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Lipschitz stability estimate for the simultaneous recovery of two coefficients in the anisotropic Schrödinger type equation via local Cauchy data

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ABSTRACT

We consider the inverse problem of the simultaneous identification of the coefficients σ and q of the equation $\mathrm{div}(\sigma\nabla u) + qu = 0$ from the knowledge of the Cauchy data set. We assume that $\sigma = \gamma A$, where A is a given matrix function and γ and q are unknown piecewise affine scalar functions. No sign, nor spectrum condition on q is assumed. We derive a result of global Lipschitz stability in dimension $n \geq 3$. The proof relies on the method of singular solutions and on the quantitative estimates of unique continuation.

1. Introduction

In this paper, we deal with the inverse problem of the *coefficient identification* in a Schrödinger type equation. Let Ω be a bounded domain of $\mathbb{R}^n$ and Σ be a non-empty portion of the boundary $\partial\Omega$. Let us denote by $u \in H^1(\Omega)$ a weak solution to the equation

$$\mathrm{div}(\sigma\nabla u) + q\, u = 0 \qquad \text{in } \Omega. \tag{1}$$

We denote with $u|_{\partial\Omega}$ the trace of the solution u to (1) at the boundary $\partial\Omega$ and $\sigma\nabla u \cdot \nu|_{\Omega}$ the trace of the conormal derivative of u at $\partial\Omega$. Here, ν is the outward unit normal of $\partial\Omega$, which is well-defined under appropriate assumptions of regularity at the boundary. The inverse problem consists in the simultaneous determination of the pair of coefficients σ and q from the knowledge of all the possible pairs of Cauchy data $(u|_{\partial\Omega}, \sigma\nabla u \cdot \nu|_{\partial\Omega})$ on the boundary Ω.

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The boundary value problem associated to (1) comprises a large class of inverse problems that are characterised by their ill-posed nature. Let us briefly review a selected collection that has motivated our interest in the stability issue for (1). The inverse problem of recovering only the coefficient σ when $q = 0$ from the knowledge of the Dirichlet to Neumann map is known as the Calderón problem, which was first introduced by A. Calderón in [23]. The uniqueness issue has been treated by Sylvester and Uhlmann in [45] for conductivities of class C^2 (see [46] for a complete survey). The stability issue was first investigated by Alessandrini in [3] for isotropic conductivities belonging to $H^s(\Omega)$ for $s > \frac{n}{2} + 2$. Under these assumptions, the author proved a stability estimate with logarithmic modulus of continuity. Mandache [39] proved that this estimate is indeed optimal under very general hypotheses. The ill-posed character in the inverse conductivity problem is a common denominator in this field and it constitutes an obstruction in numerical reconstructions. To reduce the ill-posed nature, it is convenient to restrict the space of admissible conductivities by imposing appropriate *a-priori* assumptions on the conductivity. In [9], Alessandrini and Vessella proved a Lipschitz stability estimate for piecewise constant conductivities defined on a finite partition of the domain Ω that satisfy certain *a-priori* bounds. Rondi [40] has proved that the Lipschitz constant appearing in the stability estimate [9, Theorem 2.7] behaves exponentially with respect to the number N of subdomains of the partition. This result was subsequently extended by Di Cristo and Rondi [25] for the inverse scattering problem and by Sincich [43] for the corrosion detection problem. Recently, in [2], Alberti et al. have extended these ideas by proving that for coefficients belonging to finite dimensional manifolds, uniqueness and stability are guaranteed. In this direction, Lipschitz stability estimates have been proved for real and complex finite dimensional isotropic coefficients ([9,6,17]), for a special type of anisotropic conductivities ([30,27]), for polyhedral inclusions in a conductive medium ([19,14,18]), for the non local operator ([41]), and for the elasticity case ([26]). As a disclaimer, we would like to remark that this list is far from being a complete collection of stability results that have been proved in the last decades. However, we would like to underline the fact that these results are all based on the singular solution method and unique continuation techniques.

When σ is the identity matrix, equation (1) is the Schrödinger equation. Lipschitz stability has been proved both when the Dirichlet to Neumann map is defined (hence under suitable spectral conditions) and when only Cauchy data are available, in the case of finite dimensional potential q (see [21,7,42]). When q has positive sign, (1) is the reduced wave equation or the Helmholtz equation. In [20], the authors succeeded in proving the conditional Lipschitz stability at selected frequencies, using the Dirichlet to Neumann map. See also [5] for the related numerical experiments.

When q is a non-positive scalar function, the boundary value problem associated to (1) models the propagation of light in a body and corresponds to the diffusion approximation of the radiative transfer equation in the frequency domain. In this framework, the coefficients σ and q model the diffusive and absorption coefficients, respectively. The corresponding application is the diffusive optical tomography (DOT), a novel, non-invasive technique that allows one to map the optical properties of a tissue (see [11,13,10]). In [12], Arridge and Lionheart proved that, under generic assumptions, it is not possible to simultaneously recover the diffusion and the absorption coefficients. However, later results showed that if the coefficients belong to a finite dimensional space of bounded functions, it is possible to determine the coefficients simultaneously. In [33], Harrach proved uniqueness under the assumption that the diffusion coefficient is piecewise constant and the absorption coefficient is piecewise analytic. The author used the technique of localised potentials, developed by the same author in [31], and monotonicity method also used in [34]. Recently, the method of localised potentials was successfully employed by Harrach and Lin ([35]) to recover piecewise analytic coefficients in a semilinear elliptic equation, under proper hypotheses that ensure the existence of the Dirichlet to Neumann map.

The aim of this work is to prove a Lipschitz stability estimate that holds simultaneously for both the coefficients. Lipschitz stability is derived by using a constructive approach based on the singular solution method and the quantitative estimates of unique continuation (see [9,16,7] and [47] for a recent survey).

We consider a partition $\{D_i\}_{i=1}^N$, $N \in \mathbb{N}$ of the domain Ω consisting of a finite number of bounded domains with boundary of class C^2. Notice that in previous works the boundary regularity was at most Lipschitz. In our context we need to impose a higher boundary regularity because we require that the boundary value problem (17) is well-posed and that a suitable version of the three sphere inequality proved in [24] can be applied.

We consider finite-dimensional coefficients of the form

$$\sigma(x) := \gamma(x)A(x) = \left(\sum_{j=1}^{N} \gamma_j(x)\chi_{D_j}(x)\right) A(x), \qquad q(x) := \sum_{j=1}^{N} q_j(x)\chi_{D_j}(x),$$

where γ_j, q_j are piecewise affine functions for $j = 1, \ldots, N$, and $A(x)$ is a known $C^{1,1}(\Omega, Sym_n)$ matrix function, with Sym_n the space of $n \times n$ real symmetric matrices. For simplicity, we denote with $\mathcal{C}_i$ the local Cauchy data set associated to the pairs of coefficients $\{\sigma^{(i)}, q^{(i)}\}_{i=1,2}$ and $d(\mathcal{C}_1, \mathcal{C}_2)$ denotes the distance between the sets (see equation (14)). In Theorem 2.2 we prove that

$$\|\sigma^{(1)} - \sigma^{(2)}\|_{L^\infty(\Omega)} + \|q^{(1)} - q^{(2)}\|_{L^\infty(\Omega)} \leq C\, d(\mathcal{C}_1, \mathcal{C}_2), \tag{2}$$

with $C > 0$ a constant that depends only on the a-priori data.

Our stability result is based on the method of singular solutions, whose application in the study of the stability in inverse problems dates back to [36,3].

The Cauchy problem associated with (1) could be in the eigenvalue regime, so the direct problem might not be well-posed. In [28, Lemma 4.1], the authors constructed Green's functions for a boundary value problem with prescribed complex-valued Robin data on a portion Σ_0 of the boundary. The original idea of using a complex-valued Robin type condition dates back to the work of Bamberger and Ha Duong [15]. They considered the exterior Robin problem for the reduced wave equation. The associated variational formulation satisfies the coercivity property, which is a necessary condition for well-posedness. Alessandrini et al. [7] applied a similar approach to establish the well-posedness for the Helmholtz type equation with a complex Robin condition and Dirichlet condition.

As in [4,6,7], we start the analysis by providing a boundary stability estimate of Hölder type for both the coefficients γ and q of the form

$$\|\sigma^{(1)} - \sigma^{(2)}\|_{L^\infty(\Sigma)} + \|q^{(1)} - q^{(2)}\|_{L^\infty(\Sigma)} \leq C\,(d(\mathcal{C}_1, \mathcal{C}_2) + E)^{1-\eta}\, d(\mathcal{C}_1, \mathcal{C}_2)^\eta, \tag{3}$$

for $0 < \eta < 1$, $E = \max\{\|\sigma^{(1)} - \sigma^{(2)}\|_{L^\infty(\Omega)}, \|q^{(1)} - q^{(2)}\|_{L^\infty(\Omega)}\}$, and $C > 0$ is a positive constant depending on the a-priori data only. Estimate (3) is derived by applying an Alessandrini's type argument, and the study of the blowup rate of the Green's function near the discontinuity interface.

We will now describe the iterative procedure to derive the Lipschitz stability estimate. We fix a chain of subdomains $D_0, D_1, \ldots, D_K$ of the partition of Ω so that, up to a reordering of indices, they are contiguous. The chain connects D_0 to the domain D_K where E is achieved. We adopt and generalise the iterative strategy introduced in [9] for the determination of one coefficient as follows. First, we determine the following Hölder type estimate for the two coefficients in D_1 in terms of the Cauchy data:

$$\|\gamma^{(1)} - \gamma^{(2)}\|_{L^\infty(D_1)} + \|q^{(1)} - q^{(2)}\|_{L^\infty(D_1)} \leq C\,(E + \varepsilon)\left(\frac{\varepsilon}{E + \varepsilon}\right)^{\tilde{\eta}_1},$$

where $0 < \tilde{\eta}_1 < 1$ depends on the a-priori data only and $\varepsilon = d(\mathcal{C}_1, \mathcal{C}_2)$. Then, we apply the following two step procedure.

i) We determine an upper bound for $\|\gamma_2^{(1)} - \gamma_2^{(2)}\|_{L^\infty(D_2)}$ using the asymptotic estimates of the singular solutions near the discontinuous interface and the a-priori information on the $q^{(i)}$, for $i = 1, 2$. We apply quantitative estimates of unique continuation (see Proposition 3.3), which are based on propagation of smallness estimates proved by Carstea and Wang in [24] that hold for piecewise Lipschitz coefficients.
ii) We estimate $\|q_2^{(1)} - q_2^{(2)}\|_{L^\infty(D_2)}$ by taking advantage of the stability estimate in *i*), the asymptotic estimates for the Green functions and the quantitative estimates of unique continuation.

Proceeding iteratively along the chain of subdomains up to D_K, we derive the following inequality:

$$\|\gamma^{(1)} - \gamma^{(2)}\|_{L^\infty(D_K)} + \|q^{(1)} - q^{(2)}\|_{L^\infty(D_K)} \le C\,(E + \varepsilon)\,\omega_{\tilde{\eta}_K}^{(3(K-1))}\left(\frac{\varepsilon}{E+\varepsilon}\right), \tag{4}$$

with $0 < \tilde{\eta}_K < 1$ a constant that depends on the a-priori data only, and $\omega_{\tilde{\eta}_K}$ is a modulus of continuity of logarithmic type of the form

$$\omega_{\tilde{\eta}_K}(t) \le C\,|\ln t^{-1}|^{-\tilde{\eta}_K} \quad \text{for } t \in (0,1).$$

The Lipschitz stability estimate (2) is deduced by (4) and the fact that $\omega_{\tilde{\eta}_K}$ is invertible.

The article is organized as follows. In Section 2, we introduce the a-priori assumptions on the domain Ω and the coefficients σ, q. After defining the local Cauchy data, we state the stability result (Theorem 2.2) and Corollary 2.3. In Section 3, we introduce the main tools needed to prove the theorem, namely the asymptotic estimates for the Green's function near the discontinuity interface (Proposition 3.2) and the quantitative estimates of the unique continuation (Proposition 3.3). In Section 4, we prove the Lipschitz stability estimate (Theorem 2.2). In Section 5, we give a sketch of the proofs of the technical Propositions introduced in Section 3. In the Appendix, we prove the stability at the boundary for both the coefficients σ and q.

2. Notation and main result

In this section, we recall the main definitions and summarise the *a-priori* information concerning the domain Ω and the coefficients σ, q of (1). Then, we state the Lipschitz stability estimate (Theorem 2.2).

For a point $x \in \mathbb{R}^n$, we can write $x = (x', x_n)$, where $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$. For any $x \in \mathbb{R}^n$, $B_r(x), B'_r(x')$ denote the open balls in $\mathbb{R}^n, \mathbb{R}^{n-1}$ centred in x and x' respectively with radius $r > 0$. Let $B_r = B_r(0)$ and $B'_r = B'_r(0)$. Let $\mathbb{R}^n_\pm = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \,:\, x_n \gtrless 0\}$ the positive (negative) real half-space, $B_r^\pm = B_r \cap \mathbb{R}^n_\pm$ the positive (negative) half-ball centred in the origin.

Definition 2.1. We say that $\Omega \subset \mathbb{R}^n$ has the *boundary of class C^2 with constants $r_0, L > 0$* if for each point $P \in \partial\Omega$ there exists a rigid transformation under which P coincides with the origin 0 and

$$\Omega \cap B_{r_0} = \{x \in B_{r_0} \,:\, x_n > \varphi(x')\},$$

where φ is a C^2 function on $\overline{B'_{r_0}}$ such that

$$\varphi(0) = |\nabla\varphi(0)| = 0 \quad \text{and} \quad \|\varphi\|_{C^2(B'_{r_0})} \le L r_0,$$

with

$$\|\varphi\|_{C^2(\overline{B'_{r_0}})} = \sum_{|\beta| \le 2} r_0^{|\beta|} \|\partial^\beta \varphi\|_{L^\infty(B'_{r_0})},$$

with $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}_0^n$, $\partial^\beta = \partial_{x_1}^{\beta_1} \partial_{x_2}^{\beta_2} \dots \partial_{x_n}^{\beta_n}$.

Definition 2.2. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. A boundary portion Σ of $\partial\Omega$ is said to be a *flat portion* of size $r_0 > 0$ if for each point $P \in \Sigma$ there exists a rigid transformation under which P coincides with the origin 0 and

$$\Sigma \cap B_{r_0} = \{x \in B_{r_0} : x_n = 0\}, \qquad \Omega \cap B_{r_0} = \{x \in B_{r_0} : x_n > 0\}.$$

2.1. A-priori information on the domain

Consider $\Omega \subset \mathbb{R}^n$ a bounded, measurable domain with boundary $\partial\Omega$ of class C^2 with constants r_0, L such that

$$|\Omega| \leq C r_0^n, \tag{5}$$

where $|\Omega|$ denotes the Lebesgue measure of Ω and C is a positive constant. Let $\Sigma \subset \partial\Omega$ be a flat portion of size r_0. We assume that there exists a partition of bounded domains $\{D_m\}_{m=1}^N$, $N \in \mathbb{N}$, $N > 1$, contained in Ω that satisfies the following conditions:

(D1) Each D_m for $m = 1, \dots, N$ is connected with boundaries ∂D_m of class C^2 with constants r_0, L. These domains are pairwise non-overlapping.

(D2) $\overline{\Omega} = \bigcup_{m=1}^{N} \overline{D}_m$.

(D3) There exists a region, denoted by D_1, such that the intersection $\partial D_1 \cap \Sigma$ contains a flat portion Σ_1 of size r_0. For any index $m \in \{2, \dots, N\}$ we assume that the intersection

$$\partial D_m \cap \partial D_{m+1}$$

contains a flat portion Σ_{m+1} of size r_0 such that $\Sigma_{m+1} \subset \Omega$. Furthermore, we assume that there exist a point $P_{m+1} \in \Sigma_{m+1}$ and a rigid transformation under which P_{m+1} coincides with the origin 0 and

$$\Sigma_{m+1} \cap B_{r_0/3} = \left\{x \in B_{r_0/3} : x_n = 0\right\}, \qquad D_m \cap B_{r_0/3} = \left\{x \in B_{r_0/3} : x_n < 0\right\}.$$

Notice that since the boundary is of class C^2, for each pair of contiguous subdomains one can simply consider a local diffeomorphism that flattens the boundary. However, in view of proving the stability estimate, it is convenient to give such assumption for granted.

