STABLE DETERMINATION OF A RIGID SCATTERER IN ELASTODYNAMICS*

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Abstract. We deal with an inverse elastic scattering problem for the shape determination of a rigid scatterer in the time-harmonic regime. We prove a local stability estimate of *log log* type for the identification of a scatterer by a single far-field measurement. The needed a priori condition on the closeness of the scatterers is estimated by the universal constant appearing in the Friedrichs inequality.

Key words. inverse problems, scattering, linear elasticity, elastic waves, stability

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1. Introduction. We consider the scattering of time-harmonic elastic waves by a rigid scatterer in $\mathbb{R}^N$ with $N \geq 2$. The time-harmonic elastic waves in a homogeneous and isotropic elastic medium satisfy the Navier equation

$$(1.1) \qquad \mu \Delta u + (\lambda + \mu)\nabla(\mathrm{div}(u)) + \rho\omega^2 u = 0,$$

where λ and μ are the Lamé constants such that $\mu > 0$ and $\lambda + 2\mu > 0$, $\rho > 0$ is the density, and $\omega > 0$ is the frequency. By the Helmholtz decomposition, any solution u to (1.1) is the superposition of a longitudinal wave u_p and a transversal wave u_s, which are solutions to the Helmholtz equation with wave numbers $\omega_p = \sqrt{\frac{\rho}{\lambda+2\mu}}\,\omega$ and $\omega_s = \sqrt{\frac{\rho}{\mu}}\,\omega$, respectively.

If an incident wave u^{inc}, which is usually given by an entire solution to (1.1), meets a rigid scatterer K, then it is perturbed by the formation of a scattered wave u^{scat} outside K. The total field u is the superposition of the incident and the scattered wave and, for a rigid scatterer, satisfies the following Dirichlet boundary condition on the boundary of the scatterer:

$$(1.2) \qquad u = 0 \qquad \text{on } \partial K.$$

The scattered wave u^{scat} is characterized by being a radiating solution to (1.1), namely, its longitudinal wave u_p^{scat} and transversal wave u_s^{scat} are radiating solutions to the corresponding Helmholtz equations. The radiation condition for elastic waves is usually referred to as the Kupradze radiation condition.

As incident wave u^{inc}, we take either a *longitudinal plane wave*

$$(1.3) \qquad u_p^{inc}(x) = d\,\mathrm{e}^{\mathrm{i}\omega_p d\cdot x}, \qquad x \in \mathbb{R}^N,$$

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where $d \in \mathbb{S}^{N-1}$ is the *direction of incidence*, or a *transversal plane wave*

$$(1.4) \qquad u_s^{inc}(x) = p\, \mathrm{e}^{\mathrm{i}\omega_s d \cdot x}, \qquad x \in \mathbb{R}^N,$$

where $p \in \mathbb{C}^N \backslash \{0\}$ is a unitary vector orthogonal to d. We can also consider a linear combination of longitudinal and transversal plane waves, namely,

$$(1.5) \qquad u^{inc}(x) = c_p u_p^{inc}(x) + c_s u_s^{inc}(x), \qquad x \in \mathbb{R}^N,$$

for some $c_p, c_s \in \mathbb{C}$ such that $|c_p|^2 + |c_s|^2 = 1$, in such a way as to have

$$(1.6) \qquad \|u^{inc}(x)\| = 1, \qquad x \in \mathbb{R}^N.$$

The forward scattering problem for a rigid obstacle is classical and, under mild regularity assumptions on the obstacle K, it is well-known to have a unique solution.

By the Kupradze radiation condition, the scattered wave u^{scat} has the following asymptotic behavior:

$$(1.7) \qquad u^{scat}(x;d) = \frac{\mathrm{e}^{\mathrm{i}\omega_p r}}{r^{(N-1)/2}} U_p(\hat{x};d) + \frac{\mathrm{e}^{\mathrm{i}\omega_s r}}{r^{(N-1)/2}} U_s(\hat{x};d) + O\left(\frac{1}{r^{(N+1)/2}}\right),$$

as $r = \|x\|$ goes to $+\infty$, uniformly in all directions $\hat{x} = x/\|x\| \in \mathbb{S}^{N-1}$. The vector fields U_p and U_s are called longitudinal and transversal far-field patterns, respectively. Since they characterize, respectively, the asymptotic behavior of the normal and of the tangential component, with respect to $\mathbb{S}^{N-1}$, of u^{scat}, by measuring the asymptotic behavior of u, or equivalently of u^{scat}, as r goes to $+\infty$, both the longitudinal part and the transversal part of the far-field pattern of u^{scat} can be measured.

We are concerned with the following geometrical inverse problem in the context of linear elasticity. Given an incident wave u^{inc}, one can measure the vector fields $(U_p(\cdot, d), U_s(\cdot, d))$, which are usually referred to as the corresponding scattering data. By changing the incident wave, for instance, by changing the frequency ω or the incident direction d, one can obtain different scattering data. We wish to determine the scatterer K by using as measured data the scattering data corresponding to one or more incident waves.

The unique determination of K using the measured data corresponding to all the incident directions $d \in \mathbb{S}^{N-1}$, with a fixed frequency ω, was first shown in [21]. In their work, they use both the components $U_p(\cdot, d)$ and $U_s(\cdot, d)$ of the elastic farfields. Next, it was proved that actually only one component of the farfield $U_p(\cdot, d)$ or $U_s(\cdot, d)$ is enough, meaning that either the pressure or the shear waves are enough to uniquely determine the scatterer K. This result was justified first for C^4-smooth scatterers in [20] and later it was extended to Lipschitz-smooth ones in [28]. In addition, reconstruction schemes were proposed in [24, 27] to actually reconstruct the scatterer K.

Here, we are interested in the determination of the scatterer K by the knowledge of the longitudinal and transversal far-field patterns corresponding to a single incident wave provided some suitable a priori information on the location of the scatterer is known.

A special instance of such a problem has been previously analyzed in [19] in a two-dimensional setting. Indeed the authors proved a uniqueness result if it is a priori known that possible scatterers do not deviate too much in area, or more precisely under the following closeness condition:

$$(1.8) \qquad |K \Delta K'| \leq \frac{k_{0,1}^2 \pi \mu}{\rho \omega^2},$$

where $k_{0,1} \approx 2.4048$ is the first zero of the Bessel function J_0. Their argument is strongly based on the fact that a lower estimate for the first Dirichlet eigenvalue of the negative Lamé operator in $K \Delta K'$ in terms of its Lebesgue measure can be achieved by the use of the Faber–Krahn inequality (see also [29]).

Here we study the stability issue for the same problem in any dimension $N \geq 2$. We prove a $log\,log$ type stability estimate for the unknown scatterer under a slightly stronger a priori closeness condition (see section 3 for a precise statement), namely, we assume that K and K' are both contained in a given scattered K^+ and that

$$(1.9) \qquad |K^+ \backslash (K \cap K')| < H_1 = \left(\frac{\min\{2\mu, 2\mu + \lambda\}}{64C(N)^2 \rho \omega^2} \right)^{N/2},$$

where $C(N)$ is an absolute constant depending on the dimension N only—actually it is the one of the isoperimetric inequality (see (2.35)). Just for comparison, for $N = 2$ and assuming for simplicity $\lambda > 0$, our closeness bound becomes

$$H_1 = \frac{\pi \mu}{8 \rho \omega^2}.$$

Such a slightly more restrictive a priori bound is justified by the fact that, in order to deal with stability, we are led to replace the use of the Faber–Krahn inequality with the one of the Friedrichs inequality. Besides the closeness condition, we require some a priori regularity of the unknown scatterer; in particular we require it to be of class $C^{2,\alpha}$, $0 < \alpha < 1$. We note, however, that we allow K to have more than one connected component. Our stability estimate, which is the main result of the paper, is stated in Theorem 3.1.

Let us note that even if the estimate is rather weak, being of $log\,log$ type, this is rather common for these kinds of inverse scattering or boundary value problems. Moreover, it has been shown that a single log estimate is optimal for the stability of these inverse problems even if many measurements are performed (see [35] and [13]).

We recall that analogous local uniqueness and local stability results have been previously achieved in the acoustic framework in [18, 48] and [47], respectively, by means of a spectral type approach as then extended to elasticity in [19]. Unfortunately, as already observed, these arguments cannot be applied to extend the stability result in the elastic case and hence new tools and an original strategy have to be introduced.

Stability results of $log\,log$ type related to the elasticity system are derived in [22, 39, 40] in the stationary case, that is, $\omega = 0$, with a single pair of displacement and traction fields measured on a surface surrounding the unknown scatterer K. Both rigid inclusions and cavities have been treated. More recently [42], an optimal single log estimate has been obtained for the determination of cavities in the two-dimensional case, by exploiting an optimal three-spheres inequality at the boundary, [4], or a doubling inequality at the boundary, [41], that have been established for the Kirchhoff–Love plate's equation. These kinds of inequality might allow us to improve our estimate to a single log one as well, but to the best of our knowledge none of them is already available in our setting.

Among other results of identification of targets by means of a single or finitely many far-field data under a priori geometric constraints, we wish to recall the well-known uniqueness result in [10] for small obstacles in an acoustic context and the corresponding stability estimates in [25, 26], under the additional starshapedness hypothesis of the scatterer.

Another case in which one measurement, or at least few measurements, uniquely identifies a scatterer is when the scatterer satisfies a different strong geometric condi-

tion, namely, it is assumed to be polyhedral. In the case of obstacles, this means that the objects to be determined are (a collection of) polygons in dimension 2 or polyhedra in dimension 3. In this direction, in [9] the first uniqueness result for sound-soft scatterers in the acoustic framework was proved. In the same case, an optimal uniqueness result with a single measurement was established in [3]. These results have been extended to a variety of other boundary conditions and to the electromagnetic case by several authors. The first stability result, still for sound-soft scatterers in the acoustic case, was obtained in [44], and it was followed by analogous results for the sound-hard boundary condition [32] and for the electromagnetic case [33]. This line of research has been first extended to the elastic case in [15], where the third and fourth boundary conditions are considered and uniqueness is established with two (suitable) measurements or one (suitable) measurement, respectively. For polyhedral obstacles, one (suitable) measurement is enough to determine the obstacle and the boundary condition, provided the latter is still of the third or fourth type; see [34], where a corresponding stability estimate is also proved. In [14] it is shown that one measurement is enough to uniquely determine a rigid polygon. More recently, still in dimension 2, in [12] it is showed that four measurements allow one to uniquely recover a collection of polygons as well as their mixed boundary condition, since the obstacles may, at the same time, be purely rigid or be traction-free or satisfy an impedance boundary condition on different parts of the boundary.

Finally, we mention that in [23] uniqueness results using few measurements are derived even if the scatterers neither satisfy smallness conditions nor have polygonal or polyhedral shapes. Instead, the authors assume that the boundary of the scatterers are nowhere analytic and show that for the Dirichlet boundary condition one measurement is enough while for the Neumann boundary condition $N-1$ measurements are enough (recalling that N is the space dimension). These results are proved for the Helmholtz model for any N and then generalized to a larger family of elliptic second order operators when $N \leq 3$.

In order to analyze the inverse problem we preliminarily discuss the direct one. Indeed, in section 2, we observe that the direct scattering problem is well-posed (see Theorem 2.3) recalling a classical result due to Kupradze et al [31]. In Theorem 2.4 we prove a regularity result up to the boundary ∂K for the solution u, independent on the scatterer K. The proof, which may be found in the appendix, is based on well-known regularity estimates for elliptic systems with Dirichlet boundary condition [1] and a preliminary bound for the solution obtained by a continuity argument inspired by Mosco convergence, as done in [38] for the acoustic case and in [33] for the electromagnetic one. In particular, we also obtain a uniform decay property, as $r \to +\infty$, of u^{scat}, again independent on the scatterer K. We conclude this preliminary part by reviewing, in subsection 2.2, the Friedrichs inequality. We observe that its constant depends on the dimension only and can be explicitly evaluated. This allows us to state the closeness condition with an explicit constant depending on the dimension N and the coefficients of the Navier equation only.

In section 3 we state our main stability result, Theorem 3.1, whose proof is developed in section 4.

The strategy of the proof is the following. Assuming we have two scatterers K and K' satisfying the closeness condition, we wish to estimate their Hausdorff distance $\tilde{d}$ from the difference of the corresponding far-field patterns. From the error on the far-field patterns, we estimate the error on the total field in a region surrounding the scatterers by a classical far-field to near-field estimate for the Helmholtz equation applied to the longitudinal and transversal part of the scattered wave. This estimate

has been proved first in dimension 3 in [25] (see also [8]) and then generalized to any dimension in [46]. For a suitably chosen small parameter $s > 0$, we call V_s the region outside $K \cup K'$ whose points can be reached from infinity by a suitable tube of radius s. By a standard unique continuation argument, whose main ingredient is a three-spheres inequality for the Helmholtz equation proved in [7], which is iteratively applied inside the s-tube to the longitudinal and transversal part of the field, we are able to estimate the error on the total field on the boundary of V_s (see Lemma 4.2). Up to swapping K with K', we may find A_s, a suitable connected component of $\mathbb{R}^N \backslash (K \cup \overline{V_s})$, which contains a ball B of radius proportional to $\tilde{d}$. By the regularity of K and K' and the closeness condition, we infer A_s is a set of finite perimeter (with a perimeter depending on the a regularity of the scatterers only) whose measure is strictly less than the closeness constant H_1. For details on the definition of V_s and A_s see the discussion before Lemma 4.2.

By the estimate on the error of the field on the boundary of V_s and the boundary condition, we obtain an estimate on the solution u on the boundary of A_s. It is now that the Friedrichs inequality comes into play. By using the Friedrichs inequality, the bound on the boundary of A_s, the a priori bound on the solution u, and the idea of the proof of the first Korn inequality, we are able to estimate the L^2-norm of ∇u in A_s by a quantity depending on s and the norm of u on ∂A_s (see Lemma 4.3). Actually, we can estimate the L^∞-norm of u_p and u_s on the ball B by a constant depending on the far-field error only, provided we choose a suitable value of s (see Corollary 4.4 and Remark 4.5).

From B we move toward infinity again and, by a unique continuation argument pretty similar to the one used before, we show that u has to be small even far away from the scatterers, such a smallness depending on the smallness of u_p and u_s on the ball B and on the radius of B itself. However, by our choice of the incident field and the decay of the scattered one, the total field u can not be too small far from K. Combining these two pieces of information, we are finally able to estimate the radius of B and consequently the Hausdorff distance between K and K' (see Lemma 4.6 and Remark 4.7). This second part of the proof is inspired by an analogous procedure developed in [44].

