $L^2(\mathbb{R})$ norm equal to 1 of L_1 associated to λ_j^2, setting

$$\mathbf{\Phi}_j := \begin{pmatrix} \phi_j \\ \mathrm{i}\lambda_j\phi_j \end{pmatrix} \text{ for } j = 1, \cdots, N, \tag{1.4}$$

we have

$$\mathbf{J}\mathbf{L}_1\mathbf{\Phi}_j = \mathrm{i}\lambda_j\mathbf{\Phi}_j \text{ and } \mathbf{J}\mathbf{L}_1\overline{\mathbf{\Phi}}_j = -\mathrm{i}\lambda_j\overline{\mathbf{\Phi}}_j. \tag{1.5}$$

In fact the $\mathbf{\Phi}_j$ and their complex conjugates $\overline{\mathbf{\Phi}}_j$ generate all the eigenfunctions of the linearization $\mathbf{J}\mathbf{L}_1$ of our NLKG (1.3).
Our NLKG (1.3) is a Hamiltonian system for the symplectic form

$$\Omega(\mathbf{u},\mathbf{v}) := \left\langle \mathbf{J}^{-1}\mathbf{u}, \mathbf{v} \right\rangle, \text{ where } \langle \mathbf{u},\mathbf{v}\rangle := Re\,(\mathbf{u},\overline{\mathbf{v}}) \text{ and} \tag{1.6}$$

$$(\mathbf{u},\mathbf{v}) := \int_{\mathbb{R}} {}^t\mathbf{u}(x)\mathbf{v}(x)dx, \tag{1.7}$$

and the Hamiltonian or energy function is given by

$$E(\mathbf{u}) = \frac{1}{2}\langle \mathbf{L}_1\mathbf{u},\mathbf{u}\rangle + \int_{\mathbb{R}} F(u_1)\,dx, \text{ where } F(u) = \int_0^s f(\tau)\,d\tau. \tag{1.8}$$

The local well-posedness of (1.3) is well known. From the conservation of the energy, we have that for sufficiently small $\delta > 0$, if $\|\mathbf{u}_0\|_{\boldsymbol{\mathcal{H}}^1} \le \delta$, then $\|\mathbf{u}\|_{L^\infty(\mathbb{R},\boldsymbol{\mathcal{H}}^1)} \lesssim \delta$ and in particular we obtain the global well-posedness for small data, where

$$\|\mathbf{u}\|^2_{\boldsymbol{\mathcal{H}}^1} = \|u_1\|^2_{H^1} + \|u_2\|^2_{L^2}. \tag{1.9}$$

Given a constant $a > 0$ we consider the space defined by the norm

$$\|\mathbf{u}\|_{\boldsymbol{\mathcal{H}}^1_{-a}} := \|\mathrm{sech}\,(ax)\,\mathbf{u}\|_{\boldsymbol{\mathcal{H}}^1}. \tag{1.10}$$

We denote by $\boldsymbol{\phi}[\mathbf{z}]$ the *refined profile*, introduced below in Sect. 1.1, where

$$\mathbf{z} = (z_1, ..., z_N), \tag{1.11}$$

encodes the discrete modes and where $\boldsymbol{\phi}[\mathbf{z}] = \sum z_j \boldsymbol{\Phi}_j + c.c. + O(\|\mathbf{z}\|)$, where by $g + c.c.$, we mean $g + \bar{g}$ and $\|\mathbf{z}\|^2 := \sum_{j=1}^N |z_j|^2$.

The main result in this paper is the following theorem.

Theorem 1.1. *Under Assumptions 1.3, 1.7 and 1.12 given below, for any $a > 0$ and $\epsilon > 0$ there exists $\delta_0 > 0$ such that if $\|\mathbf{u}_0\|_{\mathcal{H}^1} < \delta_0$, then we have a global representation*

$$\mathbf{u}(t) = \boldsymbol{\phi}[\mathbf{z}(t)] + \boldsymbol{\eta}(t) \text{ for appropriate } \mathbf{z} \in C^1(\mathbb{R}, \mathbb{C}^N) \text{ and } \boldsymbol{\eta} \in C^0(\mathbb{R}, \mathcal{H}^1), \tag{1.12}$$

and, for $I = \mathbb{R}$,

$$\int_I \|\boldsymbol{\eta}(t)\|^2_{\mathcal{H}^1_{-a}(\mathbb{R})} dt \le \epsilon, \tag{1.13}$$

and

$$\lim_{t\to\infty} \mathbf{z}(t) = 0\,. \tag{1.14}$$

The result of this paper is a partial extension to dimension 1 of the result, on local decay to zero for small real valued solutions of an NLKG with a trapping potential and, in particular, on the absence of small energy real valued periodic solutions, proved for dimension 3 by Bambusi and Cuccagna [1]. The latter was an extension, to cases with quite general spectral configurations, of a result proved by Soffer and Weinstein [33] under rather restrictive spectral hypotheses. There is a substantial literature on the asymptotic stability of patterns for wave like equations, partially reviewed for the case of the Nonlinear Schrödinger Equation (NLS) in [6]. In particular, in a series of papers referenced in [6], we have expanded the result of [1] to various contexts where dispersion can be proved using Strichartz estimates. The crux of these papers consisted in proving a form of radiation induced damping on the discrete modes of the system (the so called Nonlinear Fermi Golden Rule, or FGR), due to the spilling of the energy in the discrete modes in the radiation component of the solutions, where dispersion occurs because of linear dispersion. Recently, thanks to the notion introduced in [7], of *Refined Profile*, we have been able to simplify significantly the proofs, see also [8,9], eliminating the normal forms arguments required to find a coordinates system where the FGR can be seen. In fact, an ansatz involving the Refined Profile yields automatically a framework adequate to prove the FGR, as we will see later.

Lately, in the literature there has been considerable attention on low dimensional problems, especially in 1D, where, due to the relative strength of the nonlinearities, the Strichartz estimates are not sufficient to prove dispersion. Various papers like for example [2,11–32], [34] and [36] have recently dealt with asymptotic stability problems in the context of long range nonlinearities. In [4,5] use is made of the theory of Virial Inequalities developed by Kowalczyk et al. [16–20]. In this paper we will follow closely Kowalczyk and Martel [16]. So, as in [16–20], we will need two distinct sets of Virial Inequalities. We follow the Kowalczyk and Martel [16] idea of proving the FGR utilizing the initial sets of coordinates, contrary to what is done in [4,5]. In particular, in the proof of the FGR we use a functional derived from Kowalczyk and Martel [16], instead of the localized energy $E(\boldsymbol{\phi}[\mathbf{z}])$. The proof simplifies, avoiding the use of the smoothing estimates, which played a significant role in [4,5]. We highlight that our result works under a somewhat restrictive hypothesis on the potential V, specifically that the potential V_D obtained after eliminating all the eigenvalues of L_1 with a sequence of Darboux transformations, must be a repulsive potential, in the sense of Assumption 1.12.

1.1. Assumptions and refined profile

Notation 1.2. We write $a \lesssim b$ to mean that there exists a constant $C > 0$ s.t. $a \le Cb$. The positive number C omitted is called the implicit constant.

We set $\boldsymbol{\lambda} = (\lambda_1, \cdots, \lambda_N, -\lambda_1, \cdots, -\lambda_N) \in \mathbb{R}^{2N}$ and

$$\mathbf{R} := \{\mathbf{m} = (\mathbf{m}_+, \mathbf{m}_-) \in (\mathbb{N} \cup \{0\})^{2N} \mid |\mathbf{m} \cdot \boldsymbol{\lambda}| > m\},$$
$$\mathbf{R}_{\min} := \{\mathbf{m} \in \mathbf{R} \mid \nexists \mathbf{n} \in \mathbf{R} \text{ s.t. } \mathbf{n} \prec \mathbf{m}\},$$
$$\mathbf{I} := \{\mathbf{m} \in (\mathbb{N} \cup \{0\})^{2N} \mid \exists \mathbf{n} \in \mathbf{R}_{\min} \text{ s.t. } \mathbf{n} \prec \mathbf{m}\},$$

where

$$\mathbf{n} = (\mathbf{n}_+, \mathbf{n}_-) \prec \mathbf{m} = (\mathbf{m}_+, \mathbf{m}_-)$$
$$\iff \forall j = 1, \cdots, N, \ n_{+,j} + n_{-,j} \le m_{+,j} + m_{-,j} \text{ and } \|\mathbf{n}\| < \|\mathbf{m}\|,$$
$$\text{where } \|\mathbf{m}\| := \sum_{j=1}^{N} \sum_{\pm} m_{\pm,j}.$$

We also set $\mathbf{e}^j = (\delta_{1j}, \cdots, \delta_{Nj}, 0, \cdots, 0)$ where δ_{jk} is the Kronecker's delta, $\overline{\mathbf{m}} = \overline{(\mathbf{m}_+, \mathbf{m}_-)} := (\mathbf{m}_-, \mathbf{m}_+)$ and

$$\mathbf{NR} := (\mathbb{N} \cup \{0\})^{2N} \setminus (\mathbf{R}_{\min} \cup \mathbf{I}),$$
$$\boldsymbol{\Lambda}_j := \{\mathbf{m} \in \mathbf{NR} \mid \mathbf{m} \cdot \boldsymbol{\lambda} = \lambda_j\},$$
$$\overline{\boldsymbol{\Lambda}_j} := \{\overline{\mathbf{m}} \mid \mathbf{m} \in \boldsymbol{\Lambda}_j\}$$
$$\boldsymbol{\Lambda}_0 := \{\mathbf{m} \in \mathbf{NR} \setminus \{\mathbf{0}\} \mid \boldsymbol{\lambda} \cdot \mathbf{m} = 0\}.$$

We assume the following, which is true for generic L_1.

Assumption 1.3. For M the largest number in $\mathbb{N}(= \{1, 2, ...\})$ such that $(M-1)\lambda_1 < m$, then for a multi–index $\mathbf{m} \in \mathbb{N}_0^{2N}$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, we assume

$$\|\mathbf{m}\| \le M \Longrightarrow (\mathbf{m} \cdot \boldsymbol{\lambda})^2 \neq m^2. \tag{1.15}$$

We also assume that for $\mathbf{m} = (\mathbf{m}_+, \mathbf{m}_-) \in \mathbb{N}_0^{2N}$ then

$$\|\mathbf{m}\| \le 2M \text{ and } \mathbf{m} \cdot \boldsymbol{\lambda} = 0 \Longrightarrow \mathbf{m}_+ = \mathbf{m}_-. \tag{1.16}$$

Lemma 1.4. *The following facts hold.*

1. *If* $\|\mathbf{m}\| > M$*, with* M *the constant in Assumption 1.3, then* $\mathbf{m} \in \mathbf{I}$.
2. $\mathbf{R}_{\min}$ *and* $\mathbf{NR}$ *are finite sets.*
3. *If* $\mathbf{m} \in \mathbf{NR}$*, then* $|\boldsymbol{\lambda} \cdot \mathbf{m}| < m$ *and if* $\mathbf{m} \in \mathbf{R}_{\min}$*, then* $\mathbf{m}_+ = 0$ *or* $\mathbf{m}_- = 0$.
4. *If* $\mathbf{m} \in \mathbf{\Lambda}_j$ *then there is a* $\mathbf{n} \in \mathbf{\Lambda}_0$ *with* $\mathbf{m} = \mathbf{e}^j + \mathbf{n}$.

Proof. The proof is taken from [5]. If $\|\mathbf{m}\| > M$, we can write $\mathbf{m} = \boldsymbol{\alpha} + \boldsymbol{\beta}$ with $\|\boldsymbol{\alpha}\| = M$. If $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_+, \boldsymbol{\alpha}_-)$ and if we set $\mathbf{n} = (\mathbf{n}_+, \mathbf{n}_-)$ with $\mathbf{n}_+ = \boldsymbol{\alpha}_+ + \boldsymbol{\alpha}_-$ and $\mathbf{n}_- = 0$, then $\mathbf{n} \cdot \boldsymbol{\lambda} \geq M\lambda_1 > m$. This implies that $\mathbf{n} \in \mathbf{R}$ and that there exists $\mathfrak{a} \in \mathbf{R}_{\min}$ with $\mathfrak{a} \preceq \mathbf{n}$. From $\|\boldsymbol{\beta}\| \geq 1$ it follows that $\mathfrak{a} \prec \mathbf{m}$ and so $\mathbf{m} \in \mathbf{I}$.

Obviously, from the 1st claim it follows that if $\mathbf{m} \in \mathbf{R}_{\min} \cup \mathbf{NR}$ then $\|\mathbf{m}\| \leq M$. Next we observe that $\mathbf{m} \in \mathbf{NR}$ implies $\|\mathbf{m}\| \leq M$ and $|\boldsymbol{\lambda} \cdot \mathbf{m}| \leq m$ and, by Assumption 1.3, $|\boldsymbol{\lambda} \cdot \mathbf{m}| < m$. If $\mathbf{m} \in \mathbf{R}_{\min}$ with, say, $\mathbf{m} \cdot \boldsymbol{\lambda} > m$, then obviously we have $\mathbf{m}_+ \cdot \boldsymbol{\lambda} > m$ and it is elementary that $\mathbf{m} = (\mathbf{m}_+, 0)$. Finally, from the first claim we know that if $\mathbf{m} \in \mathbf{\Lambda}_j$ then $\|\mathbf{m}\| \leq M$. From $\mathbf{m} \cdot \boldsymbol{\lambda} - \lambda_j = 0$ it follows from (1.16) that we have the last claim. $\square$

For $\mathbf{z} = (z_1, \cdots, z_N) \in \mathbb{C}^N$ and $\mathbf{m} \in (\mathbb{N} \cup \{0\})^{2N}$, we set

$$\mathbf{z}^{\mathbf{m}} = \prod_{j=1}^{N} z_j^{m_{+,j}} \bar{z}_j^{m_{-,j}}.$$

Notice that we have $\overline{\mathbf{z}^{\mathbf{m}}} = \mathbf{z}^{\overline{\mathbf{m}}}$.

Notice that $\sum_{j=1}^{N} \left(z_j \mathbf{\Phi}_j + c.c.\right)$, satisfies (1.3) up to $O(\|\mathbf{z}\|^2)$ error if $\dot{z}_j = \mathrm{i}\lambda_j z_j$. The *refined profile* is a generalization of this kind of approximate solution of (1.3).

Remark 1.5. Since we think $\mathbf{z} = \mathbf{z}(t)$ (defined as a coordinate, see Lemma 2.1) approximately satisfies $\dot{z}_j = \mathrm{i}\lambda_j z_j + O(\|\mathbf{z}\|^2)$, which will be justified in some sense (see Proposition 2.3), a term like $\mathbf{z}^{\mathbf{m}}\mathbf{g}$ is considered to have a frequency $\mathbf{m} \cdot \boldsymbol{\lambda}$ because $\dot{\mathbf{z}}^{\mathbf{m}} = \mathrm{i}(\mathbf{m} \cdot \boldsymbol{\lambda})\mathbf{z}^{\mathbf{m}} + O(\|\mathbf{z}\|^{\|\mathbf{m}\|+1})$. On the other hand, since the essential spectrum of $\mathbf{JL}_1$ is $\mathrm{i}\left((-\infty, -m] \cup [m, \infty)\right)$, the frequency of $\mathbf{z}^{\mathbf{m}}$ resonates with the essential spectral component of $\mathbf{JL}_1$ if $|\mathbf{m} \cdot \boldsymbol{\lambda}| > m$. The notation $\mathbf{R}$ comes from this intuition (and $\mathbf{R}$ for resonant), and the same is true for $\mathbf{NR}$ (for non resonant). Next, since we will be only considering small solutions, $\mathbf{z}$ will also be small, see (2.3). Thus, we will only have to take care of the resonant terms with "minimal resonant frequencies" $\mathbf{R}_{\min}$ and we will ignore (thus the notation I), in the same they are small remainders, the terms of form $\mathbf{z}^{\mathbf{m}}\mathbf{g}$ with $\mathbf{m} \in \mathbf{I}$.

We set $\|\cdot\|_{\Sigma^s} := \|\cdot\|_{H^s_{a_2}} := \|e^{a_2\langle x\rangle} \cdot\|_{H^s}$ where $a_2 = \frac{1}{2}\sqrt{m^2 - \lambda_N^2}$ and denote by Σ^s the corresponding spaces. We set

$$\|\mathbf{u}\|^2_{\mathbf{\Sigma}^l} := \|u_1\|^2_{\Sigma^l} + \|u_2\|^2_{\Sigma^l}.$$

Let $\mathbf{\Sigma}^\infty = \cap_{l \in \mathbb{R}} \mathbf{\Sigma}^l$.

