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# On the Lichnerowicz operator in traversable wormhole spacetimes

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## Abstract

The evaluation of Casimir energies in curved background spacetimes is an essential ingredient to study the stability of traversable wormholes. In practice one has to calculate the contribution of the transverse-traceless component of the metric perturbation on a curved spacetime background. This implies the study of an eigenvalue equation involving a modified form of the Lichnerowicz operator. For arbitrary background spacetimes, however, such an operator does not display transverse-traceless properties, a fact that impedes the determination of the eigenvalues. Against this background, we show that the problem can be circumvented. Casimir energies can be calculated by gauging the original form of the modified Lichnerowicz operator into a transverse-traceless one.

## 1. Introduction

Traversable wormholes are spacetime geometries emerging as solutions of Einstein equations. Conventionally they are thought as fascinating configurations that might have formed in remote regions of the Universe. Nowadays such a perspective has changed: The understanding of the physics governing wormholes is instrumental for the concrete realization of laboratory devices for interstellar travels [1].

The current efforts in such a research field are focused on the conditions of traversability of wormholes. In practice, the wormhole throat can be stable only if there is a source of energy able to balance the gravitational pull. Contrary to the case of ordinary stars, such a balance implies the source to be of *exotic* type. The latter is a term used to indicate a non standard matter that violates the null energy conditions (NEC), namely the positiveness of the energy momentum tensor, $T_{\mu\nu} k^\mu k^\nu \geqslant 0$ for any null vector $k^\mu$. Wormholes necessitate such a violation of NEC since, according to the Raychaudhuri equation [2], the expansion of timelike congruence becomes negative at the throat while remains positive elsewhere. In other words, a set of world lines undergoes a contraction at the throat, indicating an inevitable collapse into a singularity unless exotic matter locally counteracts it. Although the pressure of ordinary matter can, in general, counteract a collapse, it would not be enough high to reestablish the positiveness of the congruence expansion on the other side of the throat [3]. As far as we know, the Casimir energy represents the only artificial source of exotic matter realizable in a laboratory [1, 4–7]. Alternatively, a local violation of energy conditions can occur due to the quantum fluctuation of the graviton. This fact propelled the idea of *self-sustained traversable wormholes*, that have been introduced in [8–10]. To study such wormholes one considers the semiclassical Einstein equations,

$$G_{\mu\nu} = \kappa \langle T_{\mu\nu} \rangle^{\mathrm{ren}}, \qquad \langle T_{\mu\nu} \rangle^{\mathrm{ren}} = -\frac{1}{\kappa} \langle \Delta G_{\mu\nu}(\bar{g}_{\mu\nu}, h_{\mu\nu}) \rangle^{\mathrm{ren}} \tag{1}$$

where the source term is the expectation value of the renormalized quantum stress tensor of the metric perturbation $h_{\mu\nu}$, with $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$. Equation (1) simplifies by focusing on the energy components, namely by a projection on a constant time spacelike hypersurface $\Sigma$. In such a way, one obtains Hamiltonian and energy densities, that, after integration, give the equation for the stability of the wormhole:

$$H_\Sigma^{(0)} = -E^\perp. \tag{2}$$

Here $E^\perp$ is the total regularized graviton one loop energy coming from the quantized stress tensor and $H_\Sigma^{(0)}$ is the classical term, coming from the Einstein tensor. Only the transverse traceless (TT) component of the graviton contribute to the energy $E^\perp$ and for this reason we introduced the superscript $\perp$. For a spherically symmetric line element of the form

$$ds^2 = -\exp(-2\Phi(r))dt^2 + \frac{dr^2}{1 - \frac{b(r)}{r}} + r^2 d\Omega^2, \tag{3}$$

where $\Phi(r)$ is the redshift function, $b(r)$ is the shape function and $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the line element of the unit sphere, the classical term reduces to

$$
\begin{aligned}
H_\Sigma^{(0)} &= \int_\Sigma d^3x \left[ (16\pi G) G_{ijkm} \pi^{ij} \pi^{km} - \frac{\sqrt{g}}{16\pi G}\, {}^3R \right] \\
&= -\frac{1}{16\pi G} \int_\Sigma d^3x\, \sqrt{g}\, {}^3R = -\frac{1}{2G} \int_{r_0}^\infty \frac{dr\, r^2}{\sqrt{1 - b(r)/r}} \frac{b'(r)}{r^2}.
\end{aligned} \tag{4}
$$