By this technique we obtain a stability estimate which is, however, extremely weak, being of *log log log* type. This is due to the fact that the estimate of the error on the boundary of V_s is already of *log log* type since V_s can be extremely irregular. Another *log* comes from the second part of the procedure when we move from B toward infinity. On the other hand, we can apply a refining procedure which is by now standard, see, for instance, [2]. Provided the error on the farfields is small enough, the two scatterers are close enough and, by their a priori regularity, it can be inferred that the unbounded connected component of $\mathbb{R}^N \backslash (K \cup K')$, which we call V_0, satisfies a Lipschitz type regularity; see Lemma 4.1, which is an easy consequence of [2, Lemma 8.1]. By such regularity, we can improve our unique continuation estimate up to the boundary of V_0 to a single *log* estimate. By using a suitable domain A_0 and by exactly the same procedure as before, we are then able to improve our estimate to the final one of *log log* type.

To conclude, we wish to put into evidence a delicate point of the proof that also explains the presence of the scatterer K^+ in our closeness condition. Even if K and K' are smooth, the domain V_0 can be extremely irregular, unless we know that the scatterers are close enough to apply Lemma 4.1. This implies that we are not able to estimate the error on the field up to the boundary of V_0 from the farfields error and are thus forced to introduce the set V_s, $s > 0$. In turn, this introduces another

difficulty. In fact the domain A_0, which we can construct from V_0, is contained in $K' \backslash K$, and thus its measure is bounded by the measure of $K \Delta K'$. Instead, for $s > 0$, the measure of the domain A_s, which we can construct from V_s, is not controlled by the measure of $K \Delta K'$. This is the reason why we need to introduce K^+ in the closeness condition.

2. Preliminaries. Throughout the paper the integer $N \geq 2$ will denote the space dimension. We note that we drop the dependence of any constant from the space dimension N. For any two column vectors $U = (U^1, \ldots, U^N)^T$ and $V = (V^1, \ldots, V^N)^T$ in $\mathbb{C}^N$

$$U \cdot V = U^T V = \sum_{i=1}^{N} U^i V^i.$$

Here, and in what follows, for any matrix A, A^T denotes its transpose. By I_N we denote the identity $N \times N$ matrix. For any two matrices $A = \{a_{i,j}\}_{i,j=1}^{N}$ and $B = \{b_{i,j}\}_{i,j=1}^{N}$,

$$A : B = \sum_{i,j=1}^{N} a_{ij} b_{ij}.$$

For any $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$, we denote $x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$. For any $s > 0$ and any $x \in \mathbb{R}^N$, $B_s(x)$ denotes the open ball contained in $\mathbb{R}^N$ with radius s and center x, whereas $B_s = B_s(0)$. For any $E \subset \mathbb{R}^N$, $B_s(E) = \bigcup_{x \in E} B_s(x)$. Given a point $x \in \mathbb{R}^N$, a vector $v \in \mathbb{S}^{N-1}$, and constants $r > 0$ and θ, $0 < \theta \leq \pi/2$, we call $\mathcal{C}(x, v, r, \theta)$ the open cone with vertex in x, bisecting vector given by v, radius r, and amplitude given by θ, that is,

$$\mathcal{C}(x, v, r, \theta) = \left\{ y \in \mathbb{R}^N : 0 < \|y - x\| < r \text{ and } \cos(\theta) < \frac{y - x}{\|y - x\|} \cdot v \leq 1 \right\}.$$

We remark that by a cone we always mean a bounded not empty open cone of the kind defined above.

For any measurable subset of $\mathbb{R}^N$ we call $|E|$ its N-dimensional Lebesgue measure. By $\mathcal{H}^{N-1}$ we denote the $(N-1)$-dimensional Hausdorff measure.

DEFINITION 2.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set. Let k be a nonnegative integer and $0 \leq \alpha \leq 1$.*

We say that Ω is of class $C^{k,\alpha}$ (Lipschitz if $k = 0$ and $\alpha = 1$, C^k if $\alpha = 0$) if for any $x \in \partial \Omega$ there exist a $C^{k,\alpha}$ function $\phi_x : \mathbb{R}^{N-1} \to \mathbb{R}$ and a neighborhood U_x of x such that for any $y \in U_x$ we have, up to a rigid transformation depending on x,

$$y = (y', y_N) \in \Omega \quad \text{if and only if} \quad y_N < \phi_x(y').$$

We also say that Ω is of class $C^{k,\alpha}$ (Lipschitz if $k = 0$ and $\alpha = 1$, C^k if $\alpha = 0$) with positive constants r and L if for any $x \in \partial \Omega$ we can choose $U_x = B_r(x)$ and ϕ_x such that $\|\phi_x\|_{C^{k,\alpha}(\mathbb{R}^{N-1})} \leq L$.

Remark 2.2. If $k + \alpha > 1$ and Ω is an open set of class $C^{k,\alpha}$ with constants r and L, there exists positive constants r_1 and L_1, depending on k, α, r, and L only, such that Ω is of class $C^{k,\alpha}$ with constants r_1 and L_1 with the further condition that for any $x \in \partial \Omega$ we have $\nabla \phi_x(x') = 0$. Therefore, without loss of generality, whenever $k + \alpha > 1$ we tacitly assume that this condition is satisfied all over $\partial \Omega$.

2.1. The direct scattering problem. We say that $\Omega \subset \mathbb{R}^N$ is a *domain* if it is open and connected. We say that Ω is an *exterior domain* if it is a domain containing the exterior of a ball. We say that $K \subset \mathbb{R}^N$ is a *scatterer* if K is compact and $\Omega = \mathbb{R}^N \backslash K$ is connected, that is, Ω is an exterior domain. We say that a scatterer K is an *obstacle* if $K = \overline{D}$, where D is an open set which we can pick as the interior of K.

We consider the inverse scattering problem for the Navier equation modelling time-harmonic elastic waves in a homogeneous and isotropic elastic medium under the presence of a rigid scatterer. For the direct scattering problem, which we here describe, we refer to the classical works of Kupradze and others [30, 31] and to the more recent one [6], for instance. Let us consider, in an open set $\Omega \subset \mathbb{R}^N$, $N \geq 2$, a weak solution u to the *Navier equation*

$$(2.1) \qquad \mu \Delta u + (\lambda + \mu)\nabla^T(\mathrm{div}(u)) + \rho\omega^2 u = 0 \qquad \text{in } \Omega.$$

Here λ and μ are the Lamé constants such that $\mu > 0$ and $\lambda + 2\mu > 0$, $\rho > 0$ is the density, and $\omega > 0$ is the frequency. We assume all these parameters to be constants. The function $u = (u^1, \ldots, u^N)^T$, the so-called *field* of the time-harmonic wave, is assumed to be a column vector. We note that if v is a scalar function, we often use $\nabla^T v$ to denote the column vector $(\nabla v)^T$.

A vector field $u \in H^1_{loc}(\Omega, \mathbb{C}^N)$ is a weak solution to (2.1) if for any $v \in H^1(\Omega, \mathbb{C}^N)$ with compact support in Ω we have

$$(2.2) \qquad 2\mu \int_\Omega Eu : \overline{Ev} + \lambda \int_\Omega \mathrm{div}(u)\overline{\mathrm{div}(v)} - \rho\omega^2 \int_\Omega u \cdot \overline{v} = 0.$$

Here $Eu = \frac{1}{2}(\nabla u + (\nabla u)^T)$ denotes the symmetric gradient of u. Hence, $\nabla u = \{u^i_j\}^N_{i,j=1}$ and $Eu = \frac{1}{2}\{u^i_j + u^j_i\}^N_{i,j=1}$.

In linearized elasticity Eu corresponds to the *strain tensor* and, by Hooke's law, the *stress* σ is given by

$$\sigma(u) = 2\mu Eu + \lambda \mathrm{tr}(Eu)I_N = 2\mu Eu + \lambda \mathrm{div}(u)I_N,$$

where tr denotes the trace. In particular, we have

$$(2.3) \qquad \min\{2\mu, 2\mu + \lambda\}(Eu : \overline{Eu}) \leq \sigma(u) : \overline{\sigma(u)} \leq \max\{2\mu, 2\mu + \lambda\}(Eu : \overline{Eu}).$$

We note that (2.1) can be rewritten as

$$\mathrm{div}(\sigma(u)) + \rho\omega^2 u = 0 \qquad \text{in } \Omega,$$

where the div applies row by row. In fact,

$$\mathrm{div}(\mathrm{div}(u)I_N) = \mathrm{div}((\nabla u)^T) = \nabla^T(\mathrm{div}(u)).$$

We call

$$\mathcal{K}(\Omega) = \{u \in L^2(\Omega, \mathbb{C}^N) : \ Eu \in L^2(\Omega, \mathbb{C}^{N \times N})\},$$

which is a Hilbert space with the corresponding norm

$$\|u\|_{\mathcal{K}(\Omega)} = \left(\|u\|^2_{L^2(\Omega)} + \|Eu\|^2_{L^2(\Omega)}\right)^{1/2} = \left(\int_\Omega u \cdot \overline{u} + \int_\Omega Eu : \overline{Eu}\right)^{1/2}.$$

We call $\mathcal{K}_0(\Omega)$ the closure of $C_0^\infty(\Omega, \mathbb{C}^N)$ with respect to the norm of $\mathcal{K}(\Omega)$. By the first Korn inequality (see, for instance, [43]), we have that $\mathcal{K}_0(\Omega) = H_0^1(\Omega, \mathbb{C}^N)$, with equivalent norms. In fact, first Korn inequality states that, for any open set Ω,

(2.4) $$\|\nabla u\|_{L^2(\Omega)}^2 \le 2\|Eu\|_{L^2(\Omega)}^2 \quad \text{for any } u \in H_0^1(\Omega, \mathbb{C}^N).$$

By the second Korn inequality (see again [43]), provided Ω is smooth enough, for instance, if Ω is a Lipschitz bounded open set, we also have that $\mathcal{K}(\Omega) = H^1(\Omega, \mathbb{C}^N)$, with equivalent norms. Consequently, $\mathcal{K}_{loc}(\Omega) = H_{loc}^1(\Omega, \mathbb{C}^N)$.

It is well-known that, by Helmholtz decomposition, any weak solution u to (2.1) can be written as the sum of a *longitudinal wave* u_p and a *transversal wave* u_s, where u_p and u_s are solutions to (2.1). Namely, we set

$$u_p = -\frac{\nabla^T \text{div}(u)}{\omega_p^2}, \qquad \omega_p^2 = \frac{\rho\omega^2}{\lambda + 2\mu}.$$

We note that $\chi = -\text{div}(u)$ is a scalar weak solution to the Helmholtz equation $\Delta\chi + \omega_p^2\chi = 0$ in Ω, and hence u_p is a vector-valued weak solution to the same Helmholtz equation

(2.5) $$\Delta u_p + \omega_p^2 u_p = 0 \quad \text{in } \Omega.$$

If we set
$$u_s = \frac{\nabla^T\text{div}(u) - \Delta u}{\omega_s^2} = \frac{\text{div}((\nabla u)^T - \nabla u)}{\omega_s^2}, \qquad \omega_s^2 = \frac{\rho\omega^2}{\mu},$$

it is not difficult to show that $u = u_p + u_s$ and u_s is a vector-valued weak solution to another Helmholtz equation

(2.6) $$\Delta u_s + \omega_s^2 u_s = 0 \quad \text{in } \Omega.$$

We note that

$$u_s = \frac{\text{curl}(\text{curl}(u))}{\omega_s^2} \quad \text{if } N = 3, \qquad u_s = \frac{-Q\nabla^T(\text{curl}_2(u))}{\omega_s^2} \quad \text{if } N = 2,$$

where $Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $\text{curl}_2(u) = u_1^2 - u_2^1$ is the two-dimensional curl of u.

Since $\text{div}(u_s) = 0$ and $(\nabla u_p)^T - \nabla u_p = 0$, we have $(u_p)_p = u_p$, $(u_p)_s = 0$, $(u_s)_s = u_s$, and $(u_s)_p = 0$.

If Ω is an exterior domain, we say that u, a solution to (2.1), is *radiating* or *outgoing* if it satisfies the *Kupradze radiation conditions*

(2.7)
$$\lim_{r\to+\infty} r^{(N-1)/2}\left(\frac{\partial u_p}{\partial r} - i\omega_p u_p\right) = 0,$$
$$\lim_{r\to+\infty} r^{(N-1)/2}\left(\frac{\partial u_s}{\partial r} - i\omega_s u_s\right) = 0,$$
$$r = \|x\|,$$

where the limits have to be intended as uniform in any direction. In other words, u_p and u_s satisfy the Sommerfeld radiation condition and, therefore, are radiating solutions to their corresponding Helmholtz equations.

For any bounded open set Ω and u solution to (2.1), the *surface traction* Tu is

$$Tu = \sigma(u)\nu = 2\mu Eu\nu + \lambda\text{div}(u)\nu = 2\mu\nabla u\nu + \lambda\text{div}(u)\nu + \mu((\nabla u)^T - \nabla u)\nu \quad \text{on } \partial\Omega,$$

ν being the exterior normal to Ω, which we assume to be a column vector. In particular, if u and $\partial\Omega$ are smooth enough, we have that

$$Tu = \sigma(u)\nu = 2\mu\nabla u_p\nu + 2\mu\nabla u_s\nu + \lambda\text{div}(u_p)\nu + \mu((\nabla u_s)^T - \nabla u_s)\nu \qquad \text{on } \partial\Omega.$$

For any $k > 0$, let ϕ_k be the fundamental solution to the Helmholtz equation $\Delta u + k^2 u = 0$, which is given by

$$\phi_k(x, y) = \frac{i}{4}\left(\frac{k}{2\pi\|x-y\|}\right)^{(N-2)/2} H^{(1)}_{(N-2)/2}(k\|x-y\|) \quad \text{for any } x,\, y \in \mathbb{R}^N,\, x \neq y.$$

For any real $s \geq 0$, $H^{(1)}_s$ denotes the Hankel function of first kind and order s. We also remark that for $N = 2, 3$ this reduces to the well-known formulas

$$\phi_k(x, y) = \frac{e^{ik\|x-y\|}}{4\pi\|x-y\|} \quad \text{for any } x,\, y \in \mathbb{R}^3,\, x \neq y,$$

and

$$\phi_k(x, y) = \frac{i}{4} H^{(1)}_0(k\|x-y\|) \quad \text{for any } x,\, y \in \mathbb{R}^2,\, x \neq y.$$

Then the fundamental solution to the Navier equation is given by, for any $x,\, y \in \mathbb{R}^N$, $x \neq y$,

$$(2.8) \qquad \Phi(x, y) = \frac{1}{\mu}\phi_{\omega_s}(x, y)I_N + \frac{1}{\rho\omega^2}\nabla_y\nabla_y^T\left[\phi_{\omega_s}(x, y) - \phi_{\omega_p}(x, y)\right].$$

Here derivatives are meant in the sense of distributions over the whole $\mathbb{R}^N$ and the Navier equation is applied to Φ row by row. For $x \neq y$, we have

$$\Phi(x, y) = \frac{1}{\mu}\phi_{\omega_s}(x, y)I_N + \frac{1}{\rho\omega^2}\nabla_x\nabla_x^T\left[\phi_{\omega_s}(x, y) - \phi_{\omega_p}(x, y)\right]$$

as well. We also note that $\Phi = \Phi^T$.

