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On the Geometry of Quantum Spheres and Hyperboloids

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Abstract. We study two classes of quantum spheres and hyperboloids, one class consisting of homogeneous spaces, which are $*$-quantum spaces for the quantum orthogonal group $\mathcal{O}(SO_q(3))$. We construct line bundles over the quantum homogeneous space associated with the quantum subgroup $SO(2)$ of $SO_q(3)$. The line bundles are associated to the quantum principal bundle via representations of $SO(2)$ and are described dually by finitely-generated projective modules $\mathcal{E}_n$ of rank 1 and of degree computed to be an even integer $-2n$. The corresponding idempotents, that represent classes in the K-theory of the base space, are explicitly worked out and are paired with two suitable Fredhom modules that compute the rank and the degree of the bundles. For q real, we show how to diagonalise the action (on the base space algebra) of the Casimir operator of the Hopf algebra $\mathcal{U}_{q^{1/2}}(sl_2)$ which is dual to $\mathcal{O}(SO_q(3))$.

Contents

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1. Introduction

The study of integer quantum Hall effect systems on a plane, a sphere or a hyperboloid is treated in a unified way by group-theoretical methods related to the symmetry group of the corresponding configuration space [12]. This leads to the study of gauged Laplacians on line bundles over the corresponding space, the gauging coming from a connection on the bundle. It is natural to look for models of Hall systems on noncommutative spaces. These models will enjoy symmetries coming from quantum groups and lead to potentially interesting mathematical/physics models. Motivated by this, in the present paper we consider two classes of quantum spheres and quantum hyperboloids with quantum symmetry given by the quantum orthogonal group $SO_q(3)$.

In the approach of [7], the quantized algebra of functions $\mathcal{O}(SO_q(N))$ on the quantum orthogonal group $SO_q(N)$ in any dimension is given as an algebra generated by N^2 elements subject to commutation relations that depend on the entries of a matrix R which is a solution of the quantum Yang–Baxter equation. The matrix R decomposes in terms of projections and, as we shall see in Sect. 4, this allows one to introduce quantum spaces carrying natural coactions of the Hopf algebra $\mathcal{O}(SO_q(N))$.

When restricting to $\mathcal{O}(SO_q(3))$, a first class of quantum spheres and hyperboloids is obtained as real forms of the quantum vector spaces of $\mathcal{O}(SO_q(3))$ associated with the q-symmetrizer projection P_- in the decomposition of the R-matrix alluded to before. The nature of the quantum space is determined by the $*$-structure: for $q \in \mathbb{R}$ one gets a sphere — the equatorial Podleś sphere, while for $|q| = 1$ an hyperboloid.

A second class, described in Sect. 5, is given by quantum homogeneous $SO_q(3)$-spaces arising from the coaction of the quantum subgroup $SO(2)$ of $SO_q(3)$ on the latter. Again, the $*$-structure discriminates between a quantum 2-sphere — now the standard Podleś sphere, and an hyperboloid. In both cases the quantum homogenous space is explicitly determined as the subalgebra B of coinvariants of $\mathcal{O}(SO_q(3))$ for the right coaction of $\mathcal{O}(SO(2))$. This also makes use of the identification of $SL_{\mathsf{s}}(2)$, for $\mathsf{s} = q^{\frac{1}{2}}$, as the 'double covering' of $SO_q(3)$, that is of the existence of a Hopf algebra isomorphism between the coordinate algebra $\mathcal{O}(SO_q(3))$ and the subalgebra of $\mathcal{O}(SL_{\mathsf{s}}(2))$ made of invariant elements for the action of the group $\mathbb{Z}_2$ (see Sect. 3.3). The algebra extension $B \subset \mathcal{O}(SO_q(3))$ is shown to be an $SO(2)$ quantum principal bundle (an $\mathcal{O}(SO(2))$-Galois extension). This quantum principal bundle has associated (modules of sections of) line bundles coming from the representations of $SO(2)$. The modules are given by finitely-generated projective modules $\mathcal{E}_n$ of rank 1 and degree an even integer $-2n$. The corresponding idempotents $p_n \in \mathrm{Mat}_{|2n|+1}(B)$, describing classes in the K-theory of the algebra B, are explicitly worked out. These idempotents are different from those usually used for the Podleś sphere, a fact that reflects in a simpler recursion formula for their trace and thus for an easier computation of their degree (Proposition 5.3).

For the study of Laplacian operators on the two $*$-quantum homogeneous spaces of $\mathcal{O}(SO_q(3))$ and of 'gauged' Laplacian operators on bundles over them (in the line of [9]), the last section of the paper is dedicated to the study of the quantum Casimir element of $\mathcal{U}_{q^{1/2}}(sl_2)$, the Hopf algebra dual to the Hopf algebra $\mathcal{O}(SO_q(3))$. For q real, the Casimir operator, which acts on the left on the algebra B and on modules of sections of lines bundles over the latter, is diagonalised via the commuting right action of $\mathcal{U}_{q^{1/2}}(sl_2)$.

2. The Quantum Special Orthogonal Groups $SO_q(N)$

We recall the construction of the coordinate algebra $\mathcal{O}(O_q(N))$ of the quantum orthogonal group $O_q(N)$; see e.g. [8, §9.3]. Let $q \in \mathbb{C}$, $q \neq 0$, fixed. Let N be an integer. For each index $i = 1, \ldots, N$, let $i' = N + 1 - i$ and define $\rho_i = \frac{N}{2} - i$ if $i < i'$, with $\rho_{i'} = -\rho_i$ and $\rho_i = 0$ if $i = i'$. For all indices $i, j, m, n = 1, \ldots, N$ we define complex numbers

$$R^{ij}_{mn} = q^{\delta_{ij} - \delta_{ij'}} \delta_{im} \delta_{jn} + (q - q^{-1}) \theta(i - m)(\delta_{jm} \delta_{in} - q^{-\rho_j - \rho_m} \delta_{ij'} \delta_{nm'}) \tag{2.1}$$

where θ is the Heaviside function, whose value is one for strictly positive argument and zero otherwise. One considers the free algebra $\mathbb{C}\langle u_{ij} \rangle$ generated over $\mathbb{C}$ by N^2 elements u_{ij}, $i, j = 1, \ldots, N$, modulo the two-sided ideal generated by elements

$$\sum_{k,l} \left(R^{ji}_{kl} u_{km} u_{ln} - u_{ik} u_{jl} R^{lk}_{mn} \right), \ i, j, m, n = 1, \ldots, N. \tag{2.2}$$

Explicitly, the quotient algebra is generated by elements u_{ij} subject to relations

$$\begin{aligned} q^{\delta_{ij} - \delta_{ij'}} u_{jm} u_{in} = {} & q^{\delta_{mn} - \delta_{mn'}} u_{in} u_{jm} + \lambda \left(\theta(n - m) - \theta(j - i) \right) u_{im} u_{jn} \\ & + \lambda \delta_{ij'} \sum_k \theta(j - k) q^{-\rho_i - \rho_k} u_{km} u_{k'n} \\ & - \lambda \delta_{nm'} \sum_k \theta(k - m) q^{-\rho_m - \rho_{k'}} u_{ik'} u_{jk} \, , \end{aligned} \tag{2.3}$$

where we set $\lambda := q - q^{-1}$. In concise matrix notations, it is the algebra generated by the entries of the $N \times N$ matrix $u = (u_{ij})$ with relations

$$R u_1 u_2 = u_2 u_1 R \, , \tag{2.4}$$

for R the $N^2 \times N^2$ matrix of entries $R = (R^{ij}_{mn})$ (where i, m are respectively the row and column block indices, and j, n are respectively the row and column index inside each block) and $u_1 = u \otimes \mathrm{I}$, $u_2 = \mathrm{I} \otimes u$ with I the unit matrix.

The algebra $\mathcal{O}(O_q(N))$ is obtained by requiring the generators u_{ij} to satisfy the additional orthogonality (metric) condition

$$uCu^tC = \mathrm{I} = Cu^tCu\,, \tag{2.5}$$

for u^t the transpose of u and $C = C^{-1}$ the invertible matrix of entries $C_{kj} = \delta_{kj'}q^{-\rho_k}$. In the classical limit $q = 1$, the condition (2.5) is the metric condition defining orthogonal matrices. The condition (2.5) corresponds to one single additional relation $Q_q - 1 = 0$ (see [8, page 319]), where Q_q can equivalently be expressed in terms of any index j as

$$Q_q = \sum_k C_{j'j}C_{kk'}u_{kj}u_{k'j'} = \sum_k C_{j'j}C_{kk'}u_{jk}u_{j'k'}\,. \tag{2.6}$$

The algebra $\mathcal{O}(O_q(N))$ is a Hopf algebra with coproduct Δ, counit ε and antipode S given on generators respectively by

$$\Delta(u_{kj}) = \sum_m u_{km} \otimes u_{mj}\,, \quad \varepsilon(u_{kj}) = \delta_{ij}\,, \quad S(u_{kj}) = q^{\rho_j - \rho_k}u_{j'k'} \tag{2.7}$$

or in matrix notation $\Delta(u) = u \otimes u$, $\varepsilon(u) = \mathrm{I}$, $S(u) = Cu^tC$. The Hopf algebra $\mathcal{O}(O_q(N))$ is the coordinate algebra of the quantum orthogonal group $O_q(N)$; in the limit $q = 1$ it is the coordinate algebra of the complex Lie group $O(N)$.

For $N = 2$ the construction above results into the (commutative) coordinate Hopf algebra $\mathcal{O}(O(2))$ of the classical group $O(2)$.

2.1. Real Forms

The algebra $\mathcal{O}(O_q(N))$ of the quantum orthogonal group admits different $*$-structures $* : \mathcal{O}(O_q(N)) \to \mathcal{O}(O_q(N))$, leading to different real forms (see [8, §9.3.5]). For the present paper we consider the following two choices.

For $q \in \mathbb{R}$, define

$$(u_{jk})^* := S(u_{kj}) = q^{\rho_j - \rho_k}u_{j'k'}. \tag{2.8}$$

Then the defining matrix u is unitary, $uu^\dagger = \mathrm{I} = u^\dagger u$, with $(u^\dagger)_{kj} = (u_{jk})^* = S(u_{kj})$. The resulting Hopf $*$-algebra is the coordinate algebra $\mathcal{O}(O_q(N, \mathbb{R}))$ of the compact quantum group $O_q(N, \mathbb{R})$.

For $|q| = 1$, define

$$(u_{jk})^* := u_{jk}. \tag{2.9}$$

The resulting Hopf $*$-algebra is the coordinate algebra $\mathcal{O}(O_q(n, n, \mathbb{R}))$ of the real quantum group $O_q(n, n, \mathbb{R})$ for $N = 2n$ even, or $\mathcal{O}(O_q(n, n+1, \mathbb{R}))$ of the real quantum group $O_q(n, n+1, \mathbb{R})$ for $N = 2n + 1$ odd.

The two $*$-structures above correspond respectively to the classical real groups $O(N, \mathbb{R})$ and $O(n, n, \mathbb{R})$ or $O(n, n+1, \mathbb{R})$.

The notation before will be used throughout the paper. So we stress that $O_q(N)$ denotes the complex quantum orthogonal group while $O_q(N, \mathbb{R})$ and $O_q(n, n, \mathbb{R})$ for $N = 2n$, or $O_q(n, n+1, \mathbb{R})$ for $N = 2n + 1$, are the real versions.

2.2. Quantum Spaces and Exterior Algebras

We recall from [7] (see also [8, §8.4.3, §9.3.2]) that the matrix R satisfies a cubic equation,

$$(\widehat{R}-q\mathrm{I})(\widehat{R}+q^{-1}\mathrm{I})(\widehat{R}-q^{1-N}\mathrm{I})=0$$

in terms of the matrix $\widehat{R}=(\widehat{R}^{kj}_{mn}):=(R^{jk}_{mn})$. Moreover for $N>2$, and assuming $(1+q^2)(1+q^{2-N})(1-q^{-N})\neq 0$ (a condition that in particular excludes $q=\pm 1$), the matrix $\widehat{R}$ can be decomposed as

$$\widehat{R}=qP_+-q^{-1}P_-+q^{1-N}P_0\ , \tag{2.10}$$

with P_α, $\alpha=\pm,0$ mutually orthogonal idempotents: $P_\alpha^2=P_\alpha$, and $P_\alpha P_\beta=0$, for $\alpha\neq\beta$. In the decomposition (2.10), the matrix P_- is the q-symmetrizer matrix on $\mathbb{C}^N\times\mathbb{C}^N$

$$P_-=\frac{\widehat{R}^2-(q+q^{1-N})\widehat{R}+q^{-N+2}\,\mathrm{I}}{q^{-2}(1+q^2)(1+q^{2-N})}.$$

This defines the quantum space

$$V=\mathbb{C}^N_q:=\mathbb{C}\langle z_m\rangle/\langle P_-\mathsf{z}_1\mathsf{z}_2\rangle \tag{2.11}$$

as the quotient of the free algebra $\mathbb{C}\langle z_m\rangle$ with generators z_m, $m=1,\dots N$, by the two sided ideal generated by relations $P_-\mathsf{z}_1\mathsf{z}_2$, with $\mathsf{z}_1=\mathsf{z}\otimes\mathrm{I}$ and with $\mathsf{z}_2=\mathrm{I}\otimes\mathsf{z}$ and $\mathsf{z}=(z_m)$.

The idempotents P_+ and P_0 are given by

$$P_+=\frac{\widehat{R}^2-(q^{1-N}-q^{-1})\widehat{R}-q^{-N}\mathrm{I}}{(1+q^2)(1-q^{-N})}\ ,\quad P_0=\frac{\widehat{R}^2-(q-q^{-1})\widehat{R}-\mathrm{I}}{(q^{-N}-1)(1+q^{2-N})}$$

and are used to define a quantized exterior algebra

$$\Lambda_q(V):=\mathbb{C}\langle e_m\rangle/\langle P_+\mathsf{e}_1\mathsf{e}_2,\,P_0\,\mathsf{e}_1\mathsf{e}_2\rangle \tag{2.12}$$

with generators e_m, $m=1,\dots N$ and notation as before.

Both V and $\Lambda_q(V)$ carry a left coaction of $\mathcal{O}(O_q(N))$ given by the algebra morphisms

$$z_j\mapsto\sum_k u_{jk}\otimes z_k\ ,\quad e_j\mapsto\sum_k u_{jk}\otimes e_k.$$

The subspace of $\Lambda_q(V)$ made of degree N polynomials is one-dimensional and thus there is a unique element $D_q(u)\in\mathcal{O}(O_q(N))$ such that the coaction is given by $\xi\mapsto D_q(u)\otimes\xi$ on elements ξ in $\Lambda_q(V)$ of degree N. The element $D_q(u)$ is called the quantum determinant of the matrix u. It is shown to belong to the centre of the algebra $\mathcal{O}(O_q(N))$ and to be group-like, that is $\Delta(D_q(u))=D_q(u)\otimes D_q(u)$ and $\varepsilon(D_q(u))=1$.

The two-sided ideal generated by $D_q(u)-1$ is a Hopf ideal of $\mathcal{O}(O_q(N))$ and the quotient Hopf algebra $\mathcal{O}(O_q(N))/\langle D_q(u)-1\rangle$ is the coordinate algebra $\mathcal{O}(SO_q(N))$ of the special orthogonal quantum group $SO_q(N)$.

