A Small-Gain Theory for Abstract Systems on Topological Spaces

Michelangelo Bin, Member, IEEE, and Thomas Parisini, Fellow, IEEE

Abstract—We develop a small-gain theory for systems described by set-valued maps between topological spaces. We introduce an abstract notion of stability unifying the continuity properties underlying different existing concepts, such as Lyapunov stability of equilibria, sets, or motions, (incremental) input–output stability, asymptotic gain properties, and continuity with respect to fast-switching inputs. Then, we prove that a feedback interconnection enjoying a given abstract small-gain property is stable. While, in general, the proposed small-gain property cannot be decomposed as the union of stability of the subsystems and a contractiveness condition, we show that it is implied by standard assumptions in the context of input-to-state stable systems. Finally, we provide application examples illustrating how the developed theory can be used for the analysis of interconnected systems and design of control systems.

Index Terms—Abstract systems, small-gain theorem, stability theory.

I. INTRODUCTION

IN ALL their different facets and variants, small-gain theorems constitute one of the most powerful classes of tools for the analysis of interconnected systems and the design of control schemes. The development of small-gain results can be traced back at least to the 60’s in the context of input–output operators between (extended) normed spaces of signals. See, for instance, [1, Ch. III], [2], [3], and the subsequent extensions to nonlinear [4], [5], [6], stochastic [7], and monotone [8] systems. The seminal work [9] developed a small-gain theory for input-to-state stable (ISS) systems described by nonlinear differential equations [10], [11]. The results in [9] were followed by extensive research efforts aimed at extending the small-gain theory to different domains and problems. In particular, [12] considered ISS systems with saturations, [13] provided characterizations in terms of Lyapunov functions, [14] extended the results of [9] to general “ISS operators,” [15] to integral ISS systems, and [16], [17], [18] to systems not necessarily ISS. Extensions to time-varying and possibly nonuniformly ISS systems appeared in [19] and [20], whereas [21], [22], [23], and [24] considered discrete-time systems, and [25] abstract systems satisfying a “weak semigroup property” (see also [26]). More recently, small-gain results have been developed also for switching and hybrid systems [18], [27], [28], [29], for infinite-dimensional systems described by partial differential equations [30], [31], for finite [29], [31], [32], [33], [34] and infinite [24], [35] networks, and for stochastic systems [36]. See also [37] for a review.

The aforementioned small-gain results typically differ in terms of the class of systems considered and the stability requirements for the systems involved in the interconnection, but they all share a common paradigm from which a “small-gain principle” can be drawn: the interconnection of stable systems satisfying a certain “small-gain condition” is itself stable. In qualitative terms

\[ \text{stability of the subsystems} + \text{small-gain property} \Rightarrow \text{stability of the interconnection} \]  

(1)

In this article, we develop a small-gain theory extending the small-gain principle (1) to set-valued maps between topological spaces. This leads to the following three main contributions.

1) It unifies different existing theorems developed for metric spaces of trajectories and gives new insights on the topological nature of the small-gain principle.

2) It enables the study of interconnections out of reach of existing small-gain theorems (e.g., formed by systems that do not satisfy ISS-like conditions).

3) It extends the small-gain principle to general maps between topological spaces not necessarily representing trajectories of a dynamical system (See Example 2).

In this respect, we point out that the main methodological corpus of this work is composed of Items 1) and 3), which establish a common framework for the small-gain principle. Instead, item 2) is illustrated through examples (see Section VI). Indeed, the application of the presented results to specific cases requires

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the definition of suitable topological spaces and a preliminary analysis, both of which are problem-specific and, hence, not treated here systematically.

Going into the specifics of the framework and the main result of the article, we describe systems in terms of set-valued maps between arbitrary sets. These sets can be endowed with different topologies turning them into topological spaces. For each choice of such topologies, we define “stability” as a property similar to upper semicontinuity generalizing the continuity properties implied by the usual notions of Lyapunov stability of equilibria, sets, or motions, global or local (incremental) stability, and asymptotic gain. In particular, the continuity conditions underlying any of these properties can be obtained in terms of the proposed notion of stability for a specific choice of the involved topological spaces.

Given a feedback interconnection of two systems of this kind, we introduce an abstract small-gain property, and we prove a small-gain theorem stating that such property implies stability of the interconnection. The proposed small-gain property is an abstraction of the joint condition “stability of the subsystems + small-gain property” of (1) that, however, does not admit a similar decomposition but is a unique requirement. Nevertheless, we show that, in an ordinary ISS context, “stability of the subsystems + small-gain condition” implies the proposed small-gain property.

Finally, in this connection, we emphasize that the presented results only concern the continuity conditions implied by the global stability and asymptotic gain properties [11] and not directly ISS. While for finite-dimensional systems these properties imply (local) ISS, this is not generally true, for instance, for hybrid (even of finite dimension) [38, Remark 3.3] and infinite-dimensional [39, 40] systems. Therefore, in an ISS context, the conclusions that can be drawn on the feedback interconnection from the proposed theory are in general weaker than ISS. Nevertheless, we remark that this is not necessarily a shortcoming of the proposed theory, as it deals with spaces where ordinary notions of uniform convergence or boundedness may not make sense. Moreover, stronger properties, such as “uniform asymptotic gain” [11], may be obtained by suitably redefining or extending the input and output spaces and their topologies, as discussed in Section V-E.

The article is organized as follows. In Section II, we introduce the basic notions of systems and interconnections. In Section III, we define the notion of stability and connect it to the usual global stability and asymptotic gain properties in metric spaces. In Section IV, we define the small-gain property, we establish the main result, and we show that ISS implies the proposed small-gain property. In Section V, we discuss further connections between stability and other existing notions. Finally, in Section VI, we present three examples illustrating how the proposed theory can be used to handle interconnections of systems falling outside the scope of existing small-gain theorems.

Notations and Preliminaries. We denote by \( \mathbb{R} \) and \( \mathbb{N} \) the set of real and natural numbers, respectively (\( 0 \in \mathbb{N} \)). If \( \sim \) is a relation on a set \( S \) and \( s \in S \), we let \( S_s := \{ z \in S : z \sim s \} \).

By \( F : X \to Y \) we denote a set-valued map from \( X \) to \( Y \). If \( S \subseteq X \), we let \( F(S) := \bigcup_{x \in S} F(x) \). Accordingly, \( F(\emptyset) = \emptyset \). If \( X = A \times B, S \subseteq A \), and \( Z \subseteq B \), then \( F(S, Z) = F(S \times Z) \).

We denote by \( \text{dom} F \) the set of \( x \in X \) for which \( F(x) \) is nonempty, and by \( \text{ran} F := F(\text{dom} F) \) the range of \( F \). For \( V \subseteq Y \), we denote by \( F^L(V) := \{ x \in X : F(x) \cap V \neq \emptyset \} \) and \( F^U(Y) := \{ x \in X : F(x) \subseteq Y \} \) the lower and upper inverse, respectively. If \( F(x) \) is a singleton for all \( x \in \text{dom} F \), we identify \( F \) with the function \( f : \text{dom} F \to Y \) satisfying \( f(x) \in F(x) \) for all \( x \in \text{dom} F \). The graph of a map \( F \) is defined as \( \text{graph} F := \{(x,y) \in X \times Y : y \in F(x)\} \).

A topological space is a pair \( (X, \tau) \) where \( X \) is a set and \( \tau \) is a collection of subsets of \( X \) which contains \( \emptyset \) and \( X \) itself and is closed under finite intersections and arbitrary unions. The elements of \( \tau \) are called open sets. A neighborhood of a point \( x \in X \) is a subset of \( X \) containing an open set containing \( x \). A neighborhood of a set \( X \subseteq X \) is a subset of \( X \) containing a neighborhood of every point of \( X \). The set of all the neighborhoods of \( X \subseteq X \) is denoted by \( N_\tau(X) \) (or \( N_\tau(x) \) if \( X = \{x\} \)). When \( \tau \) is clear from the context we omit it and, for instance, we write \( X \) for \( (X, \tau) \) and \( N_\tau(\cdot) \) for \( N_\tau(\cdot) \).

If not otherwise specified, we shall assume every \( X \subseteq X \) to be endowed with the subset topology \( \tau_X := \{ O \cap X : O \in \tau \} \) and, if \( x_1, \tau_{x_1}, \ldots, x_n, \tau_{x_n} \) are topological spaces, their product \( X_1 \times \cdots \times X_n \) will be assumed to be endowed with the product topology denoted by \( \tau_{X_1 \times \cdots \times X_n} \). A net on a set \( X \) is a map \( x : I \to X \) from a directed set \( I \) to \( X \). We denote nets also by \((x_j)_{j \in I}\).

For \( t > 0 \), we denote by \( C_t(X) \) the set of continuous functions \( [0, t) \to X \), and we let \( C_{(0, \infty)}(X) := C_{[0, \infty)} \cup C_t(X) \). A continuous function \( k : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) of class-K if it is strictly increasing and \( k(0) = 0 \). We denote by \( k^{-1} : \text{ran} k \to \mathbb{R}_{\geq 0} \) the inverse of \( k \). Notice that, if \( k \) is of class-K, there always exists \( \epsilon > 0 \) so that \([0, \epsilon) \subseteq \text{ran} k = \text{dom} k^{-1} \). Hence, \( k^{-1}(s) \) exists for all sufficiently small \( s > 0 \).

II. SYSTEMS

In this section, we introduce the basic notions we use to model systems and their interconnections.

A. Systems as Mappings

Throughout the article, systems are represented by set-valued maps between sets.

Definition 1 (Systems): A system is a triple \((D, \mathcal{Y}, \Psi)\) in which \( D \) and \( \mathcal{Y} \) are sets and \( \Psi : D \rightrightarrows \mathcal{Y} \) is a set-valued map.

The set \( D \) is called the input space, and its elements the inputs of the system. The set \( \mathcal{Y} \) is called the output space, and its elements the outputs of the system. A system with \( \text{dom} \Psi = \emptyset \) is called trivial. Since \( d \notin \text{dom} \Psi \) implies \( \Psi(d) = \emptyset \), then \( \Psi(D) = \Psi(D \cap \text{dom} \Psi) \), for all \( D \subseteq D \).

The notion of systems provided by Definition 1 resembles that of [2] and [6], with the difference that here \( D \) and \( \mathcal{Y} \) are generic sets, and not necessarily normed spaces of signals. Moreover, Definition 1 also fits the behavioral framework of [41], as the set graph \( \Psi \) is a behavior on \( D \times \mathcal{Y} \) in the sense of [41, Def. 1.2.1]. In this connection, we observe that seeing \( \Psi \) as a map \( D \rightrightarrows \mathcal{Y} \), instead of a map \( \Psi^L : \mathcal{Y} \rightrightarrows D \), is a matter of convention as \( \Psi \) and \( \Psi^L \) are isomorphic.

Definition 1 is sufficiently general to include most of the usual definitions of interest in control theory, such as transfer
functions, ordinary/partial differential equations or inclu-
sions, and hybrid systems, as shown in the following example.

**Example 1:** Consider the hybrid inclusions [42]

\[
\begin{align*}
\dot{x} &\in F(x, u) \quad (x, u) \in C \\
x^+ &\in G(x, u) \quad (x, u) \in D
\end{align*}
\]

with \( C, D \subseteq \mathbb{R}^n \times \mathbb{R}^m \), \( n, m \in \mathbb{N} \), and \( F, G : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n \). With \( X_0 \subseteq \{ (x, u) \in \mathbb{R}^n : \exists \Omega \in \mathbb{R}^m, (x, u) \in C \cup D \} \) and \( \mathcal{U} \) the set of hybrid inputs [38] on \( \mathbb{R}^m \), let \( D := X_0 \times \mathcal{U} \). Moreover, let \( Y \) be the set of hybrid arcs [38], [42] on \( \mathbb{R}^n \). For each \((x_0, u) \in D\), let \( \Psi(x_0, u) \subseteq Y \) be the set of all \( x \in Y \) such that \((x, u) \) is a solution pair to (2) with \( x \) originating at \( x_0 \). Then, \((D, Y, \Psi)\) is a system in the sense of Definition 1.

