On the Cauchy problem for the wave equation on time-dependent domains

Gianni Dal Maso [a,*], Rodica Toader [b]

[a] *SISSA, Via Bonomea 265, 34136 Trieste, Italy*
[b] *DMIF, Università degli Studi di Udine, Via delle Scienze 206, 33100, Udine, Italy*

Abstract

We introduce a notion of solution to the wave equation on a suitable class of time-dependent domains and compare it with a previous definition. We prove an existence result for the solution of the Cauchy problem and present some additional conditions which imply uniqueness.
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1. Introduction

The mathematical formulation of dynamic problems in fracture mechanics leads to the study of the wave equation in time-dependent domains (see [9,12,7]). The main feature of these problems is that at every time t the solution belongs to a different space V_t. In the case of fracture a typical situation is $V_t = H^1(\Omega \setminus \Gamma_t)$, where Ω is a domain in $\mathbb{R}^n$ and Γ_t is a closed $(n-1)$-dimensional subset of Ω, which represents the crack at time t (see [6,8,3,14]). The most important example of equation we consider is formally written as

* Corresponding author.
E-mail addresses: dalmaso@sissa.it (G. Dal Maso), rodica.toader@uniud.it (R. Toader).

$$
\begin{cases} \ddot{u} - \Delta u = f & \text{in } \Omega \setminus \Gamma_t\,, \\ \partial_\nu u = 0 & \text{on } \partial\Omega \cup \Gamma_t\,, \end{cases} \tag{1.1}
$$

where $\ddot{u}$ denotes the second order time derivative of u, Δu is the Laplacian of u with respect to the spatial variables, and $\partial_\nu u$ is the normal derivative of u.

In this paper we introduce and study a notion of solution to the wave equation on time-dependent domains in a sufficiently general abstract framework for the spaces V_t, which covers the case of increasing cracks Γ_t with homogeneous Neumann boundary conditions both on $\partial\Omega$ and Γ_t. The boundary condition on $\partial\Omega$ could be easily replaced by a prescribed nonhomogeneous Dirichlet condition. The same method applies also to the case of homogeneous Dirichlet boundary conditions both on $\partial\Omega$ and Γ_t, but only when $t \mapsto \Gamma_t$ is decreasing (see Example 2.3), which is not a natural assumption in fracture mechanics.

We compare this definition with the one introduced in [6], which was given under slightly stronger assumptions on the data, and we prove that they are equivalent when these assumptions are satisfied (see Theorems 2.16 and 2.17). Our definition is based on integration by parts in time and does not require a precise definition of the value at time t of the second derivative $\ddot{u}(t)$, which is a critical issue in the case of time-dependent domains (see Proposition 2.13). Actually, the boundedness assumptions of [6], which we remove in our paper, are used to simplify the definition of $\ddot{u}(t)$.

Under natural assumptions on the initial data, we prove an existence result for the solution to the Cauchy problem, which simplifies the proof of [6] because we can avoid some estimates regarding $\ddot{u}(t)$ (see Theorem 3.1). We also prove that the solution obtained in this way satisfies the energy inequality (see Corollary 3.2).

The last part of the paper contains the most relevant original result: some general conditions on V_t which imply the uniqueness of the solution to the Cauchy problem (see Theorem 4.3). These are given in terms of properties of some linear isomorphisms $Q_t\colon V_t \to V_0$ and $R_t\colon V_0 \to V_t$, as well as of their derivatives with respect to time.

To illustrate this uniqueness result let us consider the model situation of a rectilinear crack in the plane with subsonic speed. In this case we have to solve (1.1) with $\Omega = \mathbb{R}^2$ and $\Gamma_t = \{(x_1, 0) : x_1 \le \ell(t)\}$, where $\ell\colon [0, T] \to \mathbb{R}$ is a prescribed $C^{1,1}$ function such that

$$
0 \le \dot{\ell}(t) < 1 \quad \text{for every } t \in [0, T]\,. \tag{1.2}
$$

Using the Lipschitz continuity of $\dot{\ell}$ and (1.2) it is easy to see that all conditions for uniqueness are satisfied (see Example 4.1).

More general assumptions on the sets Γ_t under which the Cauchy problem for (1.1) has a unique solution can be expressed in terms of the regularity properties, with respect to space and time, of suitable diffeomorphisms of Ω into itself, mapping Γ_t into Γ_0 (see Example 4.2). These assumptions are weaker than those considered in [8] and [3].

2. Formulation of the evolution problem, notions of solution

Let H be a separable Hilbert space, let $T > 0$, and let $(V_t)_{t\in[0,T]}$ be a family of separable Hilbert spaces with the following properties:

(H1) for every $t \in [0, T]$ the space V_t is contained and dense in H with continuous embedding;

(H2) for every $s, t \in [0, T]$, with $s < t$, $V_s \subset V_t$ and the Hilbert space structure on V_s is the one induced by V_t.

The scalar product in H is denoted by $(\cdot, \cdot)$ and the corresponding norm by $\| \cdot \|$. The norm in V_t is denoted by $\| \cdot \|_t$. Note that for every $t \in [0, T]$, by (H2) we have $\|v\|_t = \|v\|_T$ for every $v \in V_t$.

The dual of H is identified with H, while for every $t \in [0, T]$ the dual of V_t is denoted by V_t^*. Note that the adjoint of the continuous embedding of V_t into H provides a continuous embedding of H into V_t^* and that H is dense in V_t^*. Let $\langle \cdot, \cdot \rangle_t$ be the duality product between V_t^* and V_t and let $\| \cdot \|_t^*$ be the corresponding dual norm. Note that $\langle \cdot, \cdot \rangle_t$ is the unique continuous bilinear map on $V_t^* \times V_t$ satisfying

$$\langle h, v \rangle_t = (h, v) \qquad \text{for every } h \in H \text{ and } v \in V_t. \tag{2.1}$$

For $0 \leq s < t \leq T$ we have $V_s \subset V_t$, but since V_s is not dense in V_t the dual space V_t^* is not embedded into V_s^*. However, it is useful to introduce the natural projection operators from V_t^* to V_s^*.

Definition 2.1. Let $s, t \in [0, T]$ with $s < t$. The projection map $\Pi_{st} : V_t^* \to V_s^*$ is defined by

$$\langle \Pi_{st} \zeta, v \rangle_s := \langle \zeta, v \rangle_t \qquad \text{for every } \zeta \in V_t^* \text{ and } v \in V_s\,.$$

It is easy to see that Π_{st} is continuous, with $\|\Pi_{st} \zeta\|_s^* \leq \|\zeta\|_t^*$ for every $\zeta \in V_t^*$. In general it is not injective. Note that by (2.1) we have

$$\Pi_{st} h = h \quad \text{for every } h \in H\,. \tag{2.2}$$

Moreover, we have

$$\Pi_{rs} \Pi_{st} = \Pi_{rt} \quad \text{for every } r < s < t\,. \tag{2.3}$$

Example 2.2. Let Ω be an open subset of $\mathbb{R}^n$ and $(\Gamma_t)_{t \in [0,T]}$ be a family of relatively closed subsets of Ω, with $\Gamma_s \subset \Gamma_t$ for every $0 \leq s < t \leq T$ and $\mathcal{H}^{n-1}(\Gamma_T) < +\infty$, where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure. Then the spaces $V_t := H^1(\Omega \setminus \Gamma_t)$ and $H := L^2(\Omega)$ satisfy (H1) and (H2).

Example 2.3. Let Ω be an open subset of $\mathbb{R}^n$ and $(\Gamma_t)_{t \in [0,T]}$ be a family of relatively closed subsets of Ω, with $\Gamma_t \subset \Gamma_s$ for every $0 \leq s < t \leq T$ and $\mathcal{H}^{n-1}(\Gamma_0) < +\infty$. Then the spaces $V_t := H_0^1(\Omega \setminus \Gamma_t)$ and $H := L^2(\Omega)$ satisfy (H1) and (H2).

Example 2.4. Let Ω be an open subset of $\mathbb{R}^n$ and $(\Gamma_t)_{t \in [0,T]}$ be a family of subsets of Ω, with $\Gamma_s \subset \Gamma_t$ for every $0 \leq s < t \leq T$ and $\mathcal{H}^{n-1}(\Gamma_T) < +\infty$. Then the spaces $V_t := GSBV_2^2(\Omega, \Gamma_t)$ introduced in [6, formula (2.1)], together with $H := L^2(\Omega)$, satisfy (H1) and (H2).

Let $a \colon V_T \times V_T \to \mathbb{R}$ be a bilinear symmetric form satisfying the following conditions:

(H3) continuity: there exists $M_0 > 0$ such that

$$|a(u, v)| \le M_0 \|u\|_T \|v\|_T \quad \text{for every } u, v \in V_T ; \tag{2.4}$$

(H4) coercivity: there exist $\lambda_0 \ge 0$ and $\nu_0 > 0$ such that

$$a(u, u) + \lambda_0 \|u\|^2 \ge \nu_0 \|u\|_T^2 \quad \text{for every } u \in V_T . \tag{2.5}$$

For every $\tau, t \in [0, T]$ let $A_\tau^t : V_t \to V_\tau^*$ be the continuous linear operator defined by

$$\langle A_\tau^t u, v\rangle_\tau := a(u, v) \quad \text{for every } u \in V_t \text{ and } v \in V_\tau . \tag{2.6}$$

Note that

$$\|A_\tau^t u\|_\tau^* \le M_0 \|u\|_t \quad \text{for every } u \in V_t . \tag{2.7}$$

Example 2.5. Under the hypotheses of Example 2.2, let (a_{ij}) be a symmetric $n \times n$ matrix of functions in $L^\infty(\Omega)$ satisfying the ellipticity condition with a constant $c_0 > 0$:

$$\sum_{ij} a_{ij}(x)\xi_j\xi_i \ge c_0|\xi|^2 \quad \text{for a.e. } x \in \Omega \text{ and every } \xi \in \mathbb{R}^n .$$

Then the bilinear form

$$a(u, v) := \int_{\Omega\setminus\Gamma_T} \Big(\sum_{ij} a_{ij} D_j u\, D_i v\Big) dx \quad \text{for } u, v \in H^1(\Omega \setminus \Gamma_T) \tag{2.8}$$

satisfies (H3) and (H4). Therefore, under suitable regularity assumptions, for every given $f \in H$ the equation $A_t^t u = f$ provides a weak formulation of the boundary value problem

$$\begin{cases} -\sum_{ij} D_i(a_{ij} D_j u) = f & \text{in } \Omega \setminus \Gamma_t \\ \frac{\partial u}{\partial \nu^a} = 0 & \text{on } \partial\Omega \cup \Gamma_t , \end{cases} \tag{2.9}$$

where ν^a is the conormal corresponding to (a_{ij}), whose components are given by $\nu_j^a = \sum_i a_{ij}\nu_i$.

Given $f \in L^2((0, T); H)$, we now study the evolution equation formally written as

$$\begin{cases} \ddot{u}(t) + A_t^t u(t) = f(t) \\ u(t) \in V_t \end{cases}$$

on the time interval $[0, T]$. In order to give a precise notion of solution we introduce a space of t-dependent functions.

Definition 2.6. $\mathcal{V}$ is the space of functions $u \in L^2((0,T); V_T) \cap H^1((0,T); H)$ such that $u(t) \in V_t$ for a.e. $t \in (0,T)$. It is a Hilbert space with the scalar product given by

$$(u,v)_{\mathcal{V}} = (u,v)_{L^2((0,T);V_T)} + (\dot{u},\dot{v})_{L^2((0,T);H)},$$

where $\dot{u}$ and $\dot{v}$ denote the distributional derivatives with respect to t.

It is well known that every function $u \in H^1((0,T); H)$ admits a representative, still denoted by u, which belongs to the space $C([0,T]; H)$. With this convention we have $\mathcal{V} \subset C([0,T]; H)$.

Definition 2.7. We say that u is a weak solution of the equation

$$\begin{cases} \ddot{u}(t) + A_t^t u(t) = f(t) \\ u(t) \in V_t \end{cases} \tag{2.10}$$

on the time interval $[0,T]$ if $u \in \mathcal{V}$ and

$$-\int_0^T (\dot{u}(t), \dot{\varphi}(t))dt + \int_0^T a(u(t), \varphi(t))dt = \int_0^T (f(t), \varphi(t))dt \tag{2.11}$$

for every $\varphi \in \mathcal{V}$ with $\varphi(T) = \varphi(0) = 0$.

Lemma 2.8. *Given $\varphi \in \mathcal{V}$ with $\varphi(T) = \varphi(0) = 0$, there exists a sequence of functions $\varphi_j \in C_c^\infty((0,T); V_T)$, with $\varphi_j(t) \in V_t$ for every $t \in (0,T)$, such that*

$$\varphi_j \to \varphi \qquad \text{strongly in } \mathcal{V}. \tag{2.12}$$

Proof. It is enough to consider $\varphi \in \mathcal{V}$ with compact support in $(0,T)$. Indeed, every $\varphi \in \mathcal{V}$ with $\varphi(T) = \varphi(0) = 0$ can be approximated by a sequence of functions $\varphi_k \in \mathcal{V}$ with compact support. For instance, we can take $\varphi_k(t) = \omega_k(t)\varphi(t)$ where ω_k is the piecewise affine function such that $\omega_k = 0$ on $[0, \frac{1}{k}] \cup [T - \frac{1}{k}, T]$, $\omega_k = 1$ on $[\frac{2}{k}, T - \frac{2}{k}]$, and ω_k is affine on $[\frac{1}{k}, \frac{2}{k}]$ and $[T - \frac{2}{k}, T - \frac{1}{k}]$. Using the fundamental theorem of calculus for H-valued functions and the Hölder inequality it can be easily seen that $\varphi_k \to \varphi$ strongly in $\mathcal{V}$.

Assume now that $\varphi \in \mathcal{V}$ has compact support in $(0,T)$. For every $\varepsilon > 0$ let ρ_ε be a C^∞ function on $\mathbb{R}$ with $\rho_\varepsilon \geq 0$, $\int_{\mathbb{R}} \rho_\varepsilon = 1$ and $\operatorname{supp}\rho_\varepsilon \subset (0,\varepsilon)$. For ε small enough the function

$$\varphi^\varepsilon := \varphi * \rho_\varepsilon \colon (0,T) \to V_T \tag{2.13}$$

is of class C^∞ and has compact support in $(0,T)$. By (H2) the asymmetry of the convolution kernel ρ_ε guarantees that $\varphi^\varepsilon(t) \in V_t$ for every $t \in (0,T)$, hence φ^ε belongs to $\mathcal{V}$. Moreover $\varphi^\varepsilon \to \varphi$ strongly in $\mathcal{V}$. $\square$

Remark 2.9. Let $u \in \mathcal{V}$ be a function such that (2.11) holds for every $\varphi \in C_c^\infty((0,T); V_T)$ with $\varphi(t) \in V_t$ for every $t \in (0,T)$. Using Lemma 2.8 it is easy to see that u is a weak solution of (2.10) according to Definition 2.7.