Here we have expressed the three dimensional scalar curvature $^{3\mathrm{R}}$ in terms of $b(r)$. The symbol $G_{ijkm}$ denotes the super-metric and $\pi^{ij}$ the super-momentum. Due to static conditions the kinetic term $G_{ijkm}\pi^{ij}\pi^{km}$ disappears. For a full derivation see [11]. At the one-loop approximation level, the energy $E^\perp$ is identified as a Casimir like energy against the fixed background. Its evaluation requires functional integration methods including the solution of an appropriate eigenvalue equation in terms of a modified Lichnerowicz operator (see [11] for the full derivation).

The conventional Lichnerowicz operator can be defined through its action on a tensor $h_{ij}$ as

$$
\begin{aligned}
(\triangle_L h)_{ij} &= \triangle h_{ij} - 2R_{ikjm}h^{km} + R_{ik}h_j^k + R_{jk}h_i^k \\
\triangle &= -\nabla^a \nabla_a,
\end{aligned} \tag{5}
$$

where Latin indexes run from 1 to 3 and $\nabla^a$ is the covariant derivative with respect to the 3-metric, $\bar{g}_{ij}$. For ease of notation we employ, in the following equations, the symbol $g_{ij}$ for the background without superscript -. As said above, we have to consider the TT component of the field $h_{ij}$ describing a spin 2 particle, namely

$$g^{ij}h_{ij}^\perp = 0, \qquad \nabla^i h_{ij}^\perp = 0. \tag{6}$$

Thus, the problem turns into the determination of the eigenvalues of the following modified Lichnerowicz operator,

$$(\tilde{\triangle}_L h^\perp)_{ij} = (\triangle_L h^\perp)_{ij} - 4R_i^k h_{kj}^\perp + {}^3R\, h_{ij}^\perp. \tag{7}$$

namely

$$(\tilde{\triangle}_L h^\perp)_{ij} = \lambda h_{ij}^\perp, \tag{8}$$

describing the energy spectrum of $h_{ij}^\perp$ resulting from (1). To have a well-posed equation in (8), one has, however, to make sure that the above operator does not change the TT properties of $h_{ij}^\perp$. This is, in general, a major issue because the l.h.s. is not a TT tensor for some kind of backgrounds [12].

The present paper aims to circumvent such difficulties and pave the way of a consistent study of wormhole stability within this formalism.

## 2. The Lichnerowicz operator for TT tensors

It is straightforward to see that the standard Lichnerowicz operator, $(\triangle_L h^\perp)_{ij}$, is traceless. This is, however, not enough to conclude that $(\tilde{\triangle}_L h^\perp)_{ij}$ is traceless too. Indeed, one has

$$\mathrm{Tr}\left[(\triangle_L h^\perp)_{ij} - 4R_{ik}h_j^{\perp\,k} + {}^3Rh_{ij}^\perp\right] = -4R_k^m h_m^{\perp\,k}. \tag{9}$$

As a first step, we aim to write such an operator in terms of a trace free part and a term determining the trace. For the line element (3), the mixed Ricci tensor $R_j^i$ is:

$$R_j^i = \left\{ \frac{b'(r)}{r^2} - \frac{b(r)}{r^3}, \frac{b'(r)}{2r^2} + \frac{b(r)}{2r^3}, \frac{b'(r)}{2r^2} + \frac{b(r)}{2r^3} \right\}. \tag{10}$$

