# Behavioural logics for configuration structures

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A B S T R A C T

We provide a behavioural logic for configuration structures, a model due to van Glabbeek and Plotkin which generalises the families of (finite) configurations of event structures. The logic is a conservative extension of a logic provided by Baldan and Crafa for prime event structures. We show that logical equivalence can be characterized as a form of hereditary history preserving bisimilarity. We compare such a notion of bisimilarity with an equivalence proposed by van Glabbeek in the setting of higher-dimensional automata, showing that, in general, it is finer, while the two notions coincide in the framework of general event structures. Finally, we explore how to restrict the general logic to capture a notion of history preserving bisimilarity.

## 0. Introduction

Event structures [22] are a classical model in the theory of concurrency. They describe the computation of a system in terms of events, which represent the (atomic) execution of computational steps, and dependencies between such events, clarifying when events are enabled. They are true concurrent semantic models, where concurrency is captured as a primitive notion, generally opposed to the interleaving models, where concurrency of actions is reduced to the non-deterministic choice among their possible linearisations.

Various kinds of event structures have been considered in the literature. The most studied event-based model is that of *prime event structures* [11] (PESs, for short), where dependencies between events are expressed in terms of causality, a partial order capturing the fact that an event is enabled after some others (its causes) have been executed, and conflict, capturing the intuition that some events cannot be executed in the same computation (e.g., because they consume shared resources). PESs have been used to provide a true concurrent semantics to a number of formalisms, ranging from Petri nets [11] to graph rewriting systems [2] and process calculi [8,20,21]. A survey on the use of such causal models can be found in [23].

In *general event structures* [22], causality is replaced by an enabling relation between (finite) sets of events and events. Binary conflict is sometimes replaced by a consistency predicate, which explicitly describes the sets of events that can occur in the same computation. *Configuration structures* [19] (CSs, for short) can be seen as a further generalisation: a CS is simply a set of events with a set of configurations which represent the legal computations. This means that the notion of configuration becomes primitive, i.e., it is not induced by relations over events.

When comparing system models, the operational description can be too concrete and behavioural equivalences are normally introduced to equate system specifications that, although syntactically different, denote the same system behaviour.

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In particular, true concurrent equivalences over prime event structures have been widely investigated (see, e.g., [16]), which take into account in different ways the concurrency features of a computation. Hereditary history preserving (hhp-)bisimilarity [6], the finest equivalence in the true concurrent spectrum of [16], has been shown to arise as a canonical behavioural equivalence when considering partially ordered computations [10]. Coarser equivalences like history preserving (hp-)bisimilarity [7,9,15], pomset and step bisimilarity have also been considered. Correspondingly, a number of logics have been studied, which represent the logical counterpart of such equivalences, in the sense that two models are equivalent if and only if they exactly satisfy the same logical formulae. In particular, some event-based logics have been introduced, where formulae include variables which can be bound to events [3,5,12] and formulae predicate about the dependencies on such events. The full logic characterises hhp-equivalence, and suitable fragments can be identified corresponding to the coarser equivalences.

The mentioned corpus of results smoothly extends to *stable event structures* [22]; intuitively, these are event structure models where causality among events can be always represented in terms of partial orders, possibly not global but defined locally to each configuration. Instead, less effort has been devoted to the study of behavioural equivalences and logics on general event structures; in this case, the extension is not obvious. The main problem is that causality, which is pivotal in many notions and results, is not available for general event structures.

The only attempt at notions of hhp- and hp-bisimilarity for non-stable models that we are aware of were provided by [18], in the setting of higher-dimensional automata [13,17]. Such definitions of behavioural equivalences rely on the idea of trace homotopy, and they can be instantiated to configuration structures and general event structures, as these can be seen as special higher-dimensional automata. However, [18] does not propose any logical characterization for such equivalences.

In this paper we focus on CSs and provide a logic to express properties on them. As a by-product, we shall discover that the induced logical equivalence coincides with a form of hhp-bisimilarity that is finer than the hhp-bisimilarity proposed in [18] and coincides with it for CSs closed under bounded union, which corresponds to general event structures.

The starting point is the observation that a simplified version of the logic $\mathcal{L}$ for hhp-bisimilarity in [3,5] can be naturally interpreted over configuration structures. The original logic had formulae that predicate about causal dependencies between events, and thus would not suite to be used on CSs. Here we consider a simplified version $\mathcal{L}_{\mathbf{0}}$, already studied in [4], where one can only quantify events and check their executability. More precisely, the logic has two main operators. The formula $(\mathtt{a}\, z)\varphi$ is satisfied in a state when an $\mathtt{a}$-labelled future event exists, which is bound to $z$ and then $\varphi$ (which can refer to $z$) holds. The formula $\langle z\rangle\, \varphi$ says that the event bound to $z$ is enabled in the current state and, after its execution, $\varphi$ holds. As an example, consider the CS formed by the configurations $\emptyset, \{\mathtt{a}\}, \{\mathtt{b}\}, \{\mathtt{a},\mathtt{c}\}, \{\mathtt{b},\mathtt{c}\}$: here, we have three events (labelled by $\mathtt{a}$, $\mathtt{b}$ and $\mathtt{c}$) such that $\mathtt{a}$ and $\mathtt{b}$ are mutually exclusive and either of them enables $\mathtt{c}$. A formula in $\mathcal{L}_{\mathbf{0}}$ that describes this behaviour is $(\mathtt{a}\, x)(\mathtt{b}\, y)(\mathtt{c}\, z)(\langle x\rangle\,\langle z\rangle\, \mathsf{T} \wedge \langle y\rangle\,\langle z\rangle\, \mathsf{T} \wedge \neg\langle x\rangle\,\langle y\rangle\, \mathsf{T} \wedge \neg\langle y\rangle\,\langle x\rangle\, \mathsf{T})$.

We define a semantics for this logic on configuration structures, conservative on the subclass of prime event structures (Proposition 1). We show that the logic is expressive enough to capture the property of a (finite bounded) CS to be a (stable) event structure (Proposition 2). Furthermore, we show that the logical equivalence induced by $\mathcal{L}_{\mathbf{0}}$ corresponds to a form of hhp-bisimilarity that we refer to as *configuration-based hhp-bisimilarity*: two CSs satisfy the same closed formulae in $\mathcal{L}_{\mathbf{0}}$ if and only if they are configuration-based hhp-bisimilar (Theorem 1).

Configuration based hhp-bisimilarity has a definition similar in spirit to those for prime and stable event structures. Roughly, hhp-bisimilarity requires that events of one system are simulated by events of the other system with the same causal history and with the "same concurrency" properties, a constraint which is often captured by means of a backtracking condition: for any two related computations, the computations obtained by backward performing a pair of related events must be related too. Since both constraints can be captured by suitably mixing a forward and a backward form of bisimilarity, this has no reference to causality and can be adapted to general CSs. We also compare configuration-based hhp-bisimilarity with a notion proposed in [18], referred to as path based hhp-bisimilarity, whose definition is based on paths and obtained instantiating a notion for higher-dimensional automata. It turns out (Proposition 3) that our notion of configuration based hhp-bisimilarity is strictly finer in the setting of CSs, and coincides with the proposal of [18] when restricting to CSs that are closed under bounded union (i.e., when the CSs are general event structures).

Finally, we also investigate a notion of *configuration-based hp-bisimilarity*. We show that, as for the hereditary version, it is in general finer than path-based hp-bisimilarity considered in [18] and coincides with it for CSs closed under bounded union, i.e., for general event structures (Proposition 4). Finally, we provide a logical characterisation of configuration-based hp-bisimilarity, i.e., we isolate a fragment $\mathcal{L}_h$ of $\mathcal{L}_{\mathbf{0}}$ whose logical equivalence is configuration-based hp-bisimilarity (Theorem 2).

The rest of the paper is structured as follows. In Section 1 we recall the definition of configuration structures. In Section 2 we define the logic $\mathcal{L}_{\mathbf{0}}$ on CSs; we show that it conservatively extends the logic of [4] and that it is able to characterize whether a CS is a (stable) event structure. In Section 3 we show that the logic induced by $\mathcal{L}_{\mathbf{0}}$ can be operationally characterized by a form of hhp-bisimilarity that is finer than the hhp-bisimilarity proposed by [18] for CSs (and coincides with it in the setting of general event structures). In Section 4 we start from the hp-bisimilarity from [18], devise an alternative definition (that coincides with it in the setting of general event structures) and provide a logical characterization of such an equivalence, by isolating a proper sublogic of $\mathcal{L}_{\mathbf{0}}$. In Section 5 we draw some conclusions and directions for future work. To streamline reading, a few proofs are relegated to the Appendix.

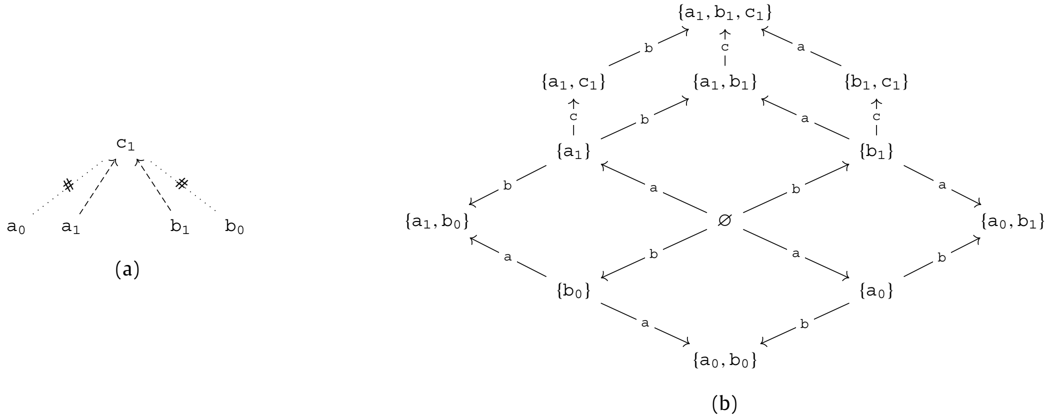



**Fig. 1.** (a): The configuration structure $\mathcal{C}_1$ and (b) its transition system.

## 1. Configuration structures

Configuration structures, introduced in [19], generalise the families of (finite) configurations of event structures.

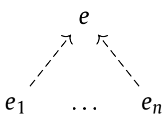
**Definition 1** *(Configuration structures).* A (labelled) configuration structure (over an alphabet $\mathcal{A}$) is a pair $\mathcal{C} = (C, l)$ where $C$ is a family of finite sets (the configurations) and $l : E_{\mathcal{C}} \to \mathcal{A}$ is a labelling function, where the set of events is defined as $E_{\mathcal{C}} = \bigcup_{X \in C} X$.

Intuitively, the events in $E_{\mathcal{C}}$ represent possible computational steps, while the elements in $C$, the configurations, represent the legal computations. A configuration can evolve into another by executing an event; this is represented by the transition relation defined below.

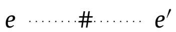
**Definition 2** *(Transition system).* Let $\mathcal{C} = (C, l)$ be a configuration structure. For $X, X' \in C$, we write $X \xrightarrow{\mathtt{a}} X'$ when $X' = X \uplus \{e\}$ and $l(e) = \mathtt{a}$. We simply write $X \to X'$ when we are not interested in the label.

A configuration structure $\mathcal{C} = (C, l)$ is called *rooted* when $\varnothing \in C$ and *connected* if $\varnothing \to^* X$, for all $X \in C$. In this paper, we shall only consider rooted and connected CSs. $\mathcal{C}$ is *closed under bounded union* if, when $X, X', Y \in C$ and $X, X' \subseteq Y$, then $X \cup X' \in C$. It is *stable* if additionally it is *closed under bounded intersection*, i.e., if, when $X, X', Y \in C$ and $X, X' \subseteq Y$, then $X \cap X' \in C$. It can be seen that the family of finite configurations of the general event structures in [22] are exactly the configuration structures which are rooted, connected and closed under bounded union. If in addition they are stable, they correspond to stable event structures.

In this paper, we will use some graphical conventions to represent CSs. First, we write




to denote that in every configuration where $e$ occurs, at least one between $e_1, \ldots, e_n$ must occur as well. Hence this is a form of "disjunctive causality". In addition, the notation




means that, in every configuration where $e$ occurs, $e'$ cannot occur, and vice versa. In the terminology of event structures, this represents a *conflict*. In order to simplify the pictures, in what follows we shall assume that events with the same label are always in conflict. Finally, events will be named by their label, possibly with subscripts.

**Example 1.** Consider the CS $\mathcal{C}_1$ arising from the diagram in Fig. 1a. The configurations are $\varnothing$, $\{\mathtt{a}_0\}$, $\{\mathtt{b}_0\}$, $\{\mathtt{a}_1\}$, $\{\mathtt{b}_1\}$, $\{\mathtt{a}_0, \mathtt{b}_0\}$, $\{\mathtt{a}_0, \mathtt{b}_1\}$, $\{\mathtt{a}_1, \mathtt{b}_0\}$, $\{\mathtt{a}_1, \mathtt{b}_1\}$, $\{\mathtt{a}_1, \mathtt{c}_1\}$, $\{\mathtt{b}_1, \mathtt{c}_1\}$, $\{\mathtt{a}_1, \mathtt{b}_1, \mathtt{c}_1\}$. According to our conventions, the labelling is $l(\mathtt{a}_0) = l(\mathtt{a}_1) = \mathtt{a}$, $l(\mathtt{b}_0) = l(\mathtt{b}_1) = \mathtt{b}$, and $l(\mathtt{c}_1) = \mathtt{c}$; furthermore, events with the same label are assumed to be in conflict, hence $\{\mathtt{a}_0, \mathtt{a}_1\}$ and $\{\mathtt{b}_0, \mathtt{b}_1\}$ are not configurations. The transition system associated with $\mathcal{C}_1$, as introduced in Definition 2, is depicted in Fig. 1b.

Notice that $\mathcal{C}_1$ is rooted, connected and closed under bounded union, hence it represents the set of configurations of some event structure. Instead, it is not stable, i.e., it is not closed under bounded intersection: the configurations $\{\mathtt{a}_1, \mathtt{c}_1\}$ and $\{\mathtt{b}_1, \mathtt{c}_1\}$ are bounded by $\{\mathtt{a}_1, \mathtt{b}_1, \mathtt{c}_1\}$ but $\{\mathtt{a}_1, \mathtt{c}_1\} \cap \{\mathtt{b}_1, \mathtt{c}_1\} = \{\mathtt{c}_1\}$ is not a configuration.

## 2. A logic for configuration structures

We shall now present a logic $\mathcal{L}_\mathbf{0}$ for expressing properties of CSs. Formulae in $\mathcal{L}_\mathbf{0}$ predicate over existence and executability of events in computations. It is a small core of the logic $\mathcal{L}$ in [3,5], which was intended for prime event structures, where there is a notion of causality and the operators explicitly refer to the dependencies between events. This core has been already studied in [4], where it is shown to characterise hhp-bisimilarity on prime event structures. After presenting its syntax and semantics, we shall prove that $\mathcal{L}_\mathbf{0}$ conservatively extends $\mathcal{L}$; moreover, we shall show that it is expressive enough for expressing properties of CSs, notably stability.

### *2.1. Syntax and semantics*

Let $\mathcal{V}$ be a countable set of variables; notationally, we use letters $x, y, z\ldots$ to range over $\mathcal{V}$, whereas letters $\mathtt{a}, \mathtt{b}, \mathtt{c}, \ldots$ are used to range over the set of actions $\mathcal{A}$.

**Definition 3** *(Syntax).* The logic $\mathcal{L}_\mathbf{0}$ is defined by the following syntax:

$$\varphi \ ::= \ \mathsf{T} \ | \ \varphi \wedge \varphi \ | \ \neg\varphi \ | \ (\mathtt{a}\, z)\, \varphi \ | \ \langle z \rangle\, \varphi$$

T, $\wedge$ and $\neg$ are standard; disjunction $\varphi \vee \psi$ is defined, as usual, by duality as the formula $\neg(\neg\varphi \wedge \neg\psi)$. The logic has two main operators, whose meaning will be formalized in Definition 6, when presenting the semantics of the logic. Intuitively, the formula $(\mathtt{a}\, z)\varphi$ is satisfied in a state when an $\mathtt{a}$-labelled future event exists such that, if such event is bound to $z$, then $\varphi$ (which normally refers to $z$) holds. A formula $(\mathtt{a}\, z)\varphi$ can be seen as a special case of the formula $(\boldsymbol{x}, \overline{\boldsymbol{y}} < \mathtt{a}\, z)\varphi$ of the logic in [3,5], stating that $z$ is causally dependent from (the events bound to) variables $\boldsymbol{x}$ and is not causally dependent from (the events bound to) variables $\boldsymbol{y}$. Since in CSs causality is not a primitive notion, in $\mathcal{L}_\mathbf{0}$ we use the less informative existential operator $(\mathtt{a}\, z)$ of [3,5]. By contrast, the formula $\langle z \rangle\, \varphi$ has the same shape as in [3,5] and it says that the event bound to $z$ is enabled in the current state and, after its execution, $\varphi$ holds. The notation for this operator is reminiscent of the diamond modality of Hennessy-Milner logic for CCS, but note that it does not have the same existential flavor: it just checks whether one specific event, viz. the one bound to variable $z$, is executable in the current configuration. The standard diamond modality rather corresponds to a combined formula $(\mathtt{a}\, z)\langle z \rangle\, \varphi$ asking for the existence and executability of an event labelled by $\mathtt{a}$ (see [5] for the formal correspondence).