3. The Quantum Orthogonal Group $SO_q(3)$

We specialize the above to the case $N = 3$. For each index $i = 1, 2, 3$, one has $i' = 4 - i$ so that $1' = 3$, $2' = 2$ and $\rho_1 = \frac{1}{2}$, $\rho_2 = 0$, $\rho_3 = -\frac{1}{2}$. The matrix $R = (R^{kj}_{mn})$ is the lower-triangular matrix

$$R = \left(\begin{array}{ccc|ccc|ccc} q & & & & & & & & \\ 0 & 1 & & & & & & & \\ 0 & 0 & q^{-1} & & & & & & \\ \hline 0 & \lambda & 0 & 1 & & & & & \\ 0 & 0 & -q^{\frac{1}{2}}\lambda & 0 & 1 & & & & \\ 0 & 0 & 0 & 0 & 0 & 1 & & & \\ \hline & & & 0 & -q^{\frac{1}{2}}\lambda & 0 & q^{-1} & & \\ & & & 0 & 0 & \lambda & 0 & 1 & \\ & & & 0 & 0 & 0 & 0 & 0 & q \end{array}\right) \tag{3.1}$$

(where $\lambda = q - q^{-1}$ as before) with non-zero entries

$$\begin{array}{llll} R^{11}_{11} = R^{33}_{33} = q & R^{13}_{13} = R^{31}_{31} = q^{-1} & R^{22}_{22} = 1 & R^{12}_{12} = R^{21}_{21} = R^{23}_{23} = R^{32}_{32} = 1 \\ R^{21}_{12} = \lambda & R^{22}_{13} = -q^{\frac{1}{2}}\lambda & R^{32}_{23} = \lambda & R^{31}_{22} = -q^{\frac{1}{2}}\lambda. \end{array}$$

According to the general theory, the Hopf algebra $\mathcal{O}(O_q(3))$ is the free algebra generated by elements u_{ij}, $i, j = 1, 2, 3$ modulo the ideal $\langle Ru_1u_2 - u_2u_1R;\ uCu^tC - \mathrm{I},\ Cu^tCu = \mathrm{I}\rangle$ giving relations (2.2) and (2.5) in the quotient. In matrix form the antipode is

$$u = (u_{ij}) \ \mapsto \ S(u) = Cu^tC = \begin{pmatrix} u_{33} & q^{-\frac{1}{2}}u_{23} & q^{-1}u_{13} \\ q^{\frac{1}{2}}u_{32} & u_{22} & q^{-\frac{1}{2}}u_{12} \\ q\ u_{31} & q^{\frac{1}{2}}u_{21} & u_{11} \end{pmatrix}.$$

3.1. The Quantum Determinant

From the decomposition (2.10) of the matrix in (3.1), one gets a quantum space $V = \mathbb{C}^3_q$, and an exterior algebra $\Lambda_q(V)$, both carrying a right coaction of $\mathcal{O}(O_q(3))$. We will return to $\mathbb{C}^3_q$ in Sect. 4 below. Here we consider the exterior algebra $\Lambda_q(V)$ in (2.12), which allows one to define the quantum determinant $D_q(u)$.

The graded algebra $\Lambda_q(V)$ is generated in degree one by elements e_1, e_2, e_3 with relations

$$\begin{aligned} &(e_1)^2 = 0\,, \qquad (e_3)^2 = 0\,, \qquad (e_2)^2 = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})e_1e_3\,, \\ &e_3e_2 = -qe_2e_3\,,\ e_3e_1 = -e_1e_3\,,\ e_2e_1 = -qe_1e_2\,, \end{aligned}$$

and coaction of $\mathcal{O}(O_q(3))$ given by $\rho : e_j \mapsto \sum_k u_{jk} \otimes e_k$ on the generators and extended to the whole $\Lambda_q(V)$ as an algebra map. From the commutation relations (3.2) it follows that in degree three all elements are proportional:

$$e_ke_me_n = \varepsilon_{kmn}\ w \qquad \text{for (say) } w := e_1e_2e_3, \quad \forall k, l, m = 1, 2, 3.$$

The only non zero components of the tensor ε are found to be

$$\begin{aligned} &\varepsilon_{123} = 1\,, \quad \varepsilon_{132} = -q\,, \quad \varepsilon_{213} = -q\,, \quad \varepsilon_{231} = q\,, \\ &\varepsilon_{312} = q\,, \quad \varepsilon_{321} = -q^2\,, \quad \varepsilon_{222} = -q(q^{\frac{1}{2}} - q^{-\frac{1}{2}})\,. \end{aligned} \tag{3.2}$$

Hence there exists a unique element $D_q(u) \in \mathcal{O}(O_q(3))$ such that $\rho(\xi) = D_q(u) \otimes \xi$ for each ξ monomial in $\Lambda_q(V)$ of degree three. For $\xi = w = e_1e_2e_3$ one obtains the following explicit formula for the quantum determinant $D_q(u)$:

$$\begin{aligned} D_q(u) = \ & u_{11}u_{22}u_{33} - qu_{12}u_{21}u_{33} - qu_{11}u_{23}u_{32} + qu_{12}u_{23}u_{31} \\ & +qu_{13}u_{21}u_{32} - q^2u_{13}u_{22}u_{31} - q(q^{\frac{1}{2}} - q^{-\frac{1}{2}})u_{12}u_{22}u_{32} \,. \end{aligned} \tag{3.3}$$

The quotient Hopf algebra $\mathcal{O}(O_q(3))/\langle D_q(u) - 1\rangle$ is the coordinate algebra $\mathcal{O}(SO_q(3))$ of the special orthogonal quantum group $SO_q(3)$.

The determinant $D_q(u)$ admits different equivalent expressions as a degree three polynomial on the generators u_{jk} of $\mathcal{O}(O_q(3))$: for each triple of indices $a, b, c = 1, 2, 3$ such that $\varepsilon_{abc} \neq 0$, being $\rho : e_j \mapsto \sum_k u_{jk} \otimes e_k$, one computes

$$\rho(e_ae_be_c) = \sum_{m,n,p} u_{am}u_{bn}u_{cp} \otimes e_me_ne_p = \sum_{m,n,p} u_{am}u_{bn}u_{cp} \otimes \varepsilon_{mnp}w$$

and therefore,

$$D_q(u) = \sum_m u_{am}\widehat{u}_{ma} \,, \qquad \text{with} \qquad \widehat{u}_{ma} := \varepsilon_{abc}^{-1} \sum_{n,p} \varepsilon_{mnp}u_{bn}u_{cp} \,. \tag{3.4}$$

We refer to this formula $D_q(u) = \sum_m u_{am}\widehat{u}_{ma}$ as the expansion of $D_q(u)$ with respect to the a-row and we call the element $\widehat{u}_{ma}$ the cofactor of u_{ma} and $\mathsf{cof}(u) := {}^t\widehat{u}$ the matrix of cofactors. Notice that each cofactor $\widehat{u}_{ma}$ admits more than one expression, one for each possible choice of indices b, c such that $\varepsilon_{abc} \neq 0$: for each $m = 1, 2, 3$ one computes

$$\begin{aligned} \widehat{u}_{m1} &= \sum_{n,p} \varepsilon_{mnp}u_{2n}u_{3p} = -q^{-1}\sum_{n,p} \varepsilon_{mnp}u_{3n}u_{2p} \\ \widehat{u}_{m2} &= -q^{-1}\sum_{n,p} \varepsilon_{mnp}u_{1n}u_{3p} = q^{-1}\sum_{n,p} \varepsilon_{mnp}u_{3n}u_{1p} \\ &= -q^{-1}(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^{-1}\sum_{n,p} \varepsilon_{mnp}u_{2n}u_{2p} \\ \widehat{u}_{m3} &= q^{-1}\sum_{n,p} \varepsilon_{mnp}u_{1n}u_{2p} = -q^{-2}\sum_{n,p} \varepsilon_{mnp}u_{2n}u_{1p} \,. \end{aligned}$$

We explicitly list all of them in Appendix C.

The matrix $\widehat{u}$ of cofactors can be identified with the antipode matrix. For this we need the following result for which we use explicit commutation relations of the type (2.3) with the matrix (3.1) as well as the orthogonality conditions.

Proposition 3.1. *Let* $\widehat{u} = (\widehat{u}_{jk})_{j,k=1,2,3}$ *be the transpose of the matrix of cofactors,* $\widehat{u}_{ma} = \mathsf{cof}(u)_{am}$*. Then* $u\widehat{u} = D_q(u)\mathrm{I}$.

Proof. The lengthy proof is in Appendix A. □

As a direct consequence of this proposition (and of the uniqueness of the antipode), in the quotient algebra $\mathcal{O}(SO_q(3)) = \mathcal{O}(O_q(3))/\langle D_q(u) - 1\rangle$ we can then identify the matrix $\widehat{u} = (\widehat{u}_{jk})_{j,k=1,2,3}$ of cofactors with the antipode matrix:

$$S(u) = \begin{pmatrix} u_{33} & q^{-\frac{1}{2}}u_{23} & q^{-1}u_{13} \\ q^{\frac{1}{2}}u_{32} & u_{22} & q^{-\frac{1}{2}}u_{12} \\ q\ u_{31} & q^{\frac{1}{2}}u_{21} & u_{11} \end{pmatrix} = S(u)u\widehat{u} = \widehat{u} = {}^{t}\mathsf{cof}(u)\,. \tag{3.5}$$

In particular, for later use in the study of coinvariant elements in Sect. 5 below, we observe we have the following identification among elements of the second column of the matrix u (or second raw of the matrix $S(u)$) and the corresponding cofactors:

$$\begin{aligned} q^{-\frac{1}{2}}u_{12} &= -u_{11}u_{23} + u_{13}u_{21} - (q^{\frac{1}{2}} - q^{-\frac{1}{2}})u_{12}u_{22} \\ &= q^{-1}u_{21}u_{13} - q^{-1}u_{23}u_{11} + q^{-1}(q^{\frac{1}{2}} - q^{-\frac{1}{2}})u_{22}u_{12} \\ u_{22} &= u_{11}u_{33} - u_{13}u_{31} + (q^{\frac{1}{2}} - q^{-\frac{1}{2}})u_{12}u_{32} \\ &= -u_{31}u_{13} + u_{33}u_{11} - (q^{\frac{1}{2}} - q^{-\frac{1}{2}})u_{32}u_{12} \\ &= (q^{\frac{1}{2}} - q^{-\frac{1}{2}})^{-1}(u_{21}u_{23} - u_{23}u_{21} + (q^{\frac{1}{2}} - q^{-\frac{1}{2}})u_{22}u_{22}) \\ q^{\frac{1}{2}}u_{32} &= -qu_{21}u_{33} + qu_{23}u_{31} - q(q^{\frac{1}{2}} - q^{-\frac{1}{2}})u_{22}u_{32} \\ &= u_{31}u_{23} - u_{33}u_{21} + (q^{\frac{1}{2}} - q^{-\frac{1}{2}})u_{32}u_{22}\,. \end{aligned} \tag{3.6}$$

3.2. Two Real Forms of $SO_q(3)$

As already mentioned above for general N, the Hopf algebra $\mathcal{O}(O_q(3))$ can be equipped with different real structures (2.8) or (2.9), depending on the deformation parameter q:

$$(u_{jk})^* = S(u_{kj}) = q^{\rho_j - \rho_k}u_{j'k'} \qquad \text{for } q \in \mathbb{R} \tag{3.7}$$

$$(u_{jk})^* = u_{jk} \qquad \text{for } |q| = 1\,. \tag{3.8}$$

These lead to the Hopf $*$-algebras $\mathcal{O}(O_q(3,\mathbb{R}))$ for $q \in \mathbb{R}$ and $\mathcal{O}(O_q(1,2,\mathbb{R}))$ for $|q| = 1$. These correspond to the classical real groups $O(3,\mathbb{R})$ and $O(1,2,\mathbb{R})$.

Moreover, by direct verification, it is easy to check the following.

Lemma 3.2. *The exterior algebra $\Lambda_q(V)$ in (3.2) is a $*$-algebra with involution $* : \Lambda_q(V) \to \Lambda_q(V)$ defined on generators e_k, $k = 1, 2, 3$ by*

$$e_k^* = q^{\rho_k}e_{k'} \qquad \textit{for } q \in \mathbb{R} \tag{3.9}$$

$$e_k^* = e_k \qquad \textit{for } |q| = 1\,. \tag{3.10}$$

Then, for $q \in \mathbb{R}$, respectively $|q| = 1$, the coaction $\rho : \Lambda_q(V) \to \mathcal{O}(O_q(3)) \otimes \Lambda_q(V)$, $e_k \mapsto \sum_j u_{kj} \otimes e_j$ is a $*$-map with respect to the $*$-structures on $\mathcal{O}(O_q(3))$ defined in (3.7), respectively (3.8).

Lemma 3.3. *For $q \in \mathbb{R}$, respectively $|q| = 1$, the quantum determinant $D_q(u)$ in (3.3) is real with respect to the $*$-structures on $\mathcal{O}(O_q(3))$ defined in (3.7), respectively (3.8).*

Proof. For each three-form $\xi \in \Lambda_q(V)$, from $\rho(\xi) = D_q(u) \otimes \xi$, it follows that $D_q(u)^* \otimes \xi^* = \rho(\xi)^* = \rho(\xi^*) = D_q(u) \otimes \xi^*$ and thus the quantum determinant is real: $D_q(u)^* = D_q(u)$. (Alternatively, the Lemma can be proved by comparing $D_q(u)^*$ computed from (3.3) with the formula for $D_q(u)$ given by expanding it with respect to the third row.) □

It follows that $\langle D_q(u) - 1 \rangle$ is a $*$-ideal. For $q \in \mathbb{R}$, we denote by $\mathcal{O}(SO_q(3,\mathbb{R}))$ the quotient Hopf $*$-algebra $\mathcal{O}(O_q(3,\mathbb{R}))/\langle D_q(u) - 1\rangle$ with $*$-structure inherited from that of $\mathcal{O}(O_q(3,\mathbb{R}))$ in (3.7). While we denote by $\mathcal{O}(SO_q(1,2,\mathbb{R}))$ the quotient Hopf $*$-algebra $\mathcal{O}(O_q(1,2,\mathbb{R}))/\langle D_q(u)-1\rangle$ with $*$-structure inherited from that of $\mathcal{O}(O_q(1,2,\mathbb{R}))$ in (3.8).