Therefore, we can write

$$R_k^i h_j^{k\,\perp} = \underbrace{f(r)h_i^{\perp j}}_{\text{traceless}} + (R_1^1 - f(r))\delta_j^1\delta_1^i h_i^{\perp j}$$

where we have used the properties $R_2^2 = R_3^3 \equiv f(r)$ and $h_1^{\perp 1} + h_2^{\perp 2} + h_3^{\perp 3} = 0$. As a result, one finds

$$(\tilde{\triangle}_L\,h^\perp)_i^j = (\tilde{\triangle}_L\,h^\perp)_i^{j\,\mathrm{T}} + \left\{\frac{1}{2}\left[\frac{3b(r)}{r^3} - \frac{b'(r)}{r^2}\right]\delta_j^1\delta_1^i\right\}h_i^{\perp\,j} \tag{11}$$

with the trace free part defined as

$$(\tilde{\triangle}_L h^\perp)_i^{j\,\mathrm{T}} \equiv (\triangle_L h^\perp)_i^j - 4f(r)h_i^{\perp j} + {}^3Rh_i^{\perp j}. \tag{12}$$

Now we compute the divergence of the above trace free part, namely [12]

$$
\begin{aligned}
\nabla^m(\tilde{\triangle}_L\,h^\perp)_m^{j\,\mathrm{T}} &= \nabla^m(\triangle_L h^\perp)_m^j - \nabla^m(4f(r)h_m^{\perp\,j} - {}^3 Rh_m^{\perp\,j}) \\
&= \triangle\,(\nabla^m h_m^{\perp\,j}) + R_k^j(\nabla^m h_m^{\perp\,k}) + (\nabla^j R_m^k)h_k^{m\,\perp} - \nabla^m(4f(r)h_m^{\perp\,j} - {}^3Rh_m^{\perp\,j}) \\
&= (\nabla^j R_m^k)h_k^{\perp m} - 4(\nabla^m f(r))h_m^{\perp\,j} + (\nabla^{m3}R)h_m^{\perp\,j}.
\end{aligned} \tag{13}
$$

where we have used the transverse property $\nabla^m h_m^{\perp\,j} = 0$ and

$$\nabla^m(\triangle_L\,h^\perp)_m^{j\,\mathrm{T}} = \triangle\,(\nabla^m h_m^{\perp\,j}) + R_k^j(\nabla^m h_m^{\perp\,k}) + (\nabla^j R_m^k)h_k^{m\,\perp}.$$

In summary one finds:

$$\nabla^m(\tilde{\triangle}_L h^\perp)_m^{j\,\mathrm{T}} = \left\{-\frac{1}{2}\,g^{11}\partial_1\left[\frac{7b(r)}{r^3} - \frac{b'(r)}{r^2}\right]\delta_j^1\delta_1^m\right\}h_m^{j\,\perp}. \tag{14}$$

The r.h.s. of the above equation vanishes provided

$$\frac{7b(r)}{r^3} - \frac{b'(r)}{r^2} = \text{constant}.$$

Thus, one obtains

$$b(r) = Ar^3 \tag{15}$$

where $A$ is a constant coefficient. Accordingly, one has $A > 0$ for de Sitter space, $A < 0$ for Anti-de Sitter space and $A = 0$ for Minkowski space. In such cases, one finds that the trace in (11) vanishes. As a result, one can conclude, from the condition (15), that the operator, $(\tilde{\triangle}_L h^\perp)_{ij}$, is a TT tensor in case of constant or vanishing curvature.

In the presence of a gravitational source, the curvature is, in general, not constant. The operator $(\tilde{\triangle}_L h^\perp)_{ij}$ can, however, satisfy the transverse condition up to negligible terms if the curvature variation is small. The reference scale in such a case is the Planck mass cubed, $M_{\mathrm{P}}^3$. The vanishing of the trace in (11) requires the curvature itself to be small with respect to $M_{\mathrm{P}}^2$, namely

$$\frac{b}{r} \ll \frac{r^2}{L_{\mathrm{P}}^2}, \quad b' \ll \frac{r^2}{L_{\mathrm{P}}^2}, \tag{16}$$

where $L_{\mathrm{P}} = 1/M_{\mathrm{P}}$. As a result, the trace freedom is a stronger condition with respect to the small variation of the curvature. Both conditions are easily met in the large distance limit. It is sufficient to assume $r \gg L_{\mathrm{P}}$ for having the TT condition fulfilled, provided $b'$ is bounded.