The operator $(\mathtt{a}\, z)$ acts as a binder for the variable $z$. Accordingly, the free variables of a formula $\varphi$ are defined as follows:

$$fv((\mathtt{a}\, z)\varphi) = fv(\varphi) \setminus \{z\} \qquad fv(\langle z \rangle\, \varphi) = fv(\varphi) \cup \{z\}$$
$$fv(\mathsf{T}) = \emptyset \quad fv(\neg\varphi) = fv(\varphi) \qquad fv(\varphi_1 \wedge \varphi_2) = fv(\varphi_1) \cup fv(\varphi_2)$$

Formulae are considered up to $\alpha$-conversion of bound variables; so we can assume that formulae do not bind the same variable multiple times.

We define a derived operator which requires the execution of a set of previously observed (quantified) events. Other than being generally useful, this also turns out to be handy for some forthcoming definitions and proofs. Given a finite set of variables $\boldsymbol{x} \subseteq \mathcal{V}$, we write $\langle \boldsymbol{x} \rangle\, \varphi$ for the formula inductively defined by

- $\langle \emptyset \rangle\, \varphi \triangleq \varphi$; and
- $\langle \boldsymbol{x} \rangle\, \varphi \triangleq \bigvee_{z \in \boldsymbol{x}} \langle z \rangle \langle \boldsymbol{x} \setminus \{z\} \rangle\, \varphi$, when $\boldsymbol{x} \neq \emptyset$.

Intuitively, $\langle \boldsymbol{x} \rangle\, \varphi$ states that, from the current state, the events bound to the variables in $\boldsymbol{x}$ can be executed, in some order, and then $\varphi$ holds.

The logic $\mathcal{L}_\mathbf{0}$ is interpreted over CSs. In particular, the satisfaction of a formula $\varphi$ is defined with respect to triples $(X, F, \eta)$, where: $X \in C$ is a configuration representing the current state of the computation; $F \in C$ represents a reachable future state, in a sense clarified below, from $X$; and $\eta : \mathcal{V} \to E_C$ is a function, called *environment*, that maps the free variables of $\varphi$ to events. It is required that $X \cup \eta(fv(\varphi)) \subseteq F$, i.e., the future $F$ must include the current state of the computation $X$ and also the events bound to variables occurring free in the formula (intuitively, these are the events that have been observed before but possibly not yet executed). Additionally, the future $F$ must be reachable from $X$, using not only forward but also backward transitions, which however must preserve events in $X \cup \eta(fv(\varphi))$.

In order to formalise the above condition, we introduce a family of relations over configurations indexed by sets of events.

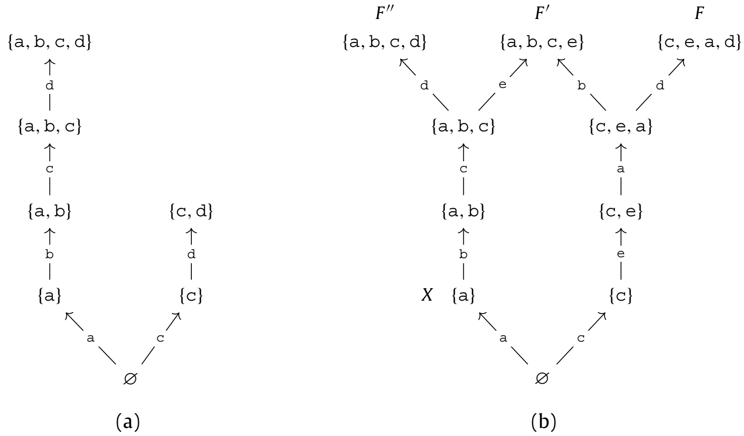



**Fig. 2.** Two configuration structures.

**Definition 4** *(Reachability).* Let $\mathcal{C} = (C, l)$ be a CS. For a fixed set of events $S \subseteq E_{\mathcal{C}}$, we say that two configurations $X, Y \in C$ are in the relation $X \underset{S}{\rightarrow} Y$ if $S \subseteq X \rightarrow \ldots \rightarrow Y$. The *back and forth reachability relation* $\overset{*}{\underset{S}{\leftrightarrow}}$ is the symmetric and transitive closure of $\underset{S}{\rightarrow}$.

Note that relation $\underset{S}{\rightarrow}$ (and, hence, $\overset{*}{\underset{S}{\leftrightarrow}}$) is reflexive, at least when restricted to the configurations including $S$. Intuitively, $X \overset{*}{\underset{S}{\leftrightarrow}} Y$ means that, from configuration $X$, it is possible to reach the configuration $Y$ by adding or removing events, via the forward or backward transitions which become available along the way, but never removing the events in $S$. For example, by looking at Fig. 1b, we have that $\{\mathtt{a}_1, \mathtt{b}_1\} \xrightarrow[\{\mathtt{a}_1\}]{} \{\mathtt{a}_1, \mathtt{b}_1, \mathtt{c}_1\}$, that $\{\mathtt{a}_1, \mathtt{b}_1, \mathtt{c}_1\} \overset{*}{\underset{\{\mathtt{a}_1\}}{\longleftrightarrow}} \{\mathtt{a}_1, \mathtt{c}_1\}$ and that $\{\mathtt{a}_1, \mathtt{b}_1, \mathtt{c}_1\} \overset{*}{\underset{\{\mathtt{a}_1\}}{\longleftrightarrow}} \{\mathtt{a}_1, \mathtt{b}_0\}$: the first relation holds thanks to the forward c-step from the two configurations; the second one holds thanks to the backward b-step from the two configurations; finally, the third one holds because $\{\mathtt{a}_1, \mathtt{b}_1, \mathtt{c}_1\}$ can move backward to $\{\mathtt{a}_1\}$ (via the b- and c-labelled transition, in either order) and $\{\mathtt{a}_1\}$ can move forward to $\{\mathtt{a}_1, \mathtt{b}_0\}$ (via the b-labelled transition). Note that in these transitions we only traverse configurations that contain the event $\mathtt{a}_1$, as prescribed by the subscript of $\underset{\{\mathtt{a}_1\}}{\longleftrightarrow}$.

We can now formalise the notion of legal triple, characterising the triples complying with the requirements hinted at above.

**Definition 5** *(Legal triple).* Given a CS $\mathcal{C}$, let $Env_{\mathcal{C}}$ denote the set of all environments, i.e., of functions $\eta : \mathcal{V} \rightarrow E_{\mathcal{C}}$. Given a formula $\varphi$ in $\mathcal{L}_{\mathbf{0}}$, a triple $(X, F, \eta) \in C \times C \times Env_{\mathcal{C}}$ is *legal* for $\varphi$ if there exists $F' \in C$ such that $X \rightarrow \ldots \rightarrow F' \overset{*}{\underset{X \cup \eta(fv(\varphi))}{\longleftrightarrow}} F$. We write $lt_{\mathcal{C}}(\varphi)$ for the set of legal triples for $\varphi$ in $\mathcal{C}$.

We omit the subscripts and write $Env$ and $lt(\varphi)$ when the CS $\mathcal{C}$ is clear from the context. The notion of a legal triple is conceptually similar to that of a *legal pair* considered for PESs in [4] (a formal definition is provided later in Definition 9), where a legal pair is a pair $(X, \eta)$ such that events in the current configuration and those planned, i.e., bound by $\eta$ to free variables in $\varphi$, are not in conflict with each other. For general CSs, there is no explicit conflict between events. Events can be enabled in different ways and different enablings can be incompatible, in the sense that their union is not included in a configuration. For instance, in the CS of Fig. 2a, event d can be enabled by either $\{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$ or by $\{\mathtt{c}\}$ and the two enablings are incompatible because, roughly speaking, they occur along conflictual computations (visibly on diverging paths). For this reason, when planning events in the future, like d, we have also to plan the future computation it will be in. For instance, in the CS at hand, the triple $(\{\mathtt{c}\}, \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\}, \eta[x \mapsto \mathtt{d}])$ is not legal for the formula $\langle x \rangle \mathsf{T}$, since $\{\mathtt{c}\} \rightarrow^* \{\mathtt{c}\} \cup \eta[x \mapsto \mathtt{d}](x) = \{\mathtt{c}, \mathtt{d}\}$, but then $\{\mathtt{c}, \mathtt{d}\}$ can reach $\{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\}$ only by retracting c and d, i.e., more precisely $\{\mathtt{c}, \mathtt{d}\} \overset{*}{\underset{\{\mathtt{c},\mathtt{d}\}}{\longleftrightarrow}} \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\}$ fails. It can be seen that, actually, all triples $(\{\mathtt{c}\}, \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\}, \eta)$ are not legal independently of $\eta$ and the formula, since $\{\mathtt{c}\}$ can reach $\{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\}$ only by retracting c.

In general, the fact that a triple is legal depends also on the environment and the free variables of the formula. Consider, for instance, the CS in Fig. 2b and let, as in the figure, $X = \{\mathtt{a}\}$, $F = \{\mathtt{c}, \mathtt{e}, \mathtt{a}, \mathtt{d}\}$ and $\varphi = \langle x \rangle \mathsf{T}$. Then $(X, F, \eta[x \mapsto \mathtt{e}])$ is legal for $\varphi$. In fact $X \cup \eta[x \mapsto \mathtt{e}](fv(\varphi)) = \{\mathtt{a}, \mathtt{e}\}$. Moreover $X \rightarrow^* F' = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{e}\}$ and $F' \overset{*}{\underset{\{\mathtt{a},\mathtt{e}\}}{\longleftrightarrow}} F$. Instead, $(X, F, \eta[x \mapsto \mathtt{d}])$ is not legal for the same formula $\varphi$, since $X \to^* F'' = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\}$, i.e., $X$ can reach a state including $X \cup \eta[x \mapsto \mathtt{d}](fv(\varphi)) = \{\mathtt{a}, \mathtt{d}\}$, but then it is not the case that $F'' \xleftrightarrow[\{\mathtt{a},\mathtt{d}\}]{*} F$.

**Definition 6** *(Semantics).* Let $\mathcal{C}$ be a CS. The denotation in $\mathcal{C}$ of a formula $\varphi$, written $\{|\varphi|\}^{\mathcal{C}} (\subseteq C \times C \times Env_{\mathcal{C}})$, is inductively defined as follows:

$$\{|\mathsf{T}|\}^{\mathcal{C}} = lt(\mathsf{T})$$

$$\{|\varphi_1 \wedge \varphi_2|\}^{\mathcal{C}} = \{|\varphi_1|\}^{\mathcal{C}} \cap \{|\varphi_2|\}^{\mathcal{C}} \cap lt(\varphi_1 \wedge \varphi_2)$$

$$\{|\neg\varphi|\}^{\mathcal{C}} = lt(\varphi) \setminus \{|\varphi|\}^{\mathcal{C}}$$

$$\{|(\mathtt{a}\, z)\, \varphi|\}^{\mathcal{C}} = \{(X, F, \eta) \in lt((\mathtt{a}\, z)\, \varphi) \mid \exists F' \in C.\ F \xleftrightarrow[X \cup \eta(fv((\mathtt{a}\, z)\varphi))]{*} F' \wedge \exists e \in F' \setminus X.\, l(e) = \mathtt{a} \wedge (X, F', \eta[z \mapsto e]) \in \{|\varphi|\}^{\mathcal{C}}\}$$

$$\{|\langle z \rangle\ \varphi|\}^{\mathcal{C}} = \{(X, F, \eta) \in lt(\langle z \rangle\ \varphi) \mid\ X \xrightarrow{l(\eta(z))} X \cup \{\eta(z)\} \wedge (X \cup \{\eta(z)\}, F, \eta) \in \{|\varphi|\}^{\mathcal{C}}\}$$

When $(X, F, \eta) \in \{|\varphi|\}^{\mathcal{C}}$, we say that the CS $\mathcal{C}$ *satisfies* the formula $\varphi$ in the configuration $X$ with planned future $F$ and environment $\eta$, and we write $\mathcal{C}, X, F \models_\eta \varphi$. For closed formulae, we write $\mathcal{C} \models \varphi$, when there exists $\eta$ such that $\mathcal{C}, \varnothing, \varnothing \models_\eta \varphi$.

In words, the formula $(\mathtt{a}\, z)\, \varphi$ holds in $(X, F, \eta)$ when the future $F$ of the configuration $X$ can reach (via back and forth transitions) another future $F'$ containing an $\mathtt{a}$-labelled event $e$ which, once bound to $z$, makes the formula $\varphi$ satisfiable. The formula $\langle z \rangle\, \varphi$ states that the event bound to $z$ is currently enabled, hence it can be executed producing a new configuration which satisfies the formula $\varphi$.

An environment $\eta$ is a total function, but it can be shown that the semantics of a formula $\varphi$ only depends on the value of the environment on the free variables $fv(\varphi)$. In particular, for closed formulae the environment is irrelevant. Moreover, it can be easily seen that $\alpha$-equivalent formulae have the same semantics.

As an example, consider the formula $\varphi = (\mathtt{a}\, x)\, (\mathtt{b}\, y)\, \langle x \rangle\, \langle y \rangle\, \neg(\mathtt{c}\, z)\, \mathsf{T}$, requiring the existence of two events, labelled $\mathtt{a}$ and $\mathtt{b}$ respectively, such that, after executing them, there will be no $\mathtt{c}$-labelled event in any possible future. Such formula is satisfied by the CS $\mathcal{C}_1$ in Fig. 1, by binding the variable $x$ to the event $\mathtt{a}_1$ and variable $y$ to $\mathtt{b}_0$. Also the formula $\psi = (\mathtt{a}\, x)\, (\mathtt{b}\, y)\, (\langle x \rangle\, (\mathtt{c}\, z)\, \mathsf{T} \wedge \langle y \rangle\, \neg(\mathtt{c}\, w)\, \mathsf{T})$, stating that there is a $\mathtt{c}$ in the future of $\mathtt{a}$ but not in that of $\mathtt{b}$, is satisfied by $\mathcal{C}_1$, with the same bindings as before. Actually, the existence of a $\mathtt{c}$ in the future of some events can be expressed even before executing them. In fact, we could have used the formulae $\varphi' = (\mathtt{a}\, x)\, (\mathtt{b}\, y)\, \neg(\mathtt{c}\, z)\, \langle x \rangle\, \langle y \rangle\, \mathsf{T}$ and $\psi' = (\mathtt{a}\, x)\, (\mathtt{b}\, y)\, ((\mathtt{c}\, z)\, \langle x \rangle\, \mathsf{T} \wedge \neg(\mathtt{c}\, w)\, \langle y \rangle\, \mathsf{T})$ instead of the previous ones, with the same results on $\mathcal{C}_1$.

A simple property of the semantics is that formulae are equally satisfied (or unsatisfied) by triples containing the same state and different, but related, futures. This can be easily proved by induction on the formula.

**Lemma 1** *(Reschedulable futures). Let $\mathcal{C}$ be a CS and $\varphi$ a formula of $\mathcal{L}_{\mathbf{0}}$. Given an environment $\eta \in Env$ and configurations $X, F_1, F_2 \in C$ such that $(X, F_1, \eta) \in lt(\varphi)$ and $F_1 \xleftrightarrow[X \cup \eta(fv(\varphi))]{*} F_2$, then $\mathcal{C}, X, F_1 \models_\eta \varphi$ iff $\mathcal{C}, X, F_2 \models_\eta \varphi$.*

**Proof.** We start by observing that, for all $(X, F_1, \eta) \in lt(\varphi)$ and $F_1 \xleftrightarrow[X \cup \eta(fv(\varphi))]{*} F_2$, we know that $(X, F_2, \eta) \in lt(\varphi)$, by definition of legal triples. Then, we proceed by induction on the shape of the formula $\varphi$. We discuss only some interesting cases and a single direction, the other being symmetric.

