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A quantum fluctuation description of charge qubits

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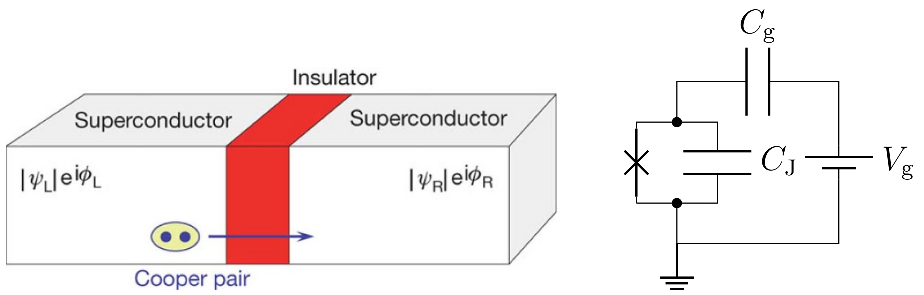
Abstract

We consider a specific instance of a superconducting circuit, the so-called charge-qubit, consisting of a capacitor and a Josephson junction that we describe by means of the BCS microscopic model in terms of two tunnelling superconducting systems in the strong-coupling quasi-spin formulation. Then, by means of collective observables we derive the Hamiltonian governing the quantum behaviour of the circuit in the limit of a large number N of quasi-spins. Our approach relies on suitable *quantum fluctuations*, i.e. on collective quasi-spin operators, different from mean-field observables, that retain a quantum character in the large-N limit. These collective operators generate the Heisenberg algebra on the circle and we show that their dynamics reproduces the phenomenological one generated by the charge qubit Hamiltonian obtained by quantizing the macroscopic classical Hamiltonian of the circuit. The microscopic derivation of the emergent, large-N behaviour provides a rigorous setting to investigate more in detail both general quantum circuits and quantum macroscopic scenarios; in particular, in the specific case of charge-qubits, it allows to explicitly obtain the temperature dependence of the critical Josephson current in the strong coupling regime, a result not accessible using standard approximation techniques.

1. Introduction

Contrary to the common wisdom according to which macroscopic systems behave classically and microsopic ones quantumly, intriguing instances of macroscopic quantum behaviour are provided by *superconducting quantum circuits* [1–13], a quite developed technology for the physical implementation of prototype quantum computers [14]. These electric circuits are based on Josephson junctions [15–17], namely on two superconducting electrodes, separated by a thin insulating barrier, able to sustain tunnelling currents depending non-linearly on the phase difference between the complex order-parameters associated with the two electrodes, as depicted in figure 1(a). In particular, figure 1(b) provides a scheme of a so-called charge-qubit, namely a superconducting circuit with a capacitive element, beside the junction, and an applied voltage. Remarkably, at low temperatures, superconducting circuits exhibit a quantum behaviour that is phenomenologically described as follows. One firstly associates the classical circuit with an effective one-dimensional anharmonic Hamiltonian which reproduces the voltage-current relations within the circuit and involves canonically conjugated position-like and momentum-like variables. Then, they are quantized, although they collectively refer to large numbers of degrees of freedom. In particular, in the case of charge qubits they act like momentum and angle variables for a particle on a circle inside an anharmonic potential [18] and correspond to the excess number of Cooper pairs on the junction and to the phase difference between the superconducting condensates.

Purpose of this manuscript is to go beyond the phenomenological approach providing, by means of a rigorous algebraic procedure, a theoretical derivation of the non-commutative behaviour of quantum circuits starting from the dynamics of their microscopic degrees of freedom. As we shall see, with respect to



(a) *Josephson junction* (b) *Charge qubit circuit*

Figure 1. On the left, a schematic drawing of a Josephson junction: $|\psi_{L,R}|^2$ are the macroscopic number of Cooper pairs on the left, respectively right electrode, while $\phi_{L,R}$ are the phases of the two corresponding quantum macrostates. On the right, a charge qubit circuit; the Josephson junction is represented by the cross on the leftmost part and has capacitance C_J while C_G is the circuit capacitance without junction and V_g is the applied voltage.

the phenomenological approach briefly summarized above, a microscopic explanation of the emergent large-scale properties allows on one hand to keep track of their dependence on the most significant physical parameters affecting the circuits, for instance their temperature. On the other hand, it permits the rigorous investigation of scopes and limitations of other purported quantum macroscopic effects like the establishment of non-local correlations among quantum-circuits and their survival in the presence of decoherence.

Investigating quantum many-body systems when the number of their constituents become large requires the use of collective observables able to capture the behaviour at macroscopic scales involving the whole set of microscopic degrees of freedom. Typical among them are the so-called mean-field observables which account for the average properties of many-body systems; if N is the number of microscopic constituents, mean-field observables are defined as the sum of N copies of single-component microscopic observables, rescaled by a factor $1/N$. Their characteristic trait is that, in the large-N limit, these observables commute, lose any quantum character and behave classically.

There exist however other types of collective observables, which still involve sums over all microscopic components, but are rescaled with different powers of N that can allow them to retain quantum properties in the large-N limit. For this reason, they are remarkably suited to model quantum phenomena at a *mesoscopic* scale, namely at an intermediate level between the purely quantum microscopic one and the purely classical macroscopic behaviour. They have been termed *quantum fluctuations* [19, 20].

In many-body systems with sufficiently short-range correlations, quantum fluctuations are defined, in analogy with the classical central limit theorem, as sums of the deviations of N copies of a single component microscopic observable from its mean-value, rescaled by $1/\sqrt{N}$. In the large-N limit, these operators generate the Heisenberg algebra of position and momentum operators.

Different, asymmetric scalings with N of conjugated collective operators, that is as $N^{-\alpha}$ and $N^{-\beta}$ with $\alpha, \beta \geqslant 0$ and $\alpha + \beta = 1$, but $\alpha \neq 1/2$, $\beta \neq 1/2$, are sometimes necessary—i.e. whenever the state of the system contains long-range correlations—and give rise to different non-commutative algebras as N becomes large. All these emerging large-N algebras are referred to as *fluctuation algebras*; they turn out to provide a natural setup for a proper investigation of the mesoscopic properties of many-body systems. More specifically, quantum fluctuations provide suitable tools to describe the dynamics at the mesoscopic level in terms of the quantum dynamics of the fluctuation algebra emerging as the large-N limit of the quantum microscopic many-body dynamics.

In the following, we focus on the specific case of the so-called charge qubit superconducting circuits and apply the quantum fluctuation approach to the microscopic dynamics of their Josephson junction as described by the *BCS* model [21–27] in the so-called strong coupling regime [28–31]. We show that the phenomenological quantum Hamiltonian usually adopted to predict their behaviour, indeed emerges as the generator of the mesoscopic dynamics of specific quantum fluctuations dictated by the structure of the *BCS* model. In the strong-coupling regime, these quantum fluctuations generate the Heisenberg algebra on the circle, with a collective observable rescaled by $1/N$ playing the role of an angle-like operator, and the canonically conjugated momentum-like one being described by a collective observable without rescaling.

Unlike in the phenomenological approach whereby the critical current is an external parameter, the quantum fluctuation approach allows to predict the dependence of the critical current on the temperature, most importantly in a regime, the strong-coupling one, where the perturbative Ambegaokar-Baratoff approach cannot be applied [32]. Since such a behaviour is in line of principle amenable to experimental verification, its confirmation would underline the usefulness of the quantum fluctuation approach for the description and prediction of further macroscopic quantum behaviours as those concerning different quantum circuits as phase qubits, flux qubits and transmons. Furthermore, starting from the microscopic coupling of the components of two Josephson junctions, the quantum fluctuation approach would allow the study of whether and how microscopic entanglement could make itself felt at the mesoscpic level and also how its properties are affected by microscopic noise and decoherence.

In section 2, we review the basic features of charge qubit superconducting circuits and discuss the *BCS* model of superconductivity [21–27] in its *strong-coupling* regime [28–31]. We adopt the so-called *quasi-spin* formulation [33–35] and physically motivate the use of the so-called *GNS*-representation of quasi-spin operators; such a technique takes its name from I. Gelfand, M. Naimark and I. Segal who devised it and provides a powerful tool in many different context of quantum statistical mechanics [36, 37].

The use of such an algebraic tool deserves a few lines of explanation: in many-body quantum systems with infinitely many constituents, usually density matrices and state vectors lose their meaning as meaningful quantum states. Indeed, in practice one has only an algebra of observables, a so-called positive functional which fixes their mean values and a dynamics which transforms them preserving their algebraic relations. The *GNS*-representation depends only on the algebra and on the assignment of mean values; it naturally leads to a Hilbert space, with a cyclic vector and pure and mixed states, together with a unitary dynamics generated by a well-defined Hamiltonian operator which need not exist at the purely algebraic level of the infinite quantum system. In a standard quantum informational context with finitely many degrees of freedom, the *GNS*-construction reduces to the so-called *purification* that identifies mixed states, that is density matrices, on a given algebra with pure states, that is Hilbert space vectors, on a larger algebra.

In sections 3–5, by means of this technical setting, we single out suitable collective quantum observables, the above-mentioned quantum fluctuations, thus providing a well-behaved algebraic structure at the mesoscopic scale, as N becomes large. This provides the basis for the description of a Josephson junction and the derivation of its large-N dynamics, in section 6. By inserting the junction in a capacitive circuit, we then directly retrieve the phenomenological charge qubit Hamiltonian as the generator of the mesoscopic dynamics. As a byproduct, the temperature dependence of the critical Josephson current in the strong coupling regime is rigorously obtained. The Appendices contain technical details and proofs.

2. Superconducting charge qubits

According to the *BCS* theory [21–27], the transition in metals from the standard to the superconducting phase is due to the presence of an attractive, phonon-mediated interaction among electrons that makes two electrons having opposite momentum and spin become correlated and form a so-called *Cooper pair*. Below a critical temperature T_c, Cooper pairs are created on a macroscopic scale, giving rise to the appearance of a temperature dependent order parameter and of an energy gap, corresponding to the energy necessary to create the first excited state of the system. The existence of an energy gap explains the absence of dissipation by resistance at temperatures smaller than those able to overcome the gap.

By pairing two superconducting electrodes separated by a thin insulating barrier, one can then form a Josephson junction [15–17]. As a consequence of Cooper pairs tunnelling between the barrier, the phases of the (complex) order parameters of the two superconductors are no longer independent and their difference leads to the generation of an electrical current, the Josephson current, even in absence of an external potential across the barrier. Nowadays, Josephson junctions are the basic tools for the construction and manipulation of qubits using superconducting quantum circuits [1–13], i.e. circuits in which electric current, voltage, charge, and flux are promoted to quantum observables.

More precisely, a quantum circuit is a network of metallic, insulating, and semiconducting elements, such as capacitors, inductors, Josephson junctions, which, combined with appropriate voltage sources, control the behaviour of the inside circulating current. The standard approach to modelling these devices consists in first identifying suitable canonically conjugated position and momentum like quantities, constructing with them a classical Hamiltonian yielding the circuit equations of motion and finally quantizing it. To resume: the circuit is built with quantum devices, as are the Josephson junctions, whose macroscopic behaviour within the circuit, though rooted in second quantization, is described by a classical, phenomenological Hamiltonian which is then re-quantized.

In the following, we show that such a phenomenological prescription can actually be rigorously derived from the microscopic *BCS* theory modelling the superconductors involved in the circuits; indeed, through the choice of suitable collective observables it will be possible to recover, in the limit of a large number of particles, the quantum circuit dynamics, without the need of introducing a phenomenologically quantized, macroscopic classical Hamiltonian.

Although the procedure is general, for sake of concreteness we shall focus on a particular class of superconducting circuits, those leading to the so-called *charge qubit* (see figure 1(b)). The latter consists of a large superconducting reservoir connected to a superconducting island through a Josephson junction. The island is attached to a voltage V_g through a gate electrode: upon controlling this external voltage, it is then possible to generate an offset charge in the island. The presence of the junction is required in order to create a non-linearity in the system energy spectrum, so that the two lowest energy levels can be used to construct a logical qubit.

The equations of motion governing the dynamics of such a circuit can be derived from a phenomenological Hamiltonian written in terms of an angle variable $\varphi \in [0, 2\pi]$, the phase drop across the Josephson junction, and its conjugate momentum p_φ, measuring the difference in number of Cooper pairs between reservoir and island:

$$H = \mathcal{E}_C \left(p_\varphi - n_g\right)^2 - \mathcal{E}_J \cos(\varphi) \ , \qquad \mathcal{E}_C \equiv \frac{(2e)^2}{2C} \ , \qquad n_g \equiv \frac{C_g V_g}{2e}, \tag{1}$$

where the total circuit capacitance $C = C_J + C_g$ is the sum of junction, C_J, and gate, C_g, contributions, while $\mathcal{E}_C$ and $\mathcal{E}_J$ are the charging and Josephson energies. The gate voltage V_g controls the induced offset charge n_g, where $2e$ is the Cooper pair charge. In the charging regime, $\mathcal{E}_C \gg \mathcal{E}_J$ only the two lowest-lying charge states, differing by one Cooper pair, are relevant, thus forming a qubit.

The phenomenologically inspired Hamiltonian (1) is classical and needs to be quantized; in order to do that, one has to promote angle and momentum variables to quantum operators $\hat{\varphi}$ and $\hat{p}_\varphi$, acting on the Hilbert space $L^2([0, 2\pi])$ of periodic square integrable functions on the circle. A naive procedure would lead one to introduce the operators:

$$(\hat{p}_\varphi \psi)(\varphi) \mapsto -i\partial_\varphi \psi(\varphi) \ , \qquad (\hat{\varphi}\psi)(\varphi) \mapsto \varphi\psi(\varphi) \, . \tag{2}$$

However, these operators suffer from the fact that the domain of self-adjointness of $\hat{p}_\varphi$ consists of periodic functions that are no longer periodic when multiplied by $\hat{\varphi}$. Therefore, one has to resort to Weyl-like unitary operators:

$$\left(e^{i\alpha\hat{p}_\varphi}\psi\right)(\varphi) = \psi(\varphi + \alpha), \quad \left(e^{in\hat{\varphi}}\psi\right)(\varphi) \mapsto e^{in\varphi}\psi(\varphi), \tag{3}$$

with $n \in \mathbb{Z}$. Then, for $\alpha, \alpha' \in \mathbb{R}$ and $m, n \in \mathbb{Z}$,

$$e^{i\alpha\hat{p}_\varphi} e^{in\hat{\varphi}} = e^{in\alpha} e^{in\hat{\varphi}} e^{i\alpha\hat{p}_\varphi}, \tag{4}$$

which can be equivalently recast as

$$\left[\hat{p}_\varphi, e^{i\hat{\varphi}}\right] = e^{i\hat{\varphi}} \ , \qquad e^{-im\hat{\varphi}} \hat{p}_\varphi e^{im\hat{\varphi}} = \hat{p}_\varphi + m \ , \quad m \in \mathbb{Z} \, . \tag{5}$$

In this way, the angular momentum operator $\hat{p}_\varphi$ and the Weyl phase operator form the *Heisenberg algebra on the circle*. It can be represented on the Hilbert space $L^2\left([0, 2\pi]\right)$ equipped with the orthonormal basis $\{|n\rangle\}$, $n \in \mathbb{Z}$, of 'plane waves',

$$\langle\theta|n\rangle = \frac{1}{\sqrt{2\pi}} e^{in\theta} \, . \tag{6}$$

These vectors are obtained by acting on the 'vacuum' state $\langle\theta|0\rangle = 1/\sqrt{2\pi}$, $\hat{p}_\varphi|0\rangle = 0$,

$$|n\rangle = e^{in\hat{\varphi}}|0\rangle \ ,$$

and, because of (5), are eigenstates of the momentum $\hat{p}_\varphi$, $\hat{p}_\varphi|n\rangle = n|n\rangle$, satisfying the orthogonality condition:

$$\langle n|m\rangle = \langle 0|e^{i(m-n)\hat{\varphi}}|0\rangle = \frac{1}{2\pi}\int_0^{2\pi} d\theta e^{i(m-n)\theta} = \delta_{m,n} \, . \tag{7}$$

3. Strong coupling limit of the *BCS* model

Superconductivity can be described as a microscopic phenomenon originated by the condensation of bounded pairs of electrons, the Cooper pairs. The low-energy pairing physics is very well captured by the *BCS* Hamiltonian [21–27]:

$$H_{BCS} = \sum_{\boldsymbol{k}} \varepsilon_{\boldsymbol{k}} \left(c^\dagger_{\boldsymbol{k}\uparrow} c_{\boldsymbol{k}\uparrow} + c^\dagger_{-\boldsymbol{k}\downarrow} c_{-\boldsymbol{k}\downarrow} \right) - \sum_{\boldsymbol{k}} V_{\boldsymbol{k}\boldsymbol{k}'}\, c^\dagger_{\boldsymbol{k}'\uparrow} c^\dagger_{-\boldsymbol{k}'\downarrow} c_{-\boldsymbol{k}\downarrow} c_{\boldsymbol{k}\uparrow} \,, \tag{8}$$

where $c_{\boldsymbol{k}\sigma}$ are fermionic annihilation operators of electronic states of momentum $\boldsymbol{k}$, spin σ and energy $\varepsilon_{\boldsymbol{k}}$, while $V_{\boldsymbol{k}\boldsymbol{k}'}$ describes an attractive two-body interaction.

