Asymptotic stability of kink with internal modes under odd perturbation

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Abstract. We give a sufficient condition, in the spirit of Kowalczyk–Martel–Munoz–Van Den Bosch (Ann PDE 7(1):Paper No. 10, 98, 2021), for the local asymptotic stability of kinks under odd perturbations. In particular, we allow the existence of quite general configuration of internal modes. The extension of our result to moving kinks remains an open problem.

1. Introduction

In this paper, we consider the problem of the asymptotic stability of kink solutions of the $(1+1)$ dimensional nonlinear scalar field model

$$\Box u_1 + W'(u_1) = 0, \quad (t,x) \in \mathbb{R}^{1+1} \text{ , where } \Box = \partial_t^2 - \partial_x^2. \tag{1.1}$$

We can write the above problem as

$$\partial_t \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \mathbf{J} \begin{pmatrix} -\partial_x^2 u_1 + W'(u_1) \\ u_2 \end{pmatrix}, \ u_1, u_2 : \mathbb{R}^{1+1} \to \mathbb{R}, \text{ where } \mathbf{J} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{1.2}$$

Our nonlinear potential W is an even C^∞ function such that

$$\exists \zeta > 0 \text{ s.t. } W(\zeta) = W'(\zeta) = 0, \ \omega^2 := W''(\zeta) > 0 \text{ and } \forall h \in (-\zeta, \zeta), \ W(h) > 0. \tag{1.3}$$

Under assumption (1.3), it is well known that an odd kink solution exists, see Lemma 1.1 of Kowalczyk et al. [26].

Proposition 1.1. *There exists odd $H \in C^\infty(\mathbb{R})$ satisfying $H'' = W'(H)$. Furthermore, we have $H'(x) > 0$, $\lim_{x\to\infty} H(x) = \zeta$, $|H(x) - \zeta| \lesssim e^{-\omega|x|}$ and*

$$\forall k \geq 1, \ |H^{(k)}(x)| \lesssim_k e^{-\omega|x|}.$$

Remark 1.2. By $A \lesssim B$, we mean that there exists $C > 0$ s.t. $A \le B$. The implicit constant C is independent of important parameters (e.g. in the claim of Proposition 1.1, the implicit constants are independent of x but depends on k).

The purpose of this paper is to study the case when the kink has internal modes, but only in the context of odd solutions of (1.2). We set $\mathbf{H} = (H, 0)$. We denote by $\mathbf{\Phi}[\mathbf{z}]$ the *refined profile*, introduced later in Sect. 1.2, where

$$\mathbf{z} = (z_1, \dots, z_N), \tag{1.4}$$

encodes the discrete modes and where $\mathbf{\Phi}[\mathbf{0}] = \mathbf{H}$. In analogy to Kowalczyk et al. [26] we set

$$\begin{aligned} \mathbf{E}_{\text{odd}} = \{\mathbf{u} \in L^1_{loc}(\mathbb{R}, \mathbb{R}^2) : \quad & u_1' \in L^2_{\text{even}}(\mathbb{R}), \\ u_2 \in L^2_{\text{odd}}(\mathbb{R}), \quad & \sqrt{W(u_1)} \in L^2_{\text{even}}(\mathbb{R})\} \text{ and} \end{aligned} \tag{1.5}$$

$$\mathbf{E}_{\mathbf{H}} = \{\mathbf{u} \in \mathbf{E}_{\text{odd}} : \quad u_1' \in L^2_{\text{even}}(\mathbb{R}), \quad u_2 \in L^2_{\text{odd}}(\mathbb{R}), \quad u_1 - H \in L^2_{\text{odd}}(\mathbb{R})\}. \tag{1.6}$$

For any $\mathbf{u}$, there is a natural identification, a natural *trivialization* in fact, of the tangent space

$$T_{\mathbf{u}}\mathbf{E}_{\mathbf{H}} = \boldsymbol{\mathcal{H}}^1 \text{ where } \boldsymbol{\mathcal{H}}^s := H^s_{\text{odd}}(\mathbb{R}, \mathbb{R}) \times H^{s-1}_{\text{odd}}(\mathbb{R}, \mathbb{R}). \tag{1.7}$$

There is a natural distance in $\mathbf{E}_{\mathbf{H}}$, given by $\|\mathbf{u} - \mathbf{v}\|_{\boldsymbol{\mathcal{H}}^1}$. Our main result is the following.

Theorem 1.3. *Under Assumptions 1.6, 1.8 and 1.12 given below, for any $\epsilon > 0$ and $a > 0$, there exists a $\delta_0 > 0$ s.t., for all odd functions*

$$\mathbf{u}(0) \in \mathbf{E}_{\mathbf{H}} \textit{ satisfying } \delta := \|\mathbf{u}(0) - \mathbf{H}\|_{\boldsymbol{\mathcal{H}}^1} < \delta_0 \tag{1.8}$$

and for the corresponding solution $\mathbf{u}$ of (1.2), we have

$$\mathbf{u}(t) = \mathbf{\Phi}[\mathbf{z}(t)] + \boldsymbol{\eta}(t) \textit{ for appropriate } \mathbf{z} \in C(\mathbb{R}, \mathbb{C}^N) \textit{ and } \boldsymbol{\eta} \in C(\mathbb{R}, \boldsymbol{\mathcal{H}}^1), \tag{1.9}$$

and , for $I = \mathbb{R}$ and $\langle x \rangle := (1 + x^2)^{1/2}$,

$$\int_I \|e^{-a\langle x \rangle} \boldsymbol{\eta}(t)\|^2_{\boldsymbol{\mathcal{H}}^1(\mathbb{R})} \le \epsilon, \tag{1.10}$$

and

$$\lim_{t \to \infty} \mathbf{z}(t) = 0 \ . \tag{1.11}$$

In Sect. 10 we show that the ϕ^8 model, when near the ϕ^4 model, satisfies our repulsivity hypothesis.

Remark 1.4.
- The local well-posedness of (1.2) in $\mathbf{E}_{\mathbf{H}}$ in a small neighborhood of the kink is known, see section 3.1 of Kowalczyk et al. [26].

- Under the assumption that $\mathbf{u}(0)$ is odd, then $\mathbf{u}(t)$ is odd for all t, by uniqueness. For general perturbations and also for boosted kinks, Kowalczyk et al. [26] gave a sufficient repulsivity hypothesis for asymptotic stability of the kinks. The repulsivity hypothesis in Kowalczyk et al. [26] implies the absence of internal modes. The purpose of this paper is to give a somewhat related repulsivity hypothesis in Assumption 1.6, which allows the existence of internal modes. Unfortunately, at this time we are unable to treat moving kinks, which were the main focus of Kowalczyk et al. [26].
- The restriction to odd solutions can simplify considerably dispersion problems, as for example is shown in Kowalczyk et al. [24]. In Germain et al. [13,14], restricting to a smaller class of odd solutions of (1.2) and assuming the absence of internal modes, but avoiding any explicit repulsivity hypothesis, there is a different kind of proof of asymptotic stability, involving rates of decay of remainder terms. Such rates of decay cannot hold with data like in (1.8), where the estimates and the asymptotic behavior on the solutions need to be invariant by time translation, like in (1.10)–(1.11). Notice, also, that the literature which uses dispersive estimates such as also [12,28] as many others, has been able so far to treat only rather simple configurations of internal modes, usually at most a single one, usually nicely positioned.
- Just before completion of this work we learned about Kowalczyk and Martel [22], which in the case of a favorably placed internal mode, encompassing the ϕ^4 model, simplifies and generalizes the proof in [23]. Our paper is independent from Kowalczyk and Martel [22]. Kowalczyk and Martel [22] make a more efficient use of dispersivity, adapting this notion to the context of odd solutions of our problem.

  By and large, it should be possible to combine [22] with our framework in the context of more general eigenvalues. Somewhat delicate, at least in our context, should be the case when L_D has a resonance at ω^2, as for example when $L_D = -\partial_x^2 + \omega^2$, for instance in the case of the ϕ^4 problem. In that case, our proof of Lemma 5.9 does not work, and perhaps the statement is wrong. On the other hand, this lemma for us is important when we bound the auxiliary variable $\mathbf{g}$ in Sect. 8, needed in our proof of the Fermi Golden Rule.

  Our proof of the Fermi Golden Rule is different from Kowalczyk and Martel [22]. From a combinatorial view point, we are dealing with a more complicated problem. In our framework, we need an expansion on the transformed variable $\mathbf{v}$, see (8.3), rather than the original variable $\mathbf{u}$, to take advantage of the cutoff factor χ_{B^2} in front of the nonlinear terms in the equation of $\mathbf{g}$. The cutoff offsets the long range nature of the nonlinearity. Unfortunately, the commutator of cutoff and linear part of the equation, generates an extra term, for example in (8.6), which is delicate. Kowalczyk and Martel [22] study the Fermi Golden Rule in the initial variable $\mathbf{u}$, so they do not have this commutator. We are neither able to generalize their FGR argument nor, exactly because of the long range nonlinearity, to perform smoothing estimates directly in the original variable $\mathbf{u}$.

- Early work on kinks by Komech and Kopylova [20, 21], treated special cases with short range nonlinearities and within the framework of Buslaev and Perelman [5]. We utilize Komech and Kopylova [19, 20] when we consider some smoothing estimates in Sects. 5 and 8.
- For a treatment of the sine-Gordon model with the Nonlinear Steepest Descent method and techniques of Integrable Systems, see [6] and therein.

This paper is very similar to our previous [8], which in turn used the dispersion theory of Kowalczyk et al. [25, 26] (see also [27] for a very recent paper related to [25]) combined with our theory on the Fermi Golden Rule (FGR). In [8] and in a number of other papers, like [9], we have produced a framework very effective in sorting out effortlessly the complexities of the transient patterns of the internal discrete modes that can occur in stability problems, and exploiting dissipation induced on the discrete modes, due to their hemorrhaging energy which, by nonlinear interaction, leaks in the radiation component where it escapes to infinity. Since Bambusi and Cuccagna [4], which can be considered the first paper in our series, we avoided the decay rates analysis of the early PDE papers on this subject, the earliest involving internal modes being Buslaev and Perelman [5]. Notice that the rarely cited paper [10] extends considerably the result in [4]. As mentioned above, decay rates cannot exist in the Energy space. Kowalczyk et al. [23, 25, 26] as well as KdV papers by Martel and Merle such as [37–39] work in energy space, presumably to achieve a maximum of generality. In fact, in the presence of discrete modes, the Energy Space framework tends to be conductive to rather simple sorting out of the discrete modes. Early in the literature, for example in [5, 43] or Komech and Kopylova [20, 21], as well as in many others such as, for example [3], there was use of dispersive estimates. This is also related to the fact that the earliest papers predate Keel and Tao [18], whose endpoint Strichartz estimate has played an important role in the theory, even though it can be replaced by smoothing estimates. Ultimately, the literature using dispersive estimates, so far has not dealt with discrete mode configurations which are not simple.

In the presence of short range nonlinearities, where it is possible to prove dispersion using Strichartz estimates, papers such as [9] provide proofs of asymptotic stability and scattering in the presence of very general discrete modes configurations. In the case of possibly long range nonlinearities, like here and [8], we use the Virial Inequalities framework originating in Merle's school, in the particular elaboration of Kowalczyk et al. We need that the linearization L_1 be, not directly dispersive, but, rather, dispersive after a sufficient number of Darboux transformations, and not just a single one like in Kowalczyk et al. [23] or Martel [36]. Darboux transformations are beautifully discussed by Deift and Trubowitz [11], although their theory would be not sufficient in the context of more general kinks than the ones discussed here.

Kowalczyk et al. [23] have been able to prove an asymptotic stability result in the absence of dissipative operators, see also [44]. Furthermore, as we remarked above, Kowalczyk and Martel [22] gave a new more general proof. We refer also to [1, 2, 44].

In the context of the dispersion theory of papers such as [29–34, 45], and the framework in [12–14, 28, 35], if it works in the absence of dissipative potentials, can obviously prove very useful. A natural problem would be to prove some form of scattering for the remainders for solutions in Energy space, with some Dollard like correction, at least in the border-like cases, see also [42].

1.1. Internal modes, Darboux transform and repulsivity assumption

We consider the Schrödinger operator

$$L_1 = -\partial_x^2 + W''(H). \tag{1.12}$$

By differentiating $H'' = W'(H)$, we obtain $H' \in \ker L_1$. Since $H' > 0$, we have $\ker L_1 = \text{span}\{H'\}$. By Proposition 1.1, $W''(H) - \omega^2$ decays exponentially. Thus, L_1 will have at most finitely many eigenvalues, which, since we are in 1D, are all simple. We label the eigenvalues corresponding to the odd eigenfunctions as follows:

- $\sigma_{\rm d}(L_1|_{L^2_{\rm odd}}) = \{\lambda_j^2 \mid j = 1, \cdots, N\}$ with $0 < \lambda_1 < \cdots < \lambda_N < \omega$.

We set $\phi_j \in L^2_{\rm odd}$ to be the corresponding (odd, normalized and $\mathbb{R}$-valued) eigenfunctions, i.e. $L_1\phi_j = \lambda_j^2\phi_j$ and $\|\phi_j\|^2_{L^2} = (2\lambda_j)^{-1}$.

Remark 1.5. The repulsivity condition in [26] implies $\sigma_{\rm p}(L_1) = \{0\}$ and, therefore, $N = 0$.

In the following, we assume $N \geq 1$. The case $N = 0$ is contained in [26].

By the Sturm-Liouville theory L_1 will have a number $\widetilde{N}$, equal to $2N$ or $2N+1$, of eigenvalues. We consider $\widetilde{\lambda}_j > 0$ so that we have $\sigma_{\rm d}(L_1) = \{\widetilde{\lambda}_j^2\}_{j=1}^{\widetilde{N}}$. In this case we have $\widetilde{\lambda}_1 = 0$ and $\lambda_j = \widetilde{\lambda}_{2j}$.

We set

$$\mathbf{L}_1 := \mathbf{J}\begin{pmatrix} L_1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -L_1 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{\Phi}_j := \begin{pmatrix} \phi_j \\ -\mathrm{i}\lambda_j\phi_j \end{pmatrix}. \tag{1.13}$$

This operator $\mathbf{L}_1$ is relevant here because it is obtained linearizing (1.2) at $\mathbf{H}$. Indeed, substituting $\mathbf{u} = \mathbf{H} + \mathbf{r}$ into (1.2), we have

$$\partial_t \mathbf{r} = \mathbf{L}_1\mathbf{r} + O(\mathbf{r}^2).$$

From now on, we will consider only odd functions. In particular $\mathbf{L}_1$ will act only on odd in x functions.

By direct computation, we see that

$$\mathbf{L}_1\mathbf{\Phi}_j = -\mathrm{i}\lambda_j\mathbf{\Phi}_j \text{ and } \mathbf{L}_1\overline{\mathbf{\Phi}_j} = \mathrm{i}\lambda_j\overline{\mathbf{\Phi}_j}. \tag{1.14}$$

We consider

$$(\mathbf{f}, \mathbf{g}) = \int_{\mathbb{R}} {}^t\mathbf{f}(x)\mathbf{g}(x)dx, \tag{1.15}$$

$$\langle \mathbf{f}, \mathbf{g}\rangle = \mathrm{Re}\,(\mathbf{f}, \overline{\mathbf{g}}) \tag{1.16}$$

and the symplectic form

$$\langle \mathbf{J}\mathbf{f}, \mathbf{g}\rangle . \tag{1.17}$$

Notice that $\left\langle \mathbf{J}\boldsymbol{\Phi}_j, \overline{\mathrm{i}\boldsymbol{\Phi}_j}\right\rangle = 1$.
It is easy to check

$$\sigma_{\mathrm{d}}(\mathbf{L}_1) = \{\pm \mathrm{i}\lambda_j \mid j = 1, \cdots, N\} \text{ and } \sigma_{\mathrm{ess}}(\mathbf{L}_1) = \mathrm{i}\left((-\infty, -\omega] \cup [\omega, \infty)\right). \tag{1.18}$$

Notice also that $\mathbf{L}_1$ leaves the following decomposition invariant,

$$L^2_{\mathrm{odd}}(\mathbb{R}, \mathbb{C}^2) = L^2_{discr} \oplus L^2_{disp} \text{ where } L^2_{discr} := \oplus_{\lambda \in \sigma_p(\mathbf{L}_1)} \ker\left(\mathbf{L}_1 - \lambda\right), \tag{1.19}$$

where L^2_{disp} is the $\langle \mathbf{J}\cdot, \cdot\rangle$-orthogonal of L^2_{discr}.

Thus, the linearized operator $\mathbf{L}_1$ has neutral eigenvalues, which will create oscillating and non-decaying solutions in the linear level. Such oscillations will last for long time in the full nonlinear problem, they will loose energy and oscillations will eventually decay. The Fermi Golden Rule (FGR) non-degeneracy condition, which will be introduce in the next subsection, guarantees such phenomenon, but it has to be combined with dispersion of the continuous modes. To prove dispersion we use virial estimates of Kowalczyk et al. [26]. For this we need to assume that the potential $W''(H)$ is "repulsive" after a series of Darboux transforms which eliminate the eigenvalues, as we explain now. The discussion is similar to [8], which was based on [11].

1.1.1. Darboux transformations. We inductively define the Schrödinger operator L_j $(j = 1, \cdots, \widetilde{N} + 1)$ and a differential operator A_j $(j = 1, \cdots, \widetilde{N})$ as follows.

1. $L_1 = -\partial_x^2 + W''(H)$ and $A_1 = (H')^{-1}\partial_x\left(H'\cdot\right)$. In this case, we have

$$L_1 = A_1 A_1^*, \tag{1.20}$$

   and we define L_2 by

$$L_2 := A_1^* A_1.$$

2. Inductively, given L_k with ψ_k the ground state of L_k, we set $A_k := \psi_k^{-1}\partial_x\left(\psi_k\cdot\right)$. Then

$$L_k = A_k A_k^* - \widetilde{\lambda}_k^2 \tag{1.21}$$

   and we define

$$L_{k+1} := A_k^* A_k - \widetilde{\lambda}_k^2$$

3. In the last step, $L_{\widetilde{N}+1} := A^*_{\widetilde{N}} A_{\widetilde{N}} - \widetilde{\lambda}^2_{\widetilde{N}}$. We set

$$L_D = L_{\widetilde{N}+1} = -\partial_x^2 + V_D \text{ where, here, } V_D - \omega^2 \in \mathcal{S}(\mathbb{R}, \mathbb{R}). \tag{1.22}$$

For the above we refer to Section 3 of [11] and Proposition 1.9 of [8]. We set

$$\mathcal{A} := A_1 \cdots A_{\widetilde{N}}. \tag{1.23}$$

Then, by simple computation we obtain.

$$\mathcal{A}^* L_1 = L_D \mathcal{A}^*. \tag{1.24}$$

We assume that V_D is repulsive, in the following sense:

Assumption 1.6. $xV_D'(x) \leq 0$ for all $x \in \mathbb{R}$ and V_D is not identically zero.

Remark 1.7. In Kowalczyk and Martel [22] the above assumption is eased into the following: there exists a $\gamma > 0$ such that the operator $-(1-\gamma)\partial_x^2 - 2^{-1/2}xV_D'(x)$ has at most one negative eigenvalue. In order to prove their result, Kowalczyk and Martel [22] modify the first virial inequality of [25]. This could be arranged here as well, but our proof, in some important special case, might face some problems, discussed in Remark 5.10 below.

1.2. Refined profile and Fermi Golden Rule assumption

As in the asymptotic stability of solitons for nonlinear Schrödinger equations [8], we introduce the notion of *refined profile*.

We introduce some notation. For $\mathbf{m} = (\mathbf{m}_+, \mathbf{m}_-) \in \mathbb{N}_0^{2N}$, where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, we write $\overline{\mathbf{m}} = (\mathbf{m}_-, \mathbf{m}_+)$ and $|\mathbf{m}| = \sum_{j=1}^{N}(m_{+j} + m_{-j})$. We set $\mathbf{e}^j = (\delta_{j1}, \cdots, \delta_{jN}, 0, \cdots, 0)$. We set

$$\boldsymbol{\lambda} := (\lambda_1, \cdots, \lambda_N, -\lambda_1, \cdots, -\lambda_N), \tag{1.25}$$

and

$$\boldsymbol{\lambda} \cdot \mathbf{m} := \sum_{j=1}^{N} \lambda_j (m_{+j} - m_{-j}). \tag{1.26}$$

We assume the following.

Assumption 1.8. For M be the largest number in $\mathbb{N}$ such that $(M-1)\lambda_1 < \omega$, then for a multi-index $\mathbf{m} \in \mathbb{N}_0^{2N}$

$$\|\mathbf{m}\| \leq M \Longrightarrow (\mathbf{m} \cdot \boldsymbol{\lambda})^2 \neq \omega^2. \tag{1.27}$$

We also assume that for $\mathbf{m} = (\mathbf{m}_+, \mathbf{m}_-) \in \mathbb{N}_0^{2N}$ then

$$\|\mathbf{m}\| \leq 2M \text{ and } \mathbf{m} \cdot \boldsymbol{\lambda} = 0 \Longrightarrow \mathbf{m}_+ = \mathbf{m}_-. \tag{1.28}$$

As in [8], we set

$$\begin{aligned}
\mathbf{R} &:= \{\mathbf{m} \in \mathbb{N}_0^{2N} \mid |\boldsymbol{\lambda} \cdot \mathbf{m}| > \omega\}, \\
\mathbf{R}_{\min} &:= \{\mathbf{m} \in \mathbf{R} \mid \nexists \mathbf{n} \in \mathbf{R} \text{ s.t. } \mathbf{n} \prec \mathbf{m}\}, \\
\mathbf{I} &:= \{\mathbf{m} \in \mathbb{N}_0^{2N} \mid \exists \mathbf{n} \in \mathbf{R}_{\min},\ \mathbf{n} \prec \mathbf{m}\} \\
\mathbf{NR} &:= \mathbb{N}_0^{2N} \setminus (\mathbf{I} \cup \mathbf{R}_{\min}), \\
\boldsymbol{\Lambda}_j &:= \{\mathbf{m} \in \mathbf{NR} \mid \boldsymbol{\lambda} \cdot \mathbf{m} = \lambda_j\} \\
\boldsymbol{\Lambda}_0 &:= \{\mathbf{m} \in \mathbf{NR} \setminus \{\mathbf{0}\} \mid \boldsymbol{\lambda} \cdot \mathbf{m} = 0\},
\end{aligned}$$

where the partial order $\prec$ is defined by

$$\mathbf{n} \prec \mathbf{m} \ \Leftrightarrow\ \forall j, n_{+j} + n_{-j} \leq m_{+j} + m_{-j} \text{ and } |\mathbf{n}| < |\mathbf{m}|.$$

Lemma 1.9. *The following facts hold.*

1. *If* $|\mathbf{m}| > M$, *with* M *the constant in Assumption 1.8, then* $\mathbf{m} \in \mathbf{I}$.

2. $\mathbf{R}_{\min}$ *and* $\mathbf{NR}$ *are finite sets.*
3. *If* $\mathbf{m} \in \mathbf{NR}$*, then* $|\boldsymbol{\lambda}\cdot\mathbf{m}| < \omega$ *and if* $\mathbf{m} \in \mathbf{R}_{\min}$*, then* $\mathbf{m}_+ = 0$ *or* $\mathbf{m}_- = 0$*.*
4. *If* $\mathbf{m} \in \boldsymbol{\Lambda}_j$ *then there is a* $\mathbf{n} \in \boldsymbol{\Lambda}_0$ *with* $\mathbf{m} = \mathbf{e}^j + \mathbf{n}$*.*

Proof. If $|\mathbf{m}| > M$, we can write $\mathbf{m} = \boldsymbol{\alpha} + \boldsymbol{\beta}$ with $|\boldsymbol{\alpha}| = M$. If $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_+, \boldsymbol{\alpha}_-)$ and if we set $\mathbf{n} = (\mathbf{n}_+, \mathbf{n}_-)$ with $\mathbf{n}_+ = \boldsymbol{\alpha}_+ + \boldsymbol{\alpha}_-$ and $\mathbf{n}_- = 0$, then $\mathbf{n}\cdot\boldsymbol{\lambda} \geq M\lambda_1 > \omega$. This implies that $\mathbf{n} \in \mathbf{R}$ and that there exists $\mathfrak{a} \in \mathbf{R}_{\min}$ with $\mathfrak{a} \preceq \mathbf{n}$. From $|\boldsymbol{\beta}| \geq 1$ it follows that $\mathfrak{a} \prec \mathbf{m}$ and so $\mathbf{m} \in \mathbf{I}$.

Obviously, from the 1st claim it follows that if $\mathbf{m} \in \mathbf{R}_{\min} \cup \mathbf{NR}$ then $|\mathbf{m}| \leq M$. Next we observe that $\mathbf{m} \in \mathbf{NR}$ implies $|\mathbf{m}| \leq M$ and $|\boldsymbol{\lambda}\cdot\mathbf{m}| \leq \omega$ and, by Assumption 1.8, $|\boldsymbol{\lambda}\cdot\mathbf{m}| < \omega$. If $\mathbf{m} \in \mathbf{R}_{\min}$ with, say, $\mathbf{m}\cdot\boldsymbol{\lambda} > \omega$, then obviously from (1.26) we have $\mathbf{m}_+\cdot\boldsymbol{\lambda} > \omega$ and it is elementary that $\mathbf{m} = (\mathbf{m}_+, 0)$. Finally, from the first claim we know that if $\mathbf{m} \in \boldsymbol{\Lambda}_j$ then $\|\mathbf{m}\| \leq M$. From $\mathbf{m}\cdot\boldsymbol{\lambda} - \lambda_j = 0$ it follows from (1.28) that we have the last claim. ☐

For $\mathbf{z} \in \mathbb{C}^N$ and $\mathbf{m} \in \mathbb{N}_0^2$, we write $\mathbf{z}^{\mathbf{m}} = \prod_{j=1}^N z_j^{m_{+j}} \overline{z_j}^{m_{-j}}$.

For $f \in C^1(\mathbb{C}^N, X)$ (differentiability is taken in the real sense), we set $D_{\mathbf{z}} f(\mathbf{z})\mathbf{w} := \frac{d}{d\epsilon} f(\mathbf{z} + \epsilon\mathbf{w})$.

Definition 1.10. We set $\|\cdot\|_{\Sigma^s} := \|\cdot\|_{H^s_{a_1}} := \|e^{a_1\langle x\rangle}\cdot\|_{H^s}$ where $a_1 = \frac{1}{2}\sqrt{\omega^2 - \lambda_N^2}$ and denote by Σ^s the corresponding spaces.

For $b \in \mathbb{R}$ we write $\|\cdot\|_{L^2_b} := \|e^{b\langle x\rangle}\cdot\|_{L^2}$

We write $\Sigma := \Sigma^1$ and denote by Σ^* its dual.

For any $s, \sigma \in \mathbb{R}$, recalling the space $\boldsymbol{\mathcal{H}}^s$ defined in (1.7), we will use also other weighted spaces, defined by the norm $\|\cdot\|_{L^{2,\sigma}} := \|\langle x\rangle^\sigma\cdot\|_{L^2}$ and spaces defined by the norm $\|\cdot\|_{\boldsymbol{\mathcal{H}}^{s,\sigma}} := \|\langle x\rangle^\sigma\cdot\|_{\boldsymbol{\mathcal{H}}^s}$.

We pick $a \in (0, a_1)$ and consider the following norm,

$$\|f\|_{\widetilde{\Sigma}}^2 = \left\langle \left(-\partial_x^2 + \operatorname{sech}^2\left(\frac{ax}{10}\right)\right) f, f \right\rangle \sim \|f\|_{\dot{H}^1}^2 + \|f\|_{L^2_{-\frac{a}{10}}}^2, \tag{1.29}$$

denoting by $\widetilde{\Sigma}$ the corresponding space. For $\mathbf{f} = (f_1, f_2)$, we will consider the norm

$$\|\mathbf{f}\|_{\widetilde{\boldsymbol{\Sigma}}} = \|f_1\|_{\widetilde{\Sigma}} + \|f_2\|_{L^2_{-\frac{a}{10}}}. \tag{1.30}$$

We observe that $H^{(n)}, \phi_j \in \Sigma^s$ for arbitrary $n \geq 1$, $s \in \mathbb{R}$ and $j = 1, \cdots N$.

The refined profile is an approximate solution of (1.2) which encodes the kink with its internal modes.