For any bounded domain Ω and u solution to (2.1), provided Ω and u are smooth enough, we have for any $x \in \Omega$

$$(2.9) \qquad u(x) = \int_{\partial\Omega}\left(\Phi(x, y)\left[Tu(y)\right] - \left[T_y\Phi(x, y)\right]u(y)\right)d\sigma(y),$$

where T is applied to Φ row by row. Regarding regularity, it is enough that Ω is of class C^2, $u \in C^2(\Omega) \cap C(\overline{\Omega})$ such that Tu exists as a uniform limit on $\partial\Omega$, namely,

$$Tu(x) = \lim_{h\to 0^+}\sigma(u)(x - h\nu(x))\nu(x) \qquad \text{for any } x \in \partial\Omega,$$

where the limit is uniform with respect to $x \in \partial\Omega$ and $\nu(x)$ is the exterior normal at x.

If Ω is an exterior domain and u is an outgoing solution to (2.1), then we still have for any $x \in \Omega$

$$(2.10) \qquad u(x) = \int_{\partial\Omega}\left(\Phi(x, y)\left[Tu(y)\right] - \left[T_y\Phi(x, y)\right]u(y)\right)d\sigma(y)$$

since the contribution at infinity is zero due to the Kupradze radiation condition satisfied by u and the corresponding properties of Φ.

By the well-known asymptotic properties of radiating solutions to Helmholtz equations applied to u_p and to u_s, we infer that $u(x) = u_p(x) + u_s(x)$ satisfies

$$(2.11) \qquad u(x) = \frac{e^{i\omega_p \|x\|}}{\|x\|^{(N-1)/2}} U_p(\hat{x}) + \frac{e^{i\omega_s \|x\|}}{\|x\|^{(N-1)/2}} U_s(\hat{x}) + O\left(\frac{1}{\|x\|^{(N+1)/2}}\right),$$

as $\|x\|$ goes to $+\infty$, uniformly in all directions $\hat{x} = x/\|x\| \in \mathbb{S}^{N-1}$. The $\mathbb{C}^N$-valued functions U_p and U_s are defined on $\mathbb{S}^{N-1}$ and are referred to as the *longitudinal part* and the *transversal part* of the *far-field pattern* $U = (U_p, U_s)$ of the field u, respectively.

By (2.10), the longitudinal part U_p is orthogonal to $\mathbb{S}^{N-1}$, that is, $U_p(\hat{x}) = u_p^\infty(\hat{x})\hat{x}$ for any $\hat{x} \in \mathbb{S}^{N-1}$ for a suitable complex-valued function u_p^∞ defined on $\mathbb{S}^{N-1}$. On the other hand, the transversal part U_s is tangential to $\mathbb{S}^{N-1}$, that is, $U_s(\hat{x}) \cdot \hat{x} = 0$ for any $\hat{x} \in \mathbb{S}^{N-1}$.

Therefore, if we consider the normal and tangential component, with respect to $\mathbb{S}^{N-1}$, of u, that is,

$$u(x) = u_N(x) + u_T(x),$$

where for any $x \in \Omega$ we have that $u_N(x)$ is proportional to $\hat{x} = x/\|x\|$ while $u_T(x)$ is orthogonal to $\hat{x}$, we conclude that

$$u_N(x) = \frac{e^{i\omega_p \|x\|}}{\|x\|^{(N-1)/2}} U_p(\hat{x}) + O\left(\frac{1}{\|x\|^{(N+1)/2}}\right)$$

and

$$u_T(x) = \frac{e^{i\omega_s \|x\|}}{\|x\|^{(N-1)/2}} U_s(\hat{x}) + O\left(\frac{1}{\|x\|^{(N+1)/2}}\right),$$

as $\|x\|$ goes to $+\infty$, uniformly in all directions. Thus, measuring the asymptotic behavior of u, as $\|x\|$ goes to $+\infty$, it corresponds to measuring both the longitudinal part and the transversal part of the far-field pattern of u.

Let us send a so-called *incident wave*, that is, a time-harmonic wave whose field u^{inc} is an entire solution to (2.1). Typically, the incident wave is a *plane* wave obtained by a linear combination of a *longitudinal plane wave*

$$(2.12) \qquad u_p^{inc}(x) = d e^{i\omega_p d \cdot x} = \frac{\nabla^T(e^{i\omega_p d \cdot x})}{i\omega_p}, \qquad x \in \mathbb{R}^N,$$

where $d \in \mathbb{S}^{N-1}$ is the *direction of propagation*, and a *transversal plane wave*

$$(2.13) \qquad u_s^{inc}(x) = p e^{i\omega_s d \cdot x}, \qquad x \in \mathbb{R}^N,$$

where $p \in \mathbb{C}^N \backslash \{0\}$ is a unitary vector orthogonal to d. For example, one can choose, if $N = 3$,

$$u_s^{inc}(x) = c e^{i\omega_s d \cdot x}(d_2 - d_3, d_3 - d_1, d_1 - d_2)^T = c \frac{\mathrm{curl}\left(e^{i\omega_s d \cdot x}(1,1,1)^T\right)}{i\omega_s}, \qquad x \in \mathbb{R}^3,$$

whereas if $N = 2$,

$$u_s^{inc}(x) = -cQd e^{i\omega_s d \cdot x} = c \frac{-Q\nabla^T(e^{i\omega_s d \cdot x})}{i\omega_s}, \qquad x \in \mathbb{R}^2,$$

where $c \in \mathbb{C} \backslash \{0\}$ is a suitable constant.

Namely, we consider

$$(2.14) \qquad u^{inc}(x) = c_p u_p^{inc}(x) + c_s u_s^{inc}(x), \qquad x \in \mathbb{R}^N,$$

for some $d \in \mathbb{S}^{N-1}$, $p \in \mathbb{C}^N \backslash \{0\}$ such that $\|p\| = 1$ and p is orthogonal to d, and $c_p, c_s \in \mathbb{C}$ such that $|c_p|^2 + |c_s|^2 = 1$, in such a way as to have

$$(2.15) \qquad \|u^{inc}(x)\| = 1, \qquad x \in \mathbb{R}^N.$$

The presence of an impenetrable object, that is, of a scatterer K, inside the medium perturbs the incident wave by creating the *scattered* or *reflected wave*, given by the field u^{scat}. The total wave is the superposition of the incident and the scattered waves and its field is denoted by u. The total field u solves (2.1) in $\Omega = \mathbb{R}^N \backslash K$ and satisfies a boundary condition on ∂K that depends on the nature of the scatterer, namely, if K is a so-called *rigid scatterer*, a homogeneous Dirichlet boundary condition

$$(2.16) \qquad u = 0 \qquad \text{on } \partial K,$$

or, if K is a so-called *cavity*, a homogeneous Neumann boundary condition

$$(2.17) \qquad Tu = \sigma(u)\nu = 0 \qquad \text{on } \partial K,$$

ν being the exterior normal to Ω.

Finally, given that $\Omega = \mathbb{R}^N \backslash K$ is unbounded, a condition at infinity has to be imposed. We require the scattered wave to be outgoing. Summarizing, the total field u solves the following exterior boundary value problem:

$$(2.18) \qquad \begin{cases} u = u^{inc} + u^{scat} & \text{in } \Omega = \mathbb{R}^N \backslash K, \\ \mu \Delta u + (\lambda + \mu) \nabla(\text{div}(u)) + \rho \omega^2 u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial \Omega = \partial K, \\ u^{scat} \text{ satisfies } (2.7) \end{cases}$$

if K is a rigid scatterer, and

$$(2.19) \qquad \begin{cases} u = u^{inc} + u^{scat} & \text{in } \Omega, \\ \mu \Delta u + (\lambda + \mu) \nabla(\text{div}(u)) + \rho \omega^2 u = 0 & \text{in } \Omega, \\ Tu = 0 & \text{on } \partial \Omega, \\ u^{scat} \text{ satisfies } (2.7) \end{cases}$$

if K is a cavity.

The weak formulation of (2.18) is the following. Assume that $K \subset B_R(0)$ for some $R > 0$. Then we look for u belonging to $H^1(B_r(0) \backslash K, \mathbb{C}^N)$ for any $r > R$ such that $u = u^{inc} + u^{scat}$ solves (2.1) in the weak sense and u^{scat} satisfies the condition at infinity given by (2.7). Finally, for what concerns the boundary condition (2.16) on ∂K, we require that $u = 0$ on ∂K in a weak sense, that is, $\chi u \in H_0^1(B_r(0) \backslash K, \mathbb{C}^N)$ for any $r > R$ and any $\chi \in C_0^\infty(B_r(0), \mathbb{R})$ such that $\chi = 1$ on $B_R(0)$.

The weak formulation of (2.19) is the following. Assume that $K \subset B_R(0)$ for some $R > 0$. Then we look for u belonging to $\mathcal{K}(B_r(0) \backslash K, \mathbb{C}^N)$ for any $r > R$ such that $u = u^{inc} + u^{scat}$ solves (2.1) in the weak sense and u^{scat} satisfies the condition at infinity given by (2.7). Finally, about the boundary condition (2.17) on ∂K, we require that for any $r > R$ and any $v \in \mathcal{K}(B_r(0) \backslash K, \mathbb{C}^N)$ with compact support contained in $B_r(0)$ we have

$$(2.20) \qquad 2\mu \int_\Omega Eu : \overline{Ev} + \lambda \int_\Omega \text{div}(u)\overline{\text{div}(v)} - \rho \omega^2 \int_\Omega u \cdot \overline{v} = 0.$$

In both cases, if $u^{inc} = 0$, then we have that for any $r > R$

$$(2.21) \qquad \Im\left(\int_{\partial B_r(0)} (Tu) \cdot \overline{u}\right) = 0,$$

and hence, by the asymptotic behavior of outgoing solutions, we infer that $u = u^{scat} = 0$ in Ω. In other words, (2.18) and (2.19) admit at most one solution, so uniqueness follows. Concerning existence, this can be established by layer potential techniques provided K, or Ω, is regular enough, say, of class C^2. In fact, for any exterior domain Ω of class C^2 and for any $\varphi \in C(\partial\Omega, \mathbb{C}^3)$ we define the single- and double-layer potentials with density φ at any $x \in \Omega$ as follows:

$$(2.22)$$
$$\mathcal{S}(\varphi)(x) = \int_{\partial\Omega} \Phi(x, y)\varphi(y)\, d\sigma(y) \quad \text{and} \quad \mathcal{D}(\varphi)(x) = \int_{\partial\Omega} [T_y\Phi(x, y)]\,\varphi(y)\, d\sigma(y).$$

We observe that $\mathcal{S}(\varphi)$ and $\mathcal{D}(\varphi)$ are outgoing solutions to (2.1) in Ω. By carefully exploiting the properties of the potentials on $\partial\Omega$, the following existence (and uniqueness) result can be proved.

THEOREM 2.3. *Assume that K is an obstacle such that $D = \overset{\circ}{K}$ is an open set of class C^2. Then, for any u^{inc} entire solution to (2.1), (2.18), and (2.19) admit one solution.*

Proof. This is a classical result (see, for instance, [31]). Actually, the regularity of D can be relaxed up to Lipschitz (see, for instance, [6, Corollary 2,3]). □

We shall need the following regularity result, whose proof is postponed to the appendix.

THEOREM 2.4. *Let us fix positive constants r, L, R, and α, $0 < \alpha < 1$. Let us also fix the coefficients $\mu > 0$, λ such that $2\mu + \lambda > 0$, $\rho > 0$, and $\omega > 0$. Assume that $K \subset B_R(0)$ is an obstacle such that $D = \overset{\circ}{K}$ is an open set of class $C^{2,\alpha}$ with constants r and L. Let u^{inc} be as in (2.14) such that (2.15) is satisfied.*

Let u be the solution to (2.18). Then there exists a constant $\tilde{C}_0$, depending on r, L, R, α, and the coefficients only, such that

$$(2.23) \qquad \|u\|_{C^2(\overline{\Omega})} \leq \tilde{C}_0.$$

Moreover, there exists a constant $\tilde{C}_1$, depending on r, L, R, α, and the coefficients only, such that

$$(2.24) \quad \|u^{scat}(x)\| \leq \|u_s^{scat}(x)\| + \|u_p^{scat}(x)\| \leq \frac{\tilde{C}_1}{\|x\|^{(N-1)/2}} \qquad \text{for any } \|x\| \geq R+1.$$

We conclude this part with the following regularity result and a three-spheres inequality for the Helmholtz equation.

LEMMA 2.5. *Let u be a solution to (2.1) in B_s with $0 < s \leq s_0$. Then there exists a constant D_0, depending on the coefficients of (2.1) and on s_0 only, such that*

$$(2.25) \qquad \|u_p\|_{L^\infty(B_{s/8})}, \ \|u_s\|_{L^\infty(B_{s/8})} \leq \frac{D_0}{s^{(N+4)/2}}\|u\|_{L^2(B_s)}$$

and

$$(2.26) \qquad \|u_p\|_{L^\infty(B_{s/4})}, \ \|u_s\|_{L^\infty(B_{s/4})} \leq \frac{D_0}{s^{(N+2)/2}}\|\nabla u\|_{L^2(B_s)}.$$

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Proof. First, we use a Caccioppoli inequality to estimate the H^1-norm of u in a smaller ball.