An often adopted approximation to this Hamiltonian assumes the pairing potential $V_{\boldsymbol{k}\boldsymbol{k}'}$ to be a positive constant $V \geqslant 0$ for all indices $\boldsymbol{k}, \boldsymbol{k}'$ in a region around the Fermi energy determined by the cutoff ω_{D}, the Debye energy of the crystal where the electrons move, and vanishing outside this region. As the interaction term acts only within the cutoff region around the Fermi energy ε_{F}, focusing on the physics relative to this energy region, one can make an additional simplification and take all the single-electron energy levels $\varepsilon_{\boldsymbol{k}}$ around the Fermi energy to be constant. This is the essence of the so-called *strong-coupling* approximation [28–31], which is justified when $\omega_{\mathrm{D}} \ll V$ so that the $\boldsymbol{k}$-dependence in $|\varepsilon_{\boldsymbol{k}} - \varepsilon_{\mathrm{F}}| < \omega_{\mathrm{D}}$ can be safely ignored.

In addition, notice that the Hamiltonian (8) involves only pair operators of the form $c_{-\boldsymbol{k}\downarrow} c_{\boldsymbol{k}\uparrow}$; they generate an $su(2)$ algebra and therefore they can be represented by quasi-spin matrices [33]:

$$c_{-\boldsymbol{k}\downarrow} c_{\boldsymbol{k}\uparrow} = \sigma_+^{(\boldsymbol{k})} \,, \quad \left(c_{-\boldsymbol{k}\downarrow} c_{\boldsymbol{k}\uparrow}\right)^\dagger = \sigma_-^{(\boldsymbol{k})} \,, \quad c^\dagger_{\boldsymbol{k}\uparrow} c_{\boldsymbol{k}\uparrow} + c^\dagger_{-\boldsymbol{k}\downarrow} c_{-\boldsymbol{k}\downarrow} = 1 - \sigma_z^{(\boldsymbol{k})} \,, \tag{9}$$

where σ_i, $i = x, y, z$, are Pauli matrices, and $\sigma_\pm = (\sigma_x \pm i\sigma_y)/2$. By enumerating the finite number N of available energy levels in the cutoff region with an integer index k, the strong-coupling limit of the *BCS* Hamiltonian (8) can finally be rewritten in the quasi-spin language as [34, 35]

$$H^N = -\varepsilon \sum_{k=1}^{N} \sigma_z^{(k)} - \frac{2\,T_c}{N} \sum_{k,k'=1}^{N} \sigma_+^{(k)} \sigma_-^{(k')}, \tag{10}$$

up to an irrelevant constant contribution. For later convenience we have set $V = 2\,T_c/N$, where the constant T_c plays the role of the critical temperature. Equivalently, in terms of collective spin operators

$$S_i^N = \frac{1}{2} \sum_{k=1}^{N} \sigma_i^{(k)} \,, \quad i = x, y, z, \qquad S_\pm^N = S_x^N \pm i S_y^N \,, \tag{11}$$

one has:

$$H^N = -2\varepsilon\, S_z^N - \frac{2\,T_c}{N}\, S_+^N S_-^N \,. \tag{12}$$

Although a simplified version of the original *BCS* Hamiltonian, the model based on (10) and (12) is able to capture the relevant characteristic features of superconducting devices, including Josephson junctions [31, 34, 38, 39].

In particular, the Hamiltonian (12) can be treated as in the standard case by considering its mean-field approximation,

$$\widetilde{H}^N = -2\varepsilon\, S_z^N - 2\,T_c \left(S_+^N \langle\langle S_-^N \rangle\rangle_\beta^N + \langle\langle S_+^N \rangle\rangle_\beta^N S_-^N \right) \equiv -\sum_{k=1}^{N} \sum_{\nu=1}^{3} \omega_\nu \cdot \sigma_\nu^{(k)} \,, \tag{13}$$

where $\omega_1 = \omega_2 = T_c$, $\omega_3 = \varepsilon$ and $\langle\langle \cdot \rangle\rangle_\beta^N$ is the expectation value with respect to the mean-field Gibbs state at the inverse temperature β:

$$\langle\langle \cdot \rangle\rangle_\beta^N = \frac{\mathrm{Tr}\left[e^{-\beta \widetilde{H}^N} \cdot \right]}{\mathrm{Tr}\left[e^{-\beta \widetilde{H}^N} \right]} \,. \tag{14}$$

The required consistency condition,

$$\beta_c\, \omega = \tanh\left(\beta\, \omega\right) \,, \qquad \omega = |\boldsymbol{\omega}| = \sqrt{\varepsilon^2 + 4\,T_c^2\, \Delta^2} \,, \tag{15}$$

allows for a non-vanishing value of the ‘gap’,

$$\frac{\langle\langle S_+^N \rangle\rangle_\beta^N}{N} \equiv \Delta\, e^{i\phi} \;, \tag{16}$$

only for temperature $T < T_c = 1/\beta_c$, thus recovering the standard result.

Notice that, as defined in (16), the gap is dimensionless and that the relation (15) fixes its modulus, Δ, but not its phase ϕ. This is to be expected as the Hamiltonian (12) is invariant under rotations around the z-axis, corresponding to the gauge invariance of the original *BCS* Hamiltonian, while the three-dimensional vector $\boldsymbol{\omega}$ in (13) points to a given direction. Remarkably, gauge invariance can be recovered, at least in the large-N limit, by averaging over ϕ; indeed, the expectation in the Gibbs state constructed with the Hamiltonian (12),

$$\langle\,\cdot\,\rangle_\beta^N = \mathrm{Tr}\left[\varrho_\beta^N\,\cdot\,\right] \;, \qquad \varrho_\beta^N = \frac{e^{-\beta H^N}}{\mathrm{Tr}\left[e^{-\beta H^N}\right]} \;, \tag{17}$$

can be obtained as [34]:

$$\lim_{N\to\infty} \langle\,\cdot\,\rangle_\beta^N = \lim_{N\to\infty} \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi\; \langle\langle\,\cdot\,\rangle\rangle_\beta^N \;, \tag{18}$$

where $\langle\langle\,\cdot\,\rangle\rangle_\beta^N$ in the r.h.s. is computed with a mean-field Hamiltonian (13) having a gap with phase ϕ.

As an application, let us compute the large-N limit of the average $\langle S_+^N S_-^N\rangle_\beta^N/N^2$, where the rescaling with N is necessary in order to obtain a finite result. Applying (18), one gets:

$$\mathfrak{c}^2 \equiv \lim_{N\to\infty} \frac{1}{N^2}\langle S_+^N S_-^N\rangle_\beta^N = \lim_{N\to\infty} \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi\, \frac{1}{N^2}\langle\langle S_+^N S_-^N\rangle\rangle_\beta^N.$$

In order to compute the mean-field expectation in the r.h.s., let us consider the splitting:

$$\frac{1}{N^2} S_+^N S_-^N = \frac{1}{N^2}\sum_{k=1}^N \sigma_+^{(k)}\sigma_-^{(k)} + \frac{1}{N^2}\sum_{\substack{k,k'=1\\ k\neq k'}}^N \sigma_+^{(k)}\sigma_-^{(k')} \;.$$

The first sum contains N terms, each one a projection $\sigma_+^{(k)}\sigma_-^{(k)} = (1+\sigma_z^{(k)})/2$; as a consequence, the factor $1/N^2$ makes this term vanish in the large-N limit. Instead, the second sum contains $N(N-1)$ terms, each one involving two different spin operators, thus making it generally non-vanishing. Since in the mean-field theory different sites are statistically independent and the Hamiltonian is permutation invariant, the mean-field average of $\sigma_i^{(k)}$ does not depend on k; therefore,

$$\frac{1}{N^2}\sum_{\substack{k,k'=1\\ k\neq k'}}^N \left\langle\!\!\left\langle \sigma_+^{(k)}\sigma_-^{(k')} \right\rangle\!\!\right\rangle_\beta^N = \langle\langle\sigma_+\rangle\rangle_\beta^N\, \langle\langle\sigma_-\rangle\rangle_\beta^N \;,$$

which in turn depends only on the gap modulus, Δ, and not on its phase; thus, finally:

$$\mathfrak{c} = \Delta \;. \tag{19}$$

This result will be useful in deriving the explicit expression of the critical Josephson current.

4. Large-N formalism

In order to implement in a mathematically controlled way the large-N limit of the quasi-spin formulation of the *BCS* model introduced in the previous section, one can resort to standard techniques developed in general for quantum systems with infinitely many degrees of freedom, based on the so-called Gelfand, Neimark and Segal (*GNS*) construction [36].

Quasi-spin systems are typical examples of quasi-local C^* algebras; given any state Ω on one such algebra $\mathcal{A}$, i.e. a functional from the elements of the algebra to the field of complex numbers $\mathbb{C}$, one can associate to it a representation on a Hilbert space endowed with a cyclic vector that reproduces all expectation values relative to the given state functional. In more precise terms, given a state Ω over $\mathcal{A}$, there always exists a cyclic representation π_Ω of $\mathcal{A}$ over an Hilbert space $\mathcal{H}_\Omega$, with a state vector Ψ_Ω, such that $\Omega(X) = \langle\Psi_\Omega|\pi_\Omega(X)|\Psi_\Omega\rangle$ for all elements $X \in \mathcal{A}$. Moreover, the representation is unique up to unitary equivalence.

As a consequence of this *GNS* construction, generic state vectors in $\mathcal{H}_\Omega$ can be obtained by acting with a suitable represented operator $\pi_\Omega(X)$ on the cyclic vector $|\Psi_\Omega\rangle$. Furthermore, any matrix element in the representation space $\mathcal{H}_\Omega$ corresponds to an expectation value with respect to Ω in the abstract formulation. Indeed, if we let $|\psi_A\rangle = \pi_\Omega(A)|\Psi_\Omega\rangle$ and $|\psi_B\rangle = \pi_\Omega(B)|\Psi_\Omega\rangle$ be arbitrary state vectors in $\mathcal{H}_\Omega$, then, for all $X \in \mathcal{A}$,

$$\begin{aligned}\langle\psi_A|\pi_\Omega(X)|\psi_B\rangle &= \langle\Psi_\Omega|\pi_\Omega^\dagger(A)\,\pi_\Omega(X)\,\pi_\Omega(B)|\Psi_\Omega\rangle \\ &= \langle\Psi_\Omega|\pi_\Omega(A^*XB)|\Psi_\Omega\rangle\,.\end{aligned}$$

When dealing with density matrices, the *GNS* construction is achieved via purification of the state. For simplicity, let us illustrate the procedure using the example of a finite-dimensional system with Hilbert space $\mathbb{C}^d$, although the construction is general. The set of observables for such a system is that of $d \times d$ complex matrices, $\mathcal{M}(d,\mathbb{C})$. Any density matrix ϱ describing a state of the system can be given in its spectral decomposition,

$$\varrho = \sum_{i=1}^{d} r_i|r_i\rangle\langle r_i|\,, \quad r_i \geqslant 0\,, \quad \sum_{i=1}^{d} r_i = 1\,, \quad \langle r_i|r_j\rangle = \delta_{ij}\,, \tag{20}$$

in terms of the basis formed by its eigenvectors $|r_i\rangle$. Let us now double the Hilbert space, and consider in it the vector $|\Psi_\varrho\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$ defined by

$$|\Psi_\varrho\rangle = \sum_{i=1}^{d} \sqrt{r_i}|r_i\rangle \otimes |r_i\rangle\,. \tag{21}$$

Then, it turns out that all expectation values of the system observables can be computed as scalar products in $\mathbb{C}^d \otimes \mathbb{C}^d$:

$$\langle X\rangle = \langle\Psi_\varrho|X \otimes \mathbb{I}|\Psi_\varrho\rangle\,, \quad \forall X \in \mathcal{M}(d,\mathbb{C})\,, \tag{22}$$

where $\mathbb{I}$ represents the identity operator. The map $\pi: X \mapsto X \otimes \mathbb{I}$ provides a *representation* of $\mathcal{M}(d,\mathbb{C})$ on the Hilbert space $\mathbb{C}^d \otimes \mathbb{C}^d$ as $\pi(\mathcal{M}(d,\mathbb{C})) = \mathcal{M}(d,\mathbb{C}) \otimes \mathbb{I} \subset \mathcal{M}(d^2,\mathbb{C})$. The advantage of this construction is that the density matrix ϱ is replaced by the pure state vector $|\Psi_\varrho\rangle$. Such a procedure is called *purification*.

Notice that the map $\pi': X \mapsto \mathbb{I} \otimes X$ also provides a representation of $\mathcal{M}(d,\mathbb{C})$, $\pi'(\mathcal{M}(d,\mathbb{C})) = \mathbb{I} \otimes \mathcal{M}(d,\mathbb{C})$): it commutes with $\pi(\mathcal{M}(d,\mathbb{C}))$ and is thus called its *commutant*.