3.3. The Double Covering of $SO_q(3)$

Classically, the complex Lie group $SL(2)$ is a double covering of $SO(3)$. The quantum analogue of this fact was proven in [4] where it was shown the existence of a Hopf algebra isomorphism between the coordinate algebra $\mathcal{O}(SO_q(3))$ and the subalgebra $\mathcal{O}(SL_{\mathsf{s}}(2))^{\mathbb{Z}_2}$ of $\mathcal{O}(SL_{\mathsf{s}}(2))$, $\mathsf{s} = q^{\frac{1}{2}}$, made of invariant elements for the action of the group $\mathbb{Z}_2$. If we denote by a, b, c, d the generators of $\mathcal{O}(SL_{\mathsf{s}}(2))$, the defining matrix and commutation relations are given by

$$v := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \begin{array}{l} ab = \mathsf{s}\, ba\,,\; ac = \mathsf{s}\, ca\,,\; bd = \mathsf{s}\, db\,, \\ cd = \mathsf{s}\, dc\,,\; bc = cb\,, \quad ad = da + (\mathsf{s} - \mathsf{s}^{-1})bc \end{array} \tag{3.11}$$

with moreover $ad - \mathsf{s}\, bc = 1$. In matrix notation, $\mathcal{O}(SL_{\mathsf{s}}(2))$ has coproduct $\Delta(v) = v \otimes v$, counit $\varepsilon(v) = \mathrm{I}$ and antipode $S(v) = \begin{pmatrix} d & -\mathsf{s}^{-1} b \\ -\mathsf{s}\, c & a \end{pmatrix}$. The algebra $\mathcal{O}(SL_{\mathsf{s}}(2))^{\mathbb{Z}_2}$ is spanned by matrix coefficients of odd-dimensional (the integer spin ones) irreducible corepresentations of $SL_{\mathsf{s}}(2)$ and is generated by the entries of the matrix

$$m := \begin{pmatrix} a^2 & (1+\mathsf{s}^2)^{\frac{1}{2}}\, ba & -b^2 \\ (1+\mathsf{s}^2)^{\frac{1}{2}}\, ca & 1 + (\mathsf{s}+\mathsf{s}^{-1})bc & -(1+\mathsf{s}^2)^{\frac{1}{2}}\, db \\ -c^2 & -(1+\mathsf{s}^2)^{\frac{1}{2}}\, dc & d^2 \end{pmatrix}.$$

With u as before the defining matrix of $\mathcal{O}(SO_q(3))$, the Hopf algebra isomorphism is

$$\mathcal{O}(SO_q(3)) \to \mathcal{O}(SL_{\mathsf{s}}(2))^{\mathbb{Z}_2}, \quad u \mapsto m\,. \tag{3.12}$$

4. The Orthogonal 2-Sphere and Hyperboloid

As mentioned in Sect. 3.1 for the general case, associated with the quantum group $SO_q(3)$ there is a quantum vector space $\mathbb{C}^3_q$. It is defined, as in (2.11), via the the q-symmetrizer matrix P_- in the decomposition (2.10) of the R-matrix. One has then the free algebra generated by three elements z_k, $k = 1, 2, 3$, modulo an ideal of relations:

$$\mathcal{O}(\mathbb{C}^3_q) := \mathbb{C}\langle z_k \rangle / \langle P_- z_1 z_2 \rangle\ . \tag{4.1}$$

With the R-matrix in (3.1), the algebra relations are given explicitly by

$$z_2 z_1 = q^{-1} z_1 z_2\ , \quad z_3 z_2 = q^{-1} z_2 z_3\ , \quad z_3 z_1 = z_1 z_3 + (q^{\frac{1}{2}} - q^{-\frac{1}{2}}) z_2^2\ . \tag{4.2}$$

By construction $\mathcal{O}(\mathbb{C}^3_q)$ carries a left coaction which is an algebra homomorphism and is given by

$$\rho : \mathcal{O}(\mathbb{C}^3_q) \to \mathcal{O}(SO_q(3)) \otimes \mathcal{O}(\mathbb{C}^3_q)\ , \quad z_k \mapsto \sum_m u_{km} \otimes z_m\ .$$

It is easy to see that the quadratic element

$$r := q^{-\frac{1}{2}} z_1 z_3 + z_2^2 + q^{\frac{1}{2}} z_3 z_1 \tag{4.3}$$

belongs to the centre of the algebra and the coaction ρ descends to a coaction on the quotient algebra $\mathcal{O}(\mathbb{C}_q^3)/\langle r-1\rangle$.

There are two $*$-structures, compatible with those of $\mathcal{O}(SO_q(3))$ in Sect. 2.1, making $\mathcal{O}(\mathbb{C}_q^3)$ a $*$-algebra. For $q \in \mathbb{R}$, the involution is $z_k^* = q^{\rho_k} z_{k'}$, or explicitly,

$$z_1^* = q^{\frac{1}{2}} z_3 \ , \quad z_2^* = z_2 \ , \quad z_3^* = q^{-\frac{1}{2}} z_1 \ , \tag{4.4}$$

while for $|q| = 1$ the algebra $\mathcal{O}(\mathbb{C}_q^3)$ becomes a $*$-algebra for $z_k^* = z_k$.

For both choices of q the central element r is real, $r^* = r$; thus the quotient algebras $\mathcal{O}(\mathbb{C}_q^3)/\langle r-1\rangle$ are left comodules $*$-algebra for the corresponding Hopf $*$-algebras obtained from $\mathcal{O}(SO_q(3))$, that is $\mathcal{O}(SO_q(3,\mathbb{R}))$ and $\mathcal{O}(SO_q(1,2,\mathbb{R}))$.

In order to understand the geometry of the quantum spaces described by the $*$-algebras $\mathcal{O}(\mathbb{C}_q^3)/\langle r-1\rangle$ we introduce cartesian coordinates. Consider the following generators:

$$x_1 := \mu\, \mathrm{i}\, \tfrac{1}{\sqrt{2}} \left(-\alpha z_1 + \beta z_3\right) \ , \quad x_2 := \gamma\, z_2 \, , \quad x_3 := \tfrac{1}{\sqrt{2}} \left(\alpha z_1 + \beta z_3\right) \tag{4.5}$$

with $\alpha, \beta, \gamma, \mu \in \mathbb{C}$ such that

$$\alpha\beta = \tfrac{1}{2}(q^{\frac{1}{2}} + q^{-\frac{1}{2}}) \ , \qquad \gamma^2 = \tfrac{1}{2}(q + q^{-1}) \, , \qquad \mu = \begin{cases} 1 & \text{if } q \in \mathbb{R} \\ -\mathrm{i} & \text{if } |q| = 1 \end{cases} .$$

Provided we choose $\beta = q^{\frac{1}{2}} \bar{\alpha}$ for $q \in \mathbb{R}$ and $\alpha = \bar{\alpha}$, $\beta = \bar{\beta}$ for $|q| = 1$, the generators x_k are real, $x_k^* = x_k$, for both $*$-structures. The quadratic identity $q^{-\frac{1}{2}} z_1 z_3 + z_2^2 + q^{\frac{1}{2}} z_3 z_1 = 1$, in terms of the real generators x_k, becomes

$$\mu^2 x_1^2 + x_2^2 + x_3^2 = 1 \, . \tag{4.6}$$

This is the equation of a two-sphere if $\mu^2 = 1$, or a hyperboloid if $\mu^2 = -1$.

For $q \in \mathbb{R}$, we denote by $\mathcal{O}(S_q^2)$ the $*$-algebra $\mathcal{O}(\mathbb{C}_q^3)/\langle r-1\rangle$, the coordinate algebra of the quantum Euclidean real unit sphere S_q^2. It is a left comodule $*$-algebra for $\mathcal{O}(SO_q(3,\mathbb{R}))$. The sphere S_q^2 is in fact the equatorial Podleś sphere of [13].

For $|q| = 1$ we denote by $\mathcal{O}(H_q^2)$ the $*$-algebra $\mathcal{O}(\mathbb{C}_q^3)/\langle r-1\rangle$, the coordinate algebra of the quantum Euclidean hyperboloid H_q^2, a left comodule $*$-algebra for $\mathcal{O}(SO_q(1,2,\mathbb{R}))$.

4.1. Pre-regular Multilinear Forms

In the spirit of [5], the algebra $\mathcal{O}(\mathbb{C}_q^3)$ in (4.1) can be given via a multilinear form. Let W be the 3-linear form on $\mathbb{C}^3$ with components

$$W(v_i, v_j, v_k) =: \varepsilon_{ijk} \tag{4.7}$$

in the canonical basis $\{v_j, \ j = 1, 2, 3\}$ of $\mathbb{C}^3$, where ε_{ijk} is the tensor in (3.2).

With reference to the theory of pre-regular multilinear forms (see [5, Def. 2]) we have the following result

Lemma 4.1. *The* 3*-linear form* W *is pre-regular, that is*

(i) *there exists an element* $T \in GL(3,\mathbb{C})$ *such that* W *is* T*-cyclic: for all* $V_1, V_2, V_3 \in \mathbb{C}^3$*,*

$$W(V_1, V_2, V_3) = W(T(V_3), V_1, V_2),$$

(ii) *if* $v \in \mathbb{C}^3$ *is such that* $W(v, e_j, e_k) = 0$ *for all indices* j, k*, then* $v = 0$*.*

Proof. Define $T \in GL(3,\mathbb{C})$ as the linear transformation $T(v_j) = \mu_j v_j$ for $\mu_1 = q$, $\mu_2 = 1$, $\mu_3 = q^{-1}$. By direct computation one verifies that W is such that $W(v_i, v_j, v_k) = W(T(v_i), v_j, v_k)$ on the elements v_i of the basis, for $i, j, k = 1, 2, 3$, being $\varepsilon_{ijk} = \mu_k \varepsilon_{kij}$. □

Lemma 4.2. *Let* $A(W, 2)$ *be the quadratic algebra generated by elements* z_i*,* $i = 1, 2, 3$*, satisfying the three relations*

$$\sum_{jk} \varepsilon_{ijk}\, z_j z_k = 0, \qquad \text{for } i = 1, 2, 3\ . \tag{4.8}$$

Then $A(W, 2)$ *coincides with the algebra* $\mathcal{O}(\mathbb{C}^3_q)$ *in* (4.1)*.*

Proof. By direct check, comparing (4.8) with relations (4.2). □

Remark 4.3. We mention that the relations (4.2) show that the algebra $\mathcal{O}(\mathbb{C}^3_q)$ is an Artin–Schelter algebra of type S_1' with $q = \alpha^{-1}$ (see table (3.11) in [2]) and thus $\mathcal{O}(\mathbb{C}^3_q)$ is a Koszul algebra. As a consequence, for $q \in \mathbb{R}$ the space $\mathbb{C}^3_q$ is a noncommutative Euclidean space for which there is a canonical generalization of Clifford algebras [6].

5. The Quantum Homogeneous Spaces

It is known that $SO(2)$ is a quantum subgroup of $SO_q(3)$ (see e.g. [14, Thm. 3.5]). Indeed, it is easily shown that $I := \langle u_{ij} | i \neq j \rangle$ is a Hopf ideal in $\mathcal{O}(SO_q(3))$. The quotient Hopf algebra $\mathcal{O}(SO_q(3))/I$ is generated by the elements $\tilde{u}_{ij} := \pi(u_{ij})$, for π the quotient map $\pi : \mathcal{O}(SO_q(3)) \to \mathcal{O}(SO_q(3))/I$, and thus has just three generators $\tilde{u}_{ii}$, $i = 1, 2, 3$. Their commutation relations are obtained via the projection π from those of $\mathcal{O}(SO_q(3))$. From the equation (2.3) one gets $\tilde{u}_{jj}\tilde{u}_{kk} = \tilde{u}_{kk}\tilde{u}_{jj}$, for $j, k = 1, 2, 3$. In addition, the metric condition (2.5) requires that $\tilde{u}_{11}\tilde{u}_{33} = 1$ and (by using also the counit ε) that $\tilde{u}_{22} = 1$. Thus the Hopf algebra $\mathcal{O}(SO_q(3))/I$ is indeed a copy of $\mathcal{O}(SO(2))$, that realises $SO(2)$ as a quantum subgroup of $SO_q(3)$.

The construction is compatible with both $*$-structures of $\mathcal{O}(SO_q(3))$, for the two cases $q \in \mathbb{R}$ or $|q| = 1$. That is, the ideal I is a $*$-ideal with respect to both of them and the quotient spaces are hence Hopf $*$-algebras. In particular, $\mathcal{O}(SO_q(3))/I$ is isomorphic to the $*$-algebra $\mathcal{O}(SO(2,\mathbb{R}))$ in the case $q \in \mathbb{R}$, with $(\tilde{u}_{kk})^* = \tilde{u}_{kk}$, for $k = 1, 2$, and to $\mathcal{O}(SO(1,1,\mathbb{R}))$, with $(\tilde{u}_{11})^* = \tilde{u}_{22}$, in the case $|q| = 1$.

By a general construction, there is then a natural (right) coaction of $SO(2)$ on $SO_q(3)$ given by restriction of the coproduct, written in matrix notation as

$$\delta = (\mathrm{id} \otimes \pi)\Delta : \mathcal{O}(SO_q(3)) \to \mathcal{O}(SO_q(3)) \otimes \mathcal{O}(SO(2))$$
$$\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{pmatrix} \mapsto \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{pmatrix} \otimes \begin{pmatrix} z & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z^{-1} \end{pmatrix}, \tag{5.1}$$

where we set $z := \tilde{u}_{11}$.

Since $\delta(u_{ij}) = \sum_k u_{ik} \otimes \pi(u_{kj}) = u_{ij} \otimes \pi(u_{jj})$, it is clear that the elements $b \in \mathcal{O}(SO_q(3))$ which are coinvariant for the coaction, $\delta(b) = b \otimes 1$, are given in degree one by the span of the elements in the second column of the defining matrix u of $\mathcal{O}(SO_q(3))$ and, in addition, in degree two by the span of products of any element of the first column with any one of the third, $u_{i1}u_{j3}$ or $u_{i3}u_{j1}$ for indices $i, j = 1, 2, 3$. Nevertheless, we next show that all the elements $u_{i1}u_{j3}$ and $u_{i3}u_{j1}$ indeed belong to the span of those of the second column.

Proposition 5.1. *The subalgebra of* $\mathcal{O}(SO_q(3))$

$$B := \mathcal{O}(SO_q(3))^{co\, \mathcal{O}(SO(2))} = \{b \in \mathcal{O}(SO_q(3)) \mid \delta(b) = b \otimes 1\}$$

of coinvariant elements for the coaction δ *of* $(\mathcal{O}(SO(2)))$ *in* (5.1) *is generated by the three elements* u_{i2}, *for* $i = 1, 2, 3$.