Proposition 1.11. *There exist* $\alpha_0 > 0$*, functions* $\{\boldsymbol{\phi}_{\mathbf{m}} : \mathbf{m} \in \mathbf{NR}\} \subset \Sigma^\infty$*,* $\widetilde{\mathbf{z}}_R \in C^\infty(\mathcal{B}_{\mathbb{C}^N}(0, \alpha_0), \mathbb{C}^N)$ *and* $\{\lambda_{\mathbf{n}j}\}_{\mathbf{n}\in\Lambda_0\cup\{\mathbf{0}\}} \subset \mathbb{R}$ *for* $j = 1, \cdots, N$ *with* $\boldsymbol{\phi}_0 = \mathbf{H}$*,* $\boldsymbol{\phi}_{\mathbf{e}^j} = \boldsymbol{\Phi}_j$*,* $\boldsymbol{\phi}_{\overline{\mathbf{m}}} = \overline{\boldsymbol{\phi}_{\mathbf{m}}}$ *and* $\lambda_{\mathbf{0}j} = \lambda_j$ *s.t. setting*

$$\boldsymbol{\phi}[\mathbf{z}] := \begin{pmatrix} \phi_1[\mathbf{z}] \\ \phi_2[\mathbf{z}] \end{pmatrix} = \boldsymbol{\phi}_0 + \widetilde{\boldsymbol{\phi}}[\mathbf{z}] := \boldsymbol{\phi}_0 + \sum_{\mathbf{m}\in\mathbf{NR}, |\mathbf{m}|\geq 1} \mathbf{z}^{\mathbf{m}} \boldsymbol{\phi}_{\mathbf{m}}, \tag{1.31}$$

$$\boldsymbol{\phi}_{\overline{\mathbf{m}}} = \overline{\boldsymbol{\phi}_{\mathbf{m}}} \tag{1.32}$$

$$\widetilde{z}_j := -\mathrm{i} \sum_{\mathbf{n}\in\Lambda_0\cup\{\mathbf{0}\}} \lambda_{\mathbf{m}j}\mathbf{z}^{\mathbf{n}} z_j + \widetilde{z}_{jR},\ \widetilde{\mathbf{z}} := (\widetilde{z}_1,\cdots,\widetilde{z}_N) \text{ and } \widetilde{\mathbf{z}}_R = (\widetilde{z}_{1R},\cdots,\widetilde{z}_{NR}), \tag{1.33}$$

$$|\widetilde{\mathbf{z}}_R| \lesssim \sum_{\mathbf{m}\in\mathbf{R}_{\min}} |\mathbf{z}^{\mathbf{m}}|, \tag{1.34}$$

$$\boldsymbol{\lambda}_{\overline{\mathbf{m}}} = \boldsymbol{\lambda}_{\mathbf{m}} \in \mathbb{R}^{2N} \tag{1.35}$$

where $\boldsymbol{\lambda}_{\mathbf{m}} := (\lambda_{\mathbf{m}1},\ldots,\lambda_{\mathbf{m}N},-\lambda_{\mathbf{m}1},\ldots,-\lambda_{\mathbf{m}N})$*, and, for the remainder function* $\boldsymbol{\mathcal{R}}[\mathbf{z}]$ *defined by*

$$\boldsymbol{\mathcal{R}}[\mathbf{z}] := \mathbf{J}\begin{pmatrix} -\partial_x^2\phi_1[\mathbf{z}] + W'(\phi_1[\mathbf{z}]) \\ \phi_2[\mathbf{z}] \end{pmatrix} - D_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}]\widetilde{\mathbf{z}}, \tag{1.36}$$

we have the expansion

$$\boldsymbol{\mathcal{R}}[\mathbf{z}] = \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}}\mathcal{R}_{\mathbf{m}} + \boldsymbol{\mathcal{R}}_1[\mathbf{z}], \tag{1.37}$$

with $\mathcal{R}_{\overline{\mathbf{m}}} = \overline{\mathcal{R}_{\mathbf{m}}} \in \boldsymbol{\Sigma}^\infty$ *and for any* $l \in \mathbb{N}$

$$\|\boldsymbol{\mathcal{R}}_1[\mathbf{z}]\|_{\boldsymbol{\Sigma}^l} \lesssim_l |\mathbf{z}| \sum_{\mathbf{m}\in\mathbf{R}_{\min}} |\mathbf{z}^{\mathbf{m}}|. \tag{1.38}$$

Furthermore,

$$\langle \mathbf{J}\boldsymbol{\mathcal{R}}[\mathbf{z}], D_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}]\boldsymbol{\zeta}\rangle = 0 \text{ for any } \boldsymbol{\zeta} \in \mathbb{C}^N. \tag{1.39}$$

Proof. We insert (1.31) in (1.36), using (1.33). We expand

$$W'(H+\widetilde{\phi}_1[\mathbf{z}]) = W'(H) + W''(H)\widetilde{\phi}_1[\mathbf{z}] + \sum_{\ell=2}^{M} \frac{W^{(1+\ell)}(H)}{\ell!}\widetilde{\phi}_1^\ell[\mathbf{z}] + O(\|\mathbf{z}\|^{M+1}),$$

where $\widetilde{\boldsymbol{\phi}}[\mathbf{z}] = (\widetilde{\phi}_1[\mathbf{z}],\widetilde{\phi}_2[\mathbf{z}])$. Then, for $\overrightarrow{\mathbf{j}} = {}^t(0,1)$,

$$\sum_{\ell=2}^{M} \frac{W^{(1+\ell)}(H)}{\ell!}\widetilde{\phi}_1^\ell[\mathbf{z}]\overrightarrow{\mathbf{j}} = \sum_{\mathbf{m}\in\mathbf{NR}} \mathbf{z}^{\mathbf{m}}\mathbf{g}_{\mathbf{m}} + \sum_{\substack{\mathbf{m}\in\mathbf{R}\cup\mathbf{I}\\ |\mathbf{m}|\le M}} \mathbf{z}^{\mathbf{m}}\mathbf{g}_{\mathbf{m}} + O(\|\mathbf{z}\|^{M+1})$$

where, for $\boldsymbol{\phi}_{\mathbf{m}} = (\phi_{1\mathbf{m}},\phi_{2\mathbf{m}})$,

$$\mathbf{g}_{\mathbf{m}} = \sum_{\ell=2}^{M} \frac{W^{(1+\ell)}(H)}{\ell!} \sum_{\substack{\mathbf{m}^1,\cdots,\mathbf{m}^\ell\in\mathbf{NR}\\ \mathbf{m}^1+\cdots+\mathbf{m}^\ell=\mathbf{m}}} \phi_{1\mathbf{m}^1}\cdots\phi_{1\mathbf{m}^\ell}\overrightarrow{\mathbf{j}}. \tag{1.40}$$

Using

$$\left(D_{\mathbf{z}}\mathbf{z}^{\mathbf{m}}\right)(\mathrm{i}\boldsymbol{\lambda}\mathbf{z}) = \mathrm{i}\mathbf{m}\cdot\boldsymbol{\lambda}\ \mathbf{z}^{\mathbf{m}},\ \text{ where } \boldsymbol{\lambda}\mathbf{z} := (\lambda_1 z_1,\ldots,\lambda_N z_N), \tag{1.41}$$

we obtain

$$D_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}]\widetilde{\mathbf{z}}[\mathbf{z}] = -\mathrm{i}\sum_{\mathbf{m}\in\mathbf{NR}} \mathbf{m}\cdot\boldsymbol{\lambda}\mathbf{z}^{\mathbf{m}}\boldsymbol{\phi}_{\mathbf{m}} - \mathrm{i}\sum_{\mathbf{m}\in\mathbf{NR},\ \mathbf{n}\in\boldsymbol{\Lambda}_0} \mathbf{m}\cdot\boldsymbol{\lambda}_{\mathbf{n}}\mathbf{z}^{\mathbf{n}}\mathbf{z}^{\mathbf{m}}\boldsymbol{\phi}_{\mathbf{m}} - D_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}]\mathrm{i}\widetilde{\mathbf{z}}_R.$$

Let us set

$$\widehat{\boldsymbol{\mathcal{R}}}[\mathbf{z}] := \mathbf{J}\begin{pmatrix} -\partial_x^2\phi_1[\mathbf{z}] + W'(\phi_1[\mathbf{z}]) \\ \phi_2[\mathbf{z}] \end{pmatrix} - D_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}](\widetilde{\mathbf{z}} - \widetilde{\mathbf{z}}_R). \tag{1.42}$$

We expand now to get

$$\widehat{\mathcal{R}}[\mathbf{z}] = \sum_{\mathbf{m}\in\mathbf{NR}} \mathbf{z}^{\mathbf{m}}\widehat{\mathcal{R}}_{\mathbf{m}} + \sum_{\substack{\mathbf{m}\in\mathbf{R}\cup\mathbf{I}\\ |\mathbf{m}|\le M}} \mathbf{z}^{\mathbf{m}}\widehat{\mathcal{R}}_{\mathbf{m}} + O(\|\mathbf{z}\|^{M+1}), \tag{1.43}$$

where

$$\begin{aligned}&\widehat{\mathcal{R}}_{\mathbf{m}} = (\mathbf{L}_1 + \mathrm{i}\boldsymbol{\lambda}\cdot\mathbf{m})\,\boldsymbol{\phi}_{\mathbf{m}} - \mathcal{E}_{\mathbf{m}} \text{ where}\\ &\mathcal{E}_{\mathbf{m}} = \mathbf{g}_{\mathbf{m}} - \sum_{\substack{\mathbf{m}'+\mathbf{n}'=\mathbf{m}\\ \mathbf{m}'\in\mathbf{NR},\ \mathbf{n}'\in\boldsymbol{\Lambda}_0}} \mathrm{i}\boldsymbol{\lambda}_{\mathbf{n}'}\cdot\mathbf{m}'\boldsymbol{\phi}_{\mathbf{m}'}.\end{aligned}$$

We seek $\widehat{\mathcal{R}}_{\mathbf{m}} \equiv 0$ for $\mathbf{m}\in\mathbf{NR}$. For $|\mathbf{m}| = 1$ the equation reduces to $(\mathbf{L}_1 + \mathrm{i}\boldsymbol{\lambda}\cdot\mathbf{m})\,\boldsymbol{\phi}_{\mathbf{m}} = 0$, so that we can set $\boldsymbol{\phi}_{\mathbf{e}^j} = \boldsymbol{\Phi}_j$ and $\boldsymbol{\phi}_{\overline{\mathbf{e}}^j} = \overline{\boldsymbol{\Phi}}_j$. Let us consider now $|\mathbf{m}|\ge 2$ with $\mathbf{m}\notin\cup_{j=1}^N\left(\boldsymbol{\Lambda}_j\cup\overline{\boldsymbol{\Lambda}}_j\right)$. In this case let us assume by induction that $\boldsymbol{\phi}_{\mathbf{m}'}$ and $\boldsymbol{\lambda}_{\mathbf{m}'}$ have been defined for $|\mathbf{m}'| < |\mathbf{m}|$ and that they satisfy (1.32)–(1.35). Then, from (1.40) we obtain $g_{\overline{\mathbf{m}}} = \overline{g}_{\mathbf{m}}$ and $\mathcal{E}_{\overline{\mathbf{m}}} = \overline{\mathcal{E}}_{\mathbf{m}}$. We can solve $\widehat{\mathcal{R}}_{\mathbf{m}} = 0$ writing $\boldsymbol{\phi}_{\mathbf{m}} = (\mathbf{L}_1 + \mathrm{i}\boldsymbol{\lambda}\cdot\mathbf{m})^{-1}\mathcal{E}_{\mathbf{m}}$. By $\boldsymbol{\lambda}\cdot\overline{\mathbf{m}} = -\boldsymbol{\lambda}\cdot\mathbf{m}$, we conclude $\boldsymbol{\phi}_{\overline{\mathbf{m}}} = \overline{\boldsymbol{\phi}}_{\mathbf{m}}$.

Let us now consider $\mathbf{m}\in\boldsymbol{\Lambda}_j$. We assume by induction $\boldsymbol{\phi}_{\mathbf{m}'}$ have been defined for $|\mathbf{m}'| < |\mathbf{m}|$ and so too $\boldsymbol{\lambda}_{\mathbf{n}'}$ for $\|\mathbf{n}'\| < \|\mathbf{m}\| - 1$. Then, for $\mathbf{m} = \mathbf{n} + \mathbf{e}^j$ where $\mathbf{n}\in\boldsymbol{\Lambda}_0$, $\widehat{\mathcal{R}}_{\mathbf{m}} = 0$ becomes

$$\begin{aligned}&(\mathbf{L}_1 + \mathrm{i}\lambda_j)\,\boldsymbol{\phi}_{\mathbf{m}} = -\mathrm{i}\boldsymbol{\lambda}_{\mathbf{n}}\cdot\mathbf{e}^j\boldsymbol{\Phi}_j - \mathcal{K}_{\mathbf{m}} \text{ with}\\ &\mathcal{K}_{\mathbf{m}} := g_{\mathbf{m}} - \sum_{\substack{\mathbf{m}'+\mathbf{n}'=\mathbf{m}\\ \mathbf{m}'\in\mathbf{NR},|\mathbf{m}'|\ge 2,\ \mathbf{n}'\in\boldsymbol{\Lambda}_0}} \mathrm{i}\boldsymbol{\lambda}_{\mathbf{n}'}\cdot\mathbf{m}'\boldsymbol{\phi}_{\mathbf{m}'}.\end{aligned} \tag{1.44}$$

This equation can be solved if we impose $\left(\mathbf{J}\mathcal{E}_{\mathbf{m}}, \overline{\boldsymbol{\Phi}}_j\right) = 0$, that is, for $\lambda_{\mathbf{n}j} := \boldsymbol{\lambda}_{\mathbf{n}}\cdot\mathbf{e}^j$,

$$-\mathrm{i}\lambda_{\mathbf{n}j}\left(\mathbf{J}\boldsymbol{\Phi}_j, \overline{\boldsymbol{\Phi}}_j\right) = -\mathrm{i}\lambda_{\mathbf{n}j}(-\mathrm{i}) = -\lambda_{\mathbf{n}j} = \left(\mathbf{J}\mathcal{K}_{\mathbf{m}}, \overline{\boldsymbol{\Phi}}_j\right),$$

which is true for $\lambda_{\mathbf{n}j} = -\left(\mathbf{J}\mathcal{K}_{\mathbf{m}}, \overline{\boldsymbol{\Phi}}_j\right)$. Then we can solve for $\boldsymbol{\phi}_{\mathbf{m}} = -(\mathbf{L}_1 + \mathrm{i}\lambda_j)^{-1}\mathcal{K}_{\mathbf{m}}$ in the complement, in (1.19), of $\ker(\mathbf{L}_1 - \mathrm{i}\lambda_j)$.
We want to show that $\lambda_{\mathbf{n}j}\in\mathbb{R}$. For the corresponding $\overline{\mathbf{m}}\in\overline{\boldsymbol{\Lambda}}_j$, we have

$$\begin{aligned}&(\mathbf{L}_1 - \mathrm{i}\lambda_j)\,\boldsymbol{\phi}_{\overline{\mathbf{m}}} = -\mathrm{i}\boldsymbol{\lambda}_{\mathbf{n}}\cdot\overline{\mathbf{e}}^j\overline{\boldsymbol{\Phi}}_j - \mathcal{K}_{\overline{\mathbf{m}}} \text{ with}\\ &\mathcal{K}_{\overline{\mathbf{m}}} := g_{\overline{\mathbf{m}}} - \sum_{\substack{\overline{\mathbf{m}}'+\mathbf{n}'=\overline{\mathbf{m}}\\ \mathbf{m}'\in\mathbf{NR_2},\ \mathbf{n}'\in\boldsymbol{\Lambda}_0}} \mathrm{i}\boldsymbol{\lambda}_{\mathbf{n}'}\cdot\overline{\mathbf{m}}'\boldsymbol{\phi}_{\overline{\mathbf{m}}'}.\end{aligned} \tag{1.45}$$

Notice that by induction $\mathcal{K}_{\overline{\mathbf{m}}} = \overline{\mathcal{K}}_{\mathbf{m}}$. Since $\boldsymbol{\lambda}_{\mathbf{n}}\cdot\overline{\mathbf{e}}^j = -\lambda_{\mathbf{n}j}$, taking the complex conjugate of (1.44) we obtain

$$\begin{aligned}&(\mathbf{L}_1 - \mathrm{i}\lambda_j)\,\boldsymbol{\phi}_{\overline{\mathbf{m}}} = \mathrm{i}\lambda_{\mathbf{n}j}\overline{\boldsymbol{\Phi}}_j - \overline{\mathcal{K}}_{\mathbf{m}} \text{ and}\\ &(\mathbf{L}_1 - \mathrm{i}\lambda_j)\,\overline{\boldsymbol{\phi}}_{\mathbf{m}} = \mathrm{i}\overline{\lambda}_{\mathbf{n}j}\overline{\boldsymbol{\Phi}}_j - \overline{\mathcal{K}}_{\mathbf{m}}.\end{aligned} \tag{1.46}$$

Applying $(\mathbf{J}\cdot, \boldsymbol{\Phi}_j)$ on both the last two equations, we obtain

$$\mathrm{i}\lambda_{\mathbf{n}j}\left(\mathbf{J}\overline{\boldsymbol{\Phi}}_j, \boldsymbol{\Phi}_j\right) = \left(\mathbf{J}\overline{\mathcal{K}}_{\mathbf{m}}, \boldsymbol{\Phi}_j\right) \text{ and } \mathrm{i}\overline{\lambda}_{\mathbf{n}j}\left(\mathbf{J}\overline{\boldsymbol{\Phi}}_j, \boldsymbol{\Phi}_j\right) = \left(\mathbf{J}\overline{\mathcal{K}}_{\mathbf{m}}, \boldsymbol{\Phi}_j\right).$$

Hence $\lambda_{\mathbf{n}j} = \overline{\lambda}_{\mathbf{n}j}$ and we have proved that $\lambda_{\mathbf{n}j} \in \mathbb{R}$.
Since the equations in (1.46) are the same, we conclude $\boldsymbol{\phi}_{\overline{\mathbf{m}}} = \overline{\boldsymbol{\phi}}_{\mathbf{m}}$.
We consider now $\boldsymbol{\mathcal{R}}[\mathbf{z}] = \widehat{\boldsymbol{\mathcal{R}}}[\mathbf{z}] - D_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}]\widetilde{\mathbf{z}}_R$ where we seek $\widetilde{\mathbf{z}}_R$ so that (1.39) is true. This will follow from

$$\left\langle \mathbf{J}\widetilde{\boldsymbol{\mathcal{R}}}[\mathbf{z}], D_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}]\boldsymbol{\zeta}\right\rangle - \langle \mathbf{J}D_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}]\widetilde{\mathbf{z}}_R, D_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}]\boldsymbol{\zeta}\rangle = 0 \text{ for the standard basis}$$
$$\boldsymbol{\zeta} = e_1, \mathrm{i}e_1, \dots, e_N, \mathrm{i}e_N.$$

Since the restriction of $\langle \mathbf{J}\cdot,\cdot\rangle$ in L^2_{discr} is a non-degenerate symplectic form and from $\boldsymbol{\phi}_{\mathbf{e}^j} = \boldsymbol{\Phi}_j$ and $\boldsymbol{\phi}_{\overline{\mathbf{e}}^j} = \overline{\boldsymbol{\Phi}}_j$, the Implicit Function Theorem guarantees the existence of $\widetilde{\mathbf{z}}_R \in C^\infty(\mathcal{B}_{\mathbb{C}^N}(0,\alpha_0),\mathbb{C}^N)$ with $\widetilde{\mathbf{z}}_R(\mathbf{0}) = \mathbf{0}$ for a sufficiently small $\alpha_0 > 0$. Furthermore, from the last formula and from the fact that in the expansion (1.43) we have $\widehat{\mathcal{R}}_{\mathbf{m}} = 0$ for all $\mathbf{m} \in \mathbf{NR}$, we obtain the bound (1.34). This in turn implies expansion (1.37) and bound (1.38). ☐

Let us consider now the expansion (1.37). An important assumption, related to the Fermi Golden Rule (FGR), is the following.

Assumption 1.12. We assume that for all $\mathbf{m} \in \mathbf{R}_{\min}$,

$$\sum_{\sigma=\pm1}\Big|\Big[-\mathrm{i}\sqrt{(\boldsymbol{\lambda}\cdot\mathbf{m})^2-\omega^2}\,\widehat{(\mathcal{R}_{\mathbf{m}})}_1(\sigma\sqrt[4]{(\boldsymbol{\lambda}\cdot\mathbf{m})^2-\omega^2}) + \widehat{(\mathcal{R}_{\mathbf{m}})}_2(\sigma\sqrt[4]{(\boldsymbol{\lambda}\cdot\mathbf{m})^2-\omega^2})\Big]\Big| > 0,$$

where $(\mathcal{R}_{\mathbf{m}})_j$ are the two components of $\mathcal{R}_{\mathbf{m}}$ for $j = 1, 2$ and we are taking the distorted Fourier transform associated to operator L_1, for which we refer to Weder [46].

2. Modulation and transformed equations

For small $\alpha \in (0, 1)$ we set

$$\mathcal{M}_\alpha = \{\boldsymbol{\phi}[\mathbf{z}] \mid \mathbf{z} \in \mathcal{B}_{\mathbb{C}^N}(0,\alpha)\}, \text{ where } \mathcal{B}_{\mathbb{C}^N}(0,\alpha) := \{\mathbf{w} \in \mathbb{C}^N \mid |\mathbf{w}| < \alpha\}$$

We first observe the following.

Lemma 2.1. *There is an $\alpha_0 \in (0, 1)$ such that for $\alpha \in (0, \alpha_0)$ the map $\mathbf{z} \to \boldsymbol{\phi}[\mathbf{z}]$ in an embedding $\mathcal{B}_{\mathbb{C}^N}(0,\alpha) \hookrightarrow \mathbf{E}_{\mathbf{H}}$.*

Proof. It is clear that the map is smooth. Next we observe that for $\alpha > 0$ sufficiently small, the above map $\mathcal{B}_{\mathbb{C}^N}(0,\alpha) \to \mathbf{E}_{\mathbf{H}}$ is an embedding. This follows from the fact that the partial derivatives computed at $\mathbf{z} = 0$ span L^2_{discr}, which is symplectic with respect to the form $\langle \mathbf{J}\cdot,\cdot\rangle$. ☐

We set

$$\mathcal{H}_c[\mathbf{z}] := \{\mathbf{u} \in \mathbf{H} + \Sigma^* \mid \forall\boldsymbol{\zeta} \in \mathbb{C}^N, \langle \mathbf{J}\mathbf{u}, D_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}]\boldsymbol{\zeta}\rangle = 0\}. \tag{2.1}$$

Lemma 2.2. (Modulation) *There exists* $\delta > 0$ *s.t. there exists* $\mathbf{z}(\cdot) \in C^\infty (B_{\mathbf{E_H}}(0, \delta), \mathbb{C}^N)$ *s.t.*

$$\boldsymbol{\eta}(\mathbf{u}) := \mathbf{u} - \boldsymbol{\phi}[\mathbf{z}(\mathbf{u})] \in \mathcal{H}_c[\mathbf{z}(\mathbf{u})] \tag{2.2}$$

and, leaving implicit the dependence of $\mathbf{z}$ *and* $\boldsymbol{\eta}$ *on* $\mathbf{u}$*,*

$$|\mathbf{z}| + \|\boldsymbol{\eta}\|_{\boldsymbol{\mathcal{H}}^1} \sim \|\mathbf{u} - \mathbf{H}\|_{\boldsymbol{\mathcal{H}}^1}. \tag{2.3}$$

Proof. For $z_{jR} = \mathrm{Re}(z_j)$ and $z_{jI} = \mathrm{Im}(z_j)$, consider a function $F(\mathbf{u}, \mathbf{z})$ with components

$$\left\langle \mathbf{J}\left(\mathbf{u} - \boldsymbol{\phi}[\mathbf{z}]\right), \partial_{z_{jJ}} \boldsymbol{\phi}[\mathbf{z}] \right\rangle \text{ for } j = 1, \dots, \widetilde{N} \text{ and } J = R, I. \tag{2.4}$$

Then $F \in C^\infty \left(\mathbf{E_H} \times \mathbb{C}^N, \mathbb{R}^{2N}\right)$, trivially we have $F(\mathbf{H}, 0) = 0$ and the Jacobian matrix $\frac{\partial F}{\partial \mathbf{z}}(\mathbf{H}, 0)$ a non-degenerate $N \times N$ matrix, exactly because, for the space in L^2_{discr} in (1.19), the form $\langle \mathbf{J}\cdot, \cdot\rangle$ is symplectic. Then, by Implicit Function Theorem, there exists the desired function $\mathbf{z}(\mathbf{u})$ such that $F(\mathbf{u}, \mathbf{z}(\mathbf{u})) = 0$, i.e. which satisfies (2.2), with $\mathbf{z}(\mathbf{H}) = 0$. The fact that $|\mathbf{z}| + \|\boldsymbol{\eta}\|_{\boldsymbol{\mathcal{H}}^1} \lesssim \|\mathbf{u} - \mathbf{H}\|_{\boldsymbol{\mathcal{H}}^1}$ follows from the Lipschitz regularity of $\mathbf{u} \to (\mathbf{z}(\mathbf{u}), \boldsymbol{\eta}(\mathbf{u}))$ at $\mathbf{H}$, while we have

$$\|\mathbf{u} - \mathbf{H}\|_{\boldsymbol{\mathcal{H}}^1} = \|\boldsymbol{\phi}[\mathbf{z}] - \mathbf{H} + \boldsymbol{\eta}\|_{\boldsymbol{\mathcal{H}}^1} \le \|\boldsymbol{\phi}[\mathbf{z}] - \mathbf{H}\|_{\boldsymbol{\mathcal{H}}^1} + \|\boldsymbol{\eta}\|_{\boldsymbol{\mathcal{H}}^1} \lesssim |\mathbf{z}| + \|\boldsymbol{\eta}\|_{\boldsymbol{\mathcal{H}}^1}.$$

□

Substituting $\mathbf{u} = \boldsymbol{\phi}[\mathbf{z}] + \boldsymbol{\eta}$, we obtain

$$\partial_t \boldsymbol{\eta} + D\boldsymbol{\phi}[\mathbf{z}](\dot{\mathbf{z}} - \widetilde{\mathbf{z}}) = \mathbf{L}_1 \boldsymbol{\eta} + (\mathbf{L}[\mathbf{z}] - \mathbf{L}_1)\boldsymbol{\eta} + \mathbf{J}\mathbf{F}[\mathbf{z}, \boldsymbol{\eta}] + \mathbf{R}[\mathbf{z}], \tag{2.5}$$

where

$$\mathbf{L}[\mathbf{z}] := \mathbf{J}\mathbf{H}[\mathbf{z}], \;\; \mathbf{H}[\mathbf{z}] = \begin{pmatrix} -\partial_x^2 + W''(\phi_1[\mathbf{z}]) & 0 \\ 0 & 1 \end{pmatrix} \tag{2.6}$$

$$\begin{aligned}\mathbf{F}[\mathbf{z}, \boldsymbol{\eta}] := \begin{pmatrix} F_1[\mathbf{z}, \eta_1] \\ 0 \end{pmatrix}, \text{ where } F_1[\mathbf{z}, \eta_1] &:= W'(\phi_1[\mathbf{z}] + \eta_1) - W'(\phi_1[\mathbf{z}]) \\ &\quad - W''(\phi_1[\mathbf{z}])\eta_1. \end{aligned} \tag{2.7}$$

We denote by P_{c} the projection onto L^2_{disp} associated to the splitting (1.19) and let $P_d = 1 - P_{\mathrm{c}}$.

Remark 2.3. Notice that $L_1 = \mathrm{diag}(L_1, L_1)$ commutes with $\mathbf{L}_1$ and with the resolvent $R_{\mathbf{L}_1}(\varsigma) := (\mathbf{L}_1 - \varsigma)^{-1}$ for ς in the resolvent set of $\mathbf{L}_1$. It then follows that L_1 commutes with the projections P_d and P_c.