Given a Hamiltonian H for a d-levels system, with spectral decomposition

$$H = \sum_{i=1}^{d} \eta_i|i\rangle\langle i|\,, \tag{23}$$

let ϱ_β be the Gibbs state at inverse temperature β, with spectral decomposition

$$\varrho_\beta = \sum_{i=1}^{d} \varrho_i|i\rangle\langle i|\,, \qquad \varrho_i = \frac{e^{-\beta\eta_i}}{\sum_{i=1}^{d} e^{-\beta\eta_i}}\,; \tag{24}$$

then, the corresponding purified state is given by:

$$|\Psi_\beta\rangle = \sum_{i=1}^{d} \sqrt{\varrho_i}|i\rangle \otimes |i\rangle\,.$$

Unlike ϱ_β which is invariant under the dynamics generated by H, the *GNS* vector $|\Psi_\beta\rangle$ is not. Indeed, one finds:

$$e^{-itH\otimes 1}|\Psi_\beta\rangle = \sum_{i=1}^{d} \sqrt{\varrho_i}\left(e^{-it\eta_i}|i\rangle\right) \otimes |i\rangle.$$

In order to make it time-invariant, one needs to find a suitable representation of the dynamics: this is given by renormalizing the Hamiltonian through the subtraction from $H\otimes 1$ of a contribution from the commutant; namely, $H\otimes 1\mapsto H\otimes 1-1\otimes H$. Then

$$e^{-it(H\otimes 1-1\otimes H)}|\Psi_\beta\rangle=\sum_{i=1}^{d}\sqrt{r_i}\left(e^{-it\eta_i}|i\rangle\right)\otimes\left(e^{it\eta_i}|i\rangle\right)=|\Psi_\beta\rangle.$$

In the case of the *BCS* quasi-spin formulation introduced in the previous section, the representation space is $\mathbb{C}^{2^{2N}}$ and, in the basis $\{|s,s_z\rangle\}$ of eigenstates of the collective Casimir operator $(\mathbf{S}^N)^2$ and spin component S_z^N in (11), the purified state associated with Gibbs state ϱ_β^N in (17) can be expressed as:

$$|\Omega_\beta^N\rangle=\sum_{s=0}^{N/2}\sum_{s_z=-s}^{s}\sqrt{\varrho_\beta^N(s,s_z)}\bigoplus_{\alpha=1}^{d(s)}|s,s_z\rangle_\alpha\otimes|s,s_z\rangle_\alpha\ , \tag{25}$$

with Boltzmann weights:

$$\varrho_\beta^N(s,s_z)=\frac{e^{-\beta\eta^N(s,s_z)}}{\sum_{s'=0}^{N/2}d(s')\sum_{s_z'=-s'}^{s'}e^{-\beta\eta^N(s',s_z')}}\ , \tag{26}$$

where η^N are the eigenvalues of H^N in (12),

$$\eta^N(s,s_z)=-2\varepsilon s_z-\frac{2T_C}{N}\left(s(s+1)-s_z(s_z-1)\right)\ , \tag{27}$$

while $d(s)$ denotes the multiplicity of the irreducible representation with orthonormal basis $|s,s_z\rangle_\alpha$, $1\leqslant\alpha\leqslant d(s)$ [34, 35]. Then, the expectation value of any spin-observable X in the Gibbs state ϱ_β^N can be expressed as the following average:

$$\langle X\rangle_\beta^N=\langle\Omega_\beta^N|X_N\otimes\mathbb{I}|\Omega_\beta^N\rangle\ . \tag{28}$$

In the following, we will always work within the *GNS* representation.

5. Algebra of fluctuations

As mentioned in the Introduction, we are interested in selecting collective quasi-spin operators having a well-defined limit as N becomes large. Specifically, we will identify collective N-spin operators p^N and $E_\pm^N$ which, in the large-N limit, become the momentum operators, p_φ, and angular exponentials, $e^{\pm i\varphi}$, respectively, obeying the commutation relations of the Heisenberg algebra on the circle given in (5).

More precisely, the momentum variable is defined by means of a collective spin component in the z direction renormalized by a contribution from the commutant, without any rescaling with powers of N:

$$p^N\equiv S_z^N\otimes\mathbb{I}-\mathbb{I}\otimes S_z^N\ . \tag{29}$$

The reason for the renormalization is that in this way p^N annihilates the Gibbs state $|\Omega_\beta^N\rangle$ introduced in the previous section:

$$p^N|\Omega_\beta^N\rangle=\sum_{s=0}^{N/2}\sum_{s_z=-s}^{s}\sqrt{\varrho_\beta^N(s,s_z)}\bigoplus_{\alpha=1}^{d(s)}(s_z-s_z)\,|s,s_z\rangle_\alpha\otimes|s,s_z\rangle_\alpha=0\ . \tag{30}$$

The relevant part of the spectrum[6] of p^N is then retrieved by acting on $|\Omega_\beta^N\rangle$ with $S_\pm^N\otimes\mathbb{I}$. Indeed, the $su(2)$ commutation relation between S_z^N and S_+^N leads to

$$\left[p^N,(S_\pm^N)^n\otimes\mathbb{I}\right]=\pm n\left[(S_\pm^N)^n\otimes\mathbb{I}\right]\ . \tag{31}$$

Then, taking into account that, by construction, $(S_\pm^N)^{N+1}=0$, one gets that the vectors $\left[(S_\pm^N)^n\otimes\mathbb{I}\right]|\Omega_\beta^N\rangle$, for $0\leqslant n\leqslant N$, are eigenvectors of p^N:

$$p^N\left[(S_\pm^N)^n\otimes\mathbb{I}\right]|\Omega_\beta^N\rangle=\pm n\left[(S_\pm^N)^n\otimes\mathbb{I}\right]|\Omega_\beta^N\rangle\ . \tag{32}$$

[6] The other part of the spectrum of p^N is associated with its component in the commutant algebra.

Thus, the thermal state $|\Omega_\beta^N\rangle$ constitutes the vacuum in the *GNS*-representation and p^N can be interpreted as the operator counting the number of quasi-spin excitations above it.

As 'conjugate' operator to p^N, it is convenient to introduce the rescaled quantities:

$$E_\pm^N \equiv \frac{S_\pm^N \otimes \mathbb{I}}{\mathfrak{c}N} \,, \tag{33}$$

with $\mathfrak{c}$ as in (19). These operators are not unitary:

$$E_-^N = \left(E_+^N\right)^\dagger \,, \quad E_+^N E_-^N = \frac{S_+^N S_-^N \otimes \mathbb{I}}{(\mathfrak{c}N)^2} \neq \mathbb{I}\,;$$

nevertheless, p^N and $E_\pm^N$ satisfy the characteristic algebraic relation for the Heisenberg algebra on the circle:

$$\left[p^N, E_\pm^N\right] = \pm E_\pm^N. \tag{34}$$

The collective operators p^N and $E_\pm^N$ are examples of quantum fluctuations. Notice that, in order to obtain the Heisenberg algebra on the circle in the large-N limit, as discussed in detail in the following, only $E_\pm^N$ need to be rescaled with N, while no such rescaling is necessary for p^N; due to this 'asymmetry', these operators are sometimes referred to as *abnormal fluctuations* [19].

Remark 1. In analogy with the fluctuations of classical stochastic variables, for which the central limit theorem can be proved, standard quantum fluctuations are defined via the scaling $1/\sqrt{N}$. In the case of quasi-spins, with respect to a state which attributes vanishing mean to the single quasi-spins along x and y the corresponding fluctuations read

$$F^N{}_{x,y} = \frac{1}{2\sqrt{N}} \sum_{k=1}^N \sigma_{x,y}^{(k)} \,. \tag{35}$$

If the many-body quasi-spin global state supports a non-vanishing mean magnetization along z, namely if, with respect to the so-called weak topology induced by the state in the *GNS*-representation,

$$S_z = \lim_{N\to+\infty} \frac{1}{2N} \sum_{i=1}^N \sigma_z \neq 0 \,,$$

then, while

$$\lim_{N\to+\infty} \left[S_x^N \,, S_y^N\right] = 0 \,,$$

in norm, the quantum fluctuations do not commute:

$$\lim_{N\to+\infty} \left[F^N{}_x \,, F^N{}_y\right] = i S_z \,.$$

In other words, the mean-field quantities give rise to classical algebras in the large-N limit, standard quantum fluctuations retain a quantum behaviour on the same limit that embodies commutation relations of the q, p type on the line. As we shall see, the abnormal fluctuations in (29) and (33) do the same but on the circle.

We want now to study the large-N behaviour of correlation functions involving p^N and $E_\pm^N$. In order to simplify the notation, we denote expectation values in the *GNS*-cyclic vector $|\Omega_\beta^N\rangle$ as:

$$\langle\cdot\rangle_\beta^N \equiv \langle\Omega_\beta^N| \cdot |\Omega_\beta^N\rangle \,.$$

Let $\{\alpha_j \in \mathbb{R},\ m_j, n_j \in \mathbb{N}\}, j = 1, 2, \ldots, r$, be a collection of (possibly null) constants; we will focus upon correlation functions of the form:

$$F_\beta^N\left(\{\alpha_j\}, \{m_j\}, \{n_j\}\right) \equiv \left\langle \prod_{j=1}^r e^{i\alpha_j p^N} \left(E_-^N\right)^{n_j} \left(E_+^N\right)^{m_j} \right\rangle_\beta^N \,. \tag{36}$$

Remark 2. Under the norm closure of their algebra, polynomials in the operators $(E_\pm^N)^m$ and $e^{i\alpha p^N}$ generate a C^* algebra. Given a state on such an algebra, a dense subspace of vectors in the associate Hilbert space of the corresponding *GNS*-representation is built with the action of any such product on the *GNS*-cyclic vector. By using the relations (34), one can then express any scalar product in such Hilbert space as a suitable correlation function $F_\beta^N\left(\{\alpha_j\}, \{m_j\}, \{n_j\}\right)$.

The following result will then give a precise meaning to the large-N limit representation of the fluctuation operators p^N and $E_\pm^N$ in terms of momentum $\hat{p}_\varphi$ and angle variable exponentials $e^{\pm i\hat{\varphi}}$, respectively; it defines correlations at the *mesoscopic* scale, in between the microscopic and the classical, macroscopic ones.

Theorem 1 (Mesoscopic correlation functions). *The large-N limit of the correlation functions in* (36) *exists and defines a state functional* Ω *on the Heisenberg algebra on the circle* $\mathcal{C}$ *such that*

$$\lim_{N\to\infty} F_\beta^N\left(\{\alpha_j\},\{m_j\},\{n_j\}\right) = \Omega\left(\prod_{j=1}^r e^{i\alpha_j\hat{p}_\varphi} e^{i\left(m_j-n_j\right)\hat{\varphi}}\right) = \delta_{m,n}\, e^{i\sum_{j=1}^r\sum_{k=1}^j \alpha_k\left(m_j-n_j\right)}\,, \tag{37}$$

where $m\equiv\sum_{j=1}^r m_j$ *and* $n\equiv\sum_{j=1}^r n_j$.

The proof of this result can be found in appendix A.

The Heisenberg algebra on the circle can be represented on the Hilbert space $L^2\left([0,2\pi]\right)$ as discussed at the end of section 2. One thus finds that the Hilbert space scalar product (7) is precisely what one obtains from (37) by setting $\alpha_j=0$, $\forall j$:

$$\lim_{N\to\infty}\left\langle\prod_j \left(E_-^N\right)^{n_j}\left(E_+^N\right)^{m_j}\right\rangle_\beta^N = \Omega\left(e^{i(m-n)\hat{\varphi}}\right) = \delta_{m,n}.$$

Notice that $p^N|\Omega\rangle_\beta^N=0$ yields $F_\beta^N\left(\{\alpha_j\},\{m_j=0\},\{n_j=0\}\right)=1$; therefore, the state functional Ω simply corresponds to taking expectation values with respect to the vacuum $|0\rangle$, and we can simply write: $\Omega(\cdot)=\langle 0|\cdot|0\rangle$. In addition, notice that in the large-N limit the ordering of the powers of $E_\pm^N$ is irrelevant

$$\lim_{N\to\infty}\left\langle\prod_j \left(E_-^N\right)^{n_j}\left(E_+^N\right)^{m_j}\right\rangle_\beta^N = \lim_{N\to\infty}\left\langle\left(E_-^N\right)^{n}\left(E_+^N\right)^{m}\right\rangle_\beta^N = \delta_{m,n}\,.$$

Theorem 1 establishes the precise meaning of the mesoscopic limit that leads from the microscopic algebra of finite size spin observables to the mesoscopic algebra on the circle. In other terms, in the large-N limit, the correlation functions involving $e^{i\alpha p^N}$ and $E_\pm^N$ can be represented as expectations of $e^{i\alpha\hat{p}_\varphi}$ and $e^{im\hat{\varphi}}$, with respect to the pure state $|0\rangle\in L^2([0,2\pi])$.

Definition 1 (Mesoscopic limit). Let $\{X^{(N)}, N\in\mathbb{N}\}$ be a sequence of observables of the quasi-spin system, each one given by a linear combination of powers of e^{ip^N} and $E_\pm^N$. We say that it converges in the *mesoscopic limit* to $X\in\mathcal{C}$ if, for all n, n', m, $m'\in\mathbb{N}$,

$$\lim_{N\to\infty}\left\langle\left(E_-^N\right)^n\left(E_+^N\right)^{n'} X^{(N)}\left(E_+^N\right)^m\left(E_-^N\right)^{m'}\right\rangle_\beta^N = \Omega\left(e^{-i\left(n-n'\right)\hat{\varphi}} X e^{i\left(m-m'\right)\hat{\varphi}}\right)\,, \tag{38}$$

and simply write

$$\underset{N\to\infty}{\text{m-lim}}\, X^{(N)} = X. \tag{39}$$

Remark 3. The left hand side of (38) has always the form of a linear combination of correlation functions such as (36), being $X^{(N)}$ a linear combination of powers of e^{ip^N}, $E_\pm^N$. The right hand side is instead the matrix element of X with respect to the vectors $|n\rangle$ and $|m\rangle$, with generic n, m, thus allowing the complete reconstruction of the operator X.

6. Dynamics of fluctuations

We now turn our attention to the emerging dynamics of the collective fluctuations at the mesoscopic scale. We will first study the fate of the microscopic Hamiltonian of a single superconductor as N becomes large, and the dynamics it induces on the Heisenberg algebra on the circle. Then we shall focus on a system made of two superconductors, coupled through a suitable tunneling term, in order to properly model the dynamics of a Josephson junction.

6.1. Single superconductor

For sake of generality, we shall allow for the presence of a non-vanishing chemical potential by adding a term proportional to S_z^N to the strong-coupling *BCS* Hamiltonian (12):

$$\mathcal{H}^N \equiv H^N + \mu \sum_{k=1}^{N} \sigma_z^{(k)} = -2\left(\varepsilon - \mu\right) S_z^N - \frac{2T_c}{N} S_+^N S_-^N \,, \tag{40}$$

so that for every Cooper pair added to the system, there is an additional energy shift -2μ. As $[H^N, S_z^N] = 0$, the spectrum of $\mathcal{H}^N$ is simply a shift of that of H^N given in (27):

$$\mathcal{H}^N |s, s_z\rangle = \left(\eta^N(s, s_z) + 2\mu s_z\right) |s, s_z\rangle \,.$$

As explained in section 4, the generator of the time evolution in the *GNS*-representation is not simply given by $\mathcal{H}^N \otimes \mathbb{I}$, but by its renormalization through the subtraction of a contribution from the commutant, $\mathcal{H}^N \otimes \mathbb{I} - \mathbb{I} \otimes \mathcal{H}^N = H^N \otimes \mathbb{I} - \mathbb{I} \otimes H^N + 2\mu p^N$; this redefinition does not affect the dynamics of any quasi-spin observable, as they are of the generic form $X \otimes \mathbb{I}$.