Proposition 1.6. *There exist* $\{\boldsymbol{\phi}_{\mathbf{m}}\}_{\mathbf{m}\in\mathbf{NR}}$ *in* $\boldsymbol{\Sigma}^\infty$, $\{\mathbf{G}_{\mathbf{m}}\}_{\mathbf{m}\in\mathbf{R}_{\min}}\subset\boldsymbol{\Sigma}^\infty$, $\{\lambda_{\mathbf{n}j}\}_{\mathbf{n}\in\Lambda_0\cup\{\mathbf{0}\}}\subset\mathbb{R}$ *for* $j=1,\cdots,N$ *with* $\boldsymbol{\phi}_{\mathbf{e}^j}=\boldsymbol{\Phi}_j$ *and* $\lambda_{\mathbf{0}j}=\lambda_j$, *a* $\delta_1>0$ *s.t. there exists* $\widetilde{\mathbf{z}}_2\in C^\infty(B_{\mathbb{C}^N}(0,\delta_1),\mathbb{C}^N)$ *satisfying*

$$\|\widetilde{\mathbf{z}}_2\|_{\mathbb{C}^N}\lesssim\sum_{\mathbf{m}\in\mathbf{R}_{\min}}|\mathbf{z}^{\mathbf{m}}|, \tag{1.17}$$

s.t. for any l

$$\|\mathbf{R}[\mathbf{z}]\|_{\boldsymbol{\Sigma}^l}\lesssim_l\|\mathbf{z}\|_{\mathbb{C}^N}\sum_{\mathbf{m}\in\mathbf{R}_{\min}}|\mathbf{z}^{\mathbf{m}}|, \tag{1.18}$$

where $\mathbf{R}[\mathbf{z}]$ *is defined by the equality*

$$D\boldsymbol{\phi}[\mathbf{z}]\widetilde{\mathbf{z}}=\mathbf{J}\left(\mathbf{L}_1\boldsymbol{\phi}[\mathbf{z}]+\mathbf{f}[\boldsymbol{\phi}[\mathbf{z}]]-\sum_{\mathbf{m}\in\mathbf{R}_{\min}}\mathbf{z}^{\mathbf{m}}\mathbf{G}_{\mathbf{m}}-\mathbf{R}[\mathbf{z}]\right) \tag{1.19}$$

(where (1.18) *and* (1.19) *define the* $\mathbf{G}_{\mathbf{m}}$*) and*

$$\boldsymbol{\phi}[\mathbf{z}]:=\begin{pmatrix}\phi_1[\mathbf{z}]\\ \phi_2[\mathbf{z}]\end{pmatrix}=\sum_{\mathbf{m}\in\mathbf{NR}}\mathbf{z}^{\mathbf{m}}\boldsymbol{\phi}_{\mathbf{m}}, \tag{1.20}$$

$$\boldsymbol{\phi}_{\overline{\mathbf{m}}}=\overline{\boldsymbol{\phi}_{\mathbf{m}}} \tag{1.21}$$

$$\widetilde{\mathbf{z}}=\widetilde{\mathbf{z}}_0+\widetilde{\mathbf{z}}_1+\widetilde{\mathbf{z}}_2 \text{ with} \tag{1.22}$$

$$\widetilde{\mathbf{z}}_0=(\mathrm{i}\lambda_1 z_1,\dots,\mathrm{i}\lambda_N z_N)=:\mathrm{i}\boldsymbol{\lambda}\mathbf{z}, \tag{1.23}$$

$$\widetilde{\mathbf{z}}_1=(\mathrm{i}\sum_{\mathbf{n}\in\Lambda_0}\lambda_{\mathbf{n}1}\mathbf{z}^{\mathbf{n}}z_1,\dots,\mathrm{i}\sum_{\mathbf{m}\in\Lambda_0}\lambda_{\mathbf{n}N}\mathbf{z}^{\mathbf{n}}z_N), \tag{1.24}$$

$$\boldsymbol{\lambda}_{\overline{\mathbf{m}}}=\boldsymbol{\lambda}_{\mathbf{m}}\in\mathbb{R}^{2N} \tag{1.25}$$

where $\boldsymbol{\lambda}_{\mathbf{m}}:=(\lambda_{\mathbf{m}1},\dots,\lambda_{\mathbf{m}N},-\lambda_{\mathbf{m}1},\dots,-\lambda_{\mathbf{m}N})$, *such that, setting*

$$\mathcal{H}_{\mathrm{c}}[\mathbf{z}]:=\{\mathbf{u}\in\mathcal{H}^1\mid\Omega(\mathbf{u},D_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}]\mathbf{w})=0\text{ for all }\mathbf{w}\in\mathbb{C}^N\} \tag{1.26}$$

and

$$\widetilde{\mathbf{R}}[\mathbf{z}]=\sum_{\mathbf{m}\in\mathbf{R}_{\min}}\mathbf{z}^{\mathbf{m}}\mathbf{G}_{\mathbf{m}}+\mathbf{R}[\mathbf{z}], \tag{1.27}$$

we have

$$\mathbf{J}\widetilde{\mathbf{R}}[\mathbf{z}]\in\mathcal{H}_{\mathrm{c}}[\mathbf{z}]. \tag{1.28}$$

Proof. We begin observing that $\mathbf{J}\mathbf{L}_1$ leaves the following decomposition invariant,

$$L^2(\mathbb{R},\mathbb{C}^2)=L^2_{discr}\oplus L^2_{disp} \text{ where } L^2_{discr}:=\oplus_{\lambda\in\sigma_p(\mathbf{J}\mathbf{L}_1)}\ker(\mathbf{L}_1-\lambda)\,, \tag{1.29}$$

where L^2_{disp} is the $\langle \mathbf{J}\cdot,\cdot\rangle$–orthogonal of L^2_{discr}.
We insert (1.20) in (1.19), using (1.22)–(1.24). We expand

$$f(\phi_1[\mathbf{z}])=\sum_{\ell=2}^{M}\frac{f^{(\ell)}(0)}{\ell!}\phi_1^{\ell}[\mathbf{z}]+O(\|\mathbf{z}\|^{M+1}).$$

Then, for $\mathbf{i}={}^t(1,0)$,

$$\sum_{\ell=2}^{M}\frac{f^{(\ell)}(0)}{\ell!}\phi_1^{\ell}[\mathbf{z}]\,\mathbf{i}=\sum_{\mathbf{m}\in\mathbf{NR}}\mathbf{z}^{\mathbf{m}}\mathbf{h}_{\mathbf{m}}+\sum_{\substack{\mathbf{m}\in\mathbf{R}\cup\mathbf{I}\\ |\mathbf{m}|\le M}}\mathbf{z}^{\mathbf{m}}\mathbf{h}_{\mathbf{m}}+O(\|\mathbf{z}\|^{M+1})$$

where, for $\boldsymbol{\phi}_{\mathbf{m}}={}^t(\phi_{1\mathbf{m}},\phi_{2\mathbf{m}})$,

$$\mathbf{h}_{\mathbf{m}}=\sum_{\ell=2}^{M}\frac{f^{(\ell)}(0)}{\ell!}\sum_{\substack{\mathbf{m}^1,\cdots,\mathbf{m}^{\ell}\in\mathbf{NR}\\ \mathbf{m}^1+\cdots+\mathbf{m}^{\ell}=\mathbf{m}}}\phi_{1\mathbf{m}^1}\cdots\phi_{1\mathbf{m}^{\ell}}\,\mathbf{i}. \tag{1.30}$$

Using

$$\left(D_{\mathbf{z}}\mathbf{z}^{\mathbf{m}}\right)\widetilde{\mathbf{z}}_0=\mathrm{i}(\mathbf{m}\cdot\boldsymbol{\lambda})\,\mathbf{z}^{\mathbf{m}},\ \text{ where } \boldsymbol{\lambda}\mathbf{z}:=(\lambda_1 z_1,...,\lambda_N z_N), \tag{1.31}$$

and recalling (1.22), we obtain

$$D_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}]\widetilde{\mathbf{z}}[\mathbf{z}]=\mathrm{i}\sum_{\mathbf{m}\in\mathbf{NR}}(\mathbf{m}\cdot\boldsymbol{\lambda})\mathbf{z}^{\mathbf{m}}\boldsymbol{\phi}_{\mathbf{m}}+\mathrm{i}\sum_{\mathbf{m}\in\mathbf{NR},\ \mathbf{n}\in\boldsymbol{\Lambda}_0}(\mathbf{m}\cdot\boldsymbol{\lambda}_{\mathbf{n}})\mathbf{z}^{\mathbf{n}}\mathbf{z}^{\mathbf{m}}\boldsymbol{\phi}_{\mathbf{m}}+D_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}]\widetilde{\mathbf{z}}_2.$$

Let us set

$$\begin{aligned}\boldsymbol{\mathcal{R}}[\mathbf{z}]&:=\mathbf{J}\left(\mathbf{L}_1\boldsymbol{\phi}[\mathbf{z}]+\mathbf{f}[\boldsymbol{\phi}[\mathbf{z}]]\right)-D_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}](\widetilde{\mathbf{z}}-\widetilde{\mathbf{z}}_2)\\ &=\mathbf{J}\begin{pmatrix}L_1\phi_1[\mathbf{z}]+f(\phi_1[\mathbf{z}])\\ \phi_2[\mathbf{z}]\end{pmatrix}-D_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}](\widetilde{\mathbf{z}}-\widetilde{\mathbf{z}}_2).\end{aligned}$$

We expand now to get

$$\boldsymbol{\mathcal{R}}[\mathbf{z}]=\sum_{\mathbf{m}\in\mathbf{NR}}\mathbf{z}^{\mathbf{m}}\mathcal{R}_{\mathbf{m}}+\sum_{\substack{\mathbf{m}\in\mathbf{R}\cup\mathbf{I}\\ |\mathbf{m}|\le M}}\mathbf{z}^{\mathbf{m}}\mathcal{R}_{\mathbf{m}}+O(\|\mathbf{z}\|^{M+1}), \tag{1.32}$$

where

$$\mathcal{R}_{\mathbf{m}} = (\mathbf{J}\mathbf{L}_1 - \mathrm{i}\boldsymbol{\lambda}\cdot\mathbf{m})\,\boldsymbol{\phi}_{\mathbf{m}} + \mathcal{E}_{\mathbf{m}} \text{ where}$$
$$\mathcal{E}_{\mathbf{m}} = \mathbf{J}\mathbf{h}_{\mathbf{m}} - \sum_{\substack{\mathbf{m}'+\mathbf{n}'=\mathbf{m}\\ \mathbf{m}'\in\mathbf{NR},\ \mathbf{n}'\in\boldsymbol{\Lambda}_0}} \mathrm{i}(\boldsymbol{\lambda}_{\mathbf{n}'}\cdot\mathbf{m}')\boldsymbol{\phi}_{\mathbf{m}'}.$$

We seek $\mathcal{R}_{\mathbf{m}} \equiv 0$ for $\mathbf{m} \in \mathbf{NR}$. For $\|\mathbf{m}\| = 1$ the equation reduces to $(\mathbf{J}\mathbf{L}_1 - \mathrm{i}\boldsymbol{\lambda}\cdot\mathbf{m})\,\boldsymbol{\phi}_{\mathbf{m}} = 0$, so that we can set $\boldsymbol{\phi}_{\mathbf{e}^j} = \boldsymbol{\Phi}_j$ and $\boldsymbol{\phi}_{\overline{\mathbf{e}}^j} = \overline{\boldsymbol{\Phi}}_j$. Let us consider now $\|\mathbf{m}\| \geq 2$ with $\mathbf{m} \notin \cup_{j=1}^{N}\left(\boldsymbol{\Lambda}_j \cup \overline{\boldsymbol{\Lambda}}_j\right)$. In this case, let us assume by induction that $\boldsymbol{\phi}_{\mathbf{m}'}$ and $\boldsymbol{\lambda}_{\mathbf{m}'}$ have been defined for $\|\mathbf{m}'\| < \|\mathbf{m}\|$ and that they satisfy (1.21)–(1.25). Then, from (1.30) we obtain $h_{\overline{\mathbf{m}}} = \overline{h}_{\mathbf{m}}$ and $\mathcal{E}_{\overline{\mathbf{m}}} = \overline{\mathcal{E}}_{\mathbf{m}}$. We can solve $\mathcal{R}_{\mathbf{m}} = 0$ writing $\boldsymbol{\phi}_{\mathbf{m}} = (\mathbf{J}\mathbf{L}_1 - \mathrm{i}\boldsymbol{\lambda}\cdot\mathbf{m})^{-1}\,\mathcal{E}_{\mathbf{m}}$. By $\boldsymbol{\lambda}\cdot\overline{\mathbf{m}} = -\boldsymbol{\lambda}\cdot\mathbf{m}$, we conclude $\boldsymbol{\phi}_{\overline{\mathbf{m}}} = \overline{\boldsymbol{\phi}}_{\mathbf{m}}$.

We now consider $\mathbf{m} \in \boldsymbol{\Lambda}_j$. We assume by induction that $\boldsymbol{\phi}_{\mathbf{m}'}$ have been defined for $\|\mathbf{m}'\| < \|\mathbf{m}\|$ and so too $\boldsymbol{\lambda}_{\mathbf{n}'}$ for $\|\mathbf{n}'\| < \|\mathbf{m}\| - 1$. Then, for $\mathbf{m} = \mathbf{n} + \mathbf{e}^j$ where $\mathbf{n} \in \boldsymbol{\Lambda}_0$, $\mathcal{R}_{\mathbf{m}} = 0$ becomes

$$\begin{aligned}&\left(\mathbf{J}\mathbf{L}_1 - \mathrm{i}\lambda_j\right)\boldsymbol{\phi}_{\mathbf{m}} = \mathcal{E}_{\mathbf{m}} = \mathrm{i}\boldsymbol{\lambda}_{\mathbf{n}}\cdot\mathbf{e}^j\,\boldsymbol{\Phi}_j - \mathcal{K}_{\mathbf{m}} \text{ with}\\ &\mathcal{K}_{\mathbf{m}} := \mathbf{J}h_{\mathbf{m}} - \sum_{\substack{\mathbf{m}'+\mathbf{n}'=\mathbf{m}\\ \mathbf{m}'\in\mathbf{NR},|\mathbf{m}'|\geq 2,\ \mathbf{n}'\in\boldsymbol{\Lambda}_0}} \mathrm{i}\boldsymbol{\lambda}_{\mathbf{n}'}\cdot\mathbf{m}'\boldsymbol{\phi}_{\mathbf{m}'}.\end{aligned} \tag{1.33}$$

This equation can be solved if we impose $\left(\mathbf{J}\mathcal{E}_{\mathbf{m}}, \overline{\boldsymbol{\Phi}}_j\right) = 0$, that is, for $\lambda_{\mathbf{n}j} := \boldsymbol{\lambda}_{\mathbf{n}}\cdot\mathbf{e}^j$, if

$$-\mathrm{i}\lambda_{\mathbf{n}j}\left(\mathbf{J}\boldsymbol{\Phi}_j, \overline{\boldsymbol{\Phi}}_j\right) = -2\lambda_{\mathbf{n}j}\lambda_j = \left(\mathbf{J}\mathcal{K}_{\mathbf{m}}, \overline{\boldsymbol{\Phi}}_j\right),$$

which is true for $\lambda_{\mathbf{n}j} = -2^{-1}\lambda_j^{-1}\left(\mathbf{J}\mathcal{K}_{\mathbf{m}}, \overline{\boldsymbol{\Phi}}_j\right)$. Then we can solve for $\boldsymbol{\phi}_{\mathbf{m}} = -\left(\mathbf{J}\mathbf{L}_1 - \mathrm{i}\lambda_j\right)^{-1}\mathcal{K}_{\mathbf{m}}$ in the complement, in (1.29), of $\ker(\mathbf{J}\mathbf{L}_1 - \mathrm{i}\lambda_j)$.
We want to show that $\lambda_{\mathbf{n}j} \in \mathbb{R}$. For the corresponding $\overline{\mathbf{m}} \in \overline{\boldsymbol{\Lambda}}_j$, we have

$$\begin{aligned}&\left(\mathbf{J}\mathbf{L}_1 + \mathrm{i}\lambda_j\right)\boldsymbol{\phi}_{\overline{\mathbf{m}}} = \mathrm{i}\boldsymbol{\lambda}_{\mathbf{n}}\cdot\overline{\mathbf{e}}^j\overline{\boldsymbol{\Phi}}_j - \mathcal{K}_{\overline{\mathbf{m}}} \text{ with}\\ &\mathcal{K}_{\overline{\mathbf{m}}} := \mathbf{J}h_{\overline{\mathbf{m}}} - \sum_{\substack{\overline{\mathbf{m}}'+\mathbf{n}'=\overline{\mathbf{m}}\\ \mathbf{m}'\in\mathbf{NR}_2,\ \mathbf{n}'\in\boldsymbol{\Lambda}_0}} \mathrm{i}\boldsymbol{\lambda}_{\mathbf{n}'}\cdot\overline{\mathbf{m}}'\boldsymbol{\phi}_{\overline{\mathbf{m}}'}.\end{aligned} \tag{1.34}$$

Notice that by induction $\mathcal{K}_{\overline{\mathbf{m}}} = \overline{\mathcal{K}}_{\mathbf{m}}$. Since $\boldsymbol{\lambda}_{\mathbf{n}}\cdot\overline{\mathbf{e}}^j = -\lambda_{\mathbf{n}j}$, taking the complex conjugate of (1.33) we obtain

$$\begin{aligned}&\left(\mathbf{J}\mathbf{L}_1 + \mathrm{i}\lambda_j\right)\boldsymbol{\phi}_{\overline{\mathbf{m}}} = \mathrm{i}\lambda_{\mathbf{n}j}\overline{\boldsymbol{\Phi}}_j - \overline{\mathcal{K}}_{\mathbf{m}} \text{ and}\\ &\left(\mathbf{J}\mathbf{L}_1 + \mathrm{i}\lambda_j\right)\overline{\boldsymbol{\phi}}_{\mathbf{m}} = \mathrm{i}\overline{\lambda}_{\mathbf{n}j}\overline{\boldsymbol{\Phi}}_j - \overline{\mathcal{K}}_{\mathbf{m}}.\end{aligned} \tag{1.35}$$

Applying $\left(\mathbf{J}\cdot, \boldsymbol{\Phi}_j\right)$ on both the last two equations, we obtain

$$\mathrm{i}\lambda_{\mathbf{n}j}\left(\mathbf{J}\overline{\boldsymbol{\Phi}}_j, \boldsymbol{\Phi}_j\right) = \left(\mathbf{J}\overline{\mathcal{K}}_{\mathbf{m}}, \boldsymbol{\Phi}_j\right) \text{ and } \mathrm{i}\overline{\lambda}_{\mathbf{n}j}\left(\mathbf{J}\overline{\boldsymbol{\Phi}}_j, \boldsymbol{\Phi}_j\right) = \left(\mathbf{J}\overline{\mathcal{K}}_{\mathbf{m}}, \boldsymbol{\Phi}_j\right).$$

Hence $\lambda_{\mathbf{n}j} = \overline{\lambda}_{\mathbf{n}j}$ and we have proved that $\lambda_{\mathbf{n}j} \in \mathbb{R}$.
Since the equations in (1.35) are the same, we conclude $\boldsymbol{\phi}_{\overline{\mathbf{m}}} = \overline{\boldsymbol{\phi}}_{\mathbf{m}}$.

We consider now

$$\mathbf{J}\widetilde{\mathbf{R}}[\mathbf{z}] = \mathcal{R}[\mathbf{z}] - D_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}]\widetilde{\mathbf{z}}_2, \tag{1.36}$$

where we seek $\widetilde{\mathbf{z}}_2$ so that (1.28) is true. This will follow from (here $\mathbf{J}^{-1} = -\mathbf{J}$)

$\langle \mathbf{J}\mathcal{R}[\mathbf{z}], D_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}]\mathbf{w}\rangle - \langle \mathbf{J}D_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}]\widetilde{\mathbf{z}}_2, D_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}]\mathbf{w}\rangle = 0$ for the standard basis $\mathbf{w} = e_1, ie_1, ..., e_N, ie_N$.