Proposition 2.10. *Let $u \in \mathcal{V}$ be a weak solution of* (2.10) *satisfying the initial conditions $u(0) = 0$ and $\dot{u}(0) = 0$, the latter in the following strong sense:*

$$\lim_{h \to 0+} \frac{1}{h} \int_0^h \|\dot{u}(t)\|^2 dt = 0\,. \tag{2.14}$$

Then

$$-\int_0^T (\dot{u}(t), \dot{\varphi}(t)) dt + \int_0^T a(u(t), \varphi(t)) dt = \int_0^T (f(t), \varphi(t)) dt \tag{2.15}$$

for every $\varphi \in \mathcal{V}$ with $\varphi(T) = 0$, even if the condition $\varphi(0) = 0$ is not satisfied.

Proof. Let us fix $\varphi \in \mathcal{V}$ with $\varphi(T) = 0$. For every $\varepsilon > 0$ set

$$\varphi_\varepsilon(t) = \begin{cases} \frac{t}{\varepsilon}\varphi(t) & t \in [0, \varepsilon] \\ \varphi(t) & t \in [\varepsilon, T]\,. \end{cases}$$

Then $\varphi_\varepsilon \in \mathcal{V}$, $\varphi_\varepsilon(0) = \varphi_\varepsilon(T) = 0$, and by (2.11)

$$-\int_\varepsilon^T (\dot{u}(t), \dot{\varphi}(t)) dt - \int_0^\varepsilon (\dot{u}(t), \dot{\varphi}_\varepsilon(t)) dt + \int_\varepsilon^T a(u(t), \varphi(t)) dt + \int_0^\varepsilon a(u(t), \varphi_\varepsilon(t)) dt$$
$$= \int_\varepsilon^T (f(t), \varphi(t)) dt + \int_0^\varepsilon (f(t), \varphi_\varepsilon(t)) dt\,.$$

For a.e. $t \in (0, \varepsilon)$ we have $\dot{\varphi}_\varepsilon(t) = \frac{1}{\varepsilon}\varphi(t) + \frac{t}{\varepsilon}\dot{\varphi}(t)$. Since $\varphi \in C([0,T]; H)$, using the Hölder Inequality and the absolute continuity of the integral by (2.14) we obtain

$$\Big|\int_0^\varepsilon (\dot{u}(t), \dot{\varphi}_\varepsilon(t)) dt\Big| \le \Big(\frac{1}{\varepsilon}\int_0^\varepsilon \|\dot{u}(t)\|^2 dt\Big)^{1/2} \Big(\frac{1}{\varepsilon}\int_0^\varepsilon \|\varphi(t)\|^2 dt\Big)^{1/2}$$
$$+ \Big(\int_0^\varepsilon \|\dot{u}(t)\|^2 dt\Big)^{1/2} \Big(\int_0^\varepsilon \|\dot{\varphi}(t)\|^2 dt\Big)^{1/2} \to 0\,,$$

$$\Big|\int_0^\varepsilon a(u(t), \varphi_\varepsilon(t)) dt\Big| \le \int_0^\varepsilon \frac{t}{\varepsilon} |a(u(t), \varphi(t))| dt \le M_0 \int_0^\varepsilon \|u(t)\|_t \|\varphi(t)\|_t dt \to 0\,,$$

and

$$\Big|\int_0^\varepsilon (f(t),\varphi_\varepsilon(t))dt\Big| \le \int_0^\varepsilon \frac{t}{\varepsilon}|(f(t),\varphi(t))|dt \le \int_0^\varepsilon \|f(t)\|\|\varphi(t)\|dt \to 0\,.$$

Therefore, passing to the limit as $\varepsilon \to 0$ we conclude that (2.15) holds. □

We now want to introduce a different notion of solution of (2.10) (see Definition 2.15), similar to the one given in [6, Definition 4.1], which does not use integration by parts with respect to time. It requires instead a precise definition of $\ddot{u}(t)$ for a.e. $t \in (0,T)$, and this is not trivial because of the time-dependent constraint $u(t) \in V_t$. We begin by introducing a new function space which will allow us to define the pointwise value of $\ddot{u}(t)$.

Definition 2.11. Given $\eta \in L^2(0,T)$, let $\mathcal{W}_\eta$ be the space of functions $u \in \mathcal{V}$ such that for every $\tau \in [0,T)$, the restriction u_τ of u to (τ,T) satisfies

$$u_\tau \in H^2((\tau,T);V_\tau^*)\,, \tag{2.16}$$

$$\|\ddot{u}_\tau(t)\|_\tau^* \le \eta(t) \quad \text{for a.e. } t \in (\tau,T)\,. \tag{2.17}$$

Note that if $\sigma,\tau \in (0,T)$ with $\sigma < \tau$ then

$$\ddot{u}_\sigma(t) = \Pi_{\sigma\tau}\ddot{u}_\tau(t) \qquad \text{for a.e. } t \in (\tau,T)\,. \tag{2.18}$$

Remark 2.12. Let $u \in \mathcal{W}_\eta$ for some $\eta \in L^2(0,T)$. For every $\tau \in [0,T)$ we consider u_τ and $\ddot{u}_\tau$ as in Definition 2.11, and note that $\dot{u}_\tau = \dot{u}$ a.e. in (τ,T). By standard properties of distributional derivatives of functions with values in Hilbert spaces (see, e.g., [2, Appendix]) there exists a negligible set N_τ in (τ,T) such that

$$\lim_{h\to 0,\ t+h\notin N_\tau} \frac{\dot{u}(t+h)-\dot{u}(t)}{h} = \ddot{u}_\tau(t) \quad \text{strongly in } V_\tau^* \text{ for every } t \in (\tau,T)\setminus N_\tau\,, \tag{2.19}$$

$$\dot{u}(t)-\dot{u}(s) = \int_s^t \ddot{u}_\tau(r)dr \quad \text{for every } s,t \in (\tau,T)\setminus N_\tau \text{ with } s<t\,, \tag{2.20}$$

where in the right-hand side we have a Bochner integral in the space V_τ^*. Hence

$$\|\dot{u}(t_2)-\dot{u}(t_1)\|_\tau^* = \|\dot{u}_\tau(t_2)-\dot{u}_\tau(t_1)\|_\tau^* \le \int_{t_1}^{t_2} \|\ddot{u}_\tau(s)\|_\tau^* ds \le \int_{t_1}^{t_2} \eta(s)ds \tag{2.21}$$

for a.e. $t_1,t_2 \in (\tau,T)$ with $t_1<t_2$. In particular, for $\tau = 0$ we have

$$\|\dot{u}(t_2)-\dot{u}(t_1)\|_0^* \le \int_{t_1}^{t_2} \eta(s)ds \tag{2.22}$$

for a.e. $t_1,t_2 \in (0,T)$ with $t_1<t_2$.

The following proposition provides a pointwise definition of $\ddot{u}(t)$ as an element of V_t^*. Similar results under slightly different hypotheses have been proved in [6, Lemma 2.2], [8, Lemma 2.2], and [14].

Proposition 2.13. *Let $\eta \in L^2(0,T)$ and let $u \in \mathcal{W}_\eta$. Then there exist a set E of full measure in $[0,T]$ and, for every $t \in E$, an element $\ddot{u}(t)$ of V_t^* such that*

$$\lim_{h\to 0+,\ t+h\in E} \frac{\dot{u}(t+h)-\dot{u}(t)}{h} = \ddot{u}(t) \quad \textit{weakly in } V_t^*, \tag{2.23}$$

$$\lim_{h\to 0,\ t+h\in E} \frac{\dot{u}(t+h)-\dot{u}(t)}{h} = \Pi_{\tau t}\ddot{u}(t) \quad \textit{strongly in } V_\tau^* \textit{ for every } \tau \in (0,t), \tag{2.24}$$

$$\|\ddot{u}(t)\|_{V_t^*} \le \eta(t)\,. \tag{2.25}$$

Moreover, for every $\tau \in [0,T]$ we have

$$\ddot{u}_\tau(t) = \Pi_{\tau t}\ddot{u}(t) \quad \textit{for a.e. } t \in (\tau,T)\,. \tag{2.26}$$

In other words, the second order distributional derivative $\ddot{u}_\tau$ in the space V_τ^* coincides a.e. on (τ,T) with the function $t \mapsto \Pi_{\tau t}\ddot{u}(t)$.

In the proof of Proposition 2.13 we shall use the following result on increasing sequences of subspaces of separable Hilbert spaces proved in [6, Lemma 2.3].

Lemma 2.14. *Let $(V_t)_{t\in[0,T]}$ be an increasing family of closed linear subspaces of a separable Hilbert space V. Then, there exists a countable set $S \subset [0,T]$ such that for all $t \in [0,T]\setminus S$, we have*

$$V_t = \overline{\bigcup_{s<t} V_s}. \tag{2.27}$$

Proof of Proposition 2.13. Let $D \subset (0,T)$ be a countable dense set. For every $\tau \in D$ we consider u_τ and $\ddot{u}_\tau$ as in Definition 2.11. By Remark 2.12 there exists a negligible set N_τ in (τ,T) such that (2.19) and (2.20) hold for every $s,t \in (\tau,T)\setminus N_\tau$ with $s<t$. Since D is countable, there exists a negligible set N in $(0,T)$ such that (2.19) holds, with N_τ replaced by N, for every $t \in (0,T)\setminus N$ and every $\tau \in D$, with $0 \le \tau < t$.

By (2.17) and (2.18), there exists a set E of full measure in $(0,T)$ such that

$$E \cap N = \emptyset, \tag{2.28}$$

$$\text{every } t \in E \text{ is a Lebesgue point of } \eta\,, \tag{2.29}$$

$$\text{every } t \in E \text{ satisfies (2.27)}, \tag{2.30}$$

$$\|\ddot{u}_\tau(t)\|_\tau^* \le \eta(t) < +\infty \quad \text{for } \tau \in D \text{ and } t \in E \cap (\tau,T)\,, \tag{2.31}$$

$$\ddot{u}_\sigma(t) = \Pi_{\sigma\tau}\ddot{u}_\tau(t) \quad \text{for } \sigma,\tau \in D \text{ with } \sigma<\tau \text{ and } t \in E\cap(\tau,T)\,. \tag{2.32}$$

Let us fix $t \in E$. By (2.27) and by the density of D we have

$$V_t = \overline{\bigcup_{\tau < t, \tau \in D} V_\tau}. \tag{2.33}$$

Therefore, for every $v \in V_t$ there exists an increasing sequence $\tau_k \to t$, with $\tau_k \in D$, and a sequence v_k converging to v strongly in V_t, with $v_k \in V_{\tau_k}$ for every k. We now define $\ddot{u}(t) \in V_t^*$ as the linear function from V_t into $\mathbb{R}$ given by

$$\langle \ddot{u}(t), v \rangle_t := \lim_{k \to \infty} \langle \ddot{u}_{\tau_k}(t), v_k \rangle_{\tau_k} \quad \text{for every } v \in V_t \, . \tag{2.34}$$

We have to show that the limit exists, that it does not depend on the approximating sequences τ_k, v_k, and that it defines a continuous linear function on V_t. As for the existence of the limit, we show that $\langle \ddot{u}_{\tau_k}(t), v_k \rangle_{\tau_k}$ satisfies the Cauchy condition. Indeed, if $k \geq h$ we have, by (2.31) and (2.32),

$$\begin{aligned} &|\langle \ddot{u}_{\tau_k}(t), v_k \rangle_{\tau_k} - \langle \ddot{u}_{\tau_h}(t), v_h \rangle_{\tau_h}| = |\langle \ddot{u}_{\tau_k}(t), v_k \rangle_{\tau_k} - \langle \Pi_{\tau_h \tau_k} \ddot{u}_{\tau_k}(t), v_h \rangle_{\tau_h}| \\ &= |\langle \ddot{u}_{\tau_k}(t), v_k - v_h \rangle_{\tau_k}| \leq \eta(t) \|v_k - v_h\|_t \, . \end{aligned}$$

A similar argument proves that the limit does not depend on the approximating sequences τ_k, v_k. This implies the linearity of the limit with respect to v. By (2.31) it follows that

$$|\langle \ddot{u}(t), v \rangle_t| \leq \eta(t) \|v\|_t \, , \tag{2.35}$$

which gives $\ddot{u}(t) \in V_t^*$ and proves the inequality (2.25).

If $v \in V_\tau$ for some $\tau \in D$ with $\tau < t$ we can take $v_k = v$ in (2.34) for every k such that $\tau \leq \tau_k$. By (2.32) this implies that

$$\langle \ddot{u}_{\tau_k}(t), v \rangle_{\tau_k} = \langle \Pi_{\tau \tau_k} \ddot{u}_{\tau_k}(t), v \rangle_\tau = \langle \ddot{u}_\tau(t), v \rangle_\tau \, ,$$

hence (2.34) yields

$$\langle \ddot{u}(t), v \rangle_t = \langle \ddot{u}_\tau(t), v \rangle_\tau \quad \text{for } \tau \in D, \ \tau < t, \text{ and } v \in V_\tau \, , \tag{2.36}$$

which gives

$$\Pi_{\tau t} \ddot{u}(t) = \ddot{u}_\tau(t) \quad \text{for } \tau \in D, \ \tau < t \, . \tag{2.37}$$

Together with (2.19) this implies that

$$\lim_{h \to 0, \ t+h \in E} \frac{\dot{u}(t+h) - \dot{u}(t)}{h} = \Pi_{\tau t} \ddot{u}(t) \quad \text{strongly in } V_\tau^* \text{ for every } \tau \in D, \ \tau < t. \tag{2.38}$$

By the density of D, for $\sigma \in (0, t)$ there exists $\tau \in D$ with $\sigma < \tau < t$. By applying $\Pi_{\sigma\tau}$ to both sides of (2.38) we obtain (2.24) (written with τ replaced by σ), thanks to (2.2) and (2.3).

Let us now prove (2.23). By (2.33) for every $\varepsilon > 0$ and for every $v \in V_t$ there exist $\tau \in D$, with $\tau < t$, and $w \in V_\tau$, with $\|v - w\|_t < \varepsilon$. Let us fix $h > 0$, with $t + h \in E$. By (2.1) we have

$$\Big|\Big\langle\frac{\dot u(t+h)-\dot u(t)}{h} - \ddot u(t), v\Big\rangle_t\Big| \le \Big|\Big\langle\frac{\dot u(t+h)-\dot u(t)}{h} - \ddot u(t), w\Big\rangle_t\Big|$$
$$+\Big|\Big(\frac{\dot u(t+h)-\dot u(t)}{h}, v-w\Big)\Big| + \Big|\big\langle\ddot u(t), v-w\big\rangle_t\Big|. \tag{2.39}$$

By (2.1), (2.19), and (2.36) we have

$$\lim_{h\to 0,\ t+h\in E}\Big\langle\frac{\dot u(t+h)-\dot u(t)}{h} - \ddot u(t), w\Big\rangle_t$$
$$= \lim_{h\to 0,\ t+h\in E}\Big\langle\frac{\dot u(t+h)-\dot u(t)}{h} - \ddot u_\tau(t), w\Big\rangle_\tau = 0. \tag{2.40}$$

Since $t \in E$ and $t + h \in E$, for every $\tau \in D$, with $\tau < t$, by (2.20) and (2.28) we have

$$\dot u(t+h) - \dot u(t) = \int_t^{t+h} \ddot u_\tau(s)ds\,.$$

By (2.17) this gives

$$\|\dot u(t+h) - \dot u(t)\|_\tau^* \le \int_t^{t+h} \eta(s)ds\,,$$

hence

$$(\dot u(t+h) - \dot u(t), z) \le \|z\|_\tau \int_t^{t+h} \eta(s)ds \quad \text{for every } z \in V_\tau\,.$$

Using (2.30) we obtain

$$(\dot u(t+h) - \dot u(t), z) \le \|z\|_t \int_t^{t+h} \eta(s)ds \quad \text{for every } z \in V_t\,.$$

Since $\|v - w\|_t < \varepsilon$, we obtain

$$\Big|\Big(\frac{\dot u(t+h)-\dot u(t)}{h}, v-w\Big)\Big| \le \frac{\varepsilon}{h}\int_t^{t+h} \eta(s)ds \tag{2.41}$$

and, by (2.35),

$$|\langle \ddot{u}(t), v - w\rangle_t| \leq \varepsilon\eta(t)\,. \tag{2.42}$$

By (2.29), (2.39), (2.40), (2.41), and (2.42) we have

$$\limsup_{h\to 0,\ t+h\in E} \Big|\Big\langle \frac{\dot{u}(t+h)-\dot{u}(t)}{h} - \ddot{u}(t), v\Big\rangle_t\Big| \leq 2\varepsilon\eta(t)$$

By (2.31), taking the limit as $\varepsilon \to 0+$ we obtain

$$\lim_{h\to 0,\ t+h\in E} \Big|\Big\langle \frac{\dot{u}(t+h)-\dot{u}(t)}{h} - \ddot{u}(t), v\Big\rangle_t\Big| = 0\,,$$

which proves (2.23).

