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# Optimal identification of cavities in the Generalized Plane Stress problem in linear elasticity

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**Abstract.** For the Generalized Plane Stress (GPS) problem in linear elasticity, we obtain an optimal stability estimate of logarithmic type for the inverse problem of determining smooth cavities inside a thin isotropic cylinder from a single boundary measurement of traction and displacement. The result is obtained by reformulating the GPS problem as a Kirchhoff–Love plate-like problem in terms of the Airy function, and by using the strong unique continuation at the boundary for a Kirchhoff–Love plate operator under homogeneous Dirichlet conditions, which has recently been obtained in [G. Alessandrini et al., Arch. Ration. Mech. Anal. 231 (2019)].

**Keywords.** Inverse problems, Generalized Plane Stress problem, stability estimates, cavity

## 1. Introduction

In this paper we consider the inverse problem of detecting cavities inside a thin isotropic elastic plate $\Omega \times (-h/2, h/2)$, where the middle plane $\Omega$ is a bounded domain in $\mathbb{R}^2$ and $h$ is the constant thickness, subject to a single experiment consisting in applying in-plane boundary loads and measuring the induced displacement at the boundary. Practical applications concern the use of non-destructive techniques for the identification of possible defects, such as cavities, inside the plate.

The static equilibrium of the plate is described in terms of the classical Generalized Plane Stress (GPS) problem, which allows one to reformulate the original three-dimensional problem in a two-dimensional setting [22]. More precisely, denoting by $D \times (-h/2, h/2)$ the cavity, with $D$ a possibly disconnected subset of $\Omega$, the in-plane

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displacement field $a = a_1 e_1 + a_2 e_2$ solving the GPS problem satisfies the following two-dimensional Neumann boundary value problem

$$\begin{cases} \operatorname{div}(\mathbb{C}\nabla a) = 0 & \text{in } \Omega \setminus \overline{D}, \quad (1.1) \\ (\mathbb{C}\nabla a)n = \widehat{N} & \text{on } \partial\Omega, \quad (1.2) \\ (\mathbb{C}\nabla a)n = 0 & \text{on } \partial D, \quad (1.3) \end{cases}$$

where $\mathbb{C}$ is the elasticity tensor of the material, which is assumed to be isotropic. Here, $\widehat{N} = \widehat{N}_1 e_1 + \widehat{N}_2 e_2$ is the in-plane load field applied to $\partial\Omega$ satisfying the compatibility condition

$$\int_{\partial\Omega} \widehat{N} \cdot r = 0 \quad \text{for every } r \in \mathcal{R}_2, \tag{1.4}$$

where $\mathcal{R}_2$ is the linear space of infinitesimal two-dimensional rigid displacements. Under suitable strong convexity assumptions on the elastic tensor of the material (see Section 3 for details), and assuming $\widehat{N} \in H^{-1/2}(\partial\Omega, \mathbb{R}^2)$, problem (1.1)–(1.4) admits a unique solution $a \in H^1(\Omega \setminus \overline{D}, \mathbb{R}^2)$ satisfying the normalization conditions

$$\int_{\Omega\setminus\overline{D}} a = 0, \quad \int_{\Omega\setminus\overline{D}} (\nabla a - \nabla^T a) = 0, \tag{1.5}$$

and such that $\|a\|_{H^1(\Omega\setminus\overline{D})} \leq C\|\widehat{N}\|_{H^{-1/2}(\partial\Omega,\mathbb{R}^2)}$.

In this work we consider the inverse problem of determining the cavity $D$ from a single pair of Cauchy data $\{a, \widehat{N}\}$ given on $\partial\Omega$. More precisely, we are interested in obtaining quantitative stability estimates, which are useful to control the effect that possible measurement errors have on the results of reconstruction procedures. The arbitrariness of the normalization conditions (1.5), which are related to the non-uniqueness of the solution to the direct problem (1.1)–(1.4), leads to the following formulation of the stability issue: given two solutions $a^{(i)} \in H^1(\Omega, \mathbb{R}^2)$, $i = 1, 2$, to the direct problem (1.1)–(1.4) with $D = D_i$, satisfying, for some $\varepsilon > 0$,

$$\min_{r\in\mathcal{R}_2} \|a^{(1)} - a^{(2)} - r\|_{L^2(\Sigma,\mathbb{R}^2)} \leq \varepsilon, \tag{1.6}$$

we wish to control the Hausdorff distance $d_H(\overline{D_1}, \overline{D_2})$ in terms of $\varepsilon$ when $\varepsilon$ goes to zero, where $\Sigma$ is an open subset of $\partial\Omega$.

Assuming $\partial D \in C^{6,\alpha}$, $0 < \alpha \leq 1$, we prove

$$d_H(\overline{D_1}, \overline{D_2}) \leq C|\log\varepsilon|^{-\eta}, \tag{1.7}$$

where $C > 0$ and $\eta > 0$ are constants only depending on the a priori data. We refer to Theorem 3.1 for a precise statement. Let us notice that, in view of the counterexamples obtained in the simpler context of electrical conductivity (see, for instance, [3], [16], [9]), we can infer the optimality of the stability estimate (1.7).