Since the restriction of $\langle \mathbf{J}\cdot,\cdot\rangle$ in L^2_{discr} is a non–degenerate symplectic form and from $\boldsymbol{\phi}_{\mathbf{e}^j} = \boldsymbol{\Phi}_j$ and $\boldsymbol{\phi}_{\overline{\mathbf{e}}^j} = \overline{\boldsymbol{\Phi}}_j$, the Implicit Function Theorem guarantees the existence of $\widetilde{\mathbf{z}}_2 \in C^\infty(\mathcal{B}_{\mathbb{C}^N}(0,\delta_1), \mathbb{C}^N)$ with $\widetilde{\mathbf{z}}_2(\mathbf{0}) = \mathbf{0}$ for a sufficiently small $\delta_1 > 0$. Furthermore, from the last formula and from the fact that in the expansion (1.32) we have $\mathcal{R}_{\mathbf{m}} = 0$ for all $\mathbf{m} \in \mathbf{NR}$, we obtain the bound (1.18).
Solving in (1.36) for $\widetilde{\mathbf{R}}[\mathbf{z}] = \mathbf{J}^{-1}\mathcal{R}[\mathbf{z}] - \mathbf{J}^{-1}D_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}]\widetilde{\mathbf{z}}_2$, exploiting the fact that we have $\mathcal{R}_{\mathbf{m}}$ for all $\mathbf{m} \in \mathbf{NR}$ and by (1.17), by Taylor expansion in the variable $\mathbf{z}$, we obtain expansion (1.27), with the estimate (1.18). □

We assume the following.

Assumption 1.7 *(Fermi Golden Rule).* For any $\mathbf{m} \in \mathbf{R}_{\min}$, there exists a bounded solution $\mathbf{g}_{\mathbf{m}}$ of $\mathbf{J}\mathbf{L}_1\mathbf{g}_{\mathbf{m}} = \mathrm{i}(\mathbf{m}\cdot\boldsymbol{\lambda})\mathbf{g}_{\mathbf{m}}$ s.t.

$$\langle \mathbf{G}_{\mathbf{m}}, \mathbf{g}_{\mathbf{m}}\rangle = \gamma_{\mathbf{m}} > 0. \tag{1.37}$$

Remark 1.8. Notice that all it matters in (1.37) is to have $\gamma_{\mathbf{m}} \neq 0$, since by replacing $\mathbf{g}_{\mathbf{m}}$ with $-\mathbf{g}_{\mathbf{m}}$, we can then obtain $\gamma_{\mathbf{m}} > 0$.

We give several examples of refined profiles.

Example 1.9 *(Refined profiles in the case $M = 2$).* We consider the case $\lambda_1, \cdots, \lambda_N \in (m/2, m)$. In this case, we have

$$\mathbf{R}_{\min} = \{\mathbf{e}^j + \mathbf{e}^k, \overline{\mathbf{e}^j + \mathbf{e}^k} \mid j,k = 1,\cdots,N\},$$
$$\mathbf{NR} = \{0, \mathbf{e}^j, \overline{\mathbf{e}^j}, \mathbf{e}^j + \overline{\mathbf{e}^k} \mid j = 1,\cdots,N\},$$

and the corresponding refined profile will have the form

$$\boldsymbol{\phi}[\mathbf{z}] = \sum_{j=1}^N \left(z_j\boldsymbol{\Phi}_j + \overline{z_j\boldsymbol{\Phi}_j}\right) + \sum_{j,k=1}^N z_j\overline{z_k}\boldsymbol{\phi}_{\mathbf{e}^j + \overline{\mathbf{e}^k}}.$$

Here, notice that even in the simplest case $N = 1$ and $M = 2$, we need a correction term $|z_1|^2\boldsymbol{\phi}_{\mathbf{e}^1+\overline{\mathbf{e}^1}}$.

Following the computation of the proof of Proposition 1.6, we will have

$$\boldsymbol{\phi}_{\mathbf{e}^j+\overline{\mathbf{e}^k}} = \begin{cases} \begin{pmatrix} -f''(0)L_1^{-1}\left(\phi_j^2\right) \\ 0 \end{pmatrix} & j=k, \\ -\frac{1}{2}f''(0)(L_1-(\lambda_j-\lambda_k)^2)^{-1}(\phi_j\phi_k)\begin{pmatrix} 1 \\ \mathrm{i}(\lambda_j-\lambda_k) \end{pmatrix} & j\neq k, \end{cases}$$

$$\mathbf{G}_{\mathbf{e}^j+\mathbf{e}^k} = \begin{pmatrix} \frac{1}{2}f''(0)\phi_j\phi_k \\ 0 \end{pmatrix}, \ \mathbf{G}_{\overline{\mathbf{e}^j+\mathbf{e}^k}} = \overline{\mathbf{G}_{\mathbf{e}^j+\mathbf{e}^k}}.$$

Example 1.10 *(Refined profiles in the case $M=3$).* We consider the case $N=2$ with $\lambda_1 \in (m/3, m/2)$, $\lambda_2 \in (m/2, m)$ and $\lambda_1+\lambda_2 > m$. In this case, we have

$$\mathbf{R}_{\min} = \{3\mathbf{e}^1, 3\overline{\mathbf{e}^1},\ \mathbf{e}^1+\mathbf{e}^2,\ \overline{\mathbf{e}^1+\mathbf{e}^2},\ 2\mathbf{e}^2,\ 2\overline{\mathbf{e}^2}\},$$
$$\mathbf{NR} = \{0, \mathbf{e}^1, \overline{\mathbf{e}^1}, \mathbf{e}^2, \overline{\mathbf{e}^2}, 2\mathbf{e}^1, 2\overline{\mathbf{e}^1}, \mathbf{e}^1+\overline{\mathbf{e}^1}, \mathbf{e}^2+\overline{\mathbf{e}^2}, \overline{\mathbf{e}^1}+\mathbf{e}^2, \mathbf{e}^1+\overline{\mathbf{e}^2}, 2\mathbf{e}^1+\overline{\mathbf{e}^1}, \mathbf{e}^1+2\overline{\mathbf{e}^1}\}.$$

The refined profile is given by

$$\begin{aligned}\boldsymbol{\phi}[z_1,z_2] =& z_1\boldsymbol{\Phi}_1 + \overline{z_1\boldsymbol{\Phi}_1} + z_2\boldsymbol{\Phi}_2 + \overline{z_2\boldsymbol{\Phi}_2} + |z_1|^2\boldsymbol{\phi}_{\mathbf{e}^1+\overline{\mathbf{e}^1}} + |z_2|^2\boldsymbol{\phi}_{\mathbf{e}^2+\overline{\mathbf{e}^2}} + z_1^2\boldsymbol{\phi}_{2\mathbf{e}^1} + \overline{z_1}^2\boldsymbol{\phi}_{2\overline{\mathbf{e}^1}} \\ &+ \overline{z_1}z_2\boldsymbol{\phi}_{\overline{\mathbf{e}^1}+\mathbf{e}^2} + z_1\overline{z_2}\boldsymbol{\phi}_{\mathbf{e}^1+\overline{\mathbf{e}^2}} + z_1|z_1|^2\boldsymbol{\phi}_{2\mathbf{e}^1+\overline{\mathbf{e}^1}} + \overline{z_1}|z_1|^2\boldsymbol{\phi}_{\mathbf{e}^1+2\overline{\mathbf{e}^1}},\end{aligned}$$

where

$$\begin{aligned}\boldsymbol{\phi}_{\mathbf{e}^j+\overline{\mathbf{e}^j}} &= \begin{pmatrix} -f''(0)L_1^{-1}\left(\phi_j^2\right) \\ 0 \end{pmatrix}, \quad j=1,2, \\ \boldsymbol{\phi}_{2\mathbf{e}^1} &= -\frac{1}{2}f''(0)(L_1-4\lambda_1^2)^{-1}(\phi_1^2)\begin{pmatrix} 1 \\ 2\mathrm{i}\lambda_1 \end{pmatrix}, \\ \boldsymbol{\phi}_{\overline{\mathbf{e}^1}+\mathbf{e}^2} &= -\frac{1}{2}f''(0)(L_1-(\lambda_2-\lambda_1)^2)^{-1}(\phi_1\phi_2)\begin{pmatrix} 1 \\ \mathrm{i}(\lambda_2-\lambda_1) \end{pmatrix}, \\ \boldsymbol{\phi}_{2\mathbf{e}^1+\overline{\mathbf{e}^1}} &= -\left.(L_1-\lambda_1^2)\right|_{\{\phi_1\}^\perp}^{-1}\left(P_1^\perp\left(f''(0)\phi_1\left(\phi_{1,\mathbf{e}^1+\overline{\mathbf{e}^1}}+\phi_{1,2\mathbf{e}^1}\right)+\frac{1}{2}f'''(0)\phi_1^3\right)\right)\begin{pmatrix} 1 \\ \mathrm{i}\lambda_1 \end{pmatrix} \\ &\quad + \frac{\mathrm{i}}{2\lambda_1}\int_{\mathbb{R}}\left(f''(0)\phi_1^2\left(\phi_{1,\mathbf{e}^1+\overline{\mathbf{e}^1}}+\phi_{1,2\mathbf{e}^1}\right)+\frac{1}{2}f'''(0)\phi_1^4\right)dx\begin{pmatrix} 0 \\ \phi_1 \end{pmatrix},\end{aligned}$$

and

$$\boldsymbol{\phi}_{2\overline{\mathbf{e}^1}} = \overline{\boldsymbol{\phi}_{2\mathbf{e}^1}},\ \boldsymbol{\phi}_{\mathbf{e}^1+\overline{\mathbf{e}^2}} = \overline{\boldsymbol{\phi}_{\overline{\mathbf{e}^1}+\mathbf{e}^2}},\ \boldsymbol{\phi}_{\mathbf{e}^1+2\overline{\mathbf{e}^1}} = \overline{\boldsymbol{\phi}_{2\mathbf{e}^1+\overline{\mathbf{e}^1}}}.$$

The main terms $\sum_{\mathbf{m}\in\mathbf{R}_{\min}}\mathbf{z}^{\mathbf{m}}\mathbf{G}_{\mathbf{m}}$ in the remainder are given by

$$\mathbf{G}_{3\mathbf{e}^1} = \begin{pmatrix} f''(0)\phi_1\phi_{1,2\mathbf{e}^1} + \frac{1}{6}f'''(0)\phi_1^3 \\ 0 \end{pmatrix},$$

$$\mathbf{G}_{\mathbf{e}^1+\mathbf{e}^2} = \begin{pmatrix} \frac{1}{2}f''(0)\phi_1\phi_2 \\ 0 \end{pmatrix},$$

$$\mathbf{G}_{2\mathbf{e}^2} = \begin{pmatrix} \frac{1}{2}f''(0)\phi_2^2 \\ 0 \end{pmatrix},$$

and

$$\mathbf{G}_{3\overline{\mathbf{e}^1}} = \overline{\mathbf{G}_{3\mathbf{e}^1}},\ \mathbf{G}_{\overline{\mathbf{e}^1}+\overline{\mathbf{e}^2}} = \overline{\mathbf{G}_{\mathbf{e}^1+\mathbf{e}^2}},\ \mathbf{G}_{2\overline{\mathbf{e}^2}} = \overline{\mathbf{G}_{2\mathbf{e}^2}}. \quad \square$$

The distinguishing feature of this paper is that we deal with an L_1 with a general configuration of eigenvalues. We will use the spectral decomposition

$$L^2(\mathbb{R}, \mathbb{C}) = \left(\oplus_{j=1}^N \ker\left(L_1 - \lambda_j^2\right)\right) \oplus L_c^2(L_1), \tag{1.38}$$

where $L_c^2(L_1)$ is the continuous spectrum component associated to L_1. We denote by P_c the orthogonal projection onto $L_c^2(L_1)$. We adopt the dispersion theory of Kowalczyk et al. [16–20], based on the following. First of all we recall, from Sect. 3 in Deift-Trubowitz [10], the following result on Darboux transformations (the A_W and A_W^* in the proposition), here stated less generally than in [10].

Proposition 1.11. *Let $W \in \mathcal{S}(\mathbb{R}, \mathbb{R})$ s.t. $\sigma_{\mathrm{d}}(-\partial_x^2 + W) \neq \emptyset$ and let $\omega = \inf \sigma_{\mathrm{d}}(-\partial_x^2 + W)$. Let ψ be a ground state of $-\partial_x^2 + W$, that is a generator of* $\ker\left(-\partial_x^2 + W - \omega\right)$*, and set $A_W = \frac{1}{\psi}\partial_x(\psi \cdot)$ (recall that $\psi(x) \neq 0$ for all $x \in \mathbb{R}$). Then, there exists $W_1 \in \mathcal{S}(\mathbb{R}, \mathbb{R})$ s.t.*

$$A_W A_W^* = -\partial_x^2 + W - \omega,\ A_W^* A_W = -\partial_x^2 + W_1 - \omega$$

and $\sigma_{\mathrm{d}}(-\partial_x^2 + W_1) = \sigma_{\mathrm{d}}(-\partial_x^2 + W) \setminus \{\omega\}$. $\square$

Armed with Proposition 1.11, following Kowalczyk et al., we inductively define $V_j \in \mathcal{S}(\mathbb{R}, \mathbb{R})$ $(j = 1, \cdots, N+1)$ by the following:

1. $V_1 := V$, $L_1 := -\partial_x^2 + V_1 + m^2$, $\psi_1 = \phi_1$ and $A_1 = A_{V_1}$;
2. given V_k, we define

$$A_k := A_{V_k} \text{ and } L_{k+1} := -\partial_x^2 + V_{k+1} + m^2 := A_k^* A_k + \lambda_k^2, \tag{1.39}$$

and, by Proposition 1.11, we have $L_k = -\partial_x^2 + V_k + m^2 = A_k A_k^* + \lambda_k^2$.

From Proposition 1.11, we have

$$\sigma_{\mathrm{d}}(L_k) = \{\lambda_j^2 \mid j = k, \cdots, N\},\ k = 1, \cdots, N, \text{ and } \sigma_{\mathrm{d}}(L_{N+1}) = \emptyset.$$

If ψ_k is the ground state of L_k and $A_k = \frac{1}{\psi_k}\partial_x(\psi_k \cdot)$ then, from

$$A_j^* L_j = A_j^*(A_j A_j^* + \lambda_j^2) = (A_j^* A_j + \lambda_j^2) A_j^* = L_{j+1} A_j^*, \tag{1.40}$$

we have the conjugation relation

$$\mathcal{A}^* L_1 = L_{N+1} \mathcal{A}^*, \tag{1.41}$$

where

$$\mathcal{A} = A_1 \cdots A_N \text{ and } \mathcal{A}^* = A_N^* \cdots A_1^*. \tag{1.42}$$

We write $L_D := L_{N+1}$ and $V_D := V_{N+1}$. Dispersion will follow from the hypothesis that V_D is repulsive with respect to the origin, specifically, the following.

Assumption 1.12. We assume $x V_{\mathrm{D}}' \le 0$ and $x V_{\mathrm{D}}'(x) \not\equiv 0$.

2. Main estimates and proof of Theorem 1.1

Using the refined profile given in Proposition 1.6, we first decompose the solution by appropriate orthogonality condition.

Lemma 2.1 *(Modulation). There exists $\delta_1 > 0$ s.t. there exists $\mathbf{z} \in C^\infty(B_{\mathcal{H}^1}(0, \delta_1), \mathbb{C}^N)$ s.t. $\mathbf{z}(\mathbf{0}) = \mathbf{0}$ and*

$$\boldsymbol{\eta}[\mathbf{u}] := \mathbf{u} - \boldsymbol{\phi}[\mathbf{z}(\mathbf{u})] \in \mathcal{H}_{\mathrm{c}}[\mathbf{z}(\boldsymbol{u})]. \tag{2.1}$$

Furthermore, we have

$$\|\mathbf{u}\|_{\mathcal{H}^1} \sim \|\boldsymbol{\eta}[\mathbf{u}]\|_{\mathcal{H}^1} + \|\mathbf{z}(\mathbf{u})\|. \tag{2.2}$$

Proof. Standard. □

In the following, we fix a solution $\mathbf{u}$ of (1.3) with $\mathbf{u}(0) = \mathbf{u}_0$ and $\|\mathbf{u}_0\|_{\mathcal{H}^1} =: \delta$ satisfying the assumption of Theorem 1.1 (with $\delta_0 > 0$ to be determined). We write $\mathbf{z}(t) = \mathbf{z}(\mathbf{u}(t))$ and $\boldsymbol{\eta}(t) = \boldsymbol{\eta}[\mathbf{u}(t)]$. By the conservation of energy and by (2.2) we have

$$\|\mathbf{z}\|_{L^\infty(\mathbb{R}, \mathbb{C}^N)} + \|\boldsymbol{\eta}\|_{L^\infty(\mathbb{R}, \mathcal{H}^1)} \lesssim \delta. \tag{2.3}$$

Substituting $\mathbf{u} = \boldsymbol{\phi}[\mathbf{z}] + \boldsymbol{\eta}$ into (1.3), we have

$$\dot{\boldsymbol{\eta}} + D_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}](\dot{\mathbf{z}} - \widetilde{\mathbf{z}}) = \mathbf{J}\left(\mathbf{L}[\mathbf{z}]\boldsymbol{\eta} + \mathbf{F}[\mathbf{z}, \boldsymbol{\eta}] + \sum_{\mathbf{m} \in \mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}} \mathbf{G}_{\mathbf{m}} + \mathbf{R}[\mathbf{z}]\right), \tag{2.4}$$

where, for $d\mathbf{f}$ the Frechét derivative of $\mathbf{f}$,

$$\mathbf{L}[\mathbf{z}] = \mathbf{L}_1 + d\mathbf{f}[\boldsymbol{\phi}[\mathbf{z}]], \tag{2.5}$$

$$\mathbf{F}[\mathbf{z}, \boldsymbol{\eta}] = \mathbf{f}[\boldsymbol{\phi}[\mathbf{z}] + \boldsymbol{\eta}] - \mathbf{f}[\boldsymbol{\phi}[\mathbf{z}]] - d\mathbf{f}[\boldsymbol{\phi}[\mathbf{z}]]\boldsymbol{\eta}. \tag{2.6}$$

Notice that $\mathbf{F}[\mathbf{z},\boldsymbol{\eta}] = {}^t(F_1[\mathbf{z},\eta_1]\ 0)$ where

$$F_1[\mathbf{z},\eta_1] = f(\phi_1[\mathbf{z}]+\eta_1) - f(\phi_1[\mathbf{z}]) - f'(\phi_1[\mathbf{z}])\eta_1. \tag{2.7}$$

We will consider constants $A, B, \varepsilon > 0$ satisfying

$$\log(\delta^{-1}) \gg \log(\epsilon^{-1}) \gg A \gg B^2 \gg B \gg \exp\left(\varepsilon^{-1}\right) \gg 1. \tag{2.8}$$

We will denote by $o_\varepsilon(1)$ constants depending on ε such that

$$o_\varepsilon(1) \xrightarrow{\varepsilon\to 0^+} 0. \tag{2.9}$$

Let

$$\kappa \in (0, \min(m-\lambda_N, a_1)/10)\,. \tag{2.10}$$

We will consider the norms

$$\|\boldsymbol{\eta}\|_{\boldsymbol{\Sigma}_A} := \left\|\operatorname{sech}\left(\frac{2}{A}x\right)\eta_1'\right\|_{L^2} + A^{-1}\left\|\operatorname{sech}\left(\frac{2}{A}x\right)\boldsymbol{\eta}\right\|_{L^2} \quad \text{and} \tag{2.11}$$

$$\|\boldsymbol{\eta}\|_{\boldsymbol{L}^2_{-\kappa}} := \|\operatorname{sech}(\kappa x)\,\boldsymbol{\eta}\|_{L^2}\,. \tag{2.12}$$

We will prove the following continuation argument.