Let $\sigma \in [0, T]$. By (2.37), for every $\tau \in D$ with $\sigma < \tau < T$ we have

$$\Pi_{\tau t}\ddot{u}(t) = \ddot{u}_\tau(t) \quad \text{for a.e. } t \in (\tau, T)\,.$$

Applying $\Pi_{\sigma\tau}$ to both sides of this equality, by (2.3) and (2.18) we obtain

$$\Pi_{\sigma t}\ddot{u}(t) = \ddot{u}_\sigma(t) \quad \text{for a.e. } t \in (\tau, T)\,,$$

which, by the density of D, gives

$$\Pi_{\sigma t}\ddot{u}(t) = \ddot{u}_\sigma(t) \quad \text{for a.e. } t \in (\sigma, T)\,,$$

thus proving (2.26). □

Having defined $\ddot{u}(t)$ as an element of V_t^* for a.e. $t \in (0, T)$, we can interpret (2.10) as an equality in V_t^* to be satisfied for a.e. $t \in (0, T)$. This leads to the following definition which extends to $\mathcal{W}_\eta$ the notion introduced in [6].

Definition 2.15. A function u is a strong-weak solution of the wave equation (2.10) on the time interval $[0, T]$ if $u \in \mathcal{W}_\eta$ for some $\eta \in L^2(0, T)$ and for a.e. $t \in [0, T]$

$$\langle \ddot{u}(t), v\rangle_t + a(u(t), v) = (f(t), v) \quad \text{for every } v \in V_t\,, \tag{2.43}$$

where for a.e. $t \in (0, T)$ the pointwise value of $\ddot{u}(t)$ is defined in Proposition 2.13.

In [6, Definition 4.1] the same notion of solution is considered assuming that the a priori bounds on $\|u(t)\|_t$, $\|\dot{u}(t)\|$, and $\|\ddot{u}(t)\|_t^*$ are uniform with respect to t. Weaker a priori bounds were considered in [14].

In the rest of this section we shall prove that the notions of weak solution and strong-weak solution coincide.

Theorem 2.16. *Every strong-weak solution according to Definition 2.15 is a weak solution according to Definition 2.7.*

Proof. Let u be a strong-weak solution of the wave equation (2.10). Since $u \in \mathcal{V}$, we only have to check that (2.11) is satisfied. Let us fix $\varphi \in \mathcal{V}$ with $\varphi(0) = \varphi(T) = 0$. We extend φ by setting $\varphi(t) = 0$ for $t < 0$. Let $\varepsilon > 0$ and let $\varphi_\varepsilon : [0, T] \to V_T$ be defined by $\varphi_\varepsilon(t) = \varphi(t - \varepsilon)$. Then $\varphi_\varepsilon \in \mathcal{V}$ by (H2),

$$\varphi_\varepsilon(t) \in V_{t-\varepsilon} \qquad \text{for a.e. } t \in [\varepsilon, T]\,, \tag{2.44}$$

and $\varphi_\varepsilon(t) = 0$ for $t \in [0, \varepsilon]$.

Let us prove that

$$t \mapsto (\dot{u}(t), \varphi_\varepsilon(t)) \qquad \text{is absolutely continuous on } [0, T], \tag{2.45}$$

$$\frac{d}{dt}(\dot{u}(t), \varphi_\varepsilon(t)) = (\dot{u}(t), \dot{\varphi}_\varepsilon(t)) + \langle \ddot{u}(t), \varphi_\varepsilon(t)\rangle_t \qquad \text{for a.e. } t \in [0, T]\,, \tag{2.46}$$

where the pointwise value of $\ddot{u}(t)$ is defined in Proposition 2.13.

First of all note that it is enough to prove that for every $s \in [0, T - \varepsilon]$ properties (2.45) and (2.46) hold with $[0, T]$ replaced by $[s, s + \varepsilon]$. By (2.44) we have $\varphi_\varepsilon(t) \in V_{t-\varepsilon} \subset V_s$ for a.e. $t \in [s, s + \varepsilon]$ and, by the definition of Π_{st}, we have also $\langle \ddot{u}(t), \varphi_\varepsilon(t)\rangle_t = \langle \Pi_{st}\ddot{u}(t), \varphi_\varepsilon(t)\rangle_s$ for a.e. $t \in [s, s + \varepsilon]$. Therefore the restriction $\varphi_\varepsilon|_{(s,s+\varepsilon)}$ belongs to $L^2((s, s + \varepsilon); V_s)$ and its distributional derivative belongs to $L^2((s, s + \varepsilon); H)$.

Let $v := u|_{(s,s+\varepsilon)}$. Then its distributional derivative $\dot{v}$ belongs to $L^2((s, s + \varepsilon); H)$, by (2.16) in Definition 2.11 its second order distributional derivative $\ddot{v}$ belongs to $L^2((s, s + \varepsilon); V_s^*)$, and by (2.26) in Proposition 2.13 it satisfies $\ddot{v}(t) = \Pi_{st}\ddot{u}(t)$ for a.e. $t \in (s, s + \varepsilon)$, hence $\langle \ddot{v}(t), \varphi_\varepsilon(t)\rangle_s = \langle \ddot{u}(t), \varphi_\varepsilon(t)\rangle_t$ for a.e. $t \in (s, s + \varepsilon)$. By Lemma A.1, with $\psi = \dot{v}$ and $\varphi = \varphi_\varepsilon$, we have that

$$t \mapsto (\dot{v}(t), \varphi_\varepsilon(t)) \qquad \text{is absolutely continuous on } [s, s + \varepsilon],$$

$$\frac{d}{dt}(\dot{v}(t), \varphi_\varepsilon(t)) = (\dot{v}(t), \dot{\varphi}_\varepsilon(t)) + \langle \ddot{v}(t), \varphi_\varepsilon(t)\rangle_s \qquad \text{for a.e. } t \in [s, s + \varepsilon]\,.$$

Since $s \in [0, T - \varepsilon]$ is arbitrary, we obtain (2.45) and (2.46).

By the continuity of translations in L^2 we have $\varphi_\varepsilon \to \varphi$ in $L^2((0, T); V_T)$ and $\dot{\varphi}_\varepsilon \to \dot{\varphi}$ in $L^2((0, T); H)$. Therefore, since $\dot{u} \in L^2((0, T); H)$, we obtain

$$(\dot{u}(\cdot), \varphi_\varepsilon(\cdot)) \to (\dot{u}(\cdot), \varphi(\cdot)) \quad \text{in } L^1((0, T))\,, \tag{2.47}$$

$$(\dot{u}(\cdot), \dot{\varphi}_\varepsilon(\cdot)) \to (\dot{u}(\cdot), \dot{\varphi}(\cdot)) \quad \text{in } L^1((0, T))\,. \tag{2.48}$$

Let us prove that

$$t \mapsto \langle \ddot{u}(t), \varphi_\varepsilon(t)\rangle_t \qquad \text{converges to} \qquad t \mapsto \langle \ddot{u}(t), \varphi(t)\rangle_t \qquad \text{in } L^1((0, T)). \tag{2.49}$$

Since $\varphi_\varepsilon \to \varphi$ in $L^2((0, T); V_T)$, for every sequence converging to zero there exists a subsequence $\varepsilon_j \to 0$ such that

$$\varphi_{\varepsilon_j}(t) \to \varphi(t) \qquad \text{strongly in } V_T \qquad \text{for a.e. } t \in (0, T)\,.$$

Since $\varphi_{\varepsilon_j}(t), \varphi(t) \in V_t$ for a.e. $t \in (0,T)$ and V_t is a subspace of V_T, we have that

$$\varphi_{\varepsilon_j}(t) \to \varphi(t) \quad \text{strongly in } V_t \quad \text{for a.e. } t \in (0,T)\,.$$

This implies that

$$\langle \ddot{u}(t), \varphi_{\varepsilon_j}(t)\rangle_t \to \langle \ddot{u}(t), \varphi(t)\rangle_t \quad \text{for a.e. } t \in (0,T)\,.$$

On the other hand, since $u \in \mathcal{W}_\eta$, by (2.25) in Proposition 2.13 we have

$$|\langle \ddot{u}(t), \varphi_{\varepsilon_j}(t)\rangle_t| \le \eta(t)\|\varphi_{\varepsilon_j}(t)\|_T \quad \text{for a.e. } t \in (0,T)\,.$$

Since $\varphi_\varepsilon \to \varphi$ in $L^2((0,T);V_T)$, our claim (2.49) follows from the Generalized Dominated Convergence Theorem and from the arbitrariness of the sequence converging to zero.

By (2.45)–(2.49) we obtain that the function $t \mapsto (\dot{u}(t), \varphi(t))$ belongs to $W^{1,1}(0,T)$ and satisfies

$$\frac{d}{dt}(\dot{u}(t), \varphi(t)) = (\dot{u}(t), \dot{\varphi}(t)) + \langle \ddot{u}(t), \varphi(t)\rangle_t \quad \text{for a.e. } t \in [0,T]\,.$$

Since $t \mapsto \frac{d}{dt}(\dot{u}(t), \varphi(t))$ and $t \mapsto (\dot{u}(t), \dot{\varphi}(t))$ belong to $L^1((0,T))$ we deduce also that $t \mapsto \langle \ddot{u}(t), \varphi(t)\rangle_t$ belongs to $L^1((0,T))$. As $\varphi(0) = \varphi(T) = 0$ we obtain

$$\int_0^T \langle \ddot{u}(t), \varphi(t)\rangle_t dt = -\int_0^T (\dot{u}(t), \dot{\varphi}(t))dt\,. \tag{2.50}$$

Since by (2.43) we have $\langle \ddot{u}(t), \varphi(t)\rangle_t + a(u(t), \varphi(t)) = (f(t), \varphi(t))$ for a.e. $t \in [0,T]$, integrating from 0 to T and using (2.50) we obtain (2.11). □

We now complete the proof of the equivalence of the two definitions.

Theorem 2.17. *Every weak solution according to Definition 2.7 is a strong-weak solution according to Definition 2.15.*

Proof. Let u be a weak solution of the wave equation (2.10). We have to show that $u \in \mathcal{W}_\eta$ for some $\eta \in L^2((0,T))$ and that (2.43) holds. To this end, let us fix $\tau \in [0,T)$, $v \in V_\tau$, and $\psi \in C^1_c((\tau,T))$. Then the function $t \mapsto \varphi(t) := \psi(t)v$ belongs to $\mathcal{V}$ and $\varphi(0) = \varphi(T) = 0$. Using this function in (2.11) we obtain

$$-\int_\tau^T (\dot{u}(t), v)\dot{\psi}(t)dt + \int_\tau^T a(u(t), v)\psi(t)dt = \int_\tau^T (f(t), v)\psi(t)dt\,. \tag{2.51}$$

For every $t \in [\tau,T)$ let $A^t_\tau : V_t \to V^*_\tau$ be the continuous linear operator defined by (2.6). Since $u \in L^2((0,T);V_T)$, it follows that $t \mapsto A^t_\tau u(t)$ from $(0,T)$ into V^*_τ is weakly measurable. Since

V_τ^* is separable, by (2.7) we have that $t \mapsto A^t_\tau u(t)$ belongs to $L^2((0,T);V_\tau^*)$. Hence, by (2.51) we have

$$\Big(\int_\tau^T \dot u(t)\dot\psi(t)dt, v\Big) = \Big\langle \int_\tau^T A^t_\tau u(t)\psi(t)dt, v\Big\rangle_\tau - \Big(\int_\tau^T f(t)\psi(t)dt, v\Big),$$

where the first and the third integrals are Bochner integrals in H, while the second one is a Bochner integral in V_τ^*. Since this equality holds for every $v \in V_\tau$ we deduce that

$$\int_\tau^T \dot u(t)\dot\psi(t)dt = \int_\tau^T A^t_\tau u(t)\psi(t)dt - \int_\tau^T f(t)\psi(t)dt\,.$$

Let u_τ be the restriction of u to (τ,T) as in Definition 2.11. The previous equality shows that $u_\tau \in H^2((\tau,T);V_\tau^*)$ and $\ddot u_\tau(t) = -A^t_\tau u(t) + f(t)$ for a.e. $t \in (\tau,T)$, which gives

$$\langle \ddot u_\tau(t), v\rangle_\tau + a(u(t),v) = (f(t),v) \quad \text{for every } v \in V_\tau\,. \tag{2.52}$$

Moreover, (2.7) gives

$$\|\ddot u_\tau(t)\|_\tau^* \le M_0\|u(t)\|_T + C\|f(t)\|\,, \quad \text{for a.e. } t \in (\tau,T)\,, \tag{2.53}$$

where C is the norm of the continuous immersion of H into V_T^*. This shows that $u \in \mathcal{W}_\eta$ with $\eta(t) := M_0\|u(t)\|_T + C\|f(t)\|$.

Let us fix a countable dense set D in $(0,T)$. By (2.26) and (2.52) for every $\tau \in D$ and for a.e. $t \in (\tau,T)$ we obtain

$$\langle \Pi_{\tau t}\ddot u(t), v\rangle_\tau + a(u(t),v) = (f(t),v) \quad \text{for every } v \in V_\tau\,.$$

By the definition of $\Pi_{\tau t}$ this implies that for a.e. $t \in (\tau,T)$ we have

$$\langle \ddot u(t), v\rangle_t + a(u(t),v) = (f(t),v) \quad \text{for every } v \in V_\tau\,. \tag{2.54}$$

By the countability of D, there exists a set E of full measure in $(0,T)$ such that (2.54) holds for every $t \in E$ and for every $\tau \in D$ with $0 < \tau < t$.