In addition, Definition 1 extends beyond dynamical systems. It can be used to model algebraic maps, solution mapping of optimization problems, or other relations capturing only some specific aspects of dynamics. For instance, Example 2 hereafter deals with limit sets, used in control to characterize the steady-state trajectories of a system [43], [44].

**Example 2:** In the setting of Example 1, fix \( u = 0 \) and let \( X_0 \subseteq \mathbb{R}^n \) be compact. Let \( S(X_0) \) be the set of all complete solutions \( x \) of (2) originating in \( X_0 \) and corresponding to \( u = 0 \). Suppose that \( S(X_0) \neq \emptyset \) and, for each \( \tau \geq 0 \), define the reachable tail \( R^\tau(X_0) := \{ x(t, j) \in \mathbb{R}^{n} : x \in S(X_0), (t, j) \in \text{dom} x, t + j \geq \tau \} \). Then, \( \{ R^\tau(X_0) : \tau \geq 0 \} \) is a filter base [45, Sec. 1.6] whose (possibly) empty set \( \Omega(X_0) := \bigcap_{\tau \geq 0} R^\tau(X_0) \) is the \( \Omega \)-limit set of (2) from \( X_0 \) (and with \( u = 0 \)). Let \( D \) be the set of all compact subsets \( X_0 \subseteq \mathbb{R}^n \), \( \Psi(X_0) = \mathbb{R}^n \), and \( \Psi : D \rightrightarrows Y \) the set-valued map \( \Psi(X_0) := \Omega(X_0) \). Then, \((D, Y, \Psi)\) is a system in the sense of Definition 1 representing the mapping between sets of initial conditions and the corresponding attractor.

**B. Interconnections**

Let \( \Sigma_1 = (D_1, Y_1, \Psi_1) \) and \( \Sigma_2 = (D_2, Y_2, \Psi_2) \) be systems. As a first case, assume that \( D_1 = D_2 \) and let \( D := D_1 = D_2 \). The parallel interconnection of \( \Sigma_1 \) and \( \Sigma_2 \) is defined as the system \((D, Y, \Psi)\) with \( Y:= \bigcap_{\tau \geq 0} R^\tau(X_0) \) is the \( \Omega \)-limit set of (2) from \( X_0 \) (and with \( u = 0 \)). Let \( D \) be the set of all compact subsets \( X_0 \subseteq \mathbb{R}^n \), \( \Psi(X_0) = \mathbb{R}^n \), and \( \Psi : D \rightrightarrows Y \) the set-valued map \( \Psi(X_0) := \Omega(X_0) \). Then, \((D, Y, \Psi)\) is a system in the sense of Definition 1 representing the mapping between sets of initial conditions and the corresponding attractor.

The series interconnection of \( \Sigma_1 \) and \( \Sigma_2 \) is defined as the system \((D, Y, \Psi)\) with \( D := D_1 \times D_2 \) and \( Y := \Psi_1 \times \Psi_2 \).

As a second case, assume that \( Y_1 \subseteq Y_2 \). The series interconnection of \( \Sigma_1 \) and \( \Sigma_2 \) is defined as the system \((D, Y, \Psi)\) with \( D := D_1 \times D_2 \) and \( Y := \Psi_1 \times \Psi_2 \).

Finally, assume that, for some sets \( V_1, V_2, D_1', \) and \( D_2' \) such that \( Y_1 \subseteq V_2 \) and \( Y_2 \subseteq V_1 \), we have \( D_1 = V_1 \times D_1' \) and \( D_2 = V_2 \times D_2' \) (see Fig. 1). Then, a feedback interconnection of \( \Sigma_1 \) and \( \Sigma_2 \) is a system \( \Sigma = (D, Y, \Psi) \) with \( D := D_1' \times D_2' \).

**Fig. 1.** Feedback interconnection of \( \Sigma_1 \) and \( \Sigma_2 \).
The following lemma, proved in Appendix A, provides an alternative characterization of $\Upsilon_1$ and $\Upsilon_2$ that plays an important role in the main result given later in Section IV.

**Lemma 2:** For all $d \in D$, $\Upsilon_1(d) \subseteq \Psi_1(\Upsilon_2(d), d_1)$, $\Upsilon_2(d) \subseteq \Psi_2(\Upsilon_1(d), d_2)$, and

$$
\Upsilon_1(d) = \{ y_1 \in \mathcal{Y}_1 : y_1 \in \Psi_1(\Upsilon_2(y_1, d_1), d_1) \},
$$

$$
\Upsilon_2(d) = \{ y_2 \in \mathcal{Y}_2 : y_2 \in \Psi_2(\Upsilon_1(y_2, d_2), d_2) \}. 
$$

**Remark 1:** In view of (7) and (8), the elements of $\Psi(d)$ are fixed points of the maps $\Psi_1(\cdot, d_1), \Psi_2(\cdot, d_2)$, and $\Psi_2(\Psi_1(\cdot, d_1), d_2)$, respectively. This, however, is only a necessary condition since, in general, $\Upsilon_1(d) \times \Upsilon_2(d) \not\subseteq \Psi(d)$. Namely, pairs of fixed points of the aforementioned maps need not be elements of $\Psi(d)$, although they always belong to $\Upsilon_1(d) \times \Upsilon_2(d)$.

### III. STABILITY

In this section, we introduce a notion of stability for systems satisfying Definition 1. We then discuss its relationship with the properties of global stability and asymptotic gain. This enables us to make a direct connection with ISS systems. Instead, connections with other notions, such as Lyapunov stability and incremental stability, are discussed later in Section V.

#### A. Stability as a Topological Notion

As for continuity, stability is defined with reference to a topology $\tau_\mathcal{D}$ defined on the input space $\mathcal{D}$ and a topology $\tau_\mathcal{Y}$ defined on the output space $\mathcal{Y}$. Different choices of these topologies lead to different notions of stability.

**Definition 2 (Stability):** Let $(\mathcal{D}, \tau_\mathcal{D})$ and $(\mathcal{Y}, \tau_\mathcal{Y})$ be topological spaces. A system $(\mathcal{D}, \mathcal{Y}, \Psi)$ is said to be stable at $D \subseteq \mathcal{D}$ with respect to $(\tau_\mathcal{D}, \tau_\mathcal{Y})$ (or, briefly, $(\tau_\mathcal{D}, \tau_\mathcal{Y})$-stable at $D$) if, for every $Y \in \mathcal{N}(\Psi(D))$, there exists $U \in \mathcal{N}(\mathcal{D})$, such that $\Psi(U) \subseteq Y$.

When $\tau_\mathcal{D}$ and $\tau_\mathcal{Y}$ are clear from the context they are omitted, and we say that the system is stable at $D$. If $D = \{d\}$ is a singleton, we say that the system is stable at $d$ instead of at $\{d\}$. Stability at $D \subseteq \mathcal{D}$ is implied by upper semicontinuity of $\Psi$ at each point of $D$. Indeed, every $Y \in \mathcal{N}(\Psi(D))$ contains a set of the form $\cup_{d \in D} Y_d$ with $Y_d \in \mathcal{N}(\Psi(d))$. If, for each $d \in D$, $\Psi$ is upper semicontinuous at $d$, we can find $O_d \in \mathcal{N}(d)$ such that $\Psi(O_d) \subseteq Y_d$. Then, $O := \cup_{d \in D} O_d \in \mathcal{N}(D)$ and $\Psi(O) \subseteq \cup_{d \in D} Y_d \subseteq Y$. Nevertheless, the converse does not hold. Namely, stability at $D$ does not imply upper semicontinuity of $\Psi$ at each point of $D$. Indeed, as a trivial counterexample, notice that every system is stable at $D$ (since $D$ is a neighborhood of itself) without any relation to upper semicontinuity of $\Psi$ at any point of $D$.

**Remark 2:** We underline that the notion of stability given in Definition 2 is aimed at generalizing the continuity properties implied by the notion of Lyapunov stability of autonomous differential/difference equations and global stability à la [11] for systems with input, both of which are properties of the map $\Psi$, and not convergence or attractiveness, which are instead properties of the specific inputs and outputs of $\Psi$.

Remarkably, parallel interconnections are stable at a point if so are the interconnected systems, whereas series interconnections of stable systems are stable at both points and sets. Specifically, let $\Sigma_1 = (D_1, \mathcal{Y}_1, \Psi_1)$ and $\Sigma_2 = (D_2, \mathcal{Y}_2, \Psi_2)$ be systems, and let $\Psi_\Sigma = (D_\Sigma, \mathcal{Y}_\Sigma, \Psi_\Sigma)$ denote their parallel and series interconnection (see Section II-B) whenever they make sense. Let $D_1, D_2, \mathcal{Y}_1$, and $\mathcal{Y}_2$ be endowed with some topologies, and $\mathcal{Y}_\Sigma = \mathcal{Y}_1 \times \mathcal{Y}_2$ with the product topology (in the parallel case, $D_\Sigma = D_1 \cup D_2$; hence $D_1$ and $D_2$ are given the same topology). Then, the following holds.

**Proposition 1:** If $\Sigma_1$ and $\Sigma_2$ are stable at $d \in D_\Sigma$, then $\Sigma_\Sigma$ is stable at $d$. If $\Sigma_1$ is stable at $D \subseteq D_1$, and $\Sigma_2$ is stable at $\Psi_1(D_1)$, then $\Sigma_\Sigma$ is stable at $D$.

Proposition 1 is proved in Appendix B. Stability of feedback interconnections between $\Sigma_1$ and $\Sigma_2$ is instead more delicate, and it is indeed the main object of this article. We conclude this section with the following technical lemma (proved in Appendix C), which is used by several forthcoming results.

**Lemma 3:** Let $A$ and $B$ be topological spaces, and let $A \subseteq A$ and $B \subseteq B$. Every neighborhood of $A \times B$ in the product space $A \times B$ contains a neighborhood of $A \times B$ of the form $U \times V$, where $U \in \mathcal{N}(A)$ and $V \in \mathcal{N}(B)$. In particular, a system $(\mathcal{D}, \mathcal{Y}, \Psi)$ with $D = A \times B$ is stable at $A \times B$ if and only if for every $Y \in \mathcal{N}(\Psi(A \times B))$, there exist $U \in \mathcal{N}(A)$ and $V \in \mathcal{N}(B)$ such that $\Psi(U \times V) \subseteq Y$.