Proposition 2.2. *Under the Assumptions 1.3, 1.7 and 1.12, for any small $\epsilon > 0$ there exists a $\delta_0 = \delta_0(\epsilon)$ s.t. if in $I = [0,T]$ we have*

$$\|\dot{\mathbf{z}} - \widetilde{\mathbf{z}}\|_{L^2(I)} + \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \|\mathbf{z}^{\mathbf{m}}\|_{L^2(I)} + \|\boldsymbol{\eta}\|_{L^2(I,\boldsymbol{\Sigma}_A\cap\boldsymbol{L}^2_{-\kappa})} \le \epsilon \tag{2.13}$$

then for $\delta \in (0,\delta_0)$ and $\delta = \|\mathbf{u}_0\|_{\boldsymbol{\mathcal{H}}^1}$ inequality (2.13) holds for ϵ replaced by $o_\varepsilon(1)\epsilon$ where $o_\varepsilon(1) \xrightarrow{\varepsilon\to 0^+} 0$.

It is elementary that Proposition 2.2 follows from the following Propositions 2.3–2.6.

Proposition 2.3. *We have*

$$\|\dot{\mathbf{z}} - \widetilde{\mathbf{z}}\|_{L^2(I)} = o_\varepsilon(1)\|\boldsymbol{\eta}\|_{L^2(I,\boldsymbol{L}^2_{-\kappa})}. \tag{2.14}$$

Proposition 2.4 *(FGR estimate). We have*

$$\sum_{\mathbf{m}\in\mathbf{R}_{\min}} \|\mathbf{z}^{\mathbf{m}}\|_{L^2(I)} \lesssim \delta + A^{-1/4}\|\boldsymbol{\eta}\|_{L^2(I,\boldsymbol{\Sigma}_A)}. \tag{2.15}$$

Proposition 2.5 *(1st virial estimate). We have*

$$\|\boldsymbol{\eta}\|_{L^2(I,\boldsymbol{\Sigma}_A)} \lesssim \delta + \|\boldsymbol{\eta}\|_{L^2(I,\boldsymbol{L}^2_{-\kappa})} + \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \|\mathbf{z}^{\mathbf{m}}\|_{L^2}. \tag{2.16}$$

Proposition 2.6 *(2nd virial estimate). We have*

$$\|\boldsymbol{\eta}\|_{L^2(I,\boldsymbol{L}^2_{-\kappa})} \lesssim B\varepsilon^{-N}\delta + A^{-1/4}\|\boldsymbol{\eta}\|_{L^2(I,\boldsymbol{\Sigma}_A)} + \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \|\mathbf{z}^{\mathbf{m}}\|_{L^2}. \tag{2.17}$$

Proof of Theorem 1.1. By continuity, Proposition 2.2 implies that inequality (2.13) is valid with $I = \mathbb{R}_+$. This implies (1.13) (adjusting ϵ). From the equation for $\mathbf{z}$, see (3.5) below, we have $\dot{\mathbf{z}} \in L^\infty(\mathbb{R}, \mathbb{C}^N)$. By $\mathbf{z}^{\mathbf{m}} \in L^2(\mathbb{R})$ for any $\mathbf{m} \in \mathbf{R}_{\min}$, we have $z_j^{m_j} \in L^2(\mathbb{R})$ for m_j the largest $m_j \in \mathbb{N}$ such that $(m_j - 1)\lambda_j < m$. From this we get $\lim_{t\to+\infty} \mathbf{z}(t) = 0$. □

3. Proof of Proposition 2.3

Proof of Proposition 2.3. We fix an even function $\chi \in C_0^\infty(\mathbb{R}, [0, 1])$ satisfying

$$1_{[-1,1]} \le \chi \le 1_{[-2,2]} \text{ and } x\chi'(x) \le 0 \text{ and set } \chi_A := \chi(\cdot/A). \tag{3.1}$$

Lemma 3.1. *For the F_1 in (2.7), we have*

$$\|\operatorname{sech}(\kappa x)\, F_1[\mathbf{z}, \boldsymbol{\eta}]\|_{L^2} \lesssim \delta \|\operatorname{sech}(\kappa x)\, \eta_1\|_{L^2}, \tag{3.2}$$

$$\|\chi_A F_1[\mathbf{z}, \boldsymbol{\eta}]\|_{L^1} \lesssim A^{1/2}\delta \left\|\operatorname{sech}\left(\frac{2}{A}x\right) \eta_1\right\|_{L^2}. \tag{3.3}$$

Proof. By Taylor expansion, $F_1[\mathbf{z}, \boldsymbol{\eta}] = \int_0^1 (1-t) f''(\phi_1[\mathbf{z}] + t\eta_1)\eta_1^2\, dt$. Thus,

$$\|\operatorname{sech}(\kappa x)\, F_1[\mathbf{z}, \boldsymbol{\eta}]\|_{L^2} \lesssim \sup_{|u|\le 1} |f''(u)| \|\eta_1\|_{L^\infty} \|\operatorname{sech}(\kappa x)\, \eta_1\|_{L^2} \lesssim \delta \|\operatorname{sech}(\kappa x)\, \eta_1\|_{L^2},$$

$$\|\chi_A F_1[\mathbf{z}, \boldsymbol{\eta}]\|_{L^1} \lesssim \sup_{|u|\le 1} |f''(u)| \|\eta_1\|_{L^\infty} \|\eta_1 \chi_A\|_{L^1} \lesssim A^{1/2}\delta \left\|\operatorname{sech}\left(\frac{2}{A}x\right) \eta_1\right\|_{L^2}^2,$$

where we have used $\operatorname{sech}\left(\frac{2}{A}x\right) \sim 1$ in $\operatorname{supp}\chi_A$, (2.3) and the embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$. □

Lemma 3.2. *We have*

$$\|\dot{\mathbf{z}} - \widetilde{\mathbf{z}}\| \lesssim \delta \|\operatorname{sech}(\kappa x)\, \boldsymbol{\eta}\|_{L^2}. \tag{3.4}$$

Proof. Recalling (1.27) and (2.5), differentiating (1.19) we have for $\mathbf{w} \in \mathbb{C}^N$

$$D^2_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}](\widetilde{\mathbf{z}}, \mathbf{w}) + D_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}] D_{\mathbf{z}}\widetilde{\mathbf{z}}(\mathbf{z})\mathbf{w} + \mathbf{J} D_{\mathbf{z}}\widetilde{\mathbf{R}}[\mathbf{z}]\mathbf{w} = \mathbf{J}\mathbf{L}[\mathbf{z}] D_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}]\mathbf{w}.$$

We apply $\Omega(\cdot, D_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}]\mathbf{w})$ to (2.4), obtaining

$$\Omega(\dot{\eta}, D_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}]\mathbf{w}) + \Omega(D_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}](\dot{\mathbf{z}} - \widetilde{\mathbf{z}}), D_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}]\mathbf{w})$$
$$= \langle \mathbf{L}[\mathbf{z}]\boldsymbol{\eta}, D_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}]\mathbf{w}\rangle + \langle \mathbf{F}[\mathbf{z}, \eta], D_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}]\mathbf{w}\rangle \,,$$

where we used $\Omega(\mathbf{J}\widetilde{\mathbf{R}}[\mathbf{z}], D_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}]\mathbf{w}) = 0$, that is (1.28). Using $\boldsymbol{\eta} \in \mathcal{H}_{c}[\mathbf{z}]$, we have

$$\langle \mathbf{L}[\mathbf{z}]\boldsymbol{\eta}, D_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}]\mathbf{w}\rangle = \langle \eta, \mathbf{L}[\mathbf{z}]D_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}]\mathbf{w}\rangle = \left\langle \eta, \mathbf{J}^{-1} D_{\mathbf{z}}^2\boldsymbol{\phi}[\mathbf{z}](\widetilde{\mathbf{z}}, \mathbf{w}) + D_{\mathbf{z}}\widetilde{\mathbf{R}}[\mathbf{z}]\mathbf{w}\right\rangle$$
$$= -\Omega(\eta, D_{\mathbf{z}}^2\boldsymbol{\phi}[\mathbf{z}](\widetilde{\mathbf{z}}, \mathbf{w})) + \left\langle \eta, D_{\mathbf{z}}\widetilde{\mathbf{R}}[\mathbf{z}]\mathbf{w}\right\rangle$$

and

$$\Omega(\dot{\eta}, D_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}]\mathbf{w}) = -\Omega(\eta, D_{\mathbf{z}}^2\boldsymbol{\phi}[\mathbf{z}](\dot{\mathbf{z}}, \mathbf{w})) = -\Omega(\eta, D_{\mathbf{z}}^2\boldsymbol{\phi}[\mathbf{z}](\dot{\mathbf{z}} - \widetilde{\mathbf{z}}, \mathbf{w})) - \Omega(\eta, D_{\mathbf{z}}^2\boldsymbol{\phi}[\mathbf{z}](\widetilde{\mathbf{z}}, \mathbf{w})).$$

Thus

$$\Omega(D_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}](\dot{\mathbf{z}} - \widetilde{\mathbf{z}}), D_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}]\mathbf{w}) = \Omega(\eta, D_{\mathbf{z}}^2\boldsymbol{\phi}[\mathbf{z}](\dot{\mathbf{z}} - \widetilde{\mathbf{z}}, \mathbf{w})) + \left\langle \eta, D_{\mathbf{z}}\widetilde{\mathbf{R}}[\mathbf{z}]\mathbf{w}\right\rangle + \langle \mathbf{F}[\mathbf{z}, \eta], D_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}]\mathbf{w}\rangle \,. \tag{3.5}$$

Since $\Omega(D_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}]\cdot, D_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}]\cdot)$ is a symplectic form for $\mathbb{C}^N$, taking $\|\mathbf{w}\| = 1$ in an appropriate direction we obtain

$$\|\dot{\mathbf{z}} - \widetilde{\mathbf{z}}\| \lesssim \delta \|\mathrm{sech}\,(\kappa x)\, \boldsymbol{\eta}\|_{L^2} + \|\mathrm{sech}\,(\kappa x)\, \mathbf{F}[\mathbf{z}, \boldsymbol{\eta}]\|_{L^2}.$$

By (3.2), we have the conclusion. □

Lemma 3.2 completes the proof of Proposition 2.3, recalling (2.12). □

4. Technical lemma I

The following is a slight refinement of a result in [4].

Lemma 4.1. *Let $U \geq 0$ be a non–zero potential $U \in L^1(\mathbb{R}, \mathbb{R})$. Then there exists a constant $C_U > 0$ such that for any function $0 \leq W$ such that $\langle x\rangle\, W \in L^1(\mathbb{R})$ then*

$$\langle Wf, f\rangle \leq C_U \left(\| \langle x\rangle\, W\|_{L^1(\mathbb{R})} \|f'\|^2_{L^2(\mathbb{R})} + \|W\|_{L^1(\mathbb{R})} \langle Uf, f\rangle \right). \tag{4.1}$$

In particular, we have

$$\|\mathrm{sech} \left(\frac{2}{A}x\right) f\|^2_{L^2(\mathbb{R})} \lesssim A^2 \|f'\|^2_{L^2(\mathbb{R})} + A \|\mathrm{sech}\,(\kappa x)\, f\|^2_{L^2(\mathbb{R})}. \tag{4.2}$$

Proof. Let J be a compact interval where $I_U := \int_J U(x)dx > 0$. Let then $x_0 \in J$ s.t.

$$|f(x_0)|^2 \leq I_U^{-1} \int_J |f(x)|^2 U(x)dx.$$

Then,

$$|f(x)| \le |x-x_0|^{\frac12}\|f'\|_{L^2(\mathbb{R})} + |f(x_0)| \le |x-x_0|^{\frac12}\|f'\|_{L^2(\mathbb{R})} + I_U^{-1/2}\langle Uf, f\rangle^{\frac12}.$$

Taking second power and multiplying by W it is easy to conclude the following, which after integration yields (4.1),

$$W(x)|f(x)|^2 \le 2(1+|x_0|)\langle x\rangle W(x)\|f'\|^2_{L^2(\mathbb{R})} + 2W(x)I_U^{-1}\langle Uf, f\rangle. \quad \square$$

For $A_0 = 2/\kappa$ and $A \ge A_0$ we have $\operatorname{sech}(\kappa x) \le \operatorname{sech}(\frac{2}{A}x)$ which implies the following, which we will use in Sect. 5,

$$\begin{aligned}\|\operatorname{sech}(\kappa x) f\|_{L^2} &\le A\cdot A^{-1}\left\|\operatorname{sech}\left(\frac{2}{A}x\right) f\right\|_{L^2} \\ &\le A\left(\left\|\operatorname{sech}\left(\frac{2}{A}x\right) f'\right\|_{L^2} + A^{-1}\left\|\operatorname{sech}\left(\frac{2}{A}x\right) f\right\|_{L^2}\right). \quad \square \end{aligned} \tag{4.3}$$

5. Proof of Proposition 2.4: the Fermi Golden Rule

To prove Proposition 2.4, for the $\mathbf{g_m}$ in Assumption 1.7, we consider

$$\mathcal{J}_{\mathrm{FGR}} := \Omega(\boldsymbol{\eta}, \chi_A \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}}\mathbf{g_m}). \tag{5.1}$$

Computing the time derivative of $\mathcal{J}_{\mathrm{FGR}}$, we have the following estimate.

Lemma 5.1. *We have*

$$\left|\dot{\mathcal{J}}_{\mathrm{FGR}} - \left\langle \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}}\mathbf{G_m}, \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}}\mathbf{g_m}\right\rangle\right| \lesssim A^{-1/2}\left(\sum_{\mathbf{m}\in\mathbf{R}_{\min}} |\mathbf{z}^{\mathbf{m}}|^2 + \|\boldsymbol{\eta}\|^2_{\boldsymbol{\Sigma}_A}\right). \tag{5.2}$$

Proof. Differentiating $\mathcal{J}_{\mathrm{FGR}}$ and using (2.4), we have

$$\begin{aligned}\dot{\mathcal{J}}_{\mathrm{FGR}} =& \Omega(\dot{\boldsymbol{\eta}}, \chi_A \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}}\mathbf{g_m}) + \Omega(\boldsymbol{\eta}, \chi_A \sum_{\mathbf{m}\in\mathbf{R}_{\min}} D_{\mathbf{z}}\mathbf{z}^{\mathbf{m}}\widetilde{\mathbf{z}}\,\mathbf{g_m}) \\ &+ \Omega(\boldsymbol{\eta}, \chi_A \sum_{\mathbf{m}\in\mathbf{R}_{\min}} D_{\mathbf{z}}\mathbf{z}^{\mathbf{m}}(\dot{\mathbf{z}} - \widetilde{\mathbf{z}})\,\mathbf{g_m}) =: A_1 + A_2 + A_3.\end{aligned}$$

By Lemma 3.2 and (4.3) and by (2.8), A_3 can be bounded by

$$\begin{aligned}|A_3| &\lesssim \|\boldsymbol{\eta}\chi_A\|_{L^1}\delta\|\dot{\mathbf{z}} - \widetilde{\mathbf{z}}\|_{\mathbb{C}^N} \lesssim \delta^2\left\|\operatorname{sech}\left(\frac{2}{A}x\right)\boldsymbol{\eta}\right\|_{L^2}\|\operatorname{sech}(\kappa x)\eta_1\|_{L^2} \\ &\lesssim \delta^2 A^2\|\boldsymbol{\eta}\|^2_{\boldsymbol{\Sigma}_A} \lesssim A^{-1/2}\|\boldsymbol{\eta}\|^2_{\boldsymbol{\Sigma}_A}.\end{aligned}$$

By Equation (2.4), we have

$$A_1 = \Omega(-D_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}](\dot{\mathbf{z}}-\widetilde{\mathbf{z}}), \chi_A \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}}\mathbf{g}_{\mathbf{m}}) + \left\langle \mathbf{L}_1\boldsymbol{\eta}, \chi_A \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}}\mathbf{g}_{\mathbf{m}} \right\rangle$$

$$+ \left\langle d\mathbf{f}[\boldsymbol{\phi}[\mathbf{z}]]\boldsymbol{\eta} + \mathbf{F}[z,\boldsymbol{\eta}] + \mathbf{R}[\mathbf{z}], \chi_A \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}}\mathbf{g}_{\mathbf{m}} \right\rangle + \left\langle \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}}\mathbf{G}_{\mathbf{m}}, \chi_A \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}}\mathbf{g}_{\mathbf{m}} \right\rangle$$

$$= A_{11} + A_{12} + A_{13} + A_{14}.$$

By Lemma 3.2 and (4.3) and by (2.8) we have

$$|A_{11}| \lesssim \|\dot{\mathbf{z}} - \widetilde{\mathbf{z}}\|_{\mathbb{C}^N} \sum_{\mathbf{m}\in\mathbf{R}_{\min}} |\mathbf{z}^{\mathbf{m}}| \lesssim \delta \left(\|\operatorname{sech}(\kappa x)\boldsymbol{\eta}\|_{L^2}^2 + \sum_{\mathbf{m}\in\mathbf{R}_{\min}} |\mathbf{z}^{\mathbf{m}}|^2 \right)$$
$$\le A^{-1/2} \left(\sum_{\mathbf{m}\in\mathbf{R}_{\min}} |\mathbf{z}^{\mathbf{m}}|^2 + \|\boldsymbol{\eta}\|_{\boldsymbol{\Sigma}_A}^2 \right).$$

By (1.18) and Lemma 3.1 we have

$$|A_{13}| \lesssim \sum_{\mathbf{m}\in\mathbf{R}_{\min}} |\mathbf{z}^{\mathbf{m}}| \left(\|d\mathbf{f}[\boldsymbol{\phi}[\mathbf{z}]]\boldsymbol{\eta}\|_{L^1} + \|\mathbf{F}[z,\boldsymbol{\eta}]\chi_A\|_{L^1} + \|\mathbf{R}[z]\|_{L^1} \right)$$
$$\lesssim A^{1/2}\delta \left(\left\|\operatorname{sech}\left(\frac{2}{A}x\right)\boldsymbol{\eta}\right\|_{L^2}^2 + \sum_{\mathbf{m}\in\mathbf{R}_{\min}} |\mathbf{z}^{\mathbf{m}}|^2 \right) \le A^{-1/2} \left(\sum_{\mathbf{m}\in\mathbf{R}_{\min}} |\mathbf{z}^{\mathbf{m}}|^2 + \|\boldsymbol{\eta}\|_{\boldsymbol{\Sigma}_A}^2 \right).$$