At finite N, the time-evolution operator is then given by

$$\mathcal{U}^N(t) \equiv e^{-it\left(\mathcal{H}^N \otimes \mathbb{I} - \mathbb{I} \otimes \mathcal{H}^N\right)} = e^{-it\left(H^N \otimes \mathbb{I} - \mathbb{I} \otimes H^N\right)}\, e^{-it\left(2\mu p^N\right)} \,, \tag{41}$$

and has the advantage of leaving invariant the *GNS*-cyclic vector:

$$\mathcal{U}^N(t)\, |\Omega_\beta^N\rangle = |\Omega_\beta^N\rangle \,. \tag{42}$$

In the large-N limit however, only the last exponential in (41) survives, as it turns out that the strong-coupling *BCS* Hamiltonian H^N does not contribute to the time evolution of fluctuation operators. Indeed,

Theorem 2 (Mesoscopic dynamics). *The evolution operator* (41) *converges in the mesoscopic limit to*

$$\underset{N\to\infty}{\text{m-lim}}\, \mathcal{U}^N(t) = \mathcal{U}(t) \equiv e^{-it\mathcal{H}} = e^{-it(2\mu\hat{p}_\varphi)} \,. \tag{43}$$

The Hamiltonian $\mathcal{H}$ is the generator of time-translations over the circle (for the proof, see the appendix A).

6.2. Josephson junction

As a Josephson junction involves two independent superconducting layers, in order to cope with this new physical situation we need to extend the previous description. In practice, we have to 'double' the construction so far used and in doing so we shall use the label L and R to distinguish quantities referring to either one of the two superconductors. The collective spin operators, now acting on the Hilbert space $\mathbb{C}^{2^N} \otimes \mathbb{C}^{2^N}$, will be denoted by $\mathcal{S}_i^N$ and $\mathcal{T}_i^N$, $i = x, y, z$, for the L and R electrode, respectively:

$$\mathcal{S}_i^N \equiv S_i^N \otimes \mathbb{I}_R \,, \qquad \mathcal{S}_\pm^N \stackrel{\text{def}}{=} \mathcal{S}_x^N \pm i\mathcal{S}_y^N \,, \tag{44}$$

$$\mathcal{T}_i^N \equiv \mathbb{I}_L \otimes S_i^N \,, \qquad \mathcal{T}_\pm^N \stackrel{\text{def}}{=} \mathcal{T}_x^N \pm i\mathcal{T}_y^N \,, \tag{45}$$

with S_i^N as in (11) and $\mathbb{I}_R$, $\mathbb{I}_L$ the identity operator in the R, L subspaces; these operators separately satisfy the commutations relations of two independent $su(2)$ algebras. The common eigenstates of the Casimir $(\boldsymbol{\mathcal{S}}^N)^2$, $(\boldsymbol{\mathcal{T}}^N)^2$, and $\mathcal{S}_z^N$, $\mathcal{T}_z^N$ operators are tensor products $|s, s_z\rangle \otimes |t, t_z\rangle$ of the two, single $su(2)$ basis vectors representations, with s, t ranging from 1 to $N/2$ and $s_z = 0, \pm 1, \cdots \pm s$, $t_z = 0, \pm 1, \cdots \pm t$.

With these quasi-spin operators one construct two strong-coupling *BCS* Hamiltonians,

$$\mathcal{H}_L^N = -2\varepsilon_L \mathcal{S}_z^N - \frac{2\,T_c^L}{N} \mathcal{S}_+^N \mathcal{S}_-^N \,, \qquad \mathcal{H}_R^N = -2\varepsilon_R \mathcal{T}_z^N - \frac{2\,T_c^R}{N} \mathcal{T}_+^N \mathcal{T}_-^N \,, \tag{46}$$

where, for generality, we assumed different energy parameters ε_L, ε_R, and critical temperatures T_c^L, T_c^R in the two layers.

Let us indicate with ϱ_β^N the Gibbs state associated with the total system Hamiltonian $\mathcal{H}_L^N + \mathcal{H}_R^N$, assuming no interactions between the two layers and a common inverse temperature β; it reduces to the tensor product of the corresponding Gibbs states pertaining to the two L- and R-subsystem, $\varrho_\beta^N = \varrho_{\beta,L}^N \otimes \varrho_{\beta,R}^N$, so that, given two observables X_L^N, X_R^N, acting separately on the two layers, their thermal expectation values factorizes:

$$\text{Tr}\left[\varrho_\beta^N X_L^N \otimes X_R^N\right] = \text{Tr}\left[\varrho_{\beta,L}^N X_L^N\right]\, \text{Tr}\left[\varrho_{\beta,R}^N X_R^N\right] \,. \tag{47}$$

The corresponding *GNS*-cyclic vector $|\Omega_\beta^N\rangle$ will then be given by

$$|\Omega_\beta^N\rangle = \sum_{s,s_z}\sum_{t,t_z}\sqrt{\varrho_\beta^N(s,s_z;t,t_z)}\bigoplus_{\alpha_L=1}^{d(s)}\bigoplus_{\alpha_R=1}^{d(t)}\left(|s,s_z\rangle_{\alpha_L}\otimes|s,s_z\rangle_{\alpha_L}\right)\otimes\left(|t,t_z\rangle_{\alpha_R}\otimes|t,t_z\rangle_{\alpha_R}\right)\,, \tag{48}$$

with $\varrho_\beta^N(s,s_z;t,t_z)=\varrho_{\beta,L}^N(s,s_z)\,\varrho_{\beta,R}^N(t,t_z)$, where the r.h.s. quantities are defined as in (26) for both the L and R part, while $d(s)$, $d(t)$ take into account as before the multiplicity of the various $su(2)$ spin-representations.

Having to deal with two superconducting layers, the fluctuation operators in the *GNS* representation will be obtained generalizing the construction of section 5 relative to a single superconductor. Recalling (29) and (33), we shall then use the following definition for the momentum and its conjugate phase operators in the L and R layers:

$$p_L^N = \left(S_z^N\otimes\mathbb{I}_L-\mathbb{I}_L\otimes S_z^N\right)\otimes\mathbb{I}_R\otimes\mathbb{I}_R\,, \tag{49}$$

$$\left(E_{L,\pm}^N\right)^m = \left(\frac{S_\pm^N}{\mathfrak{c}_L N}\right)^m\otimes\mathbb{I}_L\otimes\mathbb{I}_R\otimes\mathbb{I}_R\,, \tag{50}$$

$$p_R^N = \mathbb{I}_L\otimes\mathbb{I}_L\otimes\left(S_z^N\otimes\mathbb{I}_R-\mathbb{I}_R\otimes S_z^N\right), \tag{51}$$

$$\left(E_{R,\pm}^N\right)^m = \mathbb{I}_L\otimes\mathbb{I}_L\otimes\left(\frac{S_\pm^N}{\mathfrak{c}_R N}\right)^m\otimes\mathbb{I}_R\,,\qquad m\in\mathbb{N}\,, \tag{52}$$

with S_i^N, $i=z,\pm$, as in (11), and $\mathfrak{c}_L$, $\mathfrak{c}_R$ connected to the 'gaps' in the two layers as in (19). Since the Hilbert space for the system of two layers is $\mathbb{C}^{2^N}\otimes\mathbb{C}^{2^N}$, its *GNS*-extension will involve the tensor products of four factors. The choice of the ordering of these four spaces is arbitrary, as all these possible choices are unitarily related; the ordering adopted in (49)–(52) is the most convenient for what follows.

One can then check that the analogue of the algebraic relation (34) holds for the two separate subsystems:

$$[p_L^N, E_{L,\pm}^N] = \pm E_{L,\pm}^N\,,\qquad [p_R^N, E_{R,\pm}^N] = \pm E_{R,\pm}^N. \tag{53}$$

Since the operators of one quasi-spin system commute with those of the other one, and the *GNS*-cyclci vector $|\Omega_\beta^N\rangle$ in (48) does not carry correlations between the two quasi-spin systems, the whole analysis of section 5 can be carried over to the new setting. The new target algebra of fluctuations is the tensor product of two commuting algebras on the circle, $\mathcal{C}^L\otimes\mathcal{C}^R$. With obvious adaptation and extension of the notation in (36), correlation functions as

$$\mathcal{F}_\beta^N(\{\alpha\},\{m\},\{n\};\{\alpha'\},\{m'\},\{n'\})\equiv\left\langle\prod_{j=1}^r e^{i\alpha_j p_L^N}\left(E_{L,-}^N\right)^{n_j}\left(E_{L,+}^N\right)^{m_j}\prod_{k=1}^{r'}e^{i\alpha_k' p_R^N}\left(E_{R,-}^N\right)^{n_k'}\left(E_{R,+}^N\right)^{m_k'}\right\rangle_\beta^N,$$

factorize into products of correlation functions of the subsystems:

$$\mathcal{F}_\beta^N(\{\alpha\},\{m\},\{n\};\{\alpha'\},\{m'\},\{n'\}) = \left\langle\prod_{j=1}^r e^{i\alpha_j p_L^N}\left(E_{L,-}^N\right)^{n_j}\left(E_{L,+}^N\right)^{m_j}\right\rangle_\beta^N \times\left\langle\prod_{k=1}^{r'}e^{i\alpha_k' p_R^N}\left(E_{R,-}^N\right)^{n_k'}\left(E_{R,+}^N\right)^{m_k'}\right\rangle_\beta^N.$$

Taking the large-N limit, we can apply theorem 1 to both factors. The above function is then reconstructed as a correlation function on the tensor product of two commuting algebras on the circle, $\mathcal{C}_L\otimes\mathcal{C}_R$, with respect to a factorized state:

$$\begin{aligned}\lim_{N\to\infty}\mathcal{F}_\beta^N(\{\alpha\},\{m\},\{n\};\{\alpha'\},\{m'\},\{n'\}) &= \Omega\left(\prod_{j=1}^r e^{i\alpha_j\hat{p}_{\varphi_L}}e^{i\left(m_j-n_j\right)\hat{\varphi}_L}\prod_{k=1}^{r'}e^{i\alpha_k'\hat{p}_{\varphi_R}}e^{i\left(m_k'-n_k'\right)\hat{\varphi}_R}\right)\\ &= \Omega_L\left(\prod_{j=1}^r e^{i\alpha_j\hat{p}_{\varphi_L}}e^{i\left(m_j-n_j\right)\hat{\varphi}_L}\right)\,\Omega_R\left(\prod_{k=1}^{r'}e^{i\alpha_k'\hat{p}_{\varphi_R}}e^{i\left(m_k'-n_k'\right)\hat{\varphi}_R}\right)\\ &= \left(e^{i\chi}\,\delta_{m,n}\right)\left(e^{i\chi'}\,\delta_{m',n'}\right)\,,\end{aligned}$$

where the phase factors χ and χ' are given by

$$\chi = \exp\left(i\sum_{j=1}^{r}\sum_{k=1}^{j}\alpha_k\left(m_j - n_j\right)\right), \quad \chi' = \exp\left(i\sum_{k=1}^{r'}\sum_{k=1}^{j}\alpha'_k\left(m'_j - n'_j\right)\right),$$

and $m = \sum_{j=1}^{r} m_j$, $n = \sum_{j=1}^{r} n_j$, $m' = \sum_{j=1}^{r'} m'_j$, $n' = \sum_{j=1}^{r'} n'_j$. We can represent $\mathcal{C}_L \otimes \mathcal{C}_R$ onto $L^2\left([0,2\pi]\right) \otimes L^2\left([0,2\pi]\right)$, with the state Ω being the expectation on the vacuum state, realized by the constant function:

$$\langle \boldsymbol{\theta}|0\rangle = \langle \theta_L|0\rangle_L \langle \theta_R|0\rangle_R = \frac{1}{2\pi}, \tag{54}$$

where $\boldsymbol{\theta} = (\theta_L, \theta_R)$. In the following, for sake of compactness, we shall always use the notation $\langle\cdot\rangle = \langle 0|\cdot|0\rangle$ in order to indicate the expectation on $\mathcal{C}_L \otimes \mathcal{C}_R$.

Finally, the definition of the mesoscopic limit of junction observables, analogue to definition 1 for the single *BCS* superconductor, reads as follows:

Definition 2 (Mesoscopic limit). Let $\{X^{(N)},\ N \in \mathbb{N}\}$ be a sequence of observables of the double quasi-spin system, each one given by a linear combination of powers of $e^{ip_L^N}$, $E_{L,\pm}^N$ and $e^{ip_R^N}$, $E_{R,\pm}^N$. For any $m_L, m'_L, m_R, m'_R \in \mathbb{N}$, let us define the operator

$$G^N\left(m_L, m'_L, m_R, m'_R\right) = \left(E_{L,-}^N\right)^{m_L}\left(E_{L,+}^N\right)^{m'_L}\left(E_{R,-}^N\right)^{m_R}\left(E_{R,+}^N\right)^{m'_R}. \tag{55}$$

Then we say that the sequence converges in the *mesoscopic limit* to $X \in \mathcal{C}_L \otimes \mathcal{C}_R$, and write $\underset{N\to\infty}{\text{m-lim}}\, X^{(N)} = X$ if

$$\begin{aligned}\lim_{N\to\infty}&\left\langle\left(G^N\left(n_L, n'_L, n_R, n'_R\right)\right)^\dagger X^{(N)}\, G^N\left(m_L, m'_L, m_R, m'_R\right)\right\rangle_\beta^N \\ &= \left\langle e^{-i\left(n_L - n'_L\right)\hat{\varphi}_L} e^{-i\left(n_R - n'_R\right)\hat{\varphi}_R}\, X\, e^{i\left(m_R - m'_R\right)\hat{\varphi}_R} e^{i\left(m_L - m'_L\right)\hat{\varphi}_L}\right\rangle,\end{aligned} \tag{56}$$

for any $n_L, n'_L, n_R, n'_R, m_L, m'_L, m_R, m'_R \in \mathbb{N}$.

6.3. Charge qubit

Using the techniques and definitions introduced above, we can now discuss the mesoscopic dynamics of the quantum circuit relative to the charge qubit, consisting of a Josephson junction together with an external voltage and a capacitor as discussed in section 2.

The sum of the two Hamiltonians in (46) generates the dynamics of two independent superconducting layers. However, in order to describe the tunneling of Cooper pairs in a Josephson junction, an interaction between the two layers is needed. Such a coupling can be conveniently described by a bilinear Hamiltonian written in terms of the collective spin operators $\mathcal{S}_i$ and $\mathcal{T}_i$ introduced in (44) and (45) as:

$$\widehat{H}^N = \frac{\lambda}{N^2}\left(\mathcal{S}_+^N \mathcal{T}_-^N + \mathcal{S}_-^N \mathcal{T}_+^N\right),$$

where λ is a suitable coupling constant and the scaling N^{-2} takes into account that the passage of Cooper pairs in the junction is a surface effect.