The general scheme of our proof is inspired by the seminal paper [4], which established the first optimal logarithmic estimate for the determination of unknown boundaries in electrostatics. The key tool in [4] was the polynomial vanishing rate for solutions to the second order elliptic equation of electrostatics, satisfying either homogeneous Dirichlet or homogeneous Neumann boundary conditions, ensured by a doubling inequality at the boundary established in [2]. Aiming at obtaining a strong unique continuation property at the boundary (SUCB) for solutions to the GPS elliptic system, in this paper we have exploited the two-dimensional character of the problem (1.1)–(1.4) by using the classical Airy transformation, which (locally) reduces the GPS system with homogeneous Neumann boundary conditions to a scalar fourth order Kirchhoff–Love plate's equation under homogeneous Dirichlet boundary conditions. This reformulation allows us to use the finite vanishing rate at the boundary for homogeneous Dirichlet boundary conditions recently obtained in [6] in the form of a three spheres inequality at the boundary with optimal exponent, and in [20] in the form of a doubling inequality at the boundary, under the hypothesis that the boundary is of class $C^{6,\alpha}$.

It is worth noticing that the present approach, here applied to the GPS problem, allows one also to cover the analogous inverse problem of detecting cavities in a two-dimensional elastic body made of inhomogeneous Lamé material, thus improving the log-log stability result previously obtained in [17]. An optimal log-type estimate in dimension 3 remains a challenging open problem. In fact, the strong unique continuation estimates at the boundary derived in [6] are based, among other techniques, on the construction of a suitable conformal map, which is used to flatten the boundary of $D$. Let us mention that the Airy transformation has been used in [14] to prove global identifiability of the viscosity in an incompressible fluid governed by the Stokes and the Navier–Stokes equations in the plane by using boundary measurements.

The paper is organized as follows. Notation is presented in Section 2. Section 3 contains the formulation of the inverse problem and the statement of our stability result. The Airy transformation is illustrated in Section 4. The proof of the main result, given in Section 5, is based on a series of auxiliary propositions concerning Lipschitz propagation of smallness (Proposition 5.1), finite vanishing rate in the interior (Proposition 5.2), finite vanishing rate at the boundary (Proposition 5.3), and a stability estimate from Cauchy data (Proposition 5.4). Finally, for the sake of completeness, in Section 6 we recall the derivation of the GPS problem from the corresponding three-dimensional elasticity problem for a thin plate subject to in-plane boundary loads.

## 2. Notation

Let $P = (x_1(P), x_2(P))$ be a point of $\mathbb{R}^2$. We shall denote by $B_r(P)$ the disk in $\mathbb{R}^2$ of radius $r$ and center $P$ and by $R_{a,b}(P)$ the rectangle of center $P$ and sides parallel to the coordinate axes, of length $2a$ and $2b$,

$$R_{a,b}(P) = \{x = (x_1, x_2) \mid |x_1 - x_1(P)| < a,\ |x_2 - x_2(P)| < b\}. \tag{2.1}$$

**Definition 2.1** ($C^{k,\alpha}$ regularity). Let $\Omega$ be a bounded domain in $\mathbb{R}^2$. Given $k, \alpha$ with $k \in \mathbb{N}$, $0 < \alpha \leq 1$, we say that a portion $S$ of $\partial\Omega$ is of *class* $C^{k,\alpha}$ *with constants* $r_0$,

$M_0 > 0$ if, for any $P \in S$, there exists a rigid transformation of coordinates under which we have $P = 0$ and

$$\Omega \cap R_{r_0,2M_0r_0} = \{x \in R_{r_0,2M_0r_0} \mid x_2 > g(x_1)\},$$

where $g$ is a $C^{k,\alpha}$ function on $[-r_0, r_0]$ satisfying

$$g(0) = g'(0) = 0, \quad \|g\|_{C^{k,\alpha}([-r_0,r_0])} \leq M_0r_0,$$

where

$$\|g\|_{C^{k,\alpha}([-r_0,r_0])} = \sum_{i=0}^{k} r_0^i \sup_{[-r_0,r_0]} |g^{(i)}| + r_0^{k+\alpha}|g|_{k,\alpha},$$
$$|g|_{k,\alpha} = \sup_{\substack{t,s\in[-r_0,r_0]\\ t\neq s}} \frac{|g^{(k)}(t) - g^{(k)}(s)|}{|t-s|^\alpha}.$$

We normalize all norms in such a way that their terms are dimensionally homogeneous and coincide with the standard definition when the dimensional parameter equals 1. For instance,

$$\|f\|_{H^1(\Omega)} = r_0^{-1}\left(\int_\Omega f^2 + r_0^2\int_\Omega |\nabla f|^2\right)^{1/2},$$

and so on for boundary and trace norms.