The term A_{12} can be further decomposed as

$$A_{12} = \left\langle \boldsymbol{\eta}, \chi_A \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}}\mathbf{L}_1\mathbf{g}_{\mathbf{m}} \right\rangle + \left\langle \boldsymbol{\eta}, [\mathbf{L}_1, \chi_A] \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}}\mathbf{g}_{\mathbf{m}} \right\rangle =: A_{121} + A_{122}.$$

By $[\mathbf{L}_1, \chi_A] = \begin{pmatrix} -\chi_A'' - 2\chi_A'\partial_x & 0 \\ 0 & 0 \end{pmatrix}$, we have the bound

$$|A_{122}| \lesssim \sum_{\mathbf{m}\in\mathbf{R}_{\min}} |\mathbf{z}^{\mathbf{m}}| \left(\|\chi_A''\eta_1\|_{L^1} + \|\chi_A'\eta_1'\|_{L^1} \right)$$
$$\lesssim \sum_{\mathbf{m}\in\mathbf{R}_{\min}} |\mathbf{z}^{\mathbf{m}}| (A^{-3/2} \left\|\operatorname{sech}\left(\frac{2}{A}x\right)\eta_1\right\|_{L^2} + A^{-1/2}\left\|\operatorname{sech}\left(\frac{2}{A}x\right)\eta_1'\right\|_{L^2})$$
$$\lesssim A^{-1/2} \left(\sum_{\mathbf{m}\in\mathbf{R}_{\min}} |\mathbf{z}^{\mathbf{m}}|^2 + \left\|\operatorname{sech}\left(\frac{2}{A}x\right)\eta_1'\right\|_{L^2}^2 + A^{-2}\left\|\operatorname{sech}\left(\frac{2}{A}x\right)\eta_1\right\|_{L^2}^2 \right),$$

while we have, see Assumption 1.7,

$$A_{121} = \left\langle \eta, \chi_A \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}} \mathrm{i}(\mathbf{m}\cdot\lambda)\mathbf{J}^{-1}\mathbf{g}_{\mathbf{m}} \right\rangle. \tag{5.3}$$

The term A_{14} can be decomposed as

$$\begin{aligned} A_{14} &= \left\langle \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}}\mathbf{G}_{\mathbf{m}}, \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}}\mathbf{g}_{\mathbf{m}} \right\rangle - \left\langle \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}}\mathbf{G}_{\mathbf{m}}, (1-\chi_A) \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}}\mathbf{g}_{\mathbf{m}} \right\rangle \\ &= \left\langle \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}}\mathbf{G}_{\mathbf{m}}, \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}}\mathbf{g}_{\mathbf{m}} \right\rangle + A_{141}, \end{aligned} \tag{5.4}$$

where the 1st term of line (5.4) is the main term appearing in (5.2). Recalling $a_2 = \frac{1}{2}\sqrt{m^2 - \lambda_N^2}$,

$$|A_{141}| \lesssim e^{-a_2 A/2} \left| \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}} \right|^2 \lesssim A^{-1/2} \sum_{\mathbf{m}\in\mathbf{R}_{\min}} |\mathbf{z}^{\mathbf{m}}|^2.$$

By the elementary identity $D_{\mathbf{z}}\mathbf{z}^{\mathbf{m}}\widetilde{\mathbf{z}}_0 = \mathrm{i}\mathbf{m}\cdot\lambda\mathbf{z}^{\mathbf{m}}$, the term A_2 can be decomposed as

$$A_2 = \left\langle \mathbf{J}^{-1}\eta, \chi_A \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathrm{i}\mathbf{m}\cdot\lambda\mathbf{z}^{\mathbf{m}}\mathbf{g}_{\mathbf{m}} \right\rangle + \Omega\left(\eta, \chi_A \sum_{\mathbf{m}\in\mathbf{R}_{\min}} D_{\mathbf{z}}\mathbf{z}^{\mathbf{m}} \left(\widetilde{\mathbf{z}} - \widetilde{\mathbf{z}}_0\right) \mathbf{g}_{\mathbf{m}} \right) =: A_{21} + A_{22},$$

where

$$|A_{22}| \lesssim \delta \|\chi_A \eta\|_{L^1} \sum_{\mathbf{m}\in\mathbf{R}_{\min}} |\mathbf{z}^{\mathbf{m}}| \lesssim A^{-1/2} \left(\sum_{\mathbf{m}\in\mathbf{R}_{\min}} |\mathbf{z}^{\mathbf{m}}|^2 + A^{-2} \|\mathrm{sech}\left(\frac{2}{A}x\right)\eta\|_{L^2}^2 \right).$$

Finally, by the antisymmetry of $\mathbf{J}^{-1}(=-\mathbf{J})$ we have the cancellation $A_{121} + A_{21} = 0$. Collecting all the estimates, we obtain (5.2). □

We next take out the nonresonant terms from the main part of $\dot{\mathcal{J}}_{\mathrm{FGR}}$.

Lemma 5.2. *Let* $\mathbf{m}, \mathbf{n} \in \mathbf{R}_{\min}$ *and* $\mathbf{m} \neq \mathbf{n}$. *Then,*

$$\begin{aligned} &\mathbf{z}^{\mathbf{m}}\overline{\mathbf{z}^{\mathbf{n}}} = \frac{1}{\mathrm{i}\left(\mathbf{m}\cdot\lambda - \mathbf{n}\cdot\lambda\right)} \frac{d}{dt}\left(\mathbf{z}^{\mathbf{m}}\overline{\mathbf{z}^{\mathbf{n}}}\right) + r_{\mathbf{m},\mathbf{n}} \text{ where} \\ &|r_{\mathbf{m},\mathbf{n}}| \lesssim \delta \sum_{\mathbf{m}\in\mathbf{R}_{\min}} |\mathbf{z}^{\mathbf{m}}|^2 + \delta \|\dot{\mathbf{z}} - \widetilde{\mathbf{z}}\|^2. \end{aligned}$$

Proof. We have

$$\frac{d}{dt}\left(\mathbf{z}^{\mathbf{m}}\overline{\mathbf{z}^{\mathbf{n}}}\right) = \mathrm{i}\left(\mathbf{m}\cdot\lambda - \mathbf{n}\cdot\lambda\right)\mathbf{z}^{\mathbf{m}}\overline{\mathbf{z}^{\mathbf{n}}} + D_{\mathbf{z}}\left(\mathbf{z}^{\mathbf{m}}\overline{\mathbf{z}^{\mathbf{n}}}\right)\left(\widetilde{\mathbf{z}} - \widetilde{\mathbf{z}}_0\right) + D_{\mathbf{z}}\left(\mathbf{z}^{\mathbf{m}}\overline{\mathbf{z}^{\mathbf{n}}}\right)\left(\dot{\mathbf{z}} - \widetilde{\mathbf{z}}\right).$$

The estimate of $r_{\mathbf{m},\mathbf{n}}$ follows from Proposition 1.6. □

Lemma 5.3. *We have*

$$\left|\left\langle \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}}\mathbf{G}_{\mathbf{m}}, \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}}\mathbf{g}_{\mathbf{m}}\right\rangle - \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \gamma_{\mathbf{m}}|\mathbf{z}^{\mathbf{m}}|^2 - \frac{d}{dt}\Gamma\right| \lesssim \delta \sum_{\mathbf{m}\in\mathbf{R}_{\min}} |\mathbf{z}^{\mathbf{m}}|^2 \text{ where}$$

$$\Gamma := \sum_{\substack{\mathbf{m},\mathbf{n}\in\mathbf{R}_{\min}\\ \mathbf{m}\neq\mathbf{n}}} \left\langle \frac{\mathbf{z}^{\mathbf{m}}\overline{\mathbf{z}^{\mathbf{n}}}}{\mathrm{i}\,(\mathbf{m}\cdot\boldsymbol{\lambda}-\mathbf{n}\cdot\boldsymbol{\lambda})}\mathbf{G}_{\mathbf{m}}, \mathbf{g}_{\mathbf{n}}\right\rangle.$$

Proof. It is immediate from Lemma 5.2. □

Proof of Proposition 2.4. The proof follows from Lemmas 5.1 and 5.3 and the following estimates, due to (2.3),

$$|\mathcal{J}_{\mathrm{FGR}}| \lesssim \|\boldsymbol{\eta}\|_{L^2}\|\chi_A\|_{L^2}\sum_{\mathbf{m}\in\mathbf{R}_{\min}}|\mathbf{z}^{\mathbf{m}}| \lesssim \sqrt{A}\delta^3 \lesssim \delta^2 \text{ and}$$

$$|\Gamma| \lesssim \sum_{\mathbf{m}\in\mathbf{R}_{\min}}|\mathbf{z}^{\mathbf{m}}|^2 \lesssim \delta^2. \quad \square$$

6. Proof of Proposition 2.5

We set, for the χ in (3.1),

$$\zeta_A(x) := \exp\left(-\frac{|x|}{A}(1-\chi(x))\right), \ \varphi_A(x) := \int_0^x \zeta_A^2(y)\,dy \text{ and } S_A := \frac{1}{2}\varphi_A' + \varphi_A\partial_x. \quad (6.1)$$

We will consider the functionals

$$\mathcal{I}_{1\mathrm{st},1} := \frac{1}{2}\Omega(\boldsymbol{\eta}, S_A\boldsymbol{\eta}),\ \mathcal{I}_{1\mathrm{st},2} := \frac{1}{2}\Omega\left(\boldsymbol{\eta}, \sigma_3\zeta_A^4\boldsymbol{\eta}\right),$$

where both S_A and $\sigma_3\zeta_A^4$ are anti-symmetric w.r.t. Ω.

Lemma 6.1. *We have*

$$\begin{aligned}&\left\|\operatorname{sech}\left(\frac{2}{A}x\right)\eta_1'\right\|_{L^2}^2 + A^{-2}\left\|\operatorname{sech}\left(\frac{2}{A}x\right)\eta_1\right\|_{L^2}^2\\ &\lesssim -\dot{\mathcal{I}}_{1\mathrm{st},1} + A^2\delta\|\boldsymbol{\eta}\|_{\boldsymbol{\Sigma}_A}^2 + \|\operatorname{sech}(\kappa x)\,\boldsymbol{\eta}\|_{L^2}^2 + \sum_{\mathbf{m}\in\mathbf{R}_{\min}}|\mathbf{z}^{\mathbf{m}}|^2.\end{aligned} \quad (6.2)$$

Proof. We have

$$\dot{\mathcal{I}}_{1\mathrm{st},1} = -\Omega(D\boldsymbol{\phi}[\mathbf{z}](\dot{\mathbf{z}}-\widetilde{\mathbf{z}}), S_A\boldsymbol{\eta}) + \langle \mathbf{L}_1\boldsymbol{\eta}, S_A\boldsymbol{\eta}\rangle + \langle \mathbf{f}[\boldsymbol{\phi}[\mathbf{z}]+\boldsymbol{\eta}] - \mathbf{f}[\boldsymbol{\phi}[\mathbf{z}]], S_A\boldsymbol{\eta}\rangle + \left\langle \widetilde{\mathbf{R}}[\mathbf{z}], S_A\boldsymbol{\eta}\right\rangle$$
$$=: B_1 + B_2 + B_3 + B_4, \tag{6.3}$$

where $\widetilde{\mathbf{R}}$ is defined in (1.27) and

$$\widetilde{\mathbf{F}}[\mathbf{z},\boldsymbol{\eta}] := \mathbf{f}[\boldsymbol{\phi}[\mathbf{z}]+\boldsymbol{\eta}] - \mathbf{f}[\boldsymbol{\phi}[\mathbf{z}]]. \tag{6.4}$$

The main term, B_2, can be decomposed as

$$\begin{aligned} B_2 &= \langle L_1\eta_1, S_A\eta_1\rangle \\ &= -\|(\zeta_A\eta_1)'\|_{L^2}^2 - \frac{1}{2}\int \varphi_A V'\eta_1^2\,dx - \frac{1}{2}\int A^{-1}\left(\chi''|x| + 2\chi'\frac{x}{|x|}\right)\zeta_A\eta_1^2\,dx \\ &= -\|(\zeta_A\eta_1)'\|_{L^2}^2 + B_{21} + B_{22}, \end{aligned}$$

where, $|\varphi_A V'| \lesssim |xV'| \lesssim |xe^{-a_1|x|}|$ and (2.10) imply

$$|B_{21}| \lesssim \|\mathrm{sech}\,(\kappa x)\,\eta_1\|_{L^2}^2,$$

and by (3.1)

$$|B_{22}| \lesssim A^{-1}\|\mathrm{sech}\,(\kappa x)\,\eta_1\|_{L^2}^2.$$

By Lemma 3.2, we have

$$|B_1| \le \|\dot{\mathbf{z}}-\widetilde{\mathbf{z}}\|\,\|\boldsymbol{\eta}\|_{L^2_{-\kappa}} \lesssim \delta\|\boldsymbol{\eta}\|_{L^2_{-\kappa}}^2.$$

By (1.18) and (1.27) we have

$$|B_4| \lesssim \|\boldsymbol{\eta}\|_{L^2_{-\kappa}}^2 + \sum_{\mathbf{m}\in\mathbf{R}_{\min}} |\mathbf{z}^{\mathbf{m}}|^2.$$

By $f(\phi_1[\mathbf{z}]+\eta_1) - f(\phi_1[\mathbf{z}]) = \int_0^1\int_0^1 f''(s_1\phi[\mathbf{z}]_1 + s_2\eta_1)\phi_1[\mathbf{z}]\eta_1\,ds_1ds_2 + f(\eta_1)$, we have

$$B_3 = \left\langle \int_0^1\int_0^1 f''(s_1\phi_1[\mathbf{z}]+s_2\eta_1)\phi_1[\mathbf{z}]\eta_1\,ds_1ds_2, S_A\eta_1\right\rangle + \langle f(\eta_1), S_A\eta_1\rangle = B_{31} + B_{32}.$$

By integration by parts,

$$B_{31} = -\frac{1}{2}\left\langle \int_0^1\int_0^1 \partial_x\left(f''(s_1\phi_1[\mathbf{z}]+s_2\eta_1)\phi_1[\mathbf{z}]\right)\eta_1 ds_1ds_2, \varphi_A\eta_1\right\rangle.$$

Therefore, we have

$$
\begin{aligned}
|B_{31}| &\lesssim \|\cosh(\kappa x)\int_0^1\int_0^1 \partial_x\left(f''(s_1\phi_1[\mathbf{z}]+s_2\eta_1)\phi_1[\mathbf{z}]\right)ds_1ds_2\|_{L^\infty}\|\mathrm{sech}(\kappa x)\,\eta_1^2\|_{L^1}\\
&\lesssim \|\phi[\mathbf{z}]\|_{\mathbf{\Sigma}}\|\mathrm{sech}(\kappa x)\,\eta_1\|_{L^2}^2 \lesssim A^2\delta\|\boldsymbol{\eta}\|_{\mathbf{\Sigma}_A}^2,
\end{aligned}
$$

where the last inequality follows from (4.3).
For the pure in η_1 nonlinear term B_{32}, by Lemma 2.7 of [3], which follows [19], taking A sufficiently large and δ_0 sufficiently small, we have

$$
|B_{32}| \le o_\delta(1)\|(\zeta_A\eta_1)'\|_{L^2}^2.
$$

Collecting the estimates, we have

$$
\|(\zeta_A\eta_1)'\|_{L^2}^2 \lesssim -\dot{\mathcal{I}}_{1\mathrm{st},1} + \|\mathrm{sech}(\kappa x)\,\eta_1\|_{L^2}^2 + A^2\delta\|\boldsymbol{\eta}\|_A^2 + \|\mathrm{sech}(\kappa x)\,\boldsymbol{\eta}\|_{L^2}^2 + \sum_{\mathbf{m}\in\mathbf{R}_{\min}}|\mathbf{z}^{\mathbf{m}}|^2.
$$

Finally, we claim the following, which is analogous to (19) of [16],

$$
\|\mathrm{sech}\left(\frac{2}{A}x\right)\eta_1'\|_{L^2}^2 + A^{-2}\|\mathrm{sech}\left(\frac{2}{A}x\right)\eta_1\|_{L^2}^2 \lesssim \|(\zeta_A\eta_1)'\|_{L^2}^2 + A^{-1}\|\mathrm{sech}(\kappa x)\,\eta_1\|_{L^2}^2. \quad (6.5)
$$

This yields (6.2). To prove (6.5), we set $w_1 := \zeta_A\eta_1$. We have

$$
\begin{aligned}
&\int \zeta_A^2|w_1'|^2dx = \int \zeta_A^2|\zeta_A\eta_1' + \zeta_A'\eta_1|^2dx = \int\left(\zeta_A^4\eta_1'^2 + \zeta_A^3\zeta_A'(\eta_1^2)' + \zeta_A^2\zeta_A'^2\eta_1^2\right)dx\\
&= \int\left(\zeta_A^4\eta_1'^2 - \zeta_A^3\zeta_A''\eta_1^2 - 2\zeta_A^2\zeta_A'^2\eta_1^2\right)dx.
\end{aligned}
$$

This implies

$$
\int \zeta_A^4\eta_1'^2 \lesssim \int \zeta_A^2 w_1'^2dx + A^{-2}\int\zeta_A^2w_1^2dx.
$$

Since by (4.2) we have

$$
\begin{aligned}
A^{-2}\int\zeta_A^2w_1^2dx &\lesssim \|w_1'\|_{L^2(\mathbb{R})}^2 + A^{-1}\|\mathrm{sech}(2\kappa x)\,\zeta_A\eta_1\|_{L^2(\mathbb{R})}^2\\
&\lesssim \|w_1'\|_{L^2(\mathbb{R})}^2 + A^{-1}\|\mathrm{sech}(\kappa x)\,\eta_1\|_{L^2(\mathbb{R})}^2,
\end{aligned}
$$

we obtained the desired bound on the first term in the left hand side of (6.5). We have

$$
A^{-2}\|\mathrm{sech}\left(\frac{2}{A}x\right)\eta_1\|_{L^2}^2 \lesssim A^{-2}\int\zeta_A^2w_1^2dx \lesssim \|w_1'\|_{L^2(\mathbb{R})}^2 + A^{-1}\|\mathrm{sech}(\kappa x)\,\eta_1\|_{L^2(\mathbb{R})}^2
$$

and hence we conclude the proof of (6.5). □

Lemma 6.2. *There exist $\delta_0 > 0$ and $A_0 > 0$ s.t. if $\delta < \delta_0$, for any $A > A_0$, we have*

$$\begin{aligned}
&\|\operatorname{sech}\left(\frac{2}{A}x\right)\eta_2\|_{L^2}^2 \\
&\lesssim -\dot{\mathcal{I}}_{1\mathrm{st},2} + \|\operatorname{sech}\left(\frac{2}{A}x\right)\eta_1'\|_{L^2}^2 + \|\operatorname{sech}\left(\frac{2}{A}x\right)\eta_1\|_{L^2}^2 + \|\operatorname{sech}(\kappa x)\boldsymbol{\eta}\|_{L^2}^2 + \sum_{\mathbf{m}\in\mathbf{R}_{\min}} |\mathbf{z}^{\mathbf{m}}|^2.
\end{aligned} \tag{6.6}$$