Namely, let $\chi \in C_0^\infty(B_s)$ be such that $0 \le \chi \le 1$ everywhere and $\chi = 1$ on $B_{3s/4}$. We can assume that $\|\nabla\chi\| \le C/s$ everywhere for some absolute constant C. Then we apply the weak formulation of (2.1) to $v = \chi^2 u$ and obtain

$$2\mu \int_{B_s} Eu : \overline{Ev} + \lambda \int_{B_s} \text{div}(u)\overline{\text{div}(v)} = \rho\omega^2 \int_{B_s} \chi^2 \|u\|^2.$$

But

$$2\mu \int_{B_s} Eu : \overline{Ev} + \lambda \int_{B_s} \text{div}(u)\overline{\text{div}(v)} = 2\mu \int_{B_s} \chi^2 \|Eu\|^2 + \lambda \int_{B_s} \chi^2 |\text{div}(u)|^2$$
$$+ \left(2\mu \int_{B_s} \chi Eu : (\chi_j \overline{u}^i + \chi_i \overline{u}^j) + 2\lambda \int_{B_s} \chi \text{div}(u)(\chi_j \overline{u}^j) \right),$$

and hence, by Cauchy inequality, we infer that

$$2\mu \int_{B_s} \chi^2 \|Eu\|^2 + \lambda \int_{B_s} \chi^2 |\text{div}(u)|^2$$
$$\le \mu \int_{B_s} \chi^2 \|Eu\|^2 + \frac{\lambda}{2} \int_{B_s} \chi^2 |\text{div}(u)|^2 + \left(\frac{C^2}{s^2} C_1 + \rho\omega^2 \right) \int_{B_s} \|u\|^2$$

for a constant C_1 depending on μ and λ only. We conclude that

$$\frac{\min\{2\mu, 2\mu + \lambda\}}{2} \int_{B_{3s/4}} \|Eu\|^2 \le \mu \int_{B_{3s/4}} \|Eu\|^2 + \frac{\lambda}{2} \int_{B_{3s/4}} |\text{div}(u)|^2$$
$$\le \mu \int_{B_s} \chi^2 \|Eu\|^2 + \frac{\lambda}{2} \int_{B_s} \chi^2 |\text{div}(u)|^2 \le \left(\frac{C^2}{s^2} C_1 + \rho\omega^2 \right) \int_{B_s} \|u\|^2.$$

Let $\tilde{\chi} \in C_0^\infty(B_{3s/4})$ be such that $0 \le \tilde{\chi} \le 1$ everywhere and $\tilde{\chi} = 1$ on $B_{s/2}$. We can assume that $\|\nabla\tilde{\chi}\| \le C/s$ everywhere for some absolute constant C. Then $\tilde{\chi}u \in \mathcal{K}_0(B_{3s/4}) = H_0^1(B_{3s/4})$, so by (2.4),

$$\int_{B_{s/2}} \|\nabla u\|^2 \le \int_{B_{3s/4}} \|\nabla(\tilde{\chi}u)\|^2 \le 2 \int_{B_{3s/4}} \|E(\tilde{\chi}u)\|^2$$
$$= 2 \int_{B_{3s/4}} \left\| \tilde{\chi}Eu + \frac{1}{2}(\tilde{\chi}_j u^i + \tilde{\chi}_i u^j) \right\|^2 \le 4 \left(\int_{B_{3s/4}} \|Eu\|^2 + \frac{C^2}{s^2} C_2 \int_{B_{3s/4}} \|u\|^2 \right),$$

where C_2 is another absolute constant. We conclude that

(2.27)
$$\|\nabla u\|_{L^2(B_{s/2})} \le \frac{C_3}{s} \|u\|_{L^2(B_s)}$$

for a constant C_3 depending on the coefficients and on s_0 only.

Since for any $j \in \{1, \ldots, N\}$, u_j still solves (2.1), we can repeat the procedure above and prove that

(2.28)
$$\|D^2 u\|_{L^2(B_{s/4})} \le \frac{C_4}{s^2} \|u\|_{L^2(B_s)},$$

and consequently

$$(2.29) \qquad \|u_p\|_{L^2(B_{s/4})}, \|u_s\|_{L^2(B_{s/4})} \leq \frac{C_5}{s^2} \|u\|_{L^2(B_s)}$$

with C_4 and C_5 still depending on the coefficients and on s_0 only.

The last step is to estimate the L^∞-norm by the L^2-norm for a solution to a Helmholtz equation. This is a standard estimate (see, for instance, [17, Theorem 8.17]), since we have

$$(2.30) \qquad \|u_p\|_{L^\infty(B_{s/8})}, \|u_s\|_{L^\infty(B_{s/8})} \leq \frac{C_6}{s^{N/2}} \|u_p\|_{L^2(B_{s/4})}, \|u_s\|_{L^2(B_{s/4})},$$

respectively, with C_6 depending on the coefficients and on s_0 only. The proof can now be easily concluded. □

LEMMA 2.6. *There exist positive constants $\tilde{s}_0$, $\tilde{C}$, and $\tilde{c}_1$, $0 < \tilde{c}_1 < 1$, depending on k only, such that for every $0 < s_1 < s < s_2 \leq \tilde{s}_0$ and any function u such that*

$$\Delta u + k^2 u = 0 \quad in \ B_{s_2},$$

we have, for any t, $s < t < s_2$,

$$(2.31) \qquad \|u\|_{L^\infty(B_s)} \leq \tilde{C}(1 - (s/t))^{-N/2} \|u\|_{L^\infty(B_{s_2})}^{1-\beta} \|u\|_{L^\infty(B_{s_1})}^{\beta},$$

for some β such that

$$(2.32) \qquad \tilde{c}_1 \left(\log(s_2/t) \right) / \left(\log(s_2/s_1) \right) \leq \beta \leq 1 - \tilde{c}_1 \left(\log(t/s_1) \right) / \left(\log(s_2/s_1) \right).$$

Proof. It follows by the results of [7]. □

2.2. Friedrichs inequality. Let $\Omega \subset \mathbb{R}^N$ be an open and bounded set. Under suitable assumptions on Ω and u, a function defined on $\overline{\Omega}$, the inequality proved by Friedrichs, [16], is

$$(2.33) \qquad \|u\|_{L^2(\Omega)} \leq C \left[\|\nabla u\|_{L^2(\Omega)} + \|u\|_{L^2(\partial\Omega)} \right]$$

with a constant C not depending on u. This estimate is actually a straightforward consequence of a much more general estimate proved by Maz'ya [36] which is the following:
$$(2.34)$$
$$\|u\|_{L^{N/(N-1)}(\Omega)} \leq C(N) \left[\|\nabla u\|_{L^1(\Omega)} + \|u\|_{L^1(\partial\Omega)} \right] \quad \text{for any } u \in C(\overline{\Omega}) \cap W^{1,1}(\Omega).$$

The importance of this estimate is that it holds independently of the regularity of Ω and that the constant $C(N)$ is optimal and depends on N only; it is actually the one of the isoperimetric inequality, that is,

$$(2.35) \qquad C(N) = \frac{|B_1|^{(N-1)/N}}{\mathcal{H}^{N-1}(\partial B_1)}.$$

For a proof of (2.34) we refer to [37, Corollary, p. 319]. Actually, the Maz'ya inequality can even be generalized to functions of bounded variation; see [45] for an extremely general version in this direction.

Here we just point out that (2.34) implies the classical Friedrichs inequality, which we state in the next theorem.

THEOREM 2.7. *Let* $\Omega \subset \mathbb{R}^N$ *be open and bounded. Let*

$$p = \frac{2N}{N+1} \qquad and \qquad q = \frac{2N}{N-1}.$$

Then

$$(2.36) \quad \|u\|_{L^q(\Omega)} \leq 4C(N) \left[\|\nabla u\|_{L^p(\Omega)} + \|u\|_{L^2(\partial\Omega)} \right] \quad \textit{for any } u \in C(\overline{\Omega}) \cap W^{1,p}(\Omega),$$

so that for any $u \in C(\overline{\Omega}) \cap W^{1,2}(\Omega)$

$$(2.37) \qquad \|u\|_{L^2(\Omega)} \leq 4C(N)|\Omega|^{\frac{1}{2N}} \left[|\Omega|^{\frac{1}{2N}} \|\nabla u\|_{L^2(\Omega)} + \|u\|_{L^2(\partial\Omega)} \right].$$

Proof. Inequality (2.36) follows by applying (2.34) to u^2 (see the proof of [45, Corollary 2.4] for details), whereas (2.37) is an immediate consequence of (2.36) and the Hölder inequality. □

When u is vector-valued, that is, $u \in C(\overline{\Omega}, \mathbb{C}^N) \cap W^{1,2}(\Omega, \mathbb{C}^N)$, we still have

$$(2.38) \qquad \|u\|_{L^2(\Omega)} \leq 4C(N)|\Omega|^{\frac{1}{2N}} \left[|\Omega|^{\frac{1}{2N}} \|\nabla u\|_{L^2(\Omega)} + \|u\|_{L^2(\partial\Omega)} \right].$$

3. The main result. We begin by setting the hypotheses. Let us fix constants $r > 0$, $L > 0$, $R > 0$, α with $0 < \alpha < 1$, $\mu > 0$, λ with $2\mu + \lambda > 0$, $\rho > 0$, $\omega > 0$. Finally, we fix H_0 such that

$$(3.1) \qquad 0 < H_0 < H_1 = \left(\frac{\min\{2\mu, 2\mu + \lambda\}}{64C(N)^2 \rho \omega^2} \right)^{N/2} = \left(\frac{\min\{\omega_p^{-1}, \sqrt{2}\omega_s^{-1}\}}{8C(N)} \right)^N,$$

where $C(N)$ is the absolute constant appearing in (2.35). We refer to these constants as the *a priori data*.

First, we fix D^+, an open set which is Lipschitz with constants r and L and such that $K^+ = \overline{D^+}$ is an obstacle contained in $B_R(0)$. We call $\Omega^+ = \mathbb{R}^N \setminus K^+$.

Let D and D' be two open sets of class $C^{2,\alpha}$ with constants r and L such that $K = \overline{D}$ and $K' = \overline{D'}$ are obstacles contained in D^+. We call $\Omega = \mathbb{R}^N \setminus K$ and $\Omega' = \mathbb{R}^N \setminus K'$ the corresponding exterior domains. We also use the notation Ω^{ext} to denote the unbounded connected component of $\mathbb{R}^N \setminus (K \cup K')$ and call $\Gamma = \partial\Omega^{ext}$ and $\Omega^{int} = \mathbb{R}^N \setminus \overline{\Omega^{ext}}$. We clearly have $\Omega^+ \subset \Omega^{ext}$ and $\Omega^{int} \subset D^+$.

Let u^{inc} be as in (2.14) such that (2.15) is satisfied. Let u and u^{scat} be the solution to (2.18) and let u' and $(u')^{scat}$ be the solution to (2.18) with K replaced by K'.

Let $U = (U_p, U_s)$ be the far-field pattern of u^{scat} and $U' = (U'_p, U'_s)$ be the far-field pattern of $(u')^{scat}$, respectively.

We measure the difference between two obstacles K and K' by using the *Hausdorff distance* d_H, which is given by

$$d_H(K, K') = \max \left\{ \sup_{x \in K} \operatorname{dist}(x, K'), \sup_{x \in K'} \operatorname{dist}(x, K) \right\}.$$

Then we have the following stability result.

THEOREM 3.1. *Under the previous notation and assumptions, we further assume that the following* closeness condition *holds:*

$$(3.2) \qquad \left| D^+ \setminus (K \cap K') \right| \leq H_0.$$

Then there exist positive constants $\hat{\varepsilon}_0$, $0 < \hat{\varepsilon}_0 \le e^{-e}/2$, $\hat{C}$, and β, depending on the a priori data only, such that for any ε_0, $0 < \varepsilon_0 \le \hat{\varepsilon}_0$, if

$$(3.3) \qquad \|U - U'\|_{L^2(\mathbb{S}^{N-1}, \mathbb{C}^N \times \mathbb{C}^N)} \le \varepsilon_0,$$

then

$$(3.4) \qquad d_H(K, K') \le \hat{C} \left(\log \left(\log(1/\varepsilon_0) \right) \right)^{-\beta}.$$

Remark 3.2. As will be clear in the proof, about the incident wave, we just need conditions that allow (4.19) and (4.20) to be satisfied. Therefore, other suitable incident waves may be used. For instance, another common choice is to use a point source wave. However, in such a case, one needs to consider a point source which is far enough from the unknown obstacle and choose x_3 in (4.19) and (4.20) relatively close to the point source. The analysis would therefore require other technicalities that we decided not to tackle in this paper.

4. Proof of Theorem 3.1. Let K and K' be any two scatterers satisfying the hypotheses of Theorem 3.1. We state a few properties of K, as well as of K'. First we note that the number of connected components of K is bounded by a constant depending on r, L, and R only. We also have that $\mathcal{H}^{N-1}(\partial K)$ is bounded by a constant depending on r, L, and R only. Moreover, there exists a constant C_1, depending on r, L, and R only, such that for any h, $0 \le h \le 1$, we have

$$(4.1) \qquad |\overline{B_h(\partial K)}| \le C_1 h.$$

By [32, Corollary 2.3 and Proposition 2.1], there exist two positive constants c_1 and t_1, depending on r, L, and R only, such that the following holds. For any $t > 0$, if x_1, $x_2 \in \mathbb{R}^N$ are such that $B_t(x_1)$ and $B_t(x_2)$ are contained in $\mathbb{R}^N \backslash K$, then we can find a smooth (for instance, piecewise C^1) curve γ connecting x_1 to x_2 so that $\overline{B_{\delta(t)}(\gamma)}$ is contained in $\mathbb{R}^N \backslash K$ as well, where

$$(4.2) \qquad \delta(t) = \min\{c_1 t, t_1\} \quad \text{for any } t > 0.$$

We measure the distance between K and K' by

$$(4.3) \qquad d = \max \left\{ \sup_{x \in \partial K \backslash K'} \text{dist}(x, \partial K'), \ \sup_{x \in \partial K' \backslash K} \text{dist}(x, \partial K) \right\}$$

or

$$(4.4) \qquad \hat{d} = d_H(\partial K, \partial K') \quad \text{or} \quad \tilde{d} = d_H(K, K').$$

We obviously have d, $\hat{d}$, $\tilde{d} \le 2R$. The relationship between these quantities is investigated in detail in [32, section 2] under much more general conditions. Here we just use that, in particular by [32, Corollary 2.3 and Proposition 2.1], we have

$$(4.5) \qquad C_2 d \le C_2 \hat{d} \le \tilde{d} \le C_3 d \le C_3 \hat{d},$$

where C_2 and C_3 are positive constants depending on r, L, and R only.

Let us note that all the above properties are valid even if we assume that D and D' are just Lipschitz with constants r and L. In particular (4.1) holds with K replaced by K^+ as well.