Lemma 2.4. (Inverse of P_{c}) *There exists an* $\alpha_0 > 0$ *and* $R[\mathbf{z}] \in C^\infty\left(\mathcal{B}_{\mathbb{C}^N}(0, \alpha_0), \mathcal{L}(L^2)\right)$ *s.t.* $R[\mathbf{z}]P_c|_{\mathcal{H}_c[\mathbf{z}]} = 1|_{\mathcal{H}_c[\mathbf{z}]}$, $P_c R[\mathbf{z}]|_{P_c(\boldsymbol{\Sigma}^l)^*} = 1|_{P_c(\boldsymbol{\Sigma}^l)^*}$ *for all* $l \in \mathbb{N}_0$ *and*

$$\|R[\mathbf{z}] - 1\|_{(\boldsymbol{\Sigma}^l)^* \to \boldsymbol{\Sigma}^l} \lesssim_l |\mathbf{z}|. \tag{2.8}$$

Proof. Let us write, summing on repeated indexes,

$$\begin{aligned} R[\mathbf{z}] = 1 + \langle \mathbf{J}\cdot, C_{j\mathcal{A}}[\mathbf{z}]\rangle\, \partial_{z_{j\mathcal{A}}} \boldsymbol{\phi}[\mathbf{0}], \;\; &\text{with } j = 1, \dots, N, \mathcal{A} = R, I, z_{jR} := \mathrm{Re}\, z_j \\ &\text{and } z_{jI} := \mathrm{Im}\, z_j. \end{aligned}$$

Then $R[\mathbf{z}]\boldsymbol{\theta} \in \mathcal{H}_c[\mathbf{z}]$ for all $\boldsymbol{\theta} \in L^2_{\mathrm{odd}}(\mathbb{R}, \mathbb{C}^2)$ is equivalent to

$$\left\langle \mathbf{J}\boldsymbol{\theta} + \mathbf{J}\left\langle \mathbf{J}\boldsymbol{\theta}, C_{j\mathcal{A}}[\mathbf{z}]\right\rangle \partial_{z_{j\mathcal{A}}}\boldsymbol{\phi}[\mathbf{0}], \partial_{z_{j'\mathcal{A}'}}\boldsymbol{\phi}[\mathbf{z}]\right\rangle = 0 \text{ for all} j' = 1, \dots, N, \mathcal{A}' = R, I$$

or, equivalently, for all $\boldsymbol{\theta} \in L^2_{\mathrm{odd}}(\mathbb{R}, \mathbb{C}^2)$

$$\left\langle \left\langle \mathbf{J}\boldsymbol{\theta}, C_{j\mathcal{A}}[\mathbf{z}]\right\rangle \mathbf{J}\partial_{z_{j\mathcal{A}}}\boldsymbol{\phi}[\mathbf{0}], \partial_{z_{j'\mathcal{A}'}}\boldsymbol{\phi}[\mathbf{z}]\right\rangle = \left\langle \mathbf{J}\boldsymbol{\theta}, \left\langle \mathbf{J}\partial_{z_{j\mathcal{A}}}\boldsymbol{\phi}[\mathbf{0}], \partial_{z_{j'\mathcal{A}'}}\boldsymbol{\phi}[\mathbf{z}]\right\rangle C_{j\mathcal{A}}[\mathbf{z}]\right\rangle =$$
$$- \left\langle J\boldsymbol{\theta}, \partial_{z_{j'\mathcal{A}'}}\boldsymbol{\phi}[\mathbf{z}]\right\rangle,$$

that is, still summing on the repeated indexes $j = 1, \dots, N$, $\mathcal{A} = R, I$,

$$\begin{aligned}&\left\langle \mathbf{J}\partial_{z_{j\mathcal{A}}}\boldsymbol{\phi}[\mathbf{0}], \partial_{z_{j'\mathcal{A}'}}\boldsymbol{\phi}[\mathbf{z}]\right\rangle C_{j\mathcal{A}}[\mathbf{z}] \\ &\quad = -\partial_{z_{j'\mathcal{A}'}}\boldsymbol{\phi}[\mathbf{z}] \text{ for all } j' = 1, \dots, N, \mathcal{A}' = R, I.\end{aligned}$$

By the invertibility of the matrix $\left\{\left\langle \mathbf{J}\partial_{z_{j\mathcal{A}}}\boldsymbol{\phi}[\mathbf{0}], \partial_{z_{j'\mathcal{A}'}}\boldsymbol{\phi}[\mathbf{z}]\right\rangle\right\}$, this equation has a solution for $|\mathbf{z}|$ small, which is unique. So $\mathbf{z} \to C_{j\mathcal{A}}[\mathbf{z}]$ is smooth near $\mathbf{0}$ with values in $\boldsymbol{\Sigma}^l$ for all $l \in \mathbb{N}_0$. We conclude $R[\mathbf{z}] \in \mathcal{L}\left((\boldsymbol{\Sigma}^l)^*, \mathcal{H}_c[\mathbf{z}]\right)$. Now

$$P_c R[\mathbf{z}] = P_c + \langle \mathbf{J}\cdot, C_{j\mathcal{A}}[\mathbf{z}]\rangle P_c \partial_{z_{j\mathcal{A}}}\boldsymbol{\phi}[\mathbf{0}] = P_c$$

so it equals $1|_{P_c(\boldsymbol{\Sigma}^l)^*}$ when restricted in $P_c(\boldsymbol{\Sigma}^l)^*$, and in particular for $l = 1$.

Next, notice that for $\boldsymbol{\theta} \in \mathcal{H}_c[\mathbf{z}]$, we have $R[\mathbf{z}]P_c\boldsymbol{\theta} \in \mathcal{H}_c[\mathbf{z}]$ with $P_c\boldsymbol{\theta} = P_c R[\mathbf{z}]P_c\boldsymbol{\theta}$. Since, for $|\mathbf{z}|$ small, P_c is an isomorphism from $\mathcal{H}_c[\mathbf{z}]$ to L^2_{disp}, we have $R[\mathbf{z}]P_c|_{\mathcal{H}_c[\mathbf{z}]} = 1|_{\mathcal{H}_c[\mathbf{z}]}$. □

We set $\widetilde{\boldsymbol{\eta}} = P_c\boldsymbol{\eta}$ (and thus $\boldsymbol{\eta} = R[\mathbf{z}]\widetilde{\boldsymbol{\eta}}$). Then, $\widetilde{\boldsymbol{\eta}}$ satisfies

$$\partial_t \widetilde{\boldsymbol{\eta}} = \mathbf{L}_1\widetilde{\boldsymbol{\eta}} + \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}} P_c \mathcal{R}_{\mathbf{m}} + \mathbf{R}_{\widetilde{\eta}}, \tag{2.9}$$

where

$$\mathbf{R}_{\widetilde{\eta}} = P_c\boldsymbol{\mathcal{R}}_1[\mathbf{z}] + P_c\mathbf{J}\mathbf{F} + P_c(\mathbf{L}[\mathbf{z}] - \mathbf{L}_1)\boldsymbol{\eta} - P_c D\boldsymbol{\phi}[\mathbf{z}](\dot{\mathbf{z}} - \widetilde{\mathbf{z}}) + P_c\mathbf{L}_1(R[\mathbf{z}] - 1)\widetilde{\boldsymbol{\eta}}. \tag{2.10}$$

We set

$$\mathcal{T} := \langle \mathrm{i}\varepsilon\partial_x\rangle^{-\widetilde{N}} \mathcal{A}^* \text{ and} \tag{2.11}$$
$$\mathbf{v} := \mathcal{T}\chi_{B^2}\widetilde{\boldsymbol{\eta}}. \tag{2.12}$$

Then, also multiplying by the imaginary unit i, we obtain

$$\mathrm{i}\partial_t\mathbf{v} = \mathrm{i}\mathbf{L}_D\mathbf{v} + \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}}\mathrm{i}\widetilde{\mathcal{R}}_{\mathbf{m}} + \mathrm{i}\mathbf{R}_{\mathbf{v}}, \tag{2.13}$$

where

$$\mathbf{L}_D := \begin{pmatrix} 0 & 1 \\ -L_D & 0 \end{pmatrix} \tag{2.14}$$

$$\widetilde{\mathcal{R}}_{\mathbf{m}} := \mathcal{T}\chi_{B^2} P_c \mathcal{R}_{\mathbf{m}} \text{ and} \tag{2.15}$$

$$\mathbf{R}_{\mathbf{v}} = \mathcal{T}\chi_{B^2}P_c\mathbf{R}_{\widetilde{\boldsymbol{\eta}}} + \begin{pmatrix} 0 \\ -\mathcal{T}(2\chi'_{B^2}\partial_x + \chi''_{B^2})\widetilde{\eta}_1 + \langle \mathrm{i}\varepsilon\partial_x\rangle^{-\widetilde{N}}\left[V_D, \langle \mathrm{i}\varepsilon\partial_x\rangle^{\widetilde{N}}\right]v_1 \end{pmatrix}. \tag{2.16}$$

Before stating our main estimates, we state the following orbital stability result, which follows from the Modulation Lemma 2.2 and the Orbital Stability Theorem in [26], in fact also the classical [15].

Proposition 2.5. (Orbital stability) *There exist $C > 0$ and $\delta_0 > 0$ such that, for $\delta := \|\mathbf{u}(0) - \mathbf{H}\|_{\boldsymbol{\mathcal{H}}^1} < \delta_0$ the claim in line (1.9) is true for all $t \in \mathbb{R}$ and we have*

$$\|\mathbf{z}\|_{L^\infty(\mathbb{R})} + \|\boldsymbol{\eta}\|_{L^\infty(\mathbb{R},H^1)} \le C_0\delta. \tag{2.17}$$

□

Notice that the above result and (1.10), which will be proved below, along with Lemma 1.9 guarantee by elementary arguments the limit (1.11). So here the main point is (1.10) for $I = \mathbb{R}_+$.

3. Main estimates and proof of Theorem 1.3

The proof of (1.10) in Theorem 1.3 is by means of a continuation argument. In particular, we will show the following.

Proposition 3.1. *Assumptions 1.6, 1.8 and 1.12 are given. Then for any small $\epsilon > 0$ there exists a $\delta_0 = \delta_0(\epsilon)$ s.t. if (1.10) holds for $I = [0,T]$ for some $T > 0$ and for $\delta \in (0, \delta_0)$ then in fact for $I = [0,T]$ inequality (1.10) holds for ϵ replaced by $\epsilon/2$.*

Theorem 1.3 is a corollary of Proposition 3.1.

Proof. By completely routine arguments, which we skip, it is possible to show that Proposition 3.1 implies (1.10) with $I = [0,\infty)$. The time reversibility of the system, yields immediately (1.10) for $I = \mathbb{R}$. Finally, (1.11) follows from the integrabililty of $|z_j|^{2m_j}$ where m_j is the smallest integer satisfying $\omega < m_j\lambda_j$, which follows from the FGR estimate given in Proposition 3.7 below and the boundedness of $\dot{\mathbf{z}}$ which can be easily obtained from the modulation equation and orbital stability. □

We set $\chi \in C^\infty_{\mathrm{even}}(\mathbb{R})$ to satisfy $1_{|x|\le 1} \le \chi \le 1_{|x|\le 2}$ and $\chi'(x) \le 0$ for $x > 0$. For $C > 0$,

$$\zeta_C(x) := \exp\left(-\frac{|x|}{C}\left(1 - \chi(x)\right)\right), \ \varphi_C(x) := \int_0^x \zeta_C^2(y)\,dy. \tag{3.1}$$

We will consider constants $A, B, \varepsilon > 0$ satisfying

$$\log(\delta^{-1}) \gg \log(\epsilon^{-1}) \gg A \gg B^2 \gg B \gg \exp\left(\varepsilon^{-1}\right) \gg 1. \tag{3.2}$$

We will denote by $o_\varepsilon(1)$ constants depending on ε such that

$$o_\varepsilon(1) \xrightarrow{\varepsilon\to 0^+} 0. \tag{3.3}$$

We set

$$\mathbf{w} = \zeta_A \widetilde{\boldsymbol{\eta}}, \quad \boldsymbol{\xi} := \chi_{B^2}\zeta_B \mathbf{v}. \tag{3.4}$$

We will prove the following continuation argument.

Proposition 3.2. *Assumptions 1.6, 1.8 and 1.12 are given. Then for any small $\epsilon > 0$ there exists a $\delta_0 = \delta_0(\epsilon)$ s.t. if in $I = [0, T]$ we have*

$$\|\dot{\mathbf{z}} - \widetilde{\mathbf{z}}\|_{L^2(I)} + \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \|\mathbf{z}^{\mathbf{m}}\|_{L^2(I)} + \|\boldsymbol{\xi}\|_{L^2(I,\widetilde{\boldsymbol{\Sigma}})} + \|\mathbf{w}\|_{L^2(I,\widetilde{\boldsymbol{\Sigma}})} \le \epsilon \tag{3.5}$$

then for $\delta \in (0, \delta_0)$ inequality (3.5) holds for ϵ replaced by $o_\varepsilon(1)\epsilon$ where $o_\varepsilon(1) \xrightarrow{\varepsilon\to 0^+} 0$.

Notice that Proposition 3.2 implies Proposition 3.1. In the following, we always assume the assumptions of the claim of Proposition 3.2, which are true for $T > 0$ small enough.

The following is proved is Proposition 9.1 of [8].

Proposition 3.3. (Coercivity) *We have*

$$\|w_1\|_{L^2_{-\frac{a}{10}}} \lesssim \|\xi_1\|_{\widetilde{\Sigma}} + e^{-\frac{B}{20}} \|w_1'\|_{L^2}. \tag{3.6}$$

□

In analogy to [8], we now consider essentially two virial estimates, one for $\mathbf{w}$ and the other for $\boldsymbol{\xi}$. The first is based directly on the equation for $\widetilde{\boldsymbol{\eta}}$, (2.9).

Proposition 3.4. *[1st virial estimate] We have*

$$\begin{aligned} \|w_1'\|_{L^2(I,L^2)} + \|w_2\|_{L^2(I,L^2_{-\frac{a}{10}})} &\lesssim o_\varepsilon(1)\epsilon \\ &+ \|w_1\|_{L^2(I,L^2_{-\frac{a}{10}})} + \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \|\mathbf{z}^{\mathbf{m}}\|_{L^2(I)} + \delta\|\dot{\mathbf{z}} - \widetilde{\mathbf{z}}\|_{L^2(I)}. \end{aligned} \tag{3.7}$$

The second virial estimate, involves the transformed problem (2.13).

Proposition 3.5. (2nd virial) *We have*

$$\|\boldsymbol{\xi}\|_{L^2(I,\widetilde{\boldsymbol{\Sigma}})} \lesssim o_\varepsilon(1)\epsilon + \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \|\mathbf{z}^{\mathbf{m}}\|_{L^2(I)} + o_\varepsilon(1)\|\dot{\mathbf{z}} - \widetilde{\mathbf{z}}\|_{L(I)2} + o_\varepsilon(1)\|\mathbf{w}\|_{L^2,\widetilde{\boldsymbol{\Sigma}})} \tag{3.8}$$

We will also need a control of modulation parameters.

Proposition 3.6. *We have*

$$\|\dot{\mathbf{z}} - \widetilde{\mathbf{z}}\|_{L^2(I)} = o_\varepsilon(1)\|\mathbf{w}\|_{L^2(I,L^2_{-\frac{a}{10}})}. \tag{3.9}$$

The last ingredient is the FGR estimate.

Proposition 3.7. (FGR estimate) *We have*

$$\sum_{\mathbf{m}\in\mathbf{R}_{\min}} \|\mathbf{z}^{\mathbf{m}}\|_{L^2(I)} = o_\varepsilon(1)\epsilon. \tag{3.10}$$

3.1. Proof of Proposition 3.2 assuming Propositions 3.4–3.7

By (3.9)–(3.10) and by the relation between $A, B, \varepsilon, \epsilon$ and δ in (3.2), we have

$$\|\dot{\mathbf{z}} - \widetilde{\mathbf{z}}\|_{L^2(I)} + \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \|\mathbf{z}^{\mathbf{m}}\|_{L^2(I)} \le o_\varepsilon(1)\epsilon. \tag{3.11}$$

Entering this in (3.8) we get

$$\|\boldsymbol{\xi}\|_{L^2(I,\widetilde{\boldsymbol{\Sigma}})} \le o_\varepsilon(1)\epsilon \tag{3.12}$$

which, fed in (3.6), yields

$$\|w_1\|_{L^2(I,L^2_{-\frac{a}{10}})} \le o_\varepsilon(1)\epsilon.$$

Using this in (3.7), we obtain

$$\|w_1'\|_{L^2(I,L^2)} + \|w_2\|_{L^2(I,L^2_{-\frac{a}{10}})} \le o_\varepsilon(1)\epsilon.$$

This and the previous one together, yield

$$\|\mathbf{w}\|_{L^2(I,\widetilde{\boldsymbol{\Sigma}})} \le o_\varepsilon(1)\epsilon. \tag{3.13}$$

Taken together, (3.11)–(3.13) yield the improvement $o_\varepsilon(1)\epsilon$ of the statement of Proposition 3.2, concluding the proof. □

We now turn to the proofs of Propositions 3.4–3.7. The structure of the proofs is very similar to the analogous ones in [8]. In particular, Propositions 3.4–3.5 are very close to Kowalczyk et al. [25]. The proof of Proposition 3.7 requires the introduction of an additional variable $\mathbf{g}$, which, like in [8], is bounded using smoothing estimates: in particular here we borrow from Komech–Kapytula [19,20].

4. First virial estimate, for w

Recall $\langle f, g\rangle = Re(f, \overline{g})$, see (1.16). For $A \gg 1$ to be determined, we set

$$\mathcal{I}_1(\widetilde{\boldsymbol{\eta}}) := \frac{1}{2}\left\langle \mathbf{J}\widetilde{\boldsymbol{\eta}}, S_A\widetilde{\boldsymbol{\eta}}\right\rangle, \quad \mathcal{I}_2(\widetilde{\boldsymbol{\eta}}) := \frac{1}{2}\left\langle \mathbf{J}\widetilde{\boldsymbol{\eta}}, \zeta_{\widetilde{A}}^2\sigma_3\widetilde{\boldsymbol{\eta}}\right\rangle, \text{ with } \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix},$$

where $\widetilde{A}^{-1} = A^{-1} + \frac{a}{10}$ and

$$S_A := \frac{1}{2}\varphi_A' + \varphi_A\partial_x.$$

Remark 4.1. By the definition of $\widetilde{A}$ and (3.1), we have $\zeta_{\widetilde{A}} = \zeta_{\frac{10}{a}}\zeta_A$.

Lemma 4.2. *For any $c \in (0,1)$, we have*

$$\frac{d}{dt}\mathcal{I}_1(\widetilde{\boldsymbol{\eta}}) + \frac{1}{2}\|w_1'\|^2_{L^2} \lesssim \|w_1\|^2_{L^2_{-\frac{a}{10}}} + c\|w_2\|^2_{L^2_{-\frac{a}{10}}} + c^{-1}\sum_{\mathbf{m}\in\mathbf{R}_{\min}} |\mathbf{z}^{\mathbf{m}}|^2 + \delta|\dot{\mathbf{z}} - \widetilde{\mathbf{z}}|^2, \tag{4.1}$$

$$-\frac{d}{dt}\mathcal{I}_2(\widetilde{\boldsymbol{\eta}}) + \frac{1}{2}\|w_2\|^2_{L^2_{-\frac{a}{10}}} \lesssim \|w_1\|^2_{L^2_{-\frac{a}{10}}} + \|w_1'\|^2_{L^2} + \sum_{\mathbf{m}\in\mathbf{R}_{\min}} |\mathbf{z}^{\mathbf{m}}|^2 + \delta|\dot{\mathbf{z}} - \widetilde{\mathbf{z}}|^2, \tag{4.2}$$

where the implicit constants are independent of c.

Proof of Proposition 3.4. From the orbital stability bound Proposition 2.5, we have $|\mathcal{I}_1(\widetilde{\boldsymbol{\eta}})| \lesssim A\delta^2$ and $|\mathcal{I}_2(\widetilde{\boldsymbol{\eta}})| \lesssim \delta^2$. Thus, integrating (4.1) and (4.2), we have

$$\begin{aligned}\frac{1}{2}&\|w_1'\|_{L^2(I,L^2)}^2\\ &\leq C_1\Bigg(A\delta^2+\|w_1\|_{L^2(I,L^2_{-\frac{a}{10}})}^2+c\|w_2\|_{L^2(I,L^2(I))}^2\\ &\quad+c^{-1}\sum_{\mathbf{m}\in\mathbf{R}_{\min}}\|\mathbf{z}^{\mathbf{m}}\|_{L^2(I)}^2+\delta^2\|\dot{\mathbf{z}}-\widetilde{\mathbf{z}}\|_{L^2(I)}^2\Bigg),\end{aligned} \tag{4.3}$$

$$\begin{aligned}\frac{1}{2}&\|w_2\|_{L^2(I,L^2)}^2\\ &\leq C_2\Bigg(\delta^2+\|w_1\|_{L^2(I,L^2_{-\frac{a}{10}})}^2+\delta\|w_1'\|_{L^2(I,L^2(I))}^2\\ &\quad+\sum_{\mathbf{m}\in\mathbf{R}_{\min}}\|\mathbf{z}^{\mathbf{m}}\|_{L^2(I)}^2+\delta^2\|\dot{\mathbf{z}}-\widetilde{\mathbf{z}}\|_{L^2(I)}^2\Bigg).\end{aligned} \tag{4.4}$$

Taking c sufficiently small so that $4cC_1C_2 \leq 1$ and substituting (4.4) into (4.3), we can bound $\|w_1'\|_{L^2(I,L^2)}^2 \lesssim (\text{r.h.s. of } (3.7))^2$. Finally, using (4.4) again, we have the conclusion. □

The remainder of this section is devoted to the proof of Lemma 4.2. First, since both $\langle \mathbf{J}\cdot, S_A\cdot\rangle$ and $\langle \mathbf{J}\cdot, \zeta_A^2\sigma_3\cdot\rangle$ are symmetric, we have

$$\frac{d}{dt}\mathcal{I}_1(\widetilde{\boldsymbol{\eta}}) = \left\langle \mathbf{J}\left(\mathbf{L}_1\widetilde{\boldsymbol{\eta}}+\sum_{\mathbf{m}\in\mathbf{R}_{\min}}\mathbf{z}^{\mathbf{m}}G_{\mathbf{m}}+\mathbf{R}_{\widetilde{\eta}}\right), S_A\widetilde{\boldsymbol{\eta}}\right\rangle, \tag{4.5}$$

$$\frac{d}{dt}\mathcal{I}_2(\widetilde{\boldsymbol{\eta}}) = \left\langle \mathbf{J}\left(\mathbf{L}_1\widetilde{\boldsymbol{\eta}}+\sum_{\mathbf{m}\in\mathbf{R}_{\min}}\mathbf{z}^{\mathbf{m}}G_{\mathbf{m}}+\mathbf{R}_{\widetilde{\eta}}\right), \zeta_A^2\sigma_3\widetilde{\boldsymbol{\eta}}\right\rangle. \tag{4.6}$$

We will investigate each terms in the r.h.s. of (4.5) and (4.6).

Lemma 4.3. *We have*

$$\langle \mathbf{J}\mathbf{L}_1\widetilde{\boldsymbol{\eta}}, S_A\widetilde{\boldsymbol{\eta}}\rangle = -\|w_1'\|_{L^2}^2 + O\left(\|w_1\|_{L^2_{-\frac{a}{10}}}^2\right).$$

Proof. First, we have

$$\langle \mathbf{J}\mathbf{L}_1\widetilde{\boldsymbol{\eta}}, S_A\widetilde{\boldsymbol{\eta}}\rangle = -\langle L_1\widetilde{\eta}_1, S_A\widetilde{\eta}_1\rangle - \langle \widetilde{\eta}_2, S_A\widetilde{\eta}_2\rangle = -\langle L_1\widetilde{\eta}_1, S_A\widetilde{\eta}_1\rangle. \tag{4.7}$$

From [8] Lemma 4.2, we have

$$\langle L_1\widetilde{\eta}_1, S_A\widetilde{\eta}_1\rangle = \|w_1'\|_{L^2}^2 - \frac{1}{2}\int_{\mathbb{R}}\frac{\varphi_A}{\zeta_A^2}V'|w_1|^2\,dx + \frac{1}{2A}\int\left(\chi''|x|+2\chi'\frac{x}{|x|}\right)|w_1|^2\,dx, \tag{4.8}$$

where $V = W''(H)$. Since $|\varphi_A\zeta_A^{-2}V'| + A^{-1}|\chi'' x| + 2|\chi'| \lesssim e^{-\frac{2a}{10}|x|}$, we have the conclusion. □

Lemma 4.4. *For arbitrary $c \in (0,1)$, we have*

$$\left\langle \mathbf{J}\left(\sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}} P_c \mathcal{R}_{\mathbf{m}} + \mathbf{R}_{\widetilde{\eta}}\right), S_A \widetilde{\boldsymbol{\eta}} \right\rangle \lesssim \delta^{1/3} \|w_1'\|_{L^2} + c\|\mathbf{w}\|^2_{L^2_{-\frac{a}{10}}} + c^{-1} \sum_{\mathbf{m}\in\mathbf{R}_{\min}} |\mathbf{z}^{\mathbf{m}}|^2 + \delta |\dot{\mathbf{z}} - \widetilde{\mathbf{z}}|^2. \tag{4.9}$$

Here, the implicit constant is independent of c.