Given a bounded domain $\Omega$ in $\mathbb{R}^2$ such that $\partial\Omega$ is of class $C^{k,\alpha}$ with $k \geq 1$, we define the positive orientation of the boundary to be induced by the outer unit normal $n$ in the following sense. Given a point $P \in \partial\Omega$, let $\tau = \tau(P)$ denote the unit tangent at the boundary in $P$ obtained by applying to $n$ a counterclockwise rotation of angle $\pi/2$, that is,

$$\tau = e_3 \times n, \tag{2.2}$$

where $\times$ denotes the vector product in $\mathbb{R}^3$ and $\{e_1, e_2, e_3\}$ is the canonical basis in $\mathbb{R}^3$.

Given any connected component $C$ of $\partial\Omega$ and a point $P_0 \in C$, let us define the positive orientation of $C$ to be associated to an arclength parameterization $\psi(s) = (x_1(s), x_2(s))$, $s \in [0, l(C)]$, such that $\psi(0) = P_0$ and $\psi'(s) = \tau(\psi(s))$. Here $l(C)$ denotes the length of $C$.

Throughout the paper, we denote by $w_{,\alpha}$, $\alpha = 1, 2$, $w_{,s}$, and $w_{,n}$ the derivatives of a function $w$ with respect to the $x_\alpha$ variable, to the arclength $s$ and to the normal direction $n$, respectively, and similarly for higher order derivatives.

We denote by $\mathbb{M}^n$ the space of $n \times n$ real valued matrices and by $L(X, Y)$ the space of bounded linear operators between Banach spaces $X$ and $Y$.

Given $A, B \in \mathbb{M}^n$ and $\mathbb{K} \in L(\mathbb{M}^n, \mathbb{M}^n)$, we use the following notation:

$$(\mathbb{K}A)_{ij} = \sum_{k,l=1}^{n} K_{ijkl}A_{kl}, \tag{2.3}$$

$$A \cdot B = \sum_{i,j=1}^{n} A_{ij} B_{ij}, \tag{2.4}$$

$$|A| = (A \cdot A)^{1/2}, \tag{2.5}$$

$$\widehat{A} = \tfrac{1}{2}(A + A^T). \tag{2.6}$$

We denote by $I_n$ the $n \times n$ identity matrix, and by $\mathrm{tr}(A)$ the trace of $A$.

When $n = 2$, we replace Latin indices with Greek ones.

The linear space of infinitesimal rigid displacements, for $n = 2, 3$, is defined as

$$R_n = \{r(x) = c + Wx \mid c \in \mathbb{R}^n,\ W \in \mathbb{M}^n,\ W + W^T = 0\}. \tag{2.7}$$

Given a bounded domain $U \subset \mathbb{R}^2$ and $t > 0$ set

$$U_t = \{x \in U \mid \mathrm{dist}(x, \partial U) > t\}. \tag{2.8}$$

## 3. Inverse problem and main result

(i) *A priori information on the geometry.* Let $\Omega$ be a bounded domain in $\mathbb{R}^2$ and assume that the cavity $D$ is an open subset compactly contained in $\Omega$ such that

$$\Omega \setminus D \text{ is connected.} \tag{3.1}$$

Moreover, assume that, for some positive numbers $r_0$, $M_0$, $M_1$, with $M_0 \geq 1/2$, we have

$$\mathrm{diam}(\Omega) \leq M_1 r_0, \tag{3.2}$$

$$\mathrm{dist}(D, \partial\Omega) \geq 2M_0 r_0, \tag{3.3}$$

$$\partial\Omega \text{ is of class } C^{1,\alpha} \text{ with constants } r_0, M_0, \tag{3.4}$$

$$\partial D \text{ is of class } C^{6,\alpha} \text{ with constants } r_0, M_0, \tag{3.5}$$

with $\alpha$ such that $0 < \alpha \leq 1$.

Let $\Sigma$ be the open portion of $\partial\Omega$ where measurements are taken. We assume that there exists $P_0 \in \Sigma$ such that

$$\partial\Omega \cap R_{r_0, 2M_0 r_0}(P_0) \subset \Sigma, \tag{3.6}$$

$$\Sigma \text{ is of class } C^{2,\alpha} \text{ with constants } r_0, M_0. \tag{3.7}$$

Without loss of generality, we can choose $M_0 \geq 1/2$ to ensure that $B_{r_0}(P) \subset R_{r_0, 2M_0 r_0}(P)$ for every $P \in \partial\Omega$.

(ii) *A priori information on the Neumann boundary data.* We assume that

$$\widehat{N} \in H^{-1/2}(\partial\Omega, \mathbb{R}^2), \quad \widehat{N} \not\equiv 0, \tag{3.8}$$

$$\int_{\partial\Omega} \widehat{N} \cdot r = 0 \quad \text{for every } r \in \mathcal{R}_2, \tag{3.9}$$

$$\mathrm{supp}(\widehat{N}) \subset\subset \Sigma, \tag{3.10}$$

and that, for a given constant $F > 0$,

$$\frac{\|\widehat{N}\|_{H^{-1/2}(\partial\Omega,\mathbb{R}^2)}}{\|\widehat{N}\|_{H^{-1}(\partial\Omega,\mathbb{R}^2)}} \leq F, \tag{3.11}$$