2.2. A-priori information on the coefficients

Consider the elliptic equation

$$\operatorname{div}(\sigma \nabla u) + qu = 0 \quad \text{in } \Omega. \tag{6}$$

The coefficient σ is a bounded, measurable $n \times n$ real matrix function of the form

$$\sigma(x) = \gamma(x)\, A(x), \qquad x \in \Omega, \tag{7}$$

that satisfies the following conditions:

(C1) The scalar function γ is piecewise affine and has the form

$$\gamma(x) = \sum_{j=1}^{N} \gamma_j(x)\chi_{D_j}(x), \qquad \gamma_j(x) = a_j + b_j \cdot x, \qquad x \in \Omega$$

for $a_j \in \mathbb{R}$, $b_j \in \mathbb{R}^n$ and D_j for $j = 1, \dots, N$ are the given subdomains of the partition as in Section 2.1. Moreover, there exists a constant $\bar{\gamma} > 1$ such that for a.e. $x \in \Omega$,

$$\bar{\gamma}^{-1} \leq \gamma_j(x) \leq \bar{\gamma}, \qquad \text{for } j = 1, \dots, N. \tag{8}$$

(C3) The matrix function A belongs to the space $C^{1,1}(\Omega, Sym_n)$ and there is a constant $\bar{A} > 0$ such that

$$\|a_{ij}\|_{C^{1,1}(\Omega)} \leq \bar{A} \qquad \text{for } i, j = 1, \dots, n, \tag{9}$$

where

$$\|a_{ij}\|_{C^{1,1}(\Omega)} = \|a_{ij}\|_{C^1(\Omega)} + r_0 \sup_{x,y\in\Omega, x\neq y} \frac{|\nabla a_{ij}(x) - \nabla a_{ij}(y)|}{|x-y|}.$$

(C4) (*Uniform ellipticity condition*) There exists a constant $\bar{\lambda} > 1$ such that

$$\bar{\lambda}^{-1}|\xi|^2 \leq A(x)\xi \cdot \xi \leq \bar{\lambda}|\xi|^2 \qquad \text{for every } \xi \in \mathbb{R}^n, \text{ for a.e. } x \in \Omega. \tag{10}$$

(C5) The coefficient $q \in L^\infty(\Omega)$ is a piecewise affine function of the form

$$q(x) = \sum_{j=1}^{N} q_j(x)\chi_{D_j}(x), \qquad q_j(x) = c_j + d_j \cdot x, \qquad x \in \Omega,$$

for $c_j \in \mathbb{R}$, $d_j \in \mathbb{R}^n$ and D_j for $j = 1, \dots, N$ are the given subdomains of the partition as in Section 2.1.

(C6) There are $\bar{\sigma}, \bar{q} > 0$ such that

$$\|\sigma\|_{L^\infty(\Omega)} \leq \bar{\sigma}, \qquad \|q\|_{L^\infty(\Omega)} \leq \bar{q}. \tag{11}$$

The collection of constants $\{r_0, L, N, \bar{\lambda}, \bar{\gamma}, \bar{\sigma}, \bar{A}, \bar{q}\}$ along with the dimension $n \geq 3$ are called the *a-priori data*. We would like to remark here that we decide to follow the so-called *constant variable convention*, which consists in denoting with the letter C all the positive constants that depend on the a-priori data only and that may vary from line to line in the inequalities.

Remark 2.1. The class of functions $\gamma(x), q(x)$ form a finite dimensional linear subspace. The L^∞ norms of γ, q are equivalent to the following norms:

$$|||\gamma||| = \max_{j=1,\dots,N}\{|a_j| + |b_j|\}, \qquad |||q||| = \max_{j=1,\dots,N}\{|c_j| + |d_j|\},$$

modulo some constants depending on the a-priori data.

2.3. Local Cauchy data set

Before describing the local Cauchy data, we recall the definition of some useful trace spaces. Let $H^{1/2}_{co}(\Sigma)$ be the trace space of functions having compact support in Σ. The space $H^{1/2}_{00}(\Sigma)$ is the closure of $H^{1/2}_{co}(\Sigma)$

under the norm $H^{1/2}(\partial\Omega)$. The distributional space $H^{-1/2}(\partial\Omega)|_\Sigma$ is the restriction of the trace space of distributions $H^{-1/2}(\partial\Omega)$ to Σ.

For $f \in H^{1/2}_{00}(\Sigma)$, the boundary value problem

$$\begin{cases} \operatorname{div}(\sigma\nabla u) + q\,u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega, \end{cases} \tag{12}$$

may have no unique solution, since we do not make any assumption on the sign of q. In this general framework, the Dirichlet-to-Neumann map may not be defined. As in [7] (see also [37, § 5, pag. 152]), we find it convenient to introduce a set to model the pairs $(u|_{\partial\Omega}, \sigma\nabla u \cdot \nu|_{\partial\Omega})$.

Definition 2.3. The *local Cauchy data* $(u|_{\partial\Omega}, \sigma\nabla u \cdot \nu|_{\partial\Omega})$ associated to σ, q having zero first component on $\partial\Omega \setminus \overline{\Sigma}$ is the set

$$\begin{aligned} \mathcal{C}_{\sigma,q}(\Sigma) = \Big\{ (f,g) \in H^{1/2}_{00}(\Sigma) \times H^{-1/2}(\partial\Omega)|_\Sigma \,:\, & \text{there exists } u \in H^1(\Omega) \text{ such that} \\ & \operatorname{div}(\sigma\nabla u) + qu = 0 \quad \text{in } \Omega, \\ & u|_{\partial\Omega} = f, \\ & \langle \sigma\nabla u \cdot \nu|_{\partial\Omega}, \varphi\rangle = \langle g, \varphi\rangle \quad \text{for any } \varphi \in H^{1/2}_{00}(\Sigma) \Big\}. \end{aligned}$$

Notice that $\mathcal{C}_{\sigma,q}(\Sigma)$ is a subset of $H^{1/2}_{00}(\Sigma) \times H^{-1/2}(\partial\Omega)|_\Sigma$, which is a Hilbert space with the norm

$$\|(f,g)\|_{H^{1/2}_{00}(\Sigma) \bigoplus H^{-1/2}(\partial\Omega)|_\Sigma} = \left(\|f\|^2_{H^{1/2}_{00}(\Sigma)} + \|g\|^2_{H^{-1/2}(\partial\Omega)|_\Sigma} \right)^{1/2}.$$

If S_1, S_2 are two closed subspaces of a given Hilbert space $\mathcal{H}$, the distance between S_1 and S_2 is defined as

$$d(S_1,S_2) = \max\left\{ \inf_{h\in S_2\setminus\{0\}} \sup_{k\in S_1} \frac{\|h-k\|_{\mathcal{H}}}{\|h\|_{\mathcal{H}}}, \inf_{k\in S_1\setminus\{0\}} \sup_{h\in S_2} \frac{\|h-k\|_{\mathcal{H}}}{\|k\|_{\mathcal{H}}} \right\}. \tag{13}$$

If $d(S_1, S_2) < 1$ then

$$d(S_1,S_2) = \inf_{h\in S_2\setminus\{0\}} \sup_{k\in S_1} \frac{\|h-k\|_{\mathcal{H}}}{\|h\|_{\mathcal{H}}},$$

(see [38] and [7]). In our framework, for two pairs of coefficients $\{\sigma^{(k)}, q^{(k)}\}$ with $k = 1, 2$, the corresponding local Cauchy data are the sets $\mathcal{C}_1 = \mathcal{C}_{\sigma^{(1)},q^{(1)}}(\Sigma)$ and $\mathcal{C}_2 = \mathcal{C}_{\sigma^{(2)},q^{(2)}}(\Sigma)$. Since we are interested in the occurrence when $\mathcal{C}_1$ and $\mathcal{C}_2$ are rather close, the distance is given by

$$d(\mathcal{C}_1,\mathcal{C}_2) = \inf_{(f_2,g_2)\in\mathcal{C}_2\setminus\{(0,0)\}} \sup_{(f_1,g_1)\in\mathcal{C}_1} \frac{\|(f_2,g_2)-(f_1,g_1)\|_{\mathcal{H}}}{\|(f_2,g_2)\|_{\mathcal{H}}}, \tag{14}$$

with $\mathcal{H} = H^{1/2}_{00}(\Sigma) \oplus H^{-1/2}(\partial\Omega)|_\Sigma$. Notice that the above distance is computed between the closures $\overline{\mathcal{C}}_1$ and $\overline{\mathcal{C}}_2$ and it can be shown that the Cauchy data $\mathcal{C}_1, \mathcal{C}_2$ are indeed closed. Moreover, if the direct problem is well-posed, then the local Cauchy data represents the graph of the local Dirichlet to Neumann map.

We state the stability estimate, the proof of which is deferred to Section 4.

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^n$, $\Sigma \subset \partial\Omega$ be a bounded domain and a non-empty portion as in Section 2.1. Let $\{\sigma^{(i)}, q^{(i)}\}$ for $i = 1, 2$ be two pairs of parameters that satisfy the a-priori assumptions in Section 2.2. Let*

$\mathcal{C}_1, \mathcal{C}_2$ be the corresponding local Cauchy data and assume that $d(\mathcal{C}_1, \mathcal{C}_2) < 1$. Then there exists a constant $C > 0$ depending on the a-priori data only such that

$$\|\sigma^{(1)} - \sigma^{(2)}\|_{L^\infty(\Omega)} + \|q^{(1)} - q^{(2)}\|_{L^\infty(\Omega)} \leq C\, d(\mathcal{C}_1, \mathcal{C}_2). \tag{15}$$

The following corollary is a straightforward consequence of the proof of Theorem 2.2, hence we state the result and omit the proof.

Corollary 2.3. *Under the assumptions of Theorem 2.2, there exist constants $C > 0$ and $0 < \eta < 1$ depending on the a-priori data only such that*

$$\|\sigma^{(1)} - \sigma^{(2)}\|_{L^\infty(\Sigma)} + \|q^{(1)} - q^{(2)}\|_{L^\infty(\Sigma)} \leq C\, (d(\mathcal{C}_1, \mathcal{C}_2) + E)^{1-\eta}\, d(\mathcal{C}_1, \mathcal{C}_2)^\eta, \tag{16}$$

with $E = \max\{\|\sigma^{(1)} - \sigma^{(2)}\|_{L^\infty(\Sigma)}, \|q^{(1)} - q^{(2)}\|_{L^\infty(\Sigma)}\}$.

3. Auxiliary propositions

The proof of Theorem 2.2 is based on the method of singular solutions and the quantitative estimates of unique continuation. In this section, we introduce the main tools and propositions needed. In Section 3.1, we define the Green's functions and we describe their asymptotic behaviour near the discontinuity interfaces. The Green functions are weak solutions to a well-posed boundary value problem defined on an enlarged domain Ω_0 with impedance boundary data prescribed on a small portion of $\partial\Omega_0$ that is not contained in $\partial\Omega$. In Section 3.2, we introduce the singular integrals and the quantitative estimates of propagation of smallness.

3.1. Green functions and asymptotic estimates

We recall that by the a-priori assumptions on the domain (Section 2.1), there exists a point $P_1 \in \Sigma_1$ such that, up to a rigid transformation, we have that P_1 coincides with the origin. Without loss of generality, we can assume that $\Sigma = \Sigma_1$. We define

$$D_0 = \left\{ x \in (\mathbb{R}^n \setminus \Omega) \cap B_{r_0} \;:\; |x_i| < \frac{2}{3} r_0, \text{ for } i = 1, \dots, n-1, \left| x_n - \frac{r_0}{6} \right| < \frac{5}{6} r_0 \right\} .$$

The enlarged domain is defined as

$$\Omega_0 = \mathrm{Int}_{\mathbb{R}^n}(\overline{\Omega \cup D_0}) .$$

The set Ω_0 is a bounded domain with boundary of Lipschitz class of constants $r_0/3$ and $\tilde{L}$, where $\tilde{L}$ depends on L. Moreover, we introduce the following sets

$$\Sigma_0 = \left\{ x \in \partial\Omega_0 \setminus \partial\Omega \;:\; |x_i| < \frac{2}{3} r_0, \text{ for } i = 1, \dots, n-1, \; x_n = -\frac{2}{3} r_0 \right\} ,$$

$$(\Omega_0)_r = \{ x \in \Omega_0 \;:\; \mathrm{dist}(x, \partial\Omega_0) \geq r \} \qquad \text{for some } r \in (0, r_0/6) .$$

Let $\{\sigma, q\}$ be a pair of coefficients that satisfies the assumptions of Section 2.2. We extend them on D_0 by setting $\sigma|_{D_0} = Id_n$, $\gamma|_{D_0} = 1$, and $q|_{D_0} = 1$, where Id_n denotes the $n \times n$ identity matrix. With an abuse of notation, we denote with the same letters the two extended coefficients when we deal with the enlarged domain Ω_0.

Let G be the Green function associated to the elliptic operator $\mathrm{div}(\sigma\nabla\cdot) + q\cdot$. For any $y \in \Omega_0$, let $G(\cdot, y)$ be the unique distributional solution to the problem

$$\begin{cases} \mathrm{div}(\sigma\nabla G(\cdot,y)) + qG(\cdot,y) = -\delta(\cdot - y) & \text{in } \Omega_0, \\ G(\cdot,y) = 0 & \text{on } \partial\Omega_0 \setminus \Sigma_0, \\ \sigma\nabla G(\cdot,y)\cdot\nu + iG(\cdot,y) = 0 & \text{on } \Sigma_0. \end{cases} \tag{17}$$

Moreover, there exists a positive constant C that depends on λ, n only such that

$$0 < |G(x,y)| < C|x-y|^{2-n}, \qquad \text{for every } x, y \in \Omega_0,\ x \neq y. \tag{18}$$

The proof of the well-posedness of this problem can be found in [28, Lemma 4.1].

Proposition 3.1. *For all $y \in \Omega_0$ and every $r > 0$, the following inequality holds:*

$$\int_{\Omega_0 \setminus B_r(y)} |\nabla G(\cdot,y)|^2 \leq C\, r^{2-n}, \tag{19}$$

where C is a positive constant depending on the a-priori data.

Proof. The proof can be derived by combining the Caccioppoli inequality with equation (18). □

Fix an index $m \in \{0, \dots, N-1\}$, let $P_{m+1} \in \Sigma_{m+1}$ and assume that, up to a rigid transformation, P_{m+1} coincides with the origin 0 and Σ_{m+1} is a flat hyperplane of size r_0. Define the following quantities:

$$\gamma^+ = \gamma_{m+1}(0), \qquad \gamma^- = \gamma_m(0), \qquad J = \sqrt{A(0)^{-1}}, \qquad |J| = \det J.$$

The fundamental solution H associated to the elliptic operator $\mathrm{div}((\gamma^-\chi_{\mathbb{R}^n_-}(\cdot) + \gamma^+\chi_{\mathbb{R}^n_+}(\cdot))A(0)\nabla\cdot)$ in $\mathbb{R}^n$ is given by the formula

$$H(x,y) = |J| \begin{cases} \frac{1}{\gamma^+}\Gamma(Lx,Ly) + \frac{\gamma^+-\gamma^-}{\gamma^+(\gamma^++\gamma^-)}\Gamma(Lx,(Ly)^*) & \text{if } x_n, (Ly)_n > 0, \\ \frac{2}{\gamma^++\gamma^-}\Gamma(Lx,Ly) & \text{if } x_n \cdot (Ly)_n < 0, \\ \frac{1}{\gamma^-}\Gamma(Lx,Ly) + \frac{\gamma^--\gamma^+}{\gamma^-(\gamma^++\gamma^-)}\Gamma(Lx,(Ly)^*) & \text{if } x_n, (Ly)_n < 0, \end{cases} \tag{20}$$

where $L : \mathbb{R}^n \to \mathbb{R}^n$ is a linear map such that $L^{-1} \cdot (L^{-1})^T = A(0)$ (see [27] or [30]), and $y* = (y', -y_n)$.