Proof. We have

$$\begin{aligned}
&\dot{\mathcal{I}}_{1\mathrm{st},2} \\
&= -\Omega(D\boldsymbol{\phi}[\mathbf{z}](\dot{\mathbf{z}} - \widetilde{\mathbf{z}}), \sigma_3\zeta_A^4\boldsymbol{\eta}) + \left\langle \mathbf{L}_1\boldsymbol{\eta}, \sigma_3\zeta_A^4\boldsymbol{\eta}\right\rangle + \left\langle \mathbf{f}[\boldsymbol{\phi}[\mathbf{z}]+\boldsymbol{\eta}] - \mathbf{f}[\boldsymbol{\phi}[\mathbf{z}]], \sigma_3\zeta_A^4\boldsymbol{\eta}\right\rangle + \left\langle \widetilde{\mathbf{R}}[\mathbf{z}], \sigma_3\zeta_A^4\boldsymbol{\eta}\right\rangle \\
&=: C_1 + C_2 + C_3 + C_4.
\end{aligned}$$

For the main term C_2, we have

$$C_2 = -\|\zeta_A^2\eta_2\|_{L^2}^2 + \left\langle L_1\eta_1, \zeta_A^4\eta_1\right\rangle$$

and

$$|\left\langle L_1\eta_1, \zeta_A^4\eta_1\right\rangle| \lesssim \|\operatorname{sech}\left(\frac{2}{A}x\right)\eta_1'\|_{L^2}^2 + \|\operatorname{sech}\left(\frac{2}{A}x\right)\eta_1\|_{L^2}^2.$$

For the remainder terms, we have

$$\begin{aligned}
|C_1| &\lesssim \|\dot{\mathbf{z}} - \widetilde{\mathbf{z}}\| \|\operatorname{sech}(\kappa x)\boldsymbol{\eta}\|_{L^2} \lesssim \delta\|\operatorname{sech}(\kappa x)\boldsymbol{\eta}\|_{L^2}^2, \\
|C_3| &\lesssim \delta\|\operatorname{sech}\left(\frac{2}{A}x\right)\eta_1\|_{L^2}^2, \\
|C_4| &\lesssim \|\operatorname{sech}(\kappa x)\boldsymbol{\eta}\|_{L^2}^2 + \sum_{\mathbf{m}\in\mathbf{R}_{\min}} |\mathbf{z}^{\mathbf{m}}|^2
\end{aligned}$$

Collecting the estimates, we have the conclusion. □

Proof of Proposition 2.5. From $|\mathcal{I}_{1\mathrm{st},1}| \lesssim A\delta^2$, $|\mathcal{I}_{1\mathrm{st},2}| \lesssim \delta^2$, we have the conclusion from Lemmas 6.1 and 6.2. □

7. Technical lemmas II

We consider

$$\mathcal{T} := \langle \mathrm{i}\varepsilon\partial_x\rangle^{-N}\mathcal{A}^*. \tag{7.1}$$

The following lemma, where P_c is the orthogonal projection on the continuous spectrum component of L_1, see (1.38), is proved in [4, Sect. 9].

Lemma 7.1. *We have*

$$\mathbf{u} = \prod_{j=1}^{N} R_{L_1}(\lambda_j^2) P_c \mathcal{A} \langle \mathrm{i}\varepsilon\partial_x\rangle^N \mathcal{T}\mathbf{u} \text{ for all } \mathbf{u} \in L_c^2(L_1). \quad \square \tag{7.2}$$

In [4, Sect. 5] the following lemma was proved.

Lemma 7.2. *Suppose that a Schwartz function $\mathcal{V} \in \mathcal{S}(\mathbb{R}, \mathbb{C})$ has the property that for $M \geq N+1$ its Fourier transform satisfies*

$$|\widehat{\mathcal{V}}(k_1 + ik_2)| \leq C_M \langle k_1\rangle^{-M-1} \text{ for all } (k_1, k_2) \in \mathbb{R} \times [\mathbf{b}, \mathbf{b}] \text{ and} \tag{7.3}$$
$$\widehat{\mathcal{V}} \in C^0(\mathbb{R} \times [-\mathbf{b}, \mathbf{b}]) \cap H(\mathbb{R} \times (-\mathbf{b}, \mathbf{b})),$$

with $H(\Omega)$ the set of holomorphic functions in an open subset $\Omega \subseteq \mathbb{C}$ and with a number $\mathbf{b} > 0$. Then, for multiplicative operators $\cosh(\mathbf{b}x)$ and $\cosh\left(\frac{\mathbf{b}}{2}x\right)$, we have

$$\| \langle \mathrm{i}\varepsilon\partial_x\rangle^{-N} [\mathcal{V}, \langle \mathrm{i}\varepsilon\partial_x\rangle^N] \cosh(\mathbf{b}x) \|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} \leq C_{\mathbf{b}}\varepsilon, \tag{7.4}$$

$$\| \cosh\left(\frac{\mathbf{b}}{2}x\right) \langle \mathrm{i}\varepsilon\partial_x\rangle^{-N} [\mathcal{V}, \langle \mathrm{i}\varepsilon\partial_x\rangle^N] \cosh\left(\frac{\mathbf{b}}{2}x\right) \|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} \leq C_{\mathbf{b}}\varepsilon. \tag{7.5}$$

Proof. We repeat the proof because we need it for Lemma 7.3 below. We start with (7.4), repeating the proof from [4]. We have for $\sigma = 0$

$$\langle \mathrm{i}\varepsilon\partial_x\rangle^{-N} [\mathcal{V}, \langle \mathrm{i}\varepsilon\partial_x\rangle^N] f = \int_{\mathbb{R}} dy K^\sigma(x, y) f(y),$$

where we set

$$K^\sigma(x, y) = \int_{\mathbb{R}^2} e^{\mathrm{i}xk - \mathrm{i}y\ell} \langle \varepsilon k\rangle^{-\sigma} H(k, \ell) dk d\ell \text{ with} \tag{7.6}$$
$$H(k, \ell) = \langle \varepsilon k\rangle^{-N} \widehat{\mathcal{V}}(k - \ell) \left(\langle \varepsilon k\rangle^N - \langle \varepsilon \ell\rangle^N\right).$$

Notice that

$$H(k, \ell) = \varepsilon H_1(k, \ell) \text{ where } H_1(k, \ell) = \langle \varepsilon k\rangle^{-N} \widehat{\mathcal{V}}(k - \ell)(k - \ell) \frac{P(\varepsilon k, \varepsilon \ell)}{\langle \varepsilon k\rangle^N + \langle \varepsilon \ell\rangle^N}, \tag{7.7}$$

where P is a $2N - 1$ degree polynomial. Hence the generalized integral in (7.6) is absolutely convergent for $\sigma > 0$. But also for $\sigma = 0$ the operator

$$T_\sigma f(x) = \int_{\mathbb{R}} dy f(y) \int_{\mathbb{R}^2} e^{\mathrm{i}xk - \mathrm{i}y\ell} \langle \varepsilon k\rangle^{-\sigma} H_1(k, \ell) dk d\ell$$

defines an operator $L^2(\mathbb{R}) \to L^2(\mathbb{R})$ of norm uniformly bounded in $\sigma \geq 0$. Let us focus now on $k = k_1 + \mathrm{i}0$ and $\ell = \ell_1 - \mathrm{i}\mathbf{b}$

$$T_\sigma(\chi_{\mathbb{R}_+} f)(x) = \int_{\mathbb{R}_+} dy f(y) e^{-y\mathbf{b}} \int_{\mathbb{R}^2} e^{\mathrm{i}xk_1 - \mathrm{i}y\ell_1} \langle \varepsilon k_1 \rangle^{-\sigma} H_1(k_1, \ell_1 - \mathrm{i}\mathbf{b}) dk_1 d\ell_1.$$

Now we claim that there exists $C > 0$ such that

$$\|T_\sigma \chi_{\mathbb{R}_+} f\|_{L^2(\mathbb{R})} \leq C \|e^{-|x|\mathbf{b}} f\|_{L^2(\mathbb{R}_+)} \text{ for all } \sigma > 0 \text{ and for all } f. \tag{7.8}$$

Set $g(y) = \chi_{\mathbb{R}_+}(y) f(y) e^{-y\mathbf{b}}$. Then

$$T_\sigma \widehat{(\chi_{\mathbb{R}_+}} f)(k_1) = \int_{\mathbb{R}} \langle \varepsilon k_1 \rangle^{-\sigma} H_1(k_1, \ell_1 - \mathrm{i}\mathbf{b}) \widehat{g}(\ell_1) d\ell_1.$$

We claim that we have

$$\sup_{k_1 \in \mathbb{R}} \int_{\mathbb{R}} \langle \varepsilon k_1 \rangle^{-\sigma} |H_1(k_1, \ell_1 - \mathrm{i}\mathbf{b})| d\ell_1 < C, \tag{7.9}$$

$$\sup_{\ell_1 \in \mathbb{R}} \int_{\mathbb{R}} \langle \varepsilon k_1 \rangle^{-\sigma} |H_1(k_1, \ell_1 - \mathrm{i}\mathbf{b})| dk_1 < C, \tag{7.10}$$

for a fixed constant $C > 0$.
We have

$$\begin{aligned} &\int_{\mathbb{R}} |H_1(k_1, \ell_1 - \mathrm{i}\mathbf{b})| d\ell_1 \\ &\lesssim \int_{|\ell_1| \in \left[\frac{|k_1|}{2}, 2|k_1|\right]} \langle \varepsilon k_1 \rangle^{-N} \langle k_1 - \ell_1 \rangle^{-M} \left(\langle \varepsilon k_1 \rangle^{N-1} + |\langle \varepsilon \ell_1 - \mathrm{i}\varepsilon \mathbf{b} \rangle|^{N-1} \right) d\ell_1 \\ &+ \int_{|\ell_1| \notin \left[\frac{|k_1|}{2}, 2|k_1|\right]} \langle \varepsilon k_1 \rangle^{-N} \langle k_1 - \ell_1 \rangle^{-M} \left(\langle \varepsilon k_1 \rangle^{N-1} + |\langle \varepsilon \ell_1 - \mathrm{i}\varepsilon \mathbf{b} \rangle|^{N-1} \right) d\ell_1. \end{aligned}$$

The first integral can be bounded above by

$$\int_{|\ell_1| \in \left[\frac{|k_1|}{2}, 2|k_1|\right]} \langle \varepsilon k_1 \rangle^{-1} \langle k_1 - \ell_1 \rangle^{-M} d\ell_1 \leq \| \langle x \rangle^{-M} \|_{L^1(\mathbb{R})},$$

while the second can be bounded above by

$$\int_{\mathbb{R}} \langle \varepsilon k_1\rangle^{-N} \frac{\langle \varepsilon k_1\rangle^{N-1} + |\langle \varepsilon \ell_1 - \mathrm{i}\varepsilon \mathbf{b}\rangle|^{N-1}}{\langle k_1\rangle^M + \langle \ell_1\rangle^M} d\ell_1 \le \| \langle x\rangle^{-M-1+N} \|_{L^1(\mathbb{R})}.$$

So (7.9) is true for $C = \| \langle x\rangle^{-2} \|_{L^1(\mathbb{R})}$. Next we prove (7.10). We have

$$\begin{aligned}
&\int_{\mathbb{R}} |H_1(k_1, \ell_1 - \mathrm{i}\mathbf{b})| dk_1 \\
&\lesssim \int_{|k_1| \in \left[\frac{|\ell_1|}{2}, 2|\ell_1|\right]} \langle \varepsilon k_1\rangle^{-N} \langle k_1 - \ell_1\rangle^{-M} \left(\langle \varepsilon k_1\rangle^{N-1} + |\langle \varepsilon \ell_1 - \mathrm{i}\varepsilon \mathbf{b}\rangle|^{N-1} \right) dk_1 \\
&+ \int_{|k_1| \notin \left[\frac{|\ell_1|}{2}, 2|\ell_1|\right]} \langle \varepsilon k_1\rangle^{-N} \langle k_1 - \ell_1\rangle^{-M} \left(\langle \varepsilon k_1\rangle^{N-1} + |\langle \varepsilon \ell_1 - \mathrm{i}\varepsilon \mathbf{b}\rangle|^{N-1} \right) dk_1.
\end{aligned}$$

The first integral can be bounded above by

$$\int_{|k_1| \in \left[\frac{|\ell_1|}{2}, 2|\ell_1|\right]} \langle \varepsilon k_1\rangle^{-1} \langle k_1 - \ell_1\rangle^{-M} dk_1 \le \| \langle x\rangle^{-M} \|_{L^1(\mathbb{R})},$$

while the second can be bounded above by

$$\int_{\mathbb{R}} \langle \varepsilon k_1\rangle^{-N} \frac{\langle \varepsilon k_1\rangle^{N-1} + |\langle \varepsilon \ell_1 - \mathrm{i}\varepsilon \mathbf{b}\rangle|^{N-1}}{\langle k_1\rangle^M + \langle \ell_1\rangle^M} dk_1 \le \| \langle x\rangle^{-M-1+N} \|_{L^1(\mathbb{R})}.$$

So (7.10) is true for $C = \| \langle x\rangle^{-2} \|_{L^1(\mathbb{R})}$. By Young's inequality, see Theorem 0.3.1 [35], we conclude that (7.8) is true $C = \| \langle x\rangle^{-2} \|_{L^1(\mathbb{R})}$. Proceeding similarly we can show

$$\|T_\sigma \chi_{\mathbb{R}_-} f\|_{L^2(\mathbb{R})} \le C \|e^{-|x|\mathbf{b}} f\|_{L^2(\mathbb{R}_-)} \text{ for all } \sigma > 0 \text{ and for all } f,$$

concluding, for $C = \| \langle x\rangle^{-2} \|_{L^1(\mathbb{R})}$,

$$\|T_\sigma f\|_{L^2(\mathbb{R})} \le C \|e^{-|x|\mathbf{b}} f\|_{L^2(\mathbb{R})} \text{ for all } \sigma > 0 \text{ and for all } f.$$

Now we show that this remains true for $\sigma = 0$. For a sequence $\sigma_n \to 0^+$ then $T_{\sigma_n} f \xrightarrow{n \to +\infty} T_0 f$ point–wise for $f \in C_c^0(\mathbb{R})$. Then by the Fatou lemma and by the density of $C_c^0(\mathbb{R})$ in $L^2(\mathbb{R})$

$$\|T_0 f\|_{L^2(\mathbb{R})} \le C \|e^{-|x|\mathbf{b}} f\|_{L^2(\mathbb{R})} \text{ for all } f. \tag{7.11}$$

This is equivalent to (7.4).
The proof of (7.5) is similar, with the difference that for example

$$\chi_{\mathbb{R}_+} T_\sigma(\chi_{\mathbb{R}_+} f)(x)$$
$$= e^{-x\frac{\mathbf{b}}{2}} \int_{\mathbb{R}_+} dy f(y) e^{-y\frac{\mathbf{b}}{2}} \int_{\mathbb{R}^2} e^{\mathrm{i}xk_1 - \mathrm{i}y\ell_1} \left\langle \varepsilon k_1 + \mathrm{i}\varepsilon \frac{\mathbf{b}}{2} \right\rangle^{-\sigma} H_1(k_1 + \mathrm{i}\frac{\mathbf{b}}{2}, \ell_1 - \mathrm{i}\mathbf{b}) dk_1 d\ell_1,$$

and correspondingly we have there exists $C > 0$ such that

$$\|e^{x\frac{\mathbf{b}}{2}} \chi_{\mathbb{R}_+} T_\sigma \chi_{\mathbb{R}_+} f\|_{L^2(\mathbb{R})} \le C \|e^{-|x|\frac{\mathbf{b}}{2}} f\|_{L^2(\mathbb{R}_+)} \text{ for all } \sigma \ge 0 \text{ and for all } f,$$

which can be proved like (7.8), and so similarly the rest of the proof of (7.5). □

We will need the following analogue of Lemma 7.2.