#### B. Connections With Global Stability

Let $(X, | \cdot |)$, $(U, | \cdot |)$, and $(Y, | \cdot |)$ be seminormed linear spaces (for ease of notation, we denote all seminorms by $| \cdot |$). Let $(\mathcal{T}_U, \geq)$ and $(\mathcal{T}_Y, \geq)$ be (possibly different) directed sets, and let $(U, | \cdot |)$ and $(Y, | \cdot |)$ be seminormed linear spaces (under the pointwise operations) of functions $\mathcal{T}_Y \to Y$ and $\mathcal{T}_U \to U$, respectively. The elements of $X$ may represent initial/boundary conditions or parameters. The elements of $U$ represent exogenous input signals, and those of $Y$ the system’s outputs. Let $D := X \times U$, and suppose a system $\Sigma = (\mathcal{D}, \mathcal{Y}, \Psi)$ is defined such that, for some class-K functions $\alpha$ and $\kappa$,

$$
\forall (x, u, y) \in \text{graph } \Psi, \quad |y| \leq \max \{ \alpha(|x|), \kappa(|u|) \}.
$$

When $|y| := \sup_{t \in \mathcal{T}_Y} |y(t)|$ and $|u| := \sup_{t \in \mathcal{T}_U} |u(t)|$, Condition (9) is a global stability property implied by ISS. If, instead, $\mathcal{T}_U = \mathbb{R}_{\geq 0}$ and $|u| := \left( \int_0^\infty |u(t)|^2 dt \right)^{1/2}$, we obtain an “integral” variant of global stability, implied by integral ISS [46]. In general, different notions of stability can be obtained for different choices of the seminorms [46]. In any case, (9) implies stability in the sense of Definition 2, of $\Sigma$ at $D^* := \{ (x, u) \in X \times U : |x| = 0, |u| = 0 \}$ with respect to $(\tau_\mathcal{D}, \tau_\mathcal{Y})$, where $\tau_\mathcal{D}$ is the topology induced on $X$ by its seminorm, and $\tau_\mathcal{Y}$ is the product topology on $D$ induced by the seminorms on $X$ and $U$. Indeed, (9) implies $\Psi(D^*) \subseteq \{ y \in Y : |y| = 0 \}$, and every neighborhood $Y$ of $\Psi(D^*)$ contains a set of the form $B_c := \{ y \in Y : |y| < \epsilon \}$ for some $\epsilon > 0$ small enough so that $\epsilon \in \text{ran } \alpha \cap \text{ran } \kappa$. As $\tau_\mathcal{D}$ is induced by the seminorm $|x, u| := \max \{|x|, |u|\}$, then the set $U := \{ (x, u) \in D : |x, u| \in B_\epsilon \}$ is a neighborhood of $D^*$, and $\Psi(U) \subseteq B_\epsilon \subseteq Y$.

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1Proposition 1 concerns parallel and series interconnections of “input–output” systems in the sense that we do not consider initial conditions as extra inputs as in Example 1. Thus, in case of dynamical systems, Proposition 1 has to be intended for fixed initial conditions. Nevertheless, the extension to the case where initial conditions are taken into account is straightforward.
C. Connections With Asymptotic Gain

In this section, we show that the asymptotic gain property implies stability according to Definition 2 for a specific choice of the topology of the input and output spaces. In particular, let \((Z, | \cdot |)\) be a seminormed linear space, \((\mathbb{T}_Z, \geq)\) be a directed set, and \(Z\) be a linear space (under the pointwise operations) of functions \(T_Z \to Z\). For all \(\zeta \in Z\), define

\[
\limsup |\zeta| := \inf_{t \in \mathbb{T}_Z} \sup_{s \geq t} |\zeta(s)|
\]

we set \(\limsup |\zeta| := \infty\) if \(\limsup |\zeta|\) does not exist in \(\mathbb{R}\). For every \(\varepsilon \geq 0\), define the set

\[
O_\varepsilon := \{\zeta \in Z : \limsup |\zeta| < \varepsilon\}
\]

which is nonempty whenever \(\varepsilon > 0\) since it contains the zero function. The collection \(\beta := \{Z\} \cup \{O_\varepsilon : \varepsilon > 0\}\) generates a topology \(\tau_Z\) on \(Z\) that we call the \(\limsup\) topology. Moreover, \(\beta\) is a base for \((Z, \tau_Z)\) [45, Prop. 1.2.1].

In the same setting of Section III-B, suppose that all \(u \in U\) and \(y \in Y\) are bounded and there exists a class-K function \(\rho\) such that

\[
\forall (x, u, y) \in \text{graph } \Psi, \quad \limsup |y| \leq \rho(\limsup |u|).
\]  

Condition (10) is an asymptotic gain property (implies ISS). Let \(\tau_D\) and \(\tau_Y\) denote the limsup topologies on \(U\) and \(Y\), respectively, and let \(\tau_X\) be any topology on \(X\). Then, for every \(S \subseteq X\), (10) implies that \(\Psi\) is stable at \(D^* := S \times \{u \in U : \limsup |u| = 0\}\) with respect to \((\tau_X \otimes \tau_D, \tau_Y)\). Indeed, (10) implies that \(\Psi(D^*) \subseteq \{y \in Y : \limsup |y| = 0\}\). Hence, every neighborhood \(V\) of \(\Psi(D^*)\) contains a set of the form \(Y_\varepsilon := \{y \in Y : \limsup |y| < \varepsilon\}\) for a sufficiently small \(\varepsilon > 0\) satisfying \(\varepsilon \in \text{ran } \rho\). Therefore, with \(\delta := \rho^{-1}(\varepsilon)\), the set \(U_\delta := \{u \in U : \limsup |u| < \delta\}\) is such that \(X \times U_\delta \subseteq \mathcal{N}(\Psi(D^*))\) and, in view of (10), \(y \in Y_\varepsilon \subseteq V\) for all \(y \in \Psi(X \times U_\delta)\).

IV. ABSTRACT SMALL-GAIN THEORY

A. The Small-Gain Property

Consider two systems \(\Sigma_1 = (D_1, Y_1, \Psi_1)\) and \(\Sigma_2 = (D_2, Y_2, \Psi_2)\) such that \(D_1 = V_1 \times D'_1\) and \(D_2 = V_2 \times D'_2\), with \(Y_2 \subseteq Y_1\) and \(Y_1 \subseteq Y_2\). Moreover, consider a feedback interconnection \(\Sigma = (D, Y, \Psi)\) of \(\Sigma_1\) and \(\Sigma_2\) (not necessarily the maximal one) obtained as specified in Section II-B with \(D := D_1 \times D'_2, \ Y := Y_1 \times Y_2,\) and \(\Psi\) defined as in (3) (see Fig. 1). We associate with \(\Sigma\) the maps \(\Gamma_{12} : Y_1 \times D \Rightarrow Y_1\) and \(\Gamma_{21} : Y_2 \times D \Rightarrow Y_2\) defined as

\[
\Gamma_{12}(y_1, d) := \Psi_1(\Psi_2(y_1, d_2), d_1)
\]

\[
\Gamma_{21}(y_2, d) := \Psi_2(\Psi_1(y_2, d_1), d_2).
\]

Moreover, for \((i, j) = (1, 2), (2, 1)\) and \(n \in \mathbb{N}_{\geq 1}\), we define the maps \(\Gamma_{ij}^n\) and \(\Gamma_{ji}^n\) according to the following recursion:

\[
\Gamma_{ij}^n(y_i, d) := \Gamma_{ij}(\Gamma_{ij}^{n-1}(y_i, d), d), \quad \forall n \in \mathbb{N}_{\geq 1},
\]

\[
\Gamma_{ji}^n(y_j, d) := \Gamma_{ji}(\Gamma_{ji}^{n-1}(y_j, d), d), \quad \forall n \in \mathbb{N}_{\geq 1},
\]

for all \(y_i \in Y_i\) and all \(d \in D\). The maps \(\Gamma_{12}\) and \(\Gamma_{21}\) satisfy the following property.

**Lemma 4:** For every \((i, j) \in \{(1, 2), (2, 1)\}, D \subseteq D, y_i \in \mathcal{T}_i(D), \) and \(n \in \mathbb{N}_{\geq 1},\) it holds that \(y_i \in \Gamma_{ij}^n(y_i, d)\).

**Proof:** Pick arbitrarily \(D \subseteq D\) and \(y_i \in \mathcal{T}_i(D)\). Then, there exists \(d \in D\) such that \(y_i \in \mathcal{T}_i(d)\). In view of (8),

\[
y_i \in \Gamma_{ij}(y_i, d).
\]

Suppose that, for some \(n \in \mathbb{N}_{\geq 1},\)

\[
y_i \in \Gamma_{ij}^n(y_i, d).
\]

Then, (12) implies that \(\Gamma_{ij}(y_i, d) \subseteq \Gamma_{ij}(\Gamma_{ij}^{n-1}(y_i, d)) = \Gamma_{ij}^{n+1}(y_i, d)\). Thus, (11) implies that \(y_i \in \Gamma_{ij}^{n+1}(y_i, d)\). As (11) is (12) with \(n = 1\), we conclude by induction that (12), and hence \(y_i \in \Gamma_{ij}^n(y_i, d)\), hold for every \(n \in \mathbb{N}_{\geq 1}\). Since \(D\) and \(y_i \in \mathcal{T}_i(D)\) were arbitrary, the claim follows.

We endow \(D\) with a topology \(\tau_D\) and \(Y\) with a topology \(\tau_Y\). Then, the following definition formalizes the small-gain condition used in this article to establish stability, with respect to \((\tau_D, \tau_Y)\), of the feedback interconnection \(\Sigma\).

**Definition 3** (Small-gain property): The feedback interconnection \(\Sigma\) is said to satisfy the small-gain property at \(D^* \subseteq D\) with respect to \((\tau_D, \tau_Y)\) if, for every \(Y \in \mathcal{N}(\Psi(D^*))\), there exists \(D \in \mathcal{N}(D^*)\), such that

\[
\forall y = (y_1, y_2) \in Y \setminus Y, \quad \exists n_1, n_2 \in \mathbb{N}_{\geq 1},
\]

\[
\text{s.t. } \Gamma_{12}^{n_1}(y_1, d) \times \Gamma_{21}^{n_2}(y_2, d) \subseteq Y.
\]

**Remark 3:** Condition (13) in Definition 3 is a “contraction” requirement for the maps \(\Gamma_{12}\) and \(\Gamma_{21}\) that plays in our setting the role of “stability of the subsystems + small-gain property” of typical ISS contexts [cf. (1)]. However, we stress that, unlike (1), Definition 3 is a single condition that, up to the authors’ knowledge, cannot be expressed in terms of the composition of stability of \(\Sigma_1\) and \(\Sigma_2\) and a contraction property. Indeed, in general, the set \(D^*\) may not even be a product of the form \(D_1^* \times D_2^*\). Nevertheless, later in Section IV-C, we show that ISS implies Definition 3.

**Remark 4:** Computing \(\Gamma_{12}\) and \(\Gamma_{21}\) may be difficult, if not impossible, for nontrivial interconnections. Fortunately, their computation is generally not needed for checking whether the condition (13) holds. Indeed, (13) can be usually checked by using some known property of the systems involved. In the following, we provide several instances where this is the case, i.e., Propositions 2 and 3, and the examples in Section VI.

B. Main Result

In this section, we prove the main result of this article, establishing that a feedback interconnection of two systems satisfying the small-gain property of Definition 3 is stable. As in the previous section, we consider two systems \(\Sigma_1 = (D_1, Y_1, \Psi_1)\) and \(\Sigma_2 = (D_2, Y_2, \Psi_2)\), where \(D_1 = V_1 \times D'_1\) and \(D_2 = V_2 \times D'_2\), with \(Y_2 \subseteq Y_1\) and \(Y_1 \subseteq Y_2\). Then, we consider a feedback interconnection \(\Sigma = (D, Y, \Psi)\) of \(\Sigma_1\) and \(\Sigma_2\) as specified in...
Section II-B, with $D := D_1^* \times D_2^*$, $Y := Y_1 \times Y_2$, and $\Psi$ defined as in (3). Finally, we endow $D$ with a topology $\tau_D$ and $Y$ with a topology $\tau_Y$.