At short distance, namely at the wormhole throat, the r.h.s. of (14) vanishes because of the presence of the metric coefficient $g^{11}$. The conditions (16) are fulfilled for a wormhole throat $r_{\mathrm{t}}$ such that $r_{\mathrm{t}} \gg L_{\mathrm{P}}$, being $b'(r_{\mathrm{t}}) < 1$ according to the flaring out condition. Up to now, the presented analysis has been focused on differential conditions. To calculate the sought eigenvalues, one has, however, to consider matrix elements, expressed in terms of the following integral

$$\int_\Sigma d^3x \sqrt{g}\,h_j^{\perp\,i}(\tilde{\triangle}_L h^\perp)_i^j, \tag{17}$$

resulting from the one loop Hamiltonian at the r.h.s of (1),

$$H_\Sigma^\perp = \frac{1}{4V}\int_\Sigma d^3x \sqrt{\bar{g}}\,G^{ijkm}\left[(2\kappa)K^{-1\perp}(x,x)_{ijkm} + \frac{1}{(2\kappa)}(\,\tilde{\triangle}_L\,)_j^a K^\perp(x,x)_{iakm}\right], \tag{18}$$

where

$$
K^{\perp}(\vec{x},\vec{y})_{iakm} = \sum_{\tau} \frac{h^{(\tau)\perp}_{ia}(\vec{x}) h^{(\tau)\perp}_{km}(\vec{y})}{2\ell(\tau)},
\tag{19}
$$

is the graviton propagator, $\ell(\tau)$ a set of variational parameters to be determined by minimizing (18) and $\tau$ denotes a complete set of indexes (see [11]).

One can try to circumvent the problem of the TT nature of the operator $(\tilde{\triangle}_L h^{\perp})_{ij}$, since integral relations generally demand softer conditions than differential relations. To this purpose we observe that

$$
\int_{\Sigma} d^3x \sqrt{g}\; h^{\perp\,i}_j (\tilde{\triangle}_L h^{\perp})^j_i = \int_{\Sigma} d^3x \sqrt{g}\; h^{\perp\,i}_j \left[ (\tilde{\triangle}_L h^{\perp})^j_i + \frac{4}{3}\, \delta^j_i (R^{\,k}_m\, h^{\perp\,m}_k) \right] =
$$
$$
= \int_{\Sigma} d^3x \sqrt{g}\; h^{\perp\,i}_j (\tilde{\triangle}_L\, h^{\perp})^{j\;\mathrm{T}}_i.
\tag{20}
$$

The integral of the term $R^{\,k}_m\, h^{\perp\,m}_k$ at the r.h.s. identically vanishes, being $\mathrm{Tr}\; h^{\perp\,m}_k = h^{\perp\,k}_k = 0$. Due to (9) and the generic relation

$$
(T_{ij})^{\mathrm{T}} = T_{ij} - \frac{1}{3} g_{ij} [\mathrm{Tr}\; T_{km}].
\tag{21}
$$

we obtained the trace free part. The operator $(\tilde{\triangle}_L h^{\perp})_{ij}$ is not traceless, but its integral is equivalent to that of its trace free part. This property is instrumental to prove that, at level of integral relations, the transverse property is satisfied too. After gauging the trace away by means of (20), the transverse property can be analyzed by considering just the integral of the trace free part (12). The latter can be written as

$$
(\tilde{\triangle}_L h^{\perp})^{\mathrm{T}}_{ij} \equiv (\triangle_L h^{\perp})_{ij} - 4\left( R_{ik} h^{\perp\,k}_j - \frac{1}{3}\, g_{ij}(R^{\,k}_m h^{\perp\,m}_k) \right) +^3 R h^{\perp}_{ij}.
\tag{22}
$$