(iii) *A priori information on the elasticity tensor.* The elasticity tensor $\mathbb{C} = (C_{\alpha\beta\gamma\delta})$ is defined as

$$\mathbb{C}(x)A = \frac{Eh}{1-\nu^2(x)}\big((1-\nu(x))\widehat{A} + \nu(\mathrm{tr}(A))I_2\big) \tag{3.12}$$

for every $2 \times 2$ matrix $A$, where the Young modulus $E$ and the Poisson coefficient $\nu$ are given in terms of the Lamé moduli as follows:

$$E(x) = \frac{\mu(x)(2\mu(x) + 3\lambda(x))}{\mu(x) + \lambda(x)}, \quad \nu(x) = \frac{\lambda(x)}{2(\mu(x) + \lambda(x))}. \tag{3.13}$$

On the Lamé coefficients $\mu = \mu(x)$, $\lambda = \lambda(x)$, $\mu : \overline{\Omega} \to \mathbb{R}$, $\lambda : \overline{\Omega} \to \mathbb{R}$, we assume

$$\mu(x) \geq \alpha_0, \quad 2\mu(x) + 3\lambda(x) \geq \gamma_0 \quad \text{in } \overline{\Omega}, \tag{3.14}$$

for positive constants $\alpha_0$ and $\gamma_0$.

The above assumptions ensure that $\mathbb{C}$ satisfies the minor and major symmetries

$$C_{\alpha\beta\gamma\delta} = C_{\beta\alpha\gamma\delta} = C_{\alpha\beta\delta\gamma}, \quad C_{\alpha\beta\gamma\delta} = C_{\gamma\delta\alpha\beta} \quad \text{for all } \alpha, \beta, \gamma, \delta = 1, 2, \text{ in } \overline{\Omega}, \tag{3.15}$$

and that it is strongly convex in $\overline{\Omega}$, more precisely

$$\mathbb{C}A \cdot A \geq h\xi_0|A|^2 \quad \text{in } \Omega \tag{3.16}$$

for every $2 \times 2$ symmetric matrix $A$, where $\xi_0 = \min\{2\alpha_0, \gamma_0\}$ (see [18, Lemma 3.5] for details). Moreover, $E(x) > 0$ and $-1 < \nu(x) < 1/2$ in $\overline{\Omega}$.

We further assume that

$$\|\lambda\|_{C^4(\overline{\Omega})}, \|\mu\|_{C^4(\overline{\Omega})} \leq \Lambda_0 \tag{3.17}$$

for some positive constant $\Lambda_0$.

The weak formulation of (1.1)–(1.3) consists in finding $a = a(x) \in H^1(\Omega \setminus \overline{D})$ satisfying

$$\int_{\Omega\setminus\overline{D}} \mathbb{C}\nabla a \cdot \nabla v = \int_{\partial\Omega} \widehat{N} \cdot v \quad \text{for every } v \in H^1(\Omega \setminus \overline{D}). \tag{3.18}$$

Under our assumptions, there exists a unique solution to (3.18) up to addition of a rigid displacement. In order to select a single solution, we shall assume the normalization conditions

$$\int_{\Omega\setminus\overline{D}} a = 0, \quad \int_{\Omega\setminus\overline{D}} (\nabla a - \nabla^T a) = 0, \tag{3.19}$$

which imply the following stability estimate for the direct problem (1.1)–(1.2):

$$\|a\|_{H^1(\Omega\setminus\overline{D})} \leq Cr_0\|\widehat{N}\|_{H^{-1/2}(\partial\Omega,\mathbb{R}^2)}, \tag{3.20}$$

where $C > 0$ is a constant only depending on $h$, $\alpha_0$, $\gamma_0$, $M_0$ and $M_1$.

In what follows, we shall refer to the set of constants $h$, $\alpha_0$, $\gamma_0$, $\Lambda_0$, $\alpha$, $M_0$, $M_1$ and $F$ as the *a priori data*.

**Theorem 3.1** (Stability result). *Let $\Omega$ be a domain satisfying* (3.2), (3.4) *and let $\Sigma$ be an open portion of $\partial\Omega$ satisfying* (3.6)–(3.7). *Let the elasticity tensor $\mathbb{C} = \mathbb{C}(x) \in L(\mathbb{M}^2, \mathbb{M}^2)$ be given by* (3.12) *with Lamé moduli $\lambda = \lambda(x)$, $\mu = \mu(x)$ satisfy* (3.14) *and* (3.17). *Let $\widehat{N} \in H^{-1/2}(\partial\Omega, \mathbb{R}^2)$, $\widehat{N} \not\equiv 0$, satisfying* (3.9)–(3.11). *Let $D_i$, $i = 1, 2$, be two open subsets of $\Omega$ satisfying* (3.1), (3.3), (3.5), *and let $a^{(i)} \in H^1(\Omega \setminus \overline{D_i}, \mathbb{R}^2)$ be the solution to* (1.1)–(1.2) *satisfying* (3.19), *when $D = D_i$, $i = 1, 2$. If, given $\varepsilon > 0$,*

$$\min_{r \in \mathcal{R}_2} \|a^{(1)} - a^{(2)} - r\|_{L^2(\Sigma, \mathbb{R}^2)} \leq r_0 \varepsilon, \tag{3.21}$$