Proposition 3.2. *Fix $m \in \{0, \dots, N-1\}$. Let $Q_{m+1} \in B_{r_0/4}(P_{m+1}) \cap \Sigma_{m+1}$, where Σ_{m+1} is the flat portion defined in Section 2.1. For $r \in (0, r_0/8)$, set $y_{m+1} = Q_{m+1} - r\nu(Q_{m+1})$, where $\nu(Q_{m+1})$ is the outward unit normal of ∂D_m at Q_{m+1} and let $x \in B_{r_0/4}(Q_{m+1}) \cap D_{m+1}$. Then there exist C_1, C_2, C_3, C_4 positive constants, $0 < \theta_1, \theta_2, \theta_3 < 1$ that depend on the a-priori data only such that*

$$|\nabla_x G(x,y_{m+1}) - \nabla_x H(x,y_{m+1})| \leq C_1\, |x - y_{m+1}|^{1-n+\theta_1}, \tag{21}$$

$$|\nabla_x\nabla_y G(x,y_{m+1}) - \nabla_x\nabla_y H(x,y_{m+1})| \leq C_2\, |x - y_{m+1}|^{-n+\theta_2}, \tag{22}$$

$$|\nabla_y G(x,y_{m+1}) - \nabla_y H(x,y_{m+1})| \leq C_3\, |x - y_{m+1}|^{1-n+\theta_3}, \tag{23}$$

$$|\nabla_y^2 G(x,y_{m+1}) - \nabla_y^2 H(x,y_{m+1})| \leq C_4\, |x - y_{m+1}|^{1-n}. \tag{24}$$

For a proof of (21) and (22), see [28, Proposition 4.3], [27, Proposition 3.1] and [7, Proposition 3.4]. For (23) and (24), see Section 5.

3.2. Quantitative estimates of unique continuation

Consider the following sets:

$$\mathcal{U}_0 = \Omega, \qquad \mathcal{W}_k = \bigcup_{m=0}^{k} D_m, \qquad \mathcal{U}_k = \Omega \setminus \overline{\mathcal{W}_k}, \quad \text{for } k = 1, \dots, N.$$

For $y, z \in \mathcal{W}_k$, define the singular solution

$$S_k(y,z) = \int_{\mathcal{U}_k} (\sigma^{(1)} - \sigma^{(2)})(x) \nabla_x G_1(x,y) \cdot \nabla_x G_2(x,z) \, \mathrm{d}x + \\ + \int_{\mathcal{U}_k} (q^{(2)} - q^{(1)})(x) G_1(x,y) G_2(x,z) \, \mathrm{d}x, \tag{25}$$

where G_j are the weak solutions to (17). Moreover, the following partial derivatives are well defined:

$$\partial_{y_i} \partial_{z_j} S_k(y,z) = \int_{\mathcal{U}_k} (\sigma^{(1)} - \sigma^{(2)})(x) \partial_{y_i} \nabla_x G_1(x,y) \cdot \partial_{z_j} \nabla_x G_2(x,z) \, \mathrm{d}x + \\ + \int_{\mathcal{U}_k} (q^{(2)} - q^{(1)})(x) \partial_{y_i} G_1(x,y) \partial_{z_j} G_2(x,z) \, \mathrm{d}x, \tag{26}$$

and

$$\partial^2_{y_i y_j} \partial^2_{z_i z_j} S_k(y,z) = \int_{\mathcal{U}_k} (\sigma^{(1)} - \sigma^{(2)})(x) \partial^2_{y_i y_j} \nabla_x G_1(x,y) \cdot \partial^2_{z_i z_j} \nabla_x G_2(x,z) \, \mathrm{d}x + \\ + \int_{\mathcal{U}_k} (q^{(2)} - q^{(1)})(x) \partial^2_{y_i y_j} G_1(x,y) \partial^2_{z_i z_j} G_2(x,z) \, \mathrm{d}x, \tag{27}$$

for $i, j = 1, \dots, n$. For $y, z \in \mathcal{W}_k$, one can show that $S_k(\cdot, z), S_k(y, \cdot) \in H^1_{loc}(\mathcal{W}_k)$ and are weak solutions, respectively, to

$$\mathrm{div}_y(\sigma^{(1)} \nabla_y S_k(\cdot, z)) + q^{(1)} S_k(\cdot, z) = 0 \qquad \text{in } \mathcal{W}_k,$$
$$\mathrm{div}_z(\sigma^{(2)} \nabla_z S_k(y, \cdot)) + q^{(2)} S_k(y, \cdot) = 0 \qquad \text{in } \mathcal{W}_k,$$

(see [9, Proposition 3.3]). Set

$$E := \max\{\|\gamma^{(1)} - \gamma^{(2)}\|_{L^\infty(\Omega)}, \|q^{(1)} - q^{(2)}\|_{L^\infty(\Omega)}\}.$$

Notice that by Proposition 3.1, for all $y, z \in \mathcal{W}_k$,

$$|S_k(y,z)| \leq C\, E\, (d(y) d(z))^{1 - \frac{n}{2}},$$

where $d(y)$ is the distance of y from $\mathcal{U}_k$ and C is a positive constant that depends on the a-priori data.

The following proposition introduces the quantitative estimates of unique continuation for the singular integrals.

Proposition 3.3. *Suppose that for some positive ε_0 we have*

$$|S_k(y,z)| \le r_0^{2-n}\varepsilon_0, \quad \text{for every } (y,z) \in D_0 \times D_0. \tag{28}$$

Then there exist $\bar{r} > 0$, $C_5, C_6, C_7 > 0$ constants that depend on the a-priori data only such that the following inequalities hold true for every $r \in (0, \bar{r}/8)$:

$$|S_k\left(y_{k+1}, y_{k+1})\right)| \le C_5 r^{-2\tilde{\gamma}} \left(\frac{\varepsilon_0}{\varepsilon_0 + E}\right)^{\tau_r^2 \beta^{2N_1}} (\varepsilon_0 + E), \tag{29}$$

$$\left|\partial_{y_j}\partial_{z_i} S_k\left(y_{k+1}, y_{k+1}\right)\right| \le C_6 r^{-2\tilde{\gamma}-2} \left(\frac{\varepsilon_0}{\varepsilon_0 + E}\right)^{\tau_r^2 \beta^{2N_1}} (\varepsilon_0 + E), \tag{30}$$

$$|\partial^2_{y_i y_j}\partial^2_{z_i z_j} S_k(y_{k+1}, y_{k+1})| \le C_7 r^{-2\tilde{\gamma}-4} \left(\frac{\varepsilon_0}{\varepsilon_0 + E}\right)^{\tau_r^2 \beta^{2N_1}} (\varepsilon_0 + E), \tag{31}$$

for any $i, j = 1, \dots, n$, $y_{k+1} = P_{k+1} - r\nu(P_{k+1})$, where $\nu(P_{k+1})$ is the exterior unit normal to ∂D_k at the point P_{k+1}, $\tilde{\gamma} = \frac{n}{2} - 1$, $0 < \beta < 1$, $N_1 \in \mathbb{N}$ and, for $r_1 = \bar{r}/8$,

$$\tau_r = \ln\left(\frac{12r_1 - 2r}{12r_1 - 3r}\right) / \ln\left(\frac{6r_1 - r}{2r_1}\right) .$$

Remark 3.4. Notice that since

$$\frac{\tau_r}{r} \ge \frac{1}{12 r_1 \ln 3}, \tag{32}$$

one can replace τ_r with r in Proposition 3.3.

For $\eta > 0$, let $\omega_\eta(t)$ be the non-decreasing function defined on $[0, +\infty)$ as

$$\omega_\eta(t) = \begin{cases} 2^\eta e^{-2} |\ln t|^{-\eta}, & t \in (0, e^{-2}), \\ e^{-2}, & t \in [e^{-2}, +\infty). \end{cases} \tag{33}$$

Recall that

$$[0, +\infty) \ni t \to t\omega_\eta\left(\frac{1}{t}\right) \in [0, +\infty) \quad \text{is a non-decreasing function}$$

and for $\beta \in (0,1)$,

$$\omega_\eta\left(\frac{t}{\beta}\right) \le |\ln e\beta^{-1/2}|^\eta \omega_\eta(t), \qquad \omega_\eta(t^\beta) \le \left(\frac{1}{\beta}\right)^\eta \omega_\eta(t).$$

We set $\omega_\eta^{(0)} = t^\eta$ for $0 < \eta < 1$. We denote the iterated composition of ω with itself as

$$\omega_\eta^{(1)} = \omega_\eta, \quad \omega_\eta^{(j)} = \omega_\eta \circ \omega_\eta^{(j-1)} \quad \text{for } j = 2, 3, \dots$$

4. Proof of Theorem 2.2

Before proving Theorem 2.2, we recall some useful formulas. Let $u_i \in H^1(\Omega)$ for $i = 1, 2$ be two weak solutions to

$$\operatorname{div}(\sigma^{(i)}\nabla u_i) + q^{(i)}u_i = 0 \quad \text{in } \Omega,$$

with $u_i|_{\partial\Omega} \in H^{1/2}_{00}(\Sigma)$. By the weak formulation for $i = 1, 2$, one derives

$$\int_\Omega [(\sigma^{(1)} - \sigma^{(2)})(x)\nabla u_1 \cdot \nabla u_2 + (q^{(2)} - q^{(1)})(x)u_1(x)u_2(x)] \,\mathrm{d}x = $$
$$= \langle \sigma^{(2)}\nabla \bar{u}_2 \cdot \nu, u_1\rangle - \langle \sigma^{(1)}\nabla u_1 \cdot \nu, u_2\rangle. \tag{34}$$

Let $v_i \in H^1(\Omega)$ a different solution to $\operatorname{div}(\sigma^{(i)}\nabla v_i) + q^{(i)}v_i = 0$ in Ω with $v_i \in H^{1/2}_{00}(\Sigma)$, one derives the following identity:

$$\langle \sigma^{(i)}\nabla v_i \cdot \nu, \bar{u}_i - \langle \sigma^{(i)}\nabla \bar{u}_i \cdot \nu, \bar{v}_i\rangle = 0. \tag{35}$$

By setting $i = 2$ in (35) and by summing up (34) with (35) one derives

$$\int_\Omega [(\sigma^{(1)} - \sigma^{(2)})(x)\nabla u_1(x) \cdot \nabla u_2(x) + (q^{(2)} - q^{(1)})(x)u_1(x)u_2(x)] \,\mathrm{d}x =$$
$$= \langle \sigma^{(2)}\nabla \bar{u}_2 \cdot \nu, (u_1 - v_2)\rangle - \langle \sigma^{(1)}\nabla u_1 \cdot \nu - \sigma^{(2)}\nabla v_2 \cdot \nu, \bar{u}_2\rangle. \tag{36}$$

If one takes the modulus in (36), one obtains the following inequality:

$$\Big|\int_\Omega [(\sigma^{(1)} - \sigma^{(2)})(x)\nabla u_1(x) \cdot \nabla u_2(x) + (q^{(2)} - q^{(1)})(x)u_1(x)u_2(x)] \,\mathrm{d}x\Big| \leq$$
$$\leq d(\mathcal{C}_1, \mathcal{C}_2)\, \|(u_1, \sigma^{(1)}\nabla u_1 \cdot \nu)\|_{\mathcal{H}}\, \|(\bar{u}_2, \sigma^{(2)}\nabla \bar{u}_2 \cdot \nu)\|_{\mathcal{H}}. \tag{37}$$

Proof of Theorem 2.2. Let $\{\sigma^{(i)}, q^{(i)}\}$ for $i = 1, 2$ be two pairs of coefficients that satisfy the assumptions of Section 2.2 and let $\mathcal{C}_1, \mathcal{C}_2$ be the corresponding local Cauchy data. Due to the nature of the leading order coefficient, by (9), the following inequality

$$\|\sigma^{(1)} - \sigma^{(2)}\|_{L^\infty(\Omega)} \leq Cd(\mathcal{C}_1, \mathcal{C}_2)$$

is equivalent to

$$\|\gamma^{(1)} - \gamma^{(2)}\|_{L^\infty(\Omega)} \leq Cd(\mathcal{C}_1, \mathcal{C}_2),$$

where $C > 1$ is a constant that depends on $\bar{A}$ and the other a-priori data.

For $K \in \{1, \dots, N\}$, let D_K be the subdomain of the known partition of Ω such that

$$\|\gamma^{(1)} - \gamma^{(2)}\|_{L^\infty(\Omega)} = \|\gamma^{(1)}_K - \gamma^{(2)}_K\|_{L^\infty(D_K)}.$$

Similarly, for $\tilde{K} \in \{1, \dots, N\}$, let $D_{\tilde{K}}$ be such that

$$\|q^{(1)} - q^{(2)}\|_{L^\infty(\Omega)} = \|q^{(1)}_{\tilde{K}} - q^{(2)}_{\tilde{K}}\|_{L^\infty(D_{\tilde{K}})}.$$

Our goal is to prove that

$$\|q_{\tilde{K}}^{(1)} - q_{\tilde{K}}^{(2)}\|_{L^\infty(D_{\tilde{K}})} + \|\gamma_K^{(1)} - \gamma_K^{(2)}\|_{L^\infty(D_K)} \leq Cd(\mathcal{C}_1, \mathcal{C}_2).$$

Let Ω_0 be the augmented domain and let $\sigma^{(i)}$ and $q^{(i)}$ for $i = 1, 2$ be the extended coefficient on D_0, with $\sigma^{(i)}|_{D_0} = Id_n$ and $q^{(i)} = 1$. Let $D_0, D_1, \ldots, D_K$ be the chain of contiguous domains such that $\Sigma_m = \partial D_m \cap \partial D_{m+1}$ and $\Sigma_1 = \partial D_0 \cap \partial D_1$. Set

$$\begin{aligned}
\varepsilon &= d(\mathcal{C}_1, \mathcal{C}_2), \\
E &= \max\{\|\gamma_K^{(1)} - \gamma_K^{(2)}\|_{L^\infty(D_K)}, \|q_{\tilde{K}}^{(1)} - q_{\tilde{K}}^{(2)}\|_{L^\infty(D_{\tilde{K}})}\}, \\
\delta_k &= \|\gamma^{(1)} - \gamma^{(2)}\|_{L^\infty(\mathcal{W}_k)}, \\
\tilde{\delta}_k &= \|q^{(1)} - q^{(2)}\|_{L^\infty(\mathcal{W}_k)}, \\
\delta_k^* &= \max\{\delta_k, \tilde{\delta}_k\} \qquad \text{for } k = 1, \ldots, \max\{K, \tilde{K}\}.
\end{aligned}$$

Let $\{x_1, \ldots, x_n\}$ be a coordinate system with origin at P_k. Let Σ_k be the flat interface of Section 2.1. We assume that it is contained in the tangential hyperplane of $\partial D_1 \cap B_{r_0/4}$ at P_k. For a scalar function f, we denote with $D_T f(x)$ the $n-1$ dimensional vector of the tangential partial derivatives of f at x and with $\partial_\nu f(x)$ the normal partial derivative of f at x. The affine function $(\gamma_k^{(1)} - \gamma_k^{(2)})$ can be bounded from above in D_k in terms of the quantities

$$\|\gamma_k^{(1)} - \gamma_k^{(2)}\|_{L^\infty(\Sigma_k \cap B_{r_0/4}(P_k))} \quad \text{and} \quad |\partial_\nu(\gamma_k^{(1)} - \gamma_k^{(2)})(P_k)|. \tag{38}$$

Indeed, set

$$A_k + B_k \cdot x = (\gamma_k^{(1)} - \gamma_k^{(2)})(x), \qquad A_k \in \mathbb{R},\ B_k \in \mathbb{R}^n,\ x \in D_k.$$

Fix an orthonormal basis $\{e_j\}_{j=1}^{n-1}$ of Σ_k and let e_n be the direction of the normal. One can evaluate $(\gamma_k^{(1)} - \gamma_k^{(2)})$ at the points P_k and $P_k + \frac{r_0}{6} e_j$ for $j = 1, \ldots, n$ and derive

$$|A_k + B_k \cdot P_k| + \frac{r_0}{6} \sum_{j=1}^{n-1} |(B_k)_j| \leq C\|\gamma_k^{(1)} - \gamma_k^{(2)}\|_{L^\infty(\Sigma_k \cap B_{r_0/4}(P_k))}$$

and

$$|B_k \cdot e_n| = |\partial_\nu(\gamma_k^{(1)} - \gamma_k^{(2)})(P_k)|.$$

Hence, it turns out that

$$\|\gamma_k^{(1)} - \gamma_k^{(2)}\|_{L^\infty(D_k)} \leq C\left(\|\gamma_k^{(1)} - \gamma_k^{(2)}\|_{L^\infty(\Sigma_k \cap B_{r_0/4}(P_k))} + |\partial_\nu(\gamma_k^{(1)} - \gamma_k^{(2)})(P_k)|\right),$$

for $C > 0$ constant that depends on the a-priori data.