Theorem 1 (Small-gain Theorem): If $\Sigma$ satisfies the small-gain property at $D^* \subseteq D$ with respect to $(\tau_D, \tau_Y)$, then it is $(\tau_D, \tau_Y)$-stable at $D^*$.

Proof: Pick $Y \in N(\Psi(D^*))$ arbitrarily, and let $D \in N(D^*)$ be such that (13) holds. Equation (7) implies that

$$\Psi(D) \subseteq \bigcup_{d \in D} (Y_1(d) \times Y_2(d)) \subseteq Y_1(D) \times Y_2(D)$$

$$= (((Y_1(D) \times Y_2(D)) \setminus Y) \cup (((Y_1(D) \times Y_2(D)) \cap Y))$$

$$\subseteq (((Y_1(D) \times Y_2(D)) \setminus Y) \cup Y).$$

(14)

For every $(y_1, y_2) \in (Y_1(D) \times Y_2(D)) \setminus Y$, Lemma 4 implies that

$$y_1 \in \Gamma_{12}^i(D_1^*), \quad y_2 \in \Gamma_{21}^i(y_2, D), \quad \forall n_1, n_2 \in \mathbb{N}_2. \quad \text{(15)}$$

For each $y = (y_1, y_2) \in (Y_1(D) \times Y_2(D)) \setminus Y$, let $n_1$ and $n_2$ be such that (13) holds. Then, (13) and (15) imply that

$$(Y_1(D) \times Y_2(D)) \setminus Y = \bigcup_{y \in (Y_1(D) \times Y_2(D)) \setminus Y} \{y_1\} \times \{y_2\}$$

$$\subseteq \bigcup_{y \in (Y_1(D) \times Y_2(D)) \setminus Y} \Gamma_{12}^n(y_1, D) \times \Gamma_{21}^n(y_2, D)$$

$$\subseteq Y.$$

Hence, we deduce from (14) that $\Psi(D) \subseteq Y$. In this way, we have shown that, for every $Y \in N(\Psi(D^*))$, there exists $D \in N(D^*)$, such that $\Psi(D) \subseteq Y$. Namely, $\Sigma$ is stable at $D^*$.

C. Connections With Other Small-Gain Theorems

In this section, we establish a relationship between the small-gain property given in Definition 3 and some existing small-gain theorems applying to ISS systems. Specifically, we show that “stability of the subsystems + small-gain property” in (1) (with stability meaning ISS) implies the small-gain condition of Definition 3. For simplicity, we focus on stability of the origin. Nevertheless, the same arguments can be extended to stability of sets or motions, by resorting to the corresponding notions described later in Section V.

With $(i, j) \in \{(1, 2), (2, 1)\}$, let $(X_i, \{\cdot\}, (U_i, \{\cdot\})$, and $(Y_i, \{\cdot\})$ be seminormed linear spaces (as before, we denote all seminorms by $\{\cdot\}$). Let $\{T_{il}, T_{lj}\}$ be directed sets, and $U_i$ and $Y_i$ be linear spaces (under the pointwise operations) of bounded functions $T_{il} : Y_i \to Y_i$ and $T_{lj} : U_i \to U_i$, respectively. Finally, let $\Psi_i : Y_j \times X_i \times U_i \Rightarrow Y_i$, with $D_1^* := X_i \times U_i$ and $D_2^* := Y_j \times X_i \times U_i$, the triple $\Sigma_i = (D_i, Y_i, \Psi_i)$ is a system in the sense of Definition 1. The set $X_i$ may represent initial/boundary conditions in an initial value problem, while $U_i$ contains exogenous inputs.

We define $O_{X_i} := \{x_i \in X_i : \|x_i\| = 0\}$, $O_{U_i} := \{u_i \in U_i : \sup_{t \in T_{il}} u_i(t) = 0\}$, and $O_{Y_i} := \{y_i \in Y_i : \sup_{t \in T_{lj}} |y_i(t)| = 0\}$. Moreover, we assume that there exist class-K functions $\alpha_i, \varphi_i, \kappa_i$ such that

$$\sup_{t \in T_{il}} |y_i(t)| \leq \max \left\{ \alpha_i(|x_i|), \varphi_i \left( \sup_{t \in T_{lj}} |y_j(t)| \right) \right\}$$

$$\kappa_i \left( \sup_{t \in T_{lj}} |y_j(t)| \right)$$

(16)

$$\limsup_{|y_j| \to 0} |y_j| \leq \max \left\{ \varphi_i \left( \limsup_{|y_j| \to 0} |y_j| \right), \kappa_i \left( \limsup_{|u_i| \to 0} |u_i| \right) \right\}$$

(17)

hold for all $(y_j, x_i, u_i, y_i) \in \text{graph } \Psi_i$. Condition (16a) relates to global stability and (16b) to asymptotic gain. Both are implied by ISS. Finally, we suppose that the following small-gain condition holds:

$$g_i(s) := \varphi_i(g_j(s)) < s, \quad \forall s \in \mathbb{R}_{>0}. \quad \text{(18)}$$

First, we show that Conditions (16a) and (17) imply the small-gain property of Definition 3 with respect to the uniform seminorm topology. To this end, for $i = 1, 2$, we define on $\mathcal{U}_i$ and $\mathcal{Y}_i$ the seminorms $|u_i| := \sup_{t \in T_{il}} |u_i(t)|$ and $|y_i| := \sup_{t \in T_{lj}} |y_i(t)|$, respectively. The product topology $\tau_D$ on $D := D_1^* \times D_2^*$ is generated by the seminorm $|x_1, x_2, u_2| := \max_{i=1,2} \{|x_i|, |u_i|\}$. Likewise, the product topology $\tau_Y$ on $Y$ is $|y_1, y_2| := \max \{|y_1|, |y_2|\}$. Consider a feedback interconnection $\Sigma = (D, Y, \Psi)$ of $\Sigma_1$ and $\Sigma_2$, defined according to Section II-B, and let $O_D := O_{X_1} \times O_{U_1} \times O_{X_2} \times O_{U_2}$. Then, the following holds (the proof is given in Appendix D).

Proposition 2: Suppose that, for all $(i, j) \in \{(1, 2), (2, 1)\}$, (17) holds and every $(y_j, x_i, u_i, y_i) \in \text{graph } \Psi_i$ satisfies (16a). Then, $\Sigma$ satisfies the small-gain property of Definition 3 at $O_D$ with respect to $(\tau_D, \tau_Y)$.

In view of Proposition 2, we can apply Theorem 1 to conclude that the interconnection $\Sigma$ is stable at $O_D$. This implies the following continuity property:

$$\forall \varepsilon > 0, \exists \delta > 0, \forall (x_1, x_2, u_1, y_1, y_2) \in \text{graph } \Psi,$$

$$\max \{|x_1|, |x_2|, |u_1|, |u_2|\} < \delta \Rightarrow \max \{|y_1|, |y_2|\} < \varepsilon. \quad \text{(19)}$$

Next, we show that, for the previously defined feedback interconnection $\Sigma_1$ and $\Sigma_2$, the bounds (16b) and (17) also imply the small-gain condition of Definition 3 with respect to the limsup topology on $\mathcal{U}_i$ and $\mathcal{Y}_i$ (and any topology on $X_i$).

To this end, we endow $X_i$ with an arbitrary topology $\tau_{X_i}$, we give $\mathcal{U}_i$ and $\mathcal{Y}_i$ the respective limsup topologies, as described in Section III-C, and we let $\tau_D'$ and $\tau_Y'$ be the product topologies on $D$ and $Y$, respectively. For $i = 1, 2$, define $L_{U_i} := \{u_i \in \mathcal{U}_i : \limsup_{|u_i| \to 0} |u_i| = 0\}$, $L_{Y_i} := \{y_i \in \mathcal{Y}_i : \limsup_{|y_i| \to 0} |y_i| = 0\}$, and let $L_D := X_1 \times L_{U_1} \times X_2 \times L_{U_2}$. Then, the following result holds (the proof is in Appendix E).

Proposition 3: Suppose that, for all $(i, j) \in \{(1, 2), (2, 1)\}$, (17) holds and every $(y_j, x_i, u_i, y_i) \in \text{graph } \Psi_i$ satisfies (16b). Then, $\Sigma$ satisfies the small-gain property of Definition 3 at $L_D$ with respect to $(\tau_D', \tau_Y')$.

In view of Proposition 3, we can use Theorem 1 to deduce from (16b) and (17) that the interconnection $\Sigma$ is also $(\tau_D', \tau_Y')$-stable.
at $L_D$. This implies that [cf., (18)]
\[
\forall \varepsilon > 0, \exists \delta > 0, \forall (x_1, u_1, x_2, u_2, y_1, y_2) \in \text{graph } \Psi,
\max_{i=1,2} \sup \{|u_i| < \delta \Rightarrow \max_{i=1,2} \sup \{|y_i| < \varepsilon\}.
\] (19)

To summarize, Theorem 1 allows us to conclude from the global stability and asymptotic gain properties (16) (hence, from ISS), and the small-gain condition (17), that the feedback interconnection $\Sigma$ of the two subsystems $\Sigma_1$ and $\Sigma_2$ satisfies the continuity conditions (18) and (19), which are the same continuity conditions implied by ISS of $\Sigma$.

V. CONNECTIONS WITH OTHER STABILITY NOTIONS

In this section, we present some relevant cases, obtained for a specific choice of $(D, \tau_D)$, $(\varphi, \tau_Y)$, and $\Psi$, that connect the stability notion given by Definition 2 with more common notions of stability used in control and systems theory.

A. Lyapunov Stability of Motions

Consider a differential equation of the form
\[
\dot{x} = f(x),
\] (20)
with $x \in \mathbb{R}^n$, $n \in \mathbb{N}$, $f : \mathbb{R}^n \to \mathbb{R}^n$, $X_0 \subseteq \mathbb{R}^n$, and $X := C_{\rho}(0, \infty)$. Let
\[
\rho_{\infty}(x, y) := \{\sup_{t \in \text{dom } x} |x(t) - y(t)|, \text{ if dom } x = \text{dom } y,
\] otherwise,\]
and let $\rho(x, y) := \min\{\rho_{\infty}(x, y), c\}$ for an arbitrary $c > 0$.

\textbf{Lemma 5:} $\rho$ is a metric on $X$.

\textbf{Proof:} Clearly, $\rho$ is symmetric and $\rho(x, y) = 0 \iff x = y$. It remains to show that $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ for arbitrary $x, y, z \in X$. If dom $x \neq$ dom $y$, then no $z \in X$ satisfies dom $z =$ dom $x$ and dom $z = \text{dom } y$. Hence, $\rho(x, z) + \rho(z, y) \geq c = \rho(x, y)$. If, instead, dom $x = \text{dom } y$, then either dom $z = \text{dom } x = \text{dom } y$, in which case the inequality follows by the definition of $\rho_{\infty}$, or $\rho(x, z) + \rho(z, y) = 2c \geq \rho(x, y)$. \hfill $\blacksquare$

Let $X_0$ be given the Euclidean topology, and $X$ be given the topology induced by $\rho$. Moreover, let $\Psi$ be the solution map of (20), mapping initial conditions $x_0 \in X_0$ to maximal solutions $x \in X$ satisfying $x(0) = x_0$. Let $x^*_0 \in X_0$ be such that $\Psi(x^*_0)$ is single-valued and $x^* \in \Psi(x^*_0)$ is complete (i.e., dom $x^* = [0, \infty]$). Then, system $(X_0, X, \Psi)$ is stable at $x^*_0$ in the sense of Definition 2 if and only if
\[
\forall \varepsilon > 0, \exists \delta(\varepsilon) > 0,
\]
\[
|x_0 - x^*_0| < \delta(\varepsilon) \Rightarrow \forall x \in \Psi(x_0), \rho(x^*, x) < \varepsilon.
\] (21)

Notice that, by definition of $\rho$, (21) implies that, for all initial conditions inside a sufficiently small neighborhood of $x^*_0$, the corresponding solutions are complete. We observe that, when $\Psi$ is single-valued and every solution of (20) originating in $X_0$ is complete, condition (21) (which, we recall, is Definition 2 in this context) equals Lyapunov stability of the motion $x^* = \Psi(x^*_0)$.