$\varphi = (\mathtt{a}\, z)\, \psi$**:** Assume that $\mathcal{C}, X, F_1 \models_\eta (\mathtt{a}\, z)\psi$. By definition of the semantics, we know that there are $F' \in C$ and $e \in F' \setminus X$ such that $l(e) = \mathtt{a}$, $F_1 \xleftrightarrow[X \cup \eta(fv((\mathtt{a}\, z)\psi))]{*} F'$ and $\mathcal{C}, X, F' \models_{\eta[z \mapsto e]} \psi$. Since by hypothesis $F_1 \xleftrightarrow[X \cup \eta(fv((\mathtt{a}\, z)\psi))]{*} F_2$, by transitivity we also have that $F_2 \xleftrightarrow[X \cup \eta(fv((\mathtt{a}\, z)\psi))]{*} F'$. Then, again by definition of the semantics, we can immediately conclude that $\mathcal{C}, X, F_2 \models_\eta (\mathtt{a}\, z)\psi$, since we already know that $(X, F_2, \eta) \in lt((\mathtt{a}\, z)\psi)$.

$\varphi = \langle z \rangle\ \psi$**:** Assume that $\mathcal{C}, X, F_1 \models_\eta \langle z \rangle\, \psi$. By definition of the semantics, we know that $X \xrightarrow{l(\eta(z))} X \cup \{\eta(z)\} = X'$ and $\mathcal{C}, X', F_1 \models_\eta \psi$. This also means that $(X', F_1, \eta) \in lt(\psi)$. Observing that $X \cup \eta(fv(\langle z \rangle\ \psi)) = X' \cup \eta(fv(\psi))$, we immediately deduce that $F_1 \xleftrightarrow[X' \cup \eta(fv(\psi))]{*} F_2$. Then, by inductive hypothesis, we obtain that $\mathcal{C}, X', F_2 \models_\eta \psi$. Again by definition of the semantics, we can conclude that $\mathcal{C}, X, F_2 \models_\eta \langle z \rangle\, \psi$, since we already know that $(X, F_2, \eta) \in lt(\langle z \rangle\, \psi)$. $\square$

### 2.2. Comparison with the logic in [4]

We now show that the semantics of $\mathcal{L}_\mathbf{0}$ defined in this paper, when restricted to prime event structures, is equivalent to the semantics defined in [4]. To this aim, we first recall the notion of PESs and the original definition of the semantics of $\mathcal{L}_\mathbf{0}$ over PESs.

**Definition 7** *(Prime event structures [11]).* A (labelled) *prime event structure* (PES, for short) over an alphabet $\mathcal{A}$ is a tuple $\mathcal{E} = (E, <, \#, l)$ such that

- $E$ is a set of *events*;
- $< \subseteq E \times E$ is the *causality* relation, i.e. a strict partial order such that, for all $e \in E$, the set $\lceil e \rceil = \{e' : e' < e\}$ is finite;
- $\# \subseteq E \times E$ is the *conflict* relation, i.e. an irreflexive and symmetric relation such that, for all $e, e', e'' \in E$, if $e < e'$ and $e\#e''$, then $e'\#e''$;
- $l : E \to \mathcal{A}$ is the *labelling* function.

Computations in a PES are naturally captured by the notion of configuration.

**Definition 8** *(Consistent set, configurations).* Let $\mathcal{E} = (E, <, \#, l)$ be a PES. A finite set of events $X \subseteq E$ is called *consistent* if, for all $e, e' \in X$, we have that $\neg(e\#e')$. It is called *configuration* if in addition, when $e, e' \in E$, $e' < e$ and $e \in X$, then $e' \in X$. The set of configurations of $\mathcal{E}$ is denoted $\mathcal{C}_\mathcal{E}$.

It can be seen that for a PES $\mathcal{E} = (E, <, \#, l)$ the pair $(\mathcal{C}_\mathcal{E}, l)$ is a CS closed under intersection and bounded union. Conversely, every CS $\mathcal{C} = (C, l)$ closed under intersection and bounded union corresponds to a PES $\mathcal{E}(\mathcal{C}) \triangleq (E_\mathcal{C}, <, \#, l)$ where, for all $e, e' \in E_\mathcal{C}$:

- $e < e'$ when $\forall X \in C$, if $e' \in X$, then $e \in X$;
- $e\#e'$ when $\nexists X \in C$ such that $\{e, e'\} \subseteq X$.

By the definition above and the properties of $\mathcal{C}$, it can be seen that $<$ is a partial order over $E_\mathcal{C}$, and $\#$ is irreflexive, symmetric, and hereditary with respect to $<$. The set of causes of an event $e$, defined by $\lceil e \rceil = \{e' \in E_\mathcal{C} \mid e' \leq e\}$, is the smallest configuration of $\mathcal{C}$ including $e$, which is guaranteed to be unique since $\mathcal{C}$ is closed under intersection.

In [4], given a PES $\mathcal{E} = (E, <, \#, l)$, the satisfaction of a formula $\varphi$ is defined in $\mathcal{L}_\mathbf{0}$ with respect to pairs $(X, \eta)$, where $X \in C_\mathcal{E}$ and $\eta : \mathcal{V} \to E$ maps the free variables of $\varphi$ to events. Similarly to $\mathcal{L}_\mathbf{0}$, the semantics uses a notion of legality for such pairs.

**Definition 9** *(Legal pair).* Let $\mathcal{E}$ be a PES. Given a formula $\varphi$ of $\mathcal{L}_\mathbf{0}$, a pair $(X, \eta) \in C_\mathcal{E} \times Env_\mathcal{E}$ is *legal* for $\varphi$ if $X \cup \eta(fv(\varphi))$ is a consistent set of events. We write $lp_\mathcal{E}(\varphi)$ for the set of legal pairs for $\varphi$.

Then, the semantics is defined as follows.

**Definition 10** *(PES semantics).* Let $\mathcal{E}$ be a PES. The denotation in $\mathcal{E}$ of a formula $\varphi$, written $[\![\varphi]\!]^\mathcal{E} (\subseteq C_\mathcal{E} \times Env_\mathcal{E})$, is inductively defined as follows:

$$[\![\mathsf{T}]\!]^\mathcal{E} = C_\mathcal{E} \times Env_\mathcal{E}$$

$$[\![\varphi_1 \wedge \varphi_2]\!]^\mathcal{E} = [\![\varphi_1]\!]^\mathcal{E} \cap [\![\varphi_2]\!]^\mathcal{E} \cap lp_\mathcal{E}(\varphi_1 \wedge \varphi_2)$$

$$[\![\neg\varphi]\!]^\mathcal{E} = lp_\mathcal{E}(\varphi) \setminus [\![\varphi]\!]^\mathcal{E}$$

$$\begin{aligned}[\![(\mathsf{a}\, z)\, \varphi]\!]^\mathcal{E} = \{(X, \eta) \in lp_\mathcal{E}((\mathsf{a}\, z)\, \varphi) \mid\ & \exists e \in E \setminus X.\, l(e) = \mathsf{a} \\ & \wedge X \cup \{e\} \cup \eta(fv(\varphi) \setminus \{z\}) \text{ consistent} \\ & \wedge (X, \eta[z \mapsto e]) \in [\![\varphi]\!]^\mathcal{E}\}\end{aligned}$$

$$[\![\langle z \rangle\, \varphi]\!]^\mathcal{E} = \{(X, \eta) \mid X \xrightarrow{l(\eta(z))} X \cup \{\eta(z)\} \ \wedge\ (X \cup \{\eta(z)\}, \eta) \in [\![\varphi]\!]^\mathcal{E}\}$$

When $(X, \eta) \in [\![\varphi]\!]^\mathcal{E}$, we say that the PES $\mathcal{E}$ *satisfies* the formula $\varphi$ in the configuration $X$ and environment $\eta$.

We now show that the semantics introduced in Definition 6 for the logic $\mathcal{L}_\mathbf{0}$ and the one reviewed in Definition 10 for the core of the logic $\mathcal{L}$ are actually the same, when the former is interpreted over PESs. The proof is in Appendix A.1. Recall that PESs are the subclass of CSs which are closed under intersection and bounded union (see [19]); therefore, we restrict to such CSs.

**Proposition 1** *(Logics over PESs). Let $\mathcal{C}$ be a CS closed under intersection and bounded union, and $\varphi$ be a formula of $\mathcal{L}_\mathbf{0}$. For all configurations $X, F \in C$ and environment $\eta \in Env_\mathcal{C}$ such that $(X, F, \eta) \in lt(\varphi)$, it holds $\mathcal{C}, X, F \models_\eta \varphi$ iff $(X, \eta) \in [\![\varphi]\!]^{\mathcal{E}(\mathcal{C})}$, where $\mathcal{E}(\mathcal{C})$ is the PES corresponding to $\mathcal{C}$.*

### 2.3. Expressiveness of $\mathcal{L}_\mathbf{0}$

We now show that the property of being closed under bounded union, respectively intersection (and, thus, stability), can be expressed as a logical formula, at least for finite CSs. In order to simplify the writing, let $(\_\boldsymbol{x})\varphi$, with $\boldsymbol{x} = x_1 \ldots x_n$, denote the formula $\bigvee_{\mathtt{a}_1..\mathtt{a}_n \in \mathcal{A}^n} (\mathtt{a}_1\, x_1) \ldots (\mathtt{a}_n\, x_n)\varphi$.

Given a finite CS $\mathcal{C}$, with $|E_\mathcal{C}| = n$, consider the following formula of $\mathcal{L}_\mathbf{0}$:

$$\psi_\mathcal{C} \triangleq \bigwedge_{\substack{\boldsymbol{x},\boldsymbol{y},\boldsymbol{w} \neq \emptyset \\ \boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\boldsymbol{w} \text{ pairwise disjoint} \\ |\boldsymbol{x}|+|\boldsymbol{y}|+|\boldsymbol{z}|+|\boldsymbol{w}| \leq n}} \neg(\_\boldsymbol{x})\,(\_\boldsymbol{y})\,(\_\boldsymbol{z})\,(\_\boldsymbol{w})\Big(\langle \boldsymbol{x} \cup \boldsymbol{z}\rangle \mathsf{T} \wedge \langle \boldsymbol{y} \cup \boldsymbol{z}\rangle \mathsf{T} \wedge \langle \boldsymbol{x} \cup \boldsymbol{y} \cup \boldsymbol{z} \cup \boldsymbol{w}\rangle \mathsf{T} \wedge \neg\langle \boldsymbol{x} \cup \boldsymbol{y} \cup \boldsymbol{z}\rangle \mathsf{T}\Big)$$

Such a formula requires that, for every pair of non-empty sets of variables (represented by $\boldsymbol{x}\cup\boldsymbol{z}$ and $\boldsymbol{y}\cup\boldsymbol{z}$ respectively, hence with intersection $\boldsymbol{z}$) which correspond to configurations of $\mathcal{C}$ (expressed by the first two conjuncts inside the parentheses), if their union $\boldsymbol{x} \cup \boldsymbol{y} \cup \boldsymbol{z}$ is included in some configuration (viz., $\boldsymbol{x} \cup \boldsymbol{y} \cup \boldsymbol{z} \cup \boldsymbol{w}$ in the formula), then $\boldsymbol{x} \cup \boldsymbol{y} \cup \boldsymbol{z}$ itself is also a configuration of $\mathcal{C}$ (expressed via the fourth conjunct). Notice that the satisfaction of the third conjunct, requiring the executability of all the events bound to the variables in $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$ and $\boldsymbol{w}$, implies that the corresponding four sets of events are pairwise disjoint. Thus, no event is required to be executed multiple times in the fourth conjunct, which would then inadvertently hold. Thereby, $\mathcal{C}$ satisfies $\psi_\mathcal{C}$ if and only if $\mathcal{C}$ is closed under bounded union.

Closure under bounded intersection can be characterised in a similar way via the formula

$$\theta_\mathcal{C} \triangleq \bigwedge_{\substack{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z} \neq \emptyset \\ \boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\boldsymbol{w} \text{ pairwise disjoint} \\ |\boldsymbol{x}|+|\boldsymbol{y}|+|\boldsymbol{z}|+|\boldsymbol{w}| \leq n}} \neg(\_\boldsymbol{x})\,(\_\boldsymbol{y})\,(\_\boldsymbol{z})\,(\_\boldsymbol{w})\Big(\langle \boldsymbol{x} \cup \boldsymbol{z}\rangle \mathsf{T} \wedge \langle \boldsymbol{y} \cup \boldsymbol{z}\rangle \mathsf{T} \wedge \langle \boldsymbol{x} \cup \boldsymbol{y} \cup \boldsymbol{z} \cup \boldsymbol{w}\rangle \mathsf{T} \wedge \neg\langle \boldsymbol{z}\rangle \mathsf{T}\Big)$$

In this case, the two configurations will be those bound to the variables in $\boldsymbol{x} \cup \boldsymbol{z}$ and $\boldsymbol{y} \cup \boldsymbol{z}$, respectively, so that $\boldsymbol{z}$ is their intersection. Indeed, for the same reason mentioned above, the events bound to $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z}$ are ensured to be pairwise disjoint. Then, it is enough to check that, for every such configurations, their intersection is itself a configuration, which is required, similarly to before, via the last conjunct inside the parentheses.

The properties of $\psi_\mathcal{C}$ and $\theta_\mathcal{C}$ are collected in the following result, whose proof is immediate:

**Proposition 2.** *A CS $\mathcal{C}$ is an event structure if $\mathcal{C} \models \psi_\mathcal{C}$; furthermore, $\mathcal{C}$ is stable if also $\mathcal{C} \models \theta_\mathcal{C}$.*

**Example 2.** Let us now consider the CS $\mathcal{C}_1$ given in Example 1, which is clearly finite, with $|E_{\mathcal{C}_1}| = 5$. The formula

$$\begin{aligned}
\psi_{\mathcal{C}1} \triangleq\ & \neg(_x)(_y)(_w)(\langle x\rangle \mathsf{T} \wedge \langle y\rangle \mathsf{T} \wedge \langle xyw\rangle \mathsf{T} \wedge \neg\langle xy\rangle \mathsf{T}) \\
& \wedge \neg(_x)(_y)(_z)(_w)(\langle xz\rangle \mathsf{T} \wedge \langle yz\rangle \mathsf{T} \wedge \langle xyzw\rangle \mathsf{T} \wedge \neg\langle xyz\rangle \mathsf{T}) \\
& \wedge \neg(_x)(_y)(_z_1z_2)(_w)(\langle xz_1z_2\rangle \mathsf{T} \wedge \langle yz_1z_2\rangle \mathsf{T} \wedge \langle xyz_1z_2w\rangle \mathsf{T} \wedge \neg\langle xyz_1z_2\rangle \mathsf{T}) \\
& \wedge \neg(_x)(_y)(_w_1w_2)(\langle x\rangle \mathsf{T} \wedge \langle y\rangle \mathsf{T} \wedge \langle xyw_1w_2\rangle \mathsf{T} \wedge \neg\langle xy\rangle \mathsf{T}) \\
& \wedge \neg(_x)(_y)(_z)(_w_1w_2)(\langle xz\rangle \mathsf{T} \wedge \langle yz\rangle \mathsf{T} \wedge \langle xyzw_1w_2\rangle \mathsf{T} \wedge \neg\langle xyz\rangle \mathsf{T}) \\
& \wedge \neg(_x)(_y)(_w_1w_2w_3)(\langle x\rangle \mathsf{T} \wedge \langle y\rangle \mathsf{T} \wedge \langle xyw_1w_2w_3\rangle \mathsf{T} \wedge \neg\langle xy\rangle \mathsf{T}) \\
& \wedge \neg(_x)(_y_1y_2)(_w)(\langle x\rangle \mathsf{T} \wedge \langle y_1y_2\rangle \mathsf{T} \wedge \langle xy_1y_2w\rangle \mathsf{T} \wedge \neg\langle xy_1y_2\rangle \mathsf{T}) \\
& \wedge \neg(_x)(_y_1y_2)(_z)(_w)(\langle xz\rangle \mathsf{T} \wedge \langle y_1y_2z\rangle \mathsf{T} \wedge \langle xy_1y_2zw\rangle \mathsf{T} \wedge \neg\langle xy_1y_2z\rangle \mathsf{T}) \\
& \wedge \neg(_x)(_y_1y_2)(_w_1w_2)(\langle x\rangle \mathsf{T} \wedge \langle y_1y_2\rangle \mathsf{T} \wedge \langle xy_1y_2w_1w_2\rangle \mathsf{T} \wedge \neg\langle xy_1y_2\rangle \mathsf{T}) \\
& \wedge \neg(_x)(_y_1y_2y_3)(_w)(\langle x\rangle \mathsf{T} \wedge \langle y_1y_2y_3\rangle \mathsf{T} \wedge \langle xy_1y_2y_3w\rangle \mathsf{T} \wedge \neg\langle xy_1y_2y_3\rangle \mathsf{T}) \\
& \wedge \neg(_x_1x_2)(_y)(_w)(\langle x_1x_2\rangle \mathsf{T} \wedge \langle y\rangle \mathsf{T} \wedge \langle x_1x_2yw\rangle \mathsf{T} \wedge \neg\langle x_1x_2y\rangle \mathsf{T}) \\
& \wedge \neg(_x_1x_2)(_y)(_z)(_w)(\langle x_1x_2z\rangle \mathsf{T} \wedge \langle yz\rangle \mathsf{T} \wedge \langle x_1x_2yzw\rangle \mathsf{T} \wedge \neg\langle x_1x_2yz\rangle \mathsf{T}) \\
& \wedge \neg(_x_1x_2)(_y)(_w_1w_2)(\langle x_1x_2\rangle \mathsf{T} \wedge \langle y\rangle \mathsf{T} \wedge \langle x_1x_2yw_1w_2\rangle \mathsf{T} \wedge \neg\langle x_1x_2y\rangle \mathsf{T}) \\
& \wedge \neg(_x_1x_2)(_y_1y_2)(_w)(\langle x_1x_2\rangle \mathsf{T} \wedge \langle y_1y_2\rangle \mathsf{T} \wedge \langle x_1x_2y_1y_2w\rangle \mathsf{T} \wedge \neg\langle x_1x_2y_1y_2\rangle \mathsf{T}) \\
& \wedge \neg(_x_1x_2x_3)(_y)(_w)(\langle x_1x_2x_3\rangle \mathsf{T} \wedge \langle y\rangle \mathsf{T} \wedge \langle x_1x_2x_3yw\rangle \mathsf{T} \wedge \neg\langle x_1x_2x_3y\rangle \mathsf{T})
\end{aligned}$$

requiring closure under bounded union, as defined above, is satisfied by $\mathcal{C}_1$. In fact, $\mathcal{C}_1$ is closed under bounded union, since there is no pair of configurations such that their union is included in some other configuration but it is not itself one.