*then*

$$d_H(\overline{D_1}, \overline{D_2}) \leq C r_0 \left| \log\left( \frac{\varepsilon}{\|\widehat{N}\|_{H^{-1/2}(\partial\Omega, \mathbb{R}^2)}} \right) \right|^{-\eta}, \tag{3.22}$$

*where $C, \eta > 0$ only depend on the a priori data.*

**Remark 3.2.** As will be clear from the proof, the above stability result holds true also when the domain $\Omega$ contains a finite number of connected cavities $D^{(j)}$, $j = 1, \dots, J$, such that $\partial D^{(j)} \in C^{6,\alpha}$ with constants $r_0$, $M_0$, and $\mathrm{dist}(\partial D^{(j)}, \partial D^{(k)}) \geq r_0$ for $j \neq k$.

**Remark 3.3.** We can obtain analogous stability estimates for the inverse problem of determining an unknown portion of the boundary subject to homogeneous Neumann boundary conditions, by using the same techniques.

## 4. Airy transformation

It is known that the boundary value problem in plane linear elasticity can be formulated in terms of an equivalent Kirchhoff–Love *plate-like* problem involving a scalar-valued function called *Airy's function*. Although this argument is well established (see, for instance, [13] and [10]), for the reader's convenience we recall the essential points of the analysis.

For the sake of completeness, we consider a mixed boundary value problem, in order to describe the transformation of both Dirichlet and Neumann boundary conditions. Let $\mathcal{U}$ be a simply connected bounded domain in $\mathbb{R}^2$ with boundary $\partial\mathcal{U}$ of class $C^{1,1}$. Let $\partial_u\mathcal{U}$, $\partial_t\mathcal{U}$ denote two disjoint connected open subsets of $\partial\mathcal{U}$, with $\partial\mathcal{U} = \overline{\partial_u\mathcal{U} \cup \partial_t\mathcal{U}}$. Let $a = a_1 e_1 + a_2 e_2$, $a \in H^1(\mathcal{U}, \mathbb{R}^2)$, be the solution to the GPS problem (1.1)–(1.3) written in terms of Cartesian coordinates (see Section 6):

$$\begin{cases} N_{\alpha\beta,\beta} = 0 & \text{in } \mathcal{U}, & (4.1)\\ N_{\alpha\beta} n_\beta = \widehat{N}_\alpha & \text{on } \partial_t\mathcal{U}, & (4.2)\\ a_\alpha = \widehat{a}_\alpha & \text{on } \partial_u\mathcal{U}, & (4.3)\\ N_{\alpha\beta} = \frac{Eh}{1-\nu^2}\big((1-\nu)\epsilon_{\alpha\beta} + \nu(\epsilon_{\gamma\gamma})\delta_{\alpha\beta}\big) & \text{in } \mathcal{U}, & (4.4)\\ \epsilon_{\alpha\beta} = \frac{1}{2}(a_{\alpha,\beta} + a_{\beta,\alpha}) & \text{in } \mathcal{U}, & (4.5) \end{cases}$$

where $\widehat{N} \in H^{-1/2}(\partial_t \mathcal{U}, \mathbb{R}^2)$ and $\widehat{a} \in H^{1/2}(\partial_u \mathcal{U}, \mathbb{R}^2)$ are given Neumann and Dirichlet data, respectively.

The equilibrium equations (4.1) and the simple connectedness of $\mathcal{U}$ ensure the existence of a single-valued function $\varphi = \varphi(x_1, x_2)$, $\varphi \in H^2(\mathcal{U})$, such that

$$N_{\alpha\beta} = e_{\alpha\gamma} e_{\beta\delta} \varphi_{,\gamma\delta}, \tag{4.6}$$

where the matrix $e_{\alpha\gamma}$ is defined as follows: $e_{11} = e_{22} = 0$, $e_{12} = 1$, $e_{21} = -1$ (see [1]). We recall that, by construction, the function $\varphi$ and its first partial derivatives $\varphi_{,1}$, $\varphi_{,2}$ are uniquely determined up to an arbitrary additive constant.

It is convenient to introduce the *strain functions* $K_{\alpha\beta}$, $\alpha, \beta = 1, 2$, associated to the infinitesimal strain $\epsilon_{\alpha\beta}$:

$$K_{\alpha\beta} = e_{\delta\alpha} e_{\gamma\beta} \epsilon_{\delta\gamma}, \quad \alpha, \beta = 1, 2. \tag{4.7}$$

By inverting the constitutive equation (4.4), we get

$$\epsilon_{\alpha\beta} = \frac{1+\nu}{Eh} N_{\alpha\beta} - \frac{\nu}{Eh} (N_{\gamma\gamma}) \delta_{\alpha\beta}, \tag{4.8}$$

and using (4.6) we obtain

$$\epsilon_{\alpha\beta} = \frac{1+\nu}{Eh} e_{\alpha\gamma} e_{\beta\delta} \varphi_{,\gamma\delta} - \frac{\nu}{Eh} (\varphi_{,\gamma\gamma}) \delta_{\alpha\beta}. \tag{4.9}$$