By the density of D and Lemma 2.14 we may assume that for every $t \in E$ we have

$$V_t = \overline{\bigcup_{\tau<t,\tau\in D} V_\tau}\,. \tag{2.55}$$

Let us fix $t \in E$ and $v \in V_t$. By (2.55) there exists an increasing sequence τ_k in D converging to t and a sequence v_k converging to v strongly in V_t such that $v_k \in V_{\tau_k}$ for every k. By (2.54) we have

$$\langle \ddot u(t), v_k\rangle_t + a(u(t),v_k) = (f(t),v_k) \quad \text{for every } k\,. \tag{2.56}$$

Passing to the limit in k we obtain (2.43). □

We conclude this section with a result that will be used to prove the existence of a weak solution to (2.10) satisfying some continuity conditions. For every Banach space X let $C_w([0,T];X)$ be the space of all functions $u\colon [0,T]\to X$ that are continuous for the weak topology of X. By the Banach–Steinhaus Theorem we have $C_w([0,T];X)\subset L^\infty([0,T];X)$.

Proposition 2.18. *Let $u\in\mathcal{W}_\eta$ for some $\eta\in L^2((0,T))$. Assume that $u\in L^\infty((0,T);V_T)$ and $\dot u\in L^\infty((0,T);H)$. Then, after a modification on a set of measure zero, we have $u\in C_w([0,T];V_T)\cap C([0,T];H)$ and $\dot u\in C_w([0,T];H)\cap C([0,T];V_0^*)$.*

Proof. We prove only that $\dot u\in C_w([0,T];H)\cap C([0,T];V_0^*)$. By Remark 2.12 and by the assumption $\dot u\in L^\infty((0,T);H)$ there exist a set $N\subset[0,T]$ of measure zero and a constant $C>0$ such that for every $s,t\in[0,T]\setminus N$ with $s<t$ we have

$$\|\dot u(t)-\dot u(s)\|_0^*\le\int_s^t\eta(r)dr\quad\text{and}\quad\|\dot u(t)\|\le C\,. \tag{2.57}$$

Clearly the restriction of $\dot u$ to $[0,T]\setminus N$ is strongly continuous in V_0^*. Let us prove that it is also weakly continuous in H. Let t_n be a sequence in $[0,T]\setminus N$ converging to $t\in[0,T]\setminus N$. By (2.57) the sequence $\dot u(t_n)$ is bounded in H, so a subsequence converges weakly in H. Since, by (2.57), $\dot u(t_n)$ converges to $\dot u(t)$ strongly in V_0^*, we deduce that $\dot u(t_n)$ converges to $\dot u(t)$ weakly in H.

We now redefine $\dot u$ on N in such a way that $\dot u$ is weakly continuous in H and strongly continuous in V_0^*. Let us fix $s\in N$ and a sequence $s_n\in[0,T]\setminus N$ converging to s. By the first inequality in (2.57) $\dot u(s_n)$ is a Cauchy sequence in V_0^*, hence it converges to some v^* strongly in V_0^*. By the second inequality in (2.57) the sequence $\dot u(s_n)$ is bounded in H, so a subsequence converges weakly in H. Therefore $v^*\in H$ and the whole sequence $\dot u(s_n)$ converges to v^* weakly in H. We define $\dot u(s)=v^*$. A similar argument shows that $\dot u(s)$ does not depend on the sequence s_n and that the function $\dot u$ belongs to $C_w([0,T];H)\cap C([0,T];V_0^*)$. □

3. Existence

In this section we prove the existence of a weak solution to (2.10) according to Definition 2.7. The solution we construct also satisfies additional regularity properties and the energy inequality.

Theorem 3.1. *Given $u^{(0)}\in V_0$ and $u^{(1)}\in H$, there exists a weak solution u to (2.10) on $[0,T]$ satisfying the initial conditions $u(0)=u^{(0)}$ and $\dot u(0)=u^{(1)}$, in the sense that*

$$\lim_{h\to0+}\frac1h\int_0^h(\|u(t)-u^{(0)}\|_t^2+\|\dot u(t)-u^{(1)}\|^2)dt=0\,, \tag{3.1}$$

and such that

$$u\in C_w([0,T];V_T)\cap C([0,T];H)\quad\textit{and}\quad\dot u\in C_w([0,T];H)\cap C([0,T];V_0^*)\,. \tag{3.2}$$

Proof. The proof is based on a time-discretization procedure and follows closely the proof of [6, Lemma 3.3], with some simplifications due to the fact that we do not need any estimate on $\ddot{u}$.

Step 1. Construction of the discrete-time approximants. Given $n \in \mathbb{N}$, we set $\tau_n := T/n$ and $t_n^i := i\tau_n$, with $i = -1, ..., n$. For $i = 1, \dots, n$ we set

$$f_n^i := \frac{1}{\tau_n} \int_{t_n^{i-1}}^{t_n^i} f(t) dt \,. \tag{3.3}$$

We define u_n^i for $i = -1, ..., n$ inductively. First,

$$u_n^{-1} := u^{(0)} - \tau_n u^{(1)} \quad \text{and} \quad u_n^0 := u^{(0)}\,; \tag{3.4}$$

then, for $i = 1, \dots, n$, u_n^i is a minimizer in $V_{t_n^i}$ of the functional

$$u \mapsto \frac{1}{2} \left\| \frac{u - u_n^{i-1}}{\tau_n} - \frac{u_n^{i-1} - u_n^{i-2}}{\tau_n} \right\|^2 + \frac{1}{2} a(u, u) - (f_n^i, u)\,. \tag{3.5}$$

Using the coerciveness of a (see (2.5)), it is easy to see that, if $\tau_n < \lambda_0^{-1/2}$, then the functional in (3.5) is convex and bounded from below by $\frac{\nu_0}{2} \|u\|_T^2 - C_n^i$, for a suitable constant $C_n^i \geq 0$. The existence of a minimizer then follows from the lower semicontinuity of the functional with respect to the strong (and hence to the weak) convergence in $V_{t_n^i}$.

To simplify the exposition, for $i = 0, \dots, n$ we define

$$v_n^i := \frac{u_n^i - u_n^{i-1}}{\tau_n}\,. \tag{3.6}$$

Step 2. Discrete energy estimates. The Euler equation for (3.5) gives

$$\left(\frac{v_n^i - v_n^{i-1}}{\tau_n}, \zeta \right) + a(u_n^i, \zeta) = (f_n^i, \zeta) \quad \text{for every } \zeta \in V_{t_n^i}\,. \tag{3.7}$$

Taking $\zeta = u_n^i - u_n^{i-1}$ in (3.7) we obtain

$$\|v_n^i\|^2 - (v_n^i, v_n^{i-1}) + a(u_n^i, u_n^i) - a(u_n^i, u_n^{i-1}) = (f_n^i, u_n^i - u_n^{i-1})\,.$$

Since $a(u, u) - a(u, v) = \frac{1}{2} a(u, u) + \frac{1}{2} a(u - v, u - v) - \frac{1}{2} a(v, v)$, and a similar equality holds for $(\cdot, \cdot)$, we deduce that

$$\begin{aligned} \|v_n^i\|^2 + \|v_n^i - v_n^{i-1}\|^2 + a(u_n^i, u_n^i) + a(u_n^i - u_n^{i-1}, u_n^i - u_n^{i-1}) \\ = \|v_n^{i-1}\|^2 + a(u_n^{i-1}, u_n^{i-1}) + 2(f_n^i, u_n^i - u_n^{i-1})\,. \end{aligned}$$

Summing from $i = 1$ to some j and using (3.4), we get

$$\|v_n^j\|^2 + \sum_{i=1}^{j} \|v_n^i - v_n^{i-1}\|^2 + a(u_n^j, u_n^j) + \sum_{i=1}^{j} a(u_n^i - u_n^{i-1}, u_n^i - u_n^{i-1})$$

$$= \|u^{(1)}\|^2 + a(u^{(0)}, u^{(0)}) + 2\sum_{i=1}^{j} (f_n^i, u_n^i - u_n^{i-1}) . \tag{3.8}$$

Hence (2.5) implies that

$$\|v_n^j\|^2 + a(u_n^j, u_n^j) - \lambda_0 \tau_n^2 \sum_{i=1}^{j} \|v_n^i\|^2 \le \|u^{(1)}\|^2 + a(u^{(0)}, u^{(0)}) + 2\tau_n \sum_{i=1}^{j} (f_n^i, v_n^i) . \tag{3.9}$$

Step 3. Interpolating functions. We now define u_n as the piecewise affine function which satisfies $u_n(t_n^i) = u_n^i$ for $i = -1, \dots, n$ and is affine on each interval $[t_n^{i-1}, t_n^i]$ for $i = 0, \dots, n$. Therefore

$$u_n(t) = u_n^{i-1} + (t - t_n^{i-1}) v_n^i \quad \text{for } t \in [t_n^{i-1}, t_n^i], \tag{3.10}$$

$$\dot{u}_n(t) = v_n^i \quad \text{for } t \in (t_n^{i-1}, t_n^i). \tag{3.11}$$

Note that for every $t \in [t_n^{i-1}, t_n^i]$, with $i = 1, \dots, n$, we have $u_n(t - \tau_n) \in V_{t_n^{i-1}} \subset V_t$. This implies that

$$u_n(\cdot - \tau_n) \in \mathcal{V} . \tag{3.12}$$

Moreover we consider the piecewise constant function $\tilde{u}_n$ defined for $t \in (t_n^{i-1}, t_n^i]$ and $i = 0, \dots, n$ by

$$\tilde{u}_n(t) := u_n^i = u_n(t_n^i). \tag{3.13}$$

Rewriting (3.9) using these definitions and the Cauchy Inequality, for every $t \in (t_n^{j-1}, t_n^j)$ we get

$$\|\dot{u}_n(t)\|^2 + a(\tilde{u}_n(t), \tilde{u}_n(t)) - \lambda_0 \tau_n \int_0^{t_n^j} \|\dot{u}_n(s)\|^2 ds$$

$$\le \|u^{(1)}\|^2 + a(u^{(0)}, u^{(0)}) + \int_0^{t_n^j} \|f(s)\|^2 ds + \int_0^{t_n^j} \|\dot{u}_n(s)\|^2 ds . \tag{3.14}$$

Since for $t \in (t_n^{j-1}, t_n^j)$ we have $\tilde{u}_n(t) = u_n^j = u^{(0)} + \int_0^{t_n^j} \dot{u}_n(s) ds$ we obtain that for every $\varepsilon > 0$

$$\|\tilde{u}_n(t)\|^2 \le (1 + \varepsilon) \|u^{(0)}\|^2 + \frac{1 + \varepsilon}{\varepsilon} t_n^j \int_0^{t_n^j} \|\dot{u}_n(s)\|^2 ds ,$$

which together with (3.14) gives

$$\|\dot{u}_n(t)\|^2 + a(\tilde{u}_n(t), \tilde{u}_n(t)) + \lambda_0\|\tilde{u}_n(t)\|^2$$

$$\le \|u^{(1)}\|^2 + a(u^{(0)}, u^{(0)}) + \lambda_0(1+\varepsilon)\|u^{(0)}\|^2 + \int_0^{t_n^j} \|f(s)\|^2 ds + C_\varepsilon \int_0^{t_n^j} \|\dot{u}_n(s)\|^2 ds\,, \tag{3.15}$$

where $C_\varepsilon = \lambda_0 T \frac{1+2\varepsilon}{\varepsilon} + 1$, and by (2.5) we have

$$\|\dot{u}_n(t)\|^2 + \nu_0\|\tilde{u}_n(t)\|_T^2 \le B_\varepsilon + C_\varepsilon \int_0^{t_n^j} \|\dot{u}_n(s)\|^2 ds\,, \tag{3.16}$$

where $B_\varepsilon = \|u^{(1)}\|^2 + a(u^{(0)}, u^{(0)}) + \lambda_0(1+\varepsilon)\|u^{(0)}\|^2 + \int_0^T \|f(s)\|^2 ds$. Since $t \mapsto \dot{u}_n(t)$ is constant on (t_n^{j-1}, t_n^j), we obtain

$$\|\dot{u}_n(t)\|^2 + \nu_0\|\tilde{u}_n(t)\|_T^2 \le B_\varepsilon + C_\varepsilon \int_0^{t} \|\dot{u}_n(s)\|^2 ds + C_\varepsilon \tau_n \|\dot{u}_n(t)\|^2\,. \tag{3.17}$$

If $C_\varepsilon \tau_n < 1/2$ we obtain

$$\frac{1}{2}\|\dot{u}_n(t)\|^2 + \nu_0\|\tilde{u}_n(t)\|_T^2 \le B_\varepsilon + C_\varepsilon \int_0^{t} \|\dot{u}_n(s)\|^2 ds\,. \tag{3.18}$$

By the Gronwall Inequality it follows that

$$\dot{u}_n(t) \text{ is bounded in } H \text{ uniformly in } t \text{ and } n\,, \tag{3.19}$$

which, together with the fact that $u_n(0) = u^{(0)}$, implies that

$$u_n(t) \text{ and } \tilde{u}_n(t) \text{ are bounded in } H \text{ uniformly in } t \text{ and } n\,. \tag{3.20}$$

By (3.18) we also have that

$$u_n(t) \text{ and } \tilde{u}_n(t) \text{ are bounded in } V_T \text{ uniformly in } t \text{ and } n\,. \tag{3.21}$$

Step 4. Convergence of the interpolating functions. From (3.12) and from the uniform bounds (3.19)–(3.21) it follows that the sequence $u_n(\cdot - \tau_n)$ is bounded in $\mathcal{V}$, hence, there exist a subsequence, not relabelled, and a function

$$u \in \mathcal{V} \tag{3.22}$$

such that

$$u_n(\cdot - \tau_n) \rightharpoonup u \quad \text{weakly in } \mathcal{V}\,. \tag{3.23}$$

Let us prove that

$$\tilde{u}_n \rightharpoonup u \quad \text{weakly in } L^2((0,T);H)\,. \tag{3.24}$$

We begin by observing that for every $t \in [t_n^{i-1}, t_n^i]$ we have

$$\tilde{u}_n(t) - u_n(t-\tau_n) = u_n(t_n^i) - u_n(t-\tau_n) = \int_{t-\tau_n}^{t_n^i} \dot{u}_n(s)ds\,, \tag{3.25}$$

hence by the Hölder Inequality we have

$$\|\tilde{u}_n(t) - u_n(t-\tau_n)\| \le (2\tau_n)^{1/2} \|\dot{u}_n\|_{L^2((0,T);H)}\,. \tag{3.26}$$

Therefore by (3.19) we obtain that

$$\tilde{u}_n - u_n(\cdot - \tau_n) \to 0 \quad \text{strongly in } L^\infty((0,T);H)\,, \tag{3.27}$$

which together with (3.23) implies (3.24). Similarly we can prove that

$$u_n \rightharpoonup u \quad \text{weakly in } L^2((0,T);H)\,. \tag{3.28}$$

By (3.21) a subsequence of $\tilde{u}_n$ converges to some $\tilde{u}$ weakly in $L^2((0,T);V_T)$. Since the embedding of V_T in H is continuous, from (3.24) it follows that $\tilde{u} = u$, hence

$$\tilde{u}_n \rightharpoonup u \text{ weakly in } L^2((0,T);V_T). \tag{3.29}$$

By (3.19) it follows that a subsequence of $\dot{u}_n$ converges to some u^* weakly in $L^2((0,T);H)$. Using (3.28) it is easy to see that $u^* = \dot{u}$, hence

$$\dot{u}_n \rightharpoonup \dot{u} \quad \text{weakly in } L^2((0,T);H)\,. \tag{3.30}$$

Moreover, from (3.19), (3.21), (3.29), and (3.30) it follows that

$$u \in L^\infty((0,T);V_T) \quad \text{and} \quad \dot{u} \in L^\infty((0,T);H)\,. \tag{3.31}$$

Step 5. The limit function u satisfies the equation. To prove that u satisfies (2.11) it is enough to consider $\varphi \in C_c^\infty((0,T);V_T)$ with $\varphi(t) \in V_t$ for every $t \in (0,T)$, see Remark 2.9. For $i = 1,\dots,n$ we take $\varphi(t_n^i)$ as test-function in (3.7) and sum the corresponding equalities obtaining that

$$\sum_{i=1}^n \Big(\frac{v_n^i - v_n^{i-1}}{\tau_n}, \varphi(t_n^i)\Big) + \sum_{i=1}^n a(u_n^i, \varphi(t_n^i)) = \sum_{i=1}^n (f_n^i, \varphi(t_n^i))\,. \tag{3.32}$$

Since φ has compact support we can use the discrete version of the integration by parts in the first sum to obtain

$$\sum_{i=1}^{n-1}\big(v_n^i, \frac{\varphi(t_n^i)-\varphi(t_n^{i+1})}{\tau_n}\big)+\sum_{i=1}^{n} a(u_n^i,\varphi(t_n^i)) = \sum_{i=1}^{n}(f_n^i,\varphi(t_n^i)) \tag{3.33}$$

for n large enough.