Proof. We show that the elements $u_{i3}u_{j1}$ and $u_{i1}u_{j3}$ can be written as polynomials in the elements of the second column. By taking $m = 3,\ n = 1$ in (2.3), we obtain

$$\begin{aligned} q^{-1}u_{i1}u_{j3} = {} & q^{\delta_{ij} - \delta_{ij'}} u_{j3}u_{i1} + \lambda\theta(j-i)u_{i3}u_{j1} \\ & -\lambda\delta_{ij'} \sum_k \theta(j-k) q^{-\rho_i - \rho_k} u_{k3}u_{k'1} \end{aligned} \tag{5.2}$$

so it is enough to establish the result for the elements $u_{i3}u_{j1}$. (We list nevertheless the expressions of all coinvariant elements in terms of the elements u_{k2} in Appendix C.) In the proof we will use the identities

$$u_{31}u_{13} = u_{13}u_{31}\,, \quad u_{11}u_{13} = q^2 u_{13}u_{11}\,, \quad u_{31}u_{33} = q^2 u_{33}u_{31}$$

obtained from equation (5.2), for suitable choices of indices i, j, and the identification in (3.6) of the elements of the second column of the matrix u as cofactors. We will also use the relations

$$uS(u) = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{pmatrix} \begin{pmatrix} u_{33} & q^{-\frac{1}{2}}u_{23} & q^{-1}u_{13} \\ q^{\frac{1}{2}}u_{32} & u_{22} & q^{-\frac{1}{2}}u_{12} \\ q\ u_{31} & q^{\frac{1}{2}}u_{21} & u_{11} \end{pmatrix} = \mathrm{I}$$

and

$$S(u)u = \begin{pmatrix} u_{33} & q^{-\frac{1}{2}}u_{23} & q^{-1}u_{13} \\ q^{\frac{1}{2}}u_{32} & u_{22} & q^{-\frac{1}{2}}u_{12} \\ q\ u_{31} & q^{\frac{1}{2}}u_{21} & u_{11} \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{pmatrix} = \mathrm{I}\,.$$

First, by using $u_{31}u_{13} = u_{13}u_{31}$ in the equality $(uS(u))_{33} = (S(u)u)_{11}$, we get

$$u_{23}u_{21} = u_{32}u_{12}\ .$$

By comparing the expressions $(uS(u))_{11} = 1$ and $u_{22} = \widehat{u}_{22}$, we compute

$$u_{13}u_{31} = (1+q)^{-1}(1 - u_{22} - q^{-\frac{1}{2}}u_{12}u_{32}) .$$

Similarly, from $(uS(u))_{12} = 0$ and the expression $q^{-\frac{1}{2}}u_{12} = \widehat{u}_{23}$ we obtain

$$u_{13}u_{21} = q^{-\frac{1}{2}}(1+q)^{-1}(u_{12} - u_{12}u_{22}) ,$$

while from $(uS(u))_{13} = 0$ and the relation $u_{11}u_{13} = q^2 u_{13}u_{11}$ found before,

$$u_{13}u_{11} = -q^{-\frac{1}{2}}(1+q)^{-1}u_{12}u_{12} .$$

We proceed by comparing $(uS(u))_{21} = 0$ and the expression $q^{\frac{1}{2}}u_{32} = \widehat{u}_{21}$ and obtain

$$u_{23}u_{31} = q^{-\frac{1}{2}}(1+q)^{-1}(u_{32} - u_{22}u_{32})$$

while $(uS(u))_{23} = 0$ and the expression $q^{-\frac{1}{2}}u_{21} = \widehat{u}_{23}$ gives

$$u_{23}u_{11} = -q^{\frac{1}{2}}(1+q)^{-1}(u_{12} + q^{-1}u_{22}u_{12}) .$$

Then, from $(uS(u))_{31} = 0$ and the relation $u_{31}u_{33} = q^2 u_{33}u_{31}$ we get

$$u_{33}u_{31} = -q^{-\frac{1}{2}}(1+q)^{-1}u_{32}u_{32} .$$

From $(uS(u))_{32} = 0$ and the expression $q^{\frac{1}{2}}u_{32} = \widehat{u}_{21}$ we have

$$u_{33}u_{21} = -(1+q)^{-1}(q^{\frac{1}{2}}u_{32} + q^{-\frac{1}{2}}u_{32}u_{22}) .$$

Finally, from $(uS(u))_{33} = 1$ and the equality $\widehat{u}_{22} = u_{22}$ we obtain the last required relation

$$u_{33}u_{11} = (1+q)^{-1}(q + u_{22} - q^{-\frac{1}{2}}u_{32}u_{12}) .$$

□

The commutation relations among the generators u_{k2} of the subalgebra B of coinvariants are obtained from equations (2.3) for $m = n = 2$,

$$\begin{aligned} q^{\delta_{ij}-\delta_{ij'}}u_{j2}u_{i2} &= (1 - \lambda\theta(j-i))u_{i2}u_{j2} + \lambda\delta_{ij'}\sum_k \theta(j-k)q^{-\rho_i-\rho_k}u_{k2}u_{k'2} \\ &\quad - \lambda q^{-\frac{1}{2}}u_{i1}u_{j3} \end{aligned}$$

by substituting the explicit expression of the elements $u_{i1}u_{j3}$ in terms of the elements u_{k2} (as given in Appendix C). They are given by

$$\begin{aligned} &u_{32}u_{22} = q^{-1}u_{22}u_{32} + (1-q^{-1})u_{32}, \quad u_{22}u_{12} = q^{-1}u_{12}u_{22} + (1-q^{-1})u_{12}, \\ &u_{32}u_{12} = q^{-2}u_{12}u_{32} + q^{-\frac{1}{2}}(1-q^{-1})(1-u_{22}). \end{aligned} \tag{5.3}$$

Moreover from condition $(S(u)u)_{22} = 1$ we also obtain

$$q^{\frac{1}{2}}u_{32}u_{12} + q^{-\frac{1}{2}}u_{12}u_{32} + (u_{22}-1)(u_{22}+1) = 0 . \tag{5.4}$$

We will analyse the geometry of B as a quantum $*$-algebra in Sect. 5.1 below. Before we do that, we study the bundle structure of the quantum homogeneous space B.

Proposition 5.2. *The algebra extension $\mathcal{O}(SO_q(3))^{co\,\mathcal{O}(SO(2))} \subset \mathcal{O}(SO_q(3))$ is Hopf-Galois, that is the canonical map*

$$\chi : \mathcal{O}(SO_q(3)) \otimes_B \mathcal{O}(SO_q(3)) \to \mathcal{O}(SO_q(3)) \otimes \mathcal{O}(SO(2)), \quad a' \otimes a \mapsto a'\delta(a)$$

is bijective.

Proof. We prove the statement by showing that the total space algebra $\mathcal{O}(SO_q(3))$ is strongly graded (see Thm. 4.3 and Prop. 4.6 of [1]). We assign degree $+1$ to the elements of the first column of the defining matrix u, degree -1 to the elements of the third column of the matrix u and degree 0 to the elements of the central column of the matrix u. Let us denote $\mathcal{E}_{\pm 1}$ the collection of all degree $\pm$ elements respectively in $\mathcal{O}(SO_q(3))$. Clearly they are modules over $\mathcal{E}_0 = B$; *a posteriori* these are shown to be finitely generated and projective over B (see [11, Cor. I.3.3]).

In the notation of [1] we have two sequences of elements in $\mathcal{E}_{+1}$

$$\{\xi_j\}_{j=1}^3 = (u_{11}, u_{21}, u_{31}), \qquad \{\beta_j\}_{j=1}^3 = (qu_{31}, q^{\frac{1}{2}}u_{21}, u_{11}) \tag{5.5}$$

and two sequences of elements in $\mathcal{E}_{-1}$

$$\{\eta_j\}_{j=1}^3 = (u_{33}, q^{-\frac{1}{2}}u_{23}, q^{-1}u_{13}), \qquad \{\alpha_j\}_{j=1}^3 = (u_{13}, u_{23}, u_{33})\,. \tag{5.6}$$

These are such that

$$\sum_{j=1}^3 \eta_j\xi_j = (S(u)u)_{11} = u_{33}u_{11} + q^{-\frac{1}{2}}u_{23}u_{21} + q^{-1}u_{13}u_{31} = 1 \tag{5.7}$$

and

$$\sum_{j=1}^3 \beta_j\alpha_j = (S(u)u)_{33} = qu_{31}u_{13} + q^{\frac{1}{2}}u_{21}u_{23} + u_{11}u_{33} = 1\,. \tag{5.8}$$

The inverse $\chi^{-1} : \mathcal{O}(SO_q(3)) \otimes \mathcal{O}(SO(2)) \to \mathcal{O}(SO_q(3)) \otimes_B \mathcal{O}(SO_q(3))$ of the canonical map, by the general theory of [1], is then given by

$$\chi^{-1} : a \otimes z^n \mapsto \begin{cases} \sum_{J\in\{1,2,3\}^n} a\,\eta_{j_1}\cdots\eta_{j_n} \otimes_B \xi_{j_n}\cdots\xi_{j_1}\,, & \text{for } n \geq 0 \\ a \otimes_B 1 & \text{for } n = 0 \\ \sum_{I\in\{1,2,3\}^{-n}} a\,\beta_{i_1}\cdots\beta_{i_{-n}} \otimes_B \alpha_{i_{-n}}\cdots\alpha_{i_1}\,, & \text{for } n \leq 0 \end{cases}. \tag{5.9}$$

For the convenience of the reader we recall here the proof. If $n \geq 0$,

$$\begin{aligned} \chi\circ\chi^{-1}(1\otimes z^n) &= \chi\Big(\sum_{J\in\{1,2,3\}^n} a\,\eta_{j_1}\cdots\eta_{j_n} \otimes_B \xi_{j_n}\cdots\xi_{j_1}\Big) \\ &= \sum_{J\in\{1,2,3\}^n} a\,\eta_{j_1}\cdots\eta_{j_n}\,\xi_{j_n}\cdots\xi_{j_1} \otimes z^n = 1\otimes z^n, \end{aligned}$$

using (5.7) on all indices from j_n to j_1 one after the other. Conversely, if $a \in \mathcal{O}(SO_q(3))$ is of degree n, one has $\delta(a) = a \otimes z^n$ and thus

$$\chi^{-1} \circ \chi(1 \otimes_B a) = \chi^{-1}(a \otimes z^n) = \sum_{J \in \{1,2,3\}^n} a\, \eta_{j_1} \cdots \eta_{j_n} \otimes_B \xi_{j_n} \cdots \xi_{j_1}$$
$$= \sum_{J \in \{1,2,3\}^n} 1 \otimes_B a\, \eta_{j_1} \cdots \eta_{j_n} \xi_{j_n} \cdots \xi_{j_1} = 1 \otimes_B a$$

using the fact that $a\,\eta_{j_1} \cdots \eta_{j_n} \in B$, so that it can cross over the balanced tensor product, and again (5.7). One proceeds similarly for $n \le 0$. □

5.1. Two ∗-Quantum Homogeneous Spaces of $\mathcal{O}(SO_q(3))$

We rename $w_k := u_{k2}$, $k = 1, 2, 3$, the generators of the subalgebra B of coinvariant elements of $\mathcal{O}(SO_q(3))$. They have commutation relations (5.3)

$$w_3(w_2 - 1) = q^{-1}(w_2 - 1)w_3\,, \quad w_1(w_2 - 1) = q(w_2 - 1)w_1\,,$$
$$w_3 w_1 = q^{-2} w_1 w_3 + q^{-\frac{3}{2}}(1 - q)(w_2 - 1) \tag{5.10}$$

and satisfy the quadratic condition

$$q^{-\frac{1}{2}} w_1 w_3 + q^{\frac{1}{2}} w_3 w_1 + w_2^2 = 1\,. \tag{5.11}$$

This, with the last equation in (5.10), can also be written as

$$(q^{\frac{1}{2}} + q^{-\frac{1}{2}}) w_1 w_3 = (1 - w_2)(1 + q w_2),$$
$$(q^{\frac{1}{2}} + q^{-\frac{1}{2}}) w_3 w_1 = (1 - w_2)(1 + q^{-1} w_2). \tag{5.12}$$

It is easy to see that the coaction map δ in (5.1) is a ∗-map, for both $q \in \mathbb{R}$ and $|q| = 1$ and corresponding ∗-structures in Sect. 2.1. Hence B is a ∗-algebra as well with ∗-structures inherited by those of $\mathcal{O}(SO_q(3))$ and given on the generators w_k by

$$\text{for } q \in \mathbb{R}: \qquad (w_1)^* = q^{\frac{1}{2}} w_3\,, \quad (w_2)^* = w_2\,, \quad (w_3)^* = q^{-\frac{1}{2}} w_1\,,$$
$$\text{for } |q| = 1: \qquad (w_k)^* = w_k\,, \quad k = 1, 2, 3\,, \tag{5.13}$$

in parallel with those in (4.4). Moreover, the ∗-algebra B is made of coinvariant elements of the corresponding real group by a suitable real subgroup. For $q \in \mathbb{R}$, we denote $\mathcal{O}(S^2_{q,Gr})$ the ∗-algebra B of coinvariant elements of $\mathcal{O}(SO_q(3, \mathbb{R}))$ with respect to the coaction of its quantum subgroup $\mathcal{O}(SO(2, \mathbb{R}))$. We call $\mathcal{O}(S^2_{q,Gr})$ (the algebra of coordinate functions of) the quantum (Grassmannian) sphere $S^2_{q,Gr}$. When $q \to 1$ it reduces to the coordinate algebra over the Grassmannian $Gr(1, 3) \simeq SO(3)/SO(2) \simeq S^2$ of oriented lines in $\mathbb{R}^3$. In fact, the sphere $S^2_{q,Gr}$ is isomorphic to the standard Podleś sphere S^2_q of [13].

For $|q| = 1$, we denote $\mathcal{O}(H^2_{q,Gr})$ the ∗-algebra B of coinvariant elements of the algebra $\mathcal{O}(SO_q(1, 2, \mathbb{R}))$ with respect to the coaction of its quantum subgroup $\mathcal{O}(SO(1, 1, \mathbb{R}))$. We call $\mathcal{O}(H^2_{q,Gr})$ the algebra of coordinate functions of the quantum hyperboloid $H^2_{q,Gr}$. In the limit $q = 1$ it reduces to the coordinate algebra over the hyperboloid.

Again, as in Sect. 4, the reason for the names and the nature of the spaces above is made evident when using cartesian coordinates. Let us make the following change of generators:

$$y_1 := \mu\,\mathrm{i}\,\frac{1}{\sqrt{2}}\,(-\alpha w_1 + \beta w_3)\ , \quad y_2 := w_2\,, \quad y_3 := \frac{1}{\sqrt{2}}\,(\alpha w_1 + \beta w_3) \tag{5.14}$$

with $\alpha, \beta, \mu \in \mathbb{C}$ such that

$$\alpha\beta = q^{\frac{1}{2}}\,\frac{(1+q)}{(1+q^2)}\,, \qquad \mu = \begin{cases} 1 & \text{if } q \in \mathbb{R} \\ -\mathrm{i} & \text{if } |q| = 1 \end{cases}.$$

Notice that $(\alpha\beta)^* = \alpha\beta$ for both choices of q. Provided we choose $\beta = q^{\frac{1}{2}}\bar{\alpha}$ for $q \in \mathbb{R}$ and $\alpha = \bar{\alpha}$, $\beta = \bar{\beta}$ for $|q| = 1$, for both $*$-structures in (5.13), the generators y_k are real,

$$(y_k)^* = y_k, \qquad k = 1, 2, 3.$$

Using relations (5.12), we compute

$$\begin{aligned} \mu^2 y_1^2 + y_3^2 &= \frac{q^{\frac{1}{2}}(1+q)}{(1+q^2)}(w_1 w_3 + w_3 w_1) \\ &= -\frac{1}{(1+q^2)}\left[(1+q^2)w_2^2 - (1-q)^2 w_2 - 2q\right] \end{aligned}$$

and thus in terms of the real generators y_k the quadratic condition (5.11) reads

$$\mu^2 y_1^2 + y_2^2 + y_3^2 - \frac{(1-q)^2}{1+q^2}\,y_2 = \frac{2q}{1+q^2}\,. \tag{5.15}$$

In the classical limit $q \to 1$ this reduces to

$$\mu^2 y_1^2 + y_2^2 + y_3^2 = 1$$

which is a two-sphere if $\mu^2 = 1$, or a hyperboloid if $\mu^2 = -1$.

Let us finally observe (for future use in Sect. 6.1) that by construction the subalgebra B also carries a left coaction of $\mathcal{O}(SO_q(3))$ given by the restriction of the coproduct of $\mathcal{O}(SO_q(3))$ to the elements u_{k2} generating B: the map

$$\rho = \Delta_{|B} : B \to \mathcal{O}(SO_q(3)) \otimes B\ , \quad u_{k2} \mapsto \sum_m u_{km} \otimes u_{m2} \tag{5.16}$$

makes B a left $\mathcal{O}(SO_q(3))$-comodule algebra. The coaction map ρ in (5.16) is a $*$-map for both values of q and thus B is a comodule $*$-algebra, or quantum $*$-algebra, with respect to the corresponding real forms of $\mathcal{O}(SO_q(3))$.