On the other hand, $\mathcal{C}_1$ does not satisfy the formula $\theta_{\mathcal{C}_1}$. In fact, by taking $\boldsymbol{x} = (x)$, $\boldsymbol{y} = (y)$, $\boldsymbol{z} = (z)$ and $\boldsymbol{w} = \epsilon$, we can bind $x$ to the event $\mathtt{a}_1$ and $y$ to $\mathtt{b}_1$, which, paired with $z$ bound to $\mathtt{c}_1$, form the two configurations $\{\mathtt{a}_1, \mathtt{c}_1\}$ and $\{\mathtt{b}_1, \mathtt{c}_1\}$, respectively, that are both included in the configuration $\{\mathtt{a}_1, \mathtt{b}_1, \mathtt{c}_1\}$. However, the intersection $\boldsymbol{z}$ corresponds to the singleton $\{\mathtt{c}_1\}$ which is not a configuration. Hence, the conjunct of $\theta_{\mathcal{C}_1}$

$$\neg(\_x)(\_y)(\_z)(\langle xz\rangle \mathsf{T} \wedge \langle yz\rangle \mathsf{T} \wedge \langle xyz\rangle \mathsf{T} \wedge \neg\langle z\rangle \mathsf{T})$$

turns out to be false, and this fact falsifies $\theta_{\mathcal{C}_1}$; indeed, $\mathcal{C}_1$ is not closed under bounded intersection. This shows also that $\mathcal{C}_1$ is not stable, as already mentioned in Example 1.

## 3. Logics and bisimilarities

We shall now discuss the equivalence induced by $\mathcal{L}_{\mathbf{0}}$, that holds whenever two arbitrary CSs $\mathcal{C}_1$ and $\mathcal{C}_2$ satisfy the same set of $\mathcal{L}_{\mathbf{0}}$-formulae. In particular, we shall prove that such an equivalence coincides with a form of *hereditary history preserving bisimilarity* [6] (hhp-bisimilarity, for short). As usual, this result holds only for *image-finite* CSs, that are those structures $(C, l)$ such that, for every $X \in C$ and $\mathtt{a} \in \mathcal{A}$, the set of configurations reachable from $X$ via $\mathtt{a}$ is finite.

**Remark.** In the rest of the paper, all CSs will be implicitly assumed to be image-finite.

We first introduce a notion of hhp-bisimilarity (called *configuration-based hhp-bisimilarity*, chhpb for short), that strongly resembles the standard notion of hhp-bisimilarity for prime event structures. Then, we prove that chhpb coincides with the equivalence induced by $\mathcal{L}_{\mathbf{0}}$. Finally, we relate our notion of hhp-bisimilarity with one introduced in [18] (that we call *path-based hhp-bisimilarity*, phhpb for short): in particular, the two notions coincide if we work with CSs closed under bounded union (i.e., general event structures); otherwise chhpb is strictly finer than phhpb.

### *3.1. Configuration-based hhp-bisimilarity*

We start by defining forward and backward bisimulations as bisimulations between CSs where we record the correspondence between events that simulate each other. Some ideas are similar to [1], which however worked on stable CSs and relied on the causality relation defined over configurations. Since for general CSs no causality can be defined, several notions have to be changed; in particular, configuration isomorphism is simply a set bijection that respects labelling.

**Definition 11** *(Forward and backward bisimulations on CSs).* Given two CSs $\mathcal{C}_1$ and $\mathcal{C}_2$, let $\mathcal{I}(\mathcal{C}_1, \mathcal{C}_2) = \{(X_1, f, X_2) \mid X_1 \in C_1 \ \wedge\ X_2 \in C_2 \ \wedge\ f : X_1 \to X_2 \text{ isomorphism}\}$ be the set of (label-respecting) isomorphisms between configurations of $\mathcal{C}_1$ and $\mathcal{C}_2$.

A *forward (fw-)bisimulation* is $R \subseteq \mathcal{I}(\mathcal{C}_1, \mathcal{C}_2)$ such that, if $(X_1, f, X_2) \in R$ and $X_1 \xrightarrow{\mathtt{a}} X_1'$, then there exists $X_2'$ such that $X_2 \xrightarrow{\mathtt{a}} X_2'$ and $(X_1', f', X_2') \in R$ with $f = f'|_{X_1}$, and vice versa starting from $X_2$.

A *backward (bw-)bisimulation* is $R \subseteq \mathcal{I}(\mathcal{C}_1, \mathcal{C}_2)$ such that, if $(X_1, f, X_2) \in R$ and $X_1' \xrightarrow{\mathtt{a}} X_1$, then there exists $X_2'$ such that $X_2' \xrightarrow{\mathtt{a}} X_2$ and $(X_1', f', X_2') \in R$ with $f' = f|_{X_1'}$, and vice versa starting from $X_2$.

Similarly to what happens for prime event structures, forward and backward bisimulation can be suitably combined to obtain a form of hhp-bisimilarity.

**Definition 12** *(Configuration-based hhp-bisimulation).* Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be configuration structures. A *configuration-based hereditary history preserving bisimulation* (chhpb) is a relation $R$ that is both a fw- and a bw-bisimulation. We say that $\mathcal{C}_1$ and $\mathcal{C}_2$ are *configuration-based hhp-bisimilar*, and write $\mathcal{C}_1 \sim_{\text{hhb}} \mathcal{C}_2$, if there exists a chhpb $R$ such that $(\varnothing, \varnothing, \varnothing) \in R$.

We can now prove that the logical equivalence induced by $\mathcal{L}_{\mathbf{0}}$ coincides with chhpb.

**Theorem 1** *(Logic for hhp). Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be CSs. Then $\mathcal{C}_1 \sim_{\text{hhb}} \mathcal{C}_2$ iff, for all $\varphi \in \mathcal{L}_{\mathbf{0}}$ closed, $\mathcal{C}_1 \models \varphi \Leftrightarrow \mathcal{C}_2 \models \varphi$.*

**Proof.** ($\Leftarrow$). We first introduce some notation. We fix a surjective environment $\eta : \mathcal{V} \to E_{\mathcal{C}_1}$. Then, given an event $e \in E_{\mathcal{C}_1}$, we write $x_e$ to denote a fixed distinguished variable such that $\eta(x_e) = e$. Similarly, for a configuration $X = \{e_1, \ldots, e_n\}$, we denote by $\boldsymbol{x}_X$ the corresponding set of variables $\{x_{e_1}, \ldots, x_{e_n}\}$.

Assume that, for all $\varphi$ closed in $\mathcal{L}_\mathbf{0}$, it holds $\mathcal{C}_1 \models \varphi$ iff $\mathcal{C}_2 \models \varphi$. Let $R \subseteq \mathcal{I}(\mathcal{C}_1, \mathcal{C}_2)$ be defined as

$$R = \{(X, f, Y) \in \mathcal{I}(\mathcal{C}_1, \mathcal{C}_2) \mid \forall \varphi \in \mathcal{L}_\mathbf{0}.fv(\varphi) \subseteq \boldsymbol{x}_X \Rightarrow (\mathcal{C}_1, \varnothing, X \models_\eta \varphi \text{ iff } \mathcal{C}_2, \varnothing, Y \models_{f\circ\eta} \varphi)\}$$

We show that $R$ is a chhpb between $\mathcal{C}_1$ and $\mathcal{C}_2$, i.e., $R$ is both a fw-bisimulation and a bw-bisimulation such that $(\varnothing, \varnothing, \varnothing) \in R$.

We start by showing that $R$ is a fw-bisimulation, which we do by contradiction. So, suppose that $(X, f, Y) \in R$ and, without loss of generality, that $X \xrightarrow{\mathsf{a}} X'$, but for all transitions $Y \xrightarrow{\mathsf{a}} Y'$, we have $(X', f[e \mapsto e'], Y') \notin R$ where $e \in X' \setminus X$ and $e' \in Y' \setminus Y$. Since $f[e \mapsto e']$ is still an isomorphism, this can only happen because there exists a formula $\varphi$ such that $fv(\varphi) \subseteq \boldsymbol{x}_{X'}$, $\mathcal{C}_1, \varnothing, X' \models_\eta \varphi$ and $\mathcal{C}_2, \varnothing, Y' \not\models_{f[e\mapsto e']\circ\eta} \varphi$ (or vice versa, that is analogous, and so omitted).

Note that there must be at least one such transition $Y \xrightarrow{\mathsf{a}} Y'$, otherwise we would have $\mathcal{C}_1, \varnothing, X \models_\eta (\mathsf{a}\, x_e)\langle \boldsymbol{x}_X \rangle \langle x_e \rangle \mathsf{T}$ and $\mathcal{C}_2, \varnothing, Y \not\models_{f\circ\eta} (\mathsf{a}\, x_e)\langle \boldsymbol{x}_X \rangle \langle x_e \rangle \mathsf{T}$, contradicting the fact that $(X, f, Y) \in R$.

Furthermore, since $\mathcal{C}_1$ and $\mathcal{C}_2$ are image-finite, there are finitely many transitions $Y \xrightarrow{l(e_i)} Y_i = Y \cup \{e_i\}$, indexed by $i \in \{1, \ldots, h\}$, complying with the previous conditions. For each $i \in \{1, \ldots, h\}$, call $f_i$ the corresponding isomorphism defined by $f_i = f[e \mapsto e_i]$. Then, by the assumption above, we know that there is a formula $\psi^i$ such that $fv(\psi^i) \subseteq \boldsymbol{x}_{X'}$, $\mathcal{C}_1, \varnothing, X' \models_\eta \psi^i$ and $\mathcal{C}_2, \varnothing, Y_i \not\models_{f_i\circ\eta} \psi^i$.

Now consider the formula

$$\theta = (\mathsf{a}\, x_e) \left( \langle \boldsymbol{x}_X \rangle \langle x_e \rangle \mathsf{T} \wedge \bigwedge_{i\in\{1,\ldots,h\}} \psi^i \right)$$

By hypothesis, it is easy to see that $\mathcal{C}_1, \varnothing, X \models_\eta \theta$. However, for every $i \in \{1, \ldots, h\}$, we know that $\mathcal{C}_2, \varnothing, Y_i \not\models_{f_i\circ\eta} \psi^i$, and since $f_i \circ \eta = f[e \mapsto e_i] \circ \eta = (f \circ \eta)[x_e \mapsto e_i]$, we have that $\mathcal{C}_2, \varnothing, Y_i \not\models_{(f\circ\eta)[x_e\mapsto e_i]} \psi^i$. Observe that, for all $F \overset{*}{\underset{Y}{\longleftrightarrow}} Y$ such that $e_i \in F \setminus Y$ for some $i \in \{1, \ldots, h\}$, and $\mathcal{C}_2, \varnothing, F \models_{(f\circ\eta)[x_e\mapsto e_i]} \langle \boldsymbol{x}_X \rangle \langle x_e \rangle \mathsf{T}$, we must have that $\mathcal{C}_2, Y_i, F \models_{(f\circ\eta)[x_e\mapsto e_i]} \mathsf{T}$ since $f(\eta(\boldsymbol{x}_X)) \cup \{e_i\} = Y \cup \{e_i\} = Y_i$. By definition of the semantics this requires that $(Y_i, F, (f \circ \eta)[x_e \mapsto e_i]) \in lt(\mathsf{T})$, which in turn requires that $Y_i \overset{*}{\underset{Y_i}{\longleftrightarrow}} F$. Thus, recalling that $fv(\psi^i) \subseteq \boldsymbol{x}_{X'} = \boldsymbol{x}_X \cup \{e\}$, from the fact that $\mathcal{C}_2, \varnothing, Y_i \not\models_{(f\circ\eta)[x_e\mapsto e_i]} \psi^i$, by Lemma 1 we deduce that also $\mathcal{C}_2, \varnothing, F \not\models_{(f\circ\eta)[x_e\mapsto e_i]} \psi^i$. But then, by definition of the semantics, we would have that $\mathcal{C}_2, \varnothing, Y \not\models_{f\circ\eta} \theta$ contradicting the fact that $(X, f, Y) \in R$. Thus, we can conclude that $R$ is a fw-bisimulation. Moreover, observe that $(\varnothing, \varnothing, \varnothing) \in R$, since by hypothesis $\mathcal{C}_1 \models \varphi$ iff $\mathcal{C}_2 \models \varphi$ for all $\varphi$ closed in $\mathcal{L}_\mathbf{0}$.

It remains to show that $R$ is also a bw-bisimulation. Let $(X, f, Y) \in R$. Observe that, for every transition $X' \xrightarrow{l(e)} X$, by definition of the semantics, we have that $\mathcal{C}_1, \varnothing, X \models_\eta \langle \boldsymbol{x}_{X'} \rangle \langle x_e \rangle \mathsf{T}$, since $X' \in \mathcal{C}_1$. Then, by definition of $R$, we must also have that $\mathcal{C}_2, \varnothing, Y \models_{f\circ\eta} \langle \boldsymbol{x}_{X'} \rangle \langle x_e \rangle \mathsf{T}$. It follows that $f(X') = Y' \xrightarrow{l(e)} Y$ and $f|_{X'} : X' \to Y'$ is still an isomorphism. Now, by contradiction, suppose that $(X', f|_{X'}, Y') \notin R$. Then, there must be a formula $\psi$ such that $fv(\psi) \subseteq \boldsymbol{x}_{X'}$, $\mathcal{C}_1, \varnothing, X' \models_\eta \psi$ and $\mathcal{C}_2, \varnothing, Y' \not\models_{f|_{X'}\circ\eta} \psi$. Since $fv(\psi) \subseteq \boldsymbol{x}_{X'}$, $X' \to X$ and $f|_{X'}(\eta(\boldsymbol{x}_{X'})) = f(\eta(\boldsymbol{x}_{X'})) = Y'$ (hence clearly $X' \overset{*}{\underset{\eta(fv(\psi))}{\longleftrightarrow}} X$ and $Y' \overset{*}{\underset{f(\eta(fv(\psi)))}{\longleftrightarrow}} Y$), by Lemma 1 we would have that $\mathcal{C}_1, \varnothing, X \models_\eta \psi$ and $\mathcal{C}_2, \varnothing, Y \not\models_{f\circ\eta} \psi$. But this would contradict the fact that $(X, f, Y) \in R$, since $fv(\psi) \subseteq \boldsymbol{x}_{X'} \subset \boldsymbol{x}_X$. And so we conclude that $R$ is a bw-bisimulation, hence $R$ is a chhpb.