As we shall be working in the *GNS*-representation, we need to extend both the 'free' double-layer Hamiltonian, $H_L^N + H_R^N$, and the tunneling interaction $\widehat{H}^N$ above, as operators acting on the Hilbert space obtained by doubling $\mathbb{C}^{2^N} \otimes \mathbb{C}^{2^N}$, the two layers Hilbert space. Using the same ordering of the four tensor factors in this extended space already introduced in the definitions (49)–(52), one can conveniently define the 'free' Hamiltonian as:

$$H_{\text{free}}^N \equiv \left(H_L^N \otimes \mathbb{I}_L - \mathbb{I}_L \otimes H_L^N\right) \otimes \mathbb{I}_R \otimes \mathbb{I}_R + \mathbb{I}_L \otimes \mathbb{I}_L \otimes \left(H_R^N \otimes \mathbb{I}_R - \mathbb{I}_R \otimes H_R^N\right), \tag{57}$$

where H_L^N, H_R^N are exactly as in (12), but with the parameters ε and T_c replaced by ε_L, T_c^L and ε_R, T_c^R, respectively; one easily checks that H_{free}^N annihilates the *GNS*-vacuum state $|\Omega_\beta^N\rangle$ given in (48). Similarly, the tunnelling Hamiltonian $\widehat{H}^N$ can be simply rewritten in the *GNS*-representation in terms of the fluctuation operators introduced in (50) and (52) as:

$$H_{\text{int}}^N \equiv \lambda \mathfrak{c}_L \mathfrak{c}_R\left(E_{L,+}^N E_{R,-}^N + E_{L,-}^N E_{R,+}^N\right). \tag{58}$$

In order to describe the dynamics of a charge qubit, we need to add a capacitive term to the above Hamiltonian pieces, keeping track of the excess charge induced by to the tunneling of Cooper pairs and the presence of an external gate voltage. Recall that excitations above the thermal state are created by acting with the fluctuation operators $E^N_{L,\pm}$ and $E^N_{R,\pm}$ on the the *GNS*-vacuum $|\Omega^N_\beta\rangle$, the ones with positive sign adding a positive charge $2e$ on the respective superconducting layer (thus removing a Copper pair), while the ones with negative sign adding a negative charge $-2e$ (thus creating a Cooper pair). Thanks to the commutators in (53), the number of such excitations in the two layers is counted by the conjugate momenta p^N_L and p^N_R, so that multiplying them by the Cooper pair charge yields the charge operators on the electrodes. The charge excess between the two layers can then be expressed by their semidifference. Recalling that a capacitive energy is quadratic in the excess charge, the capacitive Hamiltonian term in the *GNS*-representation can be conveniently written as

$$H^N_C = \mathcal{E}_C \left(\frac{p^N_L - p^N_R}{2} - n_g \right)^2 , \tag{59}$$

where $\mathcal{E}_C$ represents the charging energy, while the constant parameter n_g gives the excess charge induced by an external gate voltage as in (1).

At finite N, the Hamiltonian describing the charge qubit circuit can then be expressed as:

$$\mathfrak{H}^N \equiv H^N_{\text{free}} + H^N_{\text{int}} + H^N_C , \tag{60}$$

so that the corresponding time evolution operator is given by

$$U^N(t) = e^{-it\mathfrak{H}^N} . \tag{61}$$

As already anticipated by theorem 2 for a single superconductor, only the last two terms in (60) induce a nontrivial mesoscopic dynamics. Indeed, we have

Theorem 3 (Charge qubit dynamics). *The microscopic time-evolution operator $U^N(t)$ has a well-defined mesoscopic limit,*

$$\underset{N\to\infty}{\text{m-lim}}\, U^N(t) = U(t) \equiv e^{-it\mathfrak{H}} , \tag{62}$$

where the Hamiltonian generating the mesoscopic dynamics on the circle is given by

$$\mathfrak{H} = \mathcal{E}_C \left(\frac{\hat{p}_{\varphi_L} - \hat{p}_{\varphi_R}}{2} - n_g \right)^2 + 2\lambda \mathfrak{c}_L \mathfrak{c}_R \cos(\hat{\varphi}_L - \hat{\varphi}_R) . \tag{63}$$

The proof is provided in appendix B.

Introducing the relative coordinate $\hat{\varphi}$ and momentum $\hat{p}_\varphi$ operators,

$$e^{i\hat{\varphi}} \equiv e^{i\hat{\varphi}_L} e^{-i\hat{\varphi}_R} , \qquad \hat{p}_\varphi \equiv \frac{\hat{p}_{\varphi_L} - \hat{p}_{\varphi_R}}{2} , \qquad \left[\hat{p}_\varphi, e^{i\hat{\varphi}}\right] = e^{i\hat{\varphi}} , \tag{64}$$

the mesoscopic Hamiltonian can be rewritten as

$$\mathfrak{H} = \mathcal{E}_C \left(\hat{p}_\varphi - n_g\right)^2 + \mathcal{E}_J \cos\hat{\varphi} , \qquad \mathcal{E}_J = 2\lambda \mathfrak{c}_L \mathfrak{c}_R , \tag{65}$$

thus retrieving the formal, phenomenological charge qubit Hamiltonian in (1). However, it should be stressed that $\mathfrak{H}$ in (65) is a fully quantum operator obtained as mesoscopic limit of the microscopic Hamiltonian (60): in contrast to the standard phenomenological procedure, $e^{\pm i\hat{\varphi}}$ appearing in (65) are Weyl-like operators, not fixed-phase exponentials.

The Josephson current operator can now be retrieved by computing the time variation of the excess charge in the junction in units of the Cooper pair charge:

$$J(t) = -\dot{\hat{p}}_\varphi = -i[\mathfrak{H}, \hat{p}_\varphi] = \mathcal{E}_J \sin\hat{\varphi} . \tag{66}$$

Recalling the result (19), one sees that the critical Josephson current in the strong coupling regime is proportional to $\mathcal{E}_J$ and therefore to the product of the modulus of the gaps relative to the L and R layers:

$$\mathcal{E}_J = 2\lambda \Delta_L \Delta_R , \tag{67}$$

a result obtained in [40] using phenomenological methods (see also [16, 41]).

Remark 4. The above dependence of the critical current differs from the one found in [32]. However, the approach there pursued is inapplicable in the strong-coupling regime, as some finite contributions diverge when the energy levels of the electrons close to the Fermi surface are independent from their momenta, $\varepsilon_{\boldsymbol{k}} = \varepsilon$. Nevertheless, assuming for simplicity identical superconductor layers, from (67) one can recover the result of [32] in the regime of very small energies, $\varepsilon \ll 2T_c\Delta$. Indeed, in this case, the consistency condition (15) reduces to

$$2\Delta = \tanh\left(2\beta\Delta/\beta_c\right) \ , \tag{68}$$

so that, in this regime, the critical current can be rewritten as:

$$\mathcal{E}_J = \frac{\lambda\beta_c}{2}\,\boldsymbol{\Delta}\tanh\left(\frac{\beta\boldsymbol{\Delta}}{2}\right) \ , \tag{69}$$

in terms of a suitably rescaled gap $\boldsymbol{\Delta} = 4T_c\Delta$. Notice that it is precisely $\boldsymbol{\Delta}$ that should be identified with the measured gap, as it has the dimension of an energy and reproduces the correct phenomenological behaviour [22–25]. Indeed, for small temperatures, $\beta \to \infty$, the r.h.s. of (68) approaches one, so that $\boldsymbol{\Delta}(0) \simeq 2T_c$. On the other hand, close to the critical temperature, $T \simeq T_c$, by expanding the hyperbolic tangent, one instead finds: $\boldsymbol{\Delta}(T) \simeq \sqrt{3}\,\boldsymbol{\Delta}(0)\left(1 - T/T_c\right)^{1/2}$, in good agreement with the known results.

Remark 5. It should be stressed that our treatment of the charge qubit circuit, and in particular of the Josephson junction in it, is fully gauge invariant, since no phase has been fixed in the choice of collective fluctuation operators, nor in the derivation of their mesoscopic limit and dynamics. This is in contrast with the usually adopted approaches, based on the mean-field approximation, in which typically the relative phase of the two-layer gaps is fixed (e.g. see [38, 39]). In particular, the Josephson current operator in (66) can have a non-vanishing expectation value only on states with a fixed phase. Indeed, using the relative coordinates, any state on the circle can be written as:

$$|\psi\rangle = \frac{1}{2}\int_0^{2\pi} \mathrm{d}\varphi\,\psi(\varphi)\,|\varphi\rangle \ , \tag{70}$$

where $\psi(\varphi)$ is a suitable weight function, while the state $|\varphi\rangle \equiv \sum_n e^{in\varphi}|n\rangle$, with $|n\rangle$ as in (6), is formally an eigenstate of the angle operator. Clearly, the average of the Josephson operator (66) on $|\psi\rangle$,

$$\langle J\rangle_\psi = \frac{\mathcal{E}_J}{2\pi}\int_0^{2\pi} \mathrm{d}\varphi\,|\psi(\varphi)|^2\,\sin\varphi \ , \tag{71}$$

will reproduce the expected value only for a smearing function $\psi(\varphi)$ peaked at a given phase $\bar{\varphi}$, for which

$$\langle J\rangle_\psi = \mathcal{E}_J\,\sin\bar{\varphi} \ , \tag{72}$$

thus recovering the standard phenomenological result.

7. Discussion

In dealing with many-body quantum systems, made of N microscopic components, the relevant observables are collective ones, consisting of suitably scaled sums of microscopic operators. Among them, macroscopic averages that scale as the inverse of N provide, in the large-N limit, a description of the emerging commutative, henceforth classical, collective features of such quantum systems. However, other relevant classes of collective observables can be constructed, the so-called quantum fluctuations, scaling with different powers of $1/N$, while retaining quantum features in the large-N limit; for instance, whenever considering states with low correlation content, fluctuations behave as bosonic operators thus obeying canonical commutation relations. These collective observables describe many-body physics at a mesoscopic scale, in between the purely quantum behaviour of microscopic observables and the purely classical one of commuting macroscopic ones.

We have shown that quantum fluctuations are the most appropriate choice of observables for describing non-perturbatively (see [18] for a perturbative approach) the quantum behaviour of superconducting circuits based on Josephson junctions. The dynamics of these systems involves collective phenomena that can not be described by looking at the behaviour of finite-N microscopic constituents, nor at a macroscopic, classical scale. Instead, we have looked at the collective behaviour of superconducting junctions by means of two fluctuation operators: one referring to the excess number of Cooper pairs on the junction and the other to the phase difference between the superconducting condensates. We have found that, in the large-N limit, these quantities behave as conjugated canonical quantum operators, acting like momentum and angle variables for a particle on a circle inside an anharmonic potential. Remarkably, their emergent mesoscopic dynamics is generated by a quantum Hamiltonian of the same form as the phenomenolgical one obtained by quantizing the macroscopic classical Hamiltonian that reproduces the circuit equations. Most importantly, our fully quantum approach provides a derivation of the temperature dependence of the critical current that, in the strong-coupling scenario, is not accessible by means of standard approximation methods. While for very small energies, the dependence on the temperature reduces to the one found perturbatively by Ambegaokar and Baratoff, outside that regime, the dependence predicted by the quantum fluctuation approach is different and in line of principle amenable to experimental verification. Although, for sake of definiteness, our considerations have been focused on a particular class of superconducting circuits, those known as charge qubits, the presented techniques, based on the so-called *GNS*-representation and a strong-coupling, quasi-spin approach, are quite general, and can be applied to model more complex circuits, e.g. phase-qubits, flux-qubits and transmons. In this respect, considering more superconductive circuits, one could study whether and how macroscopic entanglement is allowed by the microscopic constraints to emerge in the large-N limit, beyond phenomenological considerations that would not be able to access its features, like, for instance, its resilience in the presence of decoherence (see for instance [42]). We deem this perspective one of the most intriguing outcomes of our investigation. Along this line, it will be crucial to extend the fluctuation approach beyond the strong-coupling scenario whereby the differences among the quasi-energies are important [26, 27].

Data availability statement

No new data were created or analysed in this study.

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Appendix A. Single superconductor

In this appendix we provide the proofs of theorems 1 and 2: we begin with some useful estimates.

A.1. Preliminary results

Proposition 1. *Given the definition of the collective spin operators in (11), one finds the following norm estimate:*

$$\left\|S^N_{x,y,z}\right\| = \frac{N}{2}\,, \tag{73a}$$

$$\left\|S^N_{\pm}\right\| \leqslant \frac{N+1}{2}\,, \tag{73b}$$

$$\left\|\left[\left(S^N_+\right)^n, \left(S^N_-\right)^m\right]\right\| = \mathcal{O}\left(N^{n+m-1}\right)\,. \tag{73c}$$

Proof. The first result follows from the fact that the norm of an operator X on a finite-dimensional Hilbert space amounts to the square root of the maximum eigenvalue of $X^\dagger X$, while the second one by writing:

$$S^N_{\pm}S^N_{\mp} = \left(\boldsymbol{S}^N\right)^2 - \left(S^N_z\right)^2 \pm S^N_z\,.$$

The third bound can be proved by induction: for $n=1$ and $m=1$,

$$\left\|\left[S^N_+, S^N_-\right]\right\| = \|2S^N_z\| = N = \mathcal{O}(N), \tag{74}$$

while for $n=1$ and generic m, assuming (73c) to hold for $m-1$,

$$\begin{aligned}\left\|\left[S_+^N,\left(S_-^N\right)^m\right]\right\| &\leqslant \left\|S_-^N\left[S_+^N,\left(S_-^N\right)^{m-1}\right]\right\| + \left\|\left[S_+^N,S_-^N\right]\left(S_-^N\right)^{m-1}\right\| \\ &\leqslant \left\|S_-^N\right\|\left\|\left[S_+^N,\left(S_-^N\right)^{m-1}\right]\right\| + \left\|2S_z^N\right\|\left\|S_-^N\right\|^{m-1} \\ &\leqslant \frac{N+1}{2}\left\|\left[S_+^N,\left(S_-^N\right)^{m-1}\right]\right\| + N\left(\frac{N+1}{2}\right)^{m-1} = \mathcal{O}(N^m).\end{aligned}$$

For generic n, we proceed by induction in a similar way. □

Remark 6. The previous estimates yields the following bounds for the norms of the phase operators in (33), with $N\geqslant 1$ and $m,n\in\mathbb{N}$:

$$\left\|\left(E_\pm^N\right)^m\right\| \leqslant \left(\frac{N+1}{2N\mathfrak{c}}\right)^m \leqslant \left(\frac{1}{\mathfrak{c}}\right)^m = \mathcal{O}\left(N^0\right), \tag{75a}$$

$$\left\|\left[\left(E_-^N\right)^n,\left(E_+^N\right)^m\right]\right\| = \mathcal{O}\left(N^{-1}\right). \tag{75b}$$

Concerning the correlation functions in (36), one has the following useful results:

Corollary 1. *The following limit holds:*

$$\lim_{N\to\infty}\left\langle\left(\prod_{j=1}^r\left(E_-^N\right)^{n_j}\left(E_+^N\right)^{m_j}\right)\right\rangle_\beta^N = \lim_{N\to\infty}\left\langle\left(E_-^N\right)^n\left(E_+^N\right)^m\right\rangle_\beta^N, \tag{76}$$

for all choices of non negative integers $\{n_j,m_j\}_{j=1}^r$, *where* $n\equiv\sum_{j=1}^r n_j$ *and* $m\equiv\sum_{j=1}^r m_j$. *Furthermore, if a sequence of (local) observables* $\{X^N\}$ *satisfies*

$$\lim_{N\to\infty}\left\langle\left(X^N\right)^\dagger X^N\right\rangle_\beta^N = \lim_{N\to\infty}\left\|X^N|\Omega_\beta^N\rangle\right\|^2 = 0, \tag{77}$$

then,

$$\lim_{N\to\infty}\left\langle\left(E_-^N\right)^n\left(E_+^N\right)^m X^N\right\rangle_\beta^N = 0. \tag{78}$$

Proof. Equality (76) means that in the large-N limit the operators $E_\pm^N$ that alternate in the product at the left hand side of the equality can be harmlessly regrouped. Indeed, the exchange of, say, $(E_-^N)^{n_1}$ and $(E_+^N)^{m_1}$ must be compensated by their commutator. However, due to (75a) and (75b), the modulus of

$$\left\langle\left[\left(E_-^N\right)^{n_1},\left(E_+^N\right)^{m_1}\right]\left(\prod_{j=2}^r\left(E_-^N\right)^{n_j}\left(E_+^N\right)^{m_j}\right)\right\rangle_\beta^N, \tag{79}$$

is bounded from above by

$$\left\|\left[\left(E_-^N\right)^{n_1},\left(E_+^N\right)^{m_1}\right]\right\|\left(\prod_{j=2}^r\left\|\left(E_-^N\right)^{n_j}\right\|\left\|\left(E_+^N\right)^{m_j}\right\|\right)\left\||\Omega_\beta^N\rangle\right\|^2 = \mathcal{O}\left(N^{-1}\right).$$

The limit (78) follows instead from the Cauchy-Schwarz inequality and the bound (75a):

$$\begin{aligned}\left|\left\langle\left(E_-^N\right)^n\left(E_+^N\right)^m X^N\right\rangle_\beta^N\right| &\leqslant \left\|\left(E_-^N\right)^n\left(E_+^N\right)^m\right\|\left\||\Omega_\beta^N\rangle\right\|\left\|X^N|\Omega_\beta^N\rangle\right\| \\ &\leqslant \left(\frac{1}{\mathfrak{c}}\right)^{n+m}\left\|X^N|\Omega_\beta^N\rangle\right\|,\end{aligned}$$

where we also used that $|\Omega_\beta^N\rangle$ is normalized. □

A.2. Theorem 1: fluctuations in the large-N limit

The proof of theorem 1 relies on the large-N behaviour of the correlation functions in the strong-coupling *BCS* model, and their equivalence to the correlation functions evaluated in mean-field theory, and on corollary 1.