Our goal is to estimate δ_k^* for $k = 1, \ldots, \max\{K, \tilde{K}\}$.

When $k = 1$, we obtain the following Hölder estimates at the boundary:

$$\delta_1 \leq C(E + \varepsilon)\Big(\frac{\varepsilon}{\varepsilon + E}\Big)^{\eta_1} \tag{39}$$

$$\tilde{\delta}_1 \leq C(E + \varepsilon)\Big(\frac{\varepsilon}{\varepsilon + E}\Big)^{\tilde{\eta}_1}, \tag{40}$$

with $0 < \eta_1, \tilde{\eta}_1 < 1$ that depend on $\theta_1, \theta_2, \theta_3$ and C are positive constants that depend on the a-priori data only (see the Appendix for a proof).

We proceed by estimating δ_2^*. We claim that the following inequalities hold:

$$\delta_2 \leq C\,(\varepsilon + E)\,\omega_{\eta_2}^{(2)}\left(\frac{\varepsilon}{\varepsilon + E}\right), \tag{41}$$

$$\tilde{\delta}_2 \leq C\,(\varepsilon + E)\,\omega_{\tilde{\eta}_2}^{(3)}\left(\frac{\varepsilon}{\varepsilon + E}\right), \tag{42}$$

with $0 < \eta_2, \tilde{\eta}_2 < 1$.

Our idea is to first estimate the L^2 norm of $(\gamma^{(1)} - \gamma^{(2)})$ on $\mathcal{W}_1$, namely δ_2, by means of δ_1^* and then to estimate the L^2 norm of $(q^{(2)} - q^{(1)})$ on $\mathcal{W}_1$, namely $\tilde{\delta}_2$, in terms of δ_2 and δ_1*.

For $y, z \in D_0$, the following identities hold:

$$\begin{aligned}
&\int_{\Sigma} [\sigma^{(2)}(x)\nabla_x G_2(x,z)\cdot\nu\, G_1(x,y) - \sigma^{(1)}(x)\nabla_x G_1(x,y)\cdot\nu\, G_2(x,z)]\,\mathrm{d}S(x) = \\
&= S_1(y,z) + \int_{\mathcal{W}_1} (\sigma^{(1)} - \sigma^{(2)})(x)\nabla_x G_1(x,y)\cdot\nabla_x G_2(x,z)\,\mathrm{d}x \\
&+ \int_{\mathcal{W}_1} (q^{(2)} - q^{(1)})(x) G_1(x,y) G_2(x,z)\,\mathrm{d}x,
\end{aligned} \tag{43}$$

and

$$\begin{aligned}
&\int_{\Sigma} [\sigma^{(2)}(x)\nabla_x\partial_{z_n} G_2(x,z)\cdot\nu\,\partial_{y_n} G_1(x,y) - \sigma^{(1)}(x)\nabla_x\partial_{y_n} G_1(x,y)\cdot\nu\,\partial_{z_n} G_2(x,z)]\,\mathrm{d}S(x) = \\
&= \partial_{y_n}\partial_{z_n} S_1(y,z) + \int_{\mathcal{W}_1} (\sigma^{(1)} - \sigma^{(2)})(x)\nabla_x\partial_{y_n} G_1(x,y)\cdot\nabla_x\partial_{z_n} G_2(x,z)\,\mathrm{d}x + \\
&+ \int_{\mathcal{W}_1} (q^{(2)} - q^{(1)})(x)\partial_{y_n} G_1(x,y)\partial_{z_n} G_2(x,z)\,\mathrm{d}x.
\end{aligned} \tag{44}$$

By (37),

$$\begin{aligned}
&\left| \int_{\Sigma} [\sigma^{(2)}\nabla_x G_2(x,z)\cdot\nu\, G_1(x,y) - \sigma^{(1)}\nabla_x G_1(x,y)\cdot\nu\, G_2(x,z)]\,\mathrm{d}S(x) \right| \leq \\
&\leq C\varepsilon\,(d(y)d(z))^{1-\frac{n}{2}},
\end{aligned} \tag{45}$$

where $d(y)$ denotes the distance between y and Ω. Let $\rho = r_0/4$, let $r \in (0, \bar{r}/8)$, where $\bar{r}$ is the constant of Proposition 3.3, and set $w = P_2 + r\nu(P_2)$, where $\nu(P_2)$ is the outward unit normal of ∂D_2 at P_2. Consider

$$S_1(w,w) = I_1(w) + I_2(w), \tag{46}$$

with

$$I_1(w) = \int\limits_{B_\rho(P_2)\cap D_2} (\gamma_2^{(1)} - \gamma_2^{(2)})(x)\, A(x)\, \nabla_x G_1(x,w) \cdot \nabla_x G_2(x,w)\, \mathrm{d}x + \\ + \int\limits_{B_\rho(P_2)\cap D_2} (q_2^{(2)} - q_2^{(1)})(x)\, G_1(x,w) \cdot G_2(x,w)\, \mathrm{d}x,$$

and

$$I_2(w) = \int\limits_{\mathcal{U}_1\setminus(B_\rho(P_2)\cap D_2)} (\sigma^{(1)} - \sigma^{(2)})(x)\, \nabla_x G_1(x,w) \cdot \nabla_x G_2(x,w)\, \mathrm{d}x + \\ + \int\limits_{\mathcal{U}_1\setminus(B_\rho(P_2)\cap D_2)} (q^{(2)} - q^{(1)})(x)\, G_1(x,w) \cdot G_2(x,w)\, \mathrm{d}x.$$

The volume integrals of $I_2(w)$ can be bounded from above via Caccioppoli inequality (see also [9, Proposition 3.1]):

$$|I_2(w)| \le CE\rho^{2-n}. \tag{47}$$

Regarding $I_1(w)$, notice that there exists $x^* \in \overline{\Sigma_2 \cap B_{r_0/4}(P_2)}$ such that

$$(\gamma_2^{(1)} - \gamma_2^{(2)})(x^*) = \|\gamma_2^{(1)} - \gamma_2^{(2)}\|_{L^\infty(\Sigma_2\cap B_{r_0/4}(P_2))}. \tag{48}$$

By (48),

$$I_1(w) = \int\limits_{B_\rho(P_2)\cap D_2} (\gamma_2^{(1)} - \gamma_2^{(2)})(x^*)\, A(x)\, \nabla_x G_1(x,w) \cdot \nabla_x G_2(x,w)\, \mathrm{d}x + \\ + \int\limits_{B_\rho(P_2)\cap D_2} B_2 \cdot (x - x^*)\, A(x)\, \nabla_x G_1(x,w) \cdot \nabla_x G_2(x,w)\, \mathrm{d}x + \\ + \int\limits_{B_\rho(P_2)\cap D_2} (q_2^{(2)} - q_2^{(1)})(x)\, G_1(x,w) \cdot G_2(x,w)\, \mathrm{d}x.$$

By the asymptotic estimate (21), one obtains

$$I_1(w) \ge \|\gamma_2^{(1)} - \gamma_2^{(2)}\|_{L^\infty(\Sigma_2\cap B_{r_0/4}(P_2))} \Big\{ \int\limits_{B_\rho(P_2)\cap D_2} A(x) \nabla_x H_1(x,w) \cdot \nabla_x H_2(x,w)\, \mathrm{d}x - \\ - \int\limits_{B_\rho(P_2)\cap D_2} |x-w|^{2(1-n)+\theta_1}\, \mathrm{d}x - \int\limits_{B_\rho(P_2)\cap D_2} |x-w|^{2(1-n+\theta_1)}\, \mathrm{d}x \Big\} - \\ - CE \int\limits_{B_\rho(P_2)\cap D_2} |x||x-w|^{2(1-n)}\, \mathrm{d}x - CE \int\limits_{B_\rho(P_2)\cap D_2} |x-w|^{2(2-n)}\, \mathrm{d}x.$$

It turns out that

$$|I_1(w)| \ge C\|\gamma_2^{(1)} - \gamma_2^{(2)}\|_{L^\infty(\Sigma_2\cap B_{r_0/4}(P_2))} r^{2-n} - CEr^{2-n+\theta_1} - CEr^{3-n}. \tag{49}$$

Notice that for $y, z \in (D_0)_{r_0/3}$,

$$|S_1(y,z)| \le C r_0^{2-n}(\varepsilon + \delta_1^*),$$

hence, by (29),

$$|S_1(y,z)| \le C(\varepsilon + \delta_1^* + E)\Big(\frac{\varepsilon + \delta_1^*}{\varepsilon + \delta_1^* + E}\Big)^{\beta^{2N_1}\tau_r^2} r^{2-n}. \tag{50}$$

Since

$$|I_1(w)| \le |S_1(w,w)| + |I_2(w)|,$$

if we rearrange the inequalities (49) and (47) together with (50) and (32), we derive

$$\|\gamma_2^{(1)} - \gamma_2^{(2)}\|_{L^\infty(\Sigma_2\cap B_{r_0/4}(P_2))} r^{2-n} \le C(\varepsilon + \delta_1^* + E)\Big\{\Big(\frac{\varepsilon + \delta_1^*}{\varepsilon + \delta_1^* + E}\Big)^{\beta^{2N_1}(12r_1\ln 3)^{-2}r^2} r^{2-n} + r^{\theta_1}\Big\}. \tag{51}$$

Using the argument of the proof of [8, Theorem 5.3], let

$$r = \Big|\ln\Big(\frac{\varepsilon + \delta_1^*}{\varepsilon + \delta_1^* + E}\Big)^{\beta^{2N_1}(12r_1\ln 3)^{-2}}\Big|^{-\frac{1}{n+\theta_1}},$$

then it turns out that

$$\|\gamma_2^{(1)} - \gamma_2^{(2)}\|_{L^\infty(\Sigma_2\cap B_{r_0/4}(P_2))} \le C(\varepsilon + \delta_1^* + E)\Big|\ln\Big(\frac{\varepsilon + \delta_1^*}{\varepsilon + \delta_1^* + E}\Big)\Big|^{-\frac{\theta_1}{n+\theta_1}}. \tag{52}$$

By the properties of ω_η, one derives

$$\|\gamma_2^{(1)} - \gamma_2^{(2)}\|_{L^\infty(\Sigma_2\cap B_{r_0/4}(P_2))} \le C(\varepsilon + E)\omega_\eta\Big(\frac{\varepsilon}{\varepsilon + E}\Big), \tag{53}$$

with $0 < \eta < 1$ depending on θ_1.

A similar estimate can be derived for $\partial_\nu(\gamma_2^{(1)} - \gamma_2^{(2)})$. From Taylor's formula, one derives

$$(\gamma_2^{(1)} - \gamma_2^{(2)})(x) = (\gamma_2^{(1)} - \gamma_2^{(2)})(P_2) + (D_T(\gamma_2^{(1)} - \gamma_2^{(2)})(P_2))\cdot(x - P_2)' + \\ + (\partial_\nu(\gamma_2^{(1)} - \gamma_2^{(2)})(P_2))\cdot(x - P_2)_n.$$

Hence,

$$\begin{aligned}
&|\partial_{y_n}\partial_{z_n} S_1(w,w)| \ge \\
&\quad \le \Big|\int_{B_\rho(P_2)\cap D_2} \partial_\nu(\gamma_2^{(1)} - \gamma_2^{(2)})(P_2)\cdot(x - P_2)_n A(x)\nabla_x\partial_{y_n}G_1(x,w)\cdot\nabla_x\partial_{z_n}G_2(x,w)\,\mathrm{d}x\Big| \\
&\quad - \Big|\int_{B_\rho(P_2)\cap D_2} D_T(\gamma_2^{(1)} - \gamma_2^{(2)})(P_2)\cdot(x - P_2)' A(x)\nabla_x\partial_{y_n}G_1(x,w)\cdot\nabla_x\partial_{z_n}G_2(x,w)\,\mathrm{d}x\Big| \\
&\quad - \Big|\int_{B_\rho(P_2)\cap D_2} (\gamma_2^{(1)} - \gamma_2^{(2)})(P_2)A(x)\nabla_x\partial_{y_n}G_1(x,w)\cdot\nabla_x\partial_{z_n}G_2(x,w)\,\mathrm{d}x\Big| \\
&\quad - \Big|\int_{B_\rho(P_2)\cap D_2} (q_2^{(2)} - q_2^{(1)})(x)\partial_{y_n}G_1(x,w)\cdot\partial_{z_n}G_2(x,w)\,\mathrm{d}x\Big|
\end{aligned}$$

$$
\begin{aligned}
&- \Big| \int\limits_{\mathcal{U}_1 \setminus (B_\rho(P_2) \cap D_2)} (\sigma^{(1)} - \sigma^{(2)})(x) \partial_{y_n} \nabla_x G_1(x,w) \cdot \partial_{z_n} \nabla_x G_2(x,w) \, \mathrm{d}x \Big| \\
&- \Big| \int\limits_{\mathcal{U}_1 \setminus (B_\rho(P_2) \cap D_2)} (q^{(1)} - q^{(2)})(x) \partial_{y_n} G_1(x,w) \cdot \partial_{z_n} G_2(x,w) \, \mathrm{d}x \Big| \\
&= I_{11} - I_{12} - I_{13} - I_{14} - I_{15} - I_{16}.
\end{aligned}
$$

To estimate I_{11} from below, we add and subtract the fundamental solution H_i, $i = 1, 2$, and by (22), one derives

$$
I_{11} \geq C |\partial_\nu (\gamma_2^{(1)} - \gamma_2^{(2)})(P_2)| r^{1-n} - C E r^{1-n+\theta_2}. \tag{54}
$$

To estimate the terms I_{12} and I_{13}, notice that by (53), we have

$$
|(\gamma_2^{(1)} - \gamma_2^{(2)})(P_2)| + C |D_T (\gamma_2^{(1)} - \gamma_2^{(2)})(P_2)| \leq C \|\gamma_2^{(1)} - \gamma_2^{(2)}\|_{L^\infty(\Sigma_2 \cap B_{r_0/4})} \leq C(\varepsilon + E) \omega_\eta \Big(\frac{\varepsilon}{\varepsilon + E} \Big).
$$

Regarding the integral I_{14}, we add and subtract the fundamental solutions H_1 and H_2, then by the asymptotic estimate (23) and the (20), we have

$$
\begin{aligned}
I_{14} &\leq \|q_2^{(2)} - q_2^{(1)}\|_{L^\infty(D_2)} \int\limits_{D_2 \cap B_\rho} |\partial_{y_n} G_1(x,w)| |\partial_{z_n} G_2(x,w)| \, \mathrm{d}x \\
&\leq C \int\limits_{D_2 \cap B_\rho} |x - w|^{2(1-n)} \leq C \, r^{2-n}.
\end{aligned}
$$

The integral I_{15} and I_{16} can be bounded by means of [9, Proposition 3.1] as

$$
I_{15}, I_{16} \leq C E \rho^{-n}.
$$

To sum up, we have

$$
|\partial_\nu (\gamma_2^{(1)} - \gamma_2^{(2)})(P_2)| r^{1-n} \leq |\partial_{y_n} \partial_{z_n} S_1(w,w)| + C \Big\{ E r^{1-n+\theta_2} + C(\varepsilon + E) \omega_\eta \Big(\frac{\varepsilon}{\varepsilon + E} \Big) r^{-n} \Big\}. \tag{55}
$$