B. Lyapunov Stability of Sets

In the same setting of Section V-A, let $A \subseteq X_0$ be nonempty, closed, and forward-invariant (i.e., ran $x \subseteq A$ for all $x \in A$).

\[
\rho_A(x) := \min \{\sup_{t \in \text{dom } x} |x(t)|, c\},
\] where $|x| := \inf_{a \in A} |x - a|$ denotes the distance of $x$ to $A$. Then, $\rho_A$ induces a topology on $X$ as specified by the following lemma.

\textbf{Lemma 6:} For every $\varepsilon > 0$, let $O_\varepsilon := \{x \in \mathcal{X} : \rho_A(x) < \varepsilon\}$. Then, $\tau_A := \{O_\varepsilon : \varepsilon > 0\}$ is a topology on $X$.

\textbf{Proof:} Both $\emptyset = O_0$ and $X = O_{\infty}$ belong to $\tau_A$. Moreover, since $X = O_0$, for all $\varepsilon > c$, then, for every $\varepsilon_1, \varepsilon_2 \geq 0$, $O_{\varepsilon_1} \cap O_{\varepsilon_2} = O_{\min(\varepsilon_1, \varepsilon_2)} \in \tau_A$. Finally, pick $E \subseteq \mathbb{R}_{\geq 0}$ arbitrary. Since $\varepsilon \mapsto O_\varepsilon$ is increasing (in the sense of inclusion), if $E \subseteq [0, c]$, then $\cup_{\varepsilon \in E} O_\varepsilon = O_{\sup E}$; otherwise, $\cup_{\varepsilon \in E} O_\varepsilon \notin \tau_A$.

A sequence $(x_n)_{n \in \mathbb{N}}$ converges in $\tau_A$ to a limit if, for every $\varepsilon > 0$, there exists $n^*(\varepsilon) \in \mathbb{N}$ such that $\rho_A(x_n) < \rho_A(x) + \varepsilon$ for all $n \geq n^*(\varepsilon)$. Clearly, if $(x_n)_{n \in \mathbb{N}}$ converges to $x$, it converges to every other $z$ satisfying $\rho_A(z) = \rho_A(x)$. In particular, if $x_n$ converges to an $x$ satisfying $\rho_A(x) = 0$ (namely, $\rho_A(x_n) \to 0$), we say that $(x_n)_{n \in \mathbb{N}}$ converges to $A$ and write $x_n \to x_A$. \hfill $\blacksquare$

The topology $\tau_A$ has the following property, establishing the equivalence between convergence of $x$ to $A$ in the usual sense and convergence of the “tails” of $x$ to $A$.

\textbf{Proposition 4:} Let $x \in \mathcal{X}$ be such that dom $x = [0, \infty)$. Then, \lim_{t \to \infty} |x(t)| = 0 if and only if there exists a sequence $(t_n)_{n \in \mathbb{N}}$ in $\mathbb{R}_{\geq 0}$ such that the sequence $(x_n)_{n \in \mathbb{N}}$ in $X$ with terms $x_n(\cdot) := (t_n(\cdot) + x_0)$ converges to $A$ in $\tau_A$.

\textbf{Proof:} (Only If) If $|x(t)| \to 0$, for every sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ with $\varepsilon_n \to 0$, satisfying $\varepsilon_n \to 0$, there exists a sequence $(t_n)_{n \in \mathbb{N}}$ in $\mathbb{R}_{\geq 0}$ such that $|x(s)| < \varepsilon_n$ for all $s \geq t_n$. This implies $\rho_A(x_n) = \sup_{t \geq 0} |x_n(t)| = \sup_{t \geq t_n} |x(t)| \to 0$. Namely, $x_n \to x_A$.

(If) Let $(t_n)_{n \in \mathbb{N}}$ be a sequence in $\mathbb{R}_{\geq 0}$ such that $x_n \to x_A$. Then, for every $\varepsilon > 0$, there exists $n^*(\varepsilon) \in \mathbb{N}$ such that $\rho_A(x_n) < \varepsilon$ for all $n \geq n^*(\varepsilon)$. This implies that $|x(s)| < \varepsilon$ for all $s \geq t_n$ and all $n \geq n^*(\varepsilon)$. Thus, \lim_{t \to \infty} |x(t)| = 0. \hfill $\blacksquare$

Let $\tau_A^D$ be the topology on $X_0$ generated by the family \{$Q_b : \delta \geq 0\}$, in which $Q_\delta := \{x_0 \in X_0 : |x_0[0, \delta]|$. Then, $(X_0, X, \Psi)$ is $(\tau_A, \tau_A^D)$-stable at $A$ in the sense of Definition 2 if and only if for each $\varepsilon > 0$, there exists $\delta > 0$ such that $|x_0| < \delta$ implies $\sup_{t \in \text{dom } x} |x(t)| < \varepsilon$ for all $x \in \Psi(x_0)$. In turn, this coincides with the usual notion of Lyapunov stability of a closed forward invariant set $A$ [47].

C. Incremental Input–Output Stability

Given a pseudometric space $(S, d)$, for every $\varepsilon > 0$ define the set $O_\varepsilon := \{(a, b) \in S \times S : d(a, b) < \varepsilon\}$. Then, the family $\beta_{\varepsilon} := \{O_\varepsilon : \varepsilon > 0\} \cup \{S \times S\}$ generates a topology $\tau_{\varepsilon}$ on $S \times S$ that we call the incremental topology. Moreover, $\beta_{\varepsilon}$ is a base for $(S \times S, \tau_{\varepsilon})$ [45, Prop. 1.2.1].

In the same setting of previous Section III-B, suppose now that instead of (9) we have
\[
\forall (x, u, y), (x', u', y') \in \text{graph } \Psi,
\]
\[
|y - y'| \leq \max \{\alpha(|x - x'|), \kappa(|u - u'|)\}. \] (22)
Condition (22) is the equivalent of (9) for incremental ISS systems [48]. Define the system $\delta \Sigma := (\delta D, \delta Y, \delta \Psi)$, where
\[ \delta D := D \times D, \delta Y := Y \times Y, \text{ and } \delta \Psi : \delta D \rightarrow \delta Y, (d_1, d_2) \mapsto \Psi(d_1) \times \Psi(d_2). \]

Endow \( \delta D \) and \( \delta Y \) with the respective incremental topologies. Then, by the same arguments used in Section III-B, one can show that (22) implies that the system \( \delta \Sigma \) is stable at the diagonal set \( \delta \Sigma^* := \{(x, u, (x', u')) \in \delta D : |x - x'| = 0, |u - u'| = 0 \} \).

Remark 5: \{ \Omega_\epsilon : \epsilon > 0 \} is a special case of a diagonal uniformity [49, Sec. 3.15], generalizing in topology the notion of uniform continuity. Indeed, in this sense, input-output stability (9) relates to continuity in the same way as its incremental version (22) relates to uniform continuity.

\[ D \in (\delta_1 | \text{and implies } \Sigma \text{ and asymptotic gain } s \text{ weakly, then } \times III-B \text{ may be useful, for in-} \]

\[ \text{is } \epsilon = \tau = \infty = \inf \in \dom \Psi \text{ the space of absolutely con-} \]

\[ \text{s } \in \{t \in \times \text{uniformity of convergence } \}\}

\[ W \rightarrow u \text{ also the constant function equal to } x : T \times \text{uniformity of convergence } \}\]

\[ \text{co } \subseteq \in \mathbb{Q} \in \text{weak topology } s \text{ of } 0. \text{ Then, for sufficiently small } \]

\[ \frac{1}{T_i} \int_0^{T_i} u_i(\tau) d\tau \rightarrow i \ u^*. \]
methodology that can be extended to other similar contexts. In particular, in the same setting of Section III-C, suppose that, instead of (10), the following property holds:

$$\forall (x, u, y) \in \text{graph } \Psi, \lim \sup |y| \leq \rho \lim \sup |u| + b,$$

in which $b > 0$ is a bias term. While the presence of $b$ ruins stability when $\mathcal{Y}$ has the limsup topology, stability still applies if $\mathcal{Y}$ is given the topology $\tau^b_\delta$ generated by the family $\{\mathcal{Y} \cup \{O_\varepsilon : \varepsilon > b\}\}$ where, as in Section III-C, $O_\varepsilon := \{y \in \mathcal{Y} : \lim \sup |y| < \varepsilon\}$.

Indeed, in view of (25), for every $S \subseteq X$, $D^* := \{u \in U : \lim \sup |u| = 0\}$ maps to $\Psi(D^*) \subseteq \{y \in \mathcal{Y} : \lim \sup |y| \leq b\}$, and every $\tau^b_\delta$-neighborhood $\mathcal{Y}$ of $\Psi(D^*)$ contains a set of the form $O_{b+\varepsilon}$ for all sufficiently small $\varepsilon \in \mathbb{R} \cap \mathbb{R}_>0$. Then, $D := X \times \{u \in U : \lim \sup |u| < \rho^{-1}(\varepsilon)\}$ is a neighborhood of $D^*$ and $\Psi(D) \subseteq \mathcal{Y}$.

VI. APPLICATION EXAMPLES

A. RMS Rejection of Uncertain Disturbances

Consider the system

$$\dot{x} = f(x) + w + u,$$

in which $x(t) \in \mathbb{R}$ is the state variable, $w : \mathbb{R}_>0 \to \mathbb{R}$ is a bounded unmeasured disturbance, $u(t) \in \mathbb{R}$ is a control input, and $f$ is an uncertain Lipschitz function whose nominal value $f^*$ has Lipschitz constant $\ell > 0$.

Define the asymptotic root mean square of $x$ as

$$|x|_{\text{RMS}} := \limsup_{t \to \infty} \sqrt{\frac{1}{T} \int_0^T x(s)^2 \, ds}.$$

Given an arbitrary but fixed $\varepsilon > 0$, we consider the problem of designing a controller ensuring that $|x|_{\text{RMS}} \leq \varepsilon$ at front $t$ of every bounded disturbance $w$ unknown a priori and for all sufficiently small deviations of $f$ from $f^*$.

We assume to measure the state $x$ of the plant subject to a small, bounded and smooth additive disturbance $\nu$. In particular, we assume to measure $y := x + \nu$ with $\nu \in \mathcal{V}$, in which $\mathcal{V}$ denotes the set of bounded continuously differentiable functions $\mathbb{R}_>0 \to \mathbb{R}$ with bounded derivative.