Inserting this expression of $\epsilon_{\alpha\beta}$ into (4.7), we have

$$K_{\alpha\beta} = L_{\alpha\beta\gamma\delta} \varphi_{,\gamma\delta}, \tag{4.10}$$

where the Cartesian components $L_{\alpha\beta\gamma\delta}$ of the fourth order tensor $\mathbb{L}$ are

$$L_{\alpha\beta\gamma\delta} = \frac{1+\nu}{Eh} \delta_{\alpha\gamma} \delta_{\beta\delta} - \frac{\nu}{Eh} \delta_{\alpha\beta} \delta_{\gamma\delta}. \tag{4.11}$$

The strain $\epsilon_{\alpha\beta}$ obviously satisfies the well-known two-dimensional *Saint-Venant compatibility equation*

$$\epsilon_{11,22} + \epsilon_{22,11} - 2\epsilon_{12,12} = 0 \quad \text{in } \mathcal{U}. \tag{4.12}$$

Inverting (4.7), we have

$$\epsilon_{\alpha\beta} = e_{\alpha\gamma} e_{\beta\delta} K_{\gamma\delta}, \tag{4.13}$$

and equation (4.12), written in terms of $K_{\gamma\delta}$, becomes

$$\operatorname{div}(\operatorname{div}(\mathbb{L} \nabla^2 \varphi)) = 0 \quad \text{in } \mathcal{U}, \tag{4.14}$$

or, more explicitly,

$$\Delta^2 \varphi + 2Eh \nabla \left( \frac{1}{Eh} \right) \cdot \nabla(\Delta\varphi) - Eh \Delta \left( \frac{\nu}{Eh} \right) \Delta\varphi \\ + Eh \nabla^2 \left( \frac{1+\nu}{Eh} \right) \cdot \nabla^2 \varphi = 0 \quad \text{in } \mathcal{U}. \tag{4.15}$$

The above partial differential equation expresses the form assumed by the field equation (4.1) in terms of the Airy function $\varphi$.

We now consider the transformation of the Neumann boundary condition (4.2) on $\partial_t \mathcal{U}$. By (4.6), the condition on $\partial_t \mathcal{U}$ can be written as

$$e_{\alpha\gamma} e_{\beta\delta} \varphi_{,\gamma\delta} n_\beta = \widehat{N}_\alpha, \tag{4.16}$$

that is, recalling that $\tau_\delta = e_{\beta\delta} n_\beta$ on $\partial \mathcal{U}$,

$$(\varphi_{,1})_{,s} = -\widehat{N}_2, \quad (\varphi_{,2})_{,s} = \widehat{N}_1 \quad \text{on } \partial_t \mathcal{U}, \tag{4.17}$$

where $s$ is an arc length parametrization on $\partial \mathcal{U}$. By integrating the above equations with respect to $s$, from $P_0 \in \partial_t \mathcal{U}$ to $P \in \partial_t \mathcal{U}$, with $s(P_0) = 0$ and $s(P) = s$, the gradient of $\varphi$ on $\partial_t \mathcal{U}$ can be determined up to an additive constant vector $c = c_1 e_1 + c_2 e_2$, namely

$$\nabla \varphi(s) = c + \widehat{g}(s) \quad \text{on } \partial_t \mathcal{U}, \tag{4.18}$$

where $\widehat{g}(s) = \widehat{g}_1(s) e_1 + \widehat{g}_2(s) e_2$, $\widehat{g}_1(s) = -\int_0^s \widehat{N}_2(\xi)\, d\xi$, $\widehat{g}_2(s) = \int_0^s \widehat{N}_1(\xi)\, d\xi$. It follows that the normal derivative of $\varphi$ on $\partial_t \mathcal{U}$ is prescribed in terms of the Neumann data $\widehat{N}$, that is,

$$\varphi_{,n} = (c + \widehat{g}(s)) \cdot n \quad \text{on } \partial_t \mathcal{U}, \tag{4.19}$$

whereas, integrating (4.18) once more from $P_0$ to $P$, we have

$$\varphi(s) = C + \widehat{G}(s) \quad \text{on } \partial_t \mathcal{U}, \tag{4.20}$$

where $C = \varphi(0) =$ constant, and $\widehat{G}(s) = \int_0^s (c + \widehat{g}(\xi)) \cdot \tau(\xi)\, d\xi$. We notice that it is always possible to select the two arbitrary constants occurring in the construction of $\nabla\varphi$ such that $c_1 = c_2 = 0$ (see, for example, [22] for details). In particular, if the Neumann data $\widehat{N}$ vanishes on $\partial_t \mathcal{U}$, then we can also choose the third constant $C = 0$, so that $\varphi(s) = 0$ on $\partial_t \mathcal{U}$. In this case, the homogeneous Neumann boundary conditions for the GPS problem are transformed into the homogeneous Dirichlet boundary conditions for the Airy function (see also Section 6):

$$\varphi = 0, \quad \varphi_{,n} = 0 \quad \text{on } \partial_t \mathcal{U}. \tag{4.21}$$

The determination of the boundary conditions satisfied by $\varphi$ on $\partial_u \mathcal{U}$ is less obvious, since the corresponding boundary conditions in the original two-dimensional elasticity problem are not explicitly expressed in terms of the Airy function or its derivatives. We adopt a variational-like approach.