If D is Lipschitz with constants r and L, D and Ω satisfy a *uniform interior cone property*, that is, there exist constants $r_0 > 0$ and θ_0, $0 < \theta_0 < \pi/2$, depending on r and L only, such that for any $x \in \partial K$ we can find a unit vector v such that $\mathcal{C}(x, v, r_0, \theta_0) \subset D$ and $\mathcal{C}(x, -v, r_0, \theta_0) \subset \Omega$. Let us also note that v can be chosen constant for any $y \in \partial K$ in a neighborhood of $x \in \partial K$ depending on r and L only.

Another important property, for which D of class $C^{1,1}$ with constants r and L would be enough, is the following (see [11, Theorem 5.7]). There exist positive constants h_0, r_1, and L_1, depending on r and L only, such that for any h, $0 < h \leq h_0$, the set

$$D_h = \{x \in \mathbb{R}^N : \operatorname{dist}(x, K) < h\}$$

is an open set of class $C^{1,1}$ with constants r_1 and L_1. Moreover,

$$\partial(D_h) = \{x \in \mathbb{R}^N : \operatorname{dist}(x, K) = h\}.$$

We can conclude that there exists a constant C_4, depending on r, L, and R only, such that for any h, $0 \leq h \leq h_0$,

$$\mathcal{H}^{N-1}(\partial(D_h)) \leq C_4, \tag{4.6}$$

where we identify D_0 with D and $\partial(D_0)$ with ∂D.

The final property we need about the obstacles K and K' is the following.

LEMMA 4.1. *Assume that D and D' are $C^{1,1}$ with constants r and L. Then there exists a constant $\tilde{d}_0$, depending on r, L, and R only, such that if*

$$\tilde{d} = d_H(K, K') \leq \tilde{d}_0,$$

then Ω^{ext} satisfies a uniform interior cone property with constants $\tilde{r}_0$ and $\tilde{\theta}_0$ depending on r and L only.

Proof. It immediately follows from [2, Lemma 8.1]. We just note that D and D' belonging to $C^{1,\alpha}$, $0 < \alpha < 1$, with constants r and L would be enough, but in this case the constants would depend on α as well. $\square$

By Theorem 2.4, we have that

$$\|u_p(x)\| + \|u'_p(x)\| + \|u_s(x)\| + \|u'_s(x)\| \leq E \quad \text{for any } x \in \overline{\Omega^{ext}}, \tag{4.7}$$

where E depends on the a priori data only and it is assumed to be greater than or equal to 1.

Finally, we fix positive R_1 and $\tilde{s}$ such that $R + 1 + \tilde{s} \leq R_1$. Let us fix a point $x_0 \in \mathbb{R}^N$ such that $R + 1 + \tilde{s} \leq \|x_0\| \leq R_1$. For a fixed ε, $0 < \varepsilon \leq E$, let

$$\|u - u'\|_{L^\infty(B_{\tilde{s}}(x_0), \mathbb{C}^N)} \leq \varepsilon. \tag{4.8}$$

We call ε the *near-field error with limited aperture*. Let ε_1, $0 < \varepsilon_1 \leq E$, be such that

$$\|u - u'\|_{L^\infty(B_{\|x_0\|+\tilde{s}} \setminus \overline{B_{\|x_0\|-\tilde{s}}}, \mathbb{C}^N)} \leq \varepsilon_1. \tag{4.9}$$

We call ε_1 the *near-field error*. Finally, if

$$\|U - U'\|_{L^2(\mathbb{S}^{N-1}, \mathbb{C}^N \times \mathbb{C}^N)} \leq \varepsilon_0, \tag{4.10}$$

ε_0 will be referred to as the *far-field error*.

By Theorem 2.4, through (4.7), Lemma 2.5, and an iterated application of the three-spheres inequality of Lemma 2.6 to $u_p - u'_p$ and $u_s - u'_s$, we can find positive constants C_5 and $\tilde{\beta}$, $0 < \tilde{\beta} < 1$, depending on E, R, $\tilde{s}$, R_1 and the coefficients of (2.1) only, such that

$$\text{(4.11)} \qquad \varepsilon \leq \varepsilon_1 \leq C_5 \varepsilon^{\tilde{\beta}}.$$

Moreover, there exist positive constants $\tilde{\varepsilon}_0 \leq 1/(2e)$ and C_6, depending on E, R, $\tilde{s}$, R_1, and the coefficients of (2.1) only, such that if $0 < \varepsilon_0 \leq \tilde{\varepsilon}_0$, then

$$\text{(4.12)} \quad \|u - u'\|_{L^\infty(B_{\|x_0\|+\tilde{s}} \setminus \overline{B_{\|x_0\|-\tilde{s}}}, \mathbb{C}^N)}$$
$$\leq \|u_p - u'_p\|_{L^\infty(B_{\|x_0\|+\tilde{s}} \setminus \overline{B_{\|x_0\|-\tilde{s}}}, \mathbb{C}^N)} + \|u_s - u'_s\|_{L^\infty(B_{\|x_0\|+\tilde{s}} \setminus \overline{B_{\|x_0\|-\tilde{s}}}, \mathbb{C}^N)}$$
$$\leq \tilde{\eta}(\varepsilon_0) = \exp\left(-C_6(-\log \varepsilon_0)^{1/2}\right),$$

that is, possibly slightly changing $\tilde{\varepsilon}_0$

$$\text{(4.13)} \qquad \varepsilon \leq \varepsilon_1 \leq \tilde{\eta}(\varepsilon_0) = \exp\left(-C_6(-\log \varepsilon_0)^{1/2}\right) \leq \exp\left(-(\log(1/\varepsilon_0))^{1/4}\right).$$

This is a classical far-field to near-field estimate, which has been first introduced in [25] for $N = 3$, with a slight improvement in [8], and that can be generalized to any $N \geq 2$; see, for instance, Theorem 4.1 in [46].

We estimate the Hausdorff distance of K and K' in terms of ε. In this case, we need to add R_1 and $\tilde{s}$ to the a priori data. By (4.11), the estimate in terms of ε_1 is clearly the same. The estimate in terms of the far-field error ε_0 can be easily obtained by using (4.13), noting that in this case R_1 and $\tilde{s}$ can be chosen as depending on R only.

For any $s > 0$ let us call V_s the set of points $x \in \Omega^{ext}$ such that there exists a smooth, that is, piecewise C^1, curve γ connecting x_0 to x such that $\overline{B_s(\gamma)} \subset \Omega^{ext}$. (See also [5] for a related argument developed in order to circumvent the case in which a domain of interest is not reachable by a chain of balls.) It follows that V_s is an open subset of Ω^{ext} and we call Γ_s its boundary and $W_s = \mathbb{R}^N \setminus \overline{V_s}$. To keep the same notation, we identify Ω^{ext} with V_0, Γ with Γ_0, and Ω^{int} with W_0. For any $0 \leq s_1 \leq s_2$ we clearly have $V_{s_2} \subset V_{s_1}$ and $W_{s_1} \subset W_{s_2}$.

An important property of Γ_s is that

$$\text{(4.14)} \qquad \Gamma_s \subset \partial(D_s) \cup \partial(D'_s).$$

Moreover, by (4.2) applied to K^+, for any $x \in \Omega^+$ whose distance from ∂D^+ is greater than or equal to $t > 0$, we have $x \in V_{\delta(t)}$.

We can find s_0, $0 < s_0 \leq \tilde{s}/8$, depending on the a priori data only, such that the following holds. It is smaller than or equal to $\tilde{s}_0$ in Lemma 2.6 for k equal to ω_p and to ω_s. It is smaller than or equal to h_0. Finally, we require that

$$\text{(4.15)} \qquad |W_s \setminus K^+| \leq \frac{H_1 - H_0}{2} \quad \text{for any } 0 \leq s \leq s_0.$$

For this last property we use (4.2) and (4.1) applied to K^+. By (4.15), via (3.2), we infer that

$$\text{(4.16)} \qquad |W_s \setminus (K \cap K')| \leq \frac{H_0 + H_1}{2} = \tilde{H}_0 < H_1 \quad \text{for any } 0 \leq s \leq s_0.$$

Up to swapping K with K', let $x_1 \in \partial K' \backslash K$ be such that $\mathrm{dist}(x_1, \partial K) = \mathrm{dist}(x_1, K) = d$. If x_1 does not belong to Γ, we can find a smooth curve γ connecting x_1 with x_0 such that $\overline{B_{\delta(d)}(\gamma)} \subset \Omega$. But γ needs to intersect $\partial K' \cap \Gamma$ in a point. Therefore, for a positive constant c_2, $0 < c_2 < 1$, depending on r, L, and R only, we can assume, without loss of generality, that there exists $x_1 \in (\partial K' \cap \Gamma) \backslash K$ such that $\mathrm{dist}(x_1, \partial K) = \mathrm{dist}(x_1, K) \geq c_2 d$.

We call A_0 the connected component of $\Omega^{int} \backslash K$ such that $x_1 \in \partial A_0$. For any s, $0 < s \leq s_0$, we call A_s the connected component of $W_s \backslash K$ containing A_0. For any s, $0 \leq s \leq s_0$, the domain A_s satisfies the following properties. By (4.14), $\partial A_s \subset \partial K \cup \partial(D_s) \cup \partial(D'_s)$. Therefore, by (4.6) and by (4.16), we have

$$(4.17) \qquad \mathcal{H}^{N-1}(\partial A_s) \leq 3C_4 \qquad \text{and} \qquad |A_s| \leq \frac{H_0 + H_1}{2} = \tilde{H}_0.$$

Moreover, by the regularity of D and D', we infer that there exist a point x_2 and a positive constant c_3, $0 < c_3 < 1$, depending on r, L, and R only, such that

$$(4.18) \qquad B_{c_3 d}(x_2) \subset A_0.$$

By (2.15) and (2.24), we can find a constant $R_2 \geq R + 2$, depending on the a priori data only, and a point x_3 such that

$$(4.19) \qquad R + 2 \leq \|x_3\| \leq R_2$$

and

$$(4.20) \qquad \|u(x)\|, \|u'(x)\| \geq \frac{1}{2} \qquad \text{for any } x \in B_1(x_3).$$

The proof of Theorem 3.1 requires several steps. The first one is to estimate $\|u - u'\|$ on Γ_s for $0 < s \leq s_0$. This is obtained by a classical quantitative unique continuation.

LEMMA 4.2. *Assume that $\varepsilon \leq 1/(2e)$. For any $0 < s \leq s_0$, we have*

$$(4.21) \qquad \|(u - u')(x)\| \leq \eta_s(\varepsilon) = E_1 \varepsilon^{a^{m(s)}} \qquad \text{for any } x \in \Gamma_s$$

with

$$(4.22) \qquad m(s) \leq \frac{F_0}{s^N},$$

where $E_1 > 0$, $F_0 > 0$, and a, $0 < a < 1$, are constants depending on the a priori data only.

Proof. For any $x \in B_R \cap V_s$, $0 < s \leq s_0$, let γ be the curve connecting x to x_0 as in the definition of V_s. Without loss of generality, we can assume that γ is contained in B_{R_1}.

We can construct a regular chain of balls, in the sense of [44, Definition 5.1], with respect to $B_s(\gamma)$ that from x_0 reaches x. The first ball is centered at x_0 and has radius less than or equal to $\tilde{s}/8$. By Lemma 2.5, we have that

$$\|u_p - u'_p\|_{L^\infty(B_{\tilde{s}/8}(x_0))}, \ \|u_s - u'_s\|_{L^\infty(B_{\tilde{s}/8}(x_0))} \leq C\varepsilon$$

for a constant C depending on the a priori data.

Then by a repeated use of the three-spheres inequality of Lemma 2.6 applied to $u_p - u'_p$ and $u_s - u'_s$ along this regular chain of balls, we obtain that

$$\|(u - u')(x)\| \leq \|(u_p - u'_p)(x)\| + \|(u_s - u'_s)(x)\| \leq E_1 \varepsilon^{a^{m(s)}},$$

where E_1 and a, $0 < a < 1$, depend on the a priori data only, and $m(s)$ denotes the number of times we have used the three-spheres inequality. It can be shown that $m(s)$ satisfies (4.22) for a constant F_0 depending on R_1 only. Then we conclude the proof by using the continuity of u and u'. $\qquad\square$

LEMMA 4.3. *Let $0 \leq s \leq s_0$. Assume that, for some η, $0 < \eta \leq E$, we have*

(4.23) $$\|(u - u')(x)\| \leq \eta \qquad \text{for any } x \in \Gamma_s.$$

Then there exists a positive constant $\hat{C}_0$, depending on the a priori data only, such that

(4.24) $$\|\nabla u\|^2_{L^2(A_s)} \leq \hat{C}_0 \hat{\eta}_s,$$

where

(4.25) $$\hat{\eta}_s = (\eta + \tilde{C}_0 s),$$

$\tilde{C}_0$ *as in* (2.23).