We propose the following controller:

$$\dot{\eta} = \alpha \max \{0, y^2 - \varepsilon^2\} \eta, \quad (\eta(0) \neq 0),$$

$$u = -ky - \alpha y \eta^2,$$

in which $\alpha > 0$ and $k \geq \ell + 3$ are arbitrary (larger values of $\alpha$ lead to faster convergence). The closed-loop system can be written in terms of $y$ as

$$\dot{y} = f(y - \nu) - ky - \alpha y \eta^2 + w + \nu,$$

which implies that, for every $\nu \in \mathcal{V}$, the solutions of (26) obtained with $f \in \text{Lip}(\ell + 1)$ and subject to $(w, \nu) \in \mathcal{W} \times \mathcal{V}$ satisfy $|y|_{\text{RMS}} \leq \varepsilon$. Finally, since

$$\|y - \nu\|_{\text{RMS}} \leq \left(1 + \frac{\ell}{2\varepsilon}\right) |y|_{\text{RMS}} + \left(1 + \frac{2\varepsilon}{\ell}\right) |\nu|_{\text{RMS}}$$

for all $\varepsilon > 0$, we conclude that, for every $f(0), \eta(0) \in \mathbb{R}_>0 \times \mathbb{R}_0$, the solutions of (26) obtained with $f \in \text{Lip}(\ell + 1)$ and subject to $(w, \nu) \in \mathcal{W} \times \mathcal{V}$ satisfy $|y|_{\text{RMS}} \leq \varepsilon$. Finally, since

$$\|y - \nu\|_{\text{RMS}} \leq \left(1 + \frac{\ell}{2\varepsilon}\right) |y|_{\text{RMS}} + \left(1 + \frac{2\varepsilon}{\ell}\right) |\nu|_{\text{RMS}}$$

for all $\varepsilon > 0$, we conclude that, for every $\nu > 0$, there exists $\varepsilon > 0$, such that for every pair $(x(0), \eta(0)) \in \mathbb{R}_>0 \times \mathbb{R}_0$ of initial conditions, the solutions of (26) obtained with $f \in \text{Lip}(\ell + 1)$ and subject to $(w, \nu) \in \mathcal{W} \times \mathcal{V}$ with $|\nu|_{\text{RMS}} \leq \varepsilon$
satisfy $|x|_{\text{ARMS}} = |y - \nu|_{\text{ARMS}} \leq \varepsilon + \epsilon$. This implies that the property $|x|_{\text{ARMS}} \leq \varepsilon$ is obtained robustly with respect to the measurement uncertainty $\nu$. Namely, if $\nu = 0$, then $|x|_{\text{ARMS}} \leq \varepsilon$; otherwise, the above-claimed continuity property between $|\nu|_{\text{ARMS}}$ and $|x|_{\text{ARMS}}$ holds.

**B. Robust Global Attractiveness Without ISS**

Consider the forced Susceptible-Infected system

$$
\dot{S} = -\beta SI, \quad \dot{I} = \beta SI - \gamma I + \nu,
$$

with $\beta > \gamma > 0$, $S(0), I(0) \geq 0$, and $\nu \in \mathcal{V}$, where $\mathcal{V}$ is the set of bounded continuous functions $\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$. It is well known that, when $\nu = 0$, the set $A := \{(S, I) \in \mathbb{R}^2_{\geq 0} : I = 0\}$ is globally attractive. Yet $A$ is not Lyapunov stable. Moreover, the $I$ subsystem is not ISS with respect to the input $(S, \nu)$, nor is the $S$ subsystem is ISS with respect to $I$. Hence, attractiveness of $A$ cannot be concluded by means of canonical small-gain arguments for ISS systems. Instead, as detailed in the rest of this section, it can be proved by using Theorem 1. In particular, we prove following stronger “robust attractiveness” property:

$$
\forall \varepsilon > 0, \exists \hat{\nu} > 0, \forall (S(0), I(0), \nu) \in \mathbb{R}^2_{\geq 0} \times \mathcal{V},
\limsup_{t \to \infty} |v| < \hat{\nu} \Rightarrow \limsup_{t \to \infty} |(S, I)|_A < \varepsilon
$$

where $|(S, I)|_A$ denotes the distance of $(S, I)$ to $A$.

Let $\Psi_1$ be the set of bounded nonincreasing continuous functions $\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, and $\Psi_2$ be the set of continuous functions $I : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ that either are Lipschitz or satisfy $I(t) \to \infty$. Define the system $\Sigma_1 := \{(D_1, \Psi_1, \Psi_1^*), \text{ in which } D_1 = \Psi_2 \times \mathbb{R}^2_{\geq 0} \}$ and $\Psi_1$ is the solution map of $\dot{S} = -\beta SI$ mapping pairs $(I, S(0)) \in D_1$ to complete solutions $S \in \Psi_1$. Define the system $\Sigma_2 := \{(D_2, \Psi_2, \Psi_2^*), \text{ in which } D_2 = \Psi_2 \times \mathbb{R}^2_{\geq 0} \times \mathcal{V} \}$ and $\Psi_2$ the solution map of $\dot{I} = \beta SI - \gamma I + \nu$ mapping triples $(S, I(0), \nu) \in D_2$ to complete solutions $I \in \Psi_2$ Then, system (29) can be seen as a feedback interconnection $\Sigma = (D, \Psi, \Psi^*)$ of the previous two systems (see Section II-B), with $D = \mathbb{R}^2_{\geq 0} \times \mathcal{V}$, $\mathcal{V} = \Psi_2 \times \mathcal{V}$, and $\Psi$ mapping triples $(S(0), I(0), \nu)$ to complete solutions $(S, I)$ of (29). We endow $\mathbb{R}^2_{\geq 0}$ and $\Psi_1$ with the respective topological trivial topologies, and we give $\Psi_2$ and $\Psi$ the respective linsup topologies (see Section III-C).

Let $D^* := \{(S_0, I_0, \nu) : D : I_0 = 0, \nu = 0\}$. Then, every $(S, I) \in \Psi(D^*)$ satisfies $(S(t), I(t)) \in A$ for all $t \in \mathbb{R}_{\geq 0}$. Moreover, $\Sigma$ satisfies the small-gain property of Definition 3 at $D^*$. Since $\Psi_1$ and $\mathbb{R}^2_{\geq 0}$ have the trivial topology, to show this it suffices to show that for every $\varepsilon > 0$, there exists $\hat{\nu} = \hat{\nu}(\varepsilon) > 0$, and for every $I \in \Psi_2 \setminus O_{\varepsilon}$ (as in Section III-C), we have $I \in \Psi_2 : \limsup_{t \to \infty} |I| < \varepsilon$.

Therefore, by using Theorem 1, we finally conclude that $\Sigma$ is stable at $D^*$. As the set $\mathbb{R}^2_{\geq 0}$ of initial conditions has the trivial topology, and $|(S, I)|_A = |I|$, this implies (30).

**C. Automatic Frequency Regulation in PWM Control**

Consider an electrical motor described by the linear system

$$
\dot{x} = Ax + Bu, \quad y = Cx,
$$

with $x(t) \in \mathbb{R}^n$, $u(t), y(t) \in \mathbb{R}$, and $A$ Hurwitz. The output $y$ represents the rotor’s angular velocity, and $u$ is the input voltage. We consider a control system that can only generate quantized switching voltages taking the value $V$ or $-V$ at a given time instant, with $V > 0$. This is typical of controllers implemented by power converters [50]. Given an ideal input profile $u^* : \mathbb{R}_{\geq 0} \to \{-V, V\}$ which can take any value in-between $-V$ and $V$, the controller approximates $u^*$ by means of a function $\hat{u}_T : \mathbb{R}_{\geq 0} \to \{-V, V\}$ satisfying

$$
\frac{1}{T_k} \int_{t_k}^{t_{k+1}} \hat{u}_T(s) ds = u^*(t_k), \quad \forall k \in \mathbb{N},
$$

where $T_k := t_{k+1} - t_k \in [T_{\text{min}}, \infty)$, with $T_{\text{min}} > 0$. The sequence $T = (T_k)_{k \in \mathbb{N}}$ represents a time-varying switching period, and it is a degree of freedom. Moreover, due to the presence of unavoidable uncertainty in the controller implementation, we suppose that the actual generated control is $u_T := \hat{u}_T + \nu$, with $\nu : \mathbb{R}_{\geq 0} \to \mathbb{R}$ a bounded additive perturbation. The controller measures the error $e_T := y^* - y_T$ between the ideal output $y^*$ that would be produced by (32) with $u^*$ and $x(0) = 0$, and the actual output $y_T$ produced by $u_T$ for some $x(0) \in \mathbb{R}^n$. The control goal is to tune the sequence $T_k$ online to eventually reduce the error below a prespecified threshold $\varepsilon^* > 0$.

The error $e_T$ is bounded, and by means of arguments similar to those used in proving Lemma 7, it can be shown that, when $\nu = 0$, for every $\varepsilon > 0$, there exists $\bar{\sigma} = \bar{\sigma}(\varepsilon) > 0$ such that, for every $y_T$ obtained with a sequence $T$ satisfying $T_k \leq \bar{\sigma}(\varepsilon)$ for all but at most $k$, $\limsup_{t \to \infty} |e_T| \leq \varepsilon$. We assume that $T_{\text{min}} < \bar{\sigma}(\varepsilon)$, so as the set of sequences $T$ for which $\limsup_{t \to \infty} |e_T| \leq \varepsilon$ when $\nu = 0$ is nonempty.

We focus on decision strategies (continuous- or discrete-time) adapting $T_k$ iteratively in such a way that

1. $T_k \leq T_h$, for $k \geq h$;
2. there exist $\alpha \in \mathbb{N}$ and $\omega : [T_{\text{min}}, \infty) \to [T_{\text{min}}, \infty)$ satisfying $\lim_{m \to \infty} \omega^m(s) = T_{\text{min}}$ for all $s \in [T_{\text{min}}, \infty)$ such that, if $|e_T(t)| > \varepsilon^*$ for some $t \in [t_k, t_{k+1}]$, then $T_{k+1} = T_{k+\alpha} - t_{k+1} \leq \omega(T_k)$.

Let $\mathcal{T}$ denote the set of bounded sequences $(T_k)_{k \in \mathbb{N}} \in \mathbb{N}$ $[T_{\text{min}}, \infty)$, $\mathcal{E}$ denote the space of bounded continuous functions $e : \mathbb{R}_{\geq 0} \to \mathbb{R}$, and $\mathcal{V} := \mathcal{E}$. We can model the closed-loop system as in Fig. 2 in terms of the feedback interconnection $\Sigma = (D, \Psi, \Psi^*)$ between $\Sigma_1 = (D_1, \Psi_1, \Psi_1^*)$ and $\Sigma_2 = (D_2, \Psi_2, \Psi_2^*)$, mapping error signals $e_T \in \mathcal{E}$ and

Clearly, $T_k = T_{\text{min}}$, for each $k$ would be the best choice in terms of asymptotic bound. Nevertheless, higher switching frequencies are associated with higher power consumption and possibly with more significant drawbacks due to switching, such as unwanted vibrations and flickering phenomena. Hence, it makes sense to seek larger switching periods guaranteeing the desired bound on the error.
initial conditions $T_0 \in [T_{\text{min}}, \infty)$ to sequences $T \in \mathcal{T}$, and $e = \max_{N=1} \sigma$ is the uniform norm topology. We set $D^* := \{(T_0, x(0), \nu) \in \mathcal{D} : T_0 = T_{\text{min}}, x(0) = 0, \nu = 0\}$. As $T_{\text{min}} < \sigma(\epsilon^*)$, as assumed before, in view of (P1) we have $\limsup |e| \leq \epsilon^*$ for all $(T, e) \in \Psi(D^*)$. Moreover, since $\mathcal{T}, [T_{\text{min}}, \infty)$, and $\mathbb{R}^n$ have the trivial topology, to show that the interconnection satisfies the small-gain property of Definition 3 at $D^*$, it suffices to show that for every $\mu > \epsilon^*$ there exists $\tilde{\nu} = \tilde{\nu}(\epsilon) > 0$, and for every $\tilde{e} \in \mathcal{E} \setminus \mathcal{E}_\mu$, there exists $m_e \in \mathbb{N}$, such that, for every $(T_0, x(0), \nu) \in \mathcal{D}$ with $|\nu|_{\infty} < \tilde{\nu}$,

$$
e = \Gamma_{\min} (\tilde{e}, (T_0, x(0), \nu)) = \limsup |e| < \mu. \tag{34}$$