($\Rightarrow$). Assume that we have a chhpb $R$ between $\mathcal{C}_1$ and $\mathcal{C}_2$. We prove that, for all $\varphi$ in $\mathcal{L}_\mathbf{0}$, it holds $\mathcal{C}_1 \models \varphi$ iff $\mathcal{C}_2 \models \varphi$. Actually, we show that, for every configuration $X \in C_1$, triple $(F, f, G) \in R$, formula $\varphi \in \mathcal{L}_\mathbf{0}$ and environment $\eta \in Env_{\mathcal{C}_1}$ such that $(X, F, \eta) \in lt(\varphi)$ (hence $X \subseteq F$), it holds $\mathcal{C}_1, X, F \models_\eta \varphi$ if and only if $\mathcal{C}_2, f(X), G \models_{f\circ\eta} \varphi$. Observing that environments are irrelevant when $fv(\varphi) = \varnothing$, this is enough since by hypothesis $(\varnothing, \varnothing, \varnothing) \in R$ and clearly $(\varnothing, \varnothing, \eta) \in lt(\varphi)$, implying that $\mathcal{C}_1$ and $\mathcal{C}_2$ would satisfy the same closed formulae of $\mathcal{L}_\mathbf{0}$.

First of all, observe that, for all $X \in C_1$ and $(F, f, G) \in R$ such that $(X, F, \eta) \in lt(\varphi)$, by definition of legal triple we know that there exists $F_1 \in C_1$ such that $X \to \ldots \to F_1 \overset{*}{\underset{X\cup\eta(fv(\varphi))}{\longleftrightarrow}} F$. Since $(F, f, G) \in R$, which is both a fw- and bw-bisimulation, there must exist a configuration $F_2 \in C_2$ such that $F_2 \overset{*}{\underset{f(X)\cup f(\eta(fv(\varphi)))}{\longleftrightarrow}} G$ and $(F_1, g, F_2) \in R$, for some $g$ s.t. $g|_{X\cup\eta(fv(\varphi))} = f|_{X\cup\eta(fv(\varphi))}$, and thus also $f(X) \to \ldots \to F_2$; this means that $(f(X), G, f \circ \eta) \in lt(\varphi)$ and $(X, f|_X, f(X)) \in R$, since $g|_X = f|_X$.

Keeping that in mind, we proceed by induction on the shape of the formula $\varphi$. We discuss only some cases and a single direction, the other being symmetric.

**$\varphi = \psi_1 \wedge \psi_2$:** Assume that $\mathcal{C}_1, X, F \models_\eta \psi_1 \wedge \psi_2$. By definition of the semantics, we know that $\mathcal{C}_1, X, F \models_\eta \psi_1$ and $\mathcal{C}_1, X, F \models_\eta \psi_2$. Then, by inductive hypothesis we have that $\mathcal{C}_2, f(X), G \models_{f\circ\eta} \psi_1$ and $\mathcal{C}_2, f(X), G \models_{f\circ\eta} \psi_2$. Since, as observed above, $(f(X), G, f \circ \eta) \in lt(\psi_1 \wedge \psi_2)$, again by definition of the semantics, we conclude that $\mathcal{C}_2, f(X), G \models_{f\circ\eta} \psi_1 \wedge \psi_2$.

$\varphi = (\mathtt{a}\, z)\, \psi$**:** Assume that $\mathcal{C}_1, X, F \models_\eta (\mathtt{a}\, z)\psi$. By definition of the semantics, we know that there are $F_1 \in C_1$ and $e \in F_1 \setminus X$ such that $l(e) = \mathtt{a}$, $F \xleftrightarrow[X \cup \eta(fv((\mathtt{a}\, z)\psi))]{*} F_1$ and $\mathcal{C}_1, X, F_1 \models_{\eta[z \mapsto e]} \psi$ (hence $(X, F_1, \eta[z \mapsto e]) \in lt(\psi)$). Since by hypothesis $(F, f, G) \in R$, we deduce that also $(F_1, g, g(F_1)) \in R$ for some $g$ s.t. $g|_{X \cup \eta(fv((\mathtt{a}\, z)\psi))} = f|_{X \cup \eta(fv((\mathtt{a}\, z)\psi))}$. This also means that $G \xleftrightarrow[f(X) \cup f(\eta(fv((\mathtt{a}\, z)\psi)))]{*} g(F_1)$. Then, by inductive hypothesis, we have that $\mathcal{C}_2, g(X), g(F_1) \models_{g \circ (\eta[z \mapsto e])} \psi$. Since $g \circ (\eta[z \mapsto e]) = (g \circ \eta)[z \mapsto g(e)]$, we can rewrite the previous statement as $\mathcal{C}_2, g(X), g(F_1) \models_{(g \circ \eta)[z \mapsto g(e)]} \psi$. Moreover, since the semantics of $\psi$ depends only on the value of the environment on its free variables, $g|_{X \cup \eta(fv((\mathtt{a}\, z)\psi))} = f|_{X \cup \eta(fv((\mathtt{a}\, z)\psi))}$ and $fv(\psi) \subseteq fv((\mathtt{a}\, z)\psi) \cup \{z\}$, we actually have that $\mathcal{C}_2, f(X), g(F_1) \models_{(f \circ \eta)[z \mapsto g(e)]} \psi$. Recalling that $G \xleftrightarrow[f(X) \cup f(\eta(fv((\mathtt{a}\, z)\psi)))]{*} g(F_1)$, $e \in F_1 \setminus X$ and $g(X) = f(X)$, thus $g(e) \in g(F_1) \setminus f(X)$, and since, as already observed, $(f(X), G, f \circ \eta) \in lt((\mathtt{a}\, z)\psi)$, by definition of the semantics we conclude that $\mathcal{C}_2, f(X), G \models_{f \circ \eta} (\mathtt{a}\, z)\psi$.

$\varphi = \langle z \rangle\, \psi$**:** Assume that $\mathcal{C}_1, X, F \models_\eta \langle z \rangle\, \psi$. By definition of the semantics, we know that $X \xrightarrow{l(\eta(z))} X \cup \{\eta(z)\} = X'$ and $\mathcal{C}_1, X', F \models_\eta \psi$ (hence $(X', F, \eta) \in lt(\psi)$). Then, by inductive hypothesis, we have that $\mathcal{C}_2, f(X'), G \models_{f \circ \eta} \psi$. Moreover, as observed above, $(X', f|_{X'}, f(X')) \in R$. Since $X \xrightarrow{l(\eta(z))} X'$, there must exist a transition $f(X) \xrightarrow{l(\eta(z))} f(X') = f(X) \cup \{f(\eta(z))\}$. Therefore, by definition of the semantics we can conclude that $\mathcal{C}_2, f(X), G \models_{f \circ \eta} \langle z \rangle\, \psi$, since, using again the observation above, we know that $(f(X), G, f \circ \eta) \in lt(\langle z \rangle\, \psi)$. □

### 3.2. Comparing hhp-bisimilarities

A notion of hhp-bisimilarity for CSs was introduced in [18], as an instance of the notion defined in the general context of higher dimensional automata. We now compare that equivalence with chhpb; to this aim, we first recall the definition of [18], that relies on the notions of paths and traces.

**Definition 13** *(Paths and traces).* Let $\mathcal{C}$ be a configuration structure. Given $X \in C$, a *path* starting from $X$ is a sequence of events $\pi = e_1 \dots e_n$ such that $X \xrightarrow{\mathtt{a}_1} X \cup \{e_1\} \xrightarrow{\mathtt{a}_2} \dots \xrightarrow{\mathtt{a}_n} X \cup \{e_1, \dots e_n\}$, where $\mathtt{a}_i = l(e_i)$ for every $i$. The sequence $\mathtt{a}_1 \dots \mathtt{a}_n$ is called the *trace* of $\pi$ and denoted $tr(\pi)$; furthermore, $set(\pi) \triangleq \{e_1, \dots, e_n\}$.

The *adjacency relation* $\overset{k}{\leftrightarrow}$ between paths is inductively defined as follows: if $X \to X \cup \{e_1\} \to X \cup \{e_1, e_2\}$ and $X \to X \cup \{e_2\} \to X \cup \{e_1, e_2\}$, then $e_1 e_2 \overset{0}{\leftrightarrow} e_2 e_1$; if $\pi \overset{k}{\leftrightarrow} \pi'$, then $\pi_1 \pi \pi_2 \overset{|\pi_1|+k}{\longleftrightarrow} \pi_1 \pi' \pi_2$. We write $\pi_1 \leftrightarrow \pi_2$ to intend that $\pi_1 \overset{k}{\leftrightarrow} \pi_2$, for some $k$, and we denote by $\leftrightarrow^*$ the reflexive and transitive closure of $\leftrightarrow$.

The set of paths in $\mathcal{C}$ starting from $\varnothing$ is denoted by $paths(\mathcal{C})$.

For instance, for the CS $\mathcal{C}_1$ of Example 1, two possible paths are $\mathtt{a}_1 \mathtt{c}_1 \mathtt{b}_1$ and $\mathtt{a}_1 \mathtt{b}_1 \mathtt{c}_1$, and the corresponding traces are $\mathtt{acb}$ and $\mathtt{abc}$. Note also that $\mathtt{a}_1 \mathtt{c}_1 \mathtt{b}_1 \overset{1}{\leftrightarrow} \mathtt{a}_1 \mathtt{b}_1 \mathtt{c}_1$.

**Definition 14** *(Path-based hhp-bisimulation).* Given two CSs $\mathcal{C}_1$ and $\mathcal{C}_2$, a *path-based hereditary history preserving bisimulation* (shortened as phhpb) between them is a symmetric relation $R \subseteq paths(\mathcal{C}_1) \times paths(\mathcal{C}_2)$ such that:

1. $(\epsilon, \epsilon) \in R$ (the empty paths are related);
2. if $(\pi_1, \pi_2) \in R$, then $tr(\pi_1) = tr(\pi_2)$;
3. if $(\pi_1, \pi_2) \in R$ and $\pi_1 \overset{k}{\leftrightarrow} \pi_1'$, then there exists $\pi_2'$ such that $\pi_2 \overset{k}{\leftrightarrow} \pi_2'$ and $(\pi_1', \pi_2') \in R$;
4. if $(\pi_1, \pi_2) \in R$ and $\pi_1 e_1 \in paths(\mathcal{C}_1)$, then there exists $e_2$ such that $(\pi_1 e_1, \pi_2 e_2) \in R$;
5. if $(\pi_1 e_1, \pi_2 e_2) \in R$, then $(\pi_1, \pi_2) \in R$.

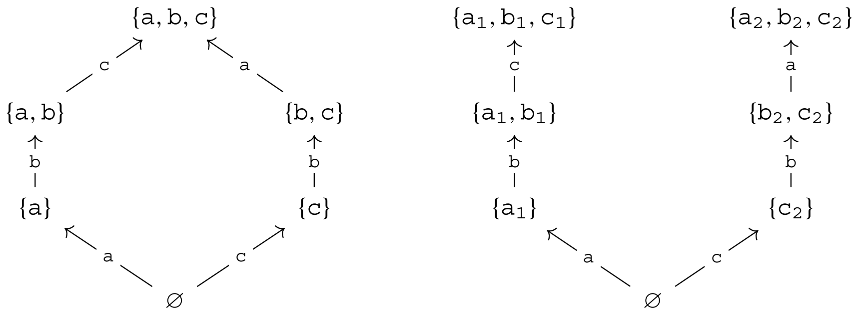
We say that $\mathcal{C}_1$ and $\mathcal{C}_2$ are *path-based hereditary history preserving bisimilar* if there exists a phhpb $R$ relating them.

We are now ready to formally relate the two notions of hhp-bisimilarity seen so far. The proof of this result is in Appendix A.2.

**Proposition 3** *(Chhp- vs phhp-bisimilarity). If two CSs are chhp-bisimilar, then they are phhp-bisimilar; the converse holds if the two CSs are closed under bounded union.*

Notice that closure under bounded union is fundamental to have that the two notions coincide. Indeed, without such a property, chhpb is an equivalence strictly finer than phhpb, as the following example shows.

**Example 3.** Let us now consider the CSs $\mathcal{C}$ and $\mathcal{C}'$, whose transition systems are, respectively, depicted at left and right below:




They are phhpb, as testified by the relation (on paths)

$$R = \{(\epsilon, \epsilon), (\mathtt{a}, \mathtt{a}_1), (\mathtt{c}, \mathtt{c}_2), (\mathtt{ab}, \mathtt{a}_1\mathtt{b}_1), (\mathtt{cb}, \mathtt{c}_2\mathtt{b}_2), (\mathtt{abc}, \mathtt{a}_1\mathtt{b}_1\mathtt{c}_1), (\mathtt{cba}, \mathtt{c}_2\mathtt{b}_2\mathtt{a}_2)\}.$$

By contrast, they are not chhpb because configuration $\{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$ of $\mathcal{C}$ cannot be related to any configuration of $\mathcal{C}'$: indeed, $\{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$ of $\mathcal{C}$ can perform two backwards step, one labelled with $\mathtt{c}$ and the other one with $\mathtt{a}$, whereas no configuration of $\mathcal{C}'$ has this property. Notice that $\mathcal{C}$ is not closed under bounded union: $\{\mathtt{a}\}$ and $\{\mathtt{c}\}$ are bounded by $\{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$, but $\{\mathtt{a}, \mathtt{c}\}$ is not a configuration.

We believe that both phhp-bisimilarity and chhp-bisimilarities can be seen as natural hhp-like equivalences in the setting CSs. Phhp-bisimilarity is based on the idea of viewing the state of a computation as a trace up to adjacency, a fact that allows us to switch only events that can be executed in any order. This justifies the seemingly strange fact that, in the example above, after the trace $\mathtt{abc}$ that leads $\emptyset$ to configuration $\{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$ in $\mathcal{C}$, event $\mathtt{a}$ cannot be "retracted" in the back-and-forth bisimulation game. Chhp-bisimilarity, instead, identifies the state of computations with configurations, hence it allows to retract event $\mathtt{a}$ from $\{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$. This also appears quite a natural choice, in line with the set-based nature of CSs.

## 4. History preserving bisimilarity

If we confine phhpb to just extending and commuting events in traces, we obtain a coarser notion of equivalence that in [18] has been named *history preserving bisimulation*. We recall here its formal definition:

**Definition 15** *(Path-based hp-bisimulation).* Given two CSs $\mathcal{C}_1$ and $\mathcal{C}_2$, a *path-based history preserving bisimulation* (shortened as phpb) between them is a symmetric relation $R \subseteq \mathit{paths}(\mathcal{C}_1) \times \mathit{paths}(\mathcal{C}_2)$ that satisfies points 1–4 of Definition 14. We say that $\mathcal{C}_1$ and $\mathcal{C}_2$ are *path-based history preserving bisimilar* if there exists a phpb $R$ relating them.

We now look for a configuration-based counterpart of this definition and a fragment of the logic $\mathcal{L}_\mathbf{0}$ whose logical equivalence coincides with it.

**Definition 16** *(Configuration-based hp-bisimulation).* Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be configuration structures. A *configuration-based history preserving bisimulation* (shortened as chpb) is a pair $(R^{\rightarrow}, R^{\leftarrow})$ where $R^{\rightarrow}$ is a fw-bisimulation, $R^{\leftarrow}$ is a bw-bisimulation and $R^{\rightarrow} \subseteq R^{\leftarrow}$. We say that $\mathcal{C}_1$ and $\mathcal{C}_2$ are *configuration-based hp-bisimilar*, and write $\mathcal{C}_1 \sim_{\mathrm{hb}} \mathcal{C}_2$, if there exists a chpb $(R^{\rightarrow}, R^{\leftarrow})$ such that $(\emptyset, \emptyset, \emptyset) \in R^{\rightarrow}$.

Notice that chpb is coarser than chhpb, since every chhpb $R$ can be seen as a chpb $(R, R)$. Moreover, the inclusion is strict, as the following example shows.

**Example 4.** Let us now consider the CS $\mathcal{C}_2$ arising from the diagram in Fig. 3 where, again, we assume that events with the same label are in conflict. Essentially, the diagram representing $\mathcal{C}_2$ is obtained from that for $\mathcal{C}_1$ in Fig. 1a by adding the red part. The transition system for $\mathcal{C}_2$ is in Fig. 4, where, again, we highlighted in red the additional part with respect to $\mathcal{C}_1$.