Proof. We have $\partial A_s \subset \partial K \cup \partial(D_s) \cup \partial(D'_s)$. We have that $u = 0$ on ∂K and, by (2.23), $\|u\| \leq \tilde{C}_0 s$ on $\partial(D_s)$. By the same reasoning, $\|u'\| \leq \tilde{C}_0 s$ on $\partial(D'_s)$. Hence

(4.26) $$\|u(x)\| \leq \hat{\eta}_s = (\eta + \tilde{C}_0 s) \qquad \text{for any } x \in \partial A_s.$$

Since $u \in C^2(\overline{\Omega})$ and A_s is a set of finite perimeter, an integration by parts leads to

$$2\mu \int_{A_s} Eu : \overline{Eu} + \lambda \int_{A_s} \operatorname{div}(u)\overline{\operatorname{div}(u)} - \rho\omega^2 \int_{A_s} u \cdot \overline{u} = \int_{\partial A_s} Tu \cdot \overline{u}.$$

Then, by (2.23), (2.38), (4.17), and (4.26), we have, for a constant M_1 depending on the a priori data only,

(4.27) $\quad M_1 \hat{\eta}_s \geq 2\mu \int_{A_s} Eu : \overline{Eu} + \lambda \int_{A_s} \operatorname{div}(u)\overline{\operatorname{div}(u)} - \rho\omega^2 \int_{A_s} u \cdot \overline{u}$

$\geq \min\{2\mu, 2\mu + \lambda\}\|Eu\|^2_{L^2(A_s)} - 32\rho\omega^2 C(N)^2 \tilde{H}_0^{\frac{1}{N}}\left[\tilde{H}_0^{\frac{1}{N}}\|\nabla u\|^2_{L^2(A_s)} + \|u\|^2_{L^2(\partial A_s)}\right]$

$\geq \min\{2\mu, 2\mu + \lambda\}\|Eu\|^2_{L^2(A_s)} - 32\rho\omega^2 C(N)^2 \left[\tilde{H}_0^{\frac{2}{N}}\|\nabla u\|^2_{L^2(A_s)} + \tilde{H}_0^{\frac{1}{N}}(3C_4 \hat{\eta}_s^2)\right].$

Then, by the idea of the proof of the first Korn inequality, we have

$$\|Eu\|^2 = \frac{1}{4}\sum_{i,j=1}^{N}\left|u_j^i + u_i^j\right|^2$$

$$= \frac{1}{2}\sum_{i,j=1}^{N}|u_j^i|^2 + \frac{1}{4}\sum_{i,j=1}^{N}\left(u_j^i \overline{u}_i^j + \overline{u}_j^i u_i^j\right) = \frac{1}{2}\sum_{i,j=1}^{N}|u_j^i|^2 + \frac{1}{2}\sum_{i,j=1}^{N}\left(u_j^i \overline{u}_i^j\right).$$

By the regularity of u and of A_s, by two integrations by parts we obtain

$$\int_{A_s} u^i_j \overline{u}^j_i = -\int_{A_s} u^i \overline{u}^j_{ij} + \int_{\partial A_s} u^i \overline{u}^j_i \nu^j = -\int_{A_s} u^i \overline{u}^j_{ji} + \int_{\partial A_s} u^i \overline{u}^j_i \nu^j$$

$$= \int_{A_s} u^i_i \overline{u}^j_j - \int_{\partial A_s} u^i \overline{u}^j_j \nu^i + \int_{\partial A_s} u^i \overline{u}^j_i \nu^j.$$

It follows that

$$\int_{A_s} \|Eu\|^2$$

$$= \frac{1}{2} \int_{A_s} \|\nabla u\|^2 + \frac{1}{2} \int_{A_s} \operatorname{div}(u) \overline{\operatorname{div}(u)} + \frac{1}{2} \sum_{i,j=1}^N \left(-\int_{\partial A_s} u^i \overline{u}^j_j \nu^i + \int_{\partial A_s} u^i \overline{u}^j_i \nu^j \right),$$

and thus

$$\int_{A_s} \|\nabla u\|^2 \leq 2 \int_{A_s} \|Eu\|^2 + \sum_{i,j=1}^N \left(\int_{\partial A_s} u^i \overline{u}^j_j \nu^i - \int_{\partial A_s} u^i \overline{u}^j_i \nu^j \right).$$

We conclude by Theorem 2.4 and (4.26) that

(4.28)
$$\int_{A_s} \|\nabla u\|^2 \leq 2 \int_{A_s} \|Eu\|^2 + 6 C_4 N^2 \tilde{C}_0 \hat{\eta}_s.$$

Coupling (4.27) and (4.28), we obtain

$$M_1 \hat{\eta}_s \geq \left(\frac{\min\{2\mu, 2\mu + \lambda\}}{2} - 32 \rho \omega^2 C(N)^2 \tilde{H}_0^{\frac{2}{N}} \right) \|\nabla u\|^2_{L^2(A_s)}$$

$$- 32 \rho \omega^2 C(N)^2 \tilde{H}_0^{\frac{1}{N}} (3 C_4 \hat{\eta}_s^2) - \min\{2\mu, 2\mu + \lambda\} 3 C_4 N^2 \tilde{C}_0 \hat{\eta}_s.$$

Since $\hat{\eta}_s$ is bounded by a constant depending on the a priori data only and, by (3.2) and (4.17), we have that $\tilde{H}_0 < H_1$, we can easily conclude the proof. $\square$

COROLLARY 4.4. *Under the same assumptions of Lemma 4.3, there exists a positive constant $\hat{C}_1$, depending on the a priori data only, such that*

(4.29)
$$\|u_p\|_{L^\infty(B_{c_3 d/4}(x_2))}, \ \|u_s\|_{L^\infty(B_{c_3 d/4}(x_2))} \leq \frac{\hat{C}_1}{d^{(N+2)/2}} \hat{\eta}_s^{1/2},$$

$\hat{\eta}_s$ *as in* (4.25).

Proof. The proof is immediate by using (2.26) and (4.24). $\square$

Remark 4.5. Assume that $\varepsilon \leq \mathrm{e}^{-\mathrm{e}}/2$. Let $\hat{\eta}_s(\varepsilon) = (\eta_s(\varepsilon) + \tilde{C}_0 s)$ as in (4.25) and with $\eta_s(\varepsilon)$ as in (4.21). Then, by taking the minimum as s varies in $(0, s_0]$, an easy computation shows that there exist positive constants $\tilde{\varepsilon}$, $\tilde{\varepsilon} \leq \mathrm{e}^{-\mathrm{e}}/2$, and $\hat{C}_2$, depending on the a priori data only, such that if $0 < \varepsilon \leq \tilde{\varepsilon}$, we have

(4.30)
$$\|u_p\|_{L^\infty(B_{c_3 d/4}(x_2))}, \ \|u_s\|_{L^\infty(B_{c_3 d/4}(x_2))} \leq \frac{\hat{C}_1}{d^{(N+2)/2}} \hat{\eta}(\varepsilon)^{1/2},$$

where

(4.31)
$$\hat{\eta}(\varepsilon) = \hat{C}_2 \left[\log(\log(1/\varepsilon)) \right]^{-1/N}.$$

In fact, let

$$s(\varepsilon) = \hat{C}_3 \left[\log(\log(1/\varepsilon)) \right]^{-1/N}$$

with $\hat{C}_3$ such that

$$\frac{\log(1/a)F_0}{\hat{C}_3^N} \leq 1/2.$$

Then

$$\varepsilon^{a^{m(s(\varepsilon))}} \leq \exp\left(-(\log(1/\varepsilon))^{1/2}\right).$$

It is enough to choose $\tilde{\varepsilon}$ such that for any $0 < \varepsilon \leq \tilde{\varepsilon}$ we have $s(\varepsilon) \leq s_0$ and

$$\exp\left(-(\log(1/\varepsilon))^{1/2}\right) \leq [\log(\log(1/\varepsilon))]^{-1/N}.$$

LEMMA 4.6. *Let us assume that for some d_1, $0 < d_1 \leq c_3 R/2$, there exists x such that $B_{d_1}(x) \subset \Omega$ and, for some $\hat{\eta} \leq \mathrm{e}^{-\mathrm{e}}/2$.*

(4.32) $$\|u_p\|_{L^\infty(B_{d_1}(x))}, \ \|u_s\|_{L^\infty(B_{d_1}(x))} \leq \hat{\eta}.$$

Then there exist a constant δ, $0 < \delta \leq \mathrm{e}^{-\mathrm{e}}/2$, and a positive constant $\hat{C}_4$, depending on the a priori data only, such that if $0 < \hat{\eta} \leq \delta$, we have

(4.33) $$d_1 \leq \varphi(\hat{\eta}),$$

where

(4.34) $$\varphi(\hat{\eta}) \leq 2\mathrm{e}R\left(\log(1/\hat{\eta})\right)^{-\hat{C}_4}.$$

Proof. We shall apply this lemma to $x = x_2$ and $d_1 = c_3 d/4$ (see Remark 4.7).

By the uniform interior cone property of Ω, we can find a direction v such that $\mathcal{C} = \mathcal{C}(x - (d_1)v, v, r_1, \theta_1) \subset \Omega$ for some $r_1 > 0$ and θ_1, $0 < \theta_1 < \pi/2$, depending on r and L only. Moreover, we can find $\tilde{x} = x + sv$ for some $s \geq 0$, and c_4, $0 < c_4 < 1$, depending on r and L only, such that $B_{c_4 d_1}(x)$ and $B_{c_4 r_1}(\tilde{x})$ are both contained in $\mathcal{C}$.

We can find γ, a piecewise C^1 curve, connecting $\tilde{x}$ with x_3 such that $B_{c_5 r_1}(\gamma) \subset \Omega$ for some c_5, $0 < c_5 < 1$, depending on r, L and R only. Without loss of generality, we can assume that $\gamma \subset B_{R_2}$ as well.

We can construct a regular chain of balls, again in the sense of [44, Definition 5.1], with respect to $\mathcal{C} \cup B_{c_5 r_1}(\gamma)$ that from x reaches x_3; see Step I of the proof of [44, Theorem 4.1] for details on this geometric construction, which we just sketch, now. The first ball is centered at x and has radius less than or equal to $c_4 d_1/8$. Then we proceed along the bisecting line of $\mathcal{C}$ until we reach $\tilde{x}$. The construction from x to $\tilde{x}$ is illustrated in Figure 1. From $\tilde{x}$ to x_3 we proceed along the curve γ; see Figure 2 for an illustration.

Again by a repeated use of the three-spheres inequality of Lemma 2.6 applied to u_p and u_s along this regular chain of balls, we obtain that

$$\|u(x_3)\| \leq \|u_p(x_3)\| + \|u_s(x_3)\| \leq E_2 \hat{\eta}^{a^n},$$

where $E_2 \geq 2\mathrm{e}$ and a, $0 < a < 1$, depend on the a priori data only, and n denotes the number of times we have used the three-spheres inequality. We can estimate n as follows:

(4.35) $$n \leq F_1 \log(2\mathrm{e}R/d_1)$$

for some constant F_1 depending on the a priori data only.

It follows by (4.20) that

$$\frac{1}{2} \leq E_2 \hat{\eta}^{a^n},$$

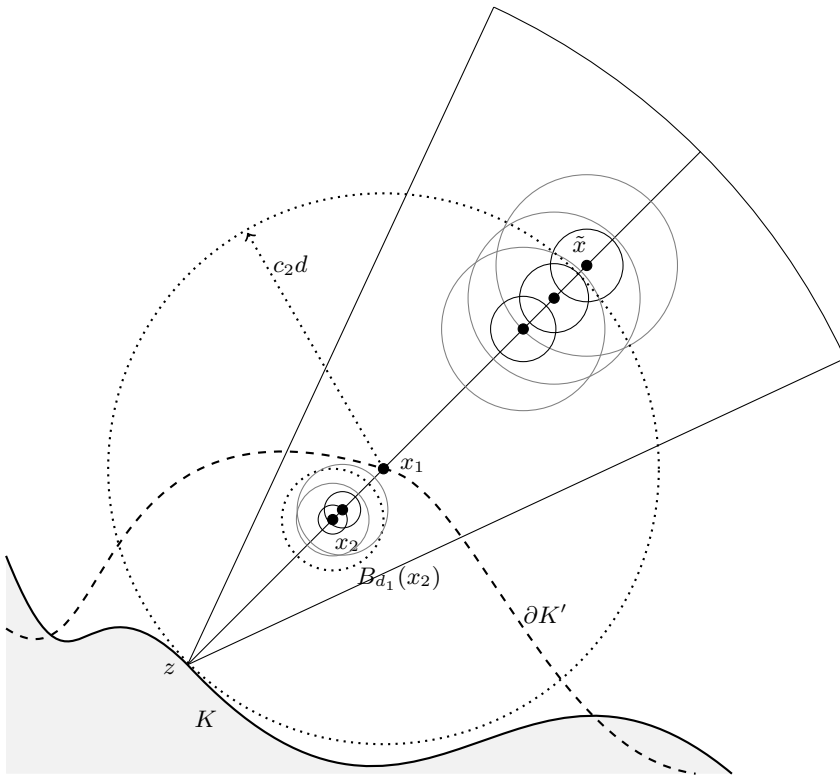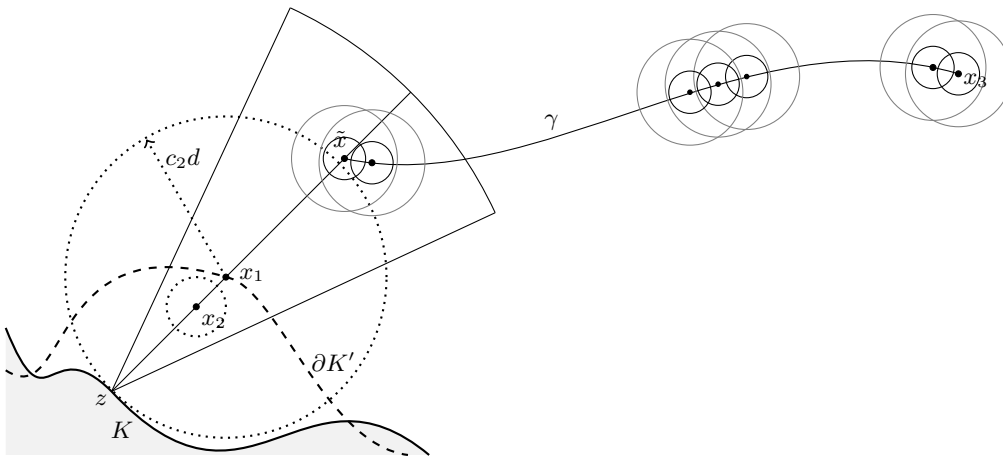FIG. 1. *The geometric construction from $x = x_2$ to $\tilde{x}$.*

FIG. 2. *From $\tilde{x}$ to x_3.*

and consequently

$$-\log(2E_2) \le -\log(1/\hat{\eta})a^n,$$

that is,

$$a^n \le \frac{\log(2E_2)}{\log(1/\hat{\eta})}.$$

So, by (4.35),

$$\log(1/a) F_1 \log(2eR/d_1) \geq n \log(1/a) \geq \log(\log(1/\hat{\eta})) - \log(\log(2E_2)).$$

The proof can now be concluded by an elementary computation. □

Remark 4.7. Let us pick, in Lemma 4.6, $x = x_2$, $d_1 = c_3 d/4$, and

$$\hat{\eta} = \frac{\hat{C}_1}{d^{(N+2)/2}} \hat{h}^{1/2}$$

with $\hat{h} = \hat{\eta}_s$ as in Corollary 4.4 or $\hat{h} = \hat{\eta}(\varepsilon)$ as in Remark 4.5, and assume that $\hat{h} \leq 1/(2e)$. We conclude that either $\hat{\eta} \geq \delta$, that is,

(4.36) $$d^{(N+2)/2} \leq \frac{\hat{C}_1}{\delta} \hat{h}^{1/2},$$

or $\hat{\eta} \leq \delta$ and, by (4.33) and (4.34),

$$d \leq \frac{8eR}{c_3} \left(\frac{N+2}{2} \log(d) - \log(\hat{C}_1) + \frac{1}{2} \log\left(\frac{1}{\hat{h}}\right) \right)^{-\hat{C}_4}.$$

In this case we have that

$$\left(\frac{8eR}{c_3}\right)^{1/\hat{C}_4} \left(\frac{1}{d}\right)^{1/\hat{C}_4} \geq \frac{N+2}{2} \log(d) - \log(\hat{C}_1) + \frac{1}{2} \log\left(\frac{1}{\hat{h}}\right),$$

that is,

$$\left(\frac{8eR}{c_3}\right)^{1/\hat{C}_4} \left(\frac{1}{d}\right)^{1/\hat{C}_4} + \frac{N+2}{2} \log\left(\frac{1}{d}\right) + \log(\hat{C}_1) \geq \frac{1}{2} \log\left(\frac{1}{\hat{h}}\right).$$

Since $d \leq 2R$, and therefore $(1/d) \geq 1/(2R)$, we can find a positive constant $\hat{C}_5$, depending on the a priori data only, such that for any $0 < d \leq 2R$ we have

$$\left(\frac{8eR}{c_3}\right)^{1/\hat{C}_4} \left(\frac{1}{d}\right)^{1/\hat{C}_4} + \frac{N+2}{2} \log\left(\frac{1}{d}\right) + \log(\hat{C}_1) \leq \frac{1}{2} \left(\frac{\hat{C}_5}{d}\right)^{1/\hat{C}_4},$$

and therefore

(4.37) $$d \leq \hat{C}_5 \left(\log\left(\frac{1}{\hat{h}}\right) \right)^{-\hat{C}_4}.$$

We can also find a positive constant $\hat{C}_6$, depending on the a priori data only, such that for any $\hat{h}$, $0 < \hat{h} \leq 1/(2e)$, we have

$$\frac{\hat{C}_1}{\delta} \hat{h}^{1/2} \leq \hat{C}_6 \left(\log\left(\frac{1}{\hat{h}}\right) \right)^{-\hat{C}_4}.$$

Coupling (4.36) and (4.37), we conclude that

(4.38) $$d \leq \hat{C}_7 \left(\log\left(\frac{1}{\hat{h}}\right) \right)^{-\hat{C}_4},$$

where $\hat{C}_7 = \max\{\hat{C}_5, \hat{C}_6\}$.