Notice that for $y, z \in (D_0)_{r_0/3}$,

$$
|\partial_{y_n} \partial_{z_n} S_1(y,z)| \leq C(\varepsilon + \delta_1^*) r^{-n},
$$

then by (30) and (32),

$$
\left| \partial_{y_j} \partial_{z_i} S_1(w,w) \right| \leq C(\varepsilon + \delta_1^* + E) \left(\frac{\varepsilon + \delta_1^*}{\varepsilon + \delta_1^* + E} \right)^{\beta^{2N_1} (12 r_1 \ln 3)^{-2} r^2} r^{-n}. \tag{56}
$$

Hence, one derives

$$
\begin{aligned}
|\partial_\nu (\gamma_2^{(1)} - \gamma_2^{(2)})(P_1)| r^{1-n} \leq C \Big\{ & E r^{1-n+\theta_2} + (\varepsilon + \delta_1^* + E) \left(\frac{\varepsilon + \delta_1^*}{\varepsilon + \delta_1^* + E} \right)^{\beta^{2N_1} (12 r_1 \ln 3)^{-2} r^2} r^{-n} + \\
& + (\varepsilon + E) \omega_\eta \Big(\frac{\varepsilon}{\varepsilon + E} \Big) r^{-n} \Big\}.
\end{aligned} \tag{57}
$$

Multiplying (57) by r^{n-1} and optimizing w.r.t. r leads to

$$|\partial_\nu(\gamma_1^{(1)} - \gamma_1^{(2)})(P_1)| \leq C(\varepsilon + E)\omega_{\eta_2}^{(2)}\left(\frac{\varepsilon}{\varepsilon + E}\right), \tag{58}$$

with $0 < \eta_2 < 1$. Hence, we conclude that

$$\delta_2 \leq C(\varepsilon + E)\omega_{\eta_2}^{(2)}\left(\frac{\varepsilon}{\varepsilon + E}\right). \tag{59}$$

Our goal is to derive a bound for δ_2^*. Notice that the norm $\|q_2^{(2)} - q_2^{(1)}\|_{L^\infty(D_2)}$ can be evaluated in terms of the following quantities:

$$\|q_2^{(2)} - q_2^{(1)}\|_{L^\infty(\Sigma_2 \cap B_{r_0/4}(P_2))} \quad \text{and} \quad |\partial_\nu(q_2^{(2)} - q_2^{(1)})(P_2)|. \tag{60}$$

Let ρ, r, w be as above. Consider

$$\partial_{y_n}\partial_{z_n} S_1(w, w) = \partial_{y_n}\partial_{z_n} I_1(w) + \partial_{y_n}\partial_{z_n} I_2(w).$$

We determine a lower bound for $\partial_{y_n}\partial_{z_n} I_1(w)$ in terms of $\|q_2^{(2)} - q_2^{(1)}\|_{L^\infty(\Sigma_2 \cap B_{r_0/4}(P_2))}$. By the asymptotic estimate (22) and (59), one derives

$$\begin{aligned}
|\partial_{y_n}\partial_{z_n} I_1(w)| \geq \|q_2^{(2)} - q_2^{(1)}\|_{L^\infty(\Sigma_2 \cap B_{r_0/4}(P_2))}\Big\{ &\int_{B_\rho(P_2)\cap D_2} \partial_{y_n} H_1(x, w)\, \partial_{z_n} H_2(x, w)\, \mathrm{d}x - \\
- \int_{B_\rho(P_2)\cap D_2} |x - w|^{2(1-n)+\theta_3}\, \mathrm{d}x &- \int_{B_\rho(P_2)\cap D_2} |x - w|^{2(1-n+\theta_3)}\, \mathrm{d}x \Big\} - \\
- CE \int_{B_\rho(P_2)\cap D_2} |x||x - w|^{2(1-n)}\, \mathrm{d}x &- C(\varepsilon + E)\omega_{\eta_2}^{(2)}\Big(\frac{\varepsilon}{\varepsilon + E}\Big) \int_{B_\rho(P_2)\cap D_2} |x - w|^{-n}\, \mathrm{d}x.
\end{aligned}$$

It turns out that

$$C\|q_2^{(2)} - q_2^{(1)}\|_{L^\infty(\Sigma_2 \cap B_{r_0/4}(P_2))} r^{2-n} \leq |\partial_{y_n}\partial_{z_n} I_1(w)| + CEr^{2-n+\theta_3} + C(\varepsilon + E)\omega_{\eta_2}^{(2)}\left(\frac{\varepsilon}{\varepsilon + E}\right) r^{-n}.$$

By (56), due to the fact that

$$|\partial_{y_n}\partial_{z_n} I_1(w)| \leq |\partial_{y_n}\partial_{z_n} S_1(w, w)| + |\partial_{y_n}\partial_{z_n} I_2(w)|,$$

by the upper bound for $I_2(w)$, (30) and (32) we derive

$$\begin{aligned}
\|q_2^{(2)} - q_2^{(1)}\|_{L^\infty(\Sigma_2 \cap B_{r_0/4}(P_2))} r^{2-n} \leq C\Big\{&(\varepsilon + \delta_1^* + E)\left(\frac{\varepsilon + \delta_1^*}{\varepsilon + \delta_1^* + E}\right)^{\beta^{2N_1}(12r_1 \ln 3)^{-2} r^2} r^{-n} + \\
&+ Er^{2-n+\theta_1} + (\varepsilon + E)\omega_{\eta_2}^{(2)}\left(\frac{\varepsilon}{\varepsilon + E}\right) r^{-n}\Big\}.
\end{aligned}$$

Multiply by r^{n-2} to obtain

$$\|q_2^{(2)} - q_2^{(1)}\|_{L^\infty(\Sigma_2 \cap B_{r_0/4}(P_2))} \leq C\Big\{(\varepsilon + \delta_1^* + E)\left(\frac{\varepsilon + \delta_1^*}{\varepsilon + \delta_1^* + E}\right)^{\beta^{2N_1}(12r_1 \ln 3)^{-2} r^2} r^{-2} + $$
$$+ E r^{\theta_1} + (\varepsilon + E)\omega_{\eta_2}^{(2)}\left(\frac{\varepsilon}{\varepsilon + E}\right) r^{-2}\Big\}.$$

By optimizing with respect to r, one concludes that

$$\|q_2^{(2)} - q_2^{(1)}\|_{L^\infty(\Sigma_2 \cap B_{r_0/4}(P_2))} \leq C(\varepsilon + E)\omega_{\bar{\eta}_2}^{(3)}\left(\frac{\varepsilon}{\varepsilon + E}\right), \tag{61}$$

with $0 < \bar{\eta}_2 < 1$ that depends on $\theta_1, \theta_2, \theta_3$.

To estimate $|\partial_\nu(q_2^{(2)} - q_2^{(1)})(P_2)|$, consider the singular solution $\partial^2_{y_i y_j}\partial^2_{z_i z_j} S_1(w, w)$ and split it as the sum of the terms

$$I_1^{ij}(w) = \int\limits_{D_2 \cap B_\rho(P_2)} (\sigma_2^{(2)} - \sigma_2^{(1)})(x)\nabla_x \partial^2_{y_i y_j} G_1(x, w) \cdot \nabla_x \partial^2_{z_i z_j} G_2(x, w)\, \mathrm{d}x +$$
$$+ \int\limits_{D_2 \cap B_\rho(P_2)} (q_2^{(2)} - q_2^{(1)})(x)\partial^2_{y_i y_j} G_1(x, w) \cdot \partial^2_{z_i z_j} G_2(x, w)\, \mathrm{d}x$$

and

$$I_2^{ij}(w) = \int\limits_{\mathcal{U}_1 \setminus (D_2 \cap B_\rho(P_2))} (\sigma^{(1)} - \sigma^{(2)})(x)\nabla_x \partial^2_{y_i y_j} G_1(x, w) \cdot \nabla_x \partial^2_{z_i z_j} G_2(x, w)\, \mathrm{d}x +$$
$$+ \int\limits_{\mathcal{U}_1 \setminus (D_2 \cap B_\rho(P_2))} (q^{(2)} - q^{(1)})(x)\partial^2_{y_i y_j} G_1(x, w) \cdot \partial^2_{z_i z_j} G_2(x, w)\, \mathrm{d}x.$$

Set $I_m(w) = \{I_m^{ij}(w)\}_{i,j=1,\dots,n}$ for $m = 1, 2$. Denote by $|I_m(w)|$ the Euclidean norm of the matrix $I_m(w)$. The upper bound for $|I_2(w)|$ is given by

$$|I_2(w)| \leq C E \rho^{-(n+2)},$$

where C is a positive constant that depends on the a-priori data only. For the lower bound of $I_1(w)$, we have

$$|I_1(w)| \geq \frac{1}{n}\sum_{i,j=1}^n \Big\{\Big|\int\limits_{D_2 \cap B_\rho(P_2)} (\partial_\nu(q_2^{(2)} - q_2^{(1)})(P_2)) \cdot (x - P_2)_n \partial^2_{y_i y_j} G_1(x, w) \cdot \partial^2_{z_i z_j} G_2(x, w)\, \mathrm{d}x\Big| -$$
$$- \Big|\int\limits_{D_2 \cap B_\rho(P_2)} (D_T(q_2^{(2)} - q_2^{(1)})(P_2)) \cdot (x - P_2)' \partial^2_{y_i y_j} G_1(x, w) \cdot \partial^2_{z_i z_j} G_2(x, w)\, \mathrm{d}x\Big| -$$
$$- \Big|\int\limits_{D_2 \cap B_\rho(P_2)} (q_2^{(2)} - q_2^{(1)})(P_2)\partial^2_{y_i y_j} G_1(x, w) \cdot \partial^2_{z_i z_j} G_2(x, w)\, \mathrm{d}x\Big|\Big\} -$$
$$- \Big|\int\limits_{D_2 \cap B_\rho(P_2)} (\sigma^{(1)} - \sigma^{(2)})(x)\partial^2_{y_i y_j} \nabla_x G_1(x, w) \cdot \partial^2_{z_i z_j} \nabla_x G_2(x, w)\, \mathrm{d}x\Big|.$$

Since

$$|(q_2^{(2)} - q_2^{(1)})(P_2)| + C|(D_T(q_2^{(2)} - q_2^{(1)})(P_2))| \leq C\|q_2^{(2)} - q_2^{(1)}\|_{L^\infty(\Sigma_2 \cap B_{r_0/4}(P_2))},$$

by (61) and (24), one derives

$$\begin{aligned} |I_1(w)| \geq C|(\partial_\nu(q_2^{(2)} - q_2^{(1)})(P_2))| r^{1-n} - C(\varepsilon + E)\omega_{\tilde{\eta}_2}^{(3)}\left(\frac{\varepsilon}{\varepsilon + E}\right) r^{-n} - \\ -CEr^{1+\theta_2-n} - C(E+\varepsilon)\omega_{\eta_2}^{(2)}\Big(\frac{\varepsilon}{\varepsilon+E}\Big) r^{-2-n}. \end{aligned} \tag{62}$$

Since for $y, z \in (D_0)_{r_0/3}$,

$$\begin{aligned} &\int_\Sigma [\sigma^{(2)}(x)\nabla_x \partial_{z_n}^2 G_2(x,z) \cdot \nu\, \partial_{y_n}^2 G_1(x,y) - \sigma^{(1)}(x)\nabla_x \partial_{y_n}^2 G_1(x,y) \cdot \nu\, \partial_{z_n}^2 G_2(x,z)]\, \mathrm{d}S(x) = \\ &= \partial_{y_n}^2 \partial_{z_n}^2 S_1(y,z) + \int_{\mathcal{W}_1} (\sigma^{(1)} - \sigma^{(2)})(x)\nabla_x \partial_{y_n}^2 G_1(x,y) \cdot \nabla_x \partial_{z_n}^2 G_2(x,z)\, \mathrm{d}x + \\ &+ \int_{\mathcal{W}_1} (q^{(2)} - q^{(1)})(x)\partial_{y_n}^2 G_1(x,y)\partial_{z_n}^2 G_2(x,z)\, \mathrm{d}x, \end{aligned}$$

by (31) and (32), it turns out that

$$|\partial_{y_n}^2 \partial_{z_n}^2 S_1(y_r, y_r)| \leq C\left(\frac{\varepsilon + \delta_1^*}{\varepsilon + \delta_1^* + E}\right)^{\beta^{2N_1}(12r_1 \ln 3)^{-2} r^2} (\varepsilon + \delta_1^* + E) r^{-2-n}. \tag{63}$$

Collecting together (62) and (63), one derives

$$\begin{aligned} |(\partial_\nu(q_1^{(2)} - q_1^{(1)})(P_1))| r^{1-n} \leq C(\varepsilon + E)\omega_{\eta_2}^{(2)}\left(\frac{\varepsilon}{\varepsilon + E}\right) r^{-2-n} + CEr^{1+\theta_2-n} + \\ + C\left(\frac{\varepsilon + \delta_1^*}{\varepsilon + \delta_1^* + E}\right)^{\beta^{2N_1}(12r_1 \ln 3)^{-2} r^2} (\varepsilon + \delta_1^* + E) r^{-2-n}. \end{aligned}$$

Multiply by r^{n-1} the last equation and optimize with respect to r leads to the estimate

$$|(\partial_\nu(q_1^{(2)} - q_1^{(1)})(P_1))| \leq C(E+\varepsilon)\omega_{\tilde{\eta}_2}^{(3)}\left(\frac{\varepsilon}{\varepsilon + E}\right), \tag{64}$$

with $0 < \tilde{\eta}_2 < 1$ that depends on $\theta_1, \theta_2, \tilde{b}$.

For the general case, consider the following identity:

$$\begin{aligned} &\int_\Sigma [\sigma^{(2)}(x)\nabla_x G_2(x,z) \cdot \nu\, G_1(x,y) - \sigma^{(1)}(x)\nabla_x G_1(x,y) \cdot \nu\, G_2(x,z)]\, \mathrm{d}S(x) = \\ &= S_{k-1}(y,z) + \int_{\mathcal{W}_{k-1}} (\sigma^{(1)} - \sigma^{(2)})(x)\nabla_x G_1(x,y) \cdot \nabla_x G_2(x,z)\, \mathrm{d}x \\ &+ \int_{\mathcal{W}_{k-1}} (q^{(2)} - q^{(1)})(x) G_1(x,y) G_2(x,z)\, \mathrm{d}x. \end{aligned} \tag{65}$$

By (44), one derives

$$\begin{aligned}
&\int_{\Sigma} \sigma^{(2)}(x)\nabla_x\partial_{z_n}G_2(x,z)\cdot\nu\,\partial_{y_n}G_1(x,y)-\sigma^{(1)}(x)\nabla_x\partial_{y_n}G_1(x,y)\cdot\nu\,\partial_{z_n}G_2(x,z)]\,\mathrm{d}S(x)=\\
&=\int_{\mathcal{W}_{k-1}}(\sigma^{(1)}-\sigma^{(2)})(x)\nabla_x\partial_{y_n}G_1(x,y)\cdot\nabla_x\partial_{z_n}G_2(x,z)\,\mathrm{d}x+\\
&+\int_{\mathcal{W}_{k-1}}(q^{(2)}-q^{(1)})(x)\partial_{y_n}G_1(x,y)\partial_{z_n}G_2(x,z)\,\mathrm{d}x+\partial_{y_n}\partial_{z_n}S_{k-1}(y,z),
\end{aligned} \tag{66}$$

and

$$\begin{aligned}
&\int_{\Sigma}[\sigma^{(2)}(x)\nabla_x\partial^2_{z_n}G_2(x,z)\cdot\nu\,\partial^2_{y_n}G_1(x,y)-\sigma^{(1)}(x)\nabla_x\partial^2_{y_n}G_1(x,y)\cdot\nu\,\partial^2_{z_n}G_2(x,z)]\,\mathrm{d}S(x)=\\
&=\int_{\mathcal{W}_{k-1}}(\sigma^{(1)}-\sigma^{(2)})(x)\nabla_x\partial^2_{y_n}G_1(x,y)\cdot\nabla_x\partial^2_{z_n}G_2(x,z)\,\mathrm{d}x+\\
&+\int_{\mathcal{W}_{k-1}}(q^{(2)}-q^{(1)})(x)\partial^2_{y_n}G_1(x,y)\partial^2_{z_n}G_2(x,z)\,\mathrm{d}x+\partial^2_{y_n}\partial^2_{z_n}S_{k-1}(y,z).
\end{aligned} \tag{67}$$

Notice that

$$|S_{k-1}(y,z)|\le C(\varepsilon+\delta^*_{k-1}),\qquad \text{for } y,z\in D_0. \tag{68}$$

To estimate the norms

$$\|\gamma^{(1)}-\gamma^{(2)}\|_{L^\infty(D_k)}\quad\text{and}\quad \|q^{(1)}-q^{(2)}\|_{L^\infty(D_k)}$$

one can proceed as in step $k=2$. Consider $\rho=r_0/4$, $r\in(0,\bar{r}/8)$ and set $w=P_k+r\nu(P_k)$, then we split the integral solutions $S_{k-1}(w,w)$, $\partial_{y_n}\partial_{z_n}S_{k-1}(w,w)$ and $\partial^2_{y_n}\partial^2_{z_n}S_{k-1}(w,w)$ into the sum of two integrals over the domains $B_\rho(P_k)\cap D_k$ and $\mathcal{U}_{k-1}\setminus(B_\rho(P_k)\cap D_k)$. At this point, one determines a lower bound for the integral on the smallest domain and an upper bound for the integral on the largest domain using the estimates of Proposition 3.3. It turns out that

$$\|\gamma^{(1)}-\gamma^{(2)}\|_{L^\infty(D_k)}\le C(\varepsilon+E)\omega^{(3k-4)}_{\eta_k}\Big(\frac{\varepsilon}{\varepsilon+E}\Big), \tag{69}$$

and then by (69), we derive

$$\|q^{(1)}-q^{(2)}\|_{L^\infty(D_k)}\le C(\varepsilon+E)\omega^{(3(k-1))}_{\tilde{\eta}_k}\Big(\frac{\varepsilon}{\varepsilon+E}\Big), \tag{70}$$

with $0<\eta_k,\tilde{\eta}_k<1$ constants that depend on the a-priori data only.