First, we note that $\mathcal{C}_1$ and $\mathcal{C}_2$ are chp-bisimilar. Indeed, choose as $R^{\rightarrow}$ the relation that acts as the identity mapping from the configurations in the transition system of $\mathcal{C}_1$ into the configurations in the black part of the transition system of $\mathcal{C}_2$ and that further associates: $\{\mathtt{a}_1\}$ and $\{\mathtt{a}_2\}$; $\{\mathtt{b}_1\}$ and $\{\mathtt{b}_2\}$; $\{\mathtt{a}_1, \mathtt{b}_1\}$ and $\{\mathtt{a}_2, \mathtt{b}_2\}$; $\{\mathtt{a}_1, \mathtt{c}_1\}$ and $\{\mathtt{a}_2, \mathtt{c}_2\}$; $\{\mathtt{b}_1, \mathtt{c}_1\}$ and $\{\mathtt{b}_2, \mathtt{c}_2\}$; $\{\mathtt{a}_1, \mathtt{b}_1, \mathtt{c}_1\}$ and $\{\mathtt{a}_2, \mathtt{b}_2, \mathtt{c}_2\}$; $\{\mathtt{a}_0, \mathtt{b}_1\}$ and $\{\mathtt{a}_2, \mathtt{b}_1\}$; $\{\mathtt{a}_1, \mathtt{b}_0\}$ and $\{\mathtt{a}_1, \mathtt{b}_2\}$. To define $R^{\leftarrow}$, add to $R^{\rightarrow}$ the associations between $\{\mathtt{b}_2\}$ and $\{\mathtt{b}_0\}$ (to cope with a backward $\mathtt{a}$-transition from $\{\mathtt{a}_1, \mathtt{b}_0\}$ and $\{\mathtt{a}_1, \mathtt{b}_2\}$) and between $\{\mathtt{a}_2\}$ and $\{\mathtt{a}_0\}$ (to cope with a backward $\mathtt{b}$-transition from $\{\mathtt{a}_0, \mathtt{b}_1\}$ and $\{\mathtt{a}_2, \mathtt{b}_1\}$).

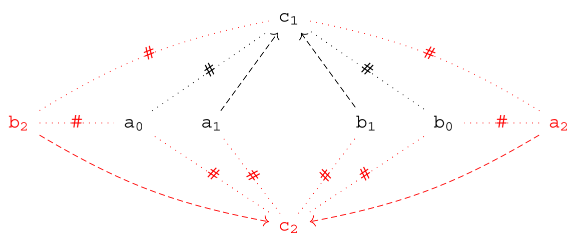



**Fig. 3.** The configuration structure $\mathcal{C}_2$. (For interpretation of the colours in the figure(s), the reader is referred to the web version of this article.)

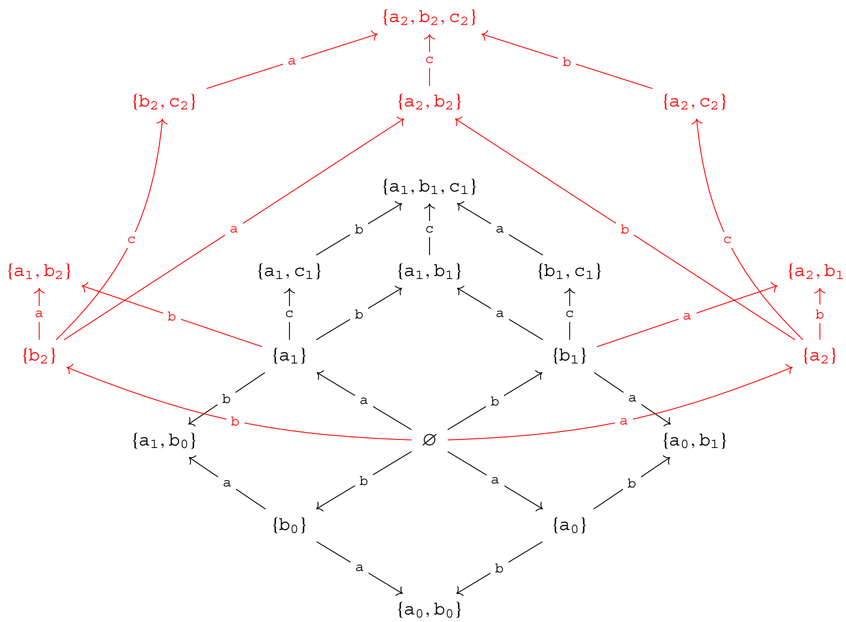



**Fig. 4.** The transition system of the configuration structure $\mathcal{C}_2$.

Second, notice that $\mathcal{C}_1$ and $\mathcal{C}_2$ are *not* chhp-bisimilar. Indeed, consider the $\mathtt{a}$-challenge from $\mathcal{C}_2$ leading to $\{\mathtt{a}_1\}$; the only possible reply from $\mathcal{C}_1$ is the one leading to $\{\mathtt{a}_1\}$. Then, consider the $\mathtt{b}$-challenge of $\mathcal{C}_2$ leading to $\{\mathtt{a}_1, \mathtt{b}_2\}$; the only possible reply from $\mathcal{C}_1$ is the one leading to $\{\mathtt{a}_1, \mathtt{b}_0\}$. Now, consider the backward $\mathtt{a}$-transition in $\mathcal{C}_2$ leading to $\{\mathtt{b}_2\}$; the only possible reply in $\mathcal{C}_1$ leads to $\{\mathtt{b}_0\}$ that cannot perform a $\mathtt{c}$, whereas $\{\mathtt{b}_2\}$ can.

Like for hhp-bisimilarities, the configuration-based version of hp implies the path-based one, and the two notions coincide only for CSs that are closed under bounded union; the proof is in Appendix A.2.

**Proposition 4** *(Chp- vs php-bisimilarity). If two CSs are chpb-bisimilar, then they are phpb-bisimilar; the converse holds if the two CSs are closed under bounded union.*

To isolate a fragment of $\mathcal{L}_0$ whose logical equivalence coincides with chpb, we first define a derived operator.

**Definition 17** *(Executability check).* Let $\boldsymbol{x} \subseteq \mathcal{V}$ be a (finite) set of variables. We let

$$(\mathtt{a}\, z\rangle_{\boldsymbol{x}}\, \varphi \triangleq (\mathtt{a}\, z)\, (\langle\boldsymbol{x}\rangle\, \langle z\rangle\, \mathsf{T} \wedge \varphi)$$

Intuitively, $(\mathtt{a}\, z\rangle_{\boldsymbol{x}}\, \varphi$ states that there is an $\mathtt{a}$-labelled event that could be executed after the events in $\boldsymbol{x}$ and, if we bind such event to $z$ *without executing it*, the formula $\varphi$ holds. By relying on this, we can now identify a fragment of $\mathcal{L}_0$ that corresponds to hp-bisimilarity.

**Definition 18** *(Logic fragment for chpb).* For a set of variables $\boldsymbol{x} \subseteq \mathcal{V}$, define

$$\varphi_{\boldsymbol{x}} ::= \mathsf{T} \mid \neg\, \varphi_{\boldsymbol{x}} \mid \varphi_{\boldsymbol{x}} \wedge \varphi_{\boldsymbol{x}} \mid (\mathtt{a}\, z\rangle_{\boldsymbol{x}}\, \varphi_{\boldsymbol{x}\cup\{z\}} \mid \langle x_1\rangle \ldots \langle x_n\rangle\, \mathsf{T}$$

where it is intended that $z \notin \boldsymbol{x}$ and $\{x_1, \ldots, x_n\} = \boldsymbol{x}$. Then, $\mathcal{L}_h$ is the set of formulae arising with $\boldsymbol{x}$ empty, i.e., $\varphi_\varnothing$.

From the definition above, it follows that every formula of the fragment $\mathcal{L}_h$ is closed; this is different from the formulae of the whole logic $\mathcal{L}_{\mathbf{0}}$ which, instead, may contain free variables. We can now prove that the logical equivalence induced by the sublogic $\mathcal{L}_h$ coincides with chpb.

**Theorem 2** *(Logic for hp). Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be CSs. Then $\mathcal{C}_1 \sim_{\text{hb}} \mathcal{C}_2$ iff, for all $\varphi \in \mathcal{L}_h$, $\mathcal{C}_1 \models \varphi \Leftrightarrow \mathcal{C}_2 \models \varphi$.*

**Proof.** ($\Leftarrow$). Fix a surjective environment $\eta : \mathcal{V} \to E_{\mathcal{C}_1}$, as we did in the proof of Theorem 1, such that $\eta(x_e) = e$ for all $e \in E_{\mathcal{C}_1}$ and the corresponding fixed variable $x_e$.

Assume that, for all $\varphi$ in $\mathcal{L}_h$, it holds $\mathcal{C}_1 \models \varphi$ iff $\mathcal{C}_2 \models \varphi$. Let $R, Q \subseteq \mathcal{I}(\mathcal{C}_1, \mathcal{C}_2)$ be defined as

$$R = \{(X, f, Y) \in \mathcal{I}(\mathcal{C}_1, \mathcal{C}_2) \mid \forall \varphi_{\boldsymbol{x}_X}.\, \mathcal{C}_1, \varnothing, X \models_\eta \varphi_{\boldsymbol{x}_X} \text{ iff } \mathcal{C}_2, \varnothing, Y \models_{f\circ\eta} \varphi_{\boldsymbol{x}_X}\}$$

and

$$Q = \{(X, f, Y) \in \mathcal{I}(\mathcal{C}_1, \mathcal{C}_2) \mid \forall \{x_1, \ldots, x_n\} = \boldsymbol{x}_X.\, \mathcal{C}_1, \varnothing, X \models_\eta \langle x_1\rangle \ldots \langle x_n\rangle \mathsf{T} \text{ iff } \mathcal{C}_2, \varnothing, Y \models_{f\circ\eta} \langle x_1\rangle \ldots \langle x_n\rangle \mathsf{T}\}$$

We show that $(R, Q)$ is a chpb between $\mathcal{C}_1$ and $\mathcal{C}_2$, i.e., $R$ is a fw-bisimulation, $Q$ is a bw-bisimulation, $(\varnothing, \varnothing, \varnothing) \in R$, and $R \subseteq Q$. The latter holds immediately by definition, since the formulae in the definition of $Q$ are a subset of those in $R$ (see Definition 18).

We start by showing that $R$ is a fw-bisimulation, which we do by contradiction. So, suppose that $(X, f, Y) \in R$ and, without loss of generality, that $X \xrightarrow{\mathsf{a}} X'$, but for all transitions $Y \xrightarrow{\mathsf{a}} Y'$, we have $(X', f[e \mapsto e'], Y') \notin R$ where $e \in X' \setminus X$ and $e' \in Y' \setminus Y$. Since $f[e \mapsto e']$ is still an isomorphism, this can only happen because there exists a formula $\varphi_{\boldsymbol{x}_{X'}}$ such that $\mathcal{C}_1, \varnothing, X' \models_\eta \varphi_{\boldsymbol{x}_{X'}}$ and $\mathcal{C}_2, \varnothing, Y' \not\models_{f[e\mapsto e']\circ\eta} \varphi_{\boldsymbol{x}_{X'}}$ (or vice versa, that is analogous, and so omitted).

Note that there must be at least one such transition $Y \xrightarrow{\mathsf{a}} Y'$, otherwise we would have $\mathcal{C}_1, \varnothing, X \models_\eta (\mathsf{a}\, x_e)_{\boldsymbol{x}_X} \mathsf{T}$ and $\mathcal{C}_2, \varnothing, Y \not\models_{f\circ\eta} (\mathsf{a}\, x_e)_{\boldsymbol{x}_X} \mathsf{T}$, contradicting the fact that $(X, f, Y) \in R$.

Furthermore, since by hypothesis $\mathcal{C}_1$ and $\mathcal{C}_2$ are image-finite, there are finitely many transitions $Y \xrightarrow{l(e_i)} Y_i = Y \cup \{e_i\}$, indexed by $i \in \{1, \ldots, h\}$, complying with the previous conditions. For each $i \in \{1, \ldots, h\}$, call $f_i$ the corresponding isomorphism defined by $f_i = f[e \mapsto e_i]$. Then, by the assumption above, we know that there is a formula $\psi^i_{\boldsymbol{x}_{X'}}$ such that $\mathcal{C}_1, \varnothing, X' \models_\eta \psi^i_{\boldsymbol{x}_{X'}}$ and $\mathcal{C}_2, \varnothing, Y_i \not\models_{f_i\circ\eta} \psi^i_{\boldsymbol{x}_{X'}}$.

Now consider the formula

$$\varphi_{\boldsymbol{x}_X} = (\mathsf{a}\, x_e)_{\boldsymbol{x}_X} \left( \bigwedge_{i\in\{1,\ldots,h\}} \psi^i_{\boldsymbol{x}_{X'}} \right) = (\mathsf{a}\, x_e) \left( \langle \boldsymbol{x}_X \rangle \langle x_e \rangle \mathsf{T} \wedge \bigwedge_{i\in\{1,\ldots,h\}} \psi^i_{\boldsymbol{x}_{X'}} \right)$$

By hypothesis, it is easy to see that $\mathcal{C}_1, \varnothing, X \models_\eta \varphi_{\boldsymbol{x}_X}$. However, for every $i \in \{1, \ldots, h\}$, we know that $\mathcal{C}_2, \varnothing, Y_i \not\models_{f_i\circ\eta} \psi^i_{\boldsymbol{x}_{X'}}$, and since $f_i \circ \eta = f[e \mapsto e_i] \circ \eta = (f \circ \eta)[x_e \mapsto e_i]$, we have that $\mathcal{C}_2, \varnothing, Y_i \not\models_{(f\circ\eta)[x_e\mapsto e_i]} \psi^i_{\boldsymbol{x}_{X'}}$. Observe that, for all $F \overset{*}{\underset{Y}{\longleftrightarrow}} Y$ such that $e_i \in F \setminus Y$ (for some $i \in \{1, \ldots, h\}$) and $\mathcal{C}_2, \varnothing, F \models_{(f\circ\eta)[x_e\mapsto e_i]} \langle \boldsymbol{x}_X \rangle \langle x_e \rangle \mathsf{T}$, we must have that $\mathcal{C}_2, Y_i, F \models_{(f\circ\eta)[x_e\mapsto e_i]} \mathsf{T}$ since $f(\eta(\boldsymbol{x}_X)) \cup \{e_i\} = Y \cup \{e_i\} = Y_i$. By definition of the semantics, this requires that $(Y_i, F, (f \circ \eta)[x_e \mapsto e_i]) \in lt(\mathsf{T})$, which in turn requires that $Y_i \overset{*}{\underset{Y_i}{\longleftrightarrow}} F$. Thus, recalling that $fv(\psi^i_{\boldsymbol{x}_{X'}}) \subseteq \boldsymbol{x}_{X'} = \boldsymbol{x}_X \cup \{e\}$, from the fact that $\mathcal{C}_2, \varnothing, Y_i \not\models_{(f\circ\eta)[x_e\mapsto e_i]} \psi^i_{\boldsymbol{x}_{X'}}$ by Lemma 1 we deduce that also $\mathcal{C}_2, \varnothing, F \not\models_{(f\circ\eta)[x_e\mapsto e_i]} \psi^i_{\boldsymbol{x}_{X'}}$. But then, by definition of the semantics, we would have that $\mathcal{C}_2, \varnothing, Y \not\models_{f\circ\eta} \varphi_{\boldsymbol{x}_X}$ contradicting the fact that $(X, f, Y) \in R$. Thus, we can conclude that $R$ is a fw-bisimulation. Moreover, observe that $(\varnothing, \varnothing, \varnothing) \in R$, since by hypothesis $\mathcal{C}_1 \models \varphi_\varnothing$ iff $\mathcal{C}_2 \models \varphi_\varnothing$ for all $\varphi_\varnothing$ in $\mathcal{L}_h$.