Proof of Theorem 3.1. We can find $\hat{\varepsilon}$, depending on the a priori data only, such that $0 < \hat{\varepsilon} \leq \tilde{\varepsilon}$ and for any ε, $0 < \varepsilon \leq \hat{\varepsilon}$, we have $\hat{\eta}(\varepsilon) \leq 1/(2\mathrm{e})$, where $\hat{\eta}(\varepsilon)$ is defined in (4.31). By the reasoning used in Remark 4.7 with $\hat{h} = \hat{\eta}(\varepsilon)$, we deduce that

$$(4.39) \qquad \tilde{d} \leq C_3 d \leq C_3 \hat{C}_7 \left(\log \left(\frac{1}{\hat{\eta}(\varepsilon)} \right) \right)^{-\hat{C}_4},$$

where we used (4.5). We have already obtained a quantitative estimate, which we can improve as follows.

Up to taking a smaller $\hat{\varepsilon} > 0$, still depending on the a priori data only, for any ε, $0 < \varepsilon \leq \hat{\varepsilon}$, we have that $\tilde{d} \leq \tilde{d}_0$, where $\tilde{d}_0$ is the constant of Lemma 4.1. Then we can improve our estimate with a by now classical technique, which we sketch now.

Let us assume that ε, $0 < \varepsilon \leq \hat{\varepsilon}$, so that $\tilde{d} \leq \tilde{d}_0$. By Lemma 4.1, we have that Ω^{ext} satisfies a uniform interior cone property, with constants $\tilde{r}_0$ and $\tilde{\theta}_0$ depending on r and L only. Let $z \in \Gamma$ and let $\mathcal{C} = \mathcal{C}(z, v, \tilde{r}_0, \tilde{\theta}_0) \subset \Omega^{ext}$ for a suitable direction v. For any s, $0 < s \leq \tilde{s}_0$, with $\tilde{s}_0 \leq 2$ small enough, let $x(s) = z + sv$. By a completely analogous construction to the one used in Lemma 4.6, just by reversing the chain of balls, we connect x_0 to $x(s)$ with a suitable regular chain of balls contained in Ω^{ext}. The construction is again illustrated in Figures 2 and 1, by replacing x_3 with x_0 and $\tilde{x}$ with x, and assuming that $z \in \Gamma$ and that the cone and γ are contained in Ω^{ext}.

The repeated use of the three-spheres inequality applied to $u_p - u_p'$ and $u_s - u_s'$ along this chain, from x_0 to $x(s)$, allows us to estimate

$$\|(u - u')(x(s))\| \leq E_3 \varepsilon^{a^{l(s)}},$$

where $E_3 \geq 2\mathrm{e}$, $0 < a < 1$, and

$$l(s) \leq F_2 \log(2\mathrm{e}/s).$$

As usual, $\tilde{s}_0$, E_3, a, and F_2 can be chosen as depending on the a priori data only. We conclude that, for any $z \in \Gamma$,

$$\|(u - u')(z)\| \leq E_3 \varepsilon^{a^{l(s)}} + \tilde{C}_0 s \quad \text{for any } 0 < s \leq \tilde{s}_0.$$

By reasoning as in Remark 4.5, we estimate the minimum as s varies in $(0, \tilde{s}_0]$. Let us further assume, without loss of generality, that for any ε, $0 < \varepsilon \leq \hat{\varepsilon}$, we have

$$s(\varepsilon) = \frac{2\mathrm{e}}{\log(1/\varepsilon)^{1/\hat{C}_8}} \leq \tilde{s}_0$$

with $\hat{C}_8$ such that

$$\frac{\log(1/a) F_2}{\hat{C}_8} \leq 1/2.$$

Then

$$\varepsilon^{a^{l(s(\varepsilon))}} \leq \exp\left(-(\log(1/\varepsilon))^{1/2} \right).$$

It is enough to choose $\hat{\varepsilon}$ such that for any $0 < \varepsilon \leq \hat{\varepsilon}$ we have $s(\varepsilon) \leq \tilde{s}_0$ and, calling $\hat{\beta} = 1/\hat{C}_8$,

$$\exp\left(-(\log(1/\varepsilon))^{1/2} \right) \leq \log(1/\varepsilon)^{-\hat{\beta}}.$$

Therefore we can find $\hat{\varepsilon}$, $0 < \hat{\varepsilon} \le \mathrm{e}^{-\mathrm{e}}/2$, and a positive constant $\hat{\beta}_0$, both depending on the a priori data only, such that for any ε, $0 < \varepsilon \le \hat{\varepsilon}$, and for any $z \in \Gamma$ we have

$$\|(u - u')(z)\| \le \log(1/\varepsilon)^{-\hat{\beta}_0} = \eta(\varepsilon) \le 1/(2\mathrm{e}).$$

By the same argument used to prove (4.39), if we replace $\hat{\eta}(\varepsilon)$ with $\eta(\varepsilon)$, we conclude that the following stability result holds.

There exist positive constants $\hat{\varepsilon}$, $0 < \hat{\varepsilon} \le \mathrm{e}^{-\mathrm{e}}/2$, C, and β, depending on the a priori data only, such that for any ε, $0 < \varepsilon \le \hat{\varepsilon}$, if

$$\|u - u'\|_{L^\infty(B_{\bar{s}}(x_0), \mathbb{C}^N)} \le \varepsilon,$$

then

(4.40) $$\tilde{d} \le C \left(\log\left(\log(1/\varepsilon)\right)\right)^{-\beta},$$

where

$$C = C_3 \hat{C}_7 \hat{\beta}_0^{-\hat{C}_4} \qquad \text{and} \qquad \beta = \hat{C}_4.$$

If we consider the far-field error ε_0 instead of the error ε, we can find $\hat{\varepsilon}_0$, $0 < \hat{\varepsilon}_0 \le \mathrm{e}^{-\mathrm{e}}/2$, such that for any ε_0, $0 < \varepsilon_0 \le \hat{\varepsilon}_0$, we have that $\tilde{\eta}(\varepsilon_0) \le \hat{\varepsilon}$, and hence by replacing ε with $\tilde{\eta}(\varepsilon_0)$ in (4.40), we conclude that

$$\tilde{d} \le \hat{C} \left(\log\left(\log(1/\varepsilon_0)\right)\right)^{-\beta},$$

where $\hat{C} = 4^\beta C$. Therefore the proof of the main theorem is concluded. $\qquad \square$

Appendix A. Proof of Theorem 2.4. In this appendix we sketch the proof of the regularity of solutions to (2.18).

Let us first observe that it is enough to prove that there exist positive constants r_1 and $\tilde{C}$, depending on r, L, R, α, and the coefficients only, such that

(A.1) $$\|u\|_{L^2(\Omega \cap B_{R+1})} \le \tilde{C}$$

and

(A.2) $$\|u\|_{C^2\left(\overline{\Omega \cap B_{r_1}(z)}\right)} \le \tilde{C} \qquad \text{for any } z \in \partial\Omega.$$

Assuming we have proved (A.1) and (A.2), we conclude the proof of Theorem 2.4. By the techniques developed in the proof of Lemma 2.5, we first show that by (A.1) we have

(A.3) $$\|u\|_{C^2\left(\overline{B_{R+3/4} \setminus B_{R+1/4}}\right)} \le \tilde{C}_2$$

and then, using also (A.2), that

(A.4) $$\|u\|_{C^2\left(\overline{\Omega \cap B_{R+3/4}}\right)} \le \tilde{C}_2$$

for a constant $\tilde{C}_2$ depending on r, L, R, α, and the coefficients only.

The estimate (A.4) implies that u_p and u_s, and thus u_p^{scat} and u_s^{scat}, are uniformly bounded in $\overline{\Omega} \cap \overline{B_{R+3/4}}$ by a constant depending on $\tilde{C}_2$ and the coefficients of (2.1) only. By standard regularity estimates for solutions to the Helmholtz equation, not very different from what we used in the proof of Lemma 2.5, we infer that

$\|u_p^{scat}\| + \|\nabla u_p^{scat}\nu\|$ is bounded on $\partial B_{R+1/2}$ by a constant depending on $\tilde{C}_2$, R, and the coefficients of (2.1) only, ν being the exterior normal to $B_{R+1/2}$. We point out that for this estimate (A.3), and thus (A.1), is enough. Then, for any x with $\|x\| > R+1/2$ we have

$$u_p^{scat}(x) = \int_{\partial B_{R+1/2}} \left(\frac{\partial \phi_{\omega_p}(x,y)}{\partial \nu(y)} u_p^{scat}(y) - \phi_{\omega_p}(x,y)\nabla u_p^{scat}(y)\nu(y) \right) d\sigma(y).$$

Then, by the regularity and decay properties of ϕ_{ω_p}, it is not difficult to prove (2.24) for what concerns u_p^{scat} and to bound $\|u_p\|_{C^2(\Omega\setminus B_{R+3/4})}$. A completely analogous argument applied to u_s completes the proof.

The proof of (A.1) and (A.2) will be done in two steps. In the first step we prove (A.1), and in the second we prove (A.2).

Step I. The estimate (A.1) is proved by a continuity argument which is inspired by Mosco convergence. We sketch the proof; for details we refer to [38], where the argument is fully developed in the acoustic case for the much harder Neumann boundary condition and for much more general classes of scatterers.

Let $\mathcal{A}$ be the class of obstacles contained in $\overline{B_R}$ whose interior is a Lipschitz open set with constants r and L. In [32, section 2] it is proved that $\mathcal{A}$ is compact with respect to the Hausdorff distance. We claim that there exists a constant $\tilde{C}$, depending on r, L, R, and the coefficients only, such that (A.1) holds for any u solution to (2.18) with $K \in \mathcal{A}$.

We argue by contradiction. Let us assume that there exists a sequence $\{K_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ such that, calling u_n the solution to (2.18) with K replaced by K_n, we have for any $n \in \mathbb{N}$

$$\|u_n\|_{L^2(\Omega_n \cap B_{R+1})} = a_n \geq n.$$

We always extend u_n to 0 in D_n, the interior of K_n, so that $u_n \in H^1(B_{R+1})$. Let $v_n = u_n/a_n$. We have that, with the usual extension to 0 outside Ω_n,

$$\|v_n\|_{L^2(B_{R+1})} = 1 \quad \text{for any } n \in \mathbb{N}.$$

By the same argument we used before, one can easily show that $\|v_n\|_{C^2(\overline{B_{R+3/4}}\setminus B_{R+1/4})}$ is bounded by a constant not depending on n. Let $\chi \in C_0^\infty(B_{R+1})$ be a cutoff function such that $0 \leq \chi \leq 1$ in B_{R+1} and $\chi = 1$ in $B_{R+1/2}$. We have that $w_n = v_n\chi \in H_0^1(B_{R+1}\setminus K_n) \subset H_0^1(B_{R+1})$ and w_n solves

$$\begin{cases} \mu\Delta w_n + (\lambda + \mu)\nabla^T(\text{div}(w_n)) + \rho\omega^2 w_n = f_n & \text{in } B_{R+1}\setminus K_n, \\ w_n = 0 & \text{on } \partial(B_{R+1}\setminus K_n) \end{cases}$$

with $\|f_n\|_{L^\infty(B_{R+1}\setminus K_n)}$ bounded by a constant not depending on n. By the weak formulation and first Korn inequality, we deduce that $\{w_n\}_{n\in\mathbb{N}}$ is bounded in $H^1(B_{R+1})$, and hence $\{v_n\}_{n\in\mathbb{N}}$ is bounded in $H^1(B_{R+1})$ as well.

Passing to subsequences, without loss of generality, we can assume that, as $n \to \infty$, K_n converges to $K \in \mathcal{A}$ in the Hausdorff distance, and v_n converges to $\tilde{v}$ weakly in $H^1(B_{R+1})$ and strongly in $L^2(B_{R+1})$. In particular,

(A.5) $$\|\tilde{v}\|_{L^2(B_{R+1})} = 1.$$

One can easily show that $\tilde{v} = 0$ in a weak sense on ∂K and that $\tilde{v}$ solves (2.1) in $B_{R+1}\setminus K$. Since $v_n^{inc} = u^{inc}/a_n$, for any $n \in \mathbb{N}$, we actually have that v_n^{scat} converges to $\tilde{v}$ weakly in $H^1(B_{R+1})$ and strongly in $L^2(B_{R+1})$. We infer that, possibly passing

to a further subsequence by using a diagonal argument, v_n^{scat} converges to a function $\tilde{v}$ in $L^2(B_r)$ for any $r > R$. Such a function $\tilde{v}$ is a radiating solution to Navier equation in $\Omega = \mathbb{R}^N \backslash K$ and $\tilde{v} = 0$ on ∂K in a weak sense. By uniqueness of the scattering problem we deduce that $\tilde{v} = 0$ and this contradicts (A.5).