Pick arbitrarily $\mu > 0$ and $\epsilon \in \mathcal{E} \setminus \mathcal{E}_\mu$ and notice that, by linearity, for all $(T_0, x(0), \nu) \in \mathcal{D}$, $e := \Gamma_{\max} (\epsilon, (T_0, x(0), \nu)) = \Psi_2 (\Psi_1 (\epsilon, T_0), x(0), \nu)$ can be written as $e = e_0 + e_\nu$ with $e_0 = \Psi_2 (\Psi_1 (\epsilon, T_0), 0, 0)$ and $\limsup |e_\nu| \leq \sigma(\epsilon^*)$ for some $c > 0$ depending only on $A, B,$ and $C$. Take $\tilde{\nu} < (\mu - \epsilon^*)/c$, and pick $(T_0, x(0), \nu) \in \mathcal{D}$ such that $|\nu|_{\infty} < \tilde{\nu}$. This implies $\limsup |e_\nu| < \mu - \epsilon^*$. As $\epsilon \notin \mathcal{E}_\mu$, $\limsup |e| \geq \mu > \epsilon^*$. Hence, for each $h, \alpha \in \mathbb{N}$ there exists $\ell > h$ such that $\limsup |e|_{[t_{\ell-1}, t_\ell]} > \epsilon^*$. Therefore, it follows from (P1) and (P2) that $e = \Psi_1 (\epsilon, T_0)$ satisfies $\epsilon < \epsilon(\epsilon^*)$ for all large enough $h$. As a consequence, we conclude that necessarily $\limsup |e| \leq \epsilon^*$, which implies $\limsup |e| < \mu$. Hence, the small-gain property (34) holds with $m_e = 1$.

Therefore, in view of Theorem 1, we claim that $\Sigma$ is stable at $D^*$. As $[T_{\text{min}}, \infty)$ and $\mathbb{R}^n$ have the trivial topology, this means that, regardless of the actual value of the initial conditions $x(0)$ and $T_0$, the controller produces an error whose linsupper is larger than $\epsilon^*$ by an amount that continuously increases with the size of the disturbance $\nu$ (in particular, if $\nu = 0$, $\limsup |e| \leq \epsilon^*$).

We underline that we made no strong stability assumption, such as ISS, to achieve the above-mentioned result. Moreover, how to characterize $\Sigma_1$ in terms of ISS is also unclear. Indeed, a map $\Psi_1$ satisfying (P1) and (P2) may not be continuous in general. For example, take $\Psi_1$ as the function mapping every $(e, T_0) \in \mathcal{D}_1$ to a sequence $T$ with initial condition $T_0$ and satisfying

$$T_k = \begin{cases} T_{k-1}/2 & \text{if } \sup_{t \in [t_{k-1}, t_k]} |e(t)| > \epsilon^* \\ T_{k-1} & \text{otherwise} \end{cases} \tag{35}$$

Clearly, the sequence generated in this way fulfills (P1) and (P2). Consider the metric $|T - T'| := \sup_{k \in \mathbb{N}} |T_k - T'_k|$ on $\mathcal{T}$, and the one induced by the uniform norm on $\mathcal{E}$. Pick $T_0 := T_{\text{min}} + 2$ and let $e \in \mathcal{E}$ be the constant function $e(t) := \epsilon^*$. Then, in view of (35), $T = \Psi_1 (T_0, e)$ satisfies $T_k = T_0 = T_{\text{min}} + 2$ for all $k \in \mathbb{N}$. Now, pick $e_{\nu} := \epsilon^* + \epsilon$ for some $\epsilon > 0$. Then, $e_{\nu} \rightarrow e$ uniformly as $\epsilon \rightarrow 0$. However, in view of (35), $T_{\nu} := \Psi_1 (T_0, e_{\nu})$ satisfies $T_{\nu} < T_{\text{min}} + 1$ for sufficiently large $k$. Hence, $|\Psi_1 (T_0, e) - \Psi_1 (T_0, e_{\nu})| \geq 1$ for all $\epsilon > 0$, which implies $\Psi_1 (T_0, e_{\nu}) \neq \Psi_1 (T_0, e)$ as $\epsilon \rightarrow 0$. Thus, $\Sigma$ is not continuous.

Fig. 3 shows a simulation obtained with the following decision policy for $T(m, \rho, \sigma)$ and $\sigma$ are auxiliary variables:

1. Start with $m(0) = p(0) = 0$ and $\sigma(t_0) = T_0$;
2. For each $k \in \mathbb{N}$,

- Define $t_{k+1} := t_k + \sigma(t_k)$.
- Integrate the following equations over $[t_k, t_{k+1}]:$
  $$\dot{m} = 0, \quad \dot{\rho} = \max \left\{0, \epsilon_{\nu} \right\}.$$
- If $t_{k+1} - m(t_{k+1}) < 3$, update the variables as
  $$\sigma(t_{k+1}) = \max \{T_{\text{min}}, 2 \sigma(t_k) \}$$
  otherwise
  $$\rho(t_{k+1}) = 0, \quad m(t_{k+1}) \leftarrow t_{k+1}.$$

The $k$th term $T_k$ of the sequence $T$ generated in this way is given by $T_k = t_{k+1} - t_k = \sigma(t_k)$. The error signal $e_T$ is defined as $e_T := C \tilde{x} - y$ with $\tilde{x} = \hat{x} + Bu^*$ and $\hat{x}(0) = 0$. Specifically, Fig. 3 shows three simulations obtained with $\hat{u} \in \mathcal{E}$ as a PWM signal with variable period (according to $T$) and, within each period, a duty cycle chosen in such a way that (33) holds, $T_{\text{min}} = 0.001, \epsilon^* = 0.05, V = 2,$ and $\nu$ given by the interpolation of a pseudorandom signal uniformly sampled from $[0, A_e]$, where $A_e$ equals, respectively, 0.01, 0.5, and 1 in the three simulations. As shown in the figure, the asymptotic amplitude of the error gradually increases with $|\nu|_{\infty}$. This is consistent with our results that claim continuity of such increase. The sequence $T$, instead suffers from an evident discontinuity (in the previously defined uniform norm) when the amplitude of $\nu$ provokes spikes of $e(t)$ above $\epsilon^*$. In such case, indeed, $T_k$ decreases to $T_{\text{min}}$ despite the actual value of $\nu$, whereas
in the other two cases with $A_{\nu} = 0.01$ and $A_{\nu} = 0.5$, only a small variation of $T_{k}$ is observed. This behavior is caused by the fact that, as for (35), also in this case $\Psi_{1}$ is discontinuous. Indeed, while discontinuity of $\Psi_{1}$ does not invalidate our results, it nevertheless implies that we have no guarantee that “small” changes of $\nu$ reflect on “small” changes of $T$ (with respect to the previously defined metric).

VII. CONCLUSIONS

In this article, we proposed a small-gain theory for interconnections of abstract systems described by set-valued maps between topological spaces. For systems of this kind, stability is defined as a continuity property generalizing and unifying the continuity conditions underlying commonly used stability notions, including Lyapunov stability of motions or sets, (incremental) input–output stability, and asymptotic gain properties. Given a feedback interconnection of two subsystems, the main result of the paper (Theorem 1) establishes the following implication:

small-gain property $\Rightarrow$ stability of the feedback interconnection

where the “small-gain property” is formally defined in Definition 3 and represents an abstraction, in the context of topological spaces, of the joint condition “stability of the subsystems + small-gain condition” of ordinary small-gain theories for input–output operators or ISS systems. While the proposed small-gain property does not admit, in general, a similar decomposition, we proved in Section III that it is always implied by ISS.

The main contribution of the article is methodological, as the presented results provide a common framework for small-gain theories and extend the “small-gain principle” beyond interconnections of systems defined between metric spaces of trajectories. Yet, the application of Theorem 1 to practical problems might not be straightforward. The examples of Section VI do suggest that the developed small-gain theory can provide a useful tool to study complex interconnections uncovered by other existing paradigms. However, its application does require the definition of suitable topological spaces and a preliminary analysis that are problem-specific and may not be easy. In this respect, further research is required.

Moreover, the developed theory focuses on continuity at a set or point, which is a local property. How to cover properties, such as uniform convergence or Lagrange stability, within the same setting of this article is still an open problem deserving further research.

APPENDIX A

Proof of Lemma 2: Let $(i, j) \in \{(1, 2), (2, 1)\}$. If $d \notin \text{dom} \Psi$, then $\Psi(d) = \emptyset$ and $\Psi_{i}(d) = \emptyset$. Hence, the claim of the lemma is vacuously true. Pick $d = (d_{1}, d_{2}) \in \text{dom} \Psi$ and $y_{i} \in \Psi_{i}(d)$. Then, there exists $y_{j} \in \Psi_{j}$ such that $y_{i} \in \Psi_{i}(y_{j}, d_{j})$ and $y_{j} \in \Psi_{j}(y_{i}, d_{i})$. This, in turn, implies that $y_{i} \in \Psi_{i}(y_{j}, d_{j})$. Thus, $y_{i} \in S_{i}(d)$. Hence, we conclude that $\Psi_{i}(d) \subseteq S_{i}(d)$. Conversely, pick $y_{i} \in S_{i}(d)$. Then, there exists $y_{j} \in \Psi_{j}(y_{i}, d_{j})$ such that $y_{i} \in \Psi_{i}(y_{j}, d_{j})$. By the definition (3), this implies that $(y_{1}, y_{2}) \in \Psi(d)$. Hence, $y_{i} \in T_{i}(d)$, which proves $S_{i}(d) \subseteq T_{i}(d)$.

APPENDIX B

Proof of Proposition 1: Pick $Y \in \mathcal{N}(\Psi_{p}(d))$. Since $\Psi_{p}(d) = \Psi_{1}(d) \times \Psi_{2}(d)$, by Lemma 3 we can find $O_{1} \in \mathcal{N}(\Psi_{1}(d))$ and $O_{2} \in \mathcal{N}(\Psi_{2}(d))$ such that $O_{1} \times O_{2} \subseteq Y$. If $\Sigma_{1}$ and $\Sigma_{2}$ are stable at $d$, we can find open sets $U_{1}, U_{2} \in \mathcal{N}(d)$ such that $\Psi_{1}(U_{1}) \subseteq \Psi_{1}(d)$ and $\Psi_{2}(U_{2}) \subseteq \Psi_{2}(d)$. Then, $U := U_{1} \cap U_{2} \in \mathcal{N}(d)$ and $\Sigma_{i}(U) = \bigcup_{d \in U} (\Psi_{1}(d) \times \Psi_{2}(d)) \subseteq \Psi_{1}(U_{1}) \times \Psi_{2}(U_{2}) \subseteq O_{1} \times O_{2} \subseteq Y$.

For the second claim, pick $Y \in \mathcal{N}(\Psi_{s}(D))$. As $\Psi_{s}(D) = \Psi_{1}(d) \times \Psi_{2}(d)$, and $\Sigma_{2}$ is stable at $d$, there exists $U_{2} \in \mathcal{N}(\Psi_{1}(d))$ such that $\Psi_{2}(U_{2}) \subseteq Y$. As $\Sigma_{1}$ is stable at $d$, there exists $U_{1} \in \mathcal{N}(d)$ such that $\Psi_{1}(U_{1}) \subseteq U_{2}$. Thus, $\Psi_{s}(U_{1}) \subseteq Y$. The claim then follows from the arbitrariness of $Y$.