5.2. Line Bundles

In general, given a right H-comodule algebra A with coaction $\delta : A \to A \otimes H$, $\delta(a) = a_{(0)} \otimes a_{(1)}$ and a left H-comodule V with coaction $\gamma : V \to H \otimes V$, $\gamma(v) = v_{(-1)} \otimes v_{(0)}$, sections of the vector bundle associated with the corepresentation γ can be identified with linear maps $\phi : V \to A$ which are H-equivariant

$$\phi(v)_{(0)} \otimes \phi(v)_{(1)} = \phi(v_{(0)}) \otimes S(v_{(-1)})\,. \tag{5.17}$$

The collection $\mathcal{E}$ of such maps is a left B-module for $B \subseteq A$ the subalgebra of coinvariant elements for the H-coaction.

For the $H = \mathcal{O}(SO(2))$ Hopf-Galois extension $B = \mathcal{O}(SO_q(3))^{co\,\mathcal{O}(SO(2))} \subset \mathcal{O}(SO_q(3))$ irreducible corepresentations of $\mathcal{O}(SO(2))$, which are one dimensional and labelled by an integer, will yield line bundles. Consider any such a corepresentation

$$\gamma_n : \mathbb{C} \to \mathcal{O}(SO(2)) \otimes \mathbb{C}, \qquad \gamma_n(1) = 1 \otimes z^{-n} \tag{5.18}$$

for any integer n. From the coaction (5.1) the first column of the matrix u will transform by z^{-n} while the last column will transform by z^n. Thus, using the generators (5.5) and (5.6), a set of generators of the corresponding B-module $\mathcal{E}_n$ of sections is given by

$$\begin{aligned} &\xi_J := \xi_{j_n} \cdots \xi_{j_1}, \qquad J = (j_1, \cdots, j_n) \in \{1,2,3\}^n \qquad \text{for} \quad n \geq 0 \\ &\alpha_I := \alpha_{i_{-n}} \cdots \alpha_{i_1}, \qquad I = (i_1, \cdots, i_n) \in \{1,2,3\}^{-n} \qquad \text{for} \quad n \leq 0\,. \end{aligned} \tag{5.19}$$

Indeed, for $n \geq 0$, one finds that

$$\begin{aligned} \delta(\xi_J) &= (\xi_{j_n} \cdots \xi_{j_1})_{(0)} \otimes (\xi_{j_n} \cdots \xi_{j_1})_{(1)} \\ &= (\xi_{j_n} \cdots \xi_{j_1}) \otimes z^n = (\xi_{j_n} \cdots \xi_{j_1}) \otimes S(z^{-n}), \end{aligned}$$

thus fulfilling condition (5.18). The case for negative n works similarly. The modules $\mathcal{E}_n$ are line bundles of degree an even integer $-2n$. To see this, one finds suitable idempotents p_n in $\mathrm{Mat}_{|2n|+1}(B)$ and identifies $\mathcal{E}_n \simeq B^{|n|+1} p_n$ as left B-modules.

The idempotents p_n are representatives of classes in the K-theory of B, $[p_n] \in K_0(B)$. One computes the corresponding rank and degree by pairing them with non-trivial elements in the dual K-homology, that is with (the class of) non-trivial Fredholm modules $[\tau] \in K^0(B)$. For this, one first calculates the corresponding Chern characters in the cyclic homology $\mathrm{ch}_\bullet(p_n) \in \mathrm{HC}_\bullet(B)$ and cyclic cohomology $\mathrm{ch}^\bullet([\tau]) \in \mathrm{HC}^\bullet(B)$ respectively, and then uses the pairing between cyclic homology and cohomology.

The Chern character of the idempotents p_n has a non-trivial component in degree zero $\mathrm{ch}_0(p_n) \in \mathrm{HC}_0(B)$ given simply by a (partial) matrix trace $\mathrm{ch}_0(p_n) := \mathrm{tr}(p_n)$ and thus $\mathrm{ch}_0(p_n) \in B$. Dually, one needs a cyclic zero-cocycle, that is a trace on B. There are indeed two such traces coming from two 1-summable even Fredholm modules for B which generate the K-homology $K^0(B)$ and that were worked out in full details in [10].

For the present paper we adapt the construction of the Fredholm modules in [3]. For this construction we need $q \in \mathbb{R}$, a condition that is also in accord with the fact that the idempotents p_n we are finding are projection for the $*$-structure (3.7) (see Remark 5.4).

The first trace comes from the counit ε of $\mathcal{O}(SO_q(3))$. Its restriction to the subalgebra $B = \mathcal{O}(S^2_q) \subset \mathcal{O}(SO_q(3))$ yields a representation $\chi_0 : B \to \mathbb{C}$ which pulls-back to B the generator of the K-homology of $\mathbb{C}$. The resulting

Fredholm module for B is given by

$$\mathcal{H}_0 = \mathbb{C} \oplus \mathbb{C}, \qquad \pi_0(b) = \chi_0(b) \oplus 0, \qquad F_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This we shall denote $[\tau^0]$ with corresponding cyclic zero-cocycle $\tau^0 = \mathrm{ch}^0\,[\tau^0]$:

$$\tau^0(b) = \chi_0(b), \quad b \in B.$$

On the generators of B it is given by $\tau^0(w_1) = \tau^0(w_3) = 0$ and $\tau^0(w_2 - 1) = 0$ (and clearly $\tau^0(1) = 1$). For the second Fredholm module one needs a second representation of B as bounded operators. It is explicitly given on $\ell^2(\mathbb{N})$ with orthonormal basis $|n\rangle$ by

$$\begin{aligned}
\chi_1(w_1)\,|n\rangle &= \beta\, q^{-\frac{n+2}{2}} \left[1 - q^{-(n+1)}\right]^{\frac{1}{2}} |n+1\rangle \\
\chi_1(w_3)\,|n\rangle &= \beta\, q^{-\frac{n+1}{2}} \left[1 - q^{-(n-1)}\right]^{\frac{1}{2}} |n-1\rangle \\
\chi_1(w_2 - 1)\,|n\rangle &= -\beta^2\, q^{-(n+1)}\,|n\rangle\,,
\end{aligned}$$

with $\beta = -(q+1)^{\frac{1}{2}}$. (The parameter q in [3] is mapped to $q^{-\frac{1}{2}}$ here.) For the above formulas to give bounded operators one needs to assume that q be such that $|q| > 0$. This is nor restrictive in that for $|q| < 0$ the appropriate formulas for the representation can be obtained from the one above by replacing the index n with $-n$: as a consequence the role of w_1 and w_3 as raising and lowering operators is exchanged.

The second even Fredholm module $[\tau^1]$ for $B = \mathcal{O}(S^2_q)$ is then given as:

$$\mathcal{H}_1 = \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N}), \qquad \pi_1(b) = \begin{pmatrix} \chi_1(b) & 0 \\ 0 & \chi_0(b)\,\mathrm{id}_{\ell^2(\mathbb{N})} \end{pmatrix} \qquad F_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Thus, the operator $\chi_1(b) - \chi_0(b)\,\mathrm{id}_{\ell^2(\mathbb{N})}$ is trace class for all $b \in B$ and one gets a second trace $\tau^1 = \mathrm{ch}^0\,[\tau^1]$ on B given by $\tau^1(b) = \mathrm{tr}_{\mathcal{H}_1}(\chi_1(b) - \chi_0(b))$. It is a trace on $B/\mathbb{C}$, that is it vanishes on $\mathbb{C} \subset B$. We need its evaluation on powers of $w_2 - 1$. One finds

$$\begin{aligned}
\tau^1((w_2 - 1)^k) &= \mathrm{tr}_{\mathcal{H}_1}(\chi_1((w_2 - 1)^k)) \\
&= (-1)^k \beta^{2k} q^{-k} \sum_n q^{-nk} = (-1)^k \beta^{2k} q^{-k} \frac{1}{1 - q^{-k}} \\
&= (-1)^k \frac{(q+1)^k}{q^k - 1}\,.
\end{aligned} \tag{5.20}$$

Let us first illustrate the above for the lowest values $n = \pm 1$. In these cases a collection of generators for the modules of sections is given by (u_{11}, u_{21}, u_{31}) and (u_{33}, u_{23}, u_{13}) respectively. The corresponding idempotents are the matrices

$$p_{+1} := \begin{pmatrix} u_{11} \\ u_{21} \\ u_{31} \end{pmatrix} \left(u_{33},\, q^{-\frac{1}{2}} u_{23},\, q^{-1} u_{13}\right), \quad p_{-1} := \begin{pmatrix} u_{33} \\ u_{23} \\ u_{13} \end{pmatrix} \left(u_{11},\, q^{\frac{1}{2}} u_{21},\, q\, u_{31}\right). \tag{5.21}$$

Since p_{+1} has entries $(p_{+1})_{ij} = u_{i1}S(u)_{1j}$, the identity (5.7) implies that p_{+1} is an idempotent $p_{+1}^2 = p_{+1}$. Similarly, for p_{-1} of components $(p_{-1})_{ij} = u_{i3}S(u)_{3j}$, the result $p_{-1}^2 = p_{-1}$ follows from (5.8). From Proposition 5.1 the entries of $p_{\pm 1}$ belong to the subalgebra B. Using the list in Appendix C for quadratic coinvariant elements and the first equality in (5.12), for the partial trace of these idempotents one computes

$$\begin{aligned}\operatorname{tr}(p_{+1}) &= u_{11}u_{33} + q^{-\frac{1}{2}}u_{21}u_{23} + q^{-1}u_{31}u_{13}\\ &= 1 + (q-1)(w_2-1) + \frac{(q-1)^2}{q+1}(w_2-1)^2. \qquad (5.22)\end{aligned}$$

Then, using $\chi_0(w_2) = 1$ one gets

$$\langle [\tau^1], [p_{+1}] \rangle := \chi_0\left(\operatorname{ch}_0(p_{+1})\right)) = 1\ .$$

Finally, using the vanishing of μ over the scalars and (5.20) one gets

$$\begin{aligned}\langle [\tau^1], [p_{+1}] \rangle &:= \tau^1(\operatorname{ch}_0(p_{+1})) = -(q-1)\frac{(q+1)}{q-1} + (q-1)\frac{(q+1)^2}{q^2-1}\\ &= -(q+1) + (q-1) = -2\,. \qquad (5.23)\end{aligned}$$

With a similar computation one gets $\langle [\tau^0], [p_{-1}] \rangle = 1$ and $\langle [\tau^1], [p_{-1}] \rangle = 2$.

For a general $n \geq 0$ consider two vector valued functions of components

$$\begin{aligned}&|\psi_n\rangle_J := \xi_J = \xi_{j_n}\cdots\xi_{j_1},\\ &\langle\phi_n|_J := \eta_J = (\eta_{j_1}\cdots\eta_{j_n}), \qquad J = (j_1,\cdots,j_n) \in \{1,2,3\}^n.\end{aligned}$$

We have already observed that from (5.7) one has

$$\langle \phi_n, \psi_n \rangle = \sum_{J\in\{1,2,3\}^n} \eta_{j_1}\cdots\eta_{j_n}\,\xi_{j_n}\cdots\xi_{j_1}\otimes z^n = 1.$$

Thus the matrix $p_n = |\psi_n\rangle\langle\phi_n|$ of components $(p_n)_{KJ} = \xi_J\eta_K$ is an idempotent. Similarly, for $n \leq 0$ we take

$$\begin{aligned}&|\psi_{-n}\rangle_I := \alpha_I = \alpha_{i_{-n}}\cdots\alpha_{i_1},\\ &\langle\phi_{-n}|_I := \beta_I = \beta_{i_1}\cdots\beta_{i_{-n}}, \quad I = (i_1,\cdots,i_n) \in \{1,2,3\}^{-n}\end{aligned}$$

and now $\langle\phi_{-n},\psi_{-n}\rangle = 1$ and the idempotent is the matrix $p_{-n} = |\psi_{-n}\rangle\langle\phi_{-n}|$.

Using an inductive argument and result (5.23), we show the following.

Proposition 5.3. *For $n \geq 0$ the modules $\mathcal{E}_n$ are line bundles of degree $-2n$, that is*

$$\langle [\tau^0], [p_n] \rangle = 1 \qquad \langle [\tau^1], [p_n] \rangle = -2n. \qquad (5.24)$$

For $n \leq 0$ one gets $\langle [\tau^0], [p_{-n}] \rangle = 1$ and positive degree $\langle [\tau^1], [p_{-n}] \rangle = -2n$.

Proof. The result rests on a recursion formula for the trace of the idempotents $\operatorname{tr}(p_n)$. For $n \geq 0$, one finds

$$\begin{aligned}&\operatorname{tr}(p_n) = \sum_J (p_n)_{JJ} = 1 + \sum_{J=1}^{2n}(q+1)^{-J}\,C_J^{(n)}(w_2-1)^J\,,\\ &C_J^{(n)} = \prod_{k=0}^{J-1}(q^{2n-k}-1). \qquad (5.25)\end{aligned}$$

We prove the formula by induction. We set here $T := (q+1)^{-1}(w_2 - 1)$ to simplify notation. Firstly, out of the commutation relations (2.3) one finds

$$u_{11}T^J = q^{2J}T^J u_{11}\ , \quad u_{21}T^J = q^J T^J u_{21}\ , \quad u_{31}T^J = T^J u_{31} \tag{5.26}$$

as well as, from the computations in Appendix C, the identities

$$\begin{aligned} &u_{11}u_{33} = 1 + (q+q^2)T + q^3T^2\ ,\ q^{-\frac{1}{2}}u_{21}u_{23} = -(q+1)(T+qT^2)\ , \\ &u_{31}u_{13} = qT^2\,. \end{aligned} \tag{5.27}$$

Formula (5.25) is verified for $n = 1$: it is just (5.22). Assume it holds for n, then

$$\begin{aligned} \mathrm{tr}(p_{n+1}) &= u_{11}\,\mathrm{tr}(p_n)u_{33} + q^{-\frac{1}{2}}u_{21}\,\mathrm{tr}(p_n)u_{23} + q^{-1}u_{31}\,\mathrm{tr}(p_n)u_{13} \\ &= \mathrm{tr}(p_1) + \sum_{J=1}^{2n} C_J^{(n)} \left(u_{11}T^J u_{33} + q^{-\frac{1}{2}}u_{21}T^J u_{23} + q^{-1}u_{31}T^J u_{13}\right) \\ &= \mathrm{tr}(p_1) + \sum_{J=1}^{2n} C_J^{(n)} T^J \Big(q^{2J}\big(1 + (q+q^2)T + q^3T^2\big) \\ &\quad - q^J(q+1)(T+qT^2) + T^2\Big) \end{aligned}$$

using (5.26) followed by (5.27) for the last identity. Then

$$\begin{aligned} \mathrm{tr}(p_{n+1}) = \mathrm{tr}(p_1) &+ \sum_{J=1}^{2n} C_J^{(n)} T^J \Big(q^{2J} + (q^{J+1}-1) \\ &(q^{J+1}+q^J)T + (q^{J+1}-1)(q^{J+2}-1)T^2\Big) \\ &= 1 + (q^2-1)T + (q^2-1)(q-1)T^2 + \sum_{J=1}^{2n} q^{2J}C_J^{(n)}T^J \\ &+ \sum_{J=2}^{2n+1} (q^J-1)(q^{J-1}+q^J)C_{J-1}^{(n)}T^J \\ &+ \sum_{J=3}^{2n+2} (q^J-1)(q^{J-1}-1)C_{J-2}^{(n)}T^J\ . \end{aligned} \tag{5.28}$$