Step II. Let $z \in \partial \Omega$ and let R_z be the unitary matrix transforming $\nu(z)$ to $(0, \ldots, 1)$ where $\nu(z)$ is the exterior normal to $\partial \Omega$ at z. We set $V(x) = R_z u(R_z^{-1}(x))$ and we note that V satisfies (2.1) in $R_z(\Omega)$. In other words, we can assume, without loss of generality, that there exists a function $\phi_z : \mathbb{R}^{N-1} \to \mathbb{R}$, with $\|\phi_z\|_{C^{2,\alpha}(\mathbb{R}^{N-1})} \leq L$, such that for any $y \in B_r(z)$ we have, without any further rigid transformation,

$$x = (x', x_N) \in \Omega \quad \text{if and only if} \quad x_N < \phi_z(x').$$

We define $\tilde{u}(\xi) = u(x)$, where $\xi = F(x)$ is defined as

(A.6)
$$\begin{cases} \xi' = x', \\ \xi_N = x_N - \phi_z(x'). \end{cases}$$

Here, and in what follows, for any $s > 0$ we denote $\Sigma_s(z) = B_s(z) \cap \{\xi_N > z_N\}$ and $\Gamma_s(z) = B_s(z) \cap \{\xi_N = z_N\}$. By the regularity properties of ϕ_z, we can infer that, for a positive constant r_2, depending on r and L only, we have that $\tilde{u}$ satisfies, for any $i = 1, \ldots, N$,

$$\mu \left[\Delta_\xi \tilde{u}^i(\xi) - 2\nabla_\xi \tilde{u}_N^i(\xi) \cdot \nabla_x \phi(x) + \tilde{u}_{NN}^i(\xi)\|\nabla_x \phi(x)\|^2 \right]$$
$$+ (\lambda+\mu)\left[(\mathrm{div}_\xi \tilde{u})_i(\xi) - (\mathrm{div}_\xi \tilde{u})_N(\xi)\phi_i(x) + \tilde{u}_{Ni}(\xi) \cdot \nabla_x \phi(x) - (\tilde{u}_{NN}(\xi) \cdot \nabla_x \phi(x))\phi_i(x)\right]$$
$$- \mu \tilde{u}_N^i(\xi)\Delta_x \phi(x) + (\lambda + \mu)\tilde{u}_N(\xi) \cdot \nabla_x(\phi_i)(x) + \rho\omega^2 \tilde{u}^i(\xi) = 0 \quad \text{in } \Sigma_{r_2}(z)$$

and

$$\tilde{u} = 0 \quad \text{on } \Gamma_{r_2}(z).$$

Here $\phi : \mathbb{R}^N \to \mathbb{R}$ is defined as $\phi(x) = \phi_z(x')$ for any $x \in \mathbb{R}^N$ and, in the formula, $x = F^{-1}(\xi)$ everywhere. By Remark 2.2, we have that $\nabla_x \phi(z) = 0$. Hence, in a suitable neighborhood of z, the principal part of the second order system solved by $\tilde{u}$ is a small perturbation of the Lamé system $\mu \Delta \tilde{u} + (\lambda + \mu)\nabla^T(\mathrm{div}(\tilde{u}))$. We can conclude that it is elliptic and satisfies all the conditions of [1, Chapter 1] and that the boundary condition $\tilde{u} = 0$ on $\Gamma_{r_2}(z)$ satisfies all the conditions of [1, Chapter 2]. Moreover, the coefficients of the elliptic system are bounded in $C^{0,\alpha}$.

As an intermediate step, we show that there exist $r_3 > 0$ and $\tilde{C}_3$, depending on r, L, R, α, and the coefficients only, such that

(A.7)
$$\|\tilde{u}\|_{L^\infty\left(\Sigma_{r_3}(z)\right)} \leq \tilde{C}_3.$$

By using [1, Theorem 10.4] with a suitable cutoff function, for any s, $0 < s \leq r_2$, and any real p, $p \geq 2$, we can find s_1, $0 < s_1 < s$, and C_1, depending on s, p, and r, L, R, α, and the coefficients only, such that

(A.8)
$$\|\tilde{u}\|_{W^{2,p}(\Sigma_{s_1}(z))} \leq C_1 \|\tilde{u}\|_{L^p(\Sigma_s(z))}.$$

By (A.1), we can control $\|\tilde{u}\|_{W^{2,2}(\Sigma_{\tilde{s}}(z))}$ for a suitable positive constant $\tilde{s}$. By Sobolev inequality, we infer that $\tilde{u}$ belongs to $L^{p_1}(\Sigma_{\tilde{s}}(z))$ for some $p_1 > 2$, and thus, repeating the argument, $\tilde{u}$ belongs to $W^{2,p_1}(\Sigma_{\tilde{s}_1}(z))$ for a smaller positive constant $\tilde{s}_1$. With a bootstrap argument, after a finite number m of steps, which depends on N only, we

obtain that $\tilde{u}$ belongs to $W^{2,p_m}(\Sigma_{\tilde{s}_m}(z))$ for a positive constant $\tilde{s}_m$ and $p_m > N$. By a final application of Sobolev inequality, we conclude that (A.7) holds.

Once (A.7) is established, by the standard estimates of [1, Theorem 9.2], we can control the $C^{2,\alpha}$-norm of $\tilde{u}$ in $\overline{\Sigma_{r_3/16}(z)}$. Going back to the usual coordinates, (A.2) can be finally proved.

REFERENCES

[1] S. AGMON, A. DOUGLIS AND L. NIRENBERG, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions.* II, Comm. Pure Appl. Math., 17 (1964), pp. 35–92.

[2] G. ALESSANDRINI, E. BERETTA, E. ROSSET, AND S. VESSELLA, *Optimal stability for inverse elliptic boundary value problem with unknown boundaries*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (4), 29 (2000), pp. 755–806.

[3] G. ALESSANDRINI AND L. RONDI, *Determining a sound-soft polyhedral scatterer by a single far-field measurement*, Proc. Amer. Math. Soc., 35 (2005), pp. 1685–1691.

[4] G. ALESSANDRINI, E. ROSSET, AND S. VESSELLA, *Optimal three spheres inequality at the boundary for the Kirchhoff-Love plate's equation with Dirichlet conditions*, Arch. Ration. Mech. Anal., 231 (2019), pp. 1455–1486.

[5] G. ALESSANDRINI AND E. SINCICH, *Cracks with impedance, stable determination from boundary data*, Indiana Univ. Math. J., 62 (2013), pp. 947–989.

[6] G. BAO, G. HU, J. SUN, AND T. YIN, *Direct and inverse elastic scattering from anisotropic media*, J. Math. Pures Appl. (9), 117 (2018), pp. 263–301.

[7] R. BRUMMELHUIS, *Three-spheres theorem for second order elliptic equations*, J. Anal. Math., 65 (1995), pp. 179–206.

[8] I. BUSHUYEV, *Stability of recovering the near-field wave from the scattering amplitude*, Inverse Problems, 12 (1996), pp. 859–867.

[9] J. CHENG AND M. YAMAMOTO, *Uniqueness in an inverse scattering problem within non-trapping polygonal obstacles with at most two incoming waves*, Inverse Problems, 19 (2003), pp. 1361–1384.

[10] D. COLTON AND B. D. SLEEMAN, *Uniqueness theorems for the inverse problem of acoustic scattering*, IMA J. Appl. Math., 31 (1983), pp. 253–259.

[11] M. C. DELFOUR AND J.-P. ZOLÉSIO, *Shape analysis via oriented distance functions*, J. Funct. Anal., 123 (1994), pp. 129–201.

[12] H. DIAO, H. LIU, AND L. WANG, *On generalized Holmgren's principle to the Lamé operator with applications to inverse elastic problems*, Calc. Var. Partial Differential Equations, 59 (2020), 179.

[13] M. DI CRISTO AND L. RONDI, *Examples of exponential instability for inverse inclusion and scattering problems*, Inverse Problems, 19 (2003), pp. 685–701.

[14] J. ELSCHNER AND G. HU, *Uniqueness and factorization method for inverse elastic scattering with a single incoming wave*, Inverse Problems, 35 (2019), 094002.

[15] J. ELSCHNER AND M. YAMAMOTO, *Uniqueness in inverse elastic scattering with finitely many incident waves*, Inverse Problems, 26 (2010), 045005.

[16] K. FRIEDRICHS, *Die Randwert-und Eigenwertprobleme aus der Theorie der elastischen Platten*, (Anwendung der direkten Methoden der Variationsrechnung.), Math. Ann., 98 (1928), pp. 205–247 (in German).

[17] D. GILBARG AND N. S. TRUDINGER, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin, 1998.

[18] D. GINTIDES, *Local uniqueness for the inverse scattering problem in acoustics via the Faber-Krahn inequality*, Inverse Problems, 21 (2005), pp. 1195–1205.

[19] D. GINTIDES AND L. MIDRINOS, *Inverse scattering problem for a rigid scatterer or a cavity in elastodynamics*, ZAMM Z. Angew. Math. Mech., 91 (2011), pp. 276–287.

[20] D. GINTIDES AND M. SINI, *Identification of obstacles using only the scattered P-waves or the scattered S-waves*, Inverse Probl. Imaging, 6 (2012), pp. 39–55.

[21] P. HÄHNER AND G. C. HSIAO, *Uniqueness theorems in inverse obstacle scattering of elastic waves*, Inverse Problems, 9 (1993), pp. 525–534.

[22] N. Higashimori, *A Conditional Stability Estimate for Identifying a Cavity by an Elastostatic Measurement*, Ph.D. Thesis, Graduate School of Informatics, Kyoto University, 2003.

[23] N. Honda, G. Nakamura, and M. Sini, *Analytic extension and reconstruction of obstacles from few measurements for elliptic second order operators*, Math. Ann., 355 (2013), pp. 401–427.

[24] G. Hu, A. Kirsch, and M. Sini, *Some inverse problems arising from elastic scattering by rigid obstacles*, Inverse Problems, 29 (2013), 015009.

[25] V. Isakov, *Stability estimates for obstacles in inverse scattering*, J. Comput. Appl. Math., 42 (1992), pp. 79–88.

[26] V. Isakov, *New stability results for soft obstacles in inverse scattering*, Inverse Problems, 9 (1993), pp. 535–543.

[27] M. Kar and M. Sini, *Reconstruction of interfaces from the elastic farfield measurements using CGO solutions*, SIAM J. Math. Anal., 46 (2014), pp. 2650–2691.

[28] M. Kar and M. Sini, *On the inverse elastic scattering by interfaces using one type of scattered waves*, J. Elasticity, 118 (2015), pp. 15–38.

[29] B. Kawohl and G. Sweers, *Remarks on eigenvalues and eigenfunctions of a special elliptic system*, Z. Angew. Math. Phys., 38 (1987), pp. 730–740.

[30] V. D. Kupradze, *Potential Methods in the Theory of Elasticity*, Daniel Davey & Co., New York, 1965.

[31] V. D. Kupradze, T. G. Gegelia, M. O. Basheleĭshvili, and T. V. Burchuladze, *Three-Dimensional Problems of the Mathematical Theory of Elasticity and Thermoelasticity*, North-Holland, Amsterdam, 1975.

[32] H. Liu, M. Petrini, L. Rondi, and J. Xiao, *Stable determination of sound-hard polyhedral scatterers by a minimal number of scattering measurements*, J. Differential Equations, 262 (2017), pp. 1631–1670.

[33] H. Liu, L. Rondi, and J. Xiao, *Mosco convergence for $H(\mathrm{curl})$ spaces, higher integrability for Maxwell's equations, and stability in direct and inverse EM scattering problems*, J. Eur. Math. Soc. (JEMS), 21 (2019), pp. 2945–2993.

[34] H. Liu and J. Xiao, *Decoupling elastic waves and its applications*, J. Differential Equations, 263 (2017), pp. 4442–4480.

[35] N. Mandache, *Exponential instability in an inverse problem for the Schrödinger equation*, Inverse Problems, 17 (2001), pp. 1435–1444.

[36] V. G. Maz'ya, *Classes of domains and imbedding theorems for function spaces*, Soviet Math. Dokl., 1 (1960), pp. 882–885 (in English).

[37] V. Maz'ya, *Sobolev Spaces*, 2nd ed., Springer-Verlag, Berlin, 2011.

[38] G. Menegatti and L. Rondi, *Stability for the acoustic scattering problem for sound-hard scatterers*, Inverse Probl. Imaging, 7 (2013), pp. 1307–1329.

[39] A. Morassi and E. Rosset, *Stable determination of cavities in elastic bodies*, Inverse Problems, 20 (2004), pp. 453–480.

[40] A. Morassi and E. Rosset, *Uniqueness and stability in determining a rigid inclusion in an elastic body*, Mem. Amer. Math. Soc., 200 (2009), n. 938.

[41] A. Morassi, E. Rosset, and S. Vessella, *Doubling inequality at the boundary for the Kirchhoff-Love plate's equation with Dirichlet conditions*, Le Matematiche, 75 (2020), pp. 27–55.

[42] A. Morassi, E. Rosset, and S. Vessella, *Optimal identification of cavities in the Generalized Plane Stress problem in linear elasticity*, J. Eur. Math. Soc. (JEMS), to appear.

[43] O. A. Oleinik, A. S. Shamaev, and G. A. Yosifian, *Mathematical Problems in Elasticity and Homogenization*, North-Holland, Amsterdam, 1992.

[44] L. Rondi, *Stable determination of sound-soft polyhedral scatterers by a single measurement*, Indiana Univ. Math. J., 57 (2008), pp. 1377–1408.

[45] L. Rondi, *A Friedrichs-Maz'ya inequality for functions of bounded variation*, Math. Nachr., 290 (2017), pp. 1830–1839.

[46] L. Rondi and M. Sini, *Stable determination of a scattered wave from its far-field pattern: The high frequency asymptotics*, Arch. Ration. Mech. Anal., 218 (2015), pp. 1–54.

[47] E. Sincich and M. Sini, *Local stability for soft obstacles by a single measurement*, Inverse Probl. Imaging, 2 (2008), pp. 301–315.

[48] P. Stefanov and G. Uhlmann, *Local uniqueness for the fixed energy fixed angle inverse problem in obstacle scattering*, Proc. Amer. Math. Soc., 132 (2004), pp. 1351–1354.