Theorem 1 (Mesoscopic correlation functions). *The large-N limit of the correlation functions in (36) exists and defines a state functional Ω on the Heisenberg algebra on the circle $\mathcal{C}$ such that*

$$\lim_{N\to\infty} F^N_\beta\left(\{\alpha_j\},\{m_j\},\{n_j\}\right) = \Omega\left(\prod_{j=1}^{r} e^{i\alpha_j\hat{p}_\varphi}\, e^{i\left(m_j-n_j\right)\hat{\varphi}}\right) = \delta_{m,n}\, e^{i\sum_{j=1}^{r}\sum_{k=1}^{j}\alpha_k\left(m_j-n_j\right)}\,, \tag{37}$$

where $m\equiv\sum_{j=1}^{r} m_j$ and $n\equiv\sum_{j=1}^{r} n_j$.

Proof. By means of the second algebraic relation in (34), we can exchange the operators inside the correlation functions on the left hand side of (37) above and bring all unitaries generated by p^N to the right. For example, consider the first term of the product:

$$\begin{aligned} e^{i\alpha_1 p^N}\left(E^N_-\right)^{n_1}\left(E^N_+\right)^{m_1} &= e^{-in_1\alpha_1}\left(E^N_-\right)^{n_1} e^{i\alpha_1 p^N}\left(E^N_+\right)^{m_1} \\ &= e^{-i(n_1-m_1)\alpha_1}\left(E^N_-\right)^{n_1}\left(E^N_+\right)^{m_1} e^{i\alpha_1 p^N}. \end{aligned} \tag{80}$$

By iteration one finds:

$$\prod_{j=1}^{r} e^{i\alpha_j p^N}\left(E^N_-\right)^{n_j}\left(E^N_+\right)^{m_j} = \exp\left(i\sum_{j=1}^{r}\sum_{k=1}^{j}\alpha_k\left(m_j-n_j\right)\right)\times\left(\prod_{j=1}^{r}\left(E^N_-\right)^{n_j}\left(E^N_+\right)^{m_j}\right) e^{i\sum_{j=1}^{r}\alpha_j p^N}, \tag{81}$$

so the phase factor in the first equality of (37) is retrieved. Moreover, due to (30), the unitaries generated by p^N leave $|\Omega^N_\beta\rangle$ invariant. Then, we just need to prove that

$$\lim_{N\to\infty}\left\langle\prod_{j=1}^{r}\left(E^N_-\right)^{n_j}\left(E^N_+\right)^{m_j}\right\rangle^N_\beta = \delta_{m,n}\,. \tag{82}$$

Recalling (76), it is sufficient to prove that:

$$\lim_{N\to\infty}\left\langle\left(E^N_-\right)^{n}\left(E^N_+\right)^{m}\right\rangle^N_\beta = \delta_{m,n}\,. \tag{83}$$

Consider $n\neq m$ with $N>n,m>0$; recalling the definition(33),

$$\left(E^N_-\right)^{n}\left(E^N_+\right)^{m}|\Omega^N_\beta\rangle \propto \sum_{s,s_z}\sqrt{\varrho^N(s,s_z)}\,\Gamma(s,s_z,m,n)\bigoplus_{\alpha=1}^{d(s)}|s,s_z+m-n\rangle_\alpha\otimes|s,s_z\rangle_\alpha\,,$$

where $|s,s_z+m-n\rangle_\alpha=0$ if either $s_z+m-n>s$ or $s_z+m-n<-s$. Moreover, the real factor $\Gamma(s,s_z,m,n)$ comes from the action of the operators $E^N_\pm$ on the spin eigenstates: thanks to (75a), it remains bounded as N becomes large. Taking the scalar product of the previous vector with $|\Omega^N_\beta\rangle$ yields

$$\sum_{s',s'_z}\sum_{s,s_z}\sum_{\alpha_1=1}^{d(s)}\sum_{\alpha_2=1}^{d(s')}\sqrt{\varrho^N(s',s'_z)\,\varrho^N(s,s_z)}\,\Gamma(s,s_z,m,n)\times\ {}_{\alpha_1}\langle s',s'_z|s,s_z+m-n\rangle_{\alpha_2}\ {}_{\alpha_1}\langle s',s'_z|s,s_z\rangle_{\alpha_2}=0.$$

Let $m=n$, then

$$\left\langle\left(E^N_-\right)^{n}\left(E^N_+\right)^{n}\right\rangle^N_\beta = \Omega^N_\beta\left(\left(\frac{1}{\mathfrak{c}N}\right)^{2n}\left(S^N_-\right)^{n}\left(S^N_+\right)^{n}\right). \tag{84}$$

Thanks to (18), the large-N limit of the right hand side of (84) reads:

$$\lim_{N\to\infty}\int_0^{2\pi}\frac{\mathrm{d}\varphi}{2\pi}\left(\frac{1}{\mathfrak{c}N}\right)^{2n}\langle\langle\left(S^N_-\right)^{n}\left(S^N_+\right)^{n}\rangle\rangle^N_\beta\,. \tag{85}$$

Recall that $\langle\langle\cdot\rangle\rangle_\beta^N$ was defined in (14) as the Gibbs state corresponding to the mean-field version of the strong-coupling *BCS* Hamiltonian. Explicitly, one has

$$\frac{\left(S_-^N\right)^n\left(S_+^N\right)^n}{N^{2n}}=\sum_{q_1,\cdots,q_n=1}^{N}\sum_{p_1,\cdots,p_n=1}^{N}\frac{\sigma_-^{(q_1)}}{N}\cdots\frac{\sigma_-^{(q_n)}}{N}\frac{\sigma_+^{(p_1)}}{N}\cdots\frac{\sigma_+^{(p_n)}}{N},$$

where all upper indices of type q, respectively p must differ, for $\sigma_\pm^2=0$. Furthermore, when $q_i=p_j$ the corresponding sum $\sum_{q_i}$ vanishes as $1/N$. Then, the mean-field expectations carry no correlations between spins at different locations and attribute site independent mean-values to single site observables. Therefore,

$$\left\langle\left\langle\left(\sigma_-^{(q_1)}\cdots\sigma_-^{(q_n)}\right)\left(\sigma_+^{(p_1)}\cdots\sigma_+^{(p_n)}\right)\right\rangle\right\rangle_\beta^N=\left(\langle\langle\sigma_-\rangle\rangle_\beta^N\right)^n\left(\langle\langle\sigma_+\rangle\rangle_\beta^N\right)^n=\Delta^{2n}=\mathfrak{c}^{2n},$$

where (19) has been used. Thus, the limit in (85) reduces to

$$\lim_{N\to\infty}\int_0^{2\pi}\frac{\mathrm{d}\varphi}{2\pi}\left(\frac{1}{\mathfrak{c}N}\right)^{2n}\frac{N!}{(N-2n)!}\mathfrak{c}^{2n}=1,$$

and (83) is proven. □

A.3. Theorem 2: fluctuations dynamics for a single superconductor

In the following we prove theorem 2 for the dynamics of the single *BCS* system. We shall deal with the evolution operator

$$\mathcal{U}^N(t)\equiv e^{-it\left(\mathcal{H}^N\otimes\mathbb{I}-\mathbb{I}\otimes\mathcal{H}^N\right)}=e^{-it\left(H^N\otimes\mathbb{I}-\mathbb{I}\otimes H^N\right)}e^{-it\left(2\mu p^N\right)}. \tag{86}$$

We show that the Hamiltonian H^N in (12) does not contribute to the time evolution of the fluctuation operators in the large-N limit. In order to do so, we introduce and characterize a new operator.

Proposition 2. *The operator $E^N(t)$ defined by*

$$E_\pm^N(t)\equiv e^{-it\left(H^N\otimes\mathbb{I}-\mathbb{I}\otimes H^N\right)}E_\pm^N e^{it\left(H^N\otimes\mathbb{I}-\mathbb{I}\otimes H^N\right)}, \tag{87}$$

is such that

$$E_\pm^N(t)=E_\pm^N W^N(t), \tag{88}$$

where $W^N(t)$ is the following unitary time-evolution generated by a Hamiltonian K^N:

$$W^N(t)\equiv e^{-itK^N},\quad K^N\equiv-2\varepsilon\,\mathbb{I}+\frac{2T_c}{N}S_z^N. \tag{89}$$

Furthermore, $W^N(t)$ satisfies the following properties for $m\in\mathbb{N}$:

$$\left\|\left[E_\pm^N,W^N(t)\right]\right\|=\mathcal{O}\left(N^{-1}\right), \tag{90a}$$

$$\lim_{N\to\infty}\left\langle\left(W^N(t)\right)^m\right\rangle_\beta^N=1, \tag{90b}$$

$$\lim_{N\to\infty}\left\|\left(\left(W^N(t)\right)^m-1\right)|\Omega_\beta^N\rangle\right\|=0. \tag{90c}$$

Proof. Since $E_-^N=(E_+^N)^\dagger$, we concentrate on E_+^N. The expression (88) for $E_+^N(t)$ can be computed by exploiting the Baker–Campbell–Hausdorff formula:

$$E_+^N(t)=E_+^N-it\left[\left(H^N\otimes\mathbb{I}-\mathbb{I}\otimes H^N\right),E_+^N\right]+\mathcal{O}(t). \tag{91}$$

Since E_+^N is proportional to $S_+^N \otimes \mathbb{I}$, all commutators with $\mathbb{I} \otimes H^N$ vanish, while

$$\left[H^N, E_+^N\right] = \frac{1}{\mathfrak{c}N}\left[H^N, S_+^N\right] = \frac{1}{\mathfrak{c}N}\left(-2\varepsilon S_+^N + \frac{2T_c}{N} S_+^N S_z^N\right) = E_+^N K^N . \tag{92}$$

Since $\left[H^N, K^N\right] = 0$, substituting (92) into (91) yields

$$\left[H^N, \left[H^N, E_+^N\right]\right] = \left[H^N, E_+^N\right] K^N = E_+^N \left(K^N\right)^2 .$$

Iterating the procedure, (91) becomes

$$E_+^N(t) = E_+^N \sum_{k=1}^{\infty} \frac{(-it)^k}{k!} \left(K^N\right)^k = E_+^N e^{-itK^N} = E_+^N W^N(t) ,$$

thus proving the first statement in (88). On the other hand, from

$$\left[E_+^N, W^N(t)\right] = E_+^N W^N(t) \left(1 - e^{-it2T_c/N}\right) ,$$

and (75), one estimates

$$\left\|\left[E_+^N, W^N(t)\right]\right\| \leqslant \underbrace{\|E_+^N\|}_{=1/\mathfrak{c}} \|W^N(t)\| \left|1 - e^{-it2T_c/N}\right| = \mathcal{O}(N^{-1}),$$

so that (90*a*) is proven. Let us now turn our attention to (90*b*). We have

$$\langle \left(W^N(t)\right)^m \rangle_\beta^N = e^{it2m\varepsilon} \Omega_\beta^N \left(e^{-it2mT_c S_z^N/N}\right) . \tag{93}$$

Thanks to the results of Thirring in [30], we know that S_z^N/N converges to ε/T_c, hence

$$\lim_{N\to\infty} \langle \left(W^N(t)\right)^m \rangle_\beta^N = e^{itm2\varepsilon} e^{-itm2\varepsilon} = 1 . \tag{94}$$

The limit in (90*c*) follows from

$$\begin{aligned} \left\| \left(\left(W^N(t)\right)^m - 1\right) |\Omega_\beta^N\rangle \right\|^2 &= \left\langle \left(\left(W^N(t)\right)^m - 1\right)^\dagger \left(\left(W^N(t)\right)^m - 1\right) \right\rangle_\beta^N \\ &= 2\left(1 - \mathrm{Re}\langle \left(W^N(t)\right)^m \rangle_\beta^N\right) , \end{aligned} \tag{95}$$

and the last term goes to zero thanks to (94). □

Proposition 2 together with corollary 1 can be used to prove the following theorem which characterises the evolution operator in the mesoscopic limit.

Theorem 2 (Mesoscopic dynamics). *The evolution operator* (41) *converges in the mesoscopic limit to*

$$\underset{N\to\infty}{\text{m-lim}} \mathcal{U}^N(t) = \mathcal{U}(t) \equiv e^{-it\mathcal{H}} = e^{-it(2\mu\hat{p}_\varphi)} . \tag{43}$$

Notice that we are going to show that the strong coupling Hamiltonian does not induce any mesoscopic dynamics, in agreement with the fact that the thermal state is an equilibrium state with respect to this evolution.

Recalling the definition of the mesoscopic limit in (38), we shall now introduce some useful short-hand notation to deal with the matrix elements appearing in the formulas. For any $n \in \mathbb{Z}$ we define the states

$$|n\rangle \equiv e^{in\hat{\varphi}}|0\rangle , \qquad |n\rangle_\beta^N \equiv \begin{cases} \left(E_+^N\right)^n |\Omega_\beta^N\rangle & \text{if } n \geqslant 0, \\ \left(E_-^N\right)^{-n} |\Omega_\beta^N\rangle & \text{if } n < 0. \end{cases} \tag{96}$$

The states $|n\rangle$ form the orthonormal basis (6) in the Hilbert space of the limit Heisenberg algebra on the circle; the states $|n\rangle_\beta^N$ are instead quasi-spin states. Notice that the integer labels of the states can be either positive or negative, while the powers of the collective quasi-spin operators operators are always non-negative. In this way for example, the main statement of theorem (2) can be recast as

$$\lim_{N\to\infty} \langle n|\mathcal{U}^N(t)|m\rangle_\beta^N = \langle n|\mathcal{U}(t)|m\rangle, \tag{97}$$

for any $n, m \in \mathbb{Z}$. We can now turn our attention to the theorem itself.

Proof. Though we need prove (97) for any $n, m \in \mathbb{Z}$, we restrict to $n,\ m \geqslant 0$ since the proofs for all other cases are analogous.