Finally, using properties

$$C_J^{(n)} = (q^{2n+1-J}-1)C_{J-1}^{(n)}\,, \qquad C_{J+2}^{(n+1)} = (q^{2n+2}-1)(q^{2n+1}-1)C_J^{(n)} \tag{5.29}$$

for the coefficients $C_J^{(n)}$, we get

$$\begin{aligned} \mathrm{tr}(p_{n+1}) = 1 + (q^{2n+2}-1)T &+ \Big((q^2-1)(q-1) + q^4C_2^{(n)} \\ &+ (q^2-1)(q+q^2)C_1^{(n)}\Big)T^2 \\ &+ \sum_{J=3}^{2n} \Big(q^{2J}(q^{2n+1-J}-1)(q^{2n+2-J}-1) \end{aligned}$$

$$
\begin{aligned}
&+ (q^J-1)(q^{J-1}+q^J)(q^{2n+2-J}-1)\\
&+ (q^J-1)(q^{J-1}-1)\Big)C^{(n)}_{J-2}T^J\\
&+ (q^{2n+1}-1)\Big((q^{2n}+q^{2n+1})(q-1)\\
&+ (q^{2n}-1)\Big)C^{(n)}_{2n-1}T^{2n+1}\\
&+ (q^{2n+2}-1)(q^{2n+1}-1)C^{(n)}_{2n}T^{2n+2}\\
={}& 1+(q^{2n+2}-1)T+(q^{2n+2}-1)(q^{2n+1}-1)T^2\\
&+ (q^{2n+2}-1)(q^{2n+1}-1)\sum_{J=3}^{2n}\Big(C^{(n)}_{J-2}T^J\\
&+ C^{(n)}_{2n-1}T^{2n+1}+C^{(n)}_{2n}T^{2n+2}\Big)\\
={}& \sum_{J=1}^{2n+2} C^{(n+1)}_J T^J \ .
\end{aligned}
$$

Being $\chi_0(w_2)=1$, or $\chi_0(T)=0$, one gets $\big\langle[\tau^0],[p_{+n}]\big\rangle=1$.
For the computation of the degree we also proceed by induction. From (5.20) one has $\tau^1(T^J)=(-1)^J\frac{1}{q^J-1}$ from which one deduces

$$
\tau^1(T^{J+1})=-\frac{q^J-1}{q^{J+1}-1}\mu(T^J)\ , \quad \tau^1(T^{J+2})=\frac{q^J-1}{q^{J+2}-1}\mu(T^J)\ .
$$

We use these formulas in the first expression in (5.28) for the trace of p_{n+1}:

$$
\begin{aligned}
\tau^1(tr(p_{n+1})) ={}& \tau^1(\mathrm{tr}(p_1))\\
&+\sum_{J=1}^{2n} C^{(n)}_J\Big(q^{2J}T^J+(q^{J+1}-1)(q^{J+1}+q^J)T^{J+1}\\
&+(q^{J+1}-1)(q^{J+2}-1)T^{J+2}\Big)\\
={}& -2+\sum_{J=1}^{2n} C^{(n)}_J\Big(q^{2J}-(q^J-1)(q^{J+1}+q^J)\\
&+(q^{J+1}-1)(q^J-1)\Big)\mu(T^J)\\
={}& -2+\sum_{J=1}^{2n} C^{(n)}_J\mu(T^J)\\
={}& -2+\tau^1(\mathrm{tr}(p_n))=-2(n+1)\ .
\end{aligned}
$$

□

Remark 5.4. For $q\in\mathbb{R}$ and $*$-structure (3.7), the idempotents $p_{\pm n}=|\psi_{\pm n}\rangle\langle\phi_{\pm n}|$ are self-adjoint, $(p_{\pm n})^*=p_{\pm n}$. This follows from the fact that $(|\psi_{\pm n}\rangle_J)^* = \langle\phi_{\pm n}|_J$, for each J, being $u^*_{11}=u_{33}$, $u^*_{21}=q^{-\frac{1}{2}}u_{23}$ and $u^*_{31}=q^{-1}u_{13}$. In contrast, the idempotents p_n are not self-adjoint for the $*$-structure (3.8) when $|q|=1$.

6. The Dual Hopf Algebra and the Casimir Element

Aiming at the study of Laplacian operators on the two $*$-quantum homogeneous spaces of $\mathcal{O}(SO_q(3))$ in Sect. 5.1, and gauged versions on bundles over them in the line of [9], in this section we study a Casimir element as a Laplacian operator acting on functions of the base space. This operator is constructed from the actions of a dual Hopf algebra $\mathcal{U}_q(sl_2)$.

6.1. The Dual Hopf Algebra $\mathcal{U}_q(sl_2)$ and Its Real Forms

From Drinfel'd–Jimbo construction of quantum universal enveloping algebras it is known that $\mathcal{U}_{q^{1/2}}(so(3)) \simeq \mathcal{U}_q(sl_2)$. On the other hand as recalled in Sect. 3.3, there is an isomorphism $\mathcal{O}(SO_q(3)) \simeq \mathcal{O}(SL_{q^{1/2}})(2)^{\mathbb{Z}_2}$. We shall then work out a dual pairing between $\mathcal{O}(SO_q(3))$ and $\mathcal{U}_{q^{1/2}}(sl_2)$.

The algebra $\mathcal{U}_{q^{1/2}}(sl_2)$ is generated by elements K, K^{-1}, E, F subject to the relations

$$K^{\pm}E = q^{\pm 1}EK^{\pm}\,, \quad K^{\pm}F = q^{\mp 1}FK^{\pm}\,, \quad EF - FE = \frac{K - K^{-1}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}$$

together with $KK^{-1} = K^{-1}K = 1$. It is a Hopf algebra with coproduct, counit and antipode given respectively by

$$\begin{aligned}
&\Delta(K^{\pm 1}) = K^{\pm 1} \otimes K^{\pm 1}\,, \quad \Delta(E) = E \otimes K + 1 \otimes E\,, \quad \Delta(F) = F \otimes 1 + K^{-1} \otimes F\,,\\
&\varepsilon(K^{\pm 1}) = 1\,, \quad \varepsilon(E) = 0\,, \quad \varepsilon(F) = 0\\
&S(K^{\pm 1}) = K^{\mp 1}\,, \quad S(E) = -EK^{-1}\,, \quad S(F) = -KF\,.
\end{aligned}$$

See e.g. [8, §3.1].

The non zero values of the pairing $\langle\cdot,\cdot\rangle : \mathcal{U}_{q^{1/2}}(sl_2) \times \mathcal{O}(SO_q(3)) \to \mathbb{C}$ on the algebra generators, besides $\langle 1, u_{kk}\rangle = 1$ for $k = 1, 2, 3$, and $\langle K^{\pm 1}, 1\rangle = 1$, are found to be

$$\begin{aligned}
&\langle K, u_{11}\rangle = q^{-1}\,, \qquad \langle K, u_{22}\rangle = 1\,, \qquad \langle K, u_{33}\rangle = q\\
&\langle K^{-1}, u_{11}\rangle = q\,, \qquad \langle K^{-1}, u_{22}\rangle = 1\,, \qquad \langle K^{-1}, u_{33}\rangle = q^{-1}\,,\\
&\langle E, u_{21}\rangle = \alpha\eta\,, \qquad \langle E, u_{32}\rangle = -\alpha q^{\frac{1}{2}}\eta\,,\\
&\langle F, u_{12}\rangle = \alpha^{-1}\eta\,, \qquad \langle F, u_{23}\rangle = -\alpha^{-1}q^{-\frac{1}{2}}\eta\,,
\end{aligned} \tag{6.1}$$

where $\eta := (q^{\frac{1}{2}} + q^{-\frac{1}{2}})^{\frac{1}{2}}$ and $\alpha \in \mathbb{C} \setminus \{0\}$. The extra parameter α in (6.1) can be re-absorbed by the Hopf algebra automorphism of $\mathcal{U}_{q^{1/2}}(sl_2)$ which rescales $E \mapsto \alpha^{-1}E$, $F \mapsto \alpha F$, $K \mapsto K$ [8, Prop. 3.6]. We hence fix $\alpha = 1$. The pairing extends to the whole algebras by the rules $\langle fg, a\rangle = \langle f \otimes g, \Delta(a)\rangle = \langle f, a_{(1)}\rangle\langle g, a_{(2)}\rangle$ and $\langle f, ab\rangle = \langle \Delta(f), a \otimes b\rangle = \langle f_{(1)}, a\rangle\langle f_{(2)} b\rangle$, for all $f, g \in \mathcal{U}_{q^{1/2}}(sl_2)$ and $a, b \in \mathcal{O}(SO_q(3))$. It satisfies $\langle 1, a\rangle = \varepsilon(a)$, $\langle f, 1\rangle = \varepsilon(f)$ and $\langle S(f), a\rangle = \langle f, S(a)\rangle$ for each $f \in \mathcal{U}_{q^{1/2}}(sl_2)$ and $a \in \mathcal{O}(SO_q(3))$.

It follows by standard arguments in Hopf algebra theory that each left (respectively right) $\mathcal{O}(SO_q(3))$-comodule algebra A carries a right representation $\triangleright$ (respectively left representation $\triangleleft$) of the dual algebra $\mathcal{U}_{q^{1/2}}(sl_2)$. In details, if A is a left comodule algebra via $\rho : A \to \mathcal{O}(SO_q(3)) \otimes A$, $a \mapsto a_{(-1)} \otimes a_{(0)}$, then A carries the right action

$$\triangleleft : A \otimes \mathcal{U}_q(sl_2) \to A\,, \quad a \triangleleft f := \langle f, a_{(-1)}\rangle a_{(0)}, \quad a \in A, f \in \mathcal{U}_q(sl_2).$$

If A is a right comodule algebra via $\delta : A \to A \otimes \mathcal{O}(SO_q(3))$, $a \mapsto a_{(0)} \otimes a_{(1)}$, then A carries the left action

$$\triangleright : \mathcal{U}_q(sl_2) \otimes A \to A \ , \quad f \triangleright a := a_{(0)} \langle f, a_{(1)} \rangle, \quad a \in A, f \in \mathcal{U}_q(sl_2).$$

For $A = \mathcal{O}(SO_q(3))$ with left and right coactions given by the coproduct, the right and left actions of $\mathcal{U}_{q^{1/2}}(sl_2)$ on generators u_{jk} of $\mathcal{O}(SO_q(3))$ read

$$u_{jk} \triangleleft f = \langle f, u_{jm} \rangle u_{mk} \qquad \text{and} \qquad f \triangleright u_{jk} = u_{jm} \langle f, u_{mk} \rangle.$$

Explicitly, the right action is

$$\begin{aligned} &u_{1k} \triangleleft K^{\pm 1} = q^{\mp 1} u_{1k} \ , \quad u_{2k} \triangleleft K^{\pm 1} = u_{2k} \ , \quad u_{3k} \triangleleft K^{\pm 1} = q^{\pm 1} u_{3k} \ , \\ &u_{1k} \triangleleft E = 0 \ , \quad u_{2k} \triangleleft E = \eta\, u_{1k} \ , \quad u_{3k} \triangleleft E = -q^{\frac{1}{2}} \eta\, u_{2k} \ , \\ &u_{1k} \triangleleft F = \eta\, u_{2k} \ , \quad u_{2k} \triangleleft F = -q^{-\frac{1}{2}} \eta\, u_{3k} \quad u_{3k} \triangleleft F = 0 \ , \end{aligned} \tag{6.2}$$

and the left action is given by

$$\begin{aligned} &K^{\pm 1} \triangleright u_{j1} = q^{\mp 1} u_{j1} \ , \quad K^{\pm 1} \triangleright u_{j2} = u_{j2} \ , \quad K^{\pm 1} \triangleright u_{j3} = q^{\pm 1} u_{j3} \ , \\ &E \triangleright u_{j1} = \eta\, u_{j2} \ , \quad E \triangleright u_{j2} = -q^{\frac{1}{2}} \eta\, u_{j3} \ , \quad E \triangleright u_{j3} = 0 \ , \\ &F \triangleright u_{j1} = 0 \ , \quad F \triangleright u_{j2} = \eta\, u_{j1} \quad F \triangleright u_{j3} = -q^{-\frac{1}{2}} \eta\, u_{j2} \ . \end{aligned} \tag{6.3}$$

Since the left coaction of $\mathcal{O}(SO_q(3))$ on itself descends to $B = \mathcal{O}(SO_q(3))^{co\,\mathcal{O}(SO(2))}$, see (5.16), the right action (6.2) preserves B. Explicitly, on the generators $w_k := u_{k2}$ of B, the action $\triangleleft : B \otimes \mathcal{U}_q(sl_2) \to B$ is given by

$$\begin{aligned} &w_1 \triangleleft K^{\pm 1} = q^{\mp 1} w_1 \ , \quad w_2 \triangleleft K^{\pm 1} = w_2 \ , \quad w_3 \triangleleft K^{\pm 1} = q^{\pm 1} w_3 \ , \\ &w_1 \triangleleft E = 0 \ , \quad w_2 \triangleleft E = \eta\, w_1 \ , \quad w_3 \triangleleft E = -q^{\frac{1}{2}} \eta\, w_2 \ , \\ &w_1 \triangleleft F = \eta\, w_2 \ , \quad w_2 \triangleleft F = -q^{-\frac{1}{2}} \eta\, w_3 \quad w_3 \triangleleft F = 0 \ . \end{aligned} \tag{6.4}$$

For the left action (6.3) this is not the case. The generators E and F do not preserve B while the generator K does and acts as the identity. Its left action is indeed dual to the right coaction in (5.1) of the generator z of $\mathcal{O}(SO(2))$ on $\mathcal{O}(SO_q(3))$ and we could equivalently define the algebra B of coinvariant elements as made by invariants for K,

$$B = \{ b \in \mathcal{O}(SO_q(3)) \,|\, K \triangleright b = b \}. \tag{6.5}$$

Depending on the values of the deformation parameter q, the Hopf algebra $\mathcal{U}_{q^{1/2}}(sl_2)$ can be equipped with the following real structures [8, §3.1.4]:

- if $q \in \mathbb{R}$, there are two (non equivalent) $*$-structures:

$$(K^{\pm 1})^* = K^{\pm 1} \ , \quad E^* = FK \ , \quad F^* = K^{-1} E \tag{6.6}$$

  with corresponding Hopf $*$-algebra $\mathcal{U}_{q^{1/2}}(su_2)$ (this is the compact real form) and

$$(K^{\pm 1})^* = K^{\pm 1} \ , \quad E^* = -FK \ , \quad F^* = -K^{-1} E \tag{6.7}$$

  with corresponding Hopf $*$-algebra $\mathcal{U}_{q^{1/2}}(su_{1,1})$;

- if $|q| = 1$ there is only one $*$-structure given by

$$(K^{\pm 1})^* = K^{\pm 1}\ , \quad E^* = -E\ , \quad F^* = -F\ . \tag{6.8}$$

The corresponding Hopf $*$-algebra is $\mathcal{U}_{q^{1/2}}(sl_2(\mathbb{R}))$. Classically the Lie algebras $su_{1,1}$ and $sl_2(\mathbb{R})$ are isomorphic.

The pairing (6.1) induces a pairing between the real forms $\mathcal{U}_{q^{1/2}}(su_2)$ and $\mathcal{O}(SO_q(3,\mathbb{R}))$ and between the real forms $\mathcal{U}_{q^{1/2}}(sl_2(\mathbb{R}))$ and $\mathcal{O}(SO_q(1,2,\mathbb{R}))$. Indeed the conditions

$$\langle f^*, a\rangle = \overline{\langle f, S(a)^*\rangle}\ , \quad \langle f, a^*\rangle = \overline{\langle S(f)^*, a\rangle} \tag{6.9}$$

are satisfied for each $f \in \mathcal{U}_{q^{1/2}}(sl_2)$ and $a \in \mathcal{O}(SO_q(3;\mathbb{R}))$ or $f \in \mathcal{U}_{q^{1/2}}(sl_2(\mathbb{R}))$ and $a \in \mathcal{O}(SO_q(1,2,\mathbb{R}))$. On the other hand, the condition (6.9) for the pairing (6.1) is not satisfied for the algebra $\mathcal{U}_{q^{1/2}}(su_{1,1})$.