We now need to show that $Q$ is a bw-bisimulation. Let $(X, f, Y) \in Q$. Observe that, for every transition $X' \xrightarrow{l(e)} X$, there must exist a formula $\theta_{\boldsymbol{x}_X} = \langle x_1\rangle \ldots \langle x_n\rangle \mathsf{T}$ such that $\{x_1, \ldots, x_{n-1}\} = \boldsymbol{x}_{X'}$, $x_n = x_e$, and $\mathcal{C}_1, \varnothing, X \models_\eta \theta_{\boldsymbol{x}_X}$, since $X' \in \mathcal{C}_1$. Then, by definition of $Q$, we must also have that $\mathcal{C}_2, \varnothing, Y \models_{f\circ\eta} \theta_{\boldsymbol{x}_X}$. By definition of the semantics, it follows that $f(X') = Y' \xrightarrow{l(e)} Y$, and $f|_{X'} : X' \to Y'$ is still an isomorphism. Now, by contradiction, suppose that $(X', f|_{X'}, Y') \notin Q$. Then, there must be a formula $\xi_{\boldsymbol{x}_{X'}} = \langle x_1\rangle \ldots \langle x_k\rangle \mathsf{T}$ such that $\{x_1, \ldots, x_k\} = \boldsymbol{x}_{X'}$, $\mathcal{C}_1, \varnothing, X' \models_\eta \xi_{\boldsymbol{x}_{X'}}$ and $\mathcal{C}_2, \varnothing, Y' \not\models_{f|_{X'}\circ\eta} \xi_{\boldsymbol{x}_{X'}}$. Since $\boldsymbol{x}_{X'} = \boldsymbol{x}_X \setminus \{x_e\}$ and $f|_{X'}(\eta(\boldsymbol{x}_{X'})) = f(\eta(\boldsymbol{x}_{X'}))$, we could build a formula $\xi_{\boldsymbol{x}_X} = \langle x_1\rangle \ldots \langle x_k\rangle \langle x_e\rangle \mathsf{T}$ such that $\mathcal{C}_1, \varnothing, X \models_\eta \xi_{\boldsymbol{x}_X}$ and $\mathcal{C}_2, \varnothing, Y \not\models_{f\circ\eta} \xi_{\boldsymbol{x}_X}$ contradicting the fact that $(X, f, Y) \in Q$. So, we conclude that $Q$ is a bw-bisimulation, hence $(R, Q)$ is a chpb.

($\Rightarrow$). Assume that we have a chpb $(R^{\rightarrow}, R^{\leftarrow})$ between $\mathcal{C}_1$ and $\mathcal{C}_2$. We prove that, for all $\varphi$ in $\mathcal{L}_h$, it holds $\mathcal{C}_1 \models \varphi$ iff $\mathcal{C}_2 \models \varphi$. Actually, we show that, for every $(X, f, Y) \in R^{\rightarrow}$, for every formula $\varphi_{\boldsymbol{x}}$ with $|\boldsymbol{x}| = |X|$ and every environment $\eta \in Env_{\mathcal{C}_1}$ such that $\eta(\boldsymbol{x}) = X$, it holds $\mathcal{C}_1, \varnothing, X \models_\eta \varphi_{\boldsymbol{x}}$ if and only if $\mathcal{C}_2, \varnothing, Y \models_{f\circ\eta} \varphi_{\boldsymbol{x}}$. Observing that environments are irrelevant when $\boldsymbol{x} = \varnothing$, i.e. when $X = \varnothing$, this is enough since by hypothesis $(\varnothing, \varnothing, \varnothing) \in R^{\rightarrow}$, implying that $\mathcal{C}_1$ and $\mathcal{C}_2$ would satisfy the same formulae of $\mathcal{L}_h$.

We proceed by induction on the shape of the formula $\varphi_{\boldsymbol{x}}$. We discuss only some cases and a single direction, the other being symmetric.

$\varphi_{\boldsymbol{x}} = \neg\,\psi_{\boldsymbol{x}}$**:** Assume that $\mathcal{C}_1, \varnothing, X \models_\eta \neg\psi_{\boldsymbol{x}}$. By definition of the semantics, we know that $\mathcal{C}_1, \varnothing, X \not\models_\eta \psi_{\boldsymbol{x}}$. Then, by inductive hypothesis, we have that $\mathcal{C}_2, \varnothing, Y \not\models_{f\circ\eta} \psi_{\boldsymbol{x}}$. Again by definition of the semantics, we conclude that $\mathcal{C}_2, \varnothing, Y \models_{f\circ\eta} \neg\psi_{\boldsymbol{x}}$.

$\varphi_{\boldsymbol{x}} = (\mathtt{a}\, z)_{\boldsymbol{x}}\, \psi_{\boldsymbol{x}'}$ **where** $\boldsymbol{x}' = \boldsymbol{x} \cup \{z\}$**:** Assume that $\mathcal{C}_1, \varnothing, X \models_\eta (\mathtt{a}\, z)_{\boldsymbol{x}} \psi_{\boldsymbol{x}'}$. By definition of the semantics, we know that there exist $X', F \in \mathcal{C}_1$ such that

1. $X \underset{X}{\leftrightarrow} F$
2. $\eta(fv(\varphi_{\boldsymbol{x}})) = \eta(\boldsymbol{x}) = X \xrightarrow{\mathtt{a}} X' = X \cup \{e\} = \eta[z \mapsto e](\boldsymbol{x}') \subseteq F$
3. $\mathcal{C}_1, \varnothing, F \models_{\eta[z\mapsto e]} \langle \boldsymbol{x} \rangle \langle z \rangle \mathsf{T}$
4. $\mathcal{C}_1, \varnothing, F \models_{\eta[z\mapsto e]} \psi_{\boldsymbol{x}'}$

By (2) and (3) above, we know that $\mathcal{C}_1, X', F \models_{\eta[z\mapsto e]} \mathsf{T}$, i.e. $(X', F, \eta[z \mapsto e]) \in lt(\mathsf{T})$, hence $X' \overset{*}{\underset{X'}{\leftrightarrow}} F$. Then, by Lemma 1 and (4), we have that $\mathcal{C}_1, \varnothing, X' \models_{\eta[z\mapsto e]} \psi_{\boldsymbol{x}'}$. Moreover, since $(X, f, Y) \in R^{\rightarrow}$, there must also exist a transition $f(\eta(\boldsymbol{x})) = Y \xrightarrow{\mathtt{a}} Y' = Y \cup \{e'\}$, for some event $e'$, such that $(X', f', Y') \in R^{\rightarrow}$ with $f = f'|_X$; hence, $f' = f[e \mapsto e']$. By inductive hypothesis, we have that $\mathcal{C}_2, \varnothing, Y' \models_{f'\circ\eta[z\mapsto e]} \psi_{\boldsymbol{x}'}$. Then, since $f' \circ \eta[z \mapsto e] = (f \circ \eta)[z \mapsto e']$, again by definition of the semantics, we can conclude that $\mathcal{C}_2, \varnothing, Y \models_{f\circ\eta} (\mathtt{a}\, z)_{\boldsymbol{x}} \psi_{\boldsymbol{x}'}$ since clearly $Y \overset{*}{\underset{Y}{\leftrightarrow}} Y'$.

$\varphi_{\boldsymbol{x}} = \langle x_1 \rangle \ldots \langle x_n \rangle \mathsf{T}$ **where** $\{x_1, \ldots, x_n\} = \boldsymbol{x}$**:** Assume that $\mathcal{C}_1, \varnothing, X \models_\eta \langle x_1 \rangle \ldots \langle x_n \rangle \mathsf{T}$. By definition of the semantics, we know that $\varnothing \xrightarrow{l(\eta(x_1))} \ldots \xrightarrow{l(\eta(x_{n-1}))} X' \xrightarrow{l(\eta(x_n))} \eta(\boldsymbol{x}) = X$. Then, since $(X, f, Y) \in R^{\rightarrow} \subseteq R^{\leftarrow}$, there must also exist a transition $Y' = f(X') \xrightarrow{l(f(\eta(x_n)))} Y = f(X)$. By iterating this argument, we obtain a sequence of transitions $\varnothing \xrightarrow{l(f(\eta(x_1)))} \ldots \xrightarrow{l(f(\eta(x_n)))} Y$; so, again by definition of the semantics, we conclude that $\mathcal{C}_2, \varnothing, Y \models_{f\circ\eta} \langle x_1 \rangle \ldots \langle x_n \rangle \mathsf{T}$. □

**Example 5.** As the logic $\mathcal{L}_{\mathbf{0}}$ fully characterises hhp-bisimilarity, it is able to distinguish between the CSs $\mathcal{C}_1$ and $\mathcal{C}_2$ presented in Examples 1 and 4, respectively, which, as already noted, are not hhp-bisimilar. Indeed, consider the formula:

$$\varphi \triangleq (\mathtt{a}\, x)\, (\mathtt{b}\, y)\, (\langle x \rangle\, (\mathtt{c}\, u)\mathsf{T} \,\wedge\, \langle y \rangle\, (\mathtt{c}\, v)\mathsf{T} \,\wedge\, \langle x \rangle\, \langle y \rangle\, \neg(\mathtt{c}\, z)\mathsf{T})$$

It requires the existence of an a-labelled event and a b-labelled one such that there is some c-labelled event in the future of each of them (possibly not the same one), but after executing both, no future c-labelled event is left. Intuitively, $\varphi$ captures the behaviour of $\mathcal{C}_2$ described in Example 4 and, indeed, it is satisfied by $\mathcal{C}_2$: take, for instance, the events $\mathtt{a}_1$ and $\mathtt{b}_2$. By contrast, such a behaviour is not present in $\mathcal{C}_1$, and $\varphi$ is not satisfied by $\mathcal{C}_1$.

Observe that $\varphi$ is not in $\mathcal{L}_h$ since it uses, in an essential way, the more general syntax of $\mathcal{L}_{\mathbf{0}}$. In particular, in order to distinguish $\mathcal{C}_1$ and $\mathcal{C}_2$, we must be able to quantify the events labelled by a and b without immediately executing them.

## 5. Conclusions

We showed how a logic already studied for prime event structures in [4] can be interpreted in the more general setting of configuration structures. We obtained a conservative extension of the original logic able to internalize concepts like closedness under bounded union or intersection. Furthermore, the logical equivalence induced by the proposed logic coincides with a notion of hereditary history preserving bisimilarity that strongly resembles the analogous notion for prime event structures. Such an equivalence is finer than the hhp-bisimilarity proposed by [18], and coincides with it in the setting of general event structures (i.e., CSs closed under bounded union). Finally, we extended the results above to the setting of history preserving bisimilarity.

There are standard questions for behavioural logics that we did not face in this paper and could represent interesting venues of research. The first one is the possibility of model checking the logic $\mathcal{L}_{\mathbf{0}}$ over suitable classes of configuration structures. Clearly, as a first step, one should identify a finitary representation for the configuration structures in the class of interest, e.g., via some notion of regularity. Another classical problem that would be worth studying for $\mathcal{L}_{\mathbf{0}}$ is satisfiability, i.e., the existence of a model (possibly in some restricted class) for a given formula in $\mathcal{L}_{\mathbf{0}}$.

A further natural question concerns the possibility of going beyond configuration structures and provide an event based behavioural logic, with forward flavour, for higher dimensional automata characterising history-preserving bisimilarities. Also the comparison with approaches based on backward modalities would be interesting to explore in this context. Particularly relevant appear the event identifier logic in [12], used to characterised various forms of true concurrent bisimilarities on stable event structures, and the logic with during/after/forward/backward modalities proposed in [14] for higher dimensional automata.

**Declaration of competing interest**

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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## Appendix A. Omitted proofs

*A.1. Proof of Proposition 1*

**Proposition 1** *(Logics over PESs). Let $\mathcal{C}$ be a CS closed under intersection and bounded union, and $\varphi$ be a formula of $\mathcal{L}_{\mathbf{0}}$. For all configurations $X, F \in C$ and environment $\eta \in Env_{\mathcal{C}}$ such that $(X, F, \eta) \in lt(\varphi)$, it holds $\mathcal{C}, X, F \models_\eta \varphi$ iff $(X, \eta) \in [\![\varphi]\!]^{\mathcal{E}(\mathcal{C})}$, where $\mathcal{E}(\mathcal{C})$ is the PES corresponding to $\mathcal{C}$.*

**Proof.** First of all, observe that $(X, F, \eta) \in lt(\varphi)$ implies $(X, \eta) \in lp_{\mathcal{E}}(\varphi)$, because there exists a configuration including $X \cup \eta(fv(\varphi))$, which is $F$. The proof proceeds by induction on the shape of the formula $\varphi$. We discuss only some cases.

$\varphi = (\mathsf{a}\, z)\, \psi$: Assume that $\mathcal{C}, X, F \models_\eta (\mathsf{a}\, z)\psi$. By Definition 6, we know that there are $F' \in C$ and $e \in F' \setminus X$ such that $l(e) = \mathsf{a}$, $F \xleftrightarrow[X\cup\eta(fv((\mathsf{a}\,z)\psi))]{*} F'$ and $\mathcal{C}, X, F' \models_{\eta[z\mapsto e]} \psi$. This also means that $(X, F', \eta[z \mapsto e]) \in lt(\psi)$. Then, by inductive hypothesis, we have that $(X, \eta[z \mapsto e]) \in [\![\psi]\!]^{\mathcal{E}}$. So, by Definition 10, we conclude that $(X, \eta) \in [\![(\mathsf{a}\, z)\psi]\!]^{\mathcal{E}}$, since as mentioned $(X, \eta) \in lp_{\mathcal{E}}((\mathsf{a}\, z)\psi)$.

Conversely, assume that $(X, \eta) \in [\![(\mathsf{a}\, z)\psi]\!]^{\mathcal{E}}$. By Definition 10 we know that there exist $F' \in C$ and $e \in F' \setminus X$ such that $l(e) = \mathsf{a}$, $X \cup \eta(fv((\mathsf{a}\, z)\psi)) \subseteq F'$, and $(X, \eta[z \mapsto e]) \in [\![\psi]\!]^{\mathcal{E}}$. Note that $X' = X \cup \lceil \eta(fv((\mathsf{a}\, z)\psi))\rceil \cup \lceil e \rceil$ is a configuration. Since $\mathcal{C}$ is closed under bounded union, the fact that $X \cup \eta[z \mapsto e](fv(\psi)) \subseteq X \cup \eta(fv((\mathsf{a}\, z)\psi)) \cup \{e\} \subseteq X' \subseteq F' \in C$ implies that $X \to \ldots \to X' \to \ldots \to F'$; hence, $(X, F', \eta[z \mapsto e]) \in lt(\psi)$. Then, by inductive hypothesis we obtain that $\mathcal{C}, X, F' \models_{\eta[z\mapsto e]} \psi$. Moreover, we also know that $X'' = X \cup \lceil \eta(fv((\mathsf{a}\, z)\psi))\rceil$ is a configuration. Thus, like before, since $\mathcal{C}$ is closed under bounded union and $X''$ is necessarily a subset of both $F$ and $F'$, we must have that $X'' \to \ldots \to F$ and $X'' \to \ldots \to F'$; hence, $F \xleftrightarrow[X'']{*} F'$ and so $F \xleftrightarrow[X\cup\eta(fv((\mathsf{a}\,z)\psi))]{*} F'$. Then, by Definition 6, we can conclude that $\mathcal{C}, X, F \models_\eta (\mathsf{a}\, z)\psi$, since by hypothesis $(X, F, \eta) \in lt((\mathsf{a}\, z)\psi)$.

$\varphi = \langle z \rangle\, \psi$: Assume that $\mathcal{C}, X, F \models_\eta \langle z \rangle\, \psi$. By Definition 6, we know that $X \xrightarrow{l(\eta(z))} X \cup \{\eta(z)\} = X'$ and $\mathcal{C}, X', F \models_\eta \psi$. This also means that $(X', F, \eta) \in lt(\psi)$. Then, by inductive hypothesis, we have that $(X', \eta) \in [\![\psi]\!]^{\mathcal{E}}$. So, by Definition 10, we can conclude that $(X, \eta) \in [\![\langle z \rangle\, \psi]\!]^{\mathcal{E}}$, since $(X, \eta) \in lp_{\mathcal{E}}(\langle z \rangle\, \psi)$.