Proof. Recall $\mathbf{R}_{\widetilde{\eta}}$ is given in (2.10). First,

$$|\langle \mathbf{J} P_c \boldsymbol{\mathcal{R}}[\mathbf{z}], S_A \boldsymbol{\eta}\rangle| \le \|\zeta_A^{-1} S_A \mathbf{J} P_c \boldsymbol{\mathcal{R}}[\mathbf{z}]\|_{L^2_{\frac{a}{10}}} \|\mathbf{w}\|_{L^2_{-\frac{a}{10}}}$$

We have $\|\zeta_A^{-1} S_A \mathbf{J} P_c\|_{\Sigma\to L^2_{\frac{a}{10}}} \lesssim 1$. Therefore, by Propositions 1.11 and 2.5,

$$|\langle \mathbf{J} P_c \boldsymbol{\mathcal{R}}[\mathbf{z}], S_A \widetilde{\boldsymbol{\eta}}\rangle| \lesssim \sum_{\mathbf{m}\in\mathbf{R}_{\min}} |\mathbf{z}^{\mathbf{m}}| \|\mathbf{w}\|_{L^2_{-\frac{a}{10}}} \lesssim c^{-1} \sum_{\mathbf{m}\in\mathbf{R}_{\min}} |\mathbf{z}^{\mathbf{m}}|^2 + c\|\mathbf{w}\|^2_{L^2_{-\frac{a}{10}}}. \tag{4.10}$$

Next, since $\|P_c D\boldsymbol{\phi}[\mathbf{z}]\|_\Sigma \lesssim \delta$ by Proposition 2.5, we have

$$|\langle \mathbf{J} P_c D\boldsymbol{\phi}[\mathbf{z}](\dot{\mathbf{z}} - \widetilde{\mathbf{z}}), S_A \widetilde{\boldsymbol{\eta}}\rangle| \lesssim \delta |\dot{\mathbf{z}} - \widetilde{\mathbf{z}}| \|\mathbf{w}\|_{L^2_{-\frac{a}{10}}} \lesssim \delta |\dot{\mathbf{z}} - \widetilde{\mathbf{z}}|^2 + \delta \|\mathbf{w}\|^2_{L^2_{-\frac{a}{10}}}. \tag{4.11}$$

Using Lemma 2.4 as well as $\|\zeta_A^{-1}\|_{L^2_{\frac{a}{10}} \to L^2_{\frac{a}{10}-\frac{1}{A}}}$, $\|S_A P_c \mathbf{L}_1\|_{\Sigma\to L^2_{\frac{a}{10}}}$, $\|\zeta_A^{-1}\|_{L^2_{-\frac{a}{10}}\to\Sigma^*}$ $\lesssim 1$, we have

$$|\langle \mathbf{J} P_c \mathbf{L}_1 (R[\mathbf{z}] - 1)\widetilde{\boldsymbol{\eta}}, S_A \widetilde{\boldsymbol{\eta}}\rangle| \lesssim \delta \|\mathbf{w}\|^2_{L^2_{-\frac{a}{10}}}. \tag{4.12}$$

For $E_1 := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $\Delta_{W''}(\mathbf{z}) := W''(\phi_1[\mathbf{z}]) - W''(H)$, we consider

$$\begin{aligned}
&\langle P_c(\mathbf{L}[\mathbf{z}] - \mathbf{L}_1)\boldsymbol{\eta}, S_A \widetilde{\boldsymbol{\eta}}\rangle = \langle \mathbf{J} P_c \Delta_{W''}(\mathbf{z}) E_1 \boldsymbol{\eta}, S_A \widetilde{\boldsymbol{\eta}}\rangle \\
&\quad = \langle \mathbf{J} E_1 \Delta_{W''}(\mathbf{z}) \widetilde{\boldsymbol{\eta}}, S_A \widetilde{\boldsymbol{\eta}}\rangle - \langle \mathbf{J} P_d \Delta_{W''}(\mathbf{z}) E_1 \widetilde{\boldsymbol{\eta}}, S_A \widetilde{\boldsymbol{\eta}}\rangle \\
&\qquad - \langle \mathbf{J} E_1 \Delta_{W''}(\mathbf{z})(R[\mathbf{z}] - 1)\widetilde{\boldsymbol{\eta}}, S_A \widetilde{\boldsymbol{\eta}}\rangle + \langle \mathbf{J} P_d \Delta_{W''}(\mathbf{z}) E_1 (R[\mathbf{z}] - 1)\widetilde{\boldsymbol{\eta}}, S_A \widetilde{\boldsymbol{\eta}}\rangle.
\end{aligned} \tag{4.13}$$

The most significant term in the right is the first. Since $\mathbf{J}E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, it follows that $\mathbf{J}E_1 S_A$ is skew-symmetric, so that

$$|\langle \mathbf{J} E_1 \Delta_{W''}(\mathbf{z}) \widetilde{\boldsymbol{\eta}}, S_A \widetilde{\boldsymbol{\eta}}\rangle| = |2^{-1} \langle [S_A, \Delta_{W''}(\mathbf{z})] \widetilde{\eta}_1, \widetilde{\eta}_1\rangle| \lesssim \delta \|w_1\|^2_{L^2_{-\frac{a}{10}}} \le \delta \|\mathbf{w}\|^2_{L^2_{-\frac{a}{10}}}.$$

The other terms in (4.13) satisfy the same estimate. For example, if we consider the 2nd term in the r.h.s. of (4.13), we have

$$|\langle \mathbf{J} S_A P_d \Delta_{W''}(\mathbf{z}) E_1 \widetilde{\boldsymbol{\eta}}, \widetilde{\boldsymbol{\eta}}\rangle| \le \|S_A P_d\|_{L^2\to\Sigma} \|\Delta_{W''}(\mathbf{z})\widetilde{\boldsymbol{\eta}}\|_{L^2} \|\widetilde{\boldsymbol{\eta}}\|_{\Sigma^*} \lesssim \delta \|\mathbf{w}\|^2_{L^2_{-\frac{a}{10}}}.$$

The other terms in the r.h.s. of (4.13) can bounded similarly, so that we can conclude

$$\langle \mathbf{J} P_c \Delta_{W''}(\mathbf{z}) E_1 \boldsymbol{\eta}, S_A \widetilde{\boldsymbol{\eta}}\rangle \lesssim \delta \|\mathbf{w}\|^2_{L^2_{-\frac{a}{10}}}. \tag{4.14}$$

For F_1 and $\mathbf{F}$ defined in (2.7), consider

$$\langle \mathbf{J}P_c\mathbf{J}\mathbf{F}, S_A\widetilde{\boldsymbol{\eta}}\rangle = -\langle \mathbf{F}, S_A\boldsymbol{\eta}\rangle - \langle \mathbf{J}P_d\mathbf{J}\mathbf{F}, S_A\boldsymbol{\eta}\rangle - \langle \mathbf{J}P_c\mathbf{J}\mathbf{F}, S_A(R[\mathbf{z}]-1)\widetilde{\boldsymbol{\eta}}\rangle . \tag{4.15}$$

Using the pointwise bound $|F_1| \lesssim |\eta_1|^2$, the fact that $P_d : \Sigma^* \to \Sigma$ and (2.8), we can bound the 2nd and the 3rd term by

$$|\langle \mathbf{F}, P_dS_A\boldsymbol{\eta}\rangle| \lesssim \|F_1\|_{\Sigma^*}\|P_dS_A\|_{\Sigma^*\to\Sigma}\|\eta_1\|_{\Sigma^*} \lesssim \|\eta_1\|_{L^\infty}\|\eta_1\|_{\Sigma^*}^2 \lesssim \delta\|w_1\|^2_{L^2_{-\frac{a}{10}}}, \tag{4.16}$$

$$|\langle \mathbf{F}, P_cS_A(R[\mathbf{z}]-1)\widetilde{\boldsymbol{\eta}}\rangle| \lesssim \|F_1\|_{\Sigma^*}\|\mathbf{J}P_cS_A(R[\mathbf{z}]-1)\|_{\Sigma^*\to\Sigma}\|\widetilde{\boldsymbol{\eta}}\|_{\Sigma^*} \lesssim \delta^2\|\mathbf{w}\|^2_{L^2_{-\frac{a}{10}}}. \tag{4.17}$$

Finally, for the 1st term of the r.h.s. of (4.15), we have

$$\begin{aligned}\langle \mathbf{F}, S_A\boldsymbol{\eta}\rangle &= \langle F_1, S_A\eta_1\rangle = 2^{-1}\left\langle F_1\eta_1, \zeta_A^2\right\rangle \\ &- \left\langle W(\phi_1[\mathbf{z}]+\eta_1) - W(\phi_1[\mathbf{z}]) - W'(\phi_1[\mathbf{z}])\eta_1 - 2^{-1}W''(\phi_1[\mathbf{z}])\eta_1^2, \zeta_A^2\right\rangle \\ &- \left\langle W'(\phi_1[\mathbf{z}]+\eta_1) - W'(\phi_1[\mathbf{z}]) - W''(\phi_1[\mathbf{z}])\eta_1 - 2^{-1}W'''(\phi_1[\mathbf{z}])\eta_1^2, \phi_1'[\mathbf{z}]\varphi_A\right\rangle .\end{aligned}$$

So

$$|\langle \mathbf{F}, S_A\boldsymbol{\eta}\rangle| \lesssim \int |\eta_1|^3\zeta_A^2\, dx$$

and, by Lemma 2.7 of [7], we have

$$|\langle \mathbf{F}, S_A\boldsymbol{\eta}\rangle| \lesssim \delta^{1/3}\|w_1'\|^2_{L^2}. \tag{4.18}$$

By (1.37) and (2.10) we have bounded all terms in the l.h.s. of (4.9). Combining, (4.10), (4.11), (4.12), (4.14), (4.16), (4.17) and (4.18) we have the conclusion. ☐

Combining Lemmas 4.3 and 4.4 we obtain (4.1).

We next prove (4.2). As (4.1), we start from examine the contribution of the 1st term in the r.h.s. of (4.6).

Lemma 4.5. *We have*

$$\left\langle \mathbf{J}\mathbf{L}_1\widetilde{\boldsymbol{\eta}}, \zeta_A^2\sigma_3\widetilde{\boldsymbol{\eta}}\right\rangle = \|\zeta_{\frac{10}{a}}w_2\|^2_{L^2} + r,$$

with r satisfying

$$|r| \lesssim \|w_1\|^2_{\widetilde{\Sigma}}. \tag{4.19}$$

Proof. We have

$$\left\langle \mathbf{J}\mathbf{L}_1\widetilde{\boldsymbol{\eta}}, \zeta_A^2\sigma_3\widetilde{\boldsymbol{\eta}}\right\rangle = \|\zeta_{\frac{10}{a}}w_2\|^2_{L^2} - \left\langle L_1\widetilde{\eta}_1, \zeta_A^2\widetilde{\eta}_1\right\rangle = \|\zeta_{\frac{10}{a}}w_2\|^2_{L^2} + r.$$

The remainder term r can be expanded as

$$-r = \left\langle L_1\widetilde{\eta}_1, \zeta_{\widetilde{A}}^2\widetilde{\eta}_1\right\rangle = \|(\zeta_{\frac{10}{a}}w_1)'\|^2_{L^2} + \int \left(\left((\log\zeta_{\widetilde{A}})'\right)^2 + W''(H)\right)|\zeta_{\frac{10}{a}}w_1|^2\, dx.$$

Thus, we have (4.19). ☐

The contribution of the remaining terms in the r.h.s. of (4.6) can be bounded as follows.

Lemma 4.6. *For arbitrary $c \in (0,1)$ we have*

$$\left|\left\langle \mathbf{J}\left(\sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}}\mathcal{R}_{\mathbf{m}} + \boldsymbol{\mathcal{R}}_{\widetilde{\eta}}\right), \zeta_{\widetilde{A}}^2\sigma_3\widetilde{\boldsymbol{\eta}}\right\rangle\right| \lesssim c\|\mathbf{w}\|^2_{L^2_{-\frac{a}{10}}} + c^{-1}\sum_{\mathbf{m}\in\mathbf{R}_{\min}} |\mathbf{z}^{\mathbf{m}}|^2 + \delta|\dot{\mathbf{z}} - \widetilde{\mathbf{z}}|^2, \tag{4.20}$$

where the implicit constant is independent of c.

Proof. The proof is similar to the proof of Lemma 4.4. Thus, we omit it. □

Combining Lemmas 4.5 and 4.6 and the fact $\|w_2\|_{L^2_{-\frac{a}{10}}} \le \|\zeta_{\frac{10}{a}} w_2\|_{L^2}$, we have (4.2). This completes the proof of Lemma 4.2.

5. Technical estimates

The following lemmas are proved in [8], to which we refer for proofs.

Lemma 5.1. *Let $U \ge 0$ be a non-zero potential $U \in L^1(\mathbb{R},\mathbb{R})$. Then there exists a constant $C_U > 0$ such that for any function $0 \le W$ such that $\langle x\rangle W \in L^1(\mathbb{R})$ then*

$$\langle Wf, f\rangle \le C_U\| \langle x\rangle W\|_{L^1(\mathbb{R})} \left\langle(-\partial_x^2 + U)f, f\right\rangle. \tag{5.1}$$

In particular, for $a > 0$ the constant in the norm $\|\cdot\|_{\widetilde{\Sigma}}$ in (1.29), there exists a constant $C(a) > 0$ such that

$$\langle Wf, f\rangle \le C(a)\| \langle x\rangle W\|_{L^1(\mathbb{R})}\|f\|^2_{\widetilde{\Sigma}}. \tag{5.2}$$

□

Lemma 5.2. *Consider a Schwartz function $\mathcal{V} \in \mathcal{S}(\mathbb{R},\mathbb{C})$. Then, for any $L \in \mathbb{N}\cup\{0\}$ there exists a constant C_L s.t. we have for all $\varepsilon \in (0,1]$ and for $L^{2,s}(\mathbb{R})$ is defined in Definition 1.10,*

$$\| \langle \mathrm{i}\varepsilon\partial_x\rangle^{-\widetilde{N}} \left[\mathcal{V}, \langle \mathrm{i}\varepsilon\partial_x\rangle^{\widetilde{N}}\right] \|_{L^{2,-L}(\mathbb{R})\to L^{2,L}(\mathbb{R})} \le C_L\varepsilon. \tag{5.3}$$

where $L^{2,s}(\mathbb{R})$ is defined in Definition 1.10. □

Lemma 5.3. *There exist constants C_0 and $C_{\widetilde{N}}$ such that for $\varepsilon >$ small enough we have*

$$\|\mathcal{T}\|_{L^2\to L^2} \le C_0\varepsilon^{-\widetilde{N}} \text{ and } \|\mathcal{T}\|_{\Sigma^{\widetilde{N}}\to\Sigma^0} \le C_{\widetilde{N}}. \tag{5.4}$$

Furthermore, let $K_\varepsilon(x,y) \in \mathcal{D}'(\mathbb{R}\times\mathbb{R})$ be the Schwartz kernel of $\mathcal{T}$. Then, we have

$$|K_\varepsilon(x,y)| \le C_0 e^{-\frac{|x-y|}{3\varepsilon}} \text{ for all } x, y \text{ with } |x-y| \ge 1. \tag{5.5}$$

□

Lemma 5.4. *We have*

$$\|w_1\|_{L^2(|x|\le 2B^2)} \lesssim B^2 \|w_1\|_{\widetilde{\Sigma}} \text{ for any } w, \tag{5.6}$$

$$\|\xi_1\|^2_{\widetilde{\Sigma}} \lesssim \langle (-\partial_x^2 - 2^{-2}\chi^2_{B^2} x V_D')\xi_1, \xi_1 \rangle \lesssim \|\xi_1\|^2_{\widetilde{\Sigma}} \text{ for any } \xi, \tag{5.7}$$

$$\|v_1\|_{L^2(\mathbb{R})} \lesssim \varepsilon^{-\widetilde{N}} B^2 \|w_1\|_{\widetilde{\Sigma}} \ , \tag{5.8}$$

$$\|v_1'\|_{L^2(\mathbb{R})} \lesssim \varepsilon^{-\widetilde{N}} \|w_1\|_{\widetilde{\Sigma}} \ , \tag{5.9}$$

$$\| \langle x \rangle^{-M} v_1 \|_{H^1(\mathbb{R})} \lesssim \|\xi_1\|_{\widetilde{\Sigma}} + \varepsilon^{-\widetilde{N}} \langle B \rangle^{-M+3} \|w_1\|_{\widetilde{\Sigma}} \text{ for } M \in \mathbb{N}, M \ge 4. \tag{5.10}$$

□

Lemma 5.5. *We have the formula*

$$P_c\left(\chi_{B^2}\widetilde{\eta}_1\right) = \prod_{j=1}^{\widetilde{N}} R_{L_1}(\widetilde{\lambda}_j^2) P_c \mathcal{A} \langle \mathrm{i}\varepsilon\partial_x \rangle^{\widetilde{N}} v_1. \tag{5.11}$$

□

We next consider a number of results on linear theory.

Lemma 5.6. ω^2 *is neither an eigenvalue nor a resonance for the operator* L_D*, that is, if* $L_D f = \omega^2 f$ *for* $f \in L^\infty(\mathbb{R})$*, then* $f = 0$*.*

Proof(sketch) If the statement is false, there exists a nonzero and bounded solution of $L_D f = \omega^2 f$. We can assume f is real valued. Now, let $[a,b]$ be an interval where $-xV_D'|_{[a,b]} > 0$ and let $\psi \in C_c^\infty((a,b),[0,+\infty))$ be a nonzero function such that $-x\left(V_D' - \alpha\psi'\right) > 0$ in $[a,b]$ for all $\alpha \in [0,1]$. Then it can be shown that for small $\alpha > 0$ the operator $L_D - \alpha\psi$ has exactly one negative eigenvalue. But it is elementary to see that this is incompatible with the fact that $V_D - \lambda\psi$ is repulsive. □

Notice that we can apply Komech–Kopylova [20, Proposition 3.3] and conclude the following.

Lemma 5.7. *Let* Λ *be a finite subset of* $(0,\infty)$ *and let* $S > 5/2$*. Then there exists a fixed* $c(S,\Lambda)$ *s.t. for every* $t \ge 0$ *and* $\lambda \in \Lambda$

$$\|e^{\mathbf{L}_D t} R^+_{\mathrm{i}\mathbf{L}_D}(\lambda)\mathbf{f}\|_{\boldsymbol{\mathcal{H}}^{1,-S}(\mathbb{R})} \le c(S,\Lambda)\langle t\rangle^{-\frac{3}{2}} \|\mathbf{f}\|_{\boldsymbol{\mathcal{H}}^{1,S}(\mathbb{R})} \text{ for all } \mathbf{f} \in \boldsymbol{\mathcal{H}}^{1,S}(\mathbb{R}). \tag{5.12}$$

□

We have the following resolvent identity, see Komech–Kopylova [19, formula (3.6)],

$$R_{\mathrm{i}\mathbf{L}_D}(\varsigma) = \begin{pmatrix} \varsigma R_{L_D}(\varsigma^2 - \omega^2) & \mathrm{i}R_{L_D}(\varsigma^2 - \omega^2) \\ -\mathrm{i}\left(1 + \varsigma^2 R_{L_D}(\varsigma^2 - \omega^2)\right) & \varsigma R_{L_D}(\varsigma^2 - \omega^2) \end{pmatrix}. \tag{5.13}$$

For the following see Komech–Kopylova [19, Sect. 3].

Lemma 5.8. *For any preassigned $r>0$ and for $S>1/2$ we have*

$$R_{\mathrm{i}\mathbf{L}_{\mathrm{Tr}}}(\varsigma\pm \mathrm{i}\epsilon)\xrightarrow{\epsilon\to 0^+} R^{\pm}_{\mathrm{i}\mathbf{L}_D}(\varsigma) \text{ in } L^\infty\left((-\infty,-r-\omega]\cup[r+\omega,\infty),\mathcal{L}\left(\boldsymbol{\mathcal{H}}^{1,S},\boldsymbol{\mathcal{H}}^{1,-S}\right)\right) \tag{5.14}$$

□

Combining Lemma 5.8 with Lemma 8.5 [8] we have the following.

Lemma 5.9. *For $S>5/2$ and $\tau>1/2$ we have*

$$\sup_{\varsigma\in\mathbb{R}}\|R^{\pm}_{\mathrm{i}\mathbf{L}_D}(\varsigma)\|_{\boldsymbol{\mathcal{H}}^{1,\tau}\to\boldsymbol{\mathcal{H}}^{1,-S}}<\infty. \tag{5.15}$$

Proof. An uniform upper bound in $|\varsigma|\ge 1+\omega$ holds by (5.14). So now we focus on $|\varsigma|\le 1+\omega$. By (5.13) it is enough to bound

$$\sup_{|\varsigma|\le 1+\omega}\|\langle x\rangle^{-S}R^{\pm}_{L_D-\omega^2}(\varsigma^2-\omega^2)\langle y\rangle^{-\tau}\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})}<\infty \text{ and} \tag{5.16}$$

$$\sup_{|\varsigma|\le 1+\omega}\|\langle x\rangle^{-S}[\partial_x,R^{\pm}_{L_D-\omega^2}(\varsigma^2-\omega^2)]\langle y\rangle^{-\tau}\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})}<\infty. \tag{5.17}$$

In turn, they are a consequence of the following bound, for $j=0,1$, for the integral kernel,

$$\sup_{|\varsigma|\le 1+\omega}\int_{\mathbb{R}^2}\langle x\rangle^{-2S}|\partial_x^j R^{\pm}_{L_D-\omega^2}(x,y,\varsigma^2-\omega^2)|^2\langle y\rangle^{-2\tau}dxdy<\infty, \tag{5.18}$$

where (5.16) follows from case $j=0$ and (5.17) follows from case $j=1$.
For $j=0$, in the $+$ case (case $-$ is similar), (5.18) is proved in Lemma 8.5 [8]. We sketch now case $j=1$. Recall that, say for $x<y$, with an analogous formula for $x>y$,

$$R^{+}_{L_D-\omega^2}(x,y,\varsigma^2-\omega^2)=\frac{T(\sqrt{\varsigma^2-\omega^2})}{2\mathrm{i}\sqrt{\varsigma^2-\omega^2}}e^{\mathrm{i}\sqrt{\varsigma^2-\omega^2}(x-y)}m_-(x,\sqrt{\varsigma^2-\omega^2})m_+(y,\sqrt{\varsigma^2-\omega^2}), \tag{5.19}$$

where the Jost functions $f_\pm(x,k)=e^{\pm\mathrm{i}kx}m_\pm(x,k)$ solve $\left(-\Delta+V_D-\omega^2\right)u=k^2u$ with

$$\lim_{x\to+\infty}m_+(x,k)=1=\lim_{x\to-\infty}m_-(x,k).$$

These functions satisfy, see Lemma 1 p. 130 [11],

$$|m_\pm(x,k)-1|\le C_1\langle\max\{0,\mp x\}\rangle\langle k\rangle^{-1}, \tag{5.20}$$

$$|\partial_x m_\pm(x,k)|\le C_1\langle k\rangle^{-1}, \tag{5.21}$$

while $T(k)=\alpha k(1+o(1))$ near $k=0$ for some $\alpha\in\mathbb{R}$ and $T(k)=1+O(1/k)$ for $k\to\infty$ and $T\in C^0(\mathbb{R})$, see Theorem 1 [11].
Now,

$$\partial_x R^{+}_{L_{D-\omega^2}}(x,y,\varsigma^2-\omega^2)=\mathrm{i}\sqrt{\varsigma^2-\omega^2}R_{L_D-\omega^2}(x,y,\varsigma^2-\omega^2)$$
$$+\frac{T(\sqrt{\varsigma^2-\omega^2})}{2\mathrm{i}\sqrt{\varsigma^2-\omega^2}}e^{\mathrm{i}\sqrt{\varsigma^2-\omega^2}(x-y)}m'_-(x,\sqrt{\varsigma^2-\omega^2})$$

$$\times m_+(y, \sqrt{\varsigma^2 - \omega^2}).$$

The first term on the right, by $|\varsigma| \le 1 + \omega$ is essentially like the kernel (5.19), so the corresponding contribution to (5.18) is like the case $j = 0$. It is easy to see, following the discussion in Lemma 8.5 [8], that the bound of the last line is simpler, basically because (5.21) is better than (5.20). $\square$

Remark 5.10. Lemma 5.9 is essential for us to get the key inequality (8.10). Notice that Lemma 5.9 is true under the repulsivity hypothesis of Kowalczyk and Martel [22], which we have recalled in Remark 1.7, if we further assume that ω^2 is not a resonance for L_D. But if it has a resonance, then the status of the Lemma 5.9 is unclear. For $L_D = -\partial_x^2 + \omega^2$, by $R_{-\partial_x^2}(x, y, \varsigma) = \frac{\mathrm{i}}{\sqrt{\varsigma}} e^{\mathrm{i}\sqrt{\varsigma}|x-y|}$ and by a cancelation due to the odd functions, we are reduced to the following opposite of (5.18)

$$\sup_{|\varsigma|\le 1} \int_{\mathbb{R}^2} \langle x \rangle^{-2S} \left| \frac{1}{\sqrt{\varsigma}} \left(e^{\mathrm{i}\sqrt{\varsigma}|x-y|} - 1 \right) \right|^2 \langle y \rangle^{-2\tau} \, dxdy = +\infty.$$

Notice that this follows from the fact that the above is larger than

$$\begin{aligned} &\sup_{|\varsigma|\le 1} \int_{|x| \ll |y| \ll 1/\sqrt{|\varsigma|}} \langle x \rangle^{-2S} \left| \frac{1}{\sqrt{\varsigma}} \left(e^{\mathrm{i}\sqrt{\varsigma}|x-y|} - 1 \right) \right|^2 \langle y \rangle^{-2\tau} \, dxdy \\ &\quad \sim \sup_{|\varsigma|\le 1} \int_{|x| \ll |y| \ll 1/\sqrt{|\varsigma|}} \langle x \rangle^{-2S} \langle y \rangle^{2-2\tau} \, dxdy \sim \sup_{|\varsigma|\le 1} |\varsigma|^{\frac{2\tau-3}{2}} = \\ &\quad + \infty \text{ for } \tau \in (1/2, 3/2) \end{aligned}$$

and is infinite also for $\tau = 3/2$. So, even though a resonance of L_D involves even functions, this still seems to affect the estimates for the resolvent acting only on odd functions. See also the resolvent expansions in Lemma 2.2 in Murata [41] or Lemma 2.2 in Jensen and Kato [16], which require increasing weights.

The following formulas can be proved following Mizumachi [40, Lemma 4.5], to which we refer for the proof.

Lemma 5.11. *Let for* $\mathbf{g} \in \mathcal{S}(\mathbb{R} \times \mathbb{R}, \mathbb{C}^2)$

$$\mathbf{U}(t, x) = \frac{1}{\sqrt{2\pi}\mathrm{i}} \int_{\mathbb{R}} e^{-\mathrm{i}\lambda t} \left(R^-_{\mathrm{i}\mathbf{L}_D}(\lambda) + R^+_{\mathrm{i}\mathbf{L}_D}(\lambda) \right) \mathcal{F}_t^{-1} \mathbf{g}(\lambda, \cdot) d\lambda,$$

where $\mathcal{F}_t^{-1}$ *is the inverse Fourier transform in* t. *Then*

$$\begin{aligned} 2 \int_0^t e^{(t-t')\mathbf{L}_D} \mathrm{i}\mathbf{L}_D(t') dt' = \mathbf{U}(t, x) - \int_{\mathbb{R}_-} e^{(t-t')\mathbf{L}_D} \mathrm{i}\mathbf{L}_D(t') dt' \\ + \int_{\mathbb{R}_+} e^{(t-t')\mathbf{L}_D} \mathrm{i}\mathbf{L}_D(t') dt'. \end{aligned} \tag{5.22}$$

$\square$

The last two lemmas give us the following smoothing estimate.

Lemma 5.12. *For $S > 5/2$ and $\tau > 1/2$ there exists a constant $C(S,\tau)$ such that we have*

$$\left\| \int_0^t e^{(t-t')\mathbf{L}_D} \mathbf{g}(t') dt' \right\|_{L^2(\mathbb{R}, \boldsymbol{\mathcal{H}}^{1,-S})} \le C(S,\tau) \|\mathbf{g}\|_{L^2(\mathbb{R}, \boldsymbol{\mathcal{H}}^{1,\tau})}. \tag{5.23}$$

Proof. We repeat verbatim the proof of [8, Lemma 8.7], which in turn is taken from Mizumachi [40], that is, can use formula (5.22) and bound $\mathbf{U}$, with the bound on the last two terms in the right hand side of (5.22) similar. So we have, taking Fourier transform in t and by Plancherel,

$$\begin{aligned} \|\mathbf{U}\|_{L^2_t \boldsymbol{\mathcal{H}}^{1,-S}} &\le 2 \sup_{\pm} \|R^{\pm}_{\mathrm{i}\mathbf{L}_D}(\lambda) \widehat{\mathbf{g}}(\lambda, \cdot)\|_{L^2_\lambda \boldsymbol{\mathcal{H}}^{1,-S}} \\ &\le 2 \sup_{\pm} \sup_{\lambda \in \mathbb{R}} \|R^{\pm}_{\mathrm{i}\mathbf{L}_D}(\lambda)\|_{\boldsymbol{\mathcal{H}}^{1,\tau} \to \boldsymbol{\mathcal{H}}^{1,-S}} \|\widehat{\mathbf{g}}(\lambda, x)\|_{L^2_\lambda \boldsymbol{\mathcal{H}}^{1,\tau}} \lesssim \|\mathbf{g}\|_{L^2_t \boldsymbol{\mathcal{H}}^{1,\tau}}. \end{aligned}$$

□

6. Second virial estimate, for $\boldsymbol{\xi}$

We set

$$\mathcal{J}_1(\mathbf{v}) := \frac{1}{2} \left\langle \mathbf{J}\mathbf{v}, \widetilde{S}_B \mathbf{v} \right\rangle, \quad \mathcal{J}_2(\mathbf{v}) := \frac{1}{2} \left\langle \mathbf{J}\mathbf{v}, \left(\chi_{B^2} \zeta_{\widetilde{B}}\right)^2 \sigma_3 \mathbf{v} \right\rangle \tag{6.1}$$

$$\text{where } \widetilde{B}^{-1} = B^{-1} + \frac{a}{10} \tag{6.2}$$

and

$$\widetilde{S}_B := \frac{\psi'_B}{2} + \psi_B \partial_x, \;\; \psi_B := \chi^2_{B^2} \varphi_B. \tag{6.3}$$

The main result of the section is the following.