Recalling the algebraic relation (34), and defining $\widetilde{\mathcal{U}}^N(t) \equiv e^{-it(H^N\otimes\mathbb{I}-\mathbb{I}\otimes H^N)}$, one has:

$$\langle n|\mathcal{U}^N(t)|m\rangle_\beta^N = e^{-it2\mu m}\left\langle \left(E_-^N\right)^n \widetilde{\mathcal{U}}^N(t)\left(E_+^N\right)^m e^{-it2\mu p^N}\right\rangle_\beta^N = e^{-it2\mu m}\langle n|\widetilde{\mathcal{U}}^N(t)|m\rangle_\beta^N,$$

where the last equality follows from $p^N|\Omega_\beta^N\rangle = 0$, as proven in (30). Similarly, using the Heisenberg algebra on the circle,

$$\langle n|\mathcal{U}(t)|m\rangle = e^{-it2\mu m}\langle n|m\rangle = e^{-it2\mu m}\delta_{n,m}\,.$$

Then, it remains to prove that

$$\lim_{N\to\infty}\langle n|\widetilde{\mathcal{U}}^N(t)|m\rangle_\beta^N = \delta_{n,m} = \lim_{N\to\infty}\left\langle \left(E_-^N\right)^n\left(E_+^N\right)^m\right\rangle,$$

where the second equality follows from (83). We now have

$$\begin{aligned}\langle n|\widetilde{\mathcal{U}}^N(t)|m\rangle_\beta^N &= \left\langle \left(E_-^N\right)^n\left(E_+^N(t)\right)^m\widetilde{\mathcal{U}}^N(t)\right\rangle_\beta^N = \left\langle \left(E_-^N\right)^n\left(E_+^N(t)\right)^m\right\rangle_\beta^N \\ &= \left\langle \left(E_-^N\right)^n\left(E_+^N W^N(t)\right)^m\right\rangle_\beta^N.\end{aligned}$$

For the first equality, we used (87) and the unitarity of $\widetilde{\mathcal{U}}^N(t)$; for the second one that $\widetilde{\mathcal{U}}^N(t)|\Omega_\beta^N\rangle = |\Omega_\beta^N\rangle$. Finally, we substituted $E_+^N(t) = E_+^N W_t^N$ from (88) in proposition 2 which asserts that $\|[E_+^N, W^N(t)]\| = \mathcal{O}(N^{-1})$. Therefore, in the large-N limit we can exchange the position of all the E_+^N and the $W^N(t)$, so that:

$$\begin{aligned}\lim_{N\to\infty}\langle n|\widetilde{\mathcal{U}}^N(t)|m\rangle_\beta^N &= \lim_{N\to\infty}\left\langle \left(E_-^N\right)^n\left(E_+^N W^N(t)\right)^m\right\rangle_\beta^N \\ &= \lim_{N\to\infty}\left\langle \left(E_-^N\right)^n\left(E_+^N\right)^m\left(W^N(t)\right)^m\right\rangle_\beta^N.\end{aligned}$$

Then, it remains to be proved that

$$\lim_{N\to\infty}\left\langle \left(E_-^N\right)^n\left(E_+^N\right)^m\left(\left(W^N(t)\right)^m - 1\right)\right\rangle_\beta^N = 0\,.$$

From (90*c*) in proposition 2, the norm $\|\left((W^N(t))^m - 1\right)|\Omega_\beta^N\rangle\|$ vanishes when $N\to\infty$. Therefore, we can directly apply (78) from corollary 1, using $X^N = (W^N(t))^m - 1$. It follows that the limit on the right hand side of the previous equation vanishes as well. □

Finally, we provide a result that will often be used in appendix B.

Proposition 3. *Let $\{m_j, n_j \in \mathbb{N}, j = 1,\dots,r\}$ be a finite sequence of non-negative integers. Then the following mesoscopic limit holds:*

$$\underset{N\to\infty}{\text{m-lim}}\left(\prod_{j=1}^r\left(E_-^N(t)\right)^{n_j}\left(E_+^N(t)\right)^{m_j}\right) = \underset{N\to\infty}{\text{m-lim}}\left(E_-^N\right)^n\left(E_+^N\right)^m = \delta_{n,m}, \tag{98}$$

where we set $m = \sum_{j=1}^r m_j$, $n = \sum_{j=1}^r n_j$.

Proof. Let us set

$$\begin{aligned}P_t^N(\{m\},\{n\}) &\equiv \left(\prod_{j=1}^r\left(E_-^N(t)\right)^{n_j}\left(E_+^N(t)\right)^{m_j}\right) = \left(\prod_{j=1}^r\left(\left(W^N(t)\right)^\dagger E_-^N\right)^{n_j}\left(E_+^N W^N(t)\right)^{m_j}\right) \\ &= \left(E_-^N\right)^n\left(E_+^N\right)^m\left(W^N(t)\right)^{m-n} + \Theta^N.\end{aligned} \tag{99}$$

In the second equality we directly substituted (88); while in the third one we exchanged the operators $W^N(t)$ and $E_\pm^N$, thus obtaining a remainder Θ^N containing the necessary commutators. Then, the following bound holds

$$\left\|P_t^N(\{m\},\{n\}) - \left(E_-^N\right)^n\left(E_+^N\right)^m\left(W^N(t)\right)^{m-n}\right\| = \|\Theta^N\| = \mathcal{O}\left(N^{-1}\right). \tag{100}$$

Indeed, thanks to (75*b*) and (90*a*), all commutators vanish as $1/N$.

In order to prove the first equality of (98), we consider the matrix element

$$\mathcal{M}^{(N)} \equiv \langle n'| \left(P_t^N(\{m\},\{n\}) - \left(E_-^N\right)^n \left(E_+^N\right)^m \right) |m'\rangle_\beta^N, \tag{101}$$

where in general $m', n' \in \mathbb{Z}$ and show that they vanish as N grows large. It is sufficient to restrict to $m', n' \geqslant 0$ as the proofs for all other cases are analogous. Thanks to the bound (100) we have

$$\begin{aligned} |\mathcal{M}^{(N)}| &\leqslant \left| \langle n'| \left(E_-^N\right)^n \left(E_+^N\right)^m \left(\left(W^N(t)\right)^{m-n} - 1 \right) |m'\rangle_\beta^N \right| + \mathcal{O}\left(N^{-1}\right) \\ &\leqslant \left| \left\langle \left(E_-^N\right)^{n+n'} \left(E_+^N\right)^{m+m'} \left(\left(W^N(t)\right)^{m-n} - 1 \right) \right\rangle_\beta^N \right| + \mathcal{O}\left(N^{-1}\right), \end{aligned} \tag{102}$$

where in the second line we further exchanged $(W^N(t))^{m-n}$ and $(E_+^N)^{m'}$ at the cost of introducing a commutator, which nonetheless vanishes in the large-N limit thanks once again to (90*a*).

Finally, from (90*c*) in proposition 2, the norm $\|\left((W^N(t))^m - 1\right)|\Omega_\beta^N\rangle\|$ vanishes when $N \to \infty$. Therefore, we can directly apply (78) from corollary 1, using $X^N = (W^N(t))^m - 1$. It follows that the limit on the right hand side of (102) vanishes as well, thus proving the mesoscopic limit in (98). □

Appendix B. Mesoscopic dynamics: charge qubits

In this appendix we provide the proof of theorem 3 for the dynamics of a charge qubit system. In section B.1 we begin by introducing the relative coordinates for the charge qubit system and some new notations.

B.1. Notation and relative coordinates

As in the previous appendix, we shall adopt the following notation for vectors in the Hilbert space on the circle, and for quasi-spin vectors at finite N; for any $n_L, n_R \in \mathbb{Z}$:

$$|n_L, n_R\rangle \equiv e^{i(n_L\hat{\varphi}_L + n_R\hat{\varphi}_R)}|0\rangle\ , \tag{103}$$

$$|n_L, n_R\rangle_\beta^N \equiv \begin{cases} \left(E_{L,+}^N\right)^{n_L} \left(E_{R,+}^N\right)^{n_R} |\Omega_\beta^N\rangle & \text{if } n_L \geqslant 0, n_R \geqslant 0 \\ \left(E_{L,-}^N\right)^{-n_L} \left(E_{R,+}^N\right)^{n_R} |\Omega_\beta^N\rangle & \text{if } n_L < 0, n_R \geqslant 0 \\ \left(E_{L,+}^N\right)^{n_L} \left(E_{R,-}^N\right)^{-n_R} |\Omega_\beta^N\rangle & \text{if } n_L \geqslant 0, n_R < 0 \\ \left(E_{L,-}^N\right)^{-n_L} \left(E_{R,-}^N\right)^{-n_R} |\Omega_\beta^N\rangle & \text{if } n_L < 0, n_R < 0\ . \end{cases} \tag{104}$$

The mesoscopic limit to be proved in theorem 3 then reads

$$\lim_{N\to\infty} \langle n_L', n_R'| U^N(t) |n_L, n_R\rangle_{N,\beta} = \langle n_L', n_R'| U(t) |n_L, n_R\rangle\ , \quad \forall n_{L/R}, n_{L/R}' \in \mathbb{Z}\ . \tag{105}$$

Let us introduce the *relative coordinates* for both the microscopic and mesoscopic system:

$$\hat{p}_\varphi \equiv \frac{\hat{p}_{\varphi_L} - \hat{p}_{\varphi_R}}{2}\ , \qquad e^{i\hat{\varphi}} \equiv e^{i\hat{\varphi}_L} e^{-i\hat{\varphi}_R}\ , \tag{106a}$$

$$p^N \equiv \frac{p_{\varphi_L^N} - p_{\varphi_R^N}}{2}\ , \qquad \left(\mathfrak{E}_\pm^N\right)^m \equiv \left(E_{\pm,L}^N\right)^m \left(E_{\mp,R}^N\right)^m\ , \quad m > 0\ . \tag{106b}$$

The usual algebraic relations for momentum-angle variables hold, that is, for $\alpha \in \mathbb{R}$ and $m \in \mathbb{N}$,

$$\left[\hat{p}_\varphi, e^{\pm i\hat{\varphi}}\right] = \pm e^{i\hat{\varphi}}\ , \qquad e^{i\alpha\hat{p}_\varphi} e^{im\hat{\varphi}} = e^{im\alpha} e^{im\hat{\varphi}} e^{i\alpha\hat{p}_\varphi}\ , \tag{107}$$

$$\left[p^N, \mathfrak{E}_\pm^N\right] = \pm\mathfrak{E}_\pm^N\ , \qquad e^{i\alpha p^N} \left(\mathfrak{E}_\pm^N\right)^m = e^{\pm im\alpha} \left(\mathfrak{E}_\pm^N\right)^m e^{i\alpha p^N}\ . \tag{108}$$

Notice that the bound (75*a*) extends directly to $\mathfrak{E}^N$:

$$\left\| \left(\mathfrak{E}_\pm^N\right)^m \right\| \leqslant \left(\frac{1}{\mathfrak{c}_L \mathfrak{c}_R} \right)^m, \quad m > 0\ . \tag{109}$$

Recall the definitions of the capacitive term in (59), of the tunnelling term in (58) and of the mesoscopic Hamiltonian (65):

$$H_C^N = \mathcal{E}_\mathrm{C} \left(p^N - n_\mathrm{g}\right)^2, \quad H_\mathrm{int}^N = \frac{\mathcal{E}_\mathrm{J}}{2} \left(\mathfrak{E}_+^N + \mathfrak{E}_-^N\right), \tag{110}$$

$$H = \mathcal{E}_{\rm C}\left(\hat{p}_\varphi - n_{\rm g}\right)^2 + \mathcal{E}_{\rm J}\cos\hat{\varphi}\,. \tag{111}$$

Let us now split both the finite-N Hamiltonian (60) and the mesoscopic one in two pieces, a free Hamiltonian and a perturbing term:

$$\mathfrak{H}^N = H_0^N + H_1^N, \quad H = H_0 + H_1. \tag{112}$$

where

$$H_0^N = H_{\rm free}^N + H_{\rm C}^N\,, \qquad H_1^N = H_{\rm int}^N\,, \tag{113a}$$

$$H_0 = 4\mathcal{E}_{\rm C}\left(\hat{p}_\varphi - n_{\rm g}\right)^2\,, \qquad H_1 = \mathcal{E}_{\rm J}\cos\hat{\varphi}\,, \tag{113b}$$

with corresponding time-evolution operators:

$$U_0^N(t) = e^{-itH_0^N}\,, \qquad U_0(t) = e^{-itH_0}\,. \tag{114}$$

Notice that the free microscopic dynamics $U_0^N(t)$ generated by $H^N{}_0$ is made of two commuting contributions: one coming from the *BCS* Hamiltonian, and the other from the capacitive part, namely

$$U_0^N(t) = U_{\rm free}^N(t)\, U_{\rm C}^N(t)\,, \tag{115}$$

where

$$U_{\rm free}^N(t) = e^{-itH_{\rm free}^N}\,, \qquad U_{\rm C}^N(t) = e^{-itH_{\rm C}^N}\,.$$

B.2. Josephson junctions

Theorem 2 states that, in the case of a single supercondutor, the term corresponding to $U_{\rm free}^N(t)$, does not contribute to the mesoscopic dynamics. Clearly, the same result holds in presence of two independent superconductors:

$$\lim_{N\to\infty}\langle n_L', n_R'|U_{\rm free}^N|n_L, n_R\rangle = \delta_{n_L,n_L'}\,\delta_{n_R,n_R'}\,, \tag{116}$$

for any $n_{L/R}, n'_{L/R}\in\mathbb{Z}$. Analogously, extending the definition in (87) to two superconductors,

$$\mathfrak{E}_\pm^N(t) \equiv U_{\rm free}^N(t)\;\mathfrak{E}_\pm^N\;\left(U_{\rm free}^N(t)\right)^\dagger\,. \tag{117}$$

Moreover, from proposition 3 it follows that

$$\underset{N\to\infty}{\text{m-lim}}\left(\prod_{j=1}^r\left(\mathfrak{E}_-^N(t)\right)^{n_j}\left(\mathfrak{E}_+^N(t)\right)^{m_j}\right) = \underset{N\to\infty}{\text{m-lim}}\left(\mathfrak{E}_-^N\right)^n\left(\mathfrak{E}_+^N\right)^m = \delta_{n,m}. \tag{118}$$

Notice also that $\mathfrak{E}_\pm^N(t)$ satisfy the usual algebraic relations, thanks to the commutativity of $H_{\rm free}^N$ and p^N:

$$\left[p^N, \mathfrak{E}_\pm^N(t)\right] = \pm\mathfrak{E}_\pm^N(t)\,, \qquad e^{i\alpha p^N}\left(\mathfrak{E}_\pm^N(t)\right)^m = e^{\pm im\alpha}\left(\mathfrak{E}_\pm^N(t)\right)^m e^{i\alpha p^N}\,, \quad \alpha\in\mathbb{R},\ m\in\mathbb{N}\,. \tag{119}$$

The capacitive Hamiltonians $H_{\rm C} = \mathcal{E}_{\rm C}(\hat{p}_\varphi - n_{\rm g})^2$ and $H_{\rm C}^N$ are instead functions only of the relative angular momenta $\hat{p}_\varphi$ and p^N. In order to compactify the notation, let

$$\xi(X) \equiv \mathcal{E}_{\rm C}\left(X - n_{\rm g}\right)^2\,, \tag{120}$$

for any operatorial or scalar quantity X. For instance, $H_{\rm C} = \xi(p_\varphi)$, $H_{\rm C}^N = \xi(p_\varphi^N)$, so that their eigenvalue equations read:

$$H_{\rm C}|n_L, n_R\rangle = \xi(n)\,|n_L, n_R\rangle\,, \tag{121}$$

$$H_{\rm C}^N|n_L, n_R\rangle_\beta^N = \xi(n)\,|n_L, n_R\rangle_\beta^N\,,$$

with

$$n = \frac{n_L - n_R}{2}\,, \tag{123}$$

where one exploits (106). Moreover, for operatorial and scalar quantities X, Y we set

$$\Delta\xi(X,Y) \equiv \xi(X+Y) - \xi(X), \tag{124}$$

so that

$$\Delta\xi(X,\pm 1) = \xi(X\pm 1) - \xi(X) = \mathcal{E}_C\left(1 \pm 2\left(X - n_g\right)\right) , \tag{125}$$

is a linear function of X. Therefore, exploiting (107), (108) and (119) we can write, for $m \in \mathbb{N}$ and $t \in \mathbb{R}$,

$$e^{-it\Delta\xi(\hat{p}_\varphi,\pm 1)} e^{im\hat{\varphi}} = e^{im\hat{\varphi}} e^{-it\Delta\xi(\hat{p}_\varphi + m,\pm 1)} , \tag{126a}$$

$$e^{-it\Delta\xi\left(p^N,\pm 1\right)} \left(\mathfrak{E}_+^N\right)^m = \left(\mathfrak{E}_+^N\right)^m e^{-it\Delta\xi\left(p^N \pm m,\pm 1\right)} , \tag{126b}$$

$$e^{-it\Delta\xi\left(p^N,\pm 1\right)} \left(\mathfrak{E}_+^N(t)\right)^m = \left(\mathfrak{E}_+^N(t)\right)^m e^{-it\Delta\xi\left(p^N \pm m,\pm 1\right)} . \tag{126c}$$

Analogous expressions hold true for the operators $\mathfrak{E}_-^N$ and $\mathfrak{E}_-^N(t)$.