Let now φ_n and $\tilde{\varphi}_n$ be the functions defined for $t\in(t_n^{i-1},t_n^i]$ and $i=1,\dots,n$ by

$$\varphi_n(t) := \varphi(t_n^{i-1})+(t-t_n^{i-1})\frac{\varphi(t_n^i)-\varphi(t_n^{i-1})}{\tau_n} \quad \text{and} \quad \tilde{\varphi}_n(t):=\varphi(t_n^i)\,.$$

Then $a(u_n^i,\varphi(t_n^i)) = a(\tilde{u}_n(t),\tilde{\varphi}_n(t))$ for every $t\in(t_n^{i-1},t_n^i]$. Hence

$$\sum_{i=1}^{n} a(u_n^i,\varphi(t_n^i)) = \frac{1}{\tau_n}\int_0^T a(\tilde{u}_n(t),\tilde{\varphi}_n(t))dt$$

and

$$\sum_{i=1}^{n}(f_n^i,\varphi(t_n^i)) = \frac{1}{\tau_n}\int_0^T (f(t),\tilde{\varphi}_n(t))dt\,.$$

As $\dot{u}_n(t)=v_n^i$ and $\dot{\varphi}_n(t)=\frac{1}{\tau_n}(\varphi(t_n^i)-\varphi(t_n^{i-1}))$ for every $t\in(t_n^{i-1},t_n^i)$, we have

$$\big(v_n^i, \frac{\varphi(t_n^i)-\varphi(t_n^{i+1})}{\tau_n}\big) = -\big(\dot{u}_n(t),\dot{\varphi}_n(t+\tau_n)\big)$$

for every $t\in(t_n^{i-1},t_n^i)$, so that

$$\sum_{i=1}^{n-1}\big(v_n^i, \frac{\varphi(t_n^i)-\varphi(t_n^{i+1})}{\tau_n}\big) = -\frac{1}{\tau_n}\int_0^{T-\tau_n}\big(\dot{u}_n(t),\dot{\varphi}_n(t+\tau_n)\big)dt\,.$$

Therefore, by (3.33) we obtain that

$$-\int_0^{T-\tau_n}\big(\dot{u}_n(t),\dot{\varphi}_n(t+\tau_n)\big)dt+\int_0^T a(\tilde{u}_n(t),\tilde{\varphi}_n(t))dt = \int_0^T (f(t),\tilde{\varphi}_n(t))dt\,. \tag{3.34}$$

Since $\varphi_n\to\varphi$ strongly in $H^1((0,T);V_T)$ and $\tilde{\varphi}_n\to\varphi$ strongly in $L^2((0,T);V_T)$ we obtain (2.11).

By Theorem 2.17 we have $u\in\mathcal{W}_\eta$ for some $\eta\in L^2((0,T))$. Hence Proposition 2.18 and (3.31) imply (3.2).

Step 6. Initial conditions. It remains to prove (3.1). To this aim it is enough to show that there exists a set N of measure zero in $[0, T]$ such that

$$\dot{u}(t_k) \to u^{(1)} \qquad \text{strongly in } H\,, \tag{3.35}$$

$$u(t_k) \to u^{(0)} \qquad \text{strongly in } V_T \tag{3.36}$$

for every sequence $t_k \in (0, T) \setminus N$ converging to 0.

To prove these properties we first claim that there exist a set N_1 of measure zero in $[0, T]$ and a positive constant M_1 such that

$$\|\dot{u}(t) - u^{(1)}\|_0^* \leq M_1 t^{1/2} \tag{3.37}$$

for every $t \in [0, T] \setminus N_1$ (we recall that $\|\cdot\|_0^*$ is the norm in V_0^* dual to the norm of V_0). To prove this estimate we use (3.7) and the fact that, by (3.21), $\|u_n^i\|_T$ is bounded uniformly with respect to n and i. This implies that there exists a positive constant C such that for every n and i

$$(v_n^i - v_n^{i-1}, \zeta) \leq C\tau_n \|\zeta\|_0 + C\tau_n \|f_n^i\| \, \|\zeta\|_0 \quad \text{for every } \zeta \in V_0\,. \tag{3.38}$$

Hence for every i we have

$$\|v_n^i - v_n^{i-1}\|_0^* \leq C\tau_n + C \int_{t_n^{i-1}}^{t_n^i} \|f(s)\| ds\,. \tag{3.39}$$

Iterating we obtain

$$\|v_n^i - v_n^0\|_0^* \leq Ct_n^i + C \int_0^{t_n^i} \|f(s)\| ds\,. \tag{3.40}$$

Taking into account (3.11) and the fact that $v_n^0 = u^{(1)}$, for a.e. $t \in (0, T)$ we get

$$\|\dot{u}_n(t) - u^{(1)}\|_0^* \leq C(t + \tau_n) + C \int_0^{t+\tau_n} \|f(s)\| ds\,, \tag{3.41}$$

where we set $f(s) = 0$ for $s > T$. Integrating with respect to t on $(\alpha, \beta) \subset [0, T]$ we obtain

$$\int_\alpha^\beta \|\dot{u}_n(t) - u^{(1)}\|_0^* dt \leq \int_\alpha^\beta \Big(C(t + \tau_n) + C \int_0^{t+\tau_n} \|f(s)\| ds \Big) dt\,. \tag{3.42}$$

Since $\dot{u}_n \rightharpoonup \dot{u}$ weakly in $L^2((0, T); H)$ we have also $\dot{u}_n \rightharpoonup \dot{u}$ weakly in $L^2((0, T); V_0^*)$. Therefore, by lower semicontinuity, from (3.42) we obtain

$$\int_\alpha^\beta \|\dot u(t) - u^{(1)}\|_0^* dt \le \int_\alpha^\beta \Big(Ct + C\int_0^t \|f(s)\|ds\Big)dt\,. \tag{3.43}$$

By the arbitrariness of α and β it follows that there exists a set N_1 of measure zero in $[0,T]$ such that for every $t \in [0,T] \setminus N_1$

$$\|\dot u(t) - u^{(1)}\|_0^* \le Ct + C\int_0^t \|f(s)\|ds \le Ct + Ct^{1/2}\|f\|_{L^2((0,T);H)}, \tag{3.44}$$

which gives (3.37).

By (3.19) we get that there exists a constant M_2 such that $\|\dot u_n(t)\| \le M_2$ for a.e. $t \in (0,T)$ and every n, hence

$$\|u_n(t) - u^{(0)}\| \le M_2 t \quad \text{for every } t \in [0,T]\,.$$

Arguing as in the proof of (3.44), from (3.28) we obtain that there exists a set N_2 of measure zero in $[0,T]$ such that

$$\|u(t) - u^{(0)}\| \le M_2 t \quad \text{for every } t \in [0,T] \setminus N_2\,. \tag{3.45}$$

Starting from (3.15), we now prove that there exists a set N_3 of measure zero in $[0,T]$ such that

$$\begin{aligned}&\|\dot u(t)\|^2 + a(u(t),u(t)) + \lambda_0\|u(t)\|^2\\ &\le \|u^{(1)}\|^2 + a(u^{(0)},u^{(0)}) + \lambda_0(1+\varepsilon)\|u^{(0)}\|^2 + \int_0^t \|f(s)\|^2 ds + C_\varepsilon M_2^2 t\,,\end{aligned} \tag{3.46}$$

for every $t \in [0,T] \setminus N_3$. We first observe that for every $(\alpha,\beta) \subset (0,T)$ the functional

$$\zeta \mapsto \int_\alpha^\beta \big(a(\zeta(t),\zeta(t)) + \lambda_0\|\zeta(t)\|^2\big)dt$$

is continuous on $L^2((0,T);V_T)$ thanks to (2.4). Since it is convex by (2.5), it is also lower semicontinuous in the weak topology of $L^2((0,T);V_T)$. Since $\dot u_n \rightharpoonup \dot u$ weakly in $L^2((0,T);H)$ and $\tilde u_n \rightharpoonup u$ weakly in $L^2((0,T);V_T)$, we can apply to (3.15) the arguments used in the proof of (3.44) and we obtain (3.46).

Let now $N = N_1 \cup N_2 \cup N_3$. Given a sequence $t_k \to 0$ with $t_k \in [0,T] \setminus N$, by (3.46) we obtain

$$\begin{aligned}&\|\dot u(t_k)\|^2 + a(u(t_k),u(t_k)) + \lambda_0\|u(t_k)\|^2\\ &\le \|u^{(1)}\|^2 + a(u^{(0)},u^{(0)}) + \lambda_0(1+\varepsilon)\|u^{(0)}\|^2 + \int_0^{t_k} \|f(s)\|^2 ds + C_\varepsilon M_2^2 t_k\,.\end{aligned} \tag{3.47}$$

By (3.19) a subsequence of $\dot{u}(t_k)$ converges weakly in H. By (3.37) it follows that $\dot{u}(t_k)$ converges to $u^{(1)}$ strongly in V_0^* and weakly in H. By (3.31) we have that a subsequence of $u(t_k)$ converges weakly in V_T. By (3.45) it follows that $u(t_k)$ converges to $u^{(0)}$ strongly in H and weakly in V_T.

On the space $H \times V_T$ we consider the norm defined by

$$(h, v) \mapsto (\|h\|^2 + a(v, v) + \lambda_0 \|v\|^2)^{1/2} \qquad \text{for every } (h, v) \in H \times V_T \,,$$

which is equivalent to the product norm by the properties of a (see (2.4) and (2.5)). Using the lower semicontinuity of the norm and (3.47), by the arbitrariness of ε we obtain

$$\|\dot{u}(t_k)\|^2 + a(u(t_k), u(t_k)) + \lambda_0 \|u(t_k)\|^2 \to \|u^{(1)}\|^2 + a(u^{(0)}, u^{(0)}) + \lambda_0 \|u^{(0)}\|^2 \,,$$

which implies (3.35) and (3.36) and concludes the proof. □

Corollary 3.2. *Assume that one of the following conditions is satisfied:*

(a) $a(u, u) \geq 0$ for every $u \in V_T$;
(b) the embedding of V_T into H is compact.

Then for every $u^{(0)} \in V_0$ and $u^{(1)} \in H$ there exists a weak solution u to (2.10) on $[0, T]$ which satisfies

(1) the initial conditions: $u(0) = u^{(0)}$ and $\dot{u}(0) = u^{(1)}$ in the sense of (3.1);
(2) the continuity conditions: $u \in C_w([0, T]; V_T) \cap C([0, T]; H)$ and $\dot{u} \in C_w([0, T]; H) \cap C([0, T]; V_0^*)$;
(3) the energy inequality:

$$\frac{1}{2}\|\dot{u}(t)\|^2 + \frac{1}{2}a(u(t), u(t)) \leq \frac{1}{2}\|u^{(1)}\|^2 + \frac{1}{2}a(u^{(0)}, u^{(0)}) + \int_0^t (f(s), \dot{u}(s))ds \qquad (3.48)$$

for every $t \in [0, T]$.

Proof. Let u_n, $\tilde{u}_n$, and u be as in the proof of Theorem 3.1. Then u satisfies conditions (1) and (2). To prove the energy inequality (3.48) we use (3.8) and we obtain

$$\begin{aligned} &\|\dot{u}_n(t)\|^2 + a(\tilde{u}_n(t), \tilde{u}_n(t)) - \lambda_0 \tau_n \int_0^{t_n(t)} \|\dot{u}_n(s)\|^2 ds \\ &\leq \|u^{(1)}\|^2 + a(u^{(0)}, u^{(0)}) + 2 \int_0^{t_n(t)} (f(s), \dot{u}_n(s))ds \,, \end{aligned} \qquad (3.49)$$

where $t_n(t) = t_n^j$ for $t \in (t_n^{j-1}, t_n^j)$.

If (a) holds, then for every $(\alpha, \beta) \subset (0, T)$ the functional

$$\zeta \mapsto \int_{\alpha}^{\beta} a(\zeta(t), \zeta(t))dt$$

is lower semicontinuous in the weak topology of $L^2((0, T); V_T)$. Therefore we can apply to (3.49) the arguments used in the proof of (3.44) and thanks to (3.19), (3.29), and (3.30) we obtain (3.48) for a.e. $t \in (0, T)$. This inequality can be extended to every $t \in [0, T]$ by using (2) and the lower semicontinuity with respect to weak convergence of the terms in the left-hand side of (3.48).