Set $\bar{K}=\max\{K,\tilde{K}\}$. Since $E=\delta^*_{\bar{K}}$, one derives

$$E\le C(\varepsilon+E)\omega^{(3(\bar{K}-1))}_{\tilde{\eta}_{\bar{K}}}\left(\frac{\varepsilon}{\varepsilon+E}\right).$$

If $E\ge e^2\varepsilon$ (otherwise, the statement holds), it turns out that

$$1\le C\omega^{(3(\bar{K}-1))}_{\tilde{\eta}_{\bar{K}}}\left(\frac{\varepsilon}{E}\right). \tag{71}$$

By applying the inverse of $\omega^{(3(\bar{K}-1))}_{\tilde{\eta}_{\bar{K}}}$ to (71), we conclude that

$$E \le C_1 \varepsilon,$$

with C_1 a positive constant that depends on the a-priori data only. This concludes the proof of Theorem 2.2. □

5. Proof of the auxiliary propositions

To prove Proposition 3.3, we apply a result of propagation of smallness for elliptic PDEs with piecewise Lipschitz coefficients.

Let $\Omega \subset \mathbb{R}^n$ be a domain that satisfies the assumptions of Section 2.1.

First, we derive a three sphere inequality in terms of L^∞ norms from the three sphere inequality proved by [24].

Lemma 5.1. *Let $u \in H^1(B_{\bar{r}})$ be a weak solution to*

$$\operatorname{div}(\sigma \nabla u) + qu = 0, \qquad \text{in } B_{\bar{r}},$$

with $B_{\bar{r}} \subset (\Omega)_{r_0/3}$, $\bar{r} > 0$. We assume that σ, q satisfy the a-priori assumptions of Section 2.2. Then, for any $0 < r_1 < r_2 < r_3 \le \bar{r}$, the following inequality holds:

$$\|u\|_{L^\infty(B_{r_2})} \le C_\infty \|u\|^{\beta}_{L^\infty(B_{r_1})} \|u\|^{1-\beta}_{L^\infty(B_{r_3})}, \tag{72}$$

where

$$\beta = \ln\left(\frac{2r_3}{r_2 + r_3}\right) / \ln\left(\frac{r_3}{r_1}\right), \qquad \beta \in (0,1)\ ,$$

and $C_\infty > 1$ depends on $\bar{q}$, $\bar{A}$, $\frac{r_1}{r_2}$, $\frac{r_2}{r_3}, r_0, L, \lambda$.

Proof. The proof of this Lemma relies on the well-known $L^\infty - L^2$ Moser-Stampacchia estimates valid for elliptic equations with zeroth order term. By [32, Theorem 8.17] (see also [22, Theorem 6.1]), there exists a constant $C > 1$ that depends only on $\lambda, \bar{q}, \bar{A}, n$ such that, for any $0 < r < \rho$,

$$\|u\|_{L^\infty(B_r)} \le \frac{C}{(\rho - r)^{\frac{n}{2}}} \|u\|_{L^2(B_\rho)}. \tag{73}$$

By [24, Theorem 4.1] and (73), if we choose $r = r_2$, $\rho = (r_2 + r_3)/2$,

$$\begin{aligned}
\|u\|_{L^\infty(B_{r_2})} &\le \frac{C}{\left(\frac{r_2+r_3}{2} - r_2\right)^{\frac{n}{2}}} \|u\|_{L^2(B_{\frac{r_2+r_3}{2}})} \\
&\le \frac{C}{\left(\frac{r_2+r_3}{2} - r_2\right)^{\frac{n}{2}}} \|u\|^{\beta}_{L^2(B_{r_1})} \|u\|^{1-\beta}_{L^2(B_{r_3})} \\
&\le \frac{C}{\left(\frac{r_2+r_3}{2} - r_2\right)^{\frac{n}{2}}} |B_{r_1}|^{\beta} |B_{r_3}|^{\frac{1-\beta}{2}} \|u\|^{\beta}_{L^\infty(B_{r_1})} \|u\|^{1-\beta}_{L^\infty(B_{r_3})}. \quad \square
\end{aligned}$$

In the following Proposition we derive a result of propagation of smallness valid in our setting (see also [7, Lemma 4.1] and [21, Proposition 3.9]).

Proposition 5.2. *For $k = 0, \dots, N-1$ assume that there is a weak solution $v \in H^1(\mathcal{W}_k)$ to*

$$\operatorname{div}(\sigma \nabla v) + q\, v = 0 \qquad \text{in } \mathcal{W}_k. \tag{74}$$

Suppose that for any given positive number $E_0, \varepsilon_0, \tilde{\gamma}$, the function v satisfies

$$|v(x)| \le \varepsilon_0 \qquad \text{for any } x \in D_0, \tag{75}$$

and

$$|v(x)| \le C(E_0 + \varepsilon_0)\big(r_0 d(x)\big)^{-\tilde{\gamma}} \qquad \text{for } x \in \mathcal{W}_k, \tag{76}$$

with $\tilde{\gamma} = n/2 - 1$. Let $\bar{r}$ be the constant of Proposition 5.1. Then, for any $r \in (0, \bar{r}/4)$, there exist constants $C > 1$ and $N_1 \in \mathbb{N}$ such that

$$|v(y_{k+1})| \le C(E_0 + \varepsilon_0)\Big(\frac{\varepsilon_0}{\varepsilon_0 + E_0}\Big)^{\tau_r \beta^{N_1}} r^{-\tilde{\gamma}}, \tag{77}$$

where C, N_1 depend on r_0, L, λ, $\bar{\sigma}, \bar{q}$ only, $y_{k+1} = P_{k+1} - r\nu(P_{k+1})$ with $\nu(P_{k+1})$ the outward unit normal of ∂D_k at P_{k+1}, $0 < \beta < 1$, $N_1 \in \mathbb{N}$, and, for $r_1 = \bar{r}/8$,

$$\tau_r = \ln\left(\frac{12r_1 - 2r}{12r_1 - 3r}\right) / \ln\left(\frac{6r_1 - r}{2r_1}\right).$$

Proof of Proposition 5.2. Our proof follows the lines of [29, Theorem 4.1] and [21, Proposition 3.9]. Let $P_0 \in (D_0)_{r_0/3}$, let $r_{00} > 0$ be such that $B_{r_{00}}(P_0) \subset (D_0)_{r_0/3}$. By (75),

$$|v(x)| \le \varepsilon_0 \qquad \text{for } x \in B_{r_{00}}(P_0).$$

Let $\bar{y}_{k+1} = P_{k+1} - 3\, r_1 \nu(P_{k+1})$, where $\nu(P_{k+1})$ is the outer unit normal of ∂D_k at P_{k+1}. For $y_0 \in B_{r_{00}}(P_0)$, there exists a Jordan curve contained in $\mathcal{W}_k$ that joins y_0 to $\bar{y}_{k+1}$. Call this curve $c(t) \in C([0,1], \mathcal{W}_k)$, so that $c(0) = y_0$ and $c(1) = \bar{y}_{k+1}$. Let

$$r_3 = \frac{\bar{r}}{2}, \quad r_2 = \frac{3}{4} r_3, \quad r_1 = \frac{r_3}{4},$$

so that $B_{r_1}(y_0) \subset B_{r_3}(y_0) \subset (D_0)_{r_0/3}$. Define $0 = t_0 < t_1 < \dots < t_{\bar{N}} = 1$ so that

$$\begin{aligned} &t_{k+1} = \max\{t \,:\, |c(t) - c(t_k)| = 2r_1\} \qquad \text{as long as} \quad |\bar{y}_{k+1} - c(t_k)| > 2r_1 \\ &\text{otherwise } \bar{N} = k+1,\ t_{\bar{N}} = 1. \end{aligned}$$

Notice that $B_{r_1}(c(t_k)) \cap B_{r_1}(c(t_{k+1})) = \emptyset$ and $B_{r_1}(c(t_{k+1})) \subset B_{r_2}(c(t_k))$ for $k = 1, \dots, \bar{N} - 1$. Thanks to Lemma 5.1, one can propagate the estimate $|v(y_0)|$ along the Jordan curve up to a ball centred at $\bar{y}_{k+1}$ of radius r_1 across the flat interfaces Σ_m for $m \in \{1, \dots, k\}$. Hence, one derives

$$|v(y_{k+1})| \le C \varepsilon_0^{\beta^{N_1}} (\varepsilon_0 + E_0)^{1 - \beta^{N_1}},$$

with $0 < \beta < 1$, $N_1 \in \mathbb{N}$ and $C > 0$ depend on the a-priori data only.

Let $r < r_1$, $y_{k+1} = P_{k+1} - r\nu(P_{k+1})$, then we can apply Lemma 5.1 to spheres centred at $\bar{y}_{k+1}$ of radii $r_1, 3r_1 - r, 3r_1 - r/2$ to obtain

$$\|v\|_{L^\infty(B_{3r_1-r}(\bar{y}_{k+1})} \leq Cr^{-(1-\tau_r)\tilde{\gamma}} \left(\frac{\varepsilon_0}{\varepsilon_0+E_0}\right)^{\tau_r \beta^{N_1}} (\varepsilon_0+E_0)$$

with

$$\tau_r = \log\left(\frac{12r_1-2r}{12r_1-3r}\right) / \log\left(\frac{6r_1-r}{2r}\right).$$

One derives (77) by observing that

$$C_1 r^{-\tilde{\gamma}} \leq r^{-(1-\tau_r)\tilde{\gamma}} \leq C_2 r^{-\tilde{\gamma}}. \quad \square$$

We are ready to prove the quantitative estimates of unique continuation for the singular solutions.

Proof of Proposition 3.3. Notice that by Proposition 3.1, for $y, z \in \mathcal{W}_k$,

$$|S_k(y,z)| \leq CE(d(y)d(z))^{1-\frac{n}{2}},$$

where $d(y)$ is the distance of y from $\mathcal{U}_k$.

Fix $z \in (D_0)_{r_0/3}$ and set $v(y) = S_k(y,z)$, then v is a weak solution to

$$\nabla_y \cdot (\sigma^{(1)} \nabla_y v) + q^{(1)} v = 0, \qquad \text{in } \mathcal{W}_k.$$

Notice that for $y \in \mathcal{W}_k$,

$$|v(y)| \leq CE(d(y))^{1-\frac{n}{2}}.$$

Thus, by applying Proposition 5.2, for $r \in (0, r_1)$, $z \in (D_0)_{r_0/3}$ and $y_{k+1} = P_{k+1} - r\nu(P_{k+1})$,

$$|S_k(y_{k+1}, z)| \leq Cr^{-\tilde{\gamma}} \left(\frac{\varepsilon_0}{\varepsilon_0+E}\right)^{\tau_r \beta^{N_1}} (\varepsilon_0+E),$$

with $\tilde{\gamma} = n/2 - 1$. Now, set $\tilde{v}(z) = S_k(y_{k+1}, z)$ for $z \in \mathcal{W}_k$. Then $\tilde{v}$ is a weak solution to

$$\nabla_z \cdot (\sigma^{(2)} \nabla_z \tilde{v}) + q^{(2)} \tilde{v} = 0, \qquad \text{in } \mathcal{W}_k.$$

Since

$$|\tilde{v}(z)| \leq C\,E\,(r\mathrm{dist}(z, \Sigma_{k+1}))^{1-\frac{n}{2}}, \qquad \text{for any } z \in \mathcal{W}_k,$$

then

$$|S_k(y_{k+1}, y_{k+1})| \leq Cr^{-2\tilde{\gamma}} \left(\frac{\varepsilon_0}{\varepsilon_0+E}\right)^{\tau_r^2 \beta^{2N_1}} (\varepsilon_0+E).$$

Let us determine the estimates for the partial derivatives of the integral solution. Since $S_k(y_1, \dots, y_n, z_1, \dots, z_n)$ is a weak solution to

$$\nabla_y \cdot (\sigma^{(1)} \nabla_y S_k(y,z)) + \nabla_z \cdot (\sigma^{(2)} \nabla_z S_k(y,z)) + q^{(1)} S_k(y,z) + q^{(2)} S_k(y,z) = 0, \qquad \text{in } D_k \times D_k,$$

one can apply the Schauder interior estimates (see [1] or [44]) at $y_{k+1} = P_{k+1} - 2r\nu(P_{k+1})$ and derive

$$\|\partial_{y_j}\partial_{z_i}S_k(y,z)\|_{L^\infty(B_{\frac{r}{2}}(y_{k+1})\times B_{\frac{r}{2}}(y_{k+1}))} \leq \frac{C}{r^2}\|S_k(y,z)\|_{L^\infty(B_r(y_{k+1})\times B_r(y_{k+1}))},$$

and

$$\|\partial^2_{y_j}\partial^2_{z_i}S_k(y,z)\|_{L^\infty(B_{\frac{r}{4}}(y_{k+1})\times B_{\frac{r}{4}}(y_{k+1}))} \leq \frac{C}{r^2}\|\partial_{y_j}\partial_{z_i}S_k(y,z)\|_{L^\infty(B_{\frac{r}{2}}(y_{k+1})\times B_{\frac{r}{2}}(y_{k+1}))}.$$

From the previous step, the thesis follows. □

Proof of Proposition 3.2. We prove (24). Fix $m \in \{0, \dots, N-1\}$ and let $Q_{m+1} \in \Sigma_{m+1} \cap B_{r_0/4}(P_{m+1})$. Up to a change of coordinates, we can assume that Q_{m+1} coincides with the origin. Let $\gamma^+ = \gamma_{m+1}(0)$, $\gamma^- = \gamma_m(0)$, $A = A(0)$, and define

$$\sigma_0(x) = (\gamma^+\chi_{\mathbb{R}^n_+}(x) + \gamma^-\chi_{\mathbb{R}^n_-}(x))A.$$

For simplicity, we write y in place of y_{m+1}. Let H be the fundamental solution associated to the elliptic operator $\mathrm{div}(\sigma_0\nabla\cdot)$. For $y \in \Omega_0$, let $G(\cdot, y)$ be the weak solution to the boundary value problem (17). Define

$$R(x,y) := G(x,y) - H(x,y).$$

For $y \in \Omega_0$, $R(\cdot, y)$ is a weak solution to

$$\begin{cases} \mathrm{div}(\sigma\nabla R(\cdot,y)) + qR(\cdot,y) = \mathrm{div}((\sigma_0-\sigma)\nabla H(\cdot,y)) - qH(\cdot,y) & \text{in } \Omega_0, \\ R(\cdot,y) = -H(\cdot,y) & \text{on } \partial\Omega_0 \setminus \Sigma_0, \\ \sigma\nabla R(\cdot,y)\cdot\nu + iR(\cdot,y) = -\sigma\nabla H(\cdot,y)\cdot\nu - iH(\cdot,y) & \text{on } \Sigma_0. \end{cases}$$