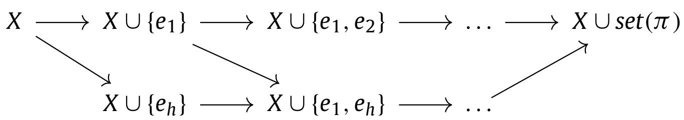
Conversely, assume that $(X, \eta) \in [\![\langle z \rangle\, \psi]\!]^{\mathcal{E}}$. By Definition 10, we know that $X \xrightarrow{l(\eta(z))} X \cup \{\eta(z)\} = X'$ and $(X', \eta) \in [\![\psi]\!]^{\mathcal{E}}$. Therefore, we must have that $F' = X' \cup \lceil \eta(fv(\psi))\rceil = X \cup \lceil \eta(fv(\langle z \rangle\, \psi))\rceil$ is a configuration. Since $\mathcal{C}$ is closed under bounded union, the fact that $X' \subseteq F' \in C$ implies that $X' \to \ldots \to F'$; hence, we have $(X, F', \eta) \in lt(\langle z \rangle\, \psi)$ and $(X', F', \eta) \in lt(\psi)$. From the latter, by inductive hypothesis we obtain that $\mathcal{C}, X', F' \models_\eta \psi$; since $X' = X \cup \{\eta(z)\}$, by Definition 6 we also know that $\mathcal{C}, X, F' \models_\eta \langle z \rangle\, \psi$. Moreover, observe that, since $\mathcal{C}$ is closed under bounded union and by hypothesis $X \cup \eta(fv(\langle z \rangle\, \psi)) \subseteq F$ (hence, by definition of causes, $F' = X \cup \lceil \eta(fv(\langle z \rangle\, \psi))\rceil \subseteq F$), we must have that $F' \to \ldots \to F$, that is, $F' \xleftrightarrow[X\cup\eta(fv(\langle z \rangle\, \psi))]{*} F$. Recalling that $(X, F', \eta) \in lt(\langle z \rangle\, \psi)$ and $\mathcal{C}, X, F' \models_\eta \langle z \rangle\, \psi$, by Lemma 1 we can conclude that $\mathcal{C}, X, F \models_\eta \langle z \rangle\, \psi$. □

*A.2. Proofs of Propositions 3 and 4*

In order to prove Propositions 3 and 4, we first need a technical result which shows that for CSs closed under bounded union, all paths that are coinitial and cofinal are adjacent. This is proved in two steps.

**Lemma 2** *(Anticipating events). Let $\mathcal{C}$ be a CS closed under bounded union, $X \in \mathcal{C}$, and $\pi = e_1 \ldots e_n$ be a path starting from $X$. If $X \to X \cup \{e_h\}$, for some $h \in \{1, \ldots, n\}$, then $\pi' = e_h e_1 \ldots e_{h-1} e_{h+1} \ldots e_n$ is a path starting from $X$ and $\pi \leftrightarrow^* \pi'$.*

**Proof.** We proceed by induction on $h$. If $h = 1$, the thesis trivially holds. If $h > 1$, observe that, since $\pi = e_1 \ldots e_n$ is a path starting from $X$, we have $X \to X \cup \{e_1\} \subseteq X \cup set(\pi)$. Moreover, by hypothesis, $X \to X \cup \{e_h\} \subseteq X \cup set(\pi)$. Hence, by closure under bounded union, we obtain that $X \cup \{e_1, e_h\} \in \mathcal{C}$. The situation is depicted in the picture below:



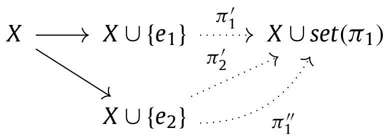

Observe that $\pi_1 = e_2 \dots e_h$ is a path starting from $X \cup \{e_1\}$ and $X \cup \{e_1\} \to X \cup \{e_1, e_h\}$. Hence, by inductive hypothesis, $\pi_1' = e_h e_2 \dots e_{h-1} e_{h+1} \dots e_n$ is a path starting from $X \cup \{e_1\}$ and $\pi_1 \leftrightarrow^* \pi_1'$. Thus

$$\pi = e_1 \pi_1 \leftrightarrow^* e_1 \pi_1' = e_1 e_h e_2 \dots e_{h-1} e_{h+1} \dots e_n \tag{A.1}$$

and $e_1 \pi_1'$ is a path starting from $X$. Additionally, $e_1 e_h \leftrightarrow^* e_h e_1$ and thus, by (A.1), we obtain $\pi \leftrightarrow^* e_h e_1 e_2 \dots e_{h-1} e_{h+1} \dots e_n$; moreover, $e_h e_1 e_2 \dots e_{h-1} e_{h+1} \dots e_n$ is a path starting from $X$, as desired. □

**Lemma 3** *(Adjacency of coinitial and cofinal paths). Let $\mathcal{C}$ be a CS closed under bounded union, $X \in \mathcal{C}$, and $\pi_1, \pi_2$ be paths starting from $X$ such that $set(\pi_1) = set(\pi_2)$. Then $\pi_1 \leftrightarrow^* \pi_2$.*

**Proof.** We proceed by induction on $n = |\pi_1| = |\pi_2|$. If $n = 0$ the thesis is trivial. If $n > 0$, let $\pi_1 = e_1 \pi_1'$ and $\pi_2 = e_2 \pi_2'$. Since $set(\pi_1) = set(\pi_2)$, we have that $e_2 \in set(\pi_1)$; hence, by Lemma 2, there is a path $e_2 \pi_1''$ starting from $X$ such that $\pi_1 \leftrightarrow^* e_2 \pi_1''$:




By inductive hypothesis, $\pi_1'' \leftrightarrow^* \pi_2'$; hence, $\pi_1 \leftrightarrow^* e_2 \pi_1'' \leftrightarrow^* e_2 \pi_2' = \pi_2$, as desired. □

Now, given a path $\pi$, we denote by $\pi(i)$ the $i$-th event in $\pi$. Furthermore, given a configuration $X \in C$, we denote with $paths(X, \mathcal{C})$ the set of all paths leading from $\varnothing$ to $X$ in $\mathcal{C}$, i.e. all the possible total orders that can be built by using events in $X$ that form a path; formally, $paths(X, \mathcal{C}) \triangleq \{\pi \in paths(\mathcal{C}) \ : \ set(\pi) = X\}$.

**Proposition 3** *(Chhp- vs phhp-bisimilarity). If two CSs are chhp-bisimilar, then they are phhp-bisimilar; the converse holds if the two CSs are closed under bounded union.*

**Proof.** (**chhp** $\Rightarrow$ **phhp**) We first show that chhp-bisimilarity implies phhp-bisimilarity. Let $S$ be a chhpb; then, we want to prove that

$$R \triangleq \bigcup_{(X_1, f, X_2) \in S} \ \bigcup_{\pi \in paths(X_1, \mathcal{C}_1)} \{(\pi, f(\pi))\}$$

is a phhpb. Trivially, $(\epsilon, \epsilon) \in R$, since $(\varnothing, \varnothing, \varnothing) \in S$. Now fix $(\pi_1, \pi_2) \in R$; this means that there exists $(X_1, f, X_2) \in S$ such that $\pi_2 = f(\pi_1)$. We have to prove that $R \subseteq paths(\mathcal{C}_1) \times paths(\mathcal{C}_2)$; we shall actually show that $\pi_2 \in paths(X_2, \mathcal{C}_2)$. To this aim, let $\pi_1 \triangleq e_1 \dots e_n$ and consider the backwards sequence of challenges $\varnothing \xrightarrow{a_1} \{e_1\} \dots \xrightarrow{a_n} \{e_1, \dots, e_n\} = X_1$, where $l(e_i) = a_i$ for all $i$s. Since $S$ is a bw-bisimulation, $\varnothing \xrightarrow{a_1} \{e_1'\} \dots \xrightarrow{a_n} \{e_1', \dots, e_n'\} = X_2$, where $e_i' = f(e_i)$ for all $i$s, and so $f(\pi_1) = \pi_2 \in paths(X_2, \mathcal{C}_2)$. Then:

- $tr(\pi_1) = tr(\pi_2)$, since $f$ is a configuration isomorphism, and so it respects labelling.
- Let $\pi_1 \overset{k}{\leftrightarrow} \pi_1'$; this means that $\pi_1 \triangleq e_1 \dots e_k e_{k+1} e_{k+2} \dots e_n$ and $\pi_1' \triangleq e_1 \dots e_{k+1} e_k e_{k+2} \dots e_n$. By letting $e_i' \triangleq f(e_i)$, we have that $\pi_2 = e_1' \dots e_k' e_{k+1}' e_{k+2}' \dots e_n'$. Now, since $set(\pi_1') = X_1$ (and so $\pi_1' \in paths(X_1, \mathcal{C}_1)$), by construction $(\pi_1', f(\pi_1')) \in R$, where $f(\pi_1') = e_1' \dots e_{k+1}' e_k' e_{k+2}' \dots e_n'$; the last equality entails that $\pi_2 \overset{k}{\leftrightarrow} f(\pi_1')$ and this suffices to conclude.
- Let $\pi_1 e \in paths(\mathcal{C}_1)$; this means that $X_1 \xrightarrow{a} X_1 \cup \{e\}$, where $a = l(e)$. Since $S$ is a fw-bisimulation, there exists $e'$ such that $X_2 \xrightarrow{a} X_2 \cup \{e'\}$ and $(X_1 \cup \{e\}, f \cup [e \mapsto e'], X_2 \cup \{e'\}) \in S$. This entails that $\pi_2 e' \in paths(\mathcal{C}_2)$ and so by construction $(\pi_1 e, \pi_2 e') \in R$, being $\pi_2 e' = (f \cup [e \mapsto e'])(\pi_1 e)$.
- Finally, we have to show that $(e_1 \dots e_{n-1}, e_1' \dots e_{n-1}') \in R$, where $\pi_1 \triangleq e_1 \dots e_n$ and $\pi_2 \triangleq e_1' \dots e_n'$ (and so $e_i' \triangleq f(e_i)$, for all $i$s). Let $X_1' \xrightarrow{a} X_1$, where $X_1' \triangleq X_1 \setminus \{e_n\}$ and $a = l(e_n)$. Since $S$ is a bw-bisimulation, there exists $X_2' \triangleq X_2 \setminus \{e\}$ such that $X_2' \xrightarrow{a} X_2$, $l(e) = a$ and $(X_1', f', X_2') \in S$, for $f = f' \cup [e_n \mapsto e]$. The very last condition entails that $e$ must be $e_n'$ and we conclude, since trivially $e_1' \dots e_{n-1}' \in paths(X_2', \mathcal{C}_2)$.

(**phhp** $\Rightarrow$ **chhp**) We next show that, if $\mathcal{C}$ is closed under bounded union, phhpb-bisimilarity implies chhpb-bisimilarity. Let $R$ be a phhpb. We want to prove that

$$S \triangleq \{(set(\pi_1), f, set(\pi_2)) \; : \; (\pi_1, \pi_2) \in R \;\wedge\; \forall i. f(\pi_1(i)) = \pi_2(i)\}$$

is a chhpb. First of all, we have that $S \subseteq \mathcal{I}(\mathcal{C}_1, \mathcal{C}_2)$ since, for every triple $(set(\pi_1), f, set(\pi_2)) \in S$, it holds that $tr(\pi_1) = tr(\pi_2)$ (see Definition 15(2)); in particular, this means that the second component of every triple in $S$ is an isomorphism of configurations (i.e. a bijection that respects labelling, since configurations are labelled sets of events). Then:

1. $(\emptyset, \emptyset, \emptyset) \in S$ since by Definition 15(1) $(\epsilon, \epsilon) \in R$.
2. Let $(set(\pi_1), f, set(\pi_2)) \in S$ and $set(\pi_1) \xrightarrow{a} set(\pi_1) \cup \{e_1\}$, for $l(e_1) = a$. Then, $\pi_1 e_1 \in paths(\mathcal{C}_1)$ and so, by Definition 15(4), there exists $e_2$ such that $\pi_2 e_2 \in paths(\mathcal{C}_2)$ and $(\pi_1 e_1, \pi_2 e_2) \in R$. This last fact implies that $l(e_2) = a$; so, $set(\pi_2) \xrightarrow{a} set(\pi_2) \cup \{e_2\}$ and, by construction, $(set(\pi_1) \cup \{e_1\}, f \cup [e_1 \mapsto e_2], set(\pi_2) \cup \{e_2\}) \in S$.
3. Let $(set(\pi_1), f, set(\pi_2)) \in S$ and $X_1 \xrightarrow{a} set(\pi_1)$. Since $\mathcal{C}_1$ is rooted and connected, this implies the existence of a path $\pi_1' e_1 \in paths(\mathcal{C}_1)$ such that $l(e_1) = a$, $X_1 = set(\pi_1')$ and $set(\pi_1) = set(\pi_1' e_1)$. Since $\mathcal{C}_1$ is closed under bounded union, by Lemma 3, $\pi_1 \leftrightarrow^* \pi_1' e_1$. By Definition 14(3), there exists $\pi_2' e_2 \in paths(\mathcal{C}_2)$ such that $\pi_2 \leftrightarrow^* \pi_2' e_2$ (hence, $set(\pi_2) = set(\pi_2' e_2)$) and $(\pi_1' e_1, \pi_2' e_2) \in R$. This implies the presence of a triple $(set(\pi_1' e_1), f', set(\pi_2', e_2)) = (set(\pi_1), f', set(\pi_2)) \in S$. Moreover, since $\pi_i' e_i$ is obtained from $\pi_i$, for $i \in \{1, 2\}$, by applying the same sequence of swappings, we have that $f = f'$. By Definition 14(5), we then have $(\pi_1', \pi_2') \in R$. Thus $X_2 = set(\pi_2') \xrightarrow{a} set(\pi_2)$ and, by construction, $(X_1, f|_{X_1}, X_2) \in S$. $\square$

In order to prove an analogous result for history-preserving bisimilarity, let us denote with $\pi|_i$ the path formed by the first $i$ events of $\pi$; that is, if $\pi = e_1 \ldots e_i \ldots e_n$ (for $0 \leq i \leq n$), then $\pi|_i = e_1 \ldots e_i$.

**Proposition 4** *(Chp- vs php-bisimilarity). If two CSs are chpb-bisimilar, then they are phpb-bisimilar; the converse holds if the two CSs are closed under bounded union.*

**Proof.** (**chhp** $\Rightarrow$ **phhp**) The fact that chpb implies phpb can be proved like Proposition 3; the only difference is that here we have a chpb $(S^{\rightarrow}, S^{\leftarrow})$ and $R$ is defined by considering all the pairs $(X_1, f, X_2) \in S^{\rightarrow}$; the proof is then identical (apart from the last item, that of course is not needed), since $S^{\rightarrow}$ is also a bw-bisimulation because, by definition, $S^{\rightarrow} \subseteq S^{\leftarrow}$.

(**phhp** $\Rightarrow$ **chhp**) Let us assume that the CSs are closed under bounded union and prove that phpb implies chpb. Let $R$ be a phpb; then, we want to prove that the pair made up by

$$S^{\rightarrow} \triangleq \{(set(\pi_1), f, set(\pi_2)) \; : \; (\pi_1, \pi_2) \in R \;\wedge\; \forall i. f(\pi_1(i)) = \pi_2(i)\}$$

$$S^{\leftarrow} \triangleq \{(set(\pi_1|_i), f, set(\pi_2|_i)) \; : \; (\pi_1, \pi_2) \in R \;\wedge\; 0 \leq i \leq |\pi_1| \;\wedge\; \forall j \leq i. f(\pi_1(j)) = \pi_2(j)\}$$

is a chpb. By construction and like in Proposition 3, we have that $S^{\rightarrow} \subseteq S^{\leftarrow} \subseteq \mathcal{I}(\mathcal{C}_1, \mathcal{C}_2)$. Trivially, $(\emptyset, \emptyset, \emptyset) \in S^{\rightarrow}$, since $(\epsilon, \epsilon) \in R$. The proof that $S^{\rightarrow}$ is a fw-bisimulation can be done like in the proof of Proposition 3 (see point 2. of the first direction of that proof). We have to prove that $S^{\leftarrow}$ is a bw-bisimulation; to this aim, let $(X_1, f, X_2) \in S^{\leftarrow}$ and $X_1' \xrightarrow{a} X_1$. By construction, $X_1 = set(\pi_1|_i)$ and $X_2 = set(\pi_2|_i)$, for some $(\pi_1, \pi_2) \in R$ and $0 \leq i \leq |\pi_1|$. Since $\mathcal{C}_1$ is rooted and connected, the fact that $X_1' \xrightarrow{a} X_1 = set(\pi_1|_i)$ implies the existence of a path $\pi_1' e_1 \in paths(\mathcal{C}_1)$ such that $set(\pi_1') = X_1'$. Since $\mathcal{C}_1$ is closed under bounded union, by Lemma 3, $\pi_1|_i \leftrightarrow^* \pi_1' e_1$. By Definition 15(3), there exists $\pi_2' e_2 \in paths(\mathcal{C}_2)$ such that $set(\pi_2|_i) = set(\pi_2' e_2)$ and $(\pi_1' e_1, \pi_2' e_2) \in R$. This implies that $set(\pi_2') \xrightarrow{a} X_2$. Moreover, by construction of $S^{\leftarrow}$, we derive the presence of a triple $(set(\pi_1'), f', set(\pi_2')) = (X_1', f', X_2') \in S$. Since $\pi_j' e_j$ is obtained from $\pi_j|_i$, for $j \in \{1, 2\}$, by applying the same sequence of swappings, we have that $f' = f|_{X_1'}$, as desired. $\square$

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