Let $\widetilde{\varphi} : \overline{\mathcal{U}} \to \mathbb{R}$ be a $C^\infty$ test function such that

$$\widetilde{\varphi} = 0, \quad \widetilde{\varphi}_{,n} = 0 \quad \text{on } \partial_t \mathcal{U}, \tag{4.22}$$

and define the *associated Airy stress field*

$$\widetilde{N}_{\alpha\beta} = e_{\alpha\gamma} e_{\beta\delta} \widetilde{\varphi}_{,\gamma\delta} \quad \text{in } \overline{\mathcal{U}}, \tag{4.23}$$

which obviously satisfies the equilibrium equations

$$\widetilde{N}_{\alpha\beta,\beta} = 0 \quad \text{in } \mathcal{U}. \tag{4.24}$$

Multiplying (4.24) by the displacement field $a = a_1 e_1 + a_2 e_2$ solving (4.1)–(4.5), and integrating by parts, we obtain

$$\int_{\mathcal{U}} \widetilde{\varphi}_{,\gamma\delta} K_{\gamma\delta} = \int_{\partial\mathcal{U}} \widetilde{N}_{\alpha\beta} n_\beta a_\alpha. \tag{4.25}$$

We first work with the integral on the left hand side of (4.25). After two integrations by parts, we obtain

$$\int_{\mathcal{U}} \widetilde{\varphi}_{,\gamma\delta} K_{\gamma\delta} = \int_{\mathcal{U}} K_{\gamma\delta,\gamma\delta}\widetilde{\varphi} + \int_{\partial\mathcal{U}} \widetilde{\varphi}_{,\gamma} K_{\gamma\delta} n_\delta - \int_{\partial\mathcal{U}} \widetilde{\varphi} K_{\gamma\delta,\delta} n_\gamma. \tag{4.26}$$

We express the second integral $I$ on the right hand side in terms of the local coordinates. Recalling that $\tau_\alpha = e_{\beta\alpha} n_\beta$ on $\partial\mathcal{U}$ and $\widetilde{\varphi}_{,\alpha} = n_\alpha \widetilde{\varphi}_{,n} + \tau_\alpha \widetilde{\varphi}_{,s}$ on $\partial\mathcal{U}$, $\alpha, \beta = 1, 2$, we have

$$I = \int_{\partial\mathcal{U}} (\widetilde{\varphi}_{,n} K_{nn} + \widetilde{\varphi}_{,s} K_{\tau n}), \tag{4.27}$$

where, to simplify the notation, we have introduced on $\partial\mathcal{U}$ the two functions

$$K_{nn} = K_{\gamma\delta} n_\delta n_\gamma, \quad K_{n\tau} = K_{\gamma\delta} n_\delta \tau_\gamma \ (= K_{\tau n}). \tag{4.28}$$

Integrating by parts the second term in (4.27) gives

$$I = \int_{\partial\mathcal{U}} \widetilde{\varphi}_{,n} K_{nn} - \widetilde{\varphi} K_{\tau n,s}. \tag{4.29}$$

Therefore, recalling (4.22), the left hand side of (4.25) takes the form

$$\int_{\mathcal{U}} \widetilde{\varphi}_{,\gamma\delta} K_{\gamma\delta} = \int_{\mathcal{U}} K_{\gamma\delta,\gamma\delta}\widetilde{\varphi} + \int_{\partial_u\mathcal{U}} K_{nn}\widetilde{\varphi}_{,n} - (K_{\gamma\delta,\delta} n_\gamma + K_{\tau n,s})\widetilde{\varphi}. \tag{4.30}$$

We next elaborate the integral appearing on the right hand side of (4.25). Let us introduce the *boundary displacement functions* associated to the Dirichlet data $\widehat{a}$:

$$\widehat{U}_\gamma = e_{\alpha\gamma} \widehat{a}_\alpha \quad \text{on } \partial_u\mathcal{U}. \tag{4.31}$$

Passing to local coordinates, after an integration by parts and using again (4.22), we have

$$\begin{aligned} \int_{\partial\mathcal{U}} \widetilde{N}_{\alpha\beta} n_\beta a_\alpha &= \int_{\partial\mathcal{U}} \widetilde{\varphi}_{,\gamma\delta} \tau_\delta e_{\alpha\gamma} a_\alpha = \int_{\partial\mathcal{U}} (\widetilde{\varphi}_{,\gamma})_{,s} e_{\alpha\gamma} a_\alpha \\ &= -\int_{\partial\mathcal{U}} \widetilde{\varphi}_{,\gamma} (e_{\alpha\gamma} a_\alpha)_{,s} = -\int_{\partial_u\mathcal{U}} \widetilde{\varphi}_{,\gamma} \widehat{U}_{\gamma,s}. \end{aligned} \tag{4.32}$$