If (b) holds, then by the Aubin–Lions Theorem (see [1, Theorem 5.1] and [11, Theorem 12.1], revisited in [13, Section 8, Corollary 4]) $\tilde{u}_n \to u$ strongly in $L^2((0, T); H)$. Adding $\lambda_0 \|\tilde{u}_n(t)\|^2$ to both sides of (3.49) we obtain

$$\begin{aligned} &\|\dot{u}_n(t)\|^2 + a(\tilde{u}_n(t), \tilde{u}_n(t)) + \lambda_0 \|\tilde{u}_n(t)\|^2 - \lambda_0 \tau_n \int_0^{t_n(t)} \|\dot{u}_n(s)\|^2 ds \\ &\leq \|u^{(1)}\|^2 + a(u^{(0)}, u^{(0)}) + \lambda_0 \|\tilde{u}_n(t)\|^2 + 2 \int_0^{t_n(t)} (f(s), \dot{u}_n(s))ds \,. \end{aligned}$$

We now argue as in the proof of (3.46) and we obtain

$$\begin{aligned} &\|\dot{u}(t)\|^2 + a(u(t), u(t)) + \lambda_0 \|u(t)\|^2 \\ &\leq \|u^{(1)}\|^2 + a(u^{(0)}, u^{(0)}) + \lambda_0 \|u(t)\|^2 + 2 \int_0^{t} (f(s), \dot{u}(s))ds \,, \end{aligned}$$

for a.e. $t \in (0, T)$. This inequality can be extended to every $t \in [0, T]$ as in case (a) and this concludes the proof of (3.48). □

4. Uniqueness

In this section we give some conditions on the family of spaces $(V_t)_{t\in[0,T]}$ which ensure the uniqueness of a weak solution to the Cauchy problem for the wave equation (2.10). These conditions describe the regular dependence of the spaces V_t on the parameter t and are expressed through the properties of some isomorphisms between V_t and V_0 and of their time derivatives. More precisely, we assume that:

(U1) for every $t \in [0, T]$ there exists a continuous linear bijective operator $Q_t \colon V_t \to V_0$ with continuous inverse $R_t \colon V_0 \to V_t$;
(U2) Q_0 and R_0 are the identity map on V_0;

(U3) there exists a constant M_1 independent of t such that

$$\|Q_t u\| \le M_1 \|u\| \text{ for every } u \in V_t \quad \text{and} \quad \|R_t v\| \le M_1 \|v\| \text{ for every } v \in V_0, \tag{4.1}$$

$$\|Q_t u\|_0 \le M_1 \|u\|_t \text{ for every } u \in V_t \quad \text{and} \quad \|R_t v\|_t \le M_1 \|v\|_0 \text{ for every } v \in V_0. \tag{4.2}$$

Since V_t is dense in H for every t, (4.1) implies that Q_t and R_t can be extended to continuous linear operators from H into itself, still denoted by Q_t and R_t.

The idea of the proof of uniqueness is to transfer a solution $u(t)$ of the wave equation (2.10) into the space V_0 by considering the function $u_0(t) := Q_t u(t)$. To study the equation satisfied by u_0 we need to control the behaviour of the operators Q_t and R_t with respect to t.

We begin with the properties of R_t, which are simpler to state because the operators R_t are defined in a space independent of t. We assume that:

(U4) for every $v \in V_0$ the function $t \mapsto R_t v$ from $[0, T]$ into H has a derivative, denoted by $\dot{R}_t v$;
(U5) there exists $\eta \in (0, 1)$ such that

$$\|\dot{R}_t Q_t v\|^2 \le \nu_0 (1 - \eta) \|v\|_t^2 \quad \text{for every } t \in [0, T] \text{ and } v \in V_t\,, \tag{4.3}$$

where ν_0 is the constant given in (2.5).

By (U4) the function $t \mapsto R_t v$ is continuous from $[0, T]$ into H. This property, together with (4.2), implies that $t \mapsto R_t v$ is weakly continuous from $[0, T]$ into V_T. By (U4) and (U5) $\dot{R}_t$ is a continuous linear operator from V_0 into H and by the Mean Value Theorem for every $0 \le s < t \le T$ and every $v \in V_0$ we have the estimate

$$\|R_t v - R_s v\| \le \nu_0^{1/2} (1 - \eta)^{1/2} M_1 \|v\|_0 (t - s)\,. \tag{4.4}$$

As for Q_t, a technical difficulty is due to the fact that its domain of definition depends on t. By analogy with (4.4) we assume that:

(U6) there exists a constant M_2 such that

$$\|Q_t v - Q_s v\| \le M_2 \|v\|_s (t - s) \quad \text{for every } 0 \le s < t \le T \text{ and every } v \in V_s; \tag{4.5}$$

(U7) for every $t \in [0, T)$ and for every $v \in V_t$ there exists an element of H, denoted by $\dot{Q}_t v$, such that

$$\lim_{h \to 0+} \frac{Q_{t+h} v - Q_t v}{h} = \dot{Q}_t v \quad \text{strongly in } H. \tag{4.6}$$

By (4.5) for every $s \in [0, T)$ and for every $v \in V_s$ the function $t \mapsto Q_t v$ is continuous from $[s, T]$ into H. This property, together with (4.2), implies that

$$t \mapsto Q_t v \quad \text{is weakly continuous from } [s, T] \text{ into } V_0\,. \tag{4.7}$$

By (4.5) we obviously have

$$\|\dot{Q}_t v\| \le M_2 \, \|v\|_t \tag{4.8}$$

for every $t \in [0, T)$ and for every $v \in V_t$. Hence $\dot{Q}_t$ is a continuous linear operator from V_t into H. We shall see in Lemma 4.5 below that properties (U6) and (U7) can be used to obtain the differentiability of $u_0(t) = Q_t u(t)$ with respect to t.

To formulate in an easier way the estimates leading to uniqueness it is convenient to introduce for every $t \in [0, T]$ the bilinear maps

$$\alpha(t)\colon V_0 \times V_0 \to \mathbb{R} \quad \text{defined by } \alpha(t)(u, v) := a(R_t u, R_t v)\,, \tag{4.9}$$

$$\beta(t)\colon V_0 \times V_0 \to \mathbb{R} \quad \text{defined by } \beta(t)(u, v) := (\dot{R}_t u, \dot{R}_t v)\,, \tag{4.10}$$

$$\gamma(t)\colon V_0 \times H \to \mathbb{R} \quad \text{defined by } \gamma(t)(u, v) := (\dot{R}_t u, R_t v)\,, \tag{4.11}$$

$$\delta(t)\colon H \times H \to \mathbb{R} \quad \text{defined by } \delta(t)(u, v) := (R_t u, R_t v) - (u, v)\,. \tag{4.12}$$

By (2.4), (4.1), (4.2), and (4.3) there exists a constant $M_3 > 0$ such that for every $t \in [0, T]$ we have

$$|\alpha(t)(u, v)| \le M_3 \|u\|_0 \|v\|_0 \quad \text{for every } u, v \in V_0\,, \tag{4.13}$$

$$|\beta(t)(u, v)| \le M_3 \|u\|_0 \|v\|_0 \quad \text{for every } u, v \in V_0\,, \tag{4.14}$$

$$|\gamma(t)(u, v)| \le M_3 \|u\|_0 \|v\| \quad \text{for every } u \in V_0,\, v \in H\,, \tag{4.15}$$

$$|\delta(t)(u, v)| \le M_3 \|u\| \|v\| \quad \text{for every } u, v \in H\,. \tag{4.16}$$

We assume that there exists a constant M_4 such that

(U8) the functions $t \mapsto \alpha(t)(u, v)$, $t \mapsto \beta(t)(u, v)$, $t \mapsto \gamma(t)(u, v)$, and $t \mapsto \delta(t)(u, v)$ are Lipschitz continuous and for a.e. $t \in (0, T)$ their derivatives satisfy

$$|\dot{\alpha}(t)(u, v)| \le M_4 \|u\|_0 \|v\|_0 \quad \text{for every } u, v \in V_0\,, \tag{4.17}$$

$$|\dot{\beta}(t)(u, v)| \le M_4 \|u\|_0 \|v\|_0 \quad \text{for every } u, v \in V_0\,, \tag{4.18}$$

$$|\dot{\gamma}(t)(u, v)| \le M_4 \|u\|_0 \|v\| \quad \text{for every } u \in V_0 \text{ and } v \in H\,, \tag{4.19}$$

$$|\dot{\delta}(t)(u, v)| \le M_4 \|u\| \|v\| \quad \text{for every } u, v \in H\,. \tag{4.20}$$

We now consider the simplest example where conditions (U1)–(U8) are satisfied.

Example 4.1. Let $\ell\colon [0, T] \to \mathbb{R}$ be a $C^{1,1}$ function such that

$$0 \le \dot{\ell}(t) < 1 \quad \text{for every } t \in [0, T]\,. \tag{4.21}$$

We set

$$\Gamma_t = \{(x_1, 0) : x_1 \le \ell(t)\}\,, \qquad V_t = H^1(\mathbb{R}^2 \setminus \Gamma_t)\,, \qquad H = L^2(\mathbb{R}^2) \tag{4.22}$$

for every $t \in [0, T]$. Then conditions (H1) and (H2) of Section 2 are satisfied.

Let $a : H^1(\mathbb{R}^2 \setminus \Gamma_T) \times H^1(\mathbb{R}^2 \setminus \Gamma_T) \to \mathbb{R}$ be defined by

$$a(u, v) = \int_{\mathbb{R}^2 \setminus \Gamma_T} \nabla u(x) \cdot \nabla v(x) dx .$$

Then conditions (H3) and (H4) of Section 2 are satisfied with $\lambda_0 = \nu_0 = 1$. For every $t \in [0, T]$ let $Q_t : H^1(\mathbb{R}^2 \setminus \Gamma_t) \to H^1(\mathbb{R}^2 \setminus \Gamma_0)$ and $R_t : H^1(\mathbb{R}^2 \setminus \Gamma_0) \to H^1(\mathbb{R}^2 \setminus \Gamma_t)$ be defined by

$$(Q_t u)(y) = u(y + \ell(t)e_1) \qquad \text{and} \qquad (R_t u)(x) = u(x - \ell(t)e_1) .$$

It is easy to see that conditions (U1)–(U7) are satisfied and that for every $t \in [0, T]$ we have

$$(\dot{R}_t u)(x) = \dot{\ell}(t) D_1 u(x - \ell(t)e_1) \quad \text{for a.e. } x \in \mathbb{R}^2 .$$

In particular (U5) follows from this equality thanks to (4.21). This formula also allows to write explicit expressions for the bilinear functions (4.9)–(4.12), which imply that (U8) is a consequence of the Lipschitz continuity of $\dot{\ell}$.

A more general situation is considered in the following example.

Example 4.2. Let Ω be a bounded open set in $\mathbb{R}^n$, let M be a C^2 manifold of dimension $n - 1$ in $\mathbb{R}^n$ with $\mathcal{H}^{n-1}(M) < \infty$, and let $(\Gamma_t)_{t\in[0,T]}$ be a family of closed subsets of $\Omega \cap M$ such that $\Gamma_s \subset \Gamma_t$ for $0 \le s < t \le T$. To impose a regular dependence on time, we assume that there exist two functions Φ, Ψ: $[0, T] \times \overline{\Omega} \to \overline{\Omega}$ of class $C^{1,1}$ such that the following properties hold for every $t \in [0, T]$:

(a) $\Phi(t, \cdot)$ and $\Psi(t, \cdot)$ are diffeomorphisms from $\overline{\Omega}$ into $\overline{\Omega}$;
(b) $\Phi(0, x) = \Psi(0, x) = x$ for every $x \in \overline{\Omega}$;
(c) $\Psi(t, \cdot)$ is the inverse of $\Phi(t, \cdot)$ on $\overline{\Omega}$;
(d) $\Phi(t, \Gamma_0) = \Gamma_t$ and $\Psi(t, \Gamma_t) = \Gamma_0$;
(e) $\det \nabla\Phi(t, x) > 0$ for every $x \in \overline{\Omega}$, where ∇ denotes the spatial gradient;
(f) $|\dot{\Phi}(t, y)|^2 < 1$ for every $y \in \overline{\Omega}$, where $\dot{\Phi}$ denotes the partial derivative of Φ with respect to t.

While conditions (a)–(e) are of qualitative nature, the quantitative condition (f) is related with the speed of the relative boundary of Γ_t in M (see the previous example and [8], [3], [4]).

For every $t \in [0, T]$ let $V_t = H^1(\Omega \setminus \Gamma_t)$ and $H = L^2(\Omega)$ as in Example 2.2. Let $a : H^1(\Omega \setminus \Gamma_T) \times H^1(\Omega \setminus \Gamma_T) \to \mathbb{R}$ be defined by

$$a(u, v) = \int_{\Omega \setminus \Gamma_T} \nabla u(x) \cdot \nabla v(x) dx .$$

Then conditions (H3) and (H4) of Section 2 are satisfied with $\lambda_0 = \nu_0 = 1$. For every $t \in [0, T]$ let $Q_t : H^1(\Omega \setminus \Gamma_t) \to H^1(\Omega \setminus \Gamma_0)$ and $R_t : H^1(\Omega \setminus \Gamma_0) \to H^1(\Omega \setminus \Gamma_t)$ be defined by

$$(Q_t u)(x) = u(\Phi(t, x)) \qquad \text{and} \qquad (R_t u)(x) = u(\Psi(t, x)) . \tag{4.23}$$

It is easy to see that conditions (U1)–(U4), (U6), and (U7) are satisfied and that for every $t \in [0,T]$ and $u \in H^1(\Omega \setminus \Gamma_0)$ we have

$$(\dot{R}_t u)(x) = \nabla u(\Psi(t,x)) \cdot \dot{\Psi}(t,x) \quad \text{for a.e. } x \in \Omega\,, \tag{4.24}$$

hence we obtain

$$\|\dot{R}_t Q_t u\|^2 \leq \int\limits_{\Omega\setminus\Gamma_t} |\nabla u(x)|^2 |\dot{\Phi}(t,\Psi(t,x))|^2 dx\,, \tag{4.25}$$

so that (U5) follows from assumption (f). To show that (U8) holds we observe that, after a change of variables, we can write the bilinear forms α , β , γ, and δ as

$$\alpha(t)(u,v) = \int\limits_{\Omega\setminus\Gamma_0} \sum_{ij} a_{ij}(t,x) D_i u(x) D_j v(x) dx \qquad \text{for } u,v \in H^1(\Omega \setminus \Gamma_0)$$

$$\beta(t)(u,v) = \int\limits_{\Omega\setminus\Gamma_0} \sum_{ij} b_{ij}(t,x) D_i u(x) D_j v(x) dx \qquad \text{for } u,v \in H^1(\Omega \setminus \Gamma_0)$$

$$\gamma(t)(u,v) = \int\limits_{\Omega\setminus\Gamma_0} \sum_{i} c_{i}(t,x) D_i u(x) v(x) dx \qquad \text{for } u \in H^1(\Omega \setminus \Gamma_0),\, v \in L^2(\Omega)$$

$$\delta(t)(u,v) = \int\limits_{\Omega\setminus\Gamma_0} d(t,x) u(x) v(x) dx \qquad \text{for } u,v \in L^2(\Omega)\,,$$

for suitable functions a_{ij}, b_{ij}, c_i, and d which are continuous on $[0,T]\times\overline{\Omega}$ and Lipschitz continuous in t uniformly with respect to x. By taking the derivatives with respect to t we obtain that (4.17)–(4.20) are satisfied.

We are now in a position to state the main result of this section.

Theorem 4.3. *Assume (U1)–(U8). Given $u^{(0)} \in V_0$, $u^{(1)} \in H$, and $f \in L^2((0,T);H)$, there exists a unique weak solution u to the wave equation* (2.10) *on* $[0,T]$ *satisfying the initial conditions $u(0) = u^{(0)}$ and $\dot{u}(0) = u^{(1)}$ in the sense that*

$$\lim_{h\to 0+} \frac{1}{h} \int\limits_0^h \big(\|u(t) - u^{(0)}\|^2 + \|\dot{u}(t) - u^{(1)}\|^2 \big) dt = 0\,. \tag{4.26}$$

Remark 4.4. By Theorem 3.1 the unique solution satisfies the initial conditions in the stronger sense

$$\lim_{h\to 0+} \frac{1}{h} \int\limits_0^h \big(\|u(t) - u^{(0)}\|_t^2 + \|\dot{u}(t) - u^{(1)}\|^2 \big) dt = 0\,.$$

To prove the theorem we need the following lemma.