B.3. The Dyson series: useful results

Let us consider ${H^N}_1, H_1$ as perturbations with respect to the free Hamiltonians ${H^N}_0$ and H_0. Let us define the time-dependent potentials

$$V^N(t) \equiv U_0^N(t) H_1^N \left(U_0^N(t)\right)^\dagger , \tag{127a}$$

$$V(t) \equiv U_0(t) H_1 U_0^\dagger(t), \tag{127b}$$

and introduce the finite N, quasi-spin and mesoscopic Dyson series

$$\mathcal{D}^N(t) \equiv 1 + \sum_{k=1}^{\infty} (-i)^k \int_0^t \mathrm{d}t_1 \cdots \int_0^{t_{k-1}} \mathrm{d}t_k V^N(t_k) \cdots V^N(t_1) , \tag{128a}$$

$$\mathcal{D}(t) \equiv 1 + \sum_{k=1}^{\infty} (-i)^k \int_0^t \mathrm{d}t_1 \cdots \int_0^{t_{k-1}} \mathrm{d}t_k V(t_k) \cdots V(t_1) . \tag{128b}$$

The convergence of the two series on their respective Hilbert spaces is the content of the following lemma.

Lemma 1. *The Dyson series $\mathcal{D}(t)$ in (128b) converges in norm to $U(t)U_0^\dagger(t)$ and $\mathcal{D}^N(t)$ in (128a) to $U^N(t)\left(U_0^N(t)\right)^\dagger$, the latter convergence being uniform with respect to N.*

Proof. The proofs of the convergence of the two series are identical, thus we provide only that of $\mathcal{D}^N(t)$. Given the partial sums

$$D_K^N(t) \equiv 1 + \sum_{k=1}^{K} (-i)^k \int_0^t \mathrm{d}t_1 \cdots \int_0^{t_{K-1}} \mathrm{d}t_K V^N(t_k) \cdots V^N(t_1) ,$$

one has to show that that $\lim_{K\to\infty} \| U^N(t)\left(U_0^N(t)\right)^\dagger - D_K^N(t) \| = 0$. Writing

$$U^N(t) - U_0^N(t) = \int_0^t \mathrm{d}t_1 \frac{\mathrm{d}}{\mathrm{d}t_1}\left(U^N(t_1) U_0^N(t-t_1)\right)$$

$$= -i \int_0^t \mathrm{d}t_1 U^N(t_1) H_1^N U_0^N(t-t_1) ,$$

and inserting (127a), one finds:

$$U^N(t)\left(U_0^N(t)\right)^\dagger = 1 - i\int_0^t \mathrm{d}t_1 U^N(t_1)\left(U_0^N(t_1)\right)^\dagger V^N(t_1) = D_K^N(t) + \Theta_{K+1}^N(t) ,$$

where the remainder takes the form

$$\Theta_{K+1}^N(t) \equiv (-i)^{K+1} \int_0^t \mathrm{d}t_1 \cdots \int_0^{t_K} \mathrm{d}t_{K+1} U^N(t_{K+1})\left(U_0^N(t_{K+1})\right)^\dagger V^N(t_{K+1}) \cdots V^N(t_1) .$$

Therefore,

$$\left\| U^N(t) U_0(t)^\dagger - D_K^N(t) \right\| = \left\| \Theta_{K+1}^N(t) \right\| \leqslant \left\| H_1^N \right\|^{K+1} \frac{t^{K+1}}{(K+1)!} . \tag{129}$$

From (110) and (109), it follows that $H_1^N = H_{\rm int}^N$ is uniformly bounded with respect to N:

$$\left\| H_1^N \right\| = \lambda \mathfrak{c}_L \mathfrak{c}_R \left\| \mathfrak{E}_+^N + \mathfrak{E}_-^N \right\| \leqslant 2\lambda \,. \tag{130}$$

Thus, when $K \to \infty$, the right hand side of the inequality (129) vanishes uniformly with respect to N. □

B.4. Conclusion of the proof of theorem 3

We can now conclude the proof of theorem 3, which states that the evolution operator $U^N(t)$ converges in the mesoscopic limit to $U(t)$ (see (62)). In order to do so, we first show that each term of the finite-size Dyson series $\mathcal{D}^N(t)$ converges to the corresponding one of $\mathcal{D}(t)$.

Proposition 4. *For any choice of $k \in \mathbb{N}$ and of times $t_1, \ldots, t_k$, one has that*

$$V^N(t_k)\cdots V^N(t_1) = \left(\frac{\mathcal{E}_J}{2}\right)^k \left[\sum_{\gamma_1=\pm}\cdots\sum_{\gamma_k=\pm}\prod_{j=1}^{k}\left(\mathfrak{E}^N_{\gamma_{k-j+1}}(t_{k-j+1})\right)\prod_{j=1}^{k} e^{-it_j\Delta\xi\left(p^N_\varphi+\overline{\gamma}_{j-1},\gamma_j\right)}\right], \tag{131a}$$

$$V(t_k)\cdots V(t_1) = \left(\frac{\mathcal{E}_J}{2}\right)^k \left[\sum_{\gamma_1=\pm}\cdots\sum_{\gamma_k=\pm}\left(\prod_{j=1}^{k} e^{i\gamma_{k-j+1}\varphi}\right)\prod_{j=1}^{k} e^{-it_j\Delta\xi\left(p_\varphi+\overline{\gamma}_{j-1},\gamma_j\right)}\right], \tag{131b}$$

where we set $\overline{\gamma}_j \stackrel{\rm def}{=} \sum_{i=1}^{j}\gamma_j$. Furthermore, the following mesoscopic limit holds:

$$\underset{N\to\infty}{\text{m-lim}}\, V^N(t_k)\cdots V^N(t_1) = V(t_k)\cdots V(t_1)\,. \tag{132}$$

Proof. We prove (131*b*), the N-dependent case in (131*a*) being analogous. The derivation relies only on (107), (108) and (119) using induction. To start, set $k=1$. Using the Baker–Campbell–Hausdorff formula and applying (107), together with the definition of $\Delta\xi$ in in (124), one obtains

$$U_0(t)\, e^{\pm i\hat{\varphi}}\, U_0^\dagger(t) = e^{\pm i\hat{\varphi}} e^{-it\Delta\xi(\hat{p}_\varphi,\pm 1)} \,,$$

where also the identity:

$$\Delta\xi(\hat{p}_\varphi - 1, 1) = \xi(\hat{p}_\varphi) - \xi(\hat{p}_\varphi - 1) = -\Delta\xi(\hat{p}_\varphi, -1) \,,$$

has been used. Substituting the expression into the definition of $V(t)$ yields

$$V(t) = \frac{\mathcal{E}_J}{2}\sum_{\gamma=\pm} e^{i\gamma\hat{\varphi}} e^{-it\Delta\xi(\hat{p}_\varphi,\gamma)},$$

which indeed coincides with what one has to prove for $k=1$. On the other hand,

$$V(t_k)\,V(t_{k-1})\ldots V(t_1) = \frac{\mathcal{E}_J}{2}\sum_{\gamma_k=\pm} e^{i\gamma_k\hat{\varphi}} e^{-it_k\Delta\xi(\hat{p}_\varphi,\gamma_k)}\, V(t_{k-1})\ldots V(t_1)\,, \tag{133}$$

where we substituted for $V(t_k)$ the previously obtained expression. The result follows by assuming $V(t_{k-1})\ldots V(t_1)$ to have the desired form and then using (126*a*) to bring the exponential generated by $\Delta\xi$ to the right of the formula.

We want now to show the validity of the mesoscopic limit in (132). Using the expression (131*b*) which we have just derived, we have for $n_L, n_R, n'_L, n'_R \in \mathbb{Z}$,

$$\langle n'_L n'_R | V(t_k)\cdots V(t_1) | n_L n_R\rangle = \left(\frac{\mathcal{E}_J}{2}\right)^k \left(\sum_{\gamma_1=\pm}\cdots\sum_{\gamma_k=\pm}\delta_{n',n+\overline{\gamma}_k}\prod_{j=1}^{k} e^{-it_j\Delta\xi\left(n+\overline{\gamma}_{j-1},\gamma_j\right)}\right)\delta_{n'_L+n'_R,n_L+n_R}\,, \tag{134}$$

where we also set, in analogy with (123),

$$n' = \frac{n'_L - n'_R}{2}\,. \tag{135}$$

On the other hand, the expression (131a) yields

$$\langle n_L' n_R' | V^N(t_k) \cdots V^N(t_1) | n_L n_R \rangle_\beta^N$$
$$= \left(\frac{\mathcal{E}_J}{2}\right)^k \left(\sum_{\gamma_1=\pm} \cdots \sum_{\gamma_k=\pm} \langle n_L' n_R' | \prod_{j=1}^{k} \mathfrak{E}^N_{\gamma_{k-j+1}}(t_{k-j+1}) | n_L n_R \rangle_\beta^N \prod_{j=1}^{k} e^{-it_j \Delta\xi(n+\overline{\gamma}_{j-1},\gamma_j)} \right) . \tag{136}$$

The limit follows then directly from (118), which states that the product of an arbitrary number of phase operators $\mathfrak{E}^N_\pm(t)$ converges to the corresponding product of Weyl operators in the Heisenberg algebra on the circle. □

Theorem 3 (Charge qubit dynamics). *The microscopic time-evolution operator $U^N(t)$ has a well-defined mesoscopic limit,*

$$\underset{N\to\infty}{\text{m-lim}}\, U^N(t) = U(t) \equiv e^{-it\mathfrak{H}} , \tag{62}$$

where the Hamiltonian generating the mesoscopic dynamics on the circle is given by

$$\mathfrak{H} = \mathcal{E}_C \left(\frac{\hat{p}_{\varphi_L} - \hat{p}_{\varphi_R}}{2} - n_g \right)^2 + 2\lambda \mathfrak{c}_L \mathfrak{c}_R \cos(\hat{\varphi}_L - \hat{\varphi}_R) . \tag{63}$$

Proof. Let us define the quantity

$$I^N(t) \equiv \left| \langle n_L' n_R' | U^N(t) | n_L n_R \rangle_\beta^N - \langle n_L' n_R' | U(t) | n_L n_R \rangle \right|,$$

where once again $n_L, n_R, n_L', n_R' \in \mathbb{Z}$. We want to show that $I^N(t)$ goes to zero as N grows large. Thanks to the results of lemma 1, we can replace the evolution operators $U^N(t)$ and $U(t)$ with their respective Dyson series, which converge in norm to the evolution operators. Therefore, for a chosen $\varepsilon > 0$, we can truncate the Dyson series at such a large K, that independently of N,

$$\left\| U^N(t) - \mathcal{D}_K^N(t)\, U_0^N(t) \right\| \leqslant \frac{\varepsilon}{3} , \quad \left\| U(t) - \mathcal{D}_K(t)\, U_0(t) \right\| \leqslant \frac{\varepsilon}{3} ,$$

and estimate $I^N(t) \leqslant I_K^N(t) + 2\,\frac{\varepsilon}{3}$, where

$$I_K^N(t) \equiv \left| \langle n_L' n_R' | \mathcal{D}_K^N(t)\, U_0^N(t) | n_L n_R \rangle_\beta^N - \langle n_L' n_R' | \mathcal{D}_K(t)\, U_0(t) | n_L n_R \rangle \right|.$$

Notice that

$$I_K^N(t) \leqslant \sum_{k=1}^{K} \int_0^t \mathrm{d}t_1 \cdots \int_0^{t_{k-1}} \mathrm{d}t_k \left| J^N(t_1, \cdots, t_k) \right| \tag{137}$$

where, since $U_0^N(t) = U_{\text{free}}^N(t)\, U_{\text{C}}^N(t)$ (see (115)),

$$\left| J^N(t_1, \cdots, t_k) \right| \equiv \left| \langle n_L' n_R' | V^N(t_k) \cdots V^N(t_1)\, U_{\text{free}}^N(t)\, U_{\text{C}}^N(t) | n_L n_R \rangle_\beta^N - \langle n_L' n_R' | V(t_k) \cdots V(t_1)\, U_0(t) | n_L n_R \rangle \right|.$$

As we shall shortly see, by choosing N large enough, one can make $|J^N(t_1, \cdots, t_k)|$ arbitrarily small so that, because of the finite sum and finite time t in the right hand side of (137), $I_K^N(t) \leqslant \frac{\varepsilon}{3}$ and thus $I^N(t) \leqslant \varepsilon$.

Indeed, acting on the corresponding vectors, $U_{\text{C}}^N(t)$ and $U_0(t)$ give rise to the same phase factor, which thus drops out. We can then proceed with adding and subtracting the same term

$$\left| J^N(t_1, \cdots, t_k) \right| = \left| J^N(t_1, \cdots, t_k) + \langle n_L' n_R' | V^N(t_k) \cdots V^N(t_1) | n_L n_R \rangle_\beta^N - \langle n_L' n_R' | V^N(t_k) \cdots V^N(t_1) | n_L n_R \rangle_\beta^N \right|$$
$$\leqslant \left| J_1^N(t_1, \cdots, t_k) \right| + \left| J_2^N(t_1, \cdots, t_k) \right|, \tag{138}$$

where, using (116) and proposition 4, the following two quantities

$$J_1^N(t_1, \cdots, t_k) \equiv \langle n_L' n_R' | V^N(t_k) \cdots V^N(t_1) \left(U_{\text{free}}^N(t) - 1 \right) | n_L n_R \rangle_\beta^N;$$
$$J_2^N(t_1, \cdots, t_k) \equiv \langle n_L' n_R' | V^N(t_k) \cdots V^N(t_1) | n_L n_R \rangle_\beta^N - \langle n_L' n_R' | V(t_k) \cdots V(t_1) | n_L n_R \rangle.$$

can be made arbitrarily small by choosing N large enough. □

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