We can further gauge the integral of $(\tilde{\triangle}_L h^{\perp})^{j\;\mathrm{T}}_i$ by adding a vanishing contribution whose integrand is traceless in order not to alter the trace free property, namely

$$
\int_{\Sigma} d^3x \sqrt{g}\; h^{\perp\,i}_j (\tilde{\triangle}_L h^{\perp})^{j\;\mathrm{T}}_i = \int_{\Sigma} d^3x \sqrt{g}\; h^{\perp\,i}_j [(\tilde{\triangle}_L h^{\perp})^{j\;\mathrm{T}}_i + (LM)^j_i],
\tag{23}
$$

where

$$
(LM)_{ij} = \nabla_i M_j + \nabla_j M_i - \frac{2}{3} g_{ij}(\nabla_k M^k).
\tag{24}
$$

One can check that the integral of $(LM)_{ij}$ is vanishing by integrating by parts the first two terms $\nabla_i M_j$ and using the transverse condition $\nabla_i h^{\perp\,i}_j = 0$. The third term vanishes because $h^{\perp\,i}_j$ is trace free.

At this point, one can suitably select $(LM)_{ij}$ to get the tranverse condition, namely

$$
0 = \nabla^i \left\{ \left[ (\triangle_L h^{\perp})_{ij} - 4\left( R_{im} h^{\perp\,m}_j - \frac{1}{3}\, g_{ij}(R_i^{\;m}\, h^{\perp\,i}_m) \right) + {}^3R h^{\perp}_{ij} \right] + (LM)_{ij} \right\} =
$$
$$
= (\nabla_j R_i^{\;m}) h^{\perp\,i}_m - \nabla^i \left[ 4 R_{im} h^{\perp\,m}_j - \frac{4}{3}\, g_{ij}(R_i^{\;m}\, h^{\perp\,i}_m) - {}^3R h^{\perp}_{ij} - (LM)_{ij} \right] =
$$
$$
= (\nabla_j R_i^{\;m}) h^{\perp\,i}_m - 4(\nabla^i f(r)) h^{\perp}_{ij} + (\nabla^{i3} R) h^{\perp}_{ij} + \nabla^i (LM)_{ij}.
\tag{25}
$$

provided (25) has solutions.

Alternatively one can consider, in place of $(LM)_{ij}$ in (23), an antisymmetric term of the kind

$$
(LN)_{ij} = \nabla_i N_j - \nabla_j N_i,
\tag{26}
$$

that is trace free and has a vanishing integral. Its covariant derivative is formally equivalent to the four current of the electromagnetic field tensor, namely

$$
\nabla^i (LN)^j_i = S^j.
\tag{27}
$$

In such a case, one can select $S^j$ provided the equivalent of (25) for $(LN)_{ij}$ has solutions.

In conclusion, even if the operator is not a TT tensor for arbitrary backgrounds, its integral (17) is equivalent to the integral of an operator (23) that display TT properties.

## 3. Final remarks

In this paper we have presented a solution to an open issue in the literature, namely the calculation of graviton energies at the one-loop approximation associated to a Lichnerowicz operator. In case of spherically symmetric spacetimes, such energies come from the TT component of the perturbation, namely

$$E^{\perp} = \frac{1}{3}\sum_{\tau}\left[\sqrt{\lambda_1^2(\tau)} + \sqrt{\lambda_2^2(\tau)}\right] \tag{28}$$

where the eingenvalues correspond to the two graviton polarizations. Unfortunately, the operator $(\tilde{\triangle}_L h^{\perp})_{ij}$ is not, in general, a TT tensor, a fact that deprives the formalism of its predictive power, apart from the case of specific spacetimes where the eigenvalue equations can be solved.

Against this background, we have shown that $E^{\perp}$ can be calculated in terms of another operator that exhibits TT properties. Such an operator is obtained by a suitable 'gauge' of the original operator $(\tilde{\triangle}_L h^{\perp})_{ij}$ in the integral relations (17) and (18).

The proposed results can pave the way to further studies based on the Lichnerowicz operator to scrutinize the conditions of stability of traversable wormholes spacetimes.

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## Data availability statement

No new data were created or analysed in this study.

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