By Green's identity, one derives

$$R(x,y) = -\int_{\Omega_0}(\sigma-\sigma_0)(z)\nabla_z H(z,y)\cdot\nabla_z G(z,x)\,\mathrm{d}z + \int_{\Omega_0} H(z,y)q(z)G(z,x)\,\mathrm{d}z + \\ + \int_{\partial\Omega_0\setminus\Sigma_0}\sigma(z)\nabla_z G(z,x)\cdot\nu H(z,y)\,\mathrm{d}S(z) - \int_{\Sigma_0}[\sigma_0\nabla_z H(z,y)\cdot\nu + iH(z,y)]G(z,x)\,\mathrm{d}S(z).$$

Set $B = B_{r_0/4}(Q_{m+1})$ and define

$$\tilde{R}(x,y) = \int_B(\sigma_0-\sigma)(z)\nabla_z H(z,y)\cdot\nabla_z G(z,x)\,\mathrm{d}z + \int_B H(z,y)q(z)G(z,x)\,\mathrm{d}z$$

Since

$$|\nabla_y(R(x,y) - \tilde{R}(x,y))| \leq C,$$

and

$$|\nabla^2_y(R(x,y) - \tilde{R}(x,y))| \leq C,$$

one has to study only the asymptotic behaviour of $\nabla_y\tilde{R}(x,y)$ and $\nabla^2_y\tilde{R}(x,y)$. Let us prove an upper bound for $\nabla_y\tilde{R}(x,y)$. Set $B' = B'_{r_0/4}$,

$$
\begin{aligned}
&B^+ = \{x \in B : x_n > 0\} \qquad B^- = \{x \in B : x_n < 0\},\\
&q^+ = q|_{B^+}, \quad q^- = q|_{B^-}, \qquad [q] = (q^+ - q^-)|_{B'},\\
&\gamma^+ = \gamma|_{B^+}, \quad \gamma^- = \gamma|_{B^-}, \qquad [\sigma] = (\sigma^+ - \sigma^-)|_{B'} = (\gamma^+ - \gamma^-)|_{B'} A|_{B'}.
\end{aligned}
$$

It turns out that for $i = 1, \dots, n$,

$$
\begin{aligned}
\partial_{y_i} \tilde{R}(x, y) &= -\int_B \partial_{y_i}((\sigma - \sigma_0)(z)\nabla_z H(z, y)) \cdot \nabla_z G(z, x) \, \mathrm{d}z + \int_B \partial_{y_i} H(z, y) q(z) G(z, x) \, \mathrm{d}z\\
&= \int_B \partial_{z_i}((\sigma - \sigma_0)(z)\nabla_z H(z, y)) \cdot \nabla_z G(z, x) \, \mathrm{d}z - \int_B \partial_{z_i} H(z, y) q(z) G(z, x) \, \mathrm{d}z\\
&= \int_{\partial B} (\sigma - \sigma_0)(z)\nabla_z H(z, y) \cdot \nabla_z G(z, x) e_i \cdot \nu \, \mathrm{d}z - \int_{\partial B} H(z, y) q(z) G(z, x) e_i \cdot \nu \, \mathrm{d}z\\
&- \int_{B'} [(\sigma - \sigma_0)(z')]\nabla_z H(z', y) \cdot \nabla_z G(z', x) e_i \cdot e_n \, \mathrm{d}z' + \int_{B'} H(z', y)[q(z')] G(z', x) e_i \cdot e_n \, \mathrm{d}z'\\
&- \int_B (\sigma - \sigma_0)(z)\nabla_z H(z, y) \cdot \partial_{z_i} \nabla_z G(z, x) \, \mathrm{d}z + \int_B H(z, y) \partial_{z_i}(q(z) G(z, x)) \, \mathrm{d}z.
\end{aligned}
\tag{78}
$$

Notice that $\partial_{z_i}(\sigma - \sigma_0)(z)$ and $\partial_{z_i} q(z)$ are well-defined on $B \setminus B'$. The first and second integrals on the right-hand side of (78) can be easily bounded by a positive constant that depends on the a-priori data only. The fifth and sixth ones are dominated by

$$
\int_B |(\sigma - \sigma_0)(z)| \, |\nabla_z H(z, y)| \, |\partial_{z_i} \nabla_z G(z, x)| \, \mathrm{d}z \le C \int_B |z||z - y|^{1-n}|z - x|^{-n} \le C|x - y|^{1-n+\theta_3},
$$

with $\theta_3 \in (0, 1)$. Since $|x - y|^2 = |x_n + r|^2 + |x'|^2 \ge r^2$, one derives

$$
\int_B |(\sigma - \sigma_0)(z)| \, |\nabla_z H(z, y)| \, |\partial_{z_i} \nabla_z G(z, x)| \, \mathrm{d}z \le C r^{1-n+\theta_3}.
$$

Notice that when $i \neq n$, the third and fourth integrals are equal to zero, hence

$$
|\partial_{y_i} \tilde{R}(x, y)| \le C|x - y|^{1-n+\theta_3}.
$$

When $i = j = n$,

$$
\begin{aligned}
&\Big| \int_{B'} [(\sigma - \sigma_0)(z')]\nabla_z H(z', y) \cdot \nabla_z G(z', x) \, \mathrm{d}z' +\\
&\qquad + \int_{B'} \partial_{y_j} H(z', y)[q(z)] G(z', x) \, \mathrm{d}z' \Big| \le\\
&\qquad \le C \int_{B'} |z'| \cdot |z' - y|^{-n} \cdot |z' - x|^{1-n} \, \mathrm{d}z \le C|x - y|^{2-n-\alpha},
\end{aligned}
$$

with $0 < \alpha < 1$. Hence we conclude that

$$|\partial_{y_n}\tilde{R}(x,y)| \le C|x-y|^{1-n+\theta_3}, \qquad \text{with } \theta_3 \in (0,1).$$

The upper bound for $\nabla_y^2\tilde{R}(x,y)$ follows by similar computations. Indeed, by further differentiation, one derives

$$\begin{aligned}
\partial_{y_j}\partial_{y_i}\tilde{R}(x,y) = &\int_{\partial B} ((\sigma-\sigma_0)(z)\partial_{y_j}\nabla_z H(z,y))\cdot\nabla_z G(z,x)e_i\cdot\nu \,\mathrm{d}z- \\
&-\int_{\partial B} \partial_{y_j}H(z,y)q(z)G(z,x)e_i\cdot\nu\,\mathrm{d}z+ \\
&-\int_{B'}[(\sigma-\sigma_0)(z')]\partial_{y_j}\nabla_z H(z',y)\cdot\nabla_z G(z',x)e_i\cdot e_n\,\mathrm{d}z'+ \\
&+\int_{B'}\partial_{y_j}H(z',y)[q(z')]G(z',x)e_i\cdot e_n\,\mathrm{d}z'+ \\
&-\int_{B}\partial_{y_j}(\sigma-\sigma_0)(z)\nabla_z H(z,y))\cdot\partial_{z_i}\nabla_z G(z,x)\,\mathrm{d}z+ \\
&+\int_{B}\partial_{y_j}H(z,y)\partial_{z_i}(q(z)G(z,x))\,\mathrm{d}z.
\end{aligned} \tag{79}$$

The first and second integrals on the righthand side of (79) can be easily bounded. The fifth and sixth ones are dominated by

$$\Big|\int_B \partial_{y_j}(\sigma-\sigma_0)(z)\,\nabla_z H(z,y)\,\partial_{z_i}\nabla_z G(z,x)\,\mathrm{d}z\Big| \le C\int_B |z-y|^{1-n}|z-x|^{-n} \le C|x-y|^{1-n}.$$

Since $|x-y|^2 = |x_n+r|^2+|x'|^2 \ge r^2$, one derives

$$\Big|\int_B \partial_{y_j}(\sigma-\sigma_0)(z)\,\nabla_z H(z,y)\,\partial_{z_i}\nabla_z G(z,x)\,\mathrm{d}z\Big| \le Cr^{1-n}.$$

Notice that when $(i,j)\neq(n,n)$, the third and fourth integrals are equal to zero, hence

$$|\partial_{y_j}\partial_{y_i}\tilde{R}(x,y)| \le C|x-y|^{1-n}.$$

When $i=j=n$,

$$\begin{aligned}
\Big|&\int_{B'}[(\sigma-\sigma_0)(z')]\partial_{y_j}\nabla_z H(z',y)\cdot\nabla_z G(z,x)e_i\cdot e_n\,\mathrm{d}z'+ \\
&+\int_{B'}\partial_{y_j}H(z',y)[q(z)]G(z',x)e_i\cdot e_n\,\mathrm{d}z'\Big| \le \\
&\le C\int_{B'}|z'|\cdot\frac{1}{|z'-y|^n}\cdot\frac{1}{|z'-x|^{1-n}}\,\mathrm{d}z \le C|x-y|^{2-n}.
\end{aligned}$$

Hence,

$$|\partial^2_{y_n}\tilde{R}(x,y)| \le C|x-y|^{1-n}. \quad \square$$

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Appendix A

Proof of Theorem 2.2 (Stability at the boundary). Let $\{x_1, \dots, x_n\}$ be a coordinate system with origin at P_1. For $y, z \in D_0$, the following identities hold:

$$\begin{aligned}&\int_\Sigma [\sigma^{(2)}(x)\nabla_x G_2(x,z)\cdot\nu\, G_1(x,y) - \sigma^{(1)}(x)\nabla_x G_1(x,y)\cdot\nu\, G_2(x,z)]\, \mathrm{d}S(x) = \\ &= \int_\Omega [(\sigma^{(1)}-\sigma^{(2)})(x)\nabla_x G_1(x,y)\cdot\nabla_x G_2(x,z) + (q^{(2)}-q^{(1)})(x)G_1(x,y)G_2(x,z)]\, \mathrm{d}x,\end{aligned} \tag{A.1}$$

and

$$\begin{aligned}&\int_\Sigma [\sigma^{(2)}(x)\nabla_x\partial_{z_n} G_2(x,z)\cdot\nu\, \partial_{y_n}G_1(x,y) - \sigma^{(1)}(x)\nabla_x\partial_{y_n} G_1(x,y)\cdot\nu\, \partial_{z_n}G_2(x,z)]\, \mathrm{d}S(x) = \\ &= \int_\Omega [(\sigma^{(1)}-\sigma^{(2)})(x)\nabla_x\partial_{y_n} G_1(x,y)\cdot\nabla_x\partial_{z_n} G_2(x,z) + (q^{(2)}-q^{(1)})(x)\partial_{y_n}G_1(x,y)\partial_{z_n}G_2(x,z)]\, \mathrm{d}x.\end{aligned} \tag{A.2}$$

By (37) and (A.1),

$$\Big|\int_\Sigma [\sigma^{(2)}\nabla_x G_2(x,z)\cdot\nu\, G_1(x,y) - \sigma^{(1)}\nabla_x G_1(x,y)\cdot\nu\, G_2(x,z)]\, \mathrm{d}S(x)\Big| \le C\varepsilon\,(d(y)d(z))^{1-\frac{n}{2}}, \tag{A.3}$$

where $d(y)$ denotes the distance between y and Ω. Notice that the norm $\|\gamma_1^{(1)}-\gamma_1^{(2)}\|_{L^\infty(D_1)}$ can be evaluated in terms of the quantities

$$\|\gamma_1^{(1)}-\gamma_1^{(2)}\|_{L^\infty(\Sigma_1\cap B_{r_0/4}(P_1))} \quad \text{and} \quad |\partial_\nu(\gamma_1^{(1)}-\gamma_1^{(2)})(P_1)|.$$

Let $\rho = r_0/4$, let $r \in (0, \bar{r}/8)$, where $\bar{r}$ is the constant of Proposition 3.3. Set $w = P_1 + r\nu(P_1)$, where $\nu(P_1)$ is the outward unit normal of ∂D_1 at P_1. Consider

$$S_0(w,w) = I_1(w) + I_2(w), \tag{A.4}$$

with

$$\begin{aligned} I_1(w) = &\int_{B_\rho(P_1)\cap D_1} (\gamma_1^{(1)}-\gamma_1^{(2)})(x)\, A(x)\,\nabla_x G_1(x,w)\cdot\nabla_x G_2(x,w)\, \mathrm{d}x + \\ &+ \int_{B_\rho(P_1)\cap D_1} (q_1^{(2)}-q_1^{(1)})(x)\, G_1(x,w)\cdot G_2(x,w)\, \mathrm{d}x,\end{aligned}$$

and

$$I_2(w) = \int\limits_{\Omega\setminus(B_\rho(P_1)\cap D_1)} (\sigma^{(1)} - \sigma^{(2)})(x)\,\nabla_x G_1(x,w)\cdot\nabla_x G_2(x,w)\,\mathrm{d}x + \\ + \int\limits_{\Omega\setminus(B_\rho(P_1)\cap D_1)} (q^{(2)} - q^{(1)})(x)\,G_1(x,w)\cdot G_2(x,w)\,\mathrm{d}x.$$

The volume integrals of $I_2(w)$ can be bounded from above via Caccioppoli inequality (see also [9, Proposition 3.1]):

$$|I_2(w)| \le CE\rho^{2-n}. \tag{A.5}$$

Regarding $I_1(w)$, notice that there exists $x^* \in \overline{\Sigma_1 \cap B_{r_0/4}(P_1)}$ such that

$$(\gamma_1^{(1)} - \gamma_1^{(2)})(x^*) = \|\gamma_1^{(1)} - \gamma_1^{(2)}\|_{L^\infty(\Sigma_1\cap B_{r_0/4}(P_1))}. \tag{A.6}$$

By (A.6),

$$I_1(w) = \int\limits_{B_\rho(P_1)\cap D_1} (\gamma_1^{(1)} - \gamma_1^{(2)})(x^*)\,A(x)\,\nabla_x G_1(x,w)\cdot\nabla_x G_2(x,w)\,\mathrm{d}x + \\ + \int\limits_{B_\rho(P_1)\cap D_1} B_1\cdot(x-x^*)\,A(x)\,\nabla_x G_1(x,w)\cdot\nabla_x G_2(x,w)\,\mathrm{d}x + \\ + \int\limits_{B_\rho(P_1)\cap D_1} (q_1^{(2)} - q_1^{(1)})(x)\,G_1(x,w)\cdot G_2(x,w)\,\mathrm{d}x.$$

By the asymptotic estimate (21), one obtains

$$I_1(w) \ge \|\gamma_1^{(1)} - \gamma_1^{(2)}\|_{L^\infty(\Sigma_1\cap B_{r_0/4})}\Big\{\int\limits_{B_\rho(P_1)\cap D_1} A(x)\nabla_x H_1(x,w)\cdot\nabla_x H_2(x,w)\,\mathrm{d}x - \\ - \int\limits_{B_\rho(P_1)\cap D_1} |x-w|^{2(1-n)+\theta_1}\,\mathrm{d}x - \int\limits_{B_\rho(P_1)\cap D_1} |x-w|^{2(1-n+\theta_1)}\,\mathrm{d}x\Big\} - \\ - CE\int\limits_{B_\rho(P_1)\cap D_1} |x||x-w|^{2(1-n)}\,\mathrm{d}x - CE\int\limits_{B_\rho(P_1)\cap D_1} |x-w|^{2(2-n)}\,\mathrm{d}x.$$

It turns out that

$$|I_1(w)| \ge C\|\gamma_1^{(1)} - \gamma_1^{(2)}\|_{L^\infty(\Sigma_1\cap B_{r_0/4}(P_1))} r^{2-n} - CEr^{2-n+\theta_1} - CEr^{3-n}. \tag{A.7}$$