Lemma 4.5. *Assume (U1)–(U3), (U6), and (U7). Let $u \in \mathcal{V}$ and for every $t \in [0,T]$ let $u_0(t) := Q_t u(t)$. Then the following properties hold:*

(a) $u_0 \in L^2((0,T); V_0)$*;*
(b) u_0 *is absolutely continuous from* $[0,T]$ *into* H*;*
(c) $\dot u_0 \in L^2((0,T); H)$ *and* $\dot u_0(t) = \dot Q_t u(t) + Q_t \dot u(t)$ *for a.e.* $t \in (0,T)$.

Proof. We begin by proving that $u_0 \colon [0,T] \to V_0$ is weakly measurable. Given $n \in \mathbb{N}$, we set $\tau_n := T/n$ and $s_n^i := i\tau_n$, with $i = 0, ..., n$. For $i = 1, \dots, n$ we define

$$u_n^i := \frac{1}{\tau_n} \int_{s_n^{i-1}}^{s_n^i} u(t) dt \tag{4.27}$$

and we set $u_n^0 = 0$. Let $u_n \colon [0,T] \to V_T$ be the step function defined by $u_n(t) = u_n^{i-1}$ for $t \in [s_n^{i-1}, s_n^i)$ and $i = 1, \dots, n$. Then $u_n(t) \in V_t$ for every $t \in [0,T]$. Since $u_n \to u$ in $L^2((0,T); V_T)$, a subsequence of u_n, not relabelled, satisfies $u_n(t) \to u(t)$ in V_T for a.e. $t \in [0,T]$. For every n the function $t \mapsto Q_t u_n(t)$ from $[0,T]$ into V_0 is weakly measurable by (4.7). Since $Q_t u_n(t) \to Q_t u(t) = u_0(t)$ for a.e. $t \in [0,T]$, we deduce that $u_0 \colon [0,T] \to V_0$ is weakly measurable. Since V_0 is separable, u_0 is measurable, so that (4.2) implies (a).

To prove (b) it is enough to show that for every $0 \le s < t \le T$ we have

$$\|u_0(t) - u_0(s)\| \le M_1 \int_s^t \|\dot u(\tau)\| d\tau + M_2 \int_s^t \|u(\tau)\|_T d\tau \,. \tag{4.28}$$

To this end we fix a sequence of partitions (t_k^i) with $s = t_k^0 < t_k^1 < \dots < t_k^k = t$ with $\max(t_k^i - t_k^{i-1}) \to 0$ such that

$$\sum_{i=1}^k \|u(t_k^{i-1})\|_T (t_k^i - t_k^{i-1}) \to \int_s^t \|u(\tau)\|_T d\tau \tag{4.29}$$

The existence of such a sequence of partitions is a consequence of the approximability of the Lebesgue integral by suitable Riemann sums (see Lemma A.2 with $X = \mathbb{R}$, $f = \|u\|_T$, and $g = 1$). We have

$$\|u_0(t) - u_0(s)\| = \|Q_t u(t) - Q_s u(s)\| \le \sum_{i=1}^k \|Q_{t_k^i} u(t_k^i) - Q_{t_k^{i-1}} u(t_k^{i-1})\|$$

$$\le \sum_{i=1}^k \|Q_{t_k^i}(u(t_k^i) - u(t_k^{i-1}))\| + \sum_{i=1}^k \|Q_{t_k^i} u(t_k^{i-1}) - Q_{t_k^{i-1}} u(t_k^{i-1})\|$$

$$\leq \sum_{i=1}^{k} M_1 \|u(t_k^i) - u(t_k^{i-1})\| + M_2 \sum_{i=1}^{k} \|u(t_k^{i-1})\|_T (t_k^i - t_k^{i-1}),$$

where the last inequality follows from (4.1) and (4.5). Hence

$$\|u_0(t) - u_0(s)\| \leq M_1 \int_s^t \|\dot{u}(\tau)\| d\tau + M_2 \sum_{i=1}^{k} \|u(t_k^{i-1})\|_T (t_k^i - t_k^{i-1}).$$

Passing to the limit for $k \to \infty$ and using (4.29) we obtain (4.28).

To prove (c) we start by the equality

$$\frac{u_0(t+h) - u_0(t)}{h} = \frac{Q_{t+h} u(t+h) - Q_t u(t)}{h}$$
$$= Q_{t+h}\Big(\frac{u(t+h) - u(t)}{h} - \dot{u}(t)\Big) + Q_{t+h}\dot{u}(t) + \frac{Q_{t+h}u(t) - Q_t u(t)}{h}.$$

For a.e. $t \in (0, T)$ the first term tends to 0 in H thanks to (4.1), while the last term tends to $\dot{Q}_t u(t)$ for every $t \in [0, T)$ as $h \to 0+$. It remains to show that

$$\lim_{h \to 0+} \|Q_{t+h}\dot{u}(t) - Q_t \dot{u}(t)\| = 0. \tag{4.30}$$

To this aim, using the density of V_t in H, for every $\varepsilon > 0$ we find $v_\varepsilon \in V_t$ such that $\|v_\varepsilon - \dot{u}(t)\| < \varepsilon$. Then we have

$$\begin{aligned}\|Q_{t+h}\dot{u}(t) - Q_t \dot{u}(t)\| &= \|Q_{t+h}(\dot{u}(t) - v_\varepsilon)\| + \|Q_{t+h}v_\varepsilon - Q_t v_\varepsilon\| + \|Q_t(v_\varepsilon - \dot{u}(t))\| \\ &\leq \|Q_{t+h}v_\varepsilon - Q_t v_\varepsilon\| + 2M_1\varepsilon\end{aligned}$$

where the inequality follows from the choice of v_ε and (4.1). Passing to the limit as $h \to 0+$, by (4.5) we get

$$\limsup_{h \to 0+} \|Q_{t+h}\dot{u}(t) - Q_t \dot{u}(t)\| \leq 2M_1\varepsilon.$$

By the arbitrariness of ε we obtain (4.30). □

Proof of Theorem 4.3. By linearity it is sufficient to prove the uniqueness in the case $f = 0$, $u^{(0)} = 0$, and $u^{(1)} = 0$. Let $u \in \mathcal{V} \subset C([0, T]; H)$ be a weak solution of the wave equation (2.10) in this case. Suppose by contradiction that there exists $t \in [0, T]$ such that $u(t) \neq 0$ and let

$$t_0 := \inf\{t \in [0, T] : u(t) \neq 0\}. \tag{4.31}$$

Then $0 \leq t_0 < T$.

Let $u_0(t) := Q_t u(t)$. By Lemma 4.5 we have that $u_0 \in L^2((0, T); V_0)$ and $\dot{u}_0 \in L^2((0, T); H)$. Since $u(t) = R_t u_0(t)$, arguing as in Lemma 4.5 we can prove that

$$\dot{u}(t) = \dot{R}_t u_0(t) + R_t \dot{u}_0(t) \quad \text{for a.e. } t \in (0, T). \tag{4.32}$$

We fix $t_1 \in (t_0, T]$ and choose

$$\varphi_0(t) = \begin{cases} \int_t^{t_1} u_0(s)ds & 0 \le t \le t_1 , \\ 0 & t_1 \le t \le T . \end{cases} \tag{4.33}$$

It is clear that $\varphi_0 \in C([0,T]; V_0)$, $\dot\varphi_0 \in L^2((0,T); V_0)$, and $\ddot\varphi_0 \in L^2((0,T); H)$. Moreover, we have

$$\dot\varphi_0(t) = \begin{cases} -u_0(t) & 0 \le t \le t_1 \\ 0 & t_1 \le t \le T \end{cases} \quad \text{and} \quad \ddot\varphi_0(t) = \begin{cases} -\dot u_0(t) & 0 \le t \le t_1 \\ 0 & t_1 \le t \le T . \end{cases} \tag{4.34}$$

By the definition of t_0 and u_0 and (4.34) it follows that

$$u_0(t) = \dot\varphi_0(t) = \ddot\varphi(t) = 0 \quad \text{for a.e. } t \in (0, t_0) . \tag{4.35}$$

For every $t \in [0,T]$ let $\varphi(t) := R_t \varphi_0(t)$. Arguing as in Lemma 4.5 we can prove that $\varphi \in L^2((0,T); V_T)$, that $\varphi \colon [0,T] \to H$ is absolutely continuous, and that

$$\dot\varphi(t) = \dot R_t \varphi_0(t) + R_t \dot\varphi_0(t) \quad \text{for a.e. } t \in (0,T), \tag{4.36}$$

hence $\dot\varphi \in L^2((0,T); H)$. By the properties of R_t we also have $\varphi(t) \in V_t$ for every $t \in [0,T]$. Therefore $\varphi \in \mathcal{V}$.

Since $\varphi(T) = 0$, in view of (4.26) and Remark 2.10 we can use φ as test function in the wave equation (2.10) satisfied by u. By (4.32) and (4.36) this leads to the equality

$$\int_0^T (\dot R_t u_0(t) + R_t \dot u_0(t), \dot R_t \varphi_0(t) + R_t \dot\varphi_0(t)) dt = \int_0^T a(R_t u_0(t), R_t \varphi_0(t)) dt ,$$

which by (4.9)–(4.12), (4.33)–(4.35) gives

$$\begin{aligned} &\int_{t_0}^{t_1} \alpha(t)(\dot\varphi_0(t), \varphi_0(t)) dt - \int_{t_0}^{t_1} \beta(t)(\dot\varphi_0(t), \varphi_0(t)) dt - \int_{t_0}^{t_1} \gamma(t)(\dot\varphi_0(t), \dot\varphi_0(t)) dt \\ &- \int_{t_0}^{t_1} \gamma(t)(\varphi_0(t), \ddot\varphi_0(t)) dt - \int_{t_0}^{t_1} (\dot u_0(t), u_0(t)) dt - \int_{t_0}^{t_1} \delta(t)(\dot u_0(t), u_0(t)) dt = 0 . \end{aligned} \tag{4.37}$$

From (U8), using (4.13)–(4.16) and the properties of u_0 and φ_0, we obtain that the functions $t \mapsto \|u_0(t)\|^2$, $t \mapsto \alpha(t)(\varphi_0(t), \varphi_0(t))$, $t \mapsto \beta(t)(\varphi_0(t), \varphi_0(t))$, $t \mapsto \gamma(t)(\varphi_0(t), \dot\varphi_0(t))$, and $t \mapsto \delta(t)(u_0(t), u_0(t))$ are absolutely continuous on $[t_0, t_1]$ and that for a.e. $t \in (t_0, t_1)$

$$\frac{1}{2}\frac{d}{dt}\|u_0(t)\|^2 = (\dot{u}_0(t), u_0(t))$$

$$\frac{1}{2}\frac{d}{dt}\Big(\alpha(t)(\varphi_0(t), \varphi_0(t))\Big) = \frac{1}{2}\dot{\alpha}(t)(\varphi_0(t), \varphi_0(t)) + \alpha(t)(\dot{\varphi}_0(t), \varphi_0(t)),$$

$$\frac{1}{2}\frac{d}{dt}\Big(\beta(t)(\varphi_0(t), \varphi_0(t))\Big) = \frac{1}{2}\dot{\beta}(t)(\varphi_0(t), \varphi_0(t)) + \beta(t)(\dot{\varphi}_0(t), \varphi_0(t)),$$

$$\frac{d}{dt}\Big(\gamma(t)(\varphi_0(t), \dot{\varphi}_0(t))\Big) = \dot{\gamma}(t)(\varphi_0(t), \dot{\varphi}_0(t)) + \gamma(t)(\dot{\varphi}_0(t), \dot{\varphi}_0(t)) + \gamma(t)(\varphi_0(t), \ddot{\varphi}_0(t)),$$

$$\frac{1}{2}\frac{d}{dt}\Big(\delta(t)(u_0(t), u_0(t))\Big) = \frac{1}{2}\dot{\delta}(t)(u_0(t), u_0(t)) + \delta(t)(\dot{u}_0(t), u_0(t)).$$

Hence, using the equalities $\varphi_0(t_1) = 0$, $\dot{\varphi}_0(t_0) = -u_0(t_0) = 0$, from (4.37) we obtain

$$\begin{aligned}
&\frac{1}{2}\alpha(t_0)(\varphi_0(t_0), \varphi_0(t_0)) + \frac{1}{2}\int_{t_0}^{t_1} \dot{\alpha}(t)(\varphi_0(t), \varphi_0(t))dt - \frac{1}{2}\beta(t_0)(\varphi_0(t_0), \varphi_0(t_0)) \\
&-\frac{1}{2}\int_{t_0}^{t_1} \dot{\beta}(t)(\varphi_0(t), \varphi_0(t))dt - \int_{t_0}^{t_1} \dot{\gamma}(t)(\varphi_0(t), \dot{\varphi}_0(t))dt + \frac{1}{2}\|u_0(t_1)\|^2 \qquad (4.38)\\
&+\frac{1}{2}\delta(t_1)(u_0(t_1), u_0(t_1)) - \frac{1}{2}\int_{t_0}^{t_1} \dot{\delta}(t)(u_0(t), u_0(t))dt = 0.
\end{aligned}$$

By (4.12) we have $\|u_0(t_1)\|^2 + \delta(t_1)(u_0(t_1), u_0(t_1)) = \|R_{t_1}u_0(t_1)\|^2 = \|u(t_1)\|^2$, where in the last equality we have used the definition of u_0 and (U1). Therefore, (4.34) and (4.38) give

$$\begin{aligned}
&\frac{1}{2}a(R_{t_0}\varphi_0(t_0), R_{t_0}\varphi_0(t_0)) - \frac{1}{2}\|\dot{R}_{t_0}\varphi_0(t_0)\|^2 + \frac{1}{2}\|u(t_1)\|^2 \\
&\le -\frac{1}{2}\int_{t_0}^{t_1} \dot{\alpha}(t)(\varphi_0(t), \varphi_0(t))dt + \frac{1}{2}\int_{t_0}^{t_1} \dot{\beta}(t)(\varphi_0(t), \varphi_0(t))dt \qquad (4.39)\\
&-\int_{t_0}^{t_1} \dot{\gamma}(t)(\varphi_0(t), u_0(t))dt + \frac{1}{2}\int_{t_0}^{t_1} \dot{\delta}(t)(u_0(t), u_0(t))dt.
\end{aligned}$$

By (2.5), (U3) and (4.3) we have

$$\begin{aligned}
a(R_{t_0}\varphi_0(t_0), R_{t_0}\varphi_0(t_0)) - \|\dot{R}_{t_0}\varphi_0(t_0)\|^2 &\ge \nu_0\eta\|R_{t_0}\varphi_0(t_0)\|_{t_0}^2 - \lambda_0\|R_{t_0}\varphi_0(t_0)\|^2 \\
&\ge \frac{\nu_0\eta}{M_1^2}\|\varphi_0(t_0)\|_0^2 - \lambda_0 M_1^2\|\varphi_0(t_0)\|^2.
\end{aligned}$$