Lemma 7.3. *Suppose that a Schwartz function $\mathcal{V} \in \mathcal{S}(\mathbb{R}, \mathbb{C})$ has the property that ts Fourier transform satisfies*

$$|\widehat{\mathcal{V}}(k_1 + ik_2)| \le C_M \langle k_1 \rangle^{-2} \text{ for all } (k_1, k_2) \in \mathbb{R} \times [\mathbf{b}, \mathbf{b}] \text{ and} \tag{7.12}$$
$$\widehat{\mathcal{V}} \in C^0(\mathbb{R} \times [-\mathbf{b}, \mathbf{b}]) \cap H(\mathbb{R} \times (-\mathbf{b}, \mathbf{b})),$$

with a number $\mathbf{b} > 0$. Then

$$\|[\mathcal{V}, \langle \mathrm{i}\varepsilon\partial_x \rangle^{-N}] \cosh(\mathbf{b}y)\|_{L^2(\mathbb{R}) \to L^2(\mathbb{R})} \le C_{\mathbf{b}}. \tag{7.13}$$

Proof. The proof is similar to that of Lemma 7.2. We have for $\sigma = 0$

$$[\mathcal{V}, \langle \mathrm{i}\varepsilon\partial_x \rangle^{-N}] f = \int_{\mathbb{R}} dy L^\sigma(x, y) f(y),$$

where we set

$$L^\sigma(x, y) = \int_{\mathbb{R}^2} e^{\mathrm{i}xk - \mathrm{i}y\ell} M_\sigma(k, \ell) dk d\ell \text{ with} \tag{7.14}$$
$$M_\sigma(k, \ell) = \widehat{\mathcal{V}}(k - \ell) \left(\langle \varepsilon k \rangle^{-N-\sigma} - \langle \varepsilon \ell \rangle^{-N-\sigma} \right).$$

Hence the generalized integral in (7.14) is absolutely convergent for $\sigma > 0$. But also for $\sigma = 0$ the operator

$$S_\sigma f(x) = \int_{\mathbb{R}} dy f(y) \int_{\mathbb{R}^2} e^{\mathrm{i}xk - \mathrm{i}y\ell} M_0(k, \ell),$$

defines an operator $L^2(\mathbb{R}) \to L^2(\mathbb{R})$, and the norm is uniformly bounded in $\sigma \ge 0$. Let us focus now on $k = k_1 + \mathrm{i}0$ and $\ell = \ell_1 - \mathrm{i}\mathbf{b}$

$$S_\sigma(\chi_{\mathbb{R}_+} f)(x) = \int_{\mathbb{R}_+} dy f(y) e^{-y\mathbf{b}} \int_{\mathbb{R}^2} e^{\mathrm{i}xk_1 - \mathrm{i}y\ell_1} M_\sigma(k_1, \ell_1 - \mathrm{i}\mathbf{b}) dk_1 d\ell_1.$$

Now we claim that there exists $C > 0$ such that

$$\|S_\sigma \chi_{\mathbb{R}_+} f\|_{L^2(\mathbb{R})} \le C \|e^{-|x|\mathbf{b}} f\|_{L^2(\mathbb{R}_+)} \text{ for all } \sigma > 0 \text{ and for all } f. \tag{7.15}$$

Set like before $g(y) = \chi_{\mathbb{R}_+}(y) f(y) e^{-y\mathbf{b}}$. Then

$$\widehat{S_\sigma(\chi_{\mathbb{R}_+} f)}(k_1) = \int_{\mathbb{R}} M_\sigma(k_1, \ell_1 - \mathrm{i}\mathbf{b}) \widehat{g}(\ell_1) d\ell_1.$$

We claim that for a fixed constant $C > 0$ we have

$$\sup_{k_1 \in \mathbb{R}} \int_{\mathbb{R}} |M_\sigma(k_1, \ell_1 - \mathrm{i}\mathbf{b})| d\ell_1 < C, \tag{7.16}$$

$$\sup_{\ell_1 \in \mathbb{R}} \int_{\mathbb{R}} |M_\sigma(k_1, \ell_1 - \mathrm{i}\mathbf{b}))| dk_1 < C. \tag{7.17}$$

We have

$$\begin{aligned} \int_{\mathbb{R}} |M_\sigma(k_1, \ell_1 - \mathrm{i}\mathbf{b}))| d\ell_1 &\lesssim \int_{\mathbb{R}} \langle k_1 - \ell_1 \rangle^{-2} \left(\langle \varepsilon k_1 \rangle^{-N-\sigma} + \left| \langle \varepsilon \ell_1 - \mathrm{i}\varepsilon \mathbf{b} \rangle^{-N-\sigma} \right| \right) d\ell_1 \\ &\lesssim \int_{\mathbb{R}} \langle k_1 - \ell_1 \rangle^{-2} d\ell_1 = \| \langle x \rangle^{-2} \|_{L^1(\mathbb{R})}. \end{aligned}$$

So (7.16) is true for $C = \| \langle x \rangle^{-2} \|_{L^1(\mathbb{R})}$. Next we prove (7.17). Proceeding as above

$$\begin{aligned} \int_{\mathbb{R}} |H_1(k_1, \ell_1 - \mathrm{i}\mathbf{b})| dk_1 &\lesssim \int_{\mathbb{R}} \langle k_1 - \ell_1 \rangle^{-2} \left(\langle \varepsilon k_1 \rangle^{-N-\sigma} + \left| \langle \varepsilon \ell_1 - \mathrm{i}\varepsilon \mathbf{b} \rangle^{-N-\sigma} \right| \right) dk_1 \\ &\lesssim \int_{\mathbb{R}} \langle k_1 - \ell_1 \rangle^{-2} dk_1 = \| \langle x \rangle^{-2} \|_{L^1(\mathbb{R})}. \end{aligned}$$

So (7.16)–(7.17) are true for $C = \| \langle x \rangle^{-2} \|_{L^1(\mathbb{R})}$ and by Young's inequality we conclude that (7.15) is true $C = \| \langle x \rangle^{-2} \|_{L^1(\mathbb{R})}$. Proceeding like above we conclude

$$\|S_\sigma f\|_{L^2(\mathbb{R})} \le C \|e^{-|x|\mathbf{b}} f\|_{L^2(\mathbb{R})} \text{ for all } \sigma > 0 \text{ and for all } f \ ,$$

which in turn, proceeding as above yields

$$\|S_0 f\|_{L^2(\mathbb{R})} \le C\|e^{-|x|\mathbf{b}} f\|_{L^2(\mathbb{R})} \text{ for all } f \ , \tag{7.18}$$

and yields (7.13). $\square$

Lemma 7.2 can be used to obtain the following result, whose proof is in [4].

Lemma 7.4. *There is a constant C_κ such that for all $0 < \varepsilon \le 1$ and $\mathbf{w} \in L^2_{-\frac{\kappa}{2}}$*

$$\| \prod_{j=1}^{N} R_{L_1}(\lambda_j^2) P_c \mathcal{A} \langle \mathrm{i}\varepsilon\partial_x \rangle^N \mathbf{w}\|_{L^2_{-\kappa}} \le C_\kappa \|\mathbf{w}\|_{L^2_{-\frac{\kappa}{2}}}. \quad \square \tag{7.19}$$

As an application of (7.13), we prove the following.

Lemma 7.5. *For any $u \in H^1$ we have*

$$\|\mathrm{sech}\left(\frac{4}{A}x\right) \mathcal{T} u\|_{L^2} \lesssim \varepsilon^{-N} \|\mathrm{sech}\left(\frac{2}{A}x\right) u\|_{L^2}, \tag{7.20}$$

$$\|\mathrm{sech}\left(\frac{4}{A}x\right) \partial_x \mathcal{T} u\|_{L^2} \lesssim \varepsilon^{-N} \|\mathrm{sech}\left(\frac{2}{A}x\right) u'\|_{L^2} + \|\mathrm{sech}\,(\kappa x)\, u\|_{L^2}. \tag{7.21}$$

Proof. We have

$$\|\mathrm{sech}\left(\frac{4}{A}x\right) \mathcal{T} u\|_{L^2} \le \| \langle \mathrm{i}\varepsilon\partial_x \rangle^{-N} \mathrm{sech}\left(\frac{4}{A}x\right) \mathcal{A}^* u\|_{L^2} + \| \left[\mathrm{sech}\left(\frac{4}{A}x\right), \langle \mathrm{i}\varepsilon\partial_x \rangle^{-N}\right] \mathcal{A}^* u\|_{L^2}$$
$$=: I + II.$$

We have

$$\mathrm{sech}\left(\frac{4}{A}x\right) \mathcal{A}^* = P_N(\partial_x) \mathrm{sech}\left(\frac{4}{A}x\right),$$

for an N–th order differential operator with smooth and bounded coefficients, uniformed bounded in $A \gg 1$, so that

$$I \le \| \langle \mathrm{i}\varepsilon\partial_x \rangle^{-N} P_N(\partial_x) \mathrm{sech}\left(\frac{4}{A}x\right) u\|_{L^2} \lesssim \varepsilon^{-N} \|\mathrm{sech}\left(\frac{4}{A}x\right) u\|_{L^2}.$$

We have

$$\begin{aligned} II &= \| \left[\mathrm{sech}\left(\frac{4}{A}x\right), \langle \mathrm{i}\varepsilon\partial_x \rangle^{-N}\right] \mathcal{A}^* u\|_{L^2} \\ &\le \| \left[\mathrm{sech}\left(\frac{4}{A}x\right), \langle \mathrm{i}\varepsilon\partial_x \rangle^{-N}\right] \cosh\left(\frac{2}{A}x\right) \|_{L^2 \to L^2} \|\mathrm{sech}\left(\frac{2}{A}x\right) \mathcal{A}^* u\|_{L^2} \\ &\lesssim \|\mathrm{sech}\left(\frac{2}{A}x\right) \mathcal{A}^* u\|_{L^2}, \end{aligned}$$

by Lemma 7.2, because $\int e^{-\mathrm{i}kx}\mathrm{sech}(x)dx = \pi\ \mathrm{sech}\left(\frac{\pi}{2}k\right)$, so that in the strip $k = k_1 + \mathrm{i}k_2$ with $|k_2| \le \mathbf{b} := 2/A$, then $\mathrm{sech}\left(\frac{\pi}{2}\frac{A}{4}k\right)$ satisfies the estimates required on $\widehat{\mathcal{V}}$ in (7.13). This completes the proof of (7.20). Now we turn to the proof of (7.21). We have

$$\mathcal{T}u = \mathcal{T}\partial_x u + \langle \mathrm{i}\varepsilon\partial_x\rangle^{-N} [\partial_x, \mathcal{A}^*]u.$$

By (7.20) we have

$$\|\mathrm{sech}\left(\frac{4}{A}x\right)\mathcal{T}\partial_x u\|_{L^2} \lesssim \varepsilon^{-N}\|\mathrm{sech}\left(\frac{2}{A}x\right)\partial_x u\|_{L^2}.$$

We have

$$[\partial_x, \mathcal{A}^*] = \sum_{j=1}^{N}\prod_{i=0}^{N-1-j} A^*_{N-i}\left(\log\psi_j\right)''\prod_{i=1}^{j-1} A^*_{j-i} = P_N(\partial_x)\mathrm{sech}(\kappa x),$$

with the convention $\prod_{i=0}^{l} B_i = B_0 \circ ... \circ B_l$, with ψ_k the ground state of L_k and with $P_N(\partial_x)$ and N–th order differential operator with bounded coefficients. We then have

$$\begin{aligned}\|\mathrm{sech}\left(\frac{2}{A}x\right)\langle \mathrm{i}\varepsilon\partial_x\rangle^{-N}[\partial_x, \mathcal{A}^*]u\|_{L^2} &\le \|\langle \mathrm{i}\varepsilon\partial_x\rangle^{-N} P_N(\partial_x)\mathrm{sech}(\kappa x)u\|_{L^2}\\ &\lesssim \varepsilon^{-N}\|\mathrm{sech}(\kappa x)u\|_{L^2}. \quad \square\end{aligned}$$

As an application of Lemma 7.3 we have the following.

Lemma 7.6. *For any $u \in H^1$,*

$$\|[\langle \mathrm{i}\varepsilon\partial_x\rangle^{-N}, V_D]\mathcal{A}^*u\|_{L^2} \lesssim \varepsilon\|\mathrm{sech}(\kappa x)\mathcal{T}u\|_{L^2}, \tag{7.22}$$

$$\|\cosh\left(\frac{\kappa}{2}x\right)[\langle \mathrm{i}\varepsilon\partial_x\rangle^{-N}, V_D]\mathcal{A}^*u\|_{L^2} \lesssim \varepsilon\|\mathrm{sech}\left(\frac{\kappa}{2}x\right)\mathcal{T}u\|_{L^2}. \tag{7.23}$$

Proof. We have

$$\|[\langle \mathrm{i}\varepsilon\partial_x\rangle^{-N}, V_D]\mathcal{A}^*u\|_{L^2} = \|\langle \mathrm{i}\varepsilon\partial_x\rangle^{-N}[V_D, \langle \mathrm{i}\varepsilon\partial_x\rangle^{N}]\mathcal{T}u\|_{L^2}.$$

Notice that

$$V_D = V - 2\sum_{j=1}^{N}\left(\log\psi_j\right)''.$$

By (1.2) and by the proof of Lemma 6, p. 156 and Theorem 2, p. 167 [10] it then follows

$$|V_D^{(l)}(x)| \le Ce^{-10\kappa|x|} \text{ for all } 0 \le l \le N+1. \tag{7.24}$$

This implies by an elementary integration by parts

$$|\widehat{V}_D(k_1+\mathrm{i}k_2)|\le C\langle k_1\rangle^{-N-1} \text{ in the strip } |k_2|\le 9\kappa. \tag{7.25}$$

Then in particular, from (7.4) we obtain

$$\|\langle \mathrm{i}\varepsilon\partial_x\rangle^{-N}[V_D,\langle \mathrm{i}\varepsilon\partial_x\rangle^{N}]\cosh(\kappa x)\mathrm{sech}(\kappa x)\mathcal{T}u\|_{L^2}\lesssim \varepsilon\|\mathrm{sech}(\kappa x)\mathcal{T}u\|_{L^2} \text{ and similarly}$$
$$\|\cosh\left(\frac{\kappa}{2}x\right)\langle \mathrm{i}\varepsilon\partial_x\rangle^{-N}[V_D,\langle \mathrm{i}\varepsilon\partial_x\rangle^{N}]\cosh\left(\frac{\kappa}{2}x\right)\mathrm{sech}\left(\frac{\kappa}{2}x\right)\mathcal{T}u\|_{L^2}$$
$$\lesssim \varepsilon\|\mathrm{sech}\left(\frac{\kappa}{2}x\right)\mathcal{T}u\|_{L^2}. \quad \square$$

8. Proof of Proposition 2.6

Using the operator $\mathcal{T}$ in (7.1), we consider the transformed variable

$$\mathbf{v}:=\mathcal{T}\boldsymbol{\eta}. \tag{8.1}$$

Then, for $\mathbf{L}_D:=\begin{pmatrix} L_D & 0\\ 0 & 1\end{pmatrix}$ the variable $\mathbf{v}$ satisfies

$$\begin{aligned}\dot{\mathbf{v}}=&-\mathcal{T}D\boldsymbol{\phi}[\mathbf{z}](\dot{\mathbf{z}}-\widetilde{\mathbf{z}})+\mathbf{J}\left(\mathbf{L}_D\mathbf{v}+\begin{pmatrix}[\langle \mathrm{i}\varepsilon\partial_x\rangle^{-N},V_D] & 0\\ 0 & 0\end{pmatrix}\mathcal{A}^*\boldsymbol{\eta}\right)\\ &+\mathbf{J}\mathcal{T}\left(\mathbf{f}[\boldsymbol{\phi}[\mathbf{z}]+\eta]-\mathbf{f}[\boldsymbol{\phi}[\mathbf{z}]]+\sum_{\mathbf{m}\in\mathbf{R}_{\min}}\mathbf{z}^{\mathbf{m}}\mathbf{G}_{\mathbf{m}}+\mathbf{R}[\mathbf{z}]\right).\end{aligned} \tag{8.2}$$

From Lemma 7.4, we have

$$\|\mathrm{sech}(\kappa x)\boldsymbol{\eta}\|_{L^2}\lesssim \|\mathrm{sech}(2^{-1}\kappa x)\mathbf{v}\|_{L^2}. \tag{8.3}$$

Set

$$\psi_{A,B}=\chi_A^2\varphi_B,\ \widetilde{S}_{A,B}=\frac{1}{2}\psi_{A,B}'+\psi_{A,B}\partial_x,$$

and consider the functionals

$$\mathcal{I}_{\mathrm{2nd},1}:=\frac{1}{2}\Omega(\mathbf{v},\widetilde{S}_{A,B}\mathbf{v}),\ \mathcal{I}_{\mathrm{2nd},2}:=\frac{1}{2}\Omega(\mathbf{v},\sigma_3e^{-\kappa\langle x\rangle}\mathbf{v}).$$

Lemma 8.1. *We have*

$$\begin{aligned}&\|\mathrm{sech}(2^{-1}\kappa x)v_1'\|_{L^2}^2+\|\mathrm{sech}(2^{-1}\kappa x)v_1\|_{L^2}^2+\dot{\mathcal{I}}_{\mathrm{2nd},1}\\ &\lesssim\left(\varepsilon^{-N}A^2\delta+A^{-1/2}\right)\|\boldsymbol{\eta}\|_{\boldsymbol{\Sigma}_A}^2+\sum_{\mathbf{m}\in\mathbf{R}_{\min}}|\mathbf{z}^{\mathbf{m}}|^2.\end{aligned} \tag{8.4}$$

Proof. We have

$$\dot{\mathcal{I}}_{2\text{nd},1} = -\Omega(\mathcal{T} D\boldsymbol{\phi}[\mathbf{z}](\dot{\mathbf{z}} - \widetilde{\mathbf{z}}), \widetilde{S}_{A,B}\mathbf{v}) + \langle \mathbf{L}_D \mathbf{v}, \widetilde{S}_{A,B}\mathbf{v}\rangle$$
$$+ \left\langle \begin{pmatrix} [\langle \mathrm{i}\varepsilon\partial_x\rangle^{-N}, V_D] & 0 \\ 0 & 0 \end{pmatrix} \mathcal{A}^* \boldsymbol{\eta}, \widetilde{S}_{A,B}\mathbf{v} \right\rangle + \langle \mathcal{T}\left(\mathbf{f}[\boldsymbol{\phi}[\mathbf{z}] + \boldsymbol{\eta}] - \mathbf{f}[\boldsymbol{\phi}[\mathbf{z}]]\right), \widetilde{S}_{A,B}\mathbf{v}\rangle + \langle \mathcal{T}\widetilde{\mathbf{R}}[\mathbf{z}], \widetilde{S}_{A,B}\mathbf{v}\rangle$$
$$=: D_1 + D_2 + D_3 + D_4 + D_5.$$

Following [16], for the main term D_2 we have

$$D_2 = \langle L_D v_1, \widetilde{S}_{A,B} v_1\rangle = -\int \left(\xi_1'^2 + V_B \xi_1\right) dx + D_{21} \text{ where } \xi_1 = \chi_A \zeta_B v_1,$$

and where

$$V_B = \frac{1}{2}\left(\frac{\zeta_B''}{\zeta_B} - \frac{(\zeta_B')^2}{\zeta_B^2}\right) - \frac{1}{2}\frac{\varphi_B}{\zeta_B^2} V_D' \text{ and}$$
$$D_{21} = \frac{1}{4}\int (\chi_A^2)'(\zeta_B^2)' v_1^2 + \frac{1}{2}\int \left(3(\chi_A')^2 + \chi_A''\chi_A\right)\zeta_B^2 v_1^2 - \int (\chi_A^2)'\varphi_B (v_1')^2 + \frac{1}{4}\int (\chi_A^2)'''\varphi_B v_1^2.$$

We claim

$$\int (\xi_1'^2 + V_B \xi_1)\, dx \gtrsim \left(\|\operatorname{sech}\left(\frac{\kappa}{2}x\right) v_1'\|_{L^2}^2 + \|\operatorname{sech}\left(\frac{\kappa}{2}x\right) v_1\|^2\right) - A^{-1}\|\boldsymbol{\eta}\|_{\boldsymbol{\Sigma}_A}^2. \tag{8.5}$$

The proof is like in [16, Lemma 3]. We have

$$\int_{|x|\le A} \operatorname{sech}(\kappa x)\, v_1^2 \le \int_{|x|\le A} \operatorname{sech}\left(\frac{\kappa}{2}x\right) \zeta_B^2 v_1^2 \le \int_{|x|\le A} \operatorname{sech}\left(\frac{\kappa}{2}x\right) \xi_1^2.$$

We have

$$\int_{|x|\le A} \operatorname{sech}(\kappa x)\, v_1'^2 \le \int_{|x|\le A} \operatorname{sech}\left(\frac{\kappa}{2}x\right) (\xi_1' - \zeta_B' v_1)^2 \lesssim \int_{|x|\le A} \operatorname{sech}\left(\frac{\kappa}{2}x\right) (\xi_1'^2 + \xi_1^2).$$

We have

$$\int_{|x|\ge A} \operatorname{sech}(\kappa x)\left(v_1'^2 + v_1^2\right) \le \operatorname{sech}\left(\frac{\kappa}{2}A\right) \int_{\mathbb{R}} \operatorname{sech}\left(\frac{8}{A}x\right)\left(v_1'^2 + v_1^2\right) dx$$
$$\lesssim \operatorname{sech}\left(\frac{\kappa}{2}A\right) \varepsilon^{-N} \int_{\mathbb{R}} \operatorname{sech}\left(\frac{4}{A}x\right)\left(\eta_1'^2 + \eta_1^2\right) dx \le A^{-1}\|\boldsymbol{\eta}\|_{\boldsymbol{\Sigma}_A}^2.$$

Finally, Lemma 4.1 and Assumption 1.12 imply

$$\int_{\mathbb{R}} \operatorname{sech}\left(\frac{\kappa}{2}x\right)(\xi_1'^2+\xi_1^2) \lesssim \int_{\mathbb{R}} (\xi_1'^2+V_B\xi_1)\,dx,$$

completing the proof of (8.5).