If we rearrange the inequalities (A.7) and (A.5) together with (A.3), we derive

$$\|\gamma_1^{(1)} - \gamma_1^{(2)}\|_{L^\infty(\Sigma_1\cap B_{r_0/4}(P_1))} r^{2-n} \le CEr^{3-n} + CEr^{2-n+\theta_1} + C\varepsilon r^{2-n} + CE\rho^{2-n}. \tag{A.8}$$

Multiply (A.8) by r^{n-2}, then for $r \to 0^+$,

$$\|\gamma_1^{(1)} - \gamma_1^{(2)}\|_{L^\infty(\Sigma_1\cap B_{r_0/4}(P_1))} \le C\varepsilon. \tag{A.9}$$

A similar estimate can be derived for the derivative of $\gamma_1^{(1)} - \gamma_1^{(2)}$ along the normal direction ν at P_1 by means of an argument analogous to [6, Theorem 2.3]. From Taylor's formula applied in a neighbourhood of the point P_1, one derives

$$(\gamma_1^{(1)} - \gamma_1^{(2)})(x) = (\gamma_1^{(1)} - \gamma_1^{(2)})(P_1) + (D_T(\gamma_1^{(1)} - \gamma_1^{(2)})(P_1)) \cdot (x - P_1)' + \\ + (\partial_\nu(\gamma_1^{(1)} - \gamma_1^{(2)})(P_1)) \cdot (x - P_1)_n.$$

Hence,

$$\begin{aligned}
&|\partial_{y_n}\partial_{z_n} S_0(w,w)| \geq \\
&\geq \Big| \int_{B_\rho(P_1)\cap D_1} \partial_\nu(\gamma_1^{(1)} - \gamma_1^{(2)})(P_1) \cdot (x - P_1)_n A(x)\nabla_x \partial_{y_n} G_1(x,w) \cdot \nabla_x \partial_{z_n} G_2(x,w)\,\mathrm{d}x\Big| - \\
&- \Big| \int_{B_\rho(P_1)\cap D_1} D_T(\gamma_1^{(1)} - \gamma_1^{(2)})(P_1) \cdot (x - P_1)' A(x)\nabla_x \partial_{y_n} G_1(x,w) \cdot \nabla_x \partial_{z_n} G_2(x,w)\,\mathrm{d}x\Big| - \\
&- \Big| \int_{B_\rho(P_1)\cap D_1} (\gamma_1^{(1)} - \gamma_1^{(2)})(P_1) A(x)\nabla_x \partial_{y_n} G_1(x,w) \cdot \nabla_x \partial_{z_n} G_2(x,w)\,\mathrm{d}x\Big| - \\
&- \Big| \int_{B_\rho(P_1)\cap D_1} (q_1^{(2)} - q_1^{(1)})(x)\partial_{y_n} G_1(x,w) \cdot \partial_{z_n} G_2(x,w)\,\mathrm{d}x\Big| \\
&- \Big| \int_{\Omega\setminus(B_\rho(P_1)\cap D_1)} (\sigma^{(1)} - \sigma^{(2)})(x)\partial_{y_n}\nabla_x G_1(x,w) \cdot \partial_{z_n}\nabla_x G_2(x,w)\,\mathrm{d}x\Big| - \\
&- \Big| \int_{\Omega\setminus(B_\rho(P_1)\cap D_1)} (q^{(1)} - q^{(2)})(x)\partial_{y_n} G_1(x,w) \cdot \partial_{z_n} G_2(x,w)\,\mathrm{d}x\Big| \\
&= I_{11} - I_{12} - I_{13} - I_{14} - I_{15} - I_{16}.
\end{aligned}$$

To estimate I_{11} from below, we add and subtract the biphase fundamental solution and by (22), one derives

$$I_{11} \geq C|\partial_\nu(\gamma_1^{(1)} - \gamma_1^{(2)})(P_1)| r^{1-n} - CEr^{1-n+\theta_2}. \tag{A.10}$$

To estimate the terms I_{12} and I_{13}, notice that

$$|(\gamma_1^{(1)} - \gamma_1^{(2)})(P_1)| + C|D_T(\gamma_1^{(1)} - \gamma_1^{(2)})(P_1)| \leq C\|\gamma_1^{(1)} - \gamma_1^{(2)}\|_{L^\infty(\Sigma_1\cap B_{r_0/4})} \leq C\varepsilon.$$

Regarding the integral I_{14}, one bounds it from above as

$$\begin{aligned}
I_{14} &\leq \|q_1^{(2)} - q_1^{(1)}\|_{L^\infty(D_1)} \int_{D_1\cap B_\rho} |\partial_{y_n} G_1(x,w)||\partial_{z_n} G_2(x,w)|\,\mathrm{d}x \\
&\leq C \int_{D_1\cap B_\rho} |x - w|^{2(1-n)} \leq C\, r^{2-n}.
\end{aligned}$$

The integral I_{15} and I_{16} can be bounded by means of [9, Proposition 3.1] as

$$I_{15}, I_{16} \leq CE\rho^{-n}.$$

To sum up, we have

$$|\partial_\nu(\gamma_1^{(1)} - \gamma_1^{(2)})(P_1)|r^{1-n} \le |\partial_{y_n}\partial_{z_n} S_0(w,w)| + C\{Er^{1-n+\theta_2} + \varepsilon r^{-n}\}. \tag{A.11}$$

Since

$$|\partial_{y_n}\partial_{z_n} S_0(w,w)| \le C\varepsilon r^{-n},$$

one derives

$$|\partial_\nu(\gamma_1^{(1)} - \gamma_1^{(2)})(P_1)|r^{1-n} \le C\{Er^{1-n+\theta_2} + \varepsilon r^{-n}\}. \tag{A.12}$$

Multiply (A.12) by r^{n-1} to obtain

$$|\partial_\nu(\gamma_1^{(1)} - \gamma_1^{(2)})(P_1)| \le C\{Er^{\theta_2} + \varepsilon r^{-1}\}.$$

By optimizing w.r.t. r, it turns out that

$$|\partial_\nu(\gamma_1^{(1)} - \gamma_1^{(2)})(P_1)| \le C(\varepsilon + E)\left(\frac{\varepsilon}{\varepsilon + E}\right)^{\frac{\theta_2}{\theta_2+1}}, \tag{A.13}$$

and we set $\eta_1 = \dfrac{\theta_2}{\theta_2 + 1}$. Hence, we conclude that

$$\|\gamma_1^{(1)} - \gamma_1^{(2)}\|_{L^\infty(D_1)} \le C(\varepsilon + E)\left(\frac{\varepsilon}{\varepsilon + E}\right)^{\eta_1}. \tag{A.14}$$

Stability at the boundary for q Our goal is to derive a bound for $\|q_1^{(1)} - q_1^{(2)}\|_{L^\infty(D_1)}$ in terms of (A.14). Notice that the norm $\|q_1^{(2)} - q^{(1)}\|_{L^\infty(D_1)}$ can be evaluated in terms of the following quantities:

$$\|q_1^{(2)} - q_1^{(1)}\|_{L^\infty(\Sigma_1 \cap B_{r_0/4}(P_1))} \quad \text{and} \quad |\partial_\nu(q_1^{(2)} - q_1^{(1)})(P_1)|. \tag{A.15}$$

Let $\rho = r_0/4$, $r \in (0, \bar{r}/8)$ and set $w = P_1 + r\,\nu(P_1)$. Consider

$$\partial_{y_n}\partial_{z_n} S_0(w,w) = \partial_{y_n}\partial_{z_n} I_1(w) + \partial_{y_n}\partial_{z_n} I_2(w),$$

with $w = P_1 + r\nu(P_1)$, as above. The term $\partial_{y_n}\partial_{z_n} I_2(w)$ can be bounded from above as

$$\partial_{y_n}\partial_{z_n} I_2(w) \le C\,E\rho^{-n}.$$

To determine a lower bound for $\partial_{y_n}\partial_{z_n} I_1(w)$, first notice that there exists a point $\bar{x} \in \overline{\Sigma_1 \cap B_\rho(P_1)}$ such that

$$(q_1^{(2)} - q_1^{(1)})(\bar{x}) = \|q_1^{(2)} - q_1^{(1)}\|_{L^\infty(\Sigma_1 \cap B_{r_0/4}(P_1))}.$$

By (21) and (A.14) one derives

$$C\|q_1^{(2)} - q_1^{(1)}\|_{L^\infty(\Sigma_1 \cap B_{r_0/4}(P_1))} r^{2-n} \le |\partial_{y_n}\partial_{z_n} I_1(w)| + CEr^{2-n+\theta_1} + C(\varepsilon + E)\left(\frac{\varepsilon}{\varepsilon + E}\right)^{\eta_1} r^{-n}.$$

By (A.2),

$$|\partial_{y_n}\partial_{z_n}S_0(w,w)|\le C\,\varepsilon\,r^{-n},$$

hence, if we collect the upper bound for $I_2(w)$ and the lower bound for $I_1(w)$, we derive

$$\|q_1^{(2)}-q_1^{(1)}\|_{L^\infty(\Sigma_1\cap B_{r_0/4}(P_1))}r^{2-n}\le C\Big\{\varepsilon\,r^{-n}+Er^{2-n+\theta_1}+(\varepsilon+E)\left(\frac{\varepsilon}{\varepsilon+E}\right)^{\eta_1}r^{-n}+E\Big\}.$$

Multiply by r^{n-2} to obtain

$$\|q_1^{(2)}-q_1^{(1)}\|_{L^\infty(\Sigma_1\cap B_{r_0/4}(P_1))}\le C(\varepsilon+E)\Big\{\left(\frac{\varepsilon}{\varepsilon+E}\right)^{\eta_1}r^{-2}+Er^{\theta_1}\Big\}.$$

By optimizing with respect to r, one concludes that

$$\|q_1^{(2)}-q_1^{(1)}\|_{L^\infty(\Sigma_1\cap B_{r_0/4}(P_1))}\le C(E+\varepsilon)\left(\frac{\varepsilon}{\varepsilon+E}\right)^{\frac{\eta_1\theta_1}{\theta_1+2}}. \tag{A.16}$$

To estimate $|\partial_\nu(q_1^{(2)}-q_1^{(1)})(P_1)|$, consider the singular solution $\partial^2_{y_iy_j}\partial^2_{z_iz_j}S_0(w,w)$ and split it as the sum of the terms

$$\begin{aligned}I_1^{ij}(w)=&\int_{D_1\cap B_\rho(P_1)}(\sigma_1^{(1)}-\sigma_1^{(2)})(x)\nabla_x\partial^2_{y_iy_j}G_1(x,w)\cdot\partial^2_{z_iz_j}G_2(x,w)\,\mathrm{d}x+\\&+\int_{D_1\cap B_\rho(P_1)}(q_1^{(2)}-q_1^{(1)})(x)\partial^2_{y_iy_j}G_1(x,w)\cdot\partial^2_{z_iz_j}G_2(x,w)\,\mathrm{d}x,\end{aligned}$$

and

$$\begin{aligned}I_2^{ij}(w)=&\int_{\Omega\setminus(D_1\cap B_\rho(P_1))}(\sigma^{(1)}-\sigma^{(2)})(x)\nabla_x\partial^2_{y_iy_j}G_1(x,w)\cdot\nabla_x\partial^2_{z_iz_j}G_2(x,w)\,\mathrm{d}x+\\&+\int_{\Omega\setminus(D_1\cap B_\rho(P_1))}(q^{(2)}-q^{(1)})(x)\partial^2_{y_iy_j}G_1(x,w)\cdot\partial^2_{z_iz_j}G_2(x,w)\,\mathrm{d}x.\end{aligned}$$

Set $I_m(w)=\{I_m^{ij}(w)\}_{i,j=1,\dots,n}$. Denote by $|I_m(w)|$ the Euclidean norm of the matrix $I_m(w)$. The upper bound for $|I_2(w)|$ is given by

$$|I_2(w)|\le CE\rho^{-(n+2)},$$

where C is a positive constant that depends on the a-priori data only. For the lower bound for $I_1(w)$,

$$\begin{aligned}|I_1(w)|\ge\frac{1}{n}\sum_{i,j=1}^{n}\Big\{&\Big|\int_{D_1\cap B_\rho(P_1)}(\partial_\nu(q_1^{(2)}-q_1^{(1)})(P_1))\cdot(x-P_1)_n\partial^2_{y_iy_j}G_1(x,w)\cdot\partial^2_{z_iz_j}G_2(x,w)\,\mathrm{d}x\Big|-\\&-\Big|\int_{D_1\cap B_\rho(P_1)}(D_T(q_1^{(2)}-q_1^{(1)})(P_1))\cdot(x-P_1)'\partial^2_{y_iy_j}G_1(x,w)\cdot\partial^2_{z_iz_j}G_2(x,w)\,\mathrm{d}x\Big|-\\&-\Big|\int_{D_1\cap B_\rho(P_1)}(q_1^{(2)}-q_1^{(1)})(P_1)\partial^2_{y_iy_j}G_1(x,w)\cdot\partial^2_{z_iz_j}G_2(x,w)\,\mathrm{d}x\Big|\Big\}-\end{aligned}$$

$$-\Big|\int\limits_{D_1\cap B_\rho(P_1)} (\sigma_1^{(2)}-\sigma_1^{(1)})(x)\partial^2_{y_iy_j}\nabla_x G_1(x,w)\cdot\partial^2_{z_iz_j}\nabla_x G_2(x,w)\,\mathrm{d}x\Big|.$$

Since

$$|(q_1^{(2)}-q_1^{(1)})(P_1)|+C|(D_T(q_1^{(2)}-q_1^{(1)})(P_1))|\le C\|q_1^{(2)}-q_1^{(1)}\|_{L^\infty(\Sigma_1\cap B_{r_0/4}(P_1))},$$

by (A.16) and (24), one derives

$$\begin{aligned}|I_1(w)|\ge C|(\partial_\nu(q_1^{(2)}-q_1^{(1)})(P_1))|r^{1-n}-C(E+\varepsilon)\left(\frac{\varepsilon}{\varepsilon+E}\right)^{\frac{\eta_1\theta_1}{\theta_1+2}}r^{-n}-\\ -CEr^{1+\theta_2-n}-C(\varepsilon+E)\left(\frac{\varepsilon}{\varepsilon+E}\right)^{\eta_1}r^{-2-n}.\end{aligned} \tag{A.17}$$

Since for $y,z\in(D_0)_{r_0/3}$,

$$\begin{aligned}&\int\limits_\Sigma[\sigma^{(2)}(x)\nabla_x\partial^2_{z_n}G_2(x,z)\cdot\nu\,\partial^2_{y_n}G_1(x,y)-\sigma^{(1)}(x)\nabla_x\partial^2_{y_n}G_1(x,y)\cdot\nu\,\partial^2_{z_n}G_2(x,z)]\,\mathrm{d}S(x)=\\ &=\int\limits_\Omega[(\sigma^{(1)}-\sigma^{(2)})(x)\nabla_x\partial^2_{y_n}G_1(x,y)\cdot\nabla_x\partial^2_{z_n}G_2(x,z)+(q^{(2)}-q^{(1)})(x)\partial^2_{y_n}G_1(x,y)\partial^2_{z_n}G_2(x,z)]\,\mathrm{d}x,\end{aligned}$$

it turns out that

$$|\partial^2_{y_n}\partial^2_{z_n}S_0(w,w)|\le C\,\varepsilon\,r^{-2-n}. \tag{A.18}$$

By (A.17) and (A.18), one derives

$$\begin{aligned}|(\partial_\nu(q_1^{(2)}-q_1^{(1)})(P_1))|r^{1-n}\le C(E+\varepsilon)\left(\frac{\varepsilon}{\varepsilon+E}\right)^{\frac{\eta_1\theta_1}{\theta_1+2}}r^{-n}+\\ +C(\varepsilon+E)\left(\frac{\varepsilon}{\varepsilon+E}\right)^{\eta_1}r^{-2-n}+CEr^{1+\theta_2-n}C\varepsilon r^{-1-n}.\end{aligned}$$

Multiply by r^{n-1} the last equation and optimize with respect to r leads to the estimate

$$|(\partial_\nu(q_1^{(2)}-q_1^{(1)})(P_1))|\le C(E+\varepsilon)\left(\frac{\varepsilon}{\varepsilon+E}\right)^{\eta_2},$$

with $\eta_2\in(0,1)$. □

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