Hence by (4.17)–(4.20) from (4.39) we obtain

$$\frac{\nu_0\eta}{2M_1^2}\|\varphi_0(t_0)\|_0^2+\frac{1}{2}\|u(t_1)\|^2\le\frac{\lambda_0M_1^2}{2}\|\varphi_0(t_0)\|^2+M_4\int_{t_0}^{t_1}\|\varphi_0(t)\|_0^2dt$$

$$+M_1M_4\int_{t_0}^{t_1}\|\varphi_0(t)\|_0\|u(t)\|dt+\frac{M_1^2M_4}{2}\int_{t_0}^{t_1}\|u(t)\|^2dt\,. \tag{4.40}$$

We now want to apply the Gronwall Lemma in order to conclude that $u=0$ on $[t_0,t_1]$ provided t_1-t_0 is small enough. To this end it is convenient to introduce the function

$$\psi_0(t):=\int_{t_0}^{t}u_0(s)ds=\int_{0}^{t_1}u_0(s)ds\quad\text{for } t\in[t_0,t_1]\,, \tag{4.41}$$

so that

$$\psi_0(t)+\varphi_0(t)=\varphi_0(t_0)=\psi_0(t_1)\quad\text{for every } t\in[t_0,t_1]\,.$$

By using the Cauchy Inequality from (4.40) we obtain

$$\frac{\nu_0\eta}{2M_1^2}\|\psi_0(t_1)\|_0^2+\frac{1}{2}\|u(t_1)\|^2\le\frac{\lambda_0M_1^2}{2}\|\psi_0(t_1)\|^2$$

$$+C\int_{t_0}^{t_1}\|\psi_0(t)-\psi_0(t_1)\|_0^2dt+C\int_{t_0}^{t_1}\|u(t)\|^2dt\,, \tag{4.42}$$

where C is a constant depending only on M_1 and M_4. By (4.1) and (4.41) we have

$$\|\psi_0(t_1)\|^2\le(t_1-t_0)\int_{t_0}^{t_1}\|u_0(t)\|^2dt\le(t_1-t_0)M_1^2\int_{t_0}^{t_1}\|u(t)\|^2dt\,,$$

therefore (4.42) gives

$$\frac{\nu_0\eta}{2M_1^2}\|\psi_0(t_1)\|_0^2+\frac{1}{2}\|u(t_1)\|^2\le 2C\int_{t_0}^{t_1}\|\psi_0(t)\|_0^2dt+2C(t_1-t_0)\|\psi_0(t_1)\|_0^2$$

$$+\left(C+\frac{\lambda_0(t_1-t_0)M_1^4}{2}\right)\int_{t_0}^{t_1}\|u(t)\|^2dt\,, \tag{4.43}$$

so that if $t_1-t_0\le\frac{\nu_0\eta}{8CM_1^2}$ we obtain

$$\frac{\nu_0\eta}{4M_1^2}\|\psi_0(t_1)\|_0^2 + \frac{1}{2}\|u_0(t_1)\|^2 \le 2C\int_{t_0}^{t_1}\|\psi_0(t)\|_0^2 dt + (C + \frac{\lambda_0\nu_0\eta M_1^2}{16C})\int_{t_0}^{t_1}\|u(t)\|^2 dt\,.$$

Let $t_0^* := \min\{T, t_0 + \frac{\nu_0\eta}{8CM_1^2}\}$. Since this inequality holds for every $t_1 \in [t_0, t_0^*]$, we can apply the Gronwall Lemma and deduce that $\psi_0(t) = 0$ and $u(t) = 0$ for every $t \in [t_0, t_0^*]$. This contradicts the definition of t_0 and concludes the proof. □

Acknowledgments

This material is based on work supported by the Italian Ministry of Education, University, and Research through the Project "Calculus of Variations" (PRIN 2015). The authors are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

Appendix A

In this section we prove two technical results that were used in the paper. Let V and H be Hilbert spaces with $V \subset H$ and V dense in H. Let V^* denote the dual of V endowed with the dual norm. As $V \subset H$ and V is dense in H, we have also that $H \subset V^*$ and H is dense in V^*. The scalar product in H is denoted by $(\cdot,\cdot)$ and the duality product between V^* and V is denoted by $\langle\cdot,\cdot\rangle$. It is obvious that

$$(u, v) = \langle u, v\rangle \quad \forall u \in H\,,\ v \in V\,. \tag{A.1}$$

The following lemma was crucial in the proof of Theorem 2.16.

Lemma A.1. *Let $\psi \in L^2((0,T); H)$ with $\dot\psi \in L^2((0,T); V^*)$ and $\varphi \in L^2((0,T); V)$ with $\dot\varphi \in L^2((0,T); H)$. Let $\omega\colon (0,T) \to \mathbb{R}$ be the function defined by*

$$\omega(t) = (\psi(t), \varphi(t)) \quad \textit{for a.e. } t \in (0,T)\,.$$

Then $\omega \in W^{1,1}((0,T))$ and

$$\dot\omega(t) = \langle\dot\psi(t), \varphi(t)\rangle + (\psi(t), \dot\varphi(t)) \quad \textit{for a.e. } t \in (0,T)\,. \tag{A.2}$$

We begin by proving the following lemma on the approximability of the Lebesgue integral by Riemann sums. The oldest result in this direction is contained in [10]. Our statement is similar to [5, Lemma 4.12].

Given a bounded closed interval $[a,b]$, for every irrational $s \in (0,1)$ we consider the finite set

$$S_k(s) := \{a + (s + \tfrac{i}{k-1})(b-a) : i \in \mathbb{Z}\} \cap (a,b)\,. \tag{A.3}$$

Since s is irrational, it is easy to see that $S_k(s)$ has $k-1$ elements. Let

$$t_k^1(s) < \cdots < t_k^{k-1}(s) \tag{A.4}$$

be an increasing enumeration of $S_k(s)$. We set

$$t_k^0(s) := a \qquad t_k^k(s) := b\,. \tag{A.5}$$

Lemma A.2. *Let $[a,b]$ be a bounded closed interval, let $(X, \|\cdot\|)$ be a Banach space with dual $(X^*, \|\cdot\|_*)$, let $f\colon [a,b] \to X$, $g\colon [a,b] \to X^*$ be Bochner measurable functions such that $\|f\|^2$ and $\|g\|_*^2$ are integrable, and let $N \subset (a,b)$ be a set of measure zero. Then for a.e. $s \in (0,1)$ the subdivisions given by (A.3)–(A.5) satisfy*

$$t_k^i(s) \notin N \quad \text{for every } i \text{ and } k\,, \tag{A.6}$$

and

$$\lim_{k\to\infty} \sum_{i=1}^{k} \int_{t_k^{i-1}(s)}^{t_k^i(s)} |\langle g(t), f(t_k^{i-1}(s)) - f(t)\rangle| dt = 0\,, \tag{A.7}$$

$$\lim_{k\to\infty} \sum_{i=1}^{k} \int_{t_k^{i-1}(s)}^{t_k^i(s)} |\langle g(t), f(t_k^{i}(s)) - f(t)\rangle| dt = 0\,, \tag{A.8}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality product between X^* and X In particular, for a.e. $s \in (0,1)$ we have

$$\sum_{i=1}^{k} \Big\langle \int_{t_k^{i-1}(s)}^{t_k^i(s)} g(t)dt\,,\, f(t_k^{i-1}(s))\Big\rangle \longrightarrow \int_a^b \langle g(t), f(t)\rangle dt\,, \tag{A.9}$$

$$\sum_{i=1}^{k} \Big\langle \int_{t_k^{i-1}(s)}^{t_k^i(s)} g(t)dt\,,\, f(t_k^{i}(s))\Big\rangle \longrightarrow \int_a^b \langle g(t), f(t)\rangle dt \tag{A.10}$$

as $k \to \infty$.

Proof. It is not restrictive to assume $a = 0$ and $b = 1$. We extend all functions to 0 outside $[0,1]$. For every $k \geq 2$ and for every $s \in (0,1)$ we have

$$\sum_{i\in\mathbb{Z}} \int_{s+\frac{i-1}{k-1}}^{s+\frac{i}{k-1}} \|g(t)\|_* \, \|f(s+\tfrac{i-1}{k-1}) - f(t)\| dt$$

$$= \sum_{i\in\mathbb{Z}} \int_0^{\frac{1}{k-1}} \|g(s+\tfrac{i-1}{k-1}+\tau)\|_* \, \|f(s+\tfrac{i-1}{k-1}) - f(s+\tfrac{i-1}{k-1}+\tau)\| d\tau\,.$$

Note that there are at most $2k$ non-zero elements in the above sums, namely those with $i \in I_k := \{i \in \mathbb{Z} : -k+1 \le i \le k\}$. Integrating with respect to s we obtain

$$\int_0^1 \Big[\sum_{i\in\mathbb{Z}} \int_{s+\frac{i-1}{k-1}}^{s+\frac{i}{k-1}} \|g(t)\|_* \, \|f(s+\tfrac{i-1}{k-1}) - f(t)\| dt\Big] ds$$

$$\le \sum_{i\in I_k} \int_0^{\frac{1}{k-1}} \Big[\int_{-\infty}^{+\infty} \|g(s+\tfrac{i-1}{k-1}+\tau)\|_* \, \|f(s+\tfrac{i-1}{k-1}) - f(s+\tfrac{i-1}{k-1}+\tau)\| \, ds\Big] d\tau \tag{A.11}$$

$$= 2k \int_0^{\frac{1}{k-1}} \Big[\int_{-\infty}^{+\infty} \|g(s)\|_* \, \|f(s-\tau) - f(s)\| ds\Big] d\tau \,.$$

By the continuity of the translations in $L^2(\mathbb{R}; X)$, for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\int_{-\infty}^{+\infty} \|f(s) - f(s-\tau)\|^2 ds < \varepsilon \tag{A.12}$$

for $0 < \tau < \delta$. Thus, from (A.11) and (A.12) we obtain

$$\lim_{k\to\infty} \int_0^1 \Big[\sum_{i\in\mathbb{Z}} \int_{s+\frac{i-1}{k-1}}^{s+\frac{i}{k-1}} \|g(t)\|_* \, \|f(s+\tfrac{i-1}{k-1}) - f(t)\| \, dt\Big] ds = 0 \,.$$

Similarly we prove that

$$\lim_{k\to\infty} \int_0^1 \Big[\sum_{i\in\mathbb{Z}} \int_{s+\frac{i-1}{k-1}}^{s+\frac{i}{k-1}} \|g(t)\|_* \, \|f(s+\tfrac{i}{k-1}) - f(t)\| dt\Big] ds = 0 \,.$$

Therefore for a.e. $s \in (0, 1)$ we have

$$\lim_{k\to\infty} \sum_{i\in\mathbb{Z}} \int_{s+\frac{i-1}{k-1}}^{s+\frac{i}{k-1}} \|g(t)\|_* \, \|f(s+\tfrac{i-1}{k-1}) - f(t)\| \, dt = 0 \,, \tag{A.13}$$

$$\lim_{k\to\infty} \sum_{i\in\mathbb{Z}} \int_{s+\frac{i-1}{k-1}}^{s+\frac{i}{k-1}} \|g(t)\|_* \, \|f(s+\tfrac{i}{k-1}) - f(t)\| \, dt = 0 \,. \tag{A.14}$$

We fix an irrational $s \in (0, 1)$ such that (A.13) and (A.14) hold, and $s + \frac{i}{k-1} \notin N$ for every $i \in \mathbb{Z}$ and every integer $k \geq 2$.

Using (A.3)–(A.5) we have

$$\sum_{i=1}^{k} \int\limits_{t_k^{i-1}(s)}^{t_k^i(s)} \|g(t)\|_* \, \|f(t_k^{i-1}(s)) - f(t)\| dt$$

$$= \sum_{i=2}^{k-1} \int\limits_{t_k^{i-1}(s)}^{t_k^i(s)} \|g(t)\|_* \, \|f(t_k^{i-1}(s)) - f(t)\| dt + \int\limits_{0}^{t_k^1(s)} \|g(t)\|_* \, \|f(0) - f(t)\| dt \tag{A.15}$$

$$+ \int\limits_{t_k^{k-1}(s)}^{1} \|g(t)\|_* \, \|f(t_k^{k-1}(s)) - f(t)\| dt$$

The first term in the right hand side of (A.15) is bounded from above by the sum in (A.13) and therefore it tends to 0. The second one tends to 0 by the absolute continuity of the integral, while the third term satisfies

$$\int\limits_{t_k^{k-1}(s)}^{1} \|g(t)\|_* \, \|f(t_k^{k-1}(s)) - f(t)\| dt = \int\limits_{t_k^{k-1}(s)}^{t_k^{k-1}(s)+\frac{1}{k-1}} \|g(t)\|_* \, \|f(t_k^{k-1}(s)) - f(t)\| dt \,,$$

and therefore it tends to 0 by (A.13). This proves that the left-hand side of (A.15) tends to zero and clearly this implies (A.7).

Similarly from (A.14) we deduce (A.8). Equalities (A.9) and (A.10) are easy consequences of (A.7) and (A.8). □

Proof of Lemma A.1. To prove that $\omega \in W^{1,1}((0, T))$ and that (A.2) holds it is enough to show that for a.e. $a, b \in (0, T)$ with $a < b$ we have

$$\omega(b) - \omega(a) = \int\limits_a^b \langle \dot\psi(t), \varphi(t) \rangle \, dt + \int\limits_a^b (\psi(t), \dot\varphi(t)) dt \,. \tag{A.16}$$

Under our hypotheses on ψ and φ, using (A.1) we obtain that there exists a set $N \subset (0, T)$ of measure zero such that

$$\begin{aligned} \omega(b) - \omega(a) &= (\psi(b), \varphi(b)) - (\psi(a), \varphi(a)) \\ &= \langle \psi(b) - \psi(a), \varphi(b) \rangle + (\psi(a), \varphi(b) - \varphi(a)) \\ &= \Big\langle \int\limits_a^b \dot\psi(t) dt \,, \varphi(b) \Big\rangle + \Big(\psi(a), \int\limits_a^b \dot\varphi(t) dt \Big) \end{aligned} \tag{A.17}$$

for every $a, b \in (0, T) \setminus N$ with $a < b$.

We fix a pair a, b with these properties. By Lemma A.2 there exists an irrational $s \in (0,1)$ such that the subdivisions $(t_k^i(s))_{0\le i\le k}$ of the interval $[a,b]$ introduced in (A.3)–(A.5) satisfy (A.6)–(A.10) simultaneously for $X = V$, $f = \varphi$, $g = \dot\psi$, and for $X = H$, $f = \psi$, $g = \dot\varphi$. By (A.6) and (A.17) we obtain

$$\omega(b) - \omega(a) = \sum_{i=1}^{k} \omega(t_k^i(s)) - \omega(t_k^{i-1}(s))$$

$$= \sum_{i=1}^{k} \Big\langle \int_{t_k^{i-1}(s)}^{t_k^i(s)} \dot\psi(t)dt\, , \varphi(t_k^i(s)) \Big\rangle + \sum_{i=1}^{k} \Big(\psi(t_k^{i-1}(s)), \int_{t_k^{i-1}(s)}^{t_k^i(s)} \dot\varphi(t)dt \Big)$$

By (A.9) and (A.10), passing to the limit as $k \to \infty$ we obtain (A.16). This concludes the proof. □

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