We need some notation. For $n \in \mathbb{N}$ the q-integer is defined as

$$[n] := [n]_{q^{\frac{1}{2}}} := \frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}. \tag{6.10}$$

One has $[n] = q^{\frac{-n+1}{2}} \sum_{j=0}^{n-1} q^j$ with property $[n] = [2][n-1] - [n-2]$.

When the deformation parameter q is not a root of unity, the centre of the algebra $\mathcal{U}_{q^{1/2}}(sl_2)$ is generated by the (quadratic) Casimir element (see [8, §3.1.1]):

$$\begin{aligned} C_q &:= EF + \frac{q^{-\frac{1}{2}}K + q^{\frac{1}{2}}K^{-1}}{(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^2} = FE + \frac{q^{\frac{1}{2}}K + q^{-\frac{1}{2}}K^{-1}}{(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^2} \\ &= \frac{1}{2}(EF + FE) + \frac{q^{\frac{1}{2}} + q^{-\frac{1}{2}}}{(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^2}(K + K^{-1}). \end{aligned} \tag{6.11}$$

We next show how to diagonalise the Casimir operator as an operator acting on the left on the algebra B. For this one uses the right action of $\mathcal{U}_{q^{1/2}}(sl_2)$ to construct a basis of eigenfunctions, since clearly $C_q \triangleright (a \triangleleft f) = (C_q \triangleright a) \triangleleft f$. As mentioned, while E and F do not preserve B, both the products EF and FE do. On the other hand, the generators K, K^{-1} act on B as the identity and hence

$$\frac{q^{-\frac{1}{2}}K + q^{\frac{1}{2}}K^{-1}}{(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^2} \triangleright b = \frac{q^{-\frac{1}{2}} + q^{\frac{1}{2}}}{(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^2}\, b, \quad b \in B.$$

Thus, we can remove from the Casimir an additive constant and consider the operator

$$\mathcal{C}_q := C_q - \frac{q^{-\frac{1}{2}} + q^{\frac{1}{2}}}{(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^2} = EF = FE \tag{6.12}$$

acting on the left on the algebra B. On the generators $w_k := u_{k2}$, $k = 1, 2, 3$, of B, the action of $\mathcal{C}_q$ is easily found to be

$$\mathcal{C}_q \triangleright w_k = [2]w_k, \quad k = 1, 2, 3. \tag{6.13}$$

Now, from [8, §4.5.2] one knows that there is a vector space decomposition

$$B = \oplus_{J \in \mathbb{N}} V_J$$

into irreducible representations V_J of $\mathcal{U}_{q^{1/2}}(sl_2)$. The spaces V_J are given by

$$V_J = \mathrm{span}\{w_3^J \triangleleft E^m\} = \mathrm{span}\{w_1^J \triangleleft F^m\}, \quad m = 0, 1, \ldots, 2J. \tag{6.14}$$

Thus w_3^J (or w_1^J) is the highest (or lowest) weight vector of the representation.

Theorem 6.1. *For any $J \in \mathbb{N}$ the space V_J is made of eigenfunctions of the operator $\mathcal{C}_q$ with eigenvalue $[J][J+1]$:*

$$\mathcal{C}_q \triangleright a = [J][J+1]\, a, \quad \forall a \in V_J. \tag{6.15}$$

Proof. In view of (6.14) it is enough to show the identity for the highest weight vector w_3^J. Clearly, if $\mathcal{C}_q \triangleright w_3^J = [J][J+1] w_3^J$, then for each $m = 0, 1, \ldots, 2J$,

$$\mathcal{C}_q \triangleright (w_3^J \triangleleft E^m) = (\mathcal{C}_q \triangleright w_3^J) \triangleleft E^m = [J][J+1](w_3^J \triangleleft E^m)\,.$$

Indeed we can show the result at once for the lowest and highest weight vectors. Using the coproduct $\Delta(EF) = EF \otimes K + K^{-1} \otimes EF + q^{-1} E K^{-1} \otimes FK + F \otimes E$ and recalling from (6.3) that K and K^{-1} act as the identity on the elements of B, the operator $\mathcal{C}_q$ acts on the product of two elements a, a' as

$$\begin{aligned}
\mathcal{C}_q \triangleright (aa') &= ((EF) \triangleright a)a' + a((EF) \triangleright a') \\
&\quad + q^{-1}(E \triangleright a)(F \triangleright a') + (F \triangleright a)(E \triangleright a') \\
&= (\mathcal{C}_q \triangleright a)a' + a(\mathcal{C}_q \triangleright a') \\
&\quad + q^{-1}(E \triangleright a)(F \triangleright a') + (F \triangleright a)(E \triangleright a')\,.
\end{aligned} \tag{6.16}$$

We hence need to compute the action of E and F on any power w_ℓ^J of w_ℓ, $\ell = 1, 3$. By induction on n one shows that

$$E \triangleright w_\ell^n = -q^{\frac{1}{2}} \left(\sum_{j=0}^{n-1} q^{-j} \right) \eta w_\ell^{n-1} u_{\ell 3} = -q^{-\frac{n}{2}+1} [n]\, \eta w_\ell^{n-1} u_{\ell 3}$$

$$F \triangleright w_\ell^n = \left(\sum_{j=0}^{n-1} q^{-j} \right) \eta w_\ell^{n-1} u_{\ell 1} = q^{\frac{n-1}{2}} [n]\, \eta w_\ell^{n-1} u_{\ell 1}\ ,$$

where $[n]$ is the $q^{\frac{1}{2}}$-number in (6.10). Next, we prove by induction that $\mathcal{C}_q \triangleright w_\ell^n = [n][n+1] w_\ell^n$. The result holds for the base case $n = 1$, as already observed in (6.13). Assume it holds for n, then, by also using (6.16), we compute

$$\begin{aligned}
\mathcal{C}_q \triangleright (w_\ell^{n+1}) &= (\mathcal{C}_q \triangleright w_\ell^n) w_\ell + w_\ell^n (\mathcal{C}_q \triangleright w_\ell) \\
&\quad + q^{-1}(E \triangleright w_\ell^n)(F \triangleright w_\ell) + (F \triangleright w_\ell^n)(E \triangleright w_\ell) \\
&= [n][n+1] w_\ell^{n+1} + [2] w_\ell^{n+1} - q^{-\frac{n}{2}} [2][n] w_\ell^{n-1} u_{\ell 3} u_{\ell 1} \\
&\quad - q^{\frac{n}{2}} [2][n] w_\ell^{n-1} u_{\ell 1} u_{\ell 3}
\end{aligned}$$

where, from Appendix C,

$$u_{\ell 1} u_{\ell 3} = -q^{\frac{3}{2}} (1+q)^{-1} w_\ell^2\,, \qquad u_{\ell 3} u_{\ell 1} = -q^{-\frac{1}{2}} (1+q)^{-1} w_\ell^2\ .$$

We hence obtain that w_ℓ^{n+1} is an eigenfunction of $\mathcal{C}_q$ with eigenvalue

$$[n][n+1] + [2] + q^{-\frac{n+1}{2}} [2][n](1+q)^{-1} + q^{\frac{n+3}{2}} [2][n](1+q)^{-1} =$$

$$= [n][n+1] + [2] + q^{-\frac{n+2}{2}}[n] + q^{\frac{n+2}{2}}[n] \, .$$

Next, by explicit computation one verifies that

$$[2] + q^{-\frac{n+2}{2}}[n] + q^{\frac{n+2}{2}}[n] = [n+1]([2][n+1] - 2[n])$$

so that, finally,

$$\mathcal{C}_q \triangleright (w_\ell^{n+1}) = [n+1]\big([n] + [2][n+1] - 2[n]\big) w_\ell^{n+1} = [n+1][n+2]\, w_\ell^{n+1}$$

where we have used the property $[2][n+1] - [n] = [n+2]$ of q-numbers. □

The above analysis is valid, when q is real, for the $*$-algebra $\mathcal{U}_{q^{1/2}}(su(2))$ acting on the algebra $B = \mathcal{O}(S^2_{q,Gr})$ of the standard Podleś sphere. The more complicated case of $\mathcal{U}_{q^{1/2}}(sl_2(\mathbb{R}))$ that involves unbounded representations (see [15]) will be studied elsewhere.

Appendix A. Proof of Proposition 3.1

From the definition (3.4), we are left to show that $\sum_m u_{dm}\widehat{u}_{ma} = 0$, for all indices $a \neq d$. Notice that for each index $a = 1, 2, 3$ (and for each m) we can always choose an expression of the cofactor $\widehat{u}_{ma} = \varepsilon^{-1}_{abc} \sum_{n,p} \varepsilon_{mnp} u_{bn} u_{cp}$ for which a, b, c are all different. So either $d = b$ or $d = c$. Without loss of generality we can take $d = b$ (that is, of the two equivalent expressions of the cofactor with $a \neq b \neq c$ we can take the one where the index b is equal to d). Thus, fixing mutually different indices $a, b = d, c$, we compute

$$\begin{aligned}
\varepsilon_{adc} \sum_m u_{dm}\widehat{u}_{ma} &= \sum_{m,n,p} \varepsilon_{mnp} u_{dm} u_{dn} u_{cp} \\
&= \sum_{m,n} \varepsilon_{mn1} u_{dm} u_{dn} u_{c1} + \sum_{m,n} \varepsilon_{mn2} u_{dm} u_{dn} u_{c2} \\
&\quad + \sum_{m,n} \varepsilon_{mn3} u_{dm} u_{dn} u_{c3} \\
&= q(u_{d2}u_{d3} - q u_{d3}u_{d2})u_{c1} - q(u_{d1}u_{d3} - u_{d3}u_{d1} \\
&\quad + (q^{\frac12} - q^{-\frac12}) u_{d2}u_{d2}) u_{c2} \\
&\quad + (u_{d1}u_{d2} - q u_{d2}u_{d1}) u_{c3} \, . \qquad \text{(A.1)}
\end{aligned}$$

We then use equation (2.3) for elements u_{dm} on the same row:

$$\begin{aligned}
q^{1-\delta_{d2}} u_{dm} u_{dn} = {} & q^{\delta_{mn} - \delta_{mn'}} u_{dn} u_{dm} + \lambda\theta(n-m) u_{dm} u_{dn} + \delta_{d2}\lambda q^{-\frac12} u_{1m} u_{3n} \\
& - \lambda \delta_{nm'} \sum_k \theta(k-m) q^{-\rho_m - \rho_{k'}} u_{dk'} u_{dk} \, . \qquad \text{(A.2)}
\end{aligned}$$

For $d \neq 2$, this yields

$$\begin{aligned}
q u_{d3} u_{d2} &= u_{d2} u_{d3} \, , \qquad q u_{d2} u_{d1} = u_{d1} u_{d2} \, , \\
q^2 u_{d3} u_{d1} &= u_{d1} u_{d3} \, , \\
(1 + q^{-1}) u_{d1} u_{d3} &= q^{-2} u_{d1} u_{d3} + q^{-1} u_{d3} u_{d1} - q^{-\frac12} \lambda u_{d2} u_{d2}.
\end{aligned}$$

The first two relations imply the vanishing of the (polynomial) coefficients of u_{c1} and u_{c3}. The last two when combined yield

$$(1+q^{-1})u_{d1}u_{d3} = (1+q^{-1})u_{d3}u_{d1} - (1+q^{-1})(q^{\frac{1}{2}} - q^{-\frac{1}{2}})u_{d2}u_{d2}$$

and the coefficient of u_{c2} vanishes as well.

For $d = 2$ the computation is more involved. Equation (A.1) becomes

$$\begin{aligned}\varepsilon_{a2c} \sum_m u_{2m}\widehat{u}_{ma} = q(u_{22}u_{23} - qu_{23}u_{22})u_{c1} - q(u_{21}u_{23} - u_{23}u_{21} \\ +(q^{\frac{1}{2}} - q^{-\frac{1}{2}})u_{22}u_{22})u_{c2} + (u_{21}u_{22} - qu_{22}u_{21})u_{c3}\ ,\end{aligned} \tag{A.3}$$

with the coefficients of the u_{cp} that do not vanish, in contrast to the case $d = 1, 3$. We hence need to proceed differently: the idea is to express the coefficients as polynomials in $u_{3k}u_{1j}$ for the case $c = 1$ or as polynomials in $u_{1k}u_{3j}$ for the case $c = 3$. We start with the coefficient of u_{c1}. The equation (A.2) yields

$$u_{23}u_{22} = u_{22}u_{23} + q^{-\frac{1}{2}}\lambda u_{13}u_{32}$$
$$(1+q^{-1})u_{22}u_{23} = u_{23}u_{22} + qu_{22}u_{23} + q^{-\frac{1}{2}}\lambda u_{12}u_{33}.$$

When combined, these yield

$$u_{22}u_{23} - qu_{23}u_{22} = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})(u_{12}u_{33} - qu_{13}u_{32}).$$

This can also be written as

$$u_{22}u_{23} - qu_{23}u_{22} = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})(qu_{33}u_{12} - u_{32}u_{13})$$

when using the commutation relations

$$q^{-1}u_{13}u_{32} = u_{32}u_{13}\ , \qquad q^{-1}u_{12}u_{33} = u_{33}u_{12} + \lambda u_{32}u_{13}$$

obtained from (2.3), for suitable choices of indices.

Analogously, for the coefficient of u_{c3}, from equation (A.2) we obtain

$$u_{22}u_{21} = u_{21}u_{22} + q^{-\frac{1}{2}}\lambda u_{12}u_{31}$$
$$(1+q^{-1})u_{21}u_{22} = u_{22}u_{21} + qu_{21}u_{22} + q^{-\frac{1}{2}}\lambda u_{11}u_{32}.$$

When combined, these yield

$$u_{21}u_{22} - qu_{22}u_{21} = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})(u_{11}u_{32} - qu_{12}u_{31}).$$

This can also be written as

$$u_{21}u_{22} - qu_{22}u_{21} = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})(qu_{32}u_{11} - u_{31}u_{12})$$

when using the commutation relations

$$q^{-1}u_{12}u_{31} = u_{31}u_{12}\ , \qquad q^{-1}u_{11}u_{32} = u_{32}u_{11} + \lambda u_{31}u_{12}$$

again obtained from (2.3), for suitable choices of indices.