Lemma 6.1.

$$\frac{d}{dt} \mathcal{J}_1(\mathbf{v}) + \frac{1}{2} \|\xi_1\|^2_{\widetilde{\Sigma}} \lesssim (c+\varepsilon) \|\xi_2\|^2_{L^2_{-\frac{a}{10}}} + c^{-1} \sum_{\mathbf{m} \in \mathbf{R}_{\min}} |\mathbf{z}^{\mathbf{m}}|^2 + \delta |\dot{\mathbf{z}} - \widetilde{\mathbf{z}}|^2 + \varepsilon \|\mathbf{w}\|^2_{\widetilde{\boldsymbol{\Sigma}}}, \tag{6.4}$$

$$-\frac{d}{dt} \mathcal{J}_2(\mathbf{v}) + \frac{1}{2} \|\xi_2\|^2_{L^2_{-\frac{a}{10}}} \lesssim \|\xi_1\|^2_{\widetilde{\Sigma}} + \sum_{\mathbf{m} \in \mathbf{R}_{\min}} |\mathbf{z}^{\mathbf{m}}|^2 + \delta |\dot{\mathbf{z}} - \widetilde{\mathbf{z}}|^2 + \varepsilon \|\mathbf{w}\|^2_{\widetilde{\boldsymbol{\Sigma}}}. \tag{6.5}$$

Proof of Proposition 3.5 assuming Lemma 6.1. We have

$$\begin{aligned} |\mathcal{J}_1(\mathbf{v})| &= |\left\langle v_2, \widetilde{S}_B v_1 \right\rangle| \le \|v_2\|_{L^2} |\widetilde{S}_B v_1| \\ &\lesssim B \| \langle \mathrm{i}\varepsilon \partial_x \rangle^{-\widetilde{N}} \|_{L^2 \to L^2} \| \langle \mathrm{i}\varepsilon \partial_x \rangle^{-\widetilde{N}} \|_{H^1 \to H^1} \|\widetilde{\eta}_2\|_{L^2} \|\widetilde{\eta}_1\|_{H^1} \lesssim B \varepsilon^{-2\widetilde{N}} \delta^2. \end{aligned}$$

Similarly, we have $|\mathcal{J}_2(\mathbf{v})| \lesssim \varepsilon^{-2\widetilde{N}} \delta^2$. Integrating (6.4) and (6.5) we have

$$\begin{aligned} \|\xi_1\|^2_{L^2 \widetilde{\Sigma}} \lesssim & B \varepsilon^{-2\widetilde{N}} \delta^2 + (c+\varepsilon) \|\xi_2\|^2_{L^2 L^2_{-\frac{a}{10}}} \\ & + c^{-1} \sum_{\mathbf{m} \in \mathbf{R}_{\min}} \|\mathbf{z}^{\mathbf{m}}\|^2_{L^2} + \delta \|\dot{\mathbf{z}} - \widetilde{\mathbf{z}}\|^2_{L^2} + \varepsilon \|\mathbf{w}\|^2_{L^2 \widetilde{\boldsymbol{\Sigma}}}, \end{aligned}$$

$$\|\xi_2\|^2_{L^2 L^2_{-\frac{a}{10}}} \lesssim \varepsilon^{-2\widetilde{N}}\delta^2 + \|\xi_1\|^2_{L^2\widetilde{\Sigma}} + \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \|\mathbf{z}^{\mathbf{m}}\|^2_{L^2} + \delta\|\dot{\mathbf{z}} - \widetilde{\mathbf{z}}\|^2_{L^2} + \varepsilon\|\mathbf{w}\|^2_{L^2\widetilde{\boldsymbol{\Sigma}}}.$$

Thus, as for the proof of Proposition 3.4, we have the conclusion. □

As in (4.5) and (4.6), we have

$$\frac{d}{dt}\mathcal{J}_1(\mathbf{v}) = \left\langle \mathbf{J}\left(\mathbf{L}_D\mathbf{v} + \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}}\widetilde{\mathcal{R}}_{\mathbf{m}} + \mathbf{R}_{\mathbf{v}}\right), \widetilde{S}_B\mathbf{v}\right\rangle, \tag{6.6}$$

$$\frac{d}{dt}\mathcal{J}_2(\mathbf{v}) = \left\langle \mathbf{J}\left(\mathbf{L}_D\mathbf{v} + \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}}\widetilde{\mathcal{R}}_{\mathbf{m}} + \mathbf{R}_{\mathbf{v}}\right), \chi^2_{B^2}\zeta^2_{\widetilde{B}}\sigma_3\mathbf{v}\right\rangle. \tag{6.7}$$

Lemma 6.2. *We have*

$$-\left\langle \mathbf{J}\mathbf{L}_D\mathbf{v}, \widetilde{S}_B\mathbf{v}\right\rangle \geq \frac{1}{2}\left\langle \left(-\partial_x^2 - \frac{1}{2}\chi^2_{B^2}xV_D'\right)\xi_1, \xi_1\right\rangle + B^{-1/2}O(\|\xi_1\|^2_{\widetilde{\Sigma}} + \|w_1\|^2_{\widetilde{\Sigma}}). \tag{6.8}$$

Proof. Since $\widetilde{S}_B$ is skew-adjoint, we have the following

$$-\left\langle \mathbf{J}\mathbf{L}_D\mathbf{v}, \widetilde{S}_B\mathbf{v}\right\rangle = \left\langle L_D v_1, \widetilde{S}_B v_1\right\rangle, \tag{6.9}$$

where by [8, Lemma 6.1] the very last term has the lower bound in the right hand side of (6.8). □

Lemma 6.3. *We have*

$$\left|\left\langle \mathbf{z}^{\mathbf{m}}\mathbf{J}\widetilde{\mathcal{R}}_{\mathbf{m}}, \widetilde{S}_B\mathbf{v}\right\rangle\right| \lesssim |\mathbf{z}^{\mathbf{m}}|\left(\|\boldsymbol{\xi}\|_{\widetilde{\boldsymbol{\Sigma}}} + e^{-B/2}\|\mathbf{w}\|_{\widetilde{\boldsymbol{\Sigma}}}\right). \tag{6.10}$$

Proof. We have

$$\left\langle \mathbf{z}^{\mathbf{m}}\mathbf{J}\widetilde{\mathcal{R}}_{\mathbf{m}}, \widetilde{S}_B\mathbf{v}\right\rangle = \left\langle \mathbf{z}^{\mathbf{m}}\widetilde{\mathcal{R}}_{\mathbf{m}2}, \widetilde{S}_B v_1\right\rangle + \left\langle \mathbf{z}^{\mathbf{m}}\widetilde{S}_B\widetilde{\mathcal{R}}_{\mathbf{m}1}, v_2\right\rangle. \tag{6.11}$$

The following inequality is the content of Lemma 6.3 in [8],

$$\left|\left\langle \mathbf{z}^{\mathbf{m}}\widetilde{\mathcal{R}}_{\mathbf{m}2}, \widetilde{S}_B v_1\right\rangle\right| \leq |\mathbf{z}^{\mathbf{m}}|\left(\|\xi_1\|_{\widetilde{\Sigma}} + e^{-B/2}\|w_1\|_{\widetilde{\Sigma}}\right).$$

We turn to the second term in the right in (6.11). Using $1 = (1-\chi_{B^2}) + \chi_{B^2}$ we split in two and bound separately the two terms. Using $\xi_2 = \chi_{B^2}\zeta_B v_2$ the contribution from χ_{B^2} is

$$\left|\left\langle \mathbf{z}^{\mathbf{m}}e^{\frac{a}{10}|x|}\zeta_B^{-1}\widetilde{S}_B\widetilde{\mathcal{R}}_{\mathbf{m}1}, e^{-\frac{a}{10}|x|}\xi_2\right\rangle\right| \leq |\mathbf{z}^{\mathbf{m}}|\|\xi_2\|_{L^2_{-\frac{a}{10}}}\|\widetilde{S}_B\widetilde{\mathcal{R}}_{\mathbf{m}1}\|_{L^2_{\frac{a}{10}+B^{-1}}}.$$

We show now that the last factor is $\lesssim 1$. The term we need to bound is

$$e^{\frac{a}{10}|x|}\zeta_B^{-1}(\chi^2_{B^2}\varphi_B)'\left\langle \mathrm{i}\varepsilon\partial_x\right\rangle^{-\widetilde{N}}\mathcal{A}^*\chi_{B^2}\mathcal{R}_{\mathbf{m}1}$$
$$+ 2e^{\frac{a}{10}|x|}\zeta_B^{-1}\chi^2_{B^2}\varphi_B\left\langle \mathrm{i}\varepsilon\partial_x\right\rangle^{-\widetilde{N}}\partial_x\mathcal{A}^*\chi_{B^2}\mathcal{R}_{\mathbf{m}1}.$$

We bound only the second term, since the first can be bounded similarly and in fact is smaller. Let us set $f := \partial_x\mathcal{A}^*\chi_{B^2}G_{\mathbf{m}1}$. We have

$$\left\langle \mathrm{i}\varepsilon\partial_x\right\rangle^{-\widetilde{N}}f(x) = \int f(y)e^{\frac{a}{5}|y|}Idy\ ,\ \text{where } I := \int \frac{e^{-\frac{a}{5}|y|}e^{\mathrm{i}(x-y)(\tau_1+\mathrm{i}\tau_2)}}{(1+\varepsilon^2\tau_1^2 - \varepsilon^2\tau_2^2 + 2\mathrm{i}\varepsilon^2\tau_1\tau_2)^{\widetilde{N}/2}}d\tau_1 \tag{6.12}$$

is a generalized integral in τ_1, well defined, using integration by parts, also for $\widetilde{N}=1$, when it is not absolutely convergent. I is constant in $|\tau_2|<\varepsilon^{-1}$. Then, for $\tau_2 = 2^{-1/2}\varepsilon^{-1}\text{sign}(x-y)$ we have

$$I = e^{-\frac{a}{5}|y|}e^{-\frac{|x-y|}{2\varepsilon}} II \text{ , where } II := \int \frac{e^{\mathrm{i}(x-y)\tau_1}}{\left(1/2+\varepsilon^2\tau_1^2+\sqrt{2}\mathrm{i}\varepsilon\tau_1\text{sign}(x-y)\right)^{\widetilde{N}/2}} d\tau_1.$$

For $\widetilde{N}>1$ we have $|II|\lesssim \varepsilon^{-1}$. Standard arguments show $|II|\lesssim \varepsilon^{-1}\log(2+\varepsilon/|x-y|)$ for $\widetilde{N}=1$. Since $e^{-\frac{a}{5}|y|}e^{-\frac{|x-y|}{4\varepsilon}} \le e^{-\frac{a}{5}|x|}$, we conclude

$$\left|\langle \mathrm{i}\varepsilon\partial_x\rangle^{-\widetilde{N}} f(x)\right| \lesssim e^{-\frac{a}{5}|x|}\int \varepsilon^{-1}K_{\widetilde{N}}\left(\frac{x-y}{\varepsilon}\right)|f(y)|e^{\frac{a}{5}|y|}dy \text{ with}$$

$$K_{\widetilde{N}}(x) = e^{-\frac{|x|}{4}}\log^{\sigma_{\widetilde{N}}}(2+/|x|),$$

where $\sigma_{\widetilde{N}}=0$ for $\widetilde{N}>1$ and $\sigma_{\widetilde{N}}=1$ for $\widetilde{N}=1$. Then

$$\begin{aligned}&\|\chi_{B^2}^2\varphi_B\langle \mathrm{i}\varepsilon\partial_x\rangle^{-\widetilde{N}} f\|_{L^2_{\frac{a}{10}+B^{-1}}}\\ &\lesssim \|e^{\frac{a}{10}|x|}\zeta_B^{-1}\varphi_B e^{-\frac{a}{5}|x|}\|_{L^\infty}\|K_{\widetilde{N}}\|_{L^1}\|e^{\frac{a}{5}|x|}\partial_x\mathcal{A}^*\chi_{B^2}\mathcal{R}_{\mathbf{m}1}\|_{L^2_{\frac{a}{10}}}\lesssim 1.\end{aligned}$$

To finish the proof we consider the following, which completes the proof,

$$\begin{aligned}&\left|\left\langle \mathbf{z}^{\mathbf{m}}e^{\frac{a}{10}}\zeta_A^{-1}(1-\chi_{B^2})\widetilde{S}_B\widetilde{\mathcal{R}}_{\mathbf{m}1}, e^{-\frac{a}{10}|y|}\zeta_A v_2\right\rangle\right|\\ &\lesssim |\mathbf{z}^{\mathbf{m}}|\ \|w_2\|_{L^2_{-\frac{a}{10}}}\|(1-\chi_{B^2})\widetilde{S}_B\widetilde{\mathcal{R}}_{\mathbf{m}1}\|_{L^2_{\frac{a}{10}+A^{-1}}}\\ &\le e^{-B}|\mathbf{z}^{\mathbf{m}}|\ \|w_2\|_{L^2_{-\frac{a}{10}}}.\end{aligned}$$

□

Lemma 6.4. *We have*

$$\left|\left\langle \mathbf{J}\mathbf{R}_{\mathbf{v}}, \widetilde{S}_B\mathbf{v}\right\rangle\right| \le o_\varepsilon(1)\left[\|\boldsymbol{\xi}\|_{\boldsymbol{\tilde\Sigma}}^2+\|\mathbf{w}\|_{\boldsymbol{\tilde\Sigma}}^2+|\dot{\mathbf{z}}-\widetilde{\mathbf{z}}|^2+\sum_{\mathbf{m}\in\mathbf{R}_{\min}}|\mathbf{z}^{\mathbf{m}}|^2\right]. \quad (6.13)$$

Proof. As the proof of Lemma 4.4 we estimate each term. First,

$$\begin{aligned}&\left\langle \mathbf{J}\begin{pmatrix}0\\ -\mathcal{T}(2\chi'_{B^2}\partial_x+\chi''_{B^2})\widetilde{\eta}_1+\langle \mathrm{i}\varepsilon\partial_x\rangle^{-\widetilde{N}}[V_D,\langle \mathrm{i}\varepsilon\partial_x\rangle^{\widetilde{N}}]v_1\end{pmatrix}, \widetilde{S}_B\mathbf{v}\right\rangle\\ &=\left\langle -\mathcal{T}(2\chi'_{B^2}\partial_x+\chi''_{B^2})\widetilde{\eta}_1+\langle \mathrm{i}\varepsilon\partial_x\rangle^{-\widetilde{N}}[V_D,\langle \mathrm{i}\varepsilon\partial_x\rangle^{\widetilde{N}}]v_1, \widetilde{S}_B v_1\right\rangle \quad (6.14)\end{aligned}$$

The last term is bounded in (6.19) and (6.20) of [8], and in particular we have

$$|(6.14)| \lesssim (\varepsilon+\varepsilon^{-\widetilde{N}}B^{-1})\left(\|\xi_1\|_{\tilde\Sigma}^2+\|w_1\|_{\tilde\Sigma}^2\right). \quad (6.15)$$

We next, we have

$$\begin{aligned}&\left|\left\langle \mathbf{J}\mathcal{T}\chi_{B^2}P_c\boldsymbol{\mathcal{R}}_1[\mathbf{z}], \widetilde{S}_B\mathbf{v}\right\rangle\right| + \left|\left\langle \mathbf{J}\mathcal{T}\chi_{B^2}P_cD_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}](\dot{\mathbf{z}}-\widetilde{\mathbf{z}}), \widetilde{S}_B\mathbf{v}\right\rangle\right|\\ &\lesssim \delta\left(|\dot{\mathbf{z}}-\widetilde{\mathbf{z}}|+\sum_{\mathbf{m}\in\mathbf{R}_{\min}}|\mathbf{z}^{\mathbf{m}}|\right)\left(\|\boldsymbol{\xi}\|_{\boldsymbol{\tilde\Sigma}}+e^{-B/2}\|\mathbf{w}\|_{\boldsymbol{\tilde\Sigma}}\right),\end{aligned}$$

because the first term in the left can be treated like (6.10), except that it is smaller because of the bound (1.38) on $\boldsymbol{\mathcal{R}}_1[\mathbf{z}]$, and a similar argument holds for the second term on the left, where additionally we use $\|P_c D_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}]\|_{\mathbb{C}^N\to\Sigma} \lesssim |\mathbf{z}|$. Next, proceeding as above

$$\left|\left\langle \mathbf{J}\mathcal{T}\chi_{B^2}P_c\mathbf{L}_1(R[\mathbf{z}]-1)\widetilde{\boldsymbol{\eta}}, \widetilde{S}_B\mathbf{v}\right\rangle\right| \lesssim \|\widetilde{S}_B\mathcal{T}\chi_{B^2}P_c\mathbf{L}_1(R[\mathbf{z}]-1)\widetilde{\boldsymbol{\eta}}\|_{L^2_{\frac{a}{10}+B^{-1}}} \times \left(\|\boldsymbol{\xi}\|_{\widetilde{\boldsymbol{\Sigma}}} + e^{-B/2}\|\mathbf{w}\|_{\widetilde{\boldsymbol{\Sigma}}}\right). \tag{6.16}$$

By

$$\|\mathcal{T}\chi_{B^2}P_c\widetilde{S}_B[\mathbf{z}]\|_{\Sigma^{\widetilde{N}+1}\to L^2_{\frac{a}{10}+B^{-1}}} \lesssim \varepsilon^{-\widetilde{N}}, \tag{6.17}$$

$$\|R[\mathbf{z}]-1\|_{\Sigma^*\to\Sigma^{\widetilde{N}+1}} \lesssim \delta, \tag{6.18}$$

$$\|\widetilde{\boldsymbol{\eta}}\|_{\Sigma^*} \lesssim \|\mathbf{w}\|_{L^2_{-\frac{a}{10}}}, \tag{6.19}$$

we conclude

$$\|\widetilde{S}_B\mathcal{T}\chi_{B^2}P_c\mathbf{L}_1(R[\mathbf{z}]-1)\widetilde{\boldsymbol{\eta}}\|_{L^2_{\frac{a}{10}+B^{-1}}} \lesssim \delta\varepsilon^{-\widetilde{N}}\|\mathbf{w}\|_{L^2_{-\frac{a}{10}}}.$$

Next, following the notation in Lemma 4.4, we consider

$$\begin{aligned}\left\langle \mathbf{J}\mathcal{T}\chi_{B^2}P_c(L[\mathbf{z}]-\mathbf{L}_1)\boldsymbol{\eta}, \widetilde{S}_B\mathbf{v}\right\rangle &= \left\langle \mathcal{T}\chi_{B^2}\Delta_{W''}(\mathbf{z})\widetilde{\eta}_1, \widetilde{S}_B v_1\right\rangle \\ &- \left\langle \mathbf{J}\mathcal{T}\chi_{B^2}P_d(L[\mathbf{z}]-\mathbf{L}_1)\widetilde{\boldsymbol{\eta}}, \widetilde{S}_B\mathbf{v}\right\rangle \\ &- \left\langle \mathbf{J}\mathcal{T}\chi_{B^2}(L[\mathbf{z}]-\mathbf{L}_1)(R[\mathbf{z}]-1)\widetilde{\boldsymbol{\eta}}, \widetilde{S}_B\mathbf{v}\right\rangle \\ &+ \left\langle \mathbf{J}\mathcal{T}\chi_{B^2}P_d(L[\mathbf{z}]-\mathbf{L}_1)(R[\mathbf{z}]-1)\widetilde{\boldsymbol{\eta}}, \widetilde{S}_B\mathbf{v}\right\rangle.\end{aligned} \tag{6.20}$$

Like in Lemma 4.4, the most significant term in the right is the first, which for brevity is the only one we bound explicitly, since the other ones are simpler. We have

$$\begin{aligned}|\left\langle \mathcal{T}\chi_{B^2}\Delta_{W''}(\mathbf{z})\widetilde{\eta}_1, \widetilde{S}_B v_1\right\rangle| &\le \varepsilon^{-\widetilde{N}}\|\chi_{B^2}\Delta_{W''}(\mathbf{z})\widetilde{\eta}_1\|_{L^1}\|\widetilde{S}_B v_1\|_{L^2} \\ &\lesssim \varepsilon^{-\widetilde{N}}\|e^{\frac{a}{10}|x|}\Delta_{W''}(\mathbf{z})\|_{L^\infty}\|w_1\|_{L^2_{-\frac{a}{10}}}\left(\varepsilon^{-\widetilde{N}}B\|w_1\|_{\widetilde{\Sigma}} + B\|\xi_1\|_{\widetilde{\Sigma}}\right) \\ &\lesssim \varepsilon^{-2\widetilde{N}}B\delta\|w_1\|_{L^2_{-\frac{a}{10}}}\left(\|w_1\|_{\widetilde{\Sigma}} + |\xi_1\|_{\widetilde{\Sigma}}\right),\end{aligned}$$

where we used,

$$\|\widetilde{S}_B v_1\|_{L^2} \lesssim \varepsilon^{-N}B\|w_1\|_{\widetilde{\Sigma}} + B\|\xi_1\|_{\widetilde{\Sigma}}. \tag{6.21}$$

The proof of (6.21) is in [25] and for completeness we write the proof in [8]. By (5.9) and $\|\psi_B\|_{L^\infty} \lesssim B$

$$\|\widetilde{S}_B v_1\|_{L^2} \lesssim \|\psi_B' v_1\|_{L^2} + \|\psi_B v_1'\|_{L^2} \lesssim \|\psi_B' v_1\|_{L^2} + \varepsilon^{-N}B\|w_1\|_{\widetilde{\Sigma}}.$$

Next, we have

$$|\psi_B'| = |2\chi_{B^2}'\chi_{B^2}\varphi_B + \chi_{B^2}^2\zeta_B^2| \lesssim B^{-1}\chi_{B^2} + \chi_{B^2}^2\zeta_B^2. \tag{6.22}$$

Then

$$B^{-1}\|v_1\|_{L^2} \lesssim B\varepsilon^{-N}B\|w_1\|_{\widetilde{\Sigma}}$$

by (5.8). By Lemma 5.1 we have

$$\|\chi_{B^2}^2\zeta_B^2 v_1\|_{L^2} = \|\chi_{B^2}\zeta_B\xi_1\|_{L^2} \lesssim \sqrt{\|\langle x\rangle \chi_{B^2}\zeta_B\|_{L^1}}\|\xi\|_{\widetilde{\Sigma}} \sim B\|\xi_1\|_{\widetilde{\Sigma}}$$

and, finally the following by (5.9), which completes the proof of (6.21),

$$\|\chi_{B^2}^2\varphi_B v_1'\|_{L^2} \lesssim B\|v_1'\|_{L^2} \lesssim B\varepsilon^{-N}\|w_1\|_{\widetilde{\Sigma}}.$$

Next, following again the notation in Lemma 4.4, we consider

$$\left\langle \mathbf{J}\mathcal{T}\chi_{B^2}P_c\mathbf{J}\mathbf{F}, \widetilde{S}_B\mathbf{v}\right\rangle = \left\langle \mathcal{T}\chi_{B^2}\mathbf{F}, \widetilde{S}_B\mathbf{v}\right\rangle - \left\langle \mathbf{J}\mathcal{T}\chi_{B^2}P_d\mathbf{J}\mathbf{F}, \widetilde{S}_B\mathbf{v}\right\rangle. \tag{6.23}$$

The main term in the right is the first, which by (6.21) can be treated as

$$\begin{aligned}
|\left\langle \mathcal{T}\chi_{B^2}F_1, \widetilde{S}_B v_1\right\rangle| &\le \|\mathcal{T}\chi_{B^2}F_1\|_{L^2}\|\widetilde{S}_B v_1\|_{L^2} \\
&\lesssim \varepsilon^{-\widetilde{N}}\|\chi_{B^2}\eta_1^2\|_{L^2}\left(\varepsilon^{-N}B\|w_1\|_{\widetilde{\Sigma}} + B\|\xi_1\|_{\widetilde{\Sigma}}\right) \\
&\lesssim \varepsilon^{-2\widetilde{N}}B\|\eta_1\|_{H^1}\left(\|\chi_{B^2}\widetilde{\eta}_1\|_{L^2} + \|(R[\mathbf{z}]-1)\widetilde{\eta}_1\|_{L^2}\right)\left(\|w_1\|_{\widetilde{\Sigma}} + \|\xi_1\|_{\widetilde{\Sigma}}\right) \\
&\lesssim \varepsilon^{-2\widetilde{N}}B\delta\left(\|w_1\|_{L^2(|x|\le 2B^2)} + \delta\|w_1\|_{\widetilde{\Sigma}}\right)\left(\|w_1\|_{\widetilde{\Sigma}} + \|\xi_1\|_{\widetilde{\Sigma}}\right) \\
&\lesssim \varepsilon^{-2\widetilde{N}}B^3\delta\|w_1\|_{\widetilde{\Sigma}}\left(\|w_1\|_{\widetilde{\Sigma}} + \|\xi_1\|_{\widetilde{\Sigma}}\right),
\end{aligned}$$

where we used (5.6), Lemma 2.4.

Turning to the second term in the right of (6.23), it is bounded from above by

$$\begin{aligned}
\|\widetilde{S}_B\mathcal{T}\chi_{B^2}P_d\mathbf{J}\mathbf{F}\|_{\Sigma}\|\mathbf{v}\|_{\Sigma^*} &\lesssim \|F_1\|_{\Sigma^*}\|\mathbf{w}\|_{L^2_{-\frac{a}{10}}} \\
&\lesssim \|\eta_1\|_{H^1}\|\eta_1\|_{\Sigma^*}\|\mathbf{w}\|_{L^2_{-\frac{a}{10}}} \lesssim \delta\|\mathbf{w}\|^2_{L^2_{-\frac{a}{10}}}.
\end{aligned}$$

□

Lemma 6.5. *For the* $\widetilde{B}$ *defined in (6.2), we have*

$$\left\langle \mathbf{J}\left(\mathbf{L}_D\mathbf{v}\right), \chi_{B^2}^2\zeta_{\widetilde{B}}^2\sigma_3\mathbf{v}\right\rangle = \int \zeta_{\frac{10}{a}}^2|\xi_2|^2\,dx - \left\langle (L_D+\omega^2)v_1, \chi_{B^2}^2\zeta_{\widetilde{B}}^2 v_1\right\rangle, \tag{6.24}$$

with

$$|\left\langle (L_D+\omega^2)v_1, \chi_{B^2}\zeta_{\widetilde{B}}^2 v_1\right\rangle| \lesssim \|\xi_1\|_{\widetilde{\Sigma}}\left(\|\xi_1\|_{\widetilde{\Sigma}} + e^{-B}\varepsilon^{\widetilde{N}}\|w_1\|_{\widetilde{\Sigma}}.\right) \tag{6.25}$$

Proof. Formula (6.24) follows from direct computation. We prove (6.25). First,

$$\left\langle (L_D+\omega^2)v_1, \chi_{B^2}\zeta_{\widetilde{B}}^2 v_1\right\rangle = \left\langle -v_1'', \chi_{B^2}\zeta_B\zeta_{\frac{10}{a}}^2\xi_1\right\rangle + \left\langle (V_D+\omega^2)\xi_1, \zeta_{\frac{10}{a}}^2\xi_1\right\rangle. \tag{6.26}$$

The 2nd term in r.h.s. of (6.26) can be bounded by

$$|\left\langle (V_D+\omega^2)\xi_1, \zeta_{\frac{10}{a}}^2\xi_1\right\rangle| \lesssim \|\xi_1\|^2_{L^2_{-\frac{a}{10}}}. \tag{6.27}$$

For the 1st term of r.h.s. of (6.26) we write

$$\begin{aligned}
\left\langle -v_1'', \chi_{B^2}\zeta_B\zeta_{\frac{10}{a}}^2\xi_1\right\rangle &= -\left\langle v_1, (\chi_{B^2}\zeta_B\zeta_{\frac{10}{a}}^2\xi_1)''\right\rangle \\
&= -\left\langle v_1, (\chi_{B^2}\zeta_B\zeta_{\frac{10}{a}}^2)''\xi_1 + 2(\chi_{B^2}\zeta_B\zeta_{\frac{10}{a}}^2)'\xi_1' + \chi_{B^2}\zeta_B\zeta_{\frac{10}{a}}^2\xi_1''\right\rangle
\end{aligned}$$

$$= -\left\langle v_1, \chi''_{B^2}\zeta_B\zeta^2_{\frac{10}{a}}\xi_1\right\rangle - \left\langle v_1, 2\chi'_{B^2}(\zeta_B\zeta_{\frac{10}{a}})'\xi_1\right\rangle - \left\langle v_1, \chi_{B^2}(\zeta_B\zeta^2_{\frac{10}{a}})''\xi_1\right\rangle \tag{6.28}$$

$$- \left\langle v_1, 2\chi'_{B^2}\zeta_B\zeta^2_{\frac{10}{a}}\xi'_1\right\rangle - \left\langle v_1, 2\chi_{B^2}(\zeta_B\zeta^2_{\frac{10}{a}})'\xi'_1\right\rangle - \left\langle v_1, \chi_{B^2}\zeta_B\zeta^2_{\frac{10}{a}}\xi''_1 v\right\rangle. \tag{6.29}$$

For the 1st term of line (6.28), we have

$$\begin{aligned} -\left\langle v_1, \chi''_{B^2}\zeta_B\zeta^2_{\frac{10}{a}}\xi_1\right\rangle &= -\left\langle \zeta_B v_1, \chi''_{B^2}\zeta_B\zeta^2_{\frac{10}{a}}\xi_1\right\rangle \\ &= -\left\langle \xi_1, \chi''_{B^2}\zeta_B\zeta^2_{\frac{10}{a}}\xi_1\right\rangle - \left\langle (1-\chi_{B^2})\zeta_B\mathcal{T}\chi_{B^2}\zeta_A^{-1}w_1, \chi''_{B^2}\zeta_B\zeta^2_{\frac{10}{a}}\xi_1\right\rangle. \end{aligned} \tag{6.30}$$

For the 1st term of line (6.30),

$$|\left\langle \xi_1, \chi''_{B^2}\zeta_B\zeta^2_{\frac{10}{a}}\xi_1\right\rangle| \lesssim B^{-4}\|\xi_1\|^2_{L^2_{-\frac{a}{10}}}. \tag{6.31}$$

For the 2nd term of line (6.30),

$$|\left\langle (1-\chi_{B^2})\zeta_B\mathcal{T}\chi_{B^2}\zeta_A^{-1}w_1, \chi''_{B^2}\zeta_B\zeta^2_{\frac{10}{a}}\xi_1\right\rangle| \lesssim e^{-B}\varepsilon^{\widetilde{N}}B^{-4}\|w_1\|_{L^2_{-\frac{a}{10}}}\|\xi_1\|_{L^2_{-\frac{a}{10}}} \tag{6.32}$$