APPENDIX C

Proof of Lemma 3: Pick $A \subseteq A$ and $B \subseteq B$. If $A = \emptyset$ or $B = \emptyset$, the first claim is vacuously true, since $A \times B = \emptyset$. Thus, we assume $A \neq \emptyset$ and $B \neq \emptyset$. As sets of the form $U \times V$, where $U$ and $V$ are open in $A$ and $B$, respectively, form a base for the product topology of $A \times B$, for every point $(a, b) \in A \times B$, every $Y \in \mathcal{N}(a, b)$ contains a set of the form $U_{a} \times V_{b}$, with $U_{a} \in \mathcal{N}(a)$ and $V_{b} \in \mathcal{N}(b)$. Then, every $Y \in \mathcal{N}(A \times B)$ contains a set of the form

$O := \bigcup_{(a, b) \in A \times B} (U_{a} \times V_{b})$, $U_{a} \in \mathcal{N}(a)$, $V_{b} \in \mathcal{N}(b)$,

which belongs itself to $\mathcal{N}(A \times B)$. Hence, the first claim follows by noticing that $O = O_{A} \times O_{B}$, where $O_{A} := \bigcup_{a \in A} U_{a}$ and $O_{B} := \bigcup_{b \in B} V_{b}$. Indeed, $(x, y) \in O$ only if there exists $(a, b) \in A \times B$ such that $(x, y) \in U_{a} \times V_{b}$, which implies $x \in U_{a} \in \mathcal{N}(a)$ and $y \in V_{b} \in \mathcal{N}(b)$. Hence, $(x, y) \in O_{A} \times O_{B}$. The converse is shown by a similar argument.

The “if” part of the second claim is obvious, since $U \in \mathcal{N}(A)$ and $V \in \mathcal{N}(B)$ imply $U \times V \in \mathcal{N}(A \times B)$. The “only if” part, instead, directly follows from the first claim of the lemma proved above. Indeed, if $Y \in \mathcal{N}(\Psi(A \times B))$ and $O \in \mathcal{N}(A \times B)$ is such that $\Psi(O) \subseteq Y$, we can find $O_{A} \in \mathcal{N}(A)$ and $O_{B} \in \mathcal{N}(B)$ such that $O_{A} \times O_{B} \in \mathcal{N}(A \times B)$ and $O_{A} \times O_{B} \subseteq O$, so that $\Psi(O_{A} \times O_{B}) \subseteq \Psi(O) \subseteq Y$.

APPENDIX D

Proof of Proposition 2: Conditions (16a) and (17) imply $\Psi(\text{OD}) \subseteq \Psi_{2}(y_{2} \times O_{2})$. Hence, for every $Y \in \mathcal{N}(\Psi(\text{OD}))$, there exists $\varepsilon > 0$, sufficiently small so that $\varepsilon \in \text{ran} \alpha_{1} \cap \text{ran} \alpha_{2} \cap \text{ran} \delta_{1} \cap \text{ran} \delta_{2}$ for all $(i, j) = (1, 2), (2, 1)$, such that $|y_{1} - y_{2}| < \varepsilon$ implies $y_{1} \in Y$. With $\delta(i, j) := \min_{p_{1}, p_{2}} \{(1, 2), (2, 1)\} \text{ran} \alpha_{1}^{-1}(\cdot), \alpha_{2}^{-1}(\cdot), \delta_{1}^{-1}(\cdot), \delta_{2}^{-1}(\cdot)$, let

$D = \{(x_{1}, u_{1}, x_{2}, u_{2}) \in D : |(x_{1}, u_{1}, x_{2}, u_{2})| < \varepsilon(\varepsilon/2)\}$.
Then $D \in \mathcal{N}(O_D)$. Next, pick $(y_1, y_2) \in \mathcal{Y}$ arbitrarily, and notice that (16a) implies that every $y'_i \in \Gamma_{ij}(y_1, D)$ satisfies $|y'_i| \leq \max \{|q_{ij}(|y_i|), \varepsilon/2\}$, with $q_{ij}$ defined in (17). Since (17) implies $\varepsilon/2 < \varepsilon/2$, by induction one obtains

$$\forall n \geq 1, \forall y'_i \in \Gamma_{ij}^n(y_1, D), \quad |y'_i| \leq \max \{|q_{ij}^n(|y_i|), \varepsilon/2\}.$$  

In view of (17), there exists $n_y$ such that $q_{ij}^n(|y_i|) < \varepsilon/2$ for both $i = 1, 2$. Hence, we conclude that, for all $y'_1 \in \Gamma_{ij}^n(y_1, D)$ and all $y'_2 \in \Gamma_{ij}^{n+1}(y_2, D)$, $|y'_1, y'_2| \leq \varepsilon/2 < \varepsilon$, i.e., $(y'_1, y'_2) \in \mathcal{Y}$. By arbitrariness of $Y$, we then conclude that the small-gain condition of Definition 3 holds at $O_D$ with respect to $(\tau_D, \tau_Y)$. 

**Appendix E**

**Proof of Proposition 3:** The proof follows the same arguments used to prove Proposition 2. In particular, (16b) and (17) imply that $\Psi(L_D) \subseteq L_1 \times L_2$. Hence, by definition of the limsup topology, every neighborhood $Y \in \mathcal{N}(\Psi(L_D))$ contains a set of the form $\{y_1, y_2 \in \mathcal{Y} : \limsup |y_i| < \varepsilon, \forall i = 1, 2\}$ for $\varepsilon > 0$ sufficiently small so that $\varepsilon \in \text{ran } \kappa_i \cap \text{ran } q_i \circ \kappa_i$ for all $(i,j) = (1,2),(2,1)$. Then, with $\delta(\cdot) := \min \{\delta(1,2), (1,2), (\varepsilon \circ \kappa_i)^{-1}(\cdot)\}$, the set

$$D := \{x_1, u_1, x_2, u_2 \in D : \limsup |u_i| < \delta(\varepsilon/2), i = 1, 2\}$$

is in $\mathcal{N}(L_D)$, and, in view of (16b) and (17), for every $i = 1, 2$ and every $y'_i \in \Gamma_{ij}(y_1, D)$, with $y'_i \in \mathcal{Y}$ arbitrary, we have

$$\limsup |y'_i| < \max \{|q_{ij}(|y_i|), \varepsilon/2\}.$$  

Thus, in view of (17), proceeding as in the proof of Proposition 2, we can find $n_y \in \mathbb{N}$ such that $\Gamma_{ij}^{n} (y_1, D) \times \Gamma_{ij}^{n} (y_2, D) \subseteq Y$, which implies (13).

**Appendix F**

**Proof of Lemma 7:** Recall that $U = L^\infty[0,t]$. Then, $u_i \to u^*$ weakly if (see [52, Def. 10.11])

$$\int_0^t \phi(s)u_i(s)ds \to \int_0^t \phi(s)u^*ds = u^* \int_0^t \phi(s)ds$$

for every integrable $\phi: [0,t] \to \mathbb{R}$. Let $\phi$ be the indicator function on a given interval $[a, b] \subseteq [0,t]$. Then, with $\omega_1(\tau) := \max \{m \in \mathbb{N} : m T_i \leq \tau\}$, we can write

$$\int_0^t \phi(s)u_i(s)ds = \sum_{k=0}^{\omega_1(b-a)-1} \int_{a+k T_i}^{a+(k+1) T_i} \int_{a}^{a + T_i} u_i(s)ds$$

$$+ \int_{a+\omega_1(b-a) T_i}^{b} u_i(s)ds.$$  

As $T_i \to 0$, we have $\omega_1(b-a) T_i \to b-a$. Hence, the second term of the right-hand side of (37) vanishes as $i \to \infty$. Since $u_i$ is $T_i$-periodic, then $\int_{a+k T_i}^{a+(k+1) T_i} u_i(s)ds = \int_{a}^{a + T_i} u_i(s)ds$. Hence, in view of (23), the first term of (37) satisfies

$$\omega_1(b-a)-1 \sum_{k=0}^{\omega_1(b-a)-1} \int_{a+k T_i}^{a+(k+1) T_i} u_i(s)ds = \frac{\omega_1(b-a) T_i}{T_i} \int_{a}^{a + T_i} u_i(s)ds \to \left(b-a\right) u^* = u^* \int_0^t \phi(s)ds.$$  

Thus, (36) holds for the indicator function $\phi$. In turn, this implies that (36) holds for all finite linear combinations of indicator functions (i.e., all simple functions).

Now, let $\phi$ be a generic integrable function. Then, there exists a sequence $\phi_k$ of simple functions such that $\int_0^t |\phi(s) - \phi_k(s)|ds \to 0$. With $q := \sup_{u \in coQ} |u|$, we have

$$\left|\int_0^t \phi(s)u_i(s)ds - u^* \int_0^t \phi(s)ds\right| \leq 2q \int_0^t |\phi(s) - \phi_k(s)|ds + \left|\int_0^t \phi_k(s)(u_i(s) - u^*)ds\right|$$

for all $k \in \mathbb{N}$, where we have used the fact that $u^* \in coQ$ and $\text{ran } u_i \subseteq Q$ for all $i$. Then, given any $\varepsilon > 0$, we can find $k$ such that $\int_0^t |\phi(s) - \phi_k(s)|ds < \varepsilon/4q$ and then $\varepsilon/2 \cdot k$ such that $\int_0^t \phi_k(s)(u_i(s) - u^*)ds < \varepsilon/2$ for all $i \geq \varepsilon/2$. Thus, it is possible since $\phi_k$ is a simple function. In turn, this implies that (36) holds for all $\varepsilon > 0$, (36) follows.

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**References**


Michelangelo Bin (Member, IEEE) received the Ph.D. degree in control theory from the University of Bologna, Bologna, Italy, in 2019. Since October 2022, he has been with the Department of Electrical, Electronic and Information Engineering, University of Bologna. He was a Research Associate with Imperial College London, London, U.K. His research interests include systems theory, nonlinear control and regulation, and adaptive systems. Dr. Bin is an Associate Editor for Systems and Control Letters.

Thomas Parisini (Fellow, IEEE) received the Ph.D. degree in electronic engineering and computer science from the University of Genoa, Genoa, Italy, in 1993, and the Honorary Doctorate in electrical and electronic engineering from University of Aalborg, Aalborg, Denmark, in 2018. He was with Politecnico di Milano, Milano, Italy, and is currently the Chair of Industrial Control and the Head of the Control and Power Research Group with Imperial College London, London, U.K. His research interests include systems theory, nonlinear control and regulation, and adaptive systems. He authored or coauthored a research monograph and more than 400 research papers in archival journals, book chapters, and international conference proceedings. Dr. Parisini was the corecipient of the IFAC Best Application Paper Prize of the *Journal of Process Control*, Elsevier, in 2011–2013 and 2014 Outstanding Paper Award of the IEEE TRANSACTIONS ON NEURAL NETWORKS. He was the recipient of the 2007 IEEE Distinguished Member Award and awarded as Principal Investigator at Imperial of the H2020 European Union flagship Teaming Project KIOS Research and Innovation Centre of Excellence, University of Cyprus, Nicosia, Cyprus. Since 2001, he has also been the Danieli Endowed Chair of Automation Engineering with the University of Trieste, Trieste, Italy. In 2009–2012, he was the Deputy Rector of the University of Trieste. He authored or coauthored a research monograph and more than 400 research papers in archival journals, book chapters, and international conference proceedings.