Finally, the coefficient of u_{c2} in (A.3) is proportional to the cofactor $\widehat{u}_{22}$:

$$\begin{aligned}u_{21}u_{23} - u_{23}u_{21} + (q^{\frac{1}{2}} - q^{-\frac{1}{2}})u_{22}u_{22} &= (q^{\frac{1}{2}} - q^{-\frac{1}{2}})\widehat{u}_{22} \\ &= (q^{\frac{1}{2}} - q^{-\frac{1}{2}})\left[u_{11}u_{33} - u_{13}u_{31} + (q^{\frac{1}{2}} - q^{-\frac{1}{2}})u_{12}u_{32}\right]\end{aligned}$$

$$= (q^{\frac{1}{2}} - q^{-\frac{1}{2}}) \Big[-u_{31}u_{13} + u_{33}u_{11} - (q^{\frac{1}{2}} - q^{-\frac{1}{2}})u_{32}u_{12} \Big] .$$

We then return to (A.3). For $c = 1$ equation (A.3) reads

$$\begin{aligned}
-q^2 \sum_m u_{2m}\widehat{u}_{m3} &= q(u_{22}u_{23} - qu_{23}u_{22})u_{11} + (u_{21}u_{22} - qu_{22}u_{21})u_{13} \\
&\quad - q\big(u_{21}u_{23} - u_{23}u_{21} + (q^{\frac{1}{2}} - q^{-\frac{1}{2}})u_{22}u_{22}\big)u_{12} \\
&= (q^{\frac{1}{2}} - q^{-\frac{1}{2}})\big[q(qu_{33}u_{12} - u_{32}u_{13})u_{11} + (qu_{32}u_{11} - u_{31}u_{12})u_{13} \\
&\quad - q\big(- u_{31}u_{13} + u_{33}u_{11} - (q^{\frac{1}{2}} - q^{-\frac{1}{2}})u_{32}u_{12}\big)u_{12}\big] \\
&= q(q^{\frac{1}{2}} - q^{-\frac{1}{2}})\, u_{32}\big[- u_{13}u_{11} + u_{11}u_{13} + (q^{\frac{1}{2}} - q^{-\frac{1}{2}})u_{12}u_{12}\big],
\end{aligned}$$

where in the last equality we have used

$$u_{12}u_{11} = q^{-1}u_{11}u_{12} \quad \text{and} \quad u_{13}u_{12} = q^{-1}u_{12}u_{13} \ ,$$

obtained once again from (2.3). From (2.3) we also obtain

$$\begin{aligned}
&u_{13}u_{11} = q^{-2}u_{11}u_{13}, \\
&(1 + q^{-1})u_{11}u_{13} = q^{-1}u_{13}u_{11} + q^{-2}u_{11}u_{13} - q^{-\frac{1}{2}}\lambda u_{12}u_{12}
\end{aligned}$$

which, when combined, give

$$u_{11}u_{13} = u_{13}u_{11} - (q^{\frac{1}{2}} - q^{-\frac{1}{2}})u_{12}u_{12}$$

and then $\sum_m u_{2m}\widehat{u}_{m3} = 0$.

Similarly, for $c = 3$ equation (A.3) reads

$$\begin{aligned}
\sum_m u_{2m}\widehat{u}_{m1} &= q(u_{22}u_{23} - qu_{23}u_{22})u_{31} + (u_{21}u_{22} - qu_{22}u_{21})u_{33} \\
&\quad - q\big(u_{21}u_{23} - u_{23}u_{21} + (q^{\frac{1}{2}} - q^{-\frac{1}{2}})u_{22}u_{22}\big)u_{32} \\
&= (q^{\frac{1}{2}} - q^{-\frac{1}{2}})\big[q(u_{12}u_{33} - qu_{13}u_{32})u_{31} + (u_{11}u_{32} - qu_{12}u_{31})u_{33} \\
&\quad - q\big(u_{11}u_{33} - u_{13}u_{31} + (q^{\frac{1}{2}} - q^{-\frac{1}{2}})u_{12}u_{32}\big)u_{32} \\
&= q(q^{\frac{1}{2}} - q^{-\frac{1}{2}})\, u_{12}\big[u_{33}u_{31} - u_{31}u_{33} - (q^{\frac{1}{2}} - q^{-\frac{1}{2}})u_{32}u_{32}\big],
\end{aligned}$$

where in the last equality we have used

$$u_{32}u_{31} = q^{-1}u_{31}u_{32} \ , \quad u_{33}u_{32} = q^{-1}u_{32}u_{33} \ ,$$

obtained once again from (2.3). From (2.3) we also obtain

$$\begin{aligned}
&u_{33}u_{31} = q^{-2}u_{31}u_{33}, \\
&(1 + q^{-1})u_{31}u_{33} = q^{-1}u_{33}u_{31} + q^{-2}u_{31}u_{33} - q^{-\frac{1}{2}}\lambda u_{32}u_{32}
\end{aligned}$$

which, when combined, give

$$u_{31}u_{33} = u_{33}u_{31} - (q^{\frac{1}{2}} - q^{-\frac{1}{2}})u_{32}u_{32}$$

and then $\sum_m u_{2m}\widehat{u}_{m1} = 0$. This concludes the proof of Prop. 3.1.

Appendix B. Commutation relations in $O_q(3)$

In this appendix we compute explicitly the commutation relations (2.3) among the generators u_{ij} of the algebra $O_q(3)$, for $j = 1, 3$, which we need for computing the coinvariant elements in Proposition 5.1.

As before $\lambda = q - q^{-1}$, and $\rho_1 = \frac{1}{2}$, $\rho_2 = 0$, $\rho_3 = -\frac{1}{2}$. Moreover, for each index $k = 1, 2, 3$, $k' = 3 - k$ so that $1' = 3$, $2' = 2$ and $3' = 1$.

Commutation Relations u_{i1} u_{j1}

For $m = n = 1$, Eq. (2.3) reduces to

$$q^{\delta_{ij}-\delta_{ij'}} u_{j1}u_{i1} = (q - \lambda\theta(j-i))u_{i1}u_{j1} + \lambda\delta_{ij'}\sum_k \theta(j-k)q^{-\rho_i-\rho_k}u_{k1}u_{k'1}\,,$$

from which

$$\begin{aligned} u_{21}u_{11} &= q^{-1}u_{11}u_{21}\,, \quad u_{31}u_{11} = q^{-2}u_{11}u_{31}\,, \\ u_{31}u_{21} &= q^{-1}u_{21}u_{31}\,, \; (u_{21})^2 = -q^{-\frac{3}{2}}(1+q)u_{11}u_{31}\,. \end{aligned} \tag{B.1}$$

Commutation Relations u_{i3} u_{j3}. For $m = n = 3$, Eq. (2.3) has an expression analogous to that for $m = n = 1$:

$$q^{\delta_{ij}-\delta_{ij'}} u_{j3}u_{i3} = (q - \lambda\theta(j-i))u_{i3}u_{j3} + \lambda\delta_{ij'}\sum_k \theta(j-k)q^{-\rho_i-\rho_k}u_{k3}u_{k'3}$$

and one has

$$\begin{aligned} u_{23}u_{13} &= q^{-1}u_{13}u_{23}\,, \qquad u_{33}u_{13} = q^{-2}u_{13}u_{33}\,, \\ u_{33}u_{23} &= q^{-1}u_{23}u_{33}\,, \qquad (u_{23})^2 = -q^{-\frac{3}{2}}(1+q)u_{13}u_{33}\,. \end{aligned} \tag{B.2}$$

Commutation Relations u_{i1} u_{j3}. For $m = 3$ and $n = 1$, Eq. (2.3) gives

$$q^{-1}u_{i1}u_{j3} = q^{\delta_{ij}-\delta_{ij'}}u_{j3}u_{i1} + \lambda\theta(j-i)u_{i3}u_{j1} - \lambda\delta_{ij'}\sum_k \theta(j-k)q^{-\rho_i-\rho_k}u_{k3}u_{k'1}$$

from which

$$\begin{aligned} &u_{13}u_{11} = q^{-2}u_{11}u_{13}, \quad u_{21}u_{13} = q\; u_{13}u_{21}, \quad u_{23}u_{11} = q^{-1}u_{11}u_{23} - \lambda u_{13}u_{21}\,, \\ &u_{23}u_{21} = q^{-1}u_{21}u_{23} + q^{-\frac{1}{2}}\lambda u_{13}u_{31}\,, \quad u_{31}u_{23} = q\; u_{23}u_{31}\,, \quad u_{31}u_{13} = u_{13}u_{31}, \\ &u_{33}u_{11} = u_{11}u_{33} + (1-q^{-1})\lambda u_{13}u_{31} + \lambda q^{-\frac{1}{2}}u_{21}u_{23}, \\ &u_{33}u_{21} = q^{-1}u_{21}u_{33} - \lambda u_{23}u_{31}\,, \qquad u_{33}u_{31} = q^{-2}u_{31}u_{33} \end{aligned} \tag{B.3}$$

The quotient algebra by the ideal generated by $Q_q - 1$ gives the algebra $\mathcal{O}(O_q(3))$, where the element Q_q in (2.6) can be written as

$$\begin{aligned} Q_q &= u_{11}u_{33} + q^{\frac{1}{2}}u_{21}u_{23} + qu_{31}u_{13} = u_{11}u_{33} + q^{\frac{1}{2}}u_{12}u_{32} + qu_{13}u_{31} \\ &= q^{-\frac{1}{2}}u_{12}u_{32} + u_{22}u_{22} + q^{\frac{1}{2}}u_{32}u_{12} = q^{-\frac{1}{2}}u_{21}u_{23} + u_{22}u_{22} + q^{\frac{1}{2}}u_{23}u_{21} \\ &= q^{-1}u_{13}u_{31} + q^{-\frac{1}{2}}u_{23}u_{21} + u_{33}u_{11} = q^{-1}u_{31}u_{13} + q^{-\frac{1}{2}}u_{32}u_{12} + u_{33}u_{11}\,, \end{aligned}$$

the diagonal entries of the matrices $S(u)u$ and $uS(u)$.

Appendix C. Cofactors and Coinvariant Elements

We list all the cofactors of the elements of the defining matrix u:

$$\begin{aligned}\widehat{u}_{11} &= u_{22}u_{33} - qu_{23}u_{32} = -q^{-1}u_{32}u_{23} + u_{33}u_{22} \\ \widehat{u}_{21} &= -qu_{21}u_{33} + qu_{23}u_{31} - q(q^{\frac{1}{2}} - q^{-\frac{1}{2}})u_{22}u_{32} \\ &= u_{31}u_{23} - u_{33}u_{21} + (q^{\frac{1}{2}} - q^{-\frac{1}{2}})u_{32}u_{22} \\ \widehat{u}_{31} &= qu_{21}u_{32} - q^2u_{22}u_{31} = -u_{31}u_{22} + qu_{32}u_{21}\end{aligned}$$

together with

$$\begin{aligned}\widehat{u}_{12} &= -q^{-1}u_{12}u_{33} + u_{13}u_{32} = q^{-1}u_{32}u_{13} - u_{33}u_{12} \\ &= -q^{-1}(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^{-1}(u_{22}u_{23} - qu_{23}u_{22}) \\ \widehat{u}_{22} &= u_{11}u_{33} - u_{13}u_{31} + (q^{\frac{1}{2}} - q^{-\frac{1}{2}})u_{12}u_{32} \\ &= -u_{31}u_{13} + u_{33}u_{11} - (q^{\frac{1}{2}} - q^{-\frac{1}{2}})u_{32}u_{12} \\ &= (q^{\frac{1}{2}} - q^{-\frac{1}{2}})^{-1}(u_{21}u_{23} - u_{23}u_{21} + (q^{\frac{1}{2}} - q^{-\frac{1}{2}})u_{22}u_{22}) \\ \widehat{u}_{32} &= -u_{11}u_{32} + qu_{12}u_{31} = q^{-1}u_{31}u_{12} - qu_{32}u_{11} \\ &= (q^{\frac{1}{2}} - q^{-\frac{1}{2}})^{-1}(-u_{21}u_{22} + qu_{22}u_{21})\end{aligned}$$

and finally

$$\begin{aligned}\widehat{u}_{13} &= q^{-1}u_{12}u_{23} - u_{13}u_{22} = -q^{-2}u_{22}u_{13} + q^{-1}u_{23}u_{12} \\ \widehat{u}_{23} &= -u_{11}u_{23} + u_{13}u_{21} - (q^{\frac{1}{2}} - q^{-\frac{1}{2}})u_{12}u_{22} \\ &= q^{-1}u_{21}u_{13} - q^{-1}u_{23}u_{11} + q^{-1}(q^{\frac{1}{2}} - q^{-\frac{1}{2}})u_{22}u_{12} \\ \widehat{u}_{33} &= u_{11}u_{22} - qu_{12}u_{21} = -q^{-2}u_{21}u_{12} + q^{-1}u_{22}u_{11}\,.\end{aligned}$$

Next, we list all quadratic coinvariant elements $u_{i3}u_{j1}$ and $u_{i1}u_{j3}$ as polynomials in the elements of the second column $u_{k2} =: w_k$. From the proof of Proposition 5.1 we have

$$\begin{aligned}&u_{13}u_{11} = -q^{-\frac{1}{2}}(1+q)^{-1}w_1^2\,, \qquad u_{13}u_{21} = q^{-\frac{1}{2}}(1+q)^{-1}\; w_1\,(1-w_2)\;, \\ &u_{13}u_{31} = (1+q)^{-1}(1 - w_2 - q^{-\frac{1}{2}}w_1w_3), \\ &u_{23}u_{11} = -q^{\frac{1}{2}}(1+q)^{-1}(1+q^{-1}w_2)w_1\,, \\ &u_{23}u_{21} = w_3w_1\,, \\ &u_{23}u_{31} = q^{-\frac{1}{2}}(1+q)^{-1}\,(1-w_2)\,w_3\,, \\ &u_{33}u_{11} = (1+q)^{-1}(q + w_2 - q^{-\frac{1}{2}}w_3w_1) \\ &u_{33}u_{21} = -q^{-\frac{1}{2}}(1+q)^{-1}\; w_3\,(q+w_2)\;, \qquad u_{33}u_{31} = -q^{-\frac{1}{2}}(1+q)^{-1}w_3^2\,.\end{aligned}$$

Formulas for the elements $u_{i1}u_{j3}$ are recovered by using (5.2), or explicitly (B.3), and also the commutation relations (5.10)

$$\begin{aligned}&w_3(w_2-1) = q^{-1}(w_2-1)w_3\,, \qquad w_1(w_2-1) = q(w_2-1)w_1\,, \\ &qw_3w_1 = q^{-1}w_1w_3 + (q^{-\frac{1}{2}} - q^{\frac{1}{2}})(w_2-1)\end{aligned}$$

or equivalently

$$
\begin{aligned}
&w_3w_2 = q^{-1}w_2w_3 + (1-q^{-1})w_3\,, \qquad w_2w_1 = q^{-1}w_1w_2 + (1-q^{-1})w_1\,,\\
&w_3w_1 = q^{-2}w_1w_3 + q^{-\frac{3}{2}}(1-q)(w_2-1)
\end{aligned}
$$

with (5.11): $q^{-\frac{1}{2}}w_1w_3 + q^{\frac{1}{2}}w_3w_1 + w_2^2 = 1$. Finally for the remaining coinvariant elements

$$
\begin{aligned}
&u_{11}u_{13} = -q^{\frac{3}{2}}(1+q)^{-1}w_1^2\,, \qquad u_{11}u_{23} = -q^{\frac{1}{2}}(1+q)^{-1}w_1\,(1+q\;w_2)\,,\\
&u_{11}u_{33} = (1+q)^{-1}(1+qw_2 - q^{\frac{3}{2}}w_1w_3)\\
&u_{21}u_{13} = \;q^{\frac{1}{2}}(1+q)^{-1}w_1\,(1-w_2)\,, \qquad u_{21}u_{23} = w_1w_3\\
&u_{21}u_{33} = -q^{\frac{1}{2}}(1+q)^{-1}(1+q\;w_2)w_3\\
&u_{31}u_{13} = (1+q)^{-1}(1-w_2-q^{-\frac{1}{2}}w_1w_3)\\
&u_{31}u_{23} = q^{\frac{1}{2}}(1+q)^{-1}\;\;(1-w_2)\,w_3\,, \qquad u_{31}u_{33} = -q^{\frac{3}{2}}(1+q)^{-1}w_3^2.
\end{aligned}
$$

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