Combining (6.31) and (6.32) we have

$$|\left\langle v_1, \chi''_{B^2}\zeta_B\zeta^2_{\frac{10}{a}}\xi_1\right\rangle| \lesssim B^{-4}\|\xi_1\|^2_{L^2_{-\frac{a}{10}}} + e^{-B}\varepsilon^{\widetilde{N}}B^{-4}\|w_1\|_{L^2_{-\frac{a}{10}}}\|\xi_1\|_{L^2_{-\frac{a}{10}}}. \tag{6.33}$$

Next, for the 2nd term of line (6.28) we have

$$\begin{aligned} -\left\langle v_1, 2\chi'_{B^2}(\zeta_B\zeta_{\frac{10}{a}})'\xi_1\right\rangle &= -\left\langle \zeta_B v_1, 2\chi'_{B^2}\zeta_B^{-1}(\zeta_B\zeta_{\frac{10}{a}})'\xi_1\right\rangle \\ &= -\left\langle \xi_1, 2\chi'_{B^2}\zeta_B^{-1}(\zeta_B\zeta_{\frac{10}{a}})'\xi_1\right\rangle \\ &\quad - \left\langle (1-\chi_{B^2})\zeta_B\mathcal{T}\chi_{B^2}\zeta_A^{-1}w_1, 2\chi'_{B^2}\zeta_B^{-1}(\zeta_B\zeta_{\frac{10}{a}})'\xi_1\right\rangle. \end{aligned} \tag{6.34}$$

The 1st term of line (6.34) can be bounded as

$$|-\left\langle \xi_1, 2\chi'_{B^2}\zeta_B^{-1}(\zeta_B\zeta_{\frac{10}{a}})'\xi_1\right\rangle| \lesssim B^{-2}\|\xi_1\|^2_{L^2_{-\frac{a}{10}}}$$

and the 2nd term of line (6.34) can be bounded as

$$\begin{aligned} &|\left\langle (1-\chi_{B^2})\zeta_B\mathcal{T}\chi_{B^2}\zeta_A^{-1}w_1, 2\chi'_{B^2}\zeta_B^{-1}(\zeta_B\zeta_{\frac{10}{a}})'\xi_1\right\rangle| \\ &\lesssim e^{-B}\varepsilon^{-\widetilde{N}}B^{-2}\|w_1\|_{L^2_{-\frac{a}{10}}}\|\xi_1\|_{L^2_{-\frac{a}{10}}}. \end{aligned}$$

Thus, we have

$$|\left\langle v_1, 2\chi'_{B^2}(\zeta_B\zeta_{\frac{10}{a}})'\xi_1\right\rangle| \lesssim B^{-2}\|\xi_1\|^2_{L^2_{-\frac{a}{10}}} + e^{-B}\varepsilon^{-\widetilde{N}}B^{-2}\|w_1\|_{L^2_{-\frac{a}{10}}}\|\xi_1\|_{L^2_{-\frac{a}{10}}}. \tag{6.35}$$

For the 3rd term of line (6.28), we have

$$|\left\langle v_1, \chi_{B^2}(\zeta_B\zeta^2_{\frac{10}{a}})''\xi_1\right\rangle| = |\left\langle \xi_1, \zeta_B^{-1}(\zeta_B\zeta^2_{\frac{10}{a}})''\xi_1\right\rangle| \lesssim \|\xi_1\|^2_{L^2_{-\frac{a}{10}}} \tag{6.36}$$

For the 1st term of line (6.29) we have

$$\begin{aligned} -\left\langle v_1, 2\chi'_{B^2}\zeta_B\zeta^2_{\frac{10}{a}}\xi'_1\right\rangle &= -\left\langle \xi_1, 2\chi'_{B^2}\zeta_B\zeta^2_{\frac{10}{a}}\xi'_1\right\rangle \\ &\quad -\left\langle (1-\chi_{B^2})\zeta_B\mathcal{T}\chi_{B^2}\zeta_A^{-1}w_1, 2\chi'_{B^2}\zeta_B\zeta^2_{\frac{10}{a}}\xi'_1\right\rangle \end{aligned} \tag{6.37}$$

For the 1st term of the r.h.s. of (6.37),

$$|\left\langle \xi_1, 2\chi'_{B^2}\zeta_B\zeta^2_{\frac{10}{a}}\xi'_1\right\rangle| \lesssim B^{-2}\|\xi_1\|_{L^2_{-\frac{a}{10}}}\|\xi'_1\|_{L^2_{-\frac{a}{10}}}, \tag{6.38}$$

and for the 2nd term of the r.h.s. of (6.37),

$$|\left\langle (1-\chi_{B^2})\zeta_B\mathcal{T}\chi_{B^2}\zeta_A^{-1}w_1, 2\chi'_{B^2}\zeta_B\zeta^2_{\frac{10}{a}}\xi'_1\right\rangle| \lesssim e^{-B}B^{-1}\varepsilon^{-\widetilde{N}}\|w_1\|_{L^2_{-\frac{a}{10}}}\|\xi'_1\|_{L^2_{-\frac{a}{10}}}. \tag{6.39}$$

Combining (6.38) and (6.39), we have

$$|\left\langle v_1, 2\chi'_{B^2}\zeta_B\zeta^2_{\frac{10}{a}}\xi'_1\right\rangle| \lesssim \left(B^{-2}\|\xi_1\|_{L^2_{-\frac{a}{10}}} + B^{-1}e^{-B}\varepsilon^{\widetilde{N}}\|w_1\|_{L^2_{-\frac{a}{10}}}\right)\|\xi'_1\|_{\widetilde{\Sigma}}. \tag{6.40}$$

For the 2nd term of line (6.29), we have

$$|\left\langle v_1, 2\chi_{B^2}(\zeta_B\zeta^2_{\frac{10}{a}})'\xi'_1\right\rangle| = 2|\left\langle \xi_1, \zeta_B^{-1}(\zeta_B\zeta^2_{\frac{10}{a}})'\xi'_1\right\rangle| \lesssim \|\xi_1\|_{L^2_{-\frac{a}{10}}}\|\xi_1\|_{\widetilde{\Sigma}}. \tag{6.41}$$

For the last term of line (6.29), we have

$$\begin{aligned} |\left\langle v_1, \chi_{B^2}\zeta_B\zeta^2_{\frac{10}{a}}\xi''_1\right\rangle| = |\left\langle \xi_1, \zeta^2_{\frac{10}{a}}\xi''_1\right\rangle| &\le |\left\langle \xi'_1, \zeta^2_{\frac{10}{a}}\xi'_1\right\rangle| \\ &\quad + |\left\langle \xi_1, \left(\zeta^2_{\frac{10}{a}}\right)'\xi'_1\right\rangle| \lesssim \|\xi_1\|^2_{\widetilde{\Sigma}}. \end{aligned} \tag{6.42}$$

Collecting (6.33), (6.35), (6.36), (6.40), (6.41) and (6.42) we have (6.25). ☐

Lemma 6.6. *We have*

$$\begin{aligned} &|\left\langle \mathbf{J}\left(\sum_{\mathbf{m}\in\boldsymbol{\mathcal{R}}_{\min}} \mathbf{z}^{\mathbf{m}}\widetilde{\mathbf{G}}_{\mathbf{m}} + \boldsymbol{\mathcal{R}}_{\mathbf{v}}\right), \chi^2_{B^2}\zeta^2_{\widetilde{B}}\sigma_3\mathbf{v}\right\rangle| \\ &\lesssim c^{-1}\sum_{\mathbf{m}\in\mathbf{R}_{\min}}|\mathbf{z}^{\mathbf{m}}|^2 + \delta|\dot{\mathbf{z}}-\widetilde{\mathbf{z}}|^2 + \|\xi_1\|^2_{\widetilde{\Sigma}} \\ &\quad + (c+\varepsilon)\|\xi_2\|^2_{L^2_{-\frac{a}{10}}} + \varepsilon\|w_1\|^2_{\widetilde{\Sigma}} + \varepsilon\|w_2\|^2_{L^2_{-\frac{a}{10}}} \end{aligned} \tag{6.43}$$

Proof. First, recalling (1.37), we have

$$|\left\langle \mathbf{J}\mathcal{T}\chi_{B^2}P_c\boldsymbol{\mathcal{R}}[\mathbf{z}], \chi^2_{B^2}\zeta^2_{\widetilde{B}}\sigma_3\mathbf{v}\right\rangle| \lesssim \sum_{\mathbf{m}\in\boldsymbol{\mathcal{R}}_{\min}}|\mathbf{z}^{\mathbf{m}}|\|\boldsymbol{\xi}\|_{L^2_{-\frac{a}{10}}}. \tag{6.44}$$

Next,

$$|\left\langle \mathbf{J}\mathcal{T}\chi_{B^2}P_cD\boldsymbol{\phi}[\mathbf{z}](\dot{\mathbf{z}}-\widetilde{\mathbf{z}}), \chi^2_{B^2}\zeta^2_{\widetilde{B}}\sigma_3\mathbf{v}\right\rangle| \lesssim \delta|\dot{\mathbf{z}}-\widetilde{\mathbf{z}}|\|\boldsymbol{\xi}\|_{L^2_{-\frac{a}{10}}}, \tag{6.45}$$

$$|\left\langle \mathbf{J}\mathcal{T}\chi_{B^2}P_c\mathbf{L}[\mathbf{z}]\left(R[\mathbf{z}]-1\right)\widetilde{\boldsymbol{\eta}},\chi_{B^2}^2\zeta_{\widetilde{B}}^2\sigma_3\mathbf{v}\right\rangle| \lesssim \delta\|\mathbf{w}\|_{L^2_{-\frac{a}{10}}}\|\boldsymbol{\xi}\|_{L^2_{-\frac{a}{10}}}. \quad (6.46)$$

We have, using the notation of Lemma 4.4,

$$|\left\langle \mathbf{J}\mathcal{T}\chi_{B^2}P_c\Delta_{W''}(\mathbf{z})E_1\boldsymbol{\eta},\chi_{B^2}^2\zeta_{\widetilde{B}}^2\sigma_3\mathbf{v}\right\rangle| \lesssim \delta\varepsilon^{-\widetilde{N}}\|\mathbf{w}\|_{L^2_{-\frac{a}{10}}}\|\boldsymbol{\xi}\|_{L^2_{-\frac{a}{10}}} \quad (6.47)$$

$$\begin{aligned}|\left\langle \mathbf{J}\mathcal{T}\chi_{B^2}P_c\mathbf{J}\mathbf{F},\chi_{B^2}^2\zeta_{\widetilde{B}}^2\sigma_3\mathbf{v}\right\rangle| &\lesssim \varepsilon^{-\widetilde{N}}\|\chi_{B^2}|\boldsymbol{\eta}|^2\|_{L^2_{-\frac{a}{10}}}\|\boldsymbol{\xi}\|_{L^2_{-\frac{a}{10}}}\\ &\lesssim \delta\varepsilon^{-\widetilde{N}}\|\mathbf{w}\|_{L^2_{-\frac{a}{10}}}\|\boldsymbol{\xi}\|_{L^2_{-\frac{a}{10}}}. \quad (6.48)\end{aligned}$$

Finally,

$$\begin{aligned}&|\left\langle \mathbf{J}\begin{pmatrix}0\\ -\mathcal{T}(2\chi'_{B^2}\partial_x+\chi''_{B^2})\widetilde{\eta}_1+\langle \mathrm{i}\varepsilon\partial_x\rangle^{-\widetilde{N}}\left[V_D,\langle \mathrm{i}\varepsilon\partial_x\rangle^{\widetilde{N}}\right]v_1\end{pmatrix},\chi_{B^2}^2\zeta_{\widetilde{B}}^2\sigma_3\mathbf{v}\right\rangle|\\ &=|\left\langle -\mathcal{T}(2\chi'_{B^2}\partial_x+\chi''_{B^2})\widetilde{\eta}_1+\langle \mathrm{i}\varepsilon\partial_x\rangle^{-\widetilde{N}}\left[V_D,\langle \mathrm{i}\varepsilon\partial_x\rangle^{\widetilde{N}}\right]v_1,\chi_{B^2}^2\zeta_{\widetilde{B}}^2v_1\right\rangle|\\ &\lesssim \varepsilon^{-\widetilde{N}}e^{-B}\|w_1\|_{\widetilde{\Sigma}}\|\xi_1\|_{\widetilde{\Sigma}}+\varepsilon\|\langle x\rangle^{-10}v_1\|_{L^2}\|\xi_1\|_{L^2_{-\frac{a}{10}}}\\ &\lesssim \varepsilon^{-\widetilde{N}}e^{-B}\|w_1\|_{\widetilde{\Sigma}}\|\xi_1\|_{\widetilde{\Sigma}}+\varepsilon\left(\|\xi_1\|_{\widetilde{\Sigma}}+\varepsilon^{-\widetilde{N}}B^{-1}\|w_1\|_{\widetilde{\Sigma}}\right)\|\xi_1\|_{L^2_{-\frac{a}{10}}}\\ &\lesssim \varepsilon\left(\|w_1\|_{\widetilde{\Sigma}}\|\xi_1\|_{\widetilde{\Sigma}}+\|\xi_1\|_{\widetilde{\Sigma}}^2\right). \quad (6.49)\end{aligned}$$

Combining (6.44)–(6.49), we obtain (6.43). □

Proof of Lemma 6.1. The inequalities (6.4) and (6.5) follows from Lemmas 6.2, 6.4, 6.5 and 6.6. □

7. Proof of Proposition 3.6

Proof of Proposition 3.6. Recalling equation (2.5), which we rewrite in an equivalent form

$$\partial_t\boldsymbol{\eta}+D_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}](\dot{\mathbf{z}}-\widetilde{\mathbf{z}})=\mathbf{L}[\mathbf{z}]\boldsymbol{\eta}+\mathbf{J}\mathbf{F}[\mathbf{z},\boldsymbol{\eta}]+\boldsymbol{\mathcal{R}}[\mathbf{z}],$$

and taking the inner product between of this equation with and $\mathbf{J}D_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}]\boldsymbol{\zeta}$, for any fixed $\boldsymbol{\zeta}\in\mathbb{C}^{\widetilde{N}}$, we have

$$\begin{aligned}\langle\partial_t\boldsymbol{\eta},\mathbf{J}D_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}]\boldsymbol{\zeta}\rangle+\langle D_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}](\dot{\mathbf{z}}-\widetilde{\mathbf{z}}),\mathbf{J}D_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}]\boldsymbol{\zeta})\rangle &= \langle\mathbf{L}[\mathbf{z}]\boldsymbol{\eta},\mathbf{J}D_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}]\boldsymbol{\zeta}\rangle\\ &+\langle\mathbf{F},D_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}]\boldsymbol{\zeta}\rangle, \quad (7.1)\end{aligned}$$

where we exploited the orthogonality (1.39), $\langle\mathbf{J}\boldsymbol{\mathcal{R}}[\mathbf{z}],D_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}]\boldsymbol{\zeta}\rangle=0$. By Leibnitz and the orthogonality condition $\langle\boldsymbol{\eta},\mathbf{J}D_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}]\boldsymbol{\zeta}\rangle=0$, we have

$$\langle\partial_t\boldsymbol{\eta},\mathbf{J}D_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}]\boldsymbol{\zeta}\rangle=-\left\langle\boldsymbol{\eta},\mathbf{J}D_{\mathbf{z}}^2\boldsymbol{\phi}[\mathbf{z}](\dot{\mathbf{z}},\boldsymbol{\zeta})\right\rangle.$$

Next, differentiating (1.36) w.r.t. $\mathbf{z}$, we have

$$\mathbf{L}[\mathbf{z}]D_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}]\boldsymbol{\zeta}=-D_{\mathbf{z}}^2\boldsymbol{\phi}[\mathbf{z}](\widetilde{\mathbf{z}},\boldsymbol{\zeta})+D_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}](D_{\mathbf{z}}\widetilde{\mathbf{z}}[\mathbf{z}]\boldsymbol{\zeta})-D_{\mathbf{z}}\boldsymbol{\mathcal{R}}[\mathbf{z}]\boldsymbol{\zeta}.$$

By the fact that $\mathbf{J}\mathbf{L}[\mathbf{z}]$ is self-adjoint and that $\boldsymbol{\eta}\in\mathcal{H}_c[\mathbf{z}]$, we have

$$\langle \mathbf{L}[\mathbf{z}]\boldsymbol{\eta}, \mathbf{J}D_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}]\boldsymbol{\zeta}\rangle = \langle \boldsymbol{\eta}, \mathbf{J}\mathbf{L}[\mathbf{z}]D_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}]\boldsymbol{\zeta}\rangle =$$
$$-\left\langle \boldsymbol{\eta}, \mathbf{J}D^2_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}](\widetilde{\mathbf{z}}, \boldsymbol{\zeta})\right\rangle - \langle \boldsymbol{\eta}, \mathbf{J}D_{\mathbf{z}}\boldsymbol{\mathcal{R}}[\mathbf{z}]\boldsymbol{\zeta}\rangle . \tag{7.2}$$

Inserting the information in (7.1), we obtain

$$\langle D_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}](\dot{\mathbf{z}} - \widetilde{\mathbf{z}}), \mathbf{J}D_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}]\boldsymbol{\zeta}\rangle + \left\langle \mathbf{J}\boldsymbol{\eta}, D^2_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}](\dot{\mathbf{z}} - \widetilde{\mathbf{z}}, \boldsymbol{\zeta})\right\rangle =$$
$$-\langle \mathbf{J}\boldsymbol{\eta}, D_{\mathbf{z}}\boldsymbol{\mathcal{R}}[\mathbf{z}]\boldsymbol{\zeta}\rangle + \langle \mathbf{F}[\mathbf{z}, \boldsymbol{\eta}], D_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}]\boldsymbol{\zeta}\rangle .$$

Now,

$$\langle D_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}](\dot{\mathbf{z}} - \widetilde{\mathbf{z}}), \mathbf{J}D_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}]\boldsymbol{\zeta}\rangle = \langle D_{\mathbf{z}}\boldsymbol{\phi}[0](\dot{\mathbf{z}} - \widetilde{\mathbf{z}}), \mathbf{J}D_{\mathbf{z}}\boldsymbol{\phi}[0]\boldsymbol{\zeta}\rangle + O(|\mathbf{z}||\dot{\mathbf{z}} - \widetilde{\mathbf{z}}|),$$

and

$$\left\langle D_{\mathbf{z}}\boldsymbol{\phi}[0](\dot{\mathbf{z}} - \widetilde{\mathbf{z}}), \mathbf{J}D\boldsymbol{\phi}[0]\mathbf{e}^j\right\rangle = 4\langle \mathrm{Re}\boldsymbol{\Phi}_j(\dot{\mathbf{z}} - \widetilde{\mathbf{z}}), \mathbf{J}\mathrm{Re}\boldsymbol{\Phi}_j\rangle = -2\mathrm{Im}(\dot{\mathbf{z}} - \widetilde{\mathbf{z}}),$$
$$\left\langle D_{\mathbf{z}}\boldsymbol{\phi}[0](\dot{\mathbf{z}} - \widetilde{\mathbf{z}}), \mathbf{J}D\boldsymbol{\phi}[0]\mathrm{i}\mathbf{e}^j\right\rangle = -4\langle \mathrm{Re}\boldsymbol{\Phi}_j(\dot{\mathbf{z}} - \widetilde{\mathbf{z}}), \mathbf{J}\mathrm{Im}\boldsymbol{\Phi}_j\rangle = -2\mathrm{Re}(\dot{\mathbf{z}} - \widetilde{\mathbf{z}}).$$

Thus, by the following, for $\boldsymbol{\zeta} = \mathbf{e}^j, \mathrm{i}\mathbf{e}^j$, we have the conclusion,

$$|\langle \mathbf{J}\boldsymbol{\eta}, D_{\mathbf{z}}\boldsymbol{\mathcal{R}}[\mathbf{z}]\boldsymbol{\zeta}\rangle| \lesssim \delta\|\mathbf{w}\|_{L^2_{-\frac{a}{10}}},$$
$$|\langle \mathbf{F}[\mathbf{z}, \boldsymbol{\eta}], D\boldsymbol{\phi}[\mathbf{z}]\boldsymbol{\zeta}\rangle| \lesssim \|\eta_1^2\|_{L^2_{-\frac{a}{10}-A^{-1}}} \lesssim \delta\|w_1\|_{L^2_{-\frac{a}{10}}}.$$

□

Our next task, is to examine the terms $\mathbf{z}^{\mathbf{m}}$ and show $\mathbf{z} \xrightarrow{t\to+\infty} 0$, that is the discrete modes are damped by nonlinear interaction with the radiation. In order to do so, we expand the variable $\mathbf{v}$, defined in (2.13), in a part resonating with the discrete modes $\mathbf{z}$, which will yield the damping, and a remainder which we denote by $\mathbf{g}$. Notice that this additional variable $\mathbf{g}$, is standard in the field, starting from [5, 43].

8. Smoothing estimate for g

Looking at the equation for $\mathbf{v}$, (2.13), we introduce the functions

$$\boldsymbol{\rho}_{\mathbf{m}} := R^{+}_{\mathrm{i}\mathbf{L}_D}(\boldsymbol{\lambda}\cdot\mathbf{m})\mathrm{i}\widetilde{\mathcal{R}}_{\mathbf{m}}, \tag{8.1}$$

which solve

$$(\mathrm{i}\mathbf{L}_D - \boldsymbol{\lambda}\cdot\mathbf{m})\boldsymbol{\rho}_{\mathbf{m}} = \mathrm{i}\widetilde{\mathcal{R}}_{\mathbf{m}} \tag{8.2}$$

and we set

$$\mathbf{g} = \mathbf{v} + Z(\mathbf{z}) \text{ where } Z(\mathbf{z}) := -\sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}}\boldsymbol{\rho}_{\mathbf{m}}. \tag{8.3}$$

An elementary computation yields

$$\mathrm{i}\partial_t\mathbf{g} = \mathrm{i}\mathbf{L}_D\mathbf{g} - \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \left(\mathrm{i}\partial_t\left(\mathbf{z}^{\mathbf{m}}\right) - \boldsymbol{\lambda}\cdot\mathbf{m}\,\mathbf{z}^{\mathbf{m}}\right)\boldsymbol{\rho}_{\mathbf{m}} + \mathrm{i}\mathbf{R}_v$$

or, equivalently,

$$\mathbf{g}(t) = e^{t\mathbf{L}_D}\mathbf{v}(0) + \mathrm{i}\sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}}(0) e^{t\mathbf{L}_D} R^{+}_{\mathrm{i}\mathbf{L}_D}(\boldsymbol{\lambda}\cdot\mathbf{m})\widetilde{\mathcal{R}}_{\mathbf{m}} \tag{8.4}$$

$$-\sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathrm{i}\int_0^t e^{(t-t')\mathbf{L}_D}\left(\partial_t\left(\mathbf{z}^{\mathbf{m}}\right) + \mathrm{i}\boldsymbol{\lambda}\cdot\mathbf{m}\,\mathbf{z}^{\mathbf{m}}\right)\boldsymbol{\rho}_{\mathbf{m}} dt' \tag{8.5}$$

$$-\mathrm{i}\int_0^t e^{(t-t')\mathbf{L}_D}\mathcal{T}\left(2\chi'_{B^2}\partial_x + \chi''_{B^2}\right)\tilde{\eta}_1\mathbf{j}dt' \tag{8.6}$$

$$-\mathrm{i}\int_0^t e^{(t-t')\mathbf{L}_D}\left(\langle\mathrm{i}\varepsilon\partial_x\rangle^{-\widetilde{N}}[V_D, \langle\mathrm{i}\varepsilon\partial_x\rangle^{\widetilde{N}}]v_1\mathbf{j} + \mathcal{T}\chi_{B^2}\mathbf{R}_{\widetilde{\eta}}\right). \tag{8.7}$$

We will prove the following, where we use the weighted spaces defined in Definition 1.10.

Proposition 8.1. *For $S > 4$ we have*

$$\|\mathbf{g}\|_{L^2(I, \boldsymbol{\mathcal{H}}^{1,-S}(\mathbb{R}))} \le o_{\varepsilon}(1)\epsilon. \tag{8.8}$$

To prove Proposition 8.1 we will need to bound one by one the terms in (8.4)–(8.7).

Lemma 8.2. *For any $S > 5/2$ there exists a fixed $c(S)$ s.t.*

$$\|e^{t\mathbf{L}_D}\mathbf{f}\|_{L^2(\mathbb{R}, \boldsymbol{\mathcal{H}}^{1,-S})} \le c(S)\|\mathbf{f}\|_{\boldsymbol{\mathcal{H}}^1} \text{ for all } f \in \boldsymbol{\mathcal{H}}^1(\mathbb{R}). \tag{8.9}$$

By Lemma 8.2 we have

$$\|\text{r.h.s. of } (8.4)\|_{L^2(I, \boldsymbol{\mathcal{H}}^{1,-S}(\mathbb{R}))} \lesssim \|v(0)\|_{\boldsymbol{\mathcal{H}}^1} + \|\mathbf{z}(0)\|^2$$

Proof of Lemma 8.2. Recall that, as a consequence of (5.15), we have

$$\sup_{0<\epsilon\le 1}\sup_{\varsigma\in\mathbb{R}} \|R_{\mathrm{i}\mathbf{L}_D}(\varsigma \pm \mathrm{i}\epsilon)\|_{\boldsymbol{\mathcal{H}}^{1,S}\to\boldsymbol{\mathcal{H}}^{1,-S}} < \infty.$$

This easily implies that

$$a_3 := \sup_{0<\epsilon\le 1}\sup_{\varsigma, \mathbf{f}} \left\langle \langle x\rangle^{-S}\left(R_{\mathrm{i}\mathbf{L}_D}(\varsigma + \mathrm{i}\epsilon) - R_{\mathrm{i}\mathbf{L}_D}(\varsigma - \mathrm{i}\epsilon)\right)(\langle x\rangle^{-S})^*\mathbf{f}, \mathbf{f}\right\rangle_{\boldsymbol{\mathcal{H}}^1},$$

where $(\langle x\rangle^{-S})^*$ is the adjoint of the multiplicative operator $\langle x\rangle^{-S}$ in $\boldsymbol{\mathcal{H}}^1$. Then by Lemma 3.6 and Lemma 5.5 [17] we have (8.9) with $C(S) = \sqrt{2\pi a_3}$. □

By Lemma 5.7 we have

$$\begin{aligned}
\|(8.15)\|_{L^2(I, \boldsymbol{\mathcal{H}}^{1,-S}(\mathbb{R}))} &\lesssim \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \int_0^t \langle t - t'\rangle^{-\frac{3}{2}} |\partial_t\left(\mathbf{z}^{\mathbf{m}}\right) + \mathrm{i}\boldsymbol{\lambda}\cdot\mathbf{m}\mathbf{z}^{\mathbf{m}}| dt' \|\widetilde{\mathcal{R}}_{\mathbf{m}}\|_{\boldsymbol{\mathcal{H}}^{1,S}} \\
&\lesssim \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \|\partial_t\left(\mathbf{z}^{\mathbf{m}}\right) + \mathrm{i}\boldsymbol{\lambda}\cdot\mathbf{m}\mathbf{z}^{\mathbf{m}}\|_{L^2(I)} \|\mathcal{R}_{\mathbf{m}}\|_{\Sigma^{\widetilde{N}+1}} \\
&\lesssim \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \|\partial_t\left(\mathbf{z}^{\mathbf{m}}\right) + \mathrm{i}\boldsymbol{\lambda}\cdot\mathbf{m}\mathbf{z}^{\mathbf{m}}\|_{L^2(I)} \\
&\lesssim \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \left(\|D_{\mathbf{z}}\mathbf{z}^{\mathbf{m}}(\dot{\mathbf{z}} - \widetilde{\mathbf{z}})\|_{L^2(I)} + \|D_{\mathbf{z}}\mathbf{z}^{\mathbf{m}}(\widetilde{\mathbf{z}} + \mathrm{i}\boldsymbol{\lambda}\mathbf{z})\|_{L^2(I)}\right)
\end{aligned}$$

$$\lesssim \|\mathbf{z}\|_{L^\infty(I)}\|\dot{\mathbf{z}}-\widetilde{\mathbf{z}}\|_{L^2(I)} + \|\mathbf{z}\|_{L^\infty(I)} \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \|\mathbf{z}^{\mathbf{m}}\|_{L^2(I)}$$

$$\lesssim \delta^2 \|\mathbf{w}\|_{L^2_{-\frac{a}{10}}} + \delta \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \|\mathbf{z}^{\mathbf{m}}\|_{L^2(I)}.$$

By Lemma 5.12 we have

$$\|(8.16)\|_{L^2(I,\boldsymbol{\mathcal{H}}^{1,-S})} \lesssim \|\mathcal{T}\left(2\chi'_{B^2}\partial_x + \chi''_{B^2}\right)\tilde{\eta}_1\|_{L^2(I,L^{2,\tau})} \lesssim B^{-\frac{1}{2}}\varepsilon = o_\varepsilon(1)\epsilon \tag{8.10}$$

where the last inequality is proved in [8, Sect. 8], in particular [8, formulas (8.23)–(8.25)].