We next claim the following, which is [16, Lemma 4],

$$|D_{21}| \lesssim A^{-1/2}\left(\|\boldsymbol{\eta}\|_{\boldsymbol{\Sigma}_A}^2+\|\operatorname{sech}(\kappa x)\,\eta_1\|_{L^2}^2\right) \\ \lesssim A^{-1/2}\left(\|\boldsymbol{\eta}\|_{\boldsymbol{\Sigma}_A}^2+\varepsilon^{-N}\|\operatorname{sech}\left(\frac{\kappa}{2}x\right)\eta_1\|_{L^2}^2\right), \tag{8.6}$$

where the 2nd inequality follows from (7.20). Now we prove the first inequality.
Notice that $\chi_A(x)$ is constant for $|x|\notin[A,2A]$, so that

$$|(\chi_A^2)'(\zeta_B^2)'| \lesssim A^{-1}B^{-1}e^{-\frac{A}{B}},\ |\left(3(\chi_A')^2+\chi_A''\chi_A\right)\zeta_B^2| \lesssim A^{-2}e^{-\frac{A}{B}}$$

and since by $|\varphi_B|\lesssim B$ we have $|(\chi_A^2)'''\varphi_B|\lesssim A^{-3}B$ and $|(\chi_A^2)'\varphi_B|\lesssim A^{-2}B$, we have

$$\left|\frac{1}{4}(\chi_A^2)'(\zeta_B^2)'v_1^2+\frac{1}{2}\left(3(\chi_A')^2+\chi_A''\chi_A\right)\zeta_B^2v_1^2-(\chi_A^2)'\varphi_B(v_1')^2+\frac{1}{4}(\chi_A^2)'''\varphi_Bv_1^2\right| \\ \lesssim \frac{B}{A}\operatorname{sech}\left(\frac{8}{A}x\right)\left(v_1'^2+\frac{1}{A^2}v_1^2\right),$$

by Lemma 7.5 we have

$$|D_{21}| \lesssim A^{-1/2}\left(\|\operatorname{sech}\left(\frac{4}{A}x\right)v_1'\|_{L^2}+A^{-2}\|\operatorname{sech}\left(\frac{4}{A}x\right)v_1\|_{L^2}\right) \\ \lesssim A^{-1/2}\varepsilon^{-N}\left(\|\operatorname{sech}\left(\frac{2}{A}x\right)\eta_1'\|_{L^2}^2+A^{-2}\|\operatorname{sech}\left(\frac{2}{A}x\right)\eta_1\|_{L^2}^2+\|\operatorname{sech}(\kappa x)\,\eta_1\|_{L^2}^2\right),$$

which yields the desired inequality (8.6).
By Lemma 3.2 and by an analogue to (7.20), we have

$$|D_1| \lesssim |\dot{\mathbf{z}}-\widetilde{\mathbf{z}}|\|\operatorname{sech}(2\kappa x)\,\mathbf{v}\|_{L^2} \lesssim \delta\|\operatorname{sech}(\kappa x)\,\eta_1\|_{L^2}\|\operatorname{sech}(2\kappa x)\,\mathbf{v}\|_{L^2} \lesssim \delta\varepsilon^{-N}\|\operatorname{sech}(\kappa x)\,\eta_1\|_{L^2}^2.$$

By Lemma 7.6, we have

$$|D_3| = |\left\langle[\langle \mathrm{i}\varepsilon\partial_x\rangle^{-N}, V_D]\mathcal{A}^*\eta_1, \widetilde{S}_{A,B}v_1\right\rangle| \\ \le \|\cosh\left(\frac{\kappa}{2}x\right)[\langle \mathrm{i}\varepsilon\partial_x\rangle^{-N}, V_D]\mathcal{A}^*\eta_1\|_{L^2}\|\operatorname{sech}\left(\frac{\kappa}{2}x\right)\widetilde{S}_{A,B}v_1\|_{L^2} \\ \le \varepsilon\|\operatorname{sech}\left(\frac{\kappa}{2}x\right)v_1\|_{L^2}\left(\|\operatorname{sech}\left(\frac{\kappa}{2}x\right)v_1'\|_{L^2}+\|\operatorname{sech}\left(\frac{\kappa}{2}x\right)v_1\|_{L^2}\right) \\ \lesssim \varepsilon\left(\|\operatorname{sech}\left(\frac{\kappa}{2}x\right)v_1'\|_{L^2}^2+\|\operatorname{sech}\left(\frac{\kappa}{2}x\right)v_1\|_{L^2}^2\right),$$

where the upper bound can be absorbed inside the left hand side of (8.4).
Like in Lemma 6.1, we have

$$D_4 = \left\langle \int_0^1 \int_0^1 f''(s_1\phi_1[\mathbf{z}] + s_2\eta_1)\phi_1[\mathbf{z}]\eta_1 \, ds_1 ds_2, \widetilde{S}_{A,B} v_1 \right\rangle + \langle f(\eta_1), \widetilde{S}_{A,B} v_1 \rangle =: D_{41} + D_{42}.$$

Ignoring the irrelevant $ds_1 ds_2$ integral, we have

$$\begin{aligned} |D_{41}| &\lesssim \| \cosh(2\kappa x) \left(f''(s_1\phi_1[\mathbf{z}] + s_2\eta_1)\phi_1[\mathbf{z}] \operatorname{sech}(\kappa x)\, \eta_1 \right) \|_{L^2} \| \operatorname{sech}(\kappa x)\, \widetilde{S}_{A,B} v_1 \|_{L^2} \\ &\lesssim \|\mathbf{z}\| \| \operatorname{sech}(\kappa x)\, \eta_1 \|_{L^2} \left(\| \operatorname{sech}(\kappa x)\, v_1' \|_{L^2} + \| \operatorname{sech}(\kappa x)\, v_1 \|_{L^2} \right) \\ &\lesssim \delta \varepsilon^{-N} \left(\| \operatorname{sech}(\kappa x)\, v_1' \|_{L^2}^2 + \| \operatorname{sech}(\kappa x)\, v_1 \|_{L^2}^2 \right), \end{aligned}$$

which can be absorbed inside the left hand side of (8.4). Next, we have

$$\begin{aligned} |D_{42}| &= \left| \left\langle \operatorname{sech}\left(\frac{2}{A}x\right) f(\eta_1), \cosh\left(\frac{2}{A}x\right) \left(\frac{1}{2} \left(\chi_A^2 \varphi_B \right)' + \chi_A^2 \varphi_B \partial_x \right) v_1 \right\rangle \right| \\ &\lesssim \|\eta_1\|_{L^\infty} \left\| \operatorname{sech}\left(\frac{2}{A}x\right) \eta_1 \right\|_{L^2} \times \\ &\left(\left\| \cosh\left(\frac{6}{A}x\right) \psi_{A,B}' \right\|_{L^\infty} \left\| \operatorname{sech}\left(\frac{4}{A}x\right) v_1 \right\|_{L^2} + \left\| \cosh\left(\frac{6}{A}x\right) \psi_{A,B} \right\|_{L^\infty} \left\| \operatorname{sech}\left(\frac{4}{A}x\right) v_1' \right\|_{L^2} \right) \\ &\lesssim A\delta \|\boldsymbol{\eta}\|_{\boldsymbol{\Sigma}_A} \left(\left\| \operatorname{sech}\left(\frac{4}{A}x\right) v_1 \right\|_{L^2} + \left\| \operatorname{sech}\left(\frac{4}{A}x\right) v_1' \right\|_{L^2} \right) \\ &\lesssim \varepsilon^{-N} A\delta \|\boldsymbol{\eta}\|_{\boldsymbol{\Sigma}_A} \left(\left\| \operatorname{sech}\left(\frac{4}{A}x\right) \eta_1 \right\|_{L^2} + \left\| \operatorname{sech}\left(\frac{4}{A}x\right) \eta_1' \right\|_{L^2} \right) \lesssim \varepsilon^{-N} A^2 \delta \|\boldsymbol{\eta}\|_{\boldsymbol{\Sigma}_A}^2. \end{aligned}$$

Finally, we consider

$$D_5 = \left\langle \sum_{\mathbf{m} \in \mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}} \mathcal{T} \mathbf{G}_{\mathbf{m}}, \widetilde{S}_{A,B} \mathbf{v} \right\rangle + \langle \mathcal{T} \mathbf{R}[\mathbf{z}], \widetilde{S}_{A,B} \mathbf{v} \rangle =: D_{51} + D_{52}.$$

We focus on D_{51} which is the main term. We have

$$\begin{aligned} |\langle \mathbf{z}^{\mathbf{m}} \mathcal{T} \mathbf{G}_{\mathbf{m}}, \widetilde{S}_{A,B} \mathbf{v} \rangle| &\le |\mathbf{z}^{\mathbf{m}}| \| \cosh(\kappa x)\, \widetilde{S}_{A,B} \mathcal{T} \mathbf{G}_{\mathbf{m}} \|_{L^2} \| \operatorname{sech}(\kappa x)\, \mathbf{v} \|_{L^2} \\ &\lesssim \frac{1}{\mu} |\mathbf{z}^{\mathbf{m}}|^2 + \mu \| \operatorname{sech}(\kappa x)\, \mathbf{v} \|_{L^2}^2, \end{aligned}$$

where for μ small enough the last term can be absorbed in the left hand side of (8.4).

Collecting the estimates, we have the conclusion. □

Lemma 8.2. *We have*

$$\|e^{-\kappa\langle x\rangle/2}v_2\|_{L^2}+\dot{\mathcal{I}}_{\mathrm{2nd},2}\lesssim \|e^{-\kappa\langle x\rangle/2}v_1'\|_{L^2}^2+\|e^{-\kappa\langle x\rangle/2}v_1\|_{L^2}^2+\sum_{\mathbf{m}\in\mathbf{R}_{\min}}|\mathbf{z}^{\mathbf{m}}|^2+\delta A\|\boldsymbol{\eta}\|_{\boldsymbol{\Sigma}_A}^2. \quad (8.7)$$

Proof. Differentiating $\mathcal{I}_{\mathrm{2nd},2}$, we have

$$\begin{aligned}\dot{\mathcal{I}}_{\mathrm{2nd},2}=&-\Omega(\mathcal{T}D\boldsymbol{\phi}[\mathbf{z}](\dot{\mathbf{z}}-\widetilde{\mathbf{z}}),\sigma_3e^{-\kappa\langle x\rangle}\mathbf{v})+\left\langle \mathbf{L}_D\mathbf{v},\sigma_3e^{-\kappa\langle x\rangle}\mathbf{v}\right\rangle\\&+\left\langle[\langle \mathrm{i}\varepsilon\partial_x\rangle^{-N},V_D]\mathcal{A}^*\eta_1,\sigma_3e^{-\kappa\langle x\rangle}v_1\right\rangle+\left\langle\mathcal{T}\left(f[\boldsymbol{\phi}[\mathbf{z}]+\boldsymbol{\eta}]-f[\boldsymbol{\phi}[\mathbf{z}]]\right),e^{-\kappa\langle x\rangle}v_1\right\rangle\\&+\left\langle\mathcal{T}\widetilde{\mathbf{R}}[\mathbf{z}],\sigma_3e^{-\kappa\langle x\rangle}\mathbf{v}\right\rangle=:E_1+E_2+E_3+E_4+E_5.\end{aligned}$$

The main term is

$$E_2=-\|e^{-\kappa\langle x\rangle/2}v_2\|_{L^2}^2+\left\langle L_Dv_1,e^{-\kappa\langle x\rangle}v_1\right\rangle=-\|e^{-\kappa\langle x\rangle/2}v_2\|_{L^2}^2+E_{21},$$

with

$$|E_{21}|\lesssim\|e^{-\kappa\langle x\rangle/2}v_1'\|_{L^2}^2+\|e^{-\kappa\langle x\rangle/2}v_1\|_{L^2}^2.$$

By Lemma 3.2, we have

$$|E_1|\lesssim\delta\|e^{-\kappa\langle x\rangle/2}\mathbf{v}\|_{L^2}\|e^{-\kappa\langle x\rangle}\boldsymbol{\eta}\|_{L^2}\lesssim\delta\varepsilon^{-N}\|e^{-\kappa\langle x\rangle/2}\mathbf{v}\|_{L^2}^2.$$

By (7.23), we have

$$|E_3|=|\left\langle[\langle \mathrm{i}\varepsilon\partial_x\rangle^{-N},V_D]\mathcal{A}^*\eta_1,\sigma_3e^{-\kappa\langle x\rangle}v_1\right\rangle|\lesssim\varepsilon\|e^{-\frac{\kappa}{2}\langle x\rangle}v_1\|_{L^2}\|e^{-\kappa\langle x\rangle}v_1\|_{L^2}\le\varepsilon\|e^{-\frac{\kappa}{2}\langle x\rangle}v_1\|_{L^2}^2.$$

We write

$$E_4=\left\langle\int_0^1\int_0^1 f''(s_1\phi_1[\mathbf{z}]+s_2\eta_1)\phi_1[\mathbf{z}]\eta_1\,ds_1ds_2,e^{-\kappa\langle x\rangle}v_1\right\rangle+\left\langle f(\eta_1),\sigma_3e^{-\kappa\langle x\rangle}v_1\right\rangle=:E_{41}+E_{42}.$$

Ignoring the irrelevant ds_1ds_2 integral, we have

$$\begin{aligned}|E_{41}|&\lesssim\|\left(f''(s_1\phi_1[\mathbf{z}]+s_2\eta_1)\cosh(\kappa x)\,\phi_1[\mathbf{z}]\mathrm{sech}\,(\kappa x)\,\eta_1\right)\|_{L^2}\|e^{-\kappa\langle x\rangle}v_1\|_{L^2}\\&\lesssim\|\mathbf{z}\|\|\mathrm{sech}\,(\kappa x)\,\eta_1\|_{L^2}\|e^{-\kappa\langle x\rangle}v_1\|_{L^2}\lesssim\delta\|\mathrm{sech}\left(\frac{\kappa}{2}x\right)v_1\|_{L^2}^2.\end{aligned}$$

We have

$$\begin{aligned}|E_{42}|&=|\left\langle f(\eta_1),e^{-\kappa\langle x\rangle}v_1\right\rangle|\\&\lesssim\|\eta_1\|_{L^\infty}\|\mathrm{sech}\left(\frac{2}{A}x\right)\eta_1\|_{L^2}\|\mathrm{sech}\left(\frac{\kappa}{2}x\right)v_1\|_{L^2}\lesssim\delta A\left(\|\mathrm{sech}\left(\frac{\kappa}{2}x\right)v_1\|_{L^2}^2+\|\boldsymbol{\eta}\|_{\boldsymbol{\Sigma}_A}^2\right).\end{aligned}$$

We have

$$E_5 = \left\langle \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}} \mathcal{T}\mathbf{G}_{\mathbf{m}}, \sigma_3 e^{-\kappa\langle x\rangle}\mathbf{v} \right\rangle + \left\langle \mathcal{T}\mathbf{R}[\mathbf{z}], \sigma_3 e^{-\kappa\langle x\rangle}\mathbf{v} \right\rangle =: E_{51} + E_{52}.$$

We focus on D_{51} which is the main term, the other being simpler. We have

$$|\left\langle \mathbf{z}^{\mathbf{m}} \mathcal{T}\mathbf{G}_{\mathbf{m}}, \sigma_3 e^{-\kappa\langle x\rangle}\mathbf{v} \right\rangle| \le |\mathbf{z}^{\mathbf{m}}| \|\mathcal{T}\mathbf{G}_{\mathbf{m}}\|_{L^2} \|\operatorname{sech}(\kappa x)\,\mathbf{v}\|_{L^2} \lesssim \frac{1}{\mu}|\mathbf{z}^{\mathbf{m}}|^2 + \mu\|\operatorname{sech}(\kappa x)\,\mathbf{v}\|_{L^2}^2$$

$$= \frac{1}{\mu}|\mathbf{z}^{\mathbf{m}}|^2 + \mu\|\operatorname{sech}(\kappa x)\,v_1\|_{L^2}^2 + \mu\|\operatorname{sech}(\kappa x)\,v_2\|_{L^2}^2,$$

where for μ small enough the very last term in v_2 can be absorbed in the left hand side of (8.7) Collecting the estimates, we have the conclusion. □

Combining Lemmas 8.1 and 8.2, we have

Lemma 8.3. *For any* $\mu > 0$*, we have*

$$\int_0^T \left(\|\operatorname{sech}\left(\frac{\kappa}{2}x\right) v_1'\|_{L^2}^2 + \|\operatorname{sech}\left(\frac{\kappa}{2}x\right)\mathbf{v}\|_{L^2}^2 \right) \lesssim B\varepsilon^{-N}\delta^2$$

$$+ \left(\varepsilon^{-1}A^2\delta + A^{-1/2}\right) \int_0^T \|\boldsymbol{\eta}\|_A^2 + \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \|\mathbf{z}^{\mathbf{m}}\|_{L^2(0,T)}^2.$$

Proof. The claim follows from Lemmas 8.1 and 8.2 and

$$|\mathcal{I}_{2\mathrm{nd},1}| \lesssim B\varepsilon^{-N}\delta^2, \ |\mathcal{I}_{2\mathrm{nd},2}| \lesssim \varepsilon^{-N}\delta^2. \quad \square \tag{8.8}$$

Proof of Proposition 2.6. It is a consequence of Lemma 8.3 and inequality (8.3). □

Data availability

No data was used for the research described in the article.

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