We next look at (8.7). Again by Lemma 5.12 we have

$$\| \int_0^t e^{(t-t')\mathbf{L}_D} \langle \mathrm{i}\varepsilon\partial_x\rangle^{-\widetilde{N}} [V_D, \langle \mathrm{i}\varepsilon\partial_x\rangle^{\widetilde{N}}] v_1 \mathbf{j}\|_{L^2(I,\boldsymbol{\mathcal{H}}^{1,-S})}$$

$$\lesssim \| \langle \mathrm{i}\varepsilon\partial_x\rangle^{-\widetilde{N}} [V_D, \langle \mathrm{i}\varepsilon\partial_x\rangle^{\widetilde{N}}] v_1 \|_{L^2(I,L^{1,\tau})} \lesssim \varepsilon\epsilon,$$

where the last inequality is proved in formula (8.26) [8]. Finally, we consider

$$\| \int_0^t e^{(t-t')\mathbf{L}_D} \mathcal{T}\chi_{B^2}\mathbf{R}_{\widetilde{\eta}}\|_{L^2(I,\boldsymbol{\mathcal{H}}^{1,-S})} \lesssim \|\mathcal{T}\chi_{B^2}\mathbf{R}_{\widetilde{\eta}}\|_{L^2(I,\boldsymbol{\mathcal{H}}^{1,\tau})}.$$

The right hand side is less than $I + II$ where

$$I = \|\chi_{8B^2}\mathcal{T}\chi_{B^2}\mathbf{R}_{\widetilde{\eta}}\|_{L^2(I,\boldsymbol{\mathcal{H}}^{1,\tau})}$$
$$II = \|(1-\chi_{8B^2})\mathcal{T}\chi_{B^2}\mathbf{R}_{\widetilde{\eta}}\|_{L^2(I,\boldsymbol{\mathcal{H}}^{1,\tau})}$$

We have

$$I \lesssim B^{2\tau}\|\mathcal{T}\chi_{B^2}\boldsymbol{\mathcal{R}}_{\widetilde{\eta}}\|_{L^2(I,\boldsymbol{\mathcal{H}}^1)}$$

with

$$\|\mathcal{T}\chi_{B^2}\boldsymbol{\mathcal{R}}_{\widetilde{\eta}}\|_{L^2(I,\boldsymbol{\mathcal{H}}^1)} \le (I_1+I_2+I_3) \text{ where}$$
$$I_1 = \|\mathcal{T}\chi_{B^2}P_c\left(\boldsymbol{\mathcal{R}}_1[\mathbf{z}] - D\boldsymbol{\phi}[\mathbf{z}](\dot{\mathbf{z}}-\widetilde{\mathbf{z}}) + \mathbf{L}_1(R[\mathbf{z}]-1)\widetilde{\boldsymbol{\eta}}\right)\|_{L^2(I,\boldsymbol{\mathcal{H}}^1)},$$
$$I_2 = \|\mathcal{T}\chi_{B^2}P_c(\mathbf{L}[\mathbf{z}]-\mathbf{L}_1)\boldsymbol{\eta}\|_{L^2(I,\boldsymbol{\mathcal{H}}^1)},$$
$$I_3 = \|\mathcal{T}\chi_{B^2}P_c\mathbf{J}\mathbf{F}\|_{L^2(I,\boldsymbol{\mathcal{H}}^1)}.$$

We have

$$I_1 \le \|\boldsymbol{\mathcal{R}}_1[\mathbf{z}]\|_{L^2(I,\Sigma^{\widetilde{N}+1})} + \|\dot{\mathbf{z}}-\widetilde{\mathbf{z}}\|_{L^2(I)} + \|(R[\mathbf{z}]-1)\widetilde{\boldsymbol{\eta}}\|_{L^2(I,\Sigma^{\widetilde{N}+1})}$$
$$\lesssim \|\mathbf{z}\|_{L^\infty(I)}[\sum_{\mathbf{m}\in\boldsymbol{\mathcal{R}}_{\min}} \|\mathbf{z}^{\mathbf{m}}\|_{L^2(I)} + \|\dot{\mathbf{z}}-\widetilde{\mathbf{z}}\|_{L^2(I)} + \|\boldsymbol{\eta}\|_{L^2(I,\Sigma^*)}]$$
$$\lesssim \delta[\sum_{\mathbf{m}\in\boldsymbol{\mathcal{R}}_{\min}} \|\mathbf{z}^{\mathbf{m}}\|_{L^2(I)} + \|\dot{\mathbf{z}}-\widetilde{\mathbf{z}}\|_{L^2(I)} + \|\mathbf{w}\|_{L^2(I,L^2_{-\frac{a}{10}})}].$$

We have

$$I_2 \le \|\mathcal{T}\chi_{B^2}(\mathbf{L}[\mathbf{z}]-\mathbf{L}_1)\boldsymbol{\eta}\|_{L^2(I,\boldsymbol{\mathcal{H}}^1)} + \|\mathcal{T}\chi_{B^2}P_d(\mathbf{L}[\mathbf{z}]-\mathbf{L}_1)\boldsymbol{\eta}\|_{L^2(I,\boldsymbol{\mathcal{H}}^1)}$$
$$\lesssim \varepsilon^{-\widetilde{N}}\|(\mathbf{L}[\mathbf{z}]-\mathbf{L}_1)\boldsymbol{\eta}\|_{L^2(I,\boldsymbol{\mathcal{H}}^1)}$$
$$\lesssim \varepsilon^{-\widetilde{N}}\|(\mathbf{L}[\mathbf{z}]-\mathbf{L}_1)\widetilde{\boldsymbol{\eta}}\|_{L^2(I,\boldsymbol{\mathcal{H}}^1)} + \varepsilon^{-\widetilde{N}}\|(\mathbf{L}[\mathbf{z}]-\mathbf{L}_1)(R[\mathbf{z}]-1)\widetilde{\boldsymbol{\eta}}\|_{L^2(I,\boldsymbol{\mathcal{H}}^1)}$$

$$
\begin{aligned}
&\lesssim \|(W''(\phi_1[\mathbf{z}]) - W''(H))\widetilde{\eta}_2\|_{L^2(I,L^2)} + \delta\|\mathbf{w}\|_{L^2(I,L^2_{-\frac{a}{10}})} \\
&\le \left(\|(W''(\phi_1[\mathbf{z}]) - W''(H))\zeta_A^{-1}e^{|x|\frac{a}{10}}\|_{L^\infty(I,L^\infty)} + \delta\right)\|\mathbf{w}\|_{L^2(I,L^2_{-\frac{a}{10}})} \\
&\lesssim \delta\|\mathbf{w}\|_{L^2(I,L^2_{-\frac{a}{10}})}.
\end{aligned}
$$

We have

$$
\begin{aligned}
I_3 &\lesssim \varepsilon^{-\widetilde{N}}\|\chi_{B^2}F_1\|_{L^2(I,L^2)} \lesssim \varepsilon^{-\widetilde{N}}\|\chi_{B^2}\eta_1^2\|_{L^2(I,L^2)} \\
&\lesssim \varepsilon^{-\widetilde{N}}\|\eta_1\|_{L^\infty(I,H^1)}\left(\|w_1\|_{L^2(I,L^2(|x|\le 2B^2))} + \|R[\mathbf{z}]-1)\eta_1\|_{L^2(I,L^2)}\right) \\
&\lesssim \varepsilon^{-\widetilde{N}}\delta\left(B^2\|w_1\|_{L^2(I,\widetilde{\Sigma})} + \|\mathbf{z}\|_{L^\infty(I)}\|w_1\|_{L^2(I,\widetilde{\Sigma})}\right) \lesssim \varepsilon^{-\widetilde{N}}\delta B^2\|w_1\|_{L^2(I,\widetilde{\Sigma})}.
\end{aligned}
$$

We conclude that

$$
\begin{aligned}
&\|\mathcal{T}\chi_{B^2}\boldsymbol{\mathcal{R}}_{\widetilde{\eta}}\|_{L^2(I,\boldsymbol{\mathcal{H}}^1)} \lesssim \text{ and} \\
&I \lesssim B^{2\tau+2}\delta^2\epsilon = o_\varepsilon(1)\epsilon. \qquad (8.11)
\end{aligned}
$$

Turning to the analysis of II, we have

$$
\begin{aligned}
II &\lesssim \|(1-\chi_{8B^2})\langle x\rangle^\tau \mathcal{T}\langle x\rangle^{-\tau}\chi_{2B^2}\|_{\boldsymbol{\mathcal{H}}^1\to\boldsymbol{\mathcal{H}}^1}\|\chi_{B^2}\mathbf{R}_{\widetilde{\eta}}\|_{L^2(I,\boldsymbol{\mathcal{H}}^{1,\tau})} \\
&\lesssim \|\chi_{B^2}\mathbf{R}_{\widetilde{\eta}}\|_{L^2(I,\boldsymbol{\mathcal{H}}^{1,\tau})} \lesssim B^{2\tau+2}\delta^2\epsilon = o_\varepsilon(1)\epsilon \qquad (8.12)
\end{aligned}
$$

where we used

$$
\|(1-\chi_{8B^2})\langle x\rangle^\tau \mathcal{T}\langle x\rangle^{-\tau}\chi_{2B^2}\|_{\boldsymbol{\mathcal{H}}^1\to\boldsymbol{\mathcal{H}}^1} \lesssim 1.
$$

Notice this will be a consequence of

$$
\|(1-\chi_{8B^2})\langle x\rangle^\tau \mathcal{T}\langle x\rangle^{-\tau}\chi_{2B^2}\|_{H^1\to H^1} \lesssim 1 \qquad (8.13)
$$

$$
\|(1-\chi_{8B^2})\langle x\rangle^\tau \mathcal{T}\langle x\rangle^{-\tau}\chi_{2B^2}\|_{L^2\to L^2} \lesssim 1. \qquad (8.14)
$$

Inequality (8.14) is proved in §8 [8]. We turn to (8.13). It is enough to bound the operator norm of

$$
\begin{aligned}
&\partial_x(1-\chi_{8B^2})\langle x\rangle^\tau \mathcal{T}\langle x\rangle^{-\tau}\chi_{2B^2} = [\partial_x,(1-\chi_{8B^2})\langle x\rangle^\tau]\mathcal{T}\langle x\rangle^{-\tau}\chi_{2B^2} \\
&+ (1-\chi_{8B^2})\langle x\rangle^\tau[\partial_x,\mathcal{T}]\langle x\rangle^{-\tau}\chi_{2B^2} \qquad (8.15) \\
&+ (1-\chi_{8B^2})\langle x\rangle^\tau \mathcal{T}[\partial_x,\langle x\rangle^{-\tau}\chi_{2B^2}] \\
&+ (1-\chi_{8B^2})\langle x\rangle^\tau \mathcal{T}\langle x\rangle^{-\tau}\chi_{2B^2}\partial_x. \qquad (8.16)
\end{aligned}
$$

All terms except the one in line (8.16) are similar to the operator in (8.13). The most interesting is the one in line (8.15). This operator equals

$$
\begin{aligned}
&(1-\chi_{8B^2})\langle x\rangle^\tau\langle \mathrm{i}\varepsilon\partial_x\rangle^{-\widetilde{N}}[\partial_x,\mathcal{A}^*]\langle x\rangle^{-\tau}\chi_{2B^2} \\
&= \sum_{j=1}^{\widetilde{N}}(1-\chi_{8B^2})\langle x\rangle^\tau\langle \mathrm{i}\varepsilon\partial_x\rangle^{-\widetilde{N}}\left(\prod_{i=0}^{\widetilde{N}-1-j}A^*_{\widetilde{N}-i}\right)(\log\psi_j)''\left(\prod_{i=1}^{j-1}A^*_{j-i}\right)\langle x\rangle^{-\tau}\chi_{2B^2}
\end{aligned}
$$

with the convention $\prod_{i=0}^{l}B_i = B_0\circ\ldots\circ B_l$ and where ψ_j is a ground state of L_j, see §1.1. The operators in the last line summation are similar to the one in (8.13) and satisfy the same estimate. Obviously for the operator in line (8.16) we have

$$\begin{aligned}&\|(1-\chi_{8B^2})\langle x\rangle^{\tau}\mathcal{T}\langle x\rangle^{-\tau}\chi_{2B^2}\partial_x\|_{H^1\to L^2}\\&\quad\le\|(1-\chi_{8B^2})\langle x\rangle^{\tau}\mathcal{T}\langle x\rangle^{-\tau}\chi_{2B^2}\|_{L^2\to L^2}\lesssim 1.\end{aligned}$$

9. Proof of Proposition 3.7: the Fermi Golden Rule

We can aptly name $E(\boldsymbol{\phi}[\mathbf{z}])$ *localized* energy, since $\boldsymbol{\eta}(t)$ is expected to disperse to infinity as $t\to+\infty$ and what remains locally of the solution is $\boldsymbol{\phi}[\mathbf{z}(t)]$. In our analysis of the FGR, $E(\boldsymbol{\phi}[\mathbf{z}])$ is like a Lyapunov function. So we compute, recall $\langle\mathbf{f},\mathbf{g}\rangle:=\mathrm{Re}\int{}^t\mathbf{f}\,\overline{\mathbf{g}}dx$,

$$\begin{aligned}\frac{d}{dt}E(\boldsymbol{\phi}[\mathbf{z}])&=\langle\nabla E(\boldsymbol{\phi}[\mathbf{z}]),D_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}]\dot{\mathbf{z}}\rangle\\&=-\left\langle\mathbf{J}\left(\boldsymbol{\mathcal{R}}[\mathbf{z}]+D_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}]\widetilde{\mathbf{z}}\right),D_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}]\dot{\mathbf{z}}\right\rangle\\&=\left\langle\mathbf{J}D_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}]\left(\dot{\mathbf{z}}-\widetilde{\mathbf{z}}\right),D_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}]\widetilde{\mathbf{z}}\right\rangle\\&=\left\langle\mathbf{J}\left(\mathbf{L}[\mathbf{z}]\boldsymbol{\eta}+\mathbf{J}\mathbf{F}[\mathbf{z},\boldsymbol{\eta}]+\cancel{\boldsymbol{\mathcal{R}}[\mathbf{z}]}-\partial_t\boldsymbol{\eta}\right),D_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}]\widetilde{\mathbf{z}}\right\rangle,\end{aligned}\tag{9.1}$$

where we have used (1.36) in the 2nd equality, the cancelation (1.39) and $\langle\mathbf{J}f,f\rangle=0$ in the 3rd equality and (2.5) in the 4th inequality and, finally, we used (1.39) for the above cancellation of the $\boldsymbol{\mathcal{R}}[\mathbf{z}]$ term. From $\boldsymbol{\eta}\in\mathcal{H}_c[\mathbf{z}]$, we have

$$-\left\langle\mathbf{J}\partial_t\boldsymbol{\eta},D_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}]\widetilde{\mathbf{z}}\right\rangle=\left\langle\mathbf{J}\boldsymbol{\eta},D^2_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}]\left(\dot{\mathbf{z}},\widetilde{\mathbf{z}}\right)\right\rangle.\tag{9.2}$$

Substituting (7.2) and (9.2) into (9.1), we have

$$\frac{d}{dt}E(\boldsymbol{\phi}[\mathbf{z}])=-\left\langle\mathbf{J}\boldsymbol{\eta},D_{\mathbf{z}}\boldsymbol{\mathcal{R}}[\mathbf{z}]\widetilde{\mathbf{z}}\right\rangle+\left\langle\mathbf{J}\boldsymbol{\eta},D^2_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}]\left(\dot{\mathbf{z}}-\widetilde{\mathbf{z}},\widetilde{\mathbf{z}}\right)\right\rangle-\left\langle\mathbf{F}[\mathbf{z},\boldsymbol{\eta}],D_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}]\widetilde{\mathbf{z}}\right\rangle\tag{9.3}$$

Claim 9.1. *For all $t\in I$*

$$\left|\int_0^t\left\langle\mathbf{J}\boldsymbol{\eta},D_{\mathbf{z}}\boldsymbol{\mathcal{R}}[\mathbf{z}]\widetilde{\mathbf{z}}\right\rangle dt'\right|=o_{\varepsilon}(1)\epsilon^2.\tag{9.4}$$

Proof. Indeed we have $E(\boldsymbol{\phi}[\mathbf{z}])]_0^t=O(\delta^2)$ from Proposition 2.5. we have

$$\left|\left\langle\mathbf{J}\boldsymbol{\eta},D^2_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}]\left(\dot{\mathbf{z}}-\widetilde{\mathbf{z}},\widetilde{\mathbf{z}}\right)\right\rangle\right|\lesssim\delta\|\mathbf{w}\|_{L^2_{-\frac{a}{10}}}|\dot{\mathbf{z}}-\widetilde{\mathbf{z}}|,$$

and

$$\left|\left\langle\mathbf{F}[\mathbf{z},\boldsymbol{\eta}],D_{\mathbf{z}}\boldsymbol{\phi}[\mathbf{z}]\widetilde{\mathbf{z}}\right\rangle\right|\lesssim\delta\|w_1\|^2_{L^2_{-\frac{a}{10}}},$$

and integrating in time, we obtain the desired bound (9.4). □

Let us focus now on the term in the left hand side of (9.4). By the expansion (1.37) of $\boldsymbol{\mathcal{R}}[\mathbf{z}]$, we have

$$\left\langle\mathbf{J}\boldsymbol{\eta},D_{\mathbf{z}}\boldsymbol{\mathcal{R}}[\mathbf{z}]\widetilde{\mathbf{z}}\right\rangle=\sum_{\mathbf{m}\in\mathbf{R}_{\min}}\left\langle\mathbf{J}\boldsymbol{\eta},D_{\mathbf{z}}\mathbf{z}^{\mathbf{m}}(-\mathrm{i}\boldsymbol{\lambda}\mathbf{z})\mathcal{R}_{\mathbf{m}}\right\rangle\tag{9.5}$$

$$+\sum_{\mathbf{m}\in\mathbf{R}_{\min}}\left\langle\mathbf{J}\boldsymbol{\eta},D\mathbf{z}^{\mathbf{m}}(\widetilde{\mathbf{z}}+\mathrm{i}\boldsymbol{\lambda}\mathbf{z})\mathcal{R}_{\mathbf{m}}\right\rangle-\left\langle\mathbf{J}\boldsymbol{\eta},D_{\mathbf{z}}\boldsymbol{\mathcal{R}}_1[\mathbf{z}]\widetilde{\mathbf{z}}\right\rangle,\tag{9.6}$$

where $\boldsymbol{\lambda}\mathbf{z} := (\lambda_1 z_1, \cdots, \lambda_N z_N)$. The 2nd line can be bounded as

$$\sum_{\mathbf{m}\in\mathbf{R}_{\min}} |\langle \mathbf{J}\boldsymbol{\eta}, D\mathbf{z}^{\mathbf{m}}(\widetilde{\mathbf{z}} + \mathrm{i}\boldsymbol{\lambda}\mathbf{z})\mathcal{R}_{\mathbf{m}}\rangle| + |\langle \mathbf{J}\boldsymbol{\eta}, D_{\mathbf{z}}\boldsymbol{\mathcal{R}}_1[\mathbf{z}]\widetilde{\mathbf{z}}\rangle| \lesssim \delta\|\mathbf{w}\|_{L^2_{-\frac{a}{10}}} \sum_{\mathbf{m}\in\mathbf{R}_{\min}} |\mathbf{z}^{\mathbf{m}}|.$$

Notice that the time integral of the last formula is of the form $o_\varepsilon(1)\epsilon^2$.
Now we focus on the term in the right in line (9.5). Using the identity $D_{\mathbf{z}}\mathbf{z}^{\mathbf{m}}(\mathrm{i}\boldsymbol{\lambda}\mathbf{z})$ $= \mathbf{i}\mathbf{m}\cdot\boldsymbol{\lambda}\,\mathbf{z}^{\mathbf{m}}$, this term equals the sum

$$-\sum_{\mathbf{m}\in\mathbf{R}_{\min}} \boldsymbol{\lambda}\cdot\mathbf{m}\,\langle \mathbf{J}P_c\boldsymbol{\eta}, \mathrm{i}\mathbf{z}^{\mathbf{m}}\mathcal{R}_{\mathbf{m}}\rangle \tag{9.7}$$

$$-\sum_{\mathbf{m}\in\mathbf{R}_{\min}} \boldsymbol{\lambda}\cdot\mathbf{m}\,\langle \mathbf{J}P_d(R[\mathbf{z}]-1)P_c\boldsymbol{\eta}, \mathrm{i}\mathbf{z}^{\mathbf{m}}\mathcal{R}_{\mathbf{m}}\rangle, \tag{9.8}$$

where, by Lemma 2.4,

$$|(9.8)| \lesssim \delta\|\mathbf{w}\|_{\widetilde{\boldsymbol{\Sigma}}} \sum_{\mathbf{m}\in\mathbf{R}_{\min}} |\mathbf{z}^{\mathbf{m}}|,$$

so that its time integral is of the form $o_\varepsilon(1)\epsilon^2$.

So now let us focus on the term in line (9.7). It equals the sum

$$-\sum_{\mathbf{m}\in\mathbf{R}_{\min}} \boldsymbol{\lambda}\cdot\mathbf{m}\,\langle \mathbf{J}P_c\chi_{B^2}\boldsymbol{\eta}, \mathrm{i}\mathbf{z}^{\mathbf{m}}\mathcal{R}_{\mathbf{m}}\rangle \tag{9.9}$$

$$-\sum_{\mathbf{m}\in\mathbf{R}_{\min}} \boldsymbol{\lambda}\cdot\mathbf{m}\,\langle \mathbf{J}P_c(1-\chi_{B^2})\boldsymbol{\eta}, \mathrm{i}\mathbf{z}^{\mathbf{m}}\mathcal{R}_{\mathbf{m}}\rangle, \tag{9.10}$$

where the terms in line (9.10) can be bounded as follows,

$$\sum_{\mathbf{m}\in\mathbf{R}_{\min}} |\boldsymbol{\lambda}\cdot\mathbf{m}\,\langle \mathbf{J}P_c(1-\chi_{B^2})\boldsymbol{\eta}, \mathrm{i}\mathbf{z}^{\mathbf{m}}\mathcal{R}_{\mathbf{m}}\rangle| \lesssim B^{-1}\|\mathbf{w}\|_{L^2_{-\frac{a}{10}}} \sum_{\mathbf{m}\in\mathbf{R}_{\min}} |\mathbf{z}^{\mathbf{m}}|,$$

and so again the time integral is of the form $o_\varepsilon(1)\epsilon^2$.

Now let us focus on (9.9). By Lemma 5.5, we have

$$\sum_{\mathbf{m}\in\mathbf{R}_{\min}} \boldsymbol{\lambda}\cdot\mathbf{m}\,\langle \mathbf{J}P_c\chi_{B^2}\boldsymbol{\eta}, \mathrm{i}\mathbf{z}^{\mathbf{m}}\mathcal{R}_{\mathbf{m}}\rangle$$
$$= \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \boldsymbol{\lambda}\cdot\mathbf{m}\left\langle \mathbf{J}\prod_{j=1}^{\widetilde{N}} R_{L_1}(\widetilde{\lambda}_j^2)P_c\mathcal{A}\,\langle \mathrm{i}\varepsilon\partial_x\rangle^{\widetilde{N}}\,\mathbf{v}, \mathrm{i}\mathbf{z}^{\mathbf{m}}\mathcal{R}_{\mathbf{m}}\right\rangle$$

We substitute $\mathbf{v} = \mathbf{g} - Z(\mathbf{z})$ using (8.1) and (8.3). Then the above term becomes

$$\sum_{\mathbf{m}\in\mathbf{R}_{\min}} \boldsymbol{\lambda}\cdot\mathbf{m}|\mathbf{z}^{\mathbf{m}}|^2\left\langle \mathbf{J}\prod_{j=1}^{\widetilde{N}} R_{L_1}(\widetilde{\lambda}_j^2)P_c\mathcal{A}\,\langle \mathrm{i}\varepsilon\partial_x\rangle^{\widetilde{N}}\, R^+_{\mathrm{i}\mathbf{L}_D}(\boldsymbol{\lambda}\cdot\mathbf{m})\mathrm{i}\widetilde{\mathcal{R}}_{\mathbf{m}}, \mathrm{i}\mathcal{R}_{\mathbf{m}}\right\rangle \tag{9.11}$$

$$+\sum_{\substack{\mathbf{m},\mathbf{n}\in\mathbf{R}_{\min}\\ \mathbf{m}\neq\mathbf{n}}} \boldsymbol{\lambda}\cdot\mathbf{m}\left\langle \mathbf{z}^{\mathbf{n}}\mathbf{J}\prod_{j=1}^{\widetilde{N}} R_{L_1}(\widetilde{\lambda}_j^2)P_c\mathcal{A}\,\langle \mathrm{i}\varepsilon\partial_x\rangle^{\widetilde{N}}\, R^+_{\mathrm{i}\mathbf{L}_D}(\boldsymbol{\lambda}\cdot\mathbf{n})\mathrm{i}\widetilde{\mathcal{R}}_{\mathbf{n}}, \mathrm{i}\mathbf{z}^{\mathbf{m}}\mathcal{R}_{\mathbf{m}}\right\rangle \tag{9.12}$$

$$+\sum_{\mathbf{m}\in\mathbf{R}_{\min}}\boldsymbol{\lambda}\cdot\mathbf{m}\left\langle\mathbf{J}\prod_{j=1}^{\widetilde{N}}R_{L_1}(\widetilde{\lambda}_j^2)P_c\mathcal{A}\left\langle\mathrm{i}\varepsilon\partial_x\right\rangle^{\widetilde{N}}\mathbf{g},\mathrm{i}\mathbf{z}^{\mathbf{m}}\mathcal{R}_{\mathbf{m}}\right\rangle. \tag{9.13}$$

The main term is the one in line (9.11) which we leave aside for a moment. We have

$$\begin{aligned}|(9.13)| &\lesssim \sum_{\mathbf{m}\in\mathbf{R}_{\min}}|\mathbf{z}^{\mathbf{m}}|\|\mathbf{g}\|_{L^{2,-S}}\|\left\langle\mathrm{i}\varepsilon\partial_x\right\rangle^{\widetilde{N}}\mathcal{A}^*P_c\prod_{j=1}^{\widetilde{N}}R_{L_1}(\widetilde{\lambda}_j^2)\mathcal{R}_{\mathbf{m}}\|_{L^{2,S}}\\ &\lesssim \|\mathbf{g}\|_{L^{2,-S}}\sum_{\mathbf{m}\in\mathbf{R}_{\min}}|\mathbf{z}^{\mathbf{m}}|,\end{aligned}$$

so that, using Proposition 8.1 and the continuation hypothesis (3.5), we have

$$\|(9.13)\|_{L^1_t}\lesssim\|\mathbf{g}\|_{L^2L^{2,-S}}\sum_{\mathbf{m}\in\mathbf{R}_{\min}}\|\mathbf{z}^{\mathbf{m}}\|_{L^2}\le o_\epsilon(1)\epsilon^2. \tag{9.14}$$

The generic bracket in line (9.12) is of the form

$$\begin{aligned}\left\langle\mathbf{z}^{\mathbf{n}}\overline{\mathbf{z}^{\mathbf{m}}},A\right\rangle &= \frac{1}{\boldsymbol{\lambda}\cdot(\mathbf{n}-\mathbf{m})}\left\langle-\mathrm{i}\boldsymbol{\lambda}\cdot(\mathbf{n}-\mathbf{m})\mathbf{z}^{\mathbf{n}}\overline{\mathbf{z}^{\mathbf{m}}},-\mathrm{i}A\right\rangle\\ &= \frac{1}{\boldsymbol{\lambda}\cdot(\mathbf{n}-\mathbf{m})}\frac{d}{dt}\left\langle\mathbf{z}^{\mathbf{n}}\overline{\mathbf{z}^{\mathbf{m}}},-\mathrm{i}A\right\rangle-\frac{1}{\boldsymbol{\lambda}\cdot(\mathbf{n}-\mathbf{m})}\\ &\quad\times\left\langle D_{\mathbf{z}}(\mathbf{z}^{\mathbf{n}}\overline{\mathbf{z}^{\mathbf{m}}})\left(\dot{\mathbf{z}}+\mathrm{i}\boldsymbol{\lambda}\mathbf{z}\right),-\mathrm{i}A\right\rangle,\end{aligned}$$

where, for $B^*=\overline{{}^tB}$, A is defined as

$$A=(R^+_{\mathrm{i}\mathbf{L}_D}(\boldsymbol{\lambda}\cdot\mathbf{n})\widetilde{\mathcal{R}}_{\mathbf{n}})^*\left\langle\mathrm{i}\varepsilon\partial_x\right\rangle^{\widetilde{N}}\mathcal{A}^*\mathbf{J}\prod_{j=1}^{\widetilde{N}}R_{L_1}(\widetilde{\lambda}_j^2)P_c\mathcal{R}_{\mathbf{m}}.$$

So we have

$$\left|\int_0^t\left\langle\mathbf{z}^{\mathbf{n}}\overline{\mathbf{z}^{\mathbf{m}}},A\right\rangle dt'\right|\lesssim\left|\left\langle\mathbf{z}^{\mathbf{n}}\overline{\mathbf{z}^{\mathbf{m}}},-\mathrm{i}A\right\rangle\right]_0^t\right|+\|D_{\mathbf{z}}(\mathbf{z}^{\mathbf{n}}\overline{\mathbf{z}^{\mathbf{m}}})\left(\dot{\mathbf{z}}+\mathrm{i}\boldsymbol{\lambda}\mathbf{z}\right)\|_{L^1(0,t)}\|A\|_{L^1_x}.$$

We have $\|A\|_{L^1_x}\lesssim 1$ uniformly in $\varepsilon\in(0,1]$. So the first term on the right is $O(\delta^2)$. We bound the second term

$$\begin{aligned}\|D_{\mathbf{z}}(\mathbf{z}^{\mathbf{n}}\overline{\mathbf{z}^{\mathbf{m}}})\left(\dot{\mathbf{z}}+\mathrm{i}\boldsymbol{\lambda}\mathbf{z}\right)\|_{L^1}&\le\|D_{\mathbf{z}}(\mathbf{z}^{\mathbf{n}}\overline{\mathbf{z}^{\mathbf{m}}})\left(\dot{\mathbf{z}}-\widetilde{\mathbf{z}}\right)\|_{L^1}+\|D_{\mathbf{z}}(\mathbf{z}^{\mathbf{n}}\overline{\mathbf{z}^{\mathbf{m}}})\left(\widetilde{\mathbf{z}}+\mathrm{i}\boldsymbol{\lambda}\mathbf{z}\right)\|_{L^1}\\ &\lesssim\|D_{\mathbf{z}}(\mathbf{z}^{\mathbf{n}}\overline{\mathbf{z}^{\mathbf{m}}})\|_{L^2}\|\dot{\mathbf{z}}-\widetilde{\mathbf{z}}\|_{L^2}+\|\mathbf{z}\|_{L^\infty}\|\mathbf{z}^{\mathbf{n}}\|_{L^2}\|\mathbf{z}^{\mathbf{m}}\|_{L^2}=o_\varepsilon(1)\epsilon^2,\end{aligned}$$

and so we conclude

$$\left|\int_0^t(9.12)dt'\right|=o_\varepsilon(1)\epsilon^2. \tag{9.15}$$

Now we focus on line (9.11), which represents the main term of formula (9.11)–(9.13). Using $\widetilde{\mathcal{R}}_{\mathbf{m}}=\mathcal{T}\chi_{B^2}P_c\mathcal{R}_{\mathbf{m}}$, the bracket in line (9.11) can be rewritten

$$\left\langle \mathbf{J} \prod_{j=1}^{\widetilde{N}} R_{L_1}(\widetilde{\lambda}_j^2) P_c \mathcal{A} R^+_{\mathrm{i}\mathbf{L}_D}(\boldsymbol{\lambda}\cdot\mathbf{m}) \mathcal{A}^* \chi_{B^2} \mathcal{R}_{\mathbf{m}}, \mathcal{R}_{\mathbf{m}} \right\rangle \qquad (a_{\mathbf{m}})$$

$$+ \left\langle \mathbf{J} \prod_{j=1}^{\widetilde{N}} R_{L_1}(\widetilde{\lambda}_j^2) P_c \mathcal{A} \langle \mathrm{i}\varepsilon\partial_x\rangle^{\widetilde{N}} [R^+_{\mathrm{i}\mathbf{L}_D}(\boldsymbol{\lambda}\cdot\mathbf{m}), \langle \mathrm{i}\varepsilon\partial_x\rangle^{-\widetilde{N}}] \mathcal{A}^* \chi_{B^2} \mathcal{R}_{\mathbf{m}}, \mathcal{R}_{\mathbf{m}} \right\rangle \tag{9.16}$$

where we will show now that the quantity in $(a_{\mathbf{m}})$ is the form $o(\varepsilon)$. This will imply that

$$\left\| \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \boldsymbol{\lambda}\cdot\mathbf{m} |\mathbf{z}^{\mathbf{m}}|^2 (a_{\mathbf{m}}) \right\|_{L^1(0,t)} = o(\varepsilon)\epsilon^2. \tag{9.17}$$

For E_1 the matrix in (4.13), the quantity in $(a_{\mathbf{m}})$ can be bounded by the product $\mathfrak{A}\cdot\mathfrak{B}$, where

$$\mathfrak{A} = \| \langle \mathrm{i}\varepsilon\partial_x\rangle^{\widetilde{N}} \mathcal{A}^* \prod_{j=1}^{\widetilde{N}} R_{L_1}(\widetilde{\lambda}_j^2) P_c \mathcal{R}_{\mathbf{m}} \|_{L^{2,\ell}} \text{ and}$$

$$\mathfrak{B} = \| R^+_{\mathrm{i}\mathbf{L}_D}(\boldsymbol{\lambda}\cdot\mathbf{m}) E_1 \left[V_D, \langle \mathrm{i}\varepsilon\partial_x\rangle^{-\widetilde{N}} \right] R^+_{\mathrm{i}\mathbf{L}_D}(\boldsymbol{\lambda}\cdot\mathbf{m}) \mathcal{A}^* \chi_{B^2} \mathcal{R}_{\mathbf{m}} \|_{L^{2,-\ell}},$$

for $\ell \geq 2$. We have

$$\begin{aligned} \mathfrak{B} \leq & \| R^+_{\mathrm{i}\mathbf{L}_D}(\boldsymbol{\lambda}\cdot\mathbf{m}) \|^2_{\boldsymbol{\mathcal{H}}^{1,\ell}\to L^{2,-\ell}} \| \langle \mathrm{i}\varepsilon\partial_x\rangle^{-\widetilde{N}} \left[V_D, \langle \mathrm{i}\varepsilon\partial_x\rangle^{\widetilde{N}} \right] \|_{\boldsymbol{\mathcal{H}}^{1,-\ell}\to\boldsymbol{\mathcal{H}}^{1,\ell}} \\ & \times \| \langle \mathrm{i}\varepsilon\partial_x\rangle^{-\widetilde{N}} \|_{\boldsymbol{\mathcal{H}}^{1,-\ell}\to\boldsymbol{\mathcal{H}}^{1,-\ell}} \| \mathcal{A}^* \chi_{B^2} \mathcal{R}_{\mathbf{m}} \|_{\boldsymbol{\mathcal{H}}^{1,\ell}} \lesssim \varepsilon, \end{aligned}$$

where the ε comes from the commutator term in the first line, by a simple adaptation of Lemma 5.2, while the other terms are uniformly bounded, with $\| \langle \mathrm{i}\varepsilon\partial_x\rangle^{-\widetilde{N}} \|_{\boldsymbol{\mathcal{H}}^{1,-\ell}\to\boldsymbol{\mathcal{H}}^{1,-\ell}} \lesssim 1$ uniformly in $\varepsilon\in(0,1]$, by the proof of the bound on (10.23) in [8]. Uniformly in $\varepsilon\in(0,1]$, we have

$$\mathfrak{A} \leq \| \langle \mathrm{i}\varepsilon\partial_x\rangle^{\widetilde{N}} \langle \mathrm{i}\partial_x\rangle^{-2\widetilde{N}} \|_{\boldsymbol{\mathcal{H}}^{1,\ell}\to\boldsymbol{\mathcal{H}}^{1,\ell}} \| \langle \mathrm{i}\partial_x\rangle^{2\widetilde{N}} \mathcal{A}^* \prod_{j=1}^{\widetilde{N}} R_{L_1}(\widetilde{\lambda}_j^2) P_c \mathcal{R}_{\mathbf{m}} \|_{\boldsymbol{\mathcal{H}}^{1,\ell}} \lesssim 1.$$

We have thus proved what was needed to obtain (9.17).
We consider (9.17), the main term. Using

$$R^+_{\mathrm{i}\mathbf{L}_D}(\boldsymbol{\lambda}\cdot\mathbf{m}) \mathcal{A}^* = \mathcal{A}^* R^+_{\mathrm{i}\mathbf{L}_1}(\boldsymbol{\lambda}\cdot\mathbf{m}),$$

which follows from (1.24) and (5.13), using the formula

$$\mathcal{A}\mathcal{A}^* = \prod_{j=1}^{\widetilde{N}} (L_1 - \widetilde{\lambda}_j^2),$$

which is an elementary consequence of the discussion in §1.1.1 and is proved in [8], and finally using the fact that L_1 commutes with P_c, see Remark 2.3, we conclude that line (9.17) equals

$$\left\langle \mathbf{J}P_c R^{+}_{\mathrm{i}\mathbf{L}_1}(\boldsymbol{\lambda}\cdot\mathbf{m})\chi_{B^2}\mathcal{R}_{\mathbf{m}},\mathcal{R}_{\mathbf{m}}\right\rangle = \left\langle \mathbf{J}P_c R^{+}_{\mathrm{i}\mathbf{L}_1}(\boldsymbol{\lambda}\cdot\mathbf{m})\mathcal{R}_{\mathbf{m}},\mathcal{R}_{\mathbf{m}}\right\rangle \tag{9.18}$$

$$- \left\langle \mathbf{J}P_c R^{+}_{\mathrm{i}\mathbf{L}_1}(\boldsymbol{\lambda}\cdot\mathbf{m})(1-\chi_{B^2})\mathcal{R}_{\mathbf{m}},\mathcal{R}_{\mathbf{m}}\right\rangle . \tag{9.19}$$

Is is elementary to show that the last line is $O(B^{-1})$, so that

$$\left\| \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \boldsymbol{\lambda}\cdot\mathbf{m}|\mathbf{z}^{\mathbf{m}}|^2 \left\langle \mathbf{J}P_c R^{+}_{\mathrm{i}\mathbf{L}_1}(\boldsymbol{\lambda}\cdot\mathbf{m})(1-\chi_{B^2})\mathcal{R}_{\mathbf{m}},\mathcal{R}_{\mathbf{m}}\right\rangle \right\|_{L^1(0,t)} = o(B^{-1})\epsilon^2. \tag{9.20}$$

Using an obvious analogue of (5.13), the term in the right hand side in line (9.18) can be rewritten as

$$- \left\langle \mathrm{i}\mathbf{A}_{\mathbf{m}} P.V.(L_1 - r_{\mathbf{m}})^{-1} P_c\mathcal{R}_{\mathbf{m}}, P_c\mathcal{R}_{\mathbf{m}}\right\rangle \tag{9.21}$$

$$+ \pi \left\langle \mathbf{A}_{\mathbf{m}}\delta(L_1 - r_{\mathbf{m}})P_c\mathcal{R}_{\mathbf{m}}, P_c\mathcal{R}_{\mathbf{m}}\right\rangle \ , \tag{9.22}$$

$$\text{where } \mathbf{A}_{\mathbf{m}} = \begin{pmatrix} r_{\mathbf{m}}^2 & \mathrm{i}r_{\mathbf{m}} \\ -\mathrm{i}r_{\mathbf{m}} & 1 \end{pmatrix} \ , \quad r_{\mathbf{m}} = \sqrt{(\boldsymbol{\lambda}\cdot\mathbf{m})^2-\omega^2}.$$

By antisymmetry, line (9.21) is equal to 0. We have

$$\mathbf{B}_{\mathbf{m}}^{-1}\mathbf{A}_{\mathbf{m}}\mathbf{B}_{\mathbf{m}} = \operatorname{diag}\left(0, 1+r_{\mathbf{m}}^2\right) \text{ where } \mathbf{B}_{\mathbf{m}} = \begin{pmatrix} 1 & \mathrm{i}r_{\mathbf{m}} \\ \mathrm{i}r_{\mathbf{m}} & 1 \end{pmatrix}.$$

Noticing that $\mathbf{B}_{\mathbf{m}}^{*} = (1+r_{\mathbf{m}}^2)\mathbf{B}_{\mathbf{m}}^{-1}$, line (9.22) equals

$$\begin{aligned} &\pi \left\langle \mathbf{B}_{\mathbf{m}}^{*}\mathbf{A}_{\mathbf{m}}\mathbf{B}_{\mathbf{m}}\delta(L_1 - r_{\mathbf{m}})\mathbf{B}_{\mathbf{m}}^{-1}P_c\mathcal{R}_{\mathbf{m}}, \mathbf{B}_{\mathbf{m}}^{-1}P_c\mathcal{R}_{\mathbf{m}}\right\rangle \\ &\quad = \pi \left\langle \delta(L_1 - r_{\mathbf{m}}), |-\mathrm{i}r_{\mathbf{m}}(P_c\mathcal{R}_{\mathbf{m}})_1 + (P_c\mathcal{R}_{\mathbf{m}})_2|^2\right\rangle \\ &\quad = \frac{\pi}{2\sqrt{r_{\mathbf{m}}}}\sum_{\pm}\left|\left[-\mathrm{i}r_{\mathbf{m}}(\widehat{P_c\mathcal{R}_{\mathbf{m}}})_1 + (\widehat{P_c\mathcal{R}_{\mathbf{m}}})_2\right](\pm\sqrt{r_{\mathbf{m}}})\right|^2 \geq 0. \end{aligned}$$

where $(P_c\mathcal{R}_{\mathbf{m}})_j$ are the two components of $P_c\mathcal{R}_{\mathbf{m}}$ for $j=1,2$ and we are taking the distorted Fourier transform associated to operator L_1, for which we refer to Weder [46] and Deift and Trubowitz [11]. By Assumption 1.12 there is a fixed $\Gamma > 0$ such that

$$\frac{\pi\boldsymbol{\lambda}\cdot\mathbf{m}}{2\sqrt{r_{\mathbf{m}}}}\sum_{\pm}\left|\left[-\mathrm{i}r_{\mathbf{m}}(\widehat{P_c\mathcal{R}_{\mathbf{m}}})_1 + (\widehat{P_c\mathcal{R}_{\mathbf{m}}})_2\right](\pm\sqrt{r_{\mathbf{m}}})\right|^2 \geq \Gamma > 0 \text{ for all } \mathbf{m}\in\mathbf{R}_{\min}. \tag{9.23}$$

Hence we conclude

$$\sum_{\mathbf{m}\in\mathbf{R}_{\min}} \boldsymbol{\lambda}\cdot\mathbf{m}|\mathbf{z}^{\mathbf{m}}|^2 \left\langle \mathbf{J}P_c R^{+}_{\mathrm{i}\mathbf{L}_1}(\boldsymbol{\lambda}\cdot\mathbf{m})\chi_{B^2}\mathcal{R}_{\mathbf{m}},\mathcal{R}_{\mathbf{m}}\right\rangle \geq \Gamma \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \boldsymbol{\lambda}\cdot\mathbf{m}|\mathbf{z}^{\mathbf{m}}|^2. \tag{9.24}$$

So we have expanded the integral in the left hand side of (9.4) as a sum of terms which are $o_{\varepsilon}(1)\epsilon^2$ plus the integral in $(0,t)$ of the left hand side of (9.24). We conclude

$$\sum_{\mathbf{m}\in\mathbf{R}_{\min}} \|\mathbf{z}^{\mathbf{m}}\|^2_{L^2(I)} = o_{\varepsilon}(1)\epsilon^2,$$

completing the proof of Proposition 3.7.

10. Repulsivity of the ϕ^8 model near the ϕ^4 model

In this section, we study that the following nonlinear potential,

$$W_\epsilon(u) := \frac{1}{4}(1+\epsilon)^2\left(u^2-1\right)^2\left(\epsilon u^2-1\right)^2,\ \epsilon\in[0,1),$$

which appears in the ϕ^8 theory. Notice that when $\epsilon=0$, W_0 is the nonlinear potential of the ϕ^4 theory. It was shown by [26] that for $2-\sqrt{3}\le\epsilon<1$, $L_{2,\epsilon}$ has repulsive potential, in the sense of the definition in [26]. Here, $L_{1,\epsilon}$ is given by $-\partial_x^2+W_\epsilon''(H_\epsilon)$ with H_ϵ the odd kink satisfying $H_\epsilon''=W'(H_\epsilon)$ and $L_{j,\epsilon}$ given by Darboux transformations in Sect. 1.1.1.

Recall that the potential $V_{2,\epsilon}$ of the 1st transformed operator $L_{2,\epsilon}=-\partial_x^2+V_{2,\epsilon}$ is given by $-W_\epsilon''(H_\epsilon)+\frac{(W_\epsilon'(H_\epsilon))^2}{W_\epsilon(H_\epsilon)}$. So, to check the repulsivity of $V_{2,\epsilon}$, one only needs to study the function $-W_\epsilon''(x)+\frac{(W_\epsilon'(x))^2}{W_\epsilon(x)}$ in the domain $x\in[-1,1]$ because H_ϵ is monotone. This was the very nice observation of [26].

On the other hand, when $\epsilon=0$, $L_{1,0}$ has two eigenvalues (0 and $\frac{3}{2}$), so $L_{2,0}$ is not repulsive and the 2nd transformed operator $L_{3,0}=-\partial_x^2+2$ has a flat potential, which lies in the boundary of repulsive potential and is not a repulsive potential in our definition, Assumption 1.6).

Since it seems that as ϵ increases, the number of eigenvalues decreases, it is natural to expect that $V_{3,\epsilon}$ is repulsive for $\epsilon\in(0,\epsilon_*)$ for the first $\epsilon_*>0$ when L_{1,ϵ_*} stops to have two eigenvalues. We will confirm this observation by computing the 1st order expansion of $V_{3,\epsilon}=2+\epsilon\widetilde{V}_3+O(\epsilon^2)$ and by numerical computation. First, $\widetilde{V}_3$ can be computed explicitly.

Proposition 10.1. *We have*

$$\widetilde{V}_3=\frac{6}{5}\operatorname{sech}^2\left(\frac{x}{\sqrt{2}}\right)+\frac{3}{5}\operatorname{sech}^4\left(\frac{x}{\sqrt{2}}\right). \tag{10.1}$$

In particular, we have $x\widetilde{V}_3'(x)<0$ for $x\neq0$.

Next, the result of numerically computation of $V_{3,\epsilon}$ is given by the following graph.

Proof of Proposition 10.1. First, by multiplying H_ϵ' to $H_\epsilon''=W_\epsilon'(H_\epsilon)$ and integrating it, we have

$$H_\epsilon'=\sqrt{2W_\epsilon(H_\epsilon)},$$

which gives a implicit representation of the kink H_ϵ by

$$x=\int_0^{H_\epsilon}\frac{dh}{\sqrt{2W_\epsilon(h)}}.$$

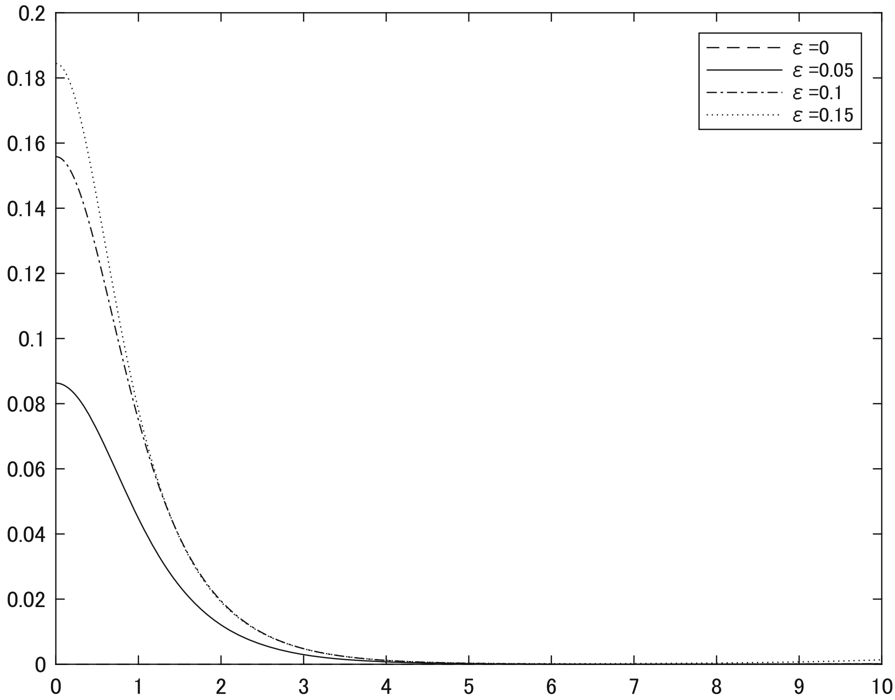

FIGURE 1. Graph of $V_{3,\epsilon} - 2 + 4\epsilon^2 - 2\epsilon^4$ generated by numerically computing H_ϵ and φ_ϵ. The case $\epsilon = 0$ is not visible because it is flat

The above formula holds for any nonlinear potential W. In our case, we can compute the integral in the right hand side and obtain

$$\sqrt{2}(1-\epsilon^2)x = \log(1+H) - \log(1-H) + \sqrt{\epsilon}\log(1-\sqrt{\epsilon}H) - \sqrt{\epsilon}\log(1+\sqrt{\epsilon}H) \tag{10.2}$$

When $\epsilon = 0$, we can solve (10.2) w.r.t. H_0 and obtain the ϕ^4-kink:

$$H_0 = \tanh\left(\frac{x}{\sqrt{2}}\right) \tag{10.3}$$

Differentiating, (10.2) w.r.t. ϵ, we have

$$\partial_\epsilon H_\epsilon = \frac{\left(1-H_\epsilon^2\right)\left(1-\epsilon H_\epsilon^2\right)}{2(1-\epsilon)}\left(-\frac{1}{2\sqrt{\epsilon}}\left(\log(1-\sqrt{\epsilon}H_\epsilon) - \log(1+\sqrt{\epsilon}H_\epsilon)\right) + \frac{H_\epsilon}{1-\epsilon H_\epsilon^2} - 2\sqrt{2}\epsilon x\right),$$

and by $\lim_{h\to 0}\frac{\log(1+hH_\epsilon)}{h} = H_\epsilon \left.\frac{d}{dh}\right|_{h=0}\log(1+h) = H_\epsilon$, we have

$$\left.\partial_\epsilon\right|_{\epsilon=0} H_\epsilon = H_0(1-H_0^2). \tag{10.4}$$

We set $\psi_\epsilon := \psi_0 + \epsilon\widetilde{\psi}_\epsilon$, with $\widetilde{\psi}_\epsilon \perp \psi_0$, to be the eigenfunction of

$$L_\epsilon := -\partial_x^2 + W_\epsilon''(H_\epsilon),$$

associated to the eigenvalue $\lambda_\epsilon = \frac{3}{2} + \epsilon\widetilde{\lambda}_\epsilon$, where

$$\psi_0(x) := \left(\frac{9}{8}\right)^{\frac{1}{4}} \frac{\sinh(\frac{x}{\sqrt{2}})}{\cosh^2(\frac{x}{\sqrt{2}})} \tag{10.5}$$

is the normalized eigenvector of L_0 satisfying $L_0\psi_0 = \frac{3}{2}\psi_0$. □

Remark 10.2. By the stability of eigenvalues, L_ϵ has a unique eigenvalue near $\frac{3}{2}$.

We set $A_{1,\epsilon} = (H_\epsilon')^{-1}\partial_x (H_\epsilon' \cdot)$ and $\varphi_\epsilon = A_{1,\epsilon}^* \psi_\epsilon$. Since the 2nd transformed potential $V_{3,\epsilon}$ is given by

$$V_{3,\epsilon} = \tilde{V}_{2,\epsilon}(H_\epsilon) - 2\left(\frac{\varphi_\epsilon'}{\varphi_\epsilon}\right)', \tag{10.6}$$

with $\tilde{V}_{2,\epsilon}$ is given by $V_{2,\epsilon} = \tilde{V}_{2,\epsilon}(H_\epsilon)$, which can be explicitly written as

$$\begin{aligned}\tilde{V}_{2,\epsilon}(x) &= -W_\epsilon(x)(\log W_\epsilon(x))'' \\ &= (1+3\epsilon+3\epsilon^2+\epsilon^3) + (1-2\epsilon-6\epsilon^2-2\epsilon^3+\epsilon^4)x^2 \\ &\quad - \epsilon(1+\epsilon)^3x^4 + 2\epsilon^2(1+\epsilon)^2x^6,\end{aligned}$$

it suffices to compute $\partial_\epsilon|_{\epsilon=0}\,\varphi_\epsilon = A_{1,0}^*\widetilde{\psi} + \left(\partial_\epsilon|_{\epsilon=0}\,A_{1,\epsilon}^*\right)\psi_0$.

Expanding $L_\epsilon\psi_\epsilon = \lambda_\epsilon\psi_\epsilon$, we have

$$\begin{aligned}\left(L_0 - \frac{3}{2}\right)\widetilde{\psi}_\epsilon &= -\frac{W_\epsilon''(H_\epsilon) - W_0''(H_0)}{\epsilon}\psi_0 + \widetilde{\lambda}_\epsilon\psi_0 \\ &\quad + \epsilon\left(-\frac{W_\epsilon''(H_\epsilon) - W_0''(H_0)}{\epsilon}\widetilde{\psi}_\epsilon + \widetilde{\lambda}_\epsilon\widetilde{\psi}_\epsilon\right).\end{aligned}$$

Thus, taking $\epsilon \to 0$, we have

$$\left(L_0 - \frac{3}{2}\right)\widetilde{\psi}_0 = -\left(W_0'''(H_0)\,\partial_\epsilon|_{\epsilon=0}\,H_\epsilon + \partial_\epsilon|_{\epsilon=0}\,W_\epsilon''(H_0)\right)\psi_0 + \widetilde{\lambda}_0\psi_0. \tag{10.7}$$

Here, $\widetilde{\lambda}_0$ is determined from the orthogonality condition:

$$\widetilde{\lambda}_0 = \left\langle \left(W'''(H_0)\,\partial_\epsilon|_{\epsilon=0}\,H_\epsilon + \partial_\epsilon|_{\epsilon=0}\,W_\epsilon''(H_0)\right)\psi_0, \psi_0\right\rangle.$$

From (10.4), we have

$$W_0'''(H_0)\,\partial_\epsilon|_{\epsilon=0}\,H_\epsilon + \partial_\epsilon|_{\epsilon=0}\,W_\epsilon''(H_0) = -3 + 24H_0^2 - 21H_0^4. \tag{10.8}$$

Therefore, from (10.3) and (10.5),

$$\widetilde{\lambda}_0 = -3 + \frac{3}{2}\int\left(24\frac{\sinh^2 x}{\cosh^2 x} - 21\frac{\sinh^4 x}{\cosh^4 x}\right)\frac{\sinh^2 x}{\cosh^4 x}\,dx = \frac{12}{5}. \tag{10.9}$$

From (10.9) and (10.8), (10.7) can be written as

$$\left(L_0 - \frac{3}{2}\right)\widetilde{\psi}_0 = \left(\frac{27}{5} - 24H_0^2 + 21H_0^4\right)\psi_0. \tag{10.10}$$

Let $A_{2,\epsilon} = \varphi_\epsilon^{-1}\partial_x(\varphi_\epsilon\cdot)$. Applying $A_{1,0}^*$ to (10.10), from $A_{1,0}^*(L_0 - 3/2) = A_{2,0}A_{2,0}^*A_{1,0}^*$we have

$$A_{2,0}A_{2,0}^*A_{1,0}^*\widetilde{\psi}_0 = A_0^*\left(\frac{27}{5} - 24H_0^2 + 21H_0^4\right)\psi_0. \tag{10.11}$$

Solving this, we have

$$A_0^*\widetilde{\psi}_0 = -\left(\frac{9}{8}\right)^{\frac{1}{4}}\sqrt{2}\frac{1}{\cosh(\frac{x}{\sqrt{2}})}\left(\frac{6}{5}\log\left(\cosh\frac{x}{\sqrt{2}}\right) - \frac{27}{10}\frac{1}{\cosh^2(\frac{x}{\sqrt{2}})} + 3\frac{1}{\cosh^4(\frac{x}{\sqrt{2}})}\right). \tag{10.12}$$

This provides all the ingredients for the computation of $\widetilde{V}_3$ by differentiating (10.6). After elementary but somewhat long computation, we obtain (10.1). □

Remark 10.3. In [22], the asymptotic stability in the odd setting for the odd kink of ϕ^8 model near the ϕ^4 model is shown. They show this result by proving ϕ^4 model is asymptotically stable and all models near ϕ^4 model are also asymptotically stable.

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