Expressing again $\nabla\widetilde{\varphi}$ in terms of local coordinates and integrating by parts, we obtain

$$\int_{\partial\mathcal{U}} \widetilde{N}_{\alpha\beta} n_\beta a_\alpha = \int_{\partial_u\mathcal{U}} -\widetilde{\varphi}_{,n} \widehat{U}_{\gamma,s} n_\gamma + \widetilde{\varphi}(\tau_\gamma \widehat{U}_{\gamma,s})_{,s}. \tag{4.33}$$

Therefore, by rewriting (4.25) using (4.30) and (4.33), the strain functions $K_{\gamma\delta}$ satisfy the condition

$$\int_{\mathcal{U}} K_{\gamma\delta,\gamma\delta}\widetilde{\varphi} + \int_{\partial_u\mathcal{U}} (K_{nn} + \widehat{U}_{\gamma,s}n_\gamma)\widetilde{\varphi}_{,n} - \int_{\partial_u\mathcal{U}} (K_{\gamma\delta,\delta}n_\gamma + K_{\tau n,s} + (\tau_\gamma\widehat{U}_{\gamma,s})_{,s})\widetilde{\varphi} = 0 \tag{4.34}$$

for every $\widetilde{\varphi} \in C^\infty(\overline{\mathcal{U}})$ with $\widetilde{\varphi} = 0, \widetilde{\varphi}_{,n} = 0$ on $\partial_t\mathcal{U}$. By the arbitrariness of the test function $\widetilde{\varphi}$, and of the traces of $\widetilde{\varphi}$ and $\widetilde{\varphi}_{,n}$ on $\partial_u\mathcal{U}$, we can determine the conditions satisfied by $K_{\gamma\delta}$, namely, the field equation

$$K_{\gamma\delta,\gamma\delta} = 0 \quad \text{in } \mathcal{U}, \tag{4.35}$$

which coincides with (4.14), and the two boundary conditions

$$K_{nn} = -\widehat{U}_{\gamma,s}n_\gamma \quad \text{on } \partial_u\mathcal{U}, \tag{4.36}$$

$$K_{\gamma\delta,\delta}n_\gamma + K_{\tau n,s} = -(\tau_\gamma\widehat{U}_{\gamma,s})_{,s} \quad \text{on } \partial_u\mathcal{U}. \tag{4.37}$$

The above equations (4.35) and (4.36), (4.37) are known as the *compatibility field equation* and the *compatibility boundary conditions* for the strain functions $K_{\gamma\delta}$, respectively. In conclusion, under the assumption $\widehat{N} = 0$ on $\partial_t\mathcal{U}$, the two-dimensional elasticity problem (4.1)–(4.5) can be formulated in terms of the Airy function as follows:

$$\begin{cases} K_{\gamma\delta,\gamma\delta} = 0 & \text{in } \mathcal{U}, & (4.38) \\ \varphi = 0 & \text{on } \partial_t\mathcal{U}, & (4.39) \\ \frac{\partial\varphi}{\partial n} = 0 & \text{on } \partial_t\mathcal{U}, & (4.40) \\ K_{\alpha\beta}n_\alpha n_\beta = -\widehat{U}_{\gamma,s}n_\gamma & \text{on } \partial_u\mathcal{U}, & (4.41) \\ K_{\alpha\beta,\beta}n_\alpha + (K_{\alpha\beta}n_\beta\tau_\alpha)_{,s} = -(\tau_\gamma\widehat{U}_{\gamma,s})_{,s} & \text{on } \partial_u\mathcal{U}, & (4.42) \\ K_{\alpha\beta} = \frac{1}{Eh}((1+\nu)\varphi_{,\alpha\beta} - \nu(\Delta\varphi)\delta_{\alpha\beta}) & \text{in } \overline{\mathcal{U}}. & (4.43) \end{cases}$$

There is an important analogy connected with the above boundary value problem. Equations (4.38)–(4.43) describe the conditions satisfied by the transversal displacement $\varphi = \varphi(x_1, x_2)$ of the middle surface $\mathcal{U}$ of a Kirchhoff–Love thin elastic plate made of isotropic material. The plate is clamped on $\partial_t\mathcal{U}$, and subject to a couple field $\widehat{M} = \widehat{M}_\tau n + \widehat{M}_n\tau$ assigned on $\partial_u\mathcal{U}$, with $\widehat{M}_n = -\widehat{U}_{\gamma,s}n_\gamma$ and $\widehat{M}_\tau = \tau_\gamma\widehat{U}_{\gamma,s}$ (see, for example, [18]). Within this analogy, the strain functions $K_{\alpha\beta} = K_{\alpha\beta}(x_1, x_2)$ play the role of the *bending moments* (for $\alpha = \beta$) and the *twisting moments* (for $\alpha \neq \beta$) of the plate at $(x_1, x_2) \in \overline{\Omega}$ (per unit length), and the *bending stiffness* of the plate is equal to $(Eh)^{-1}$.

Observe that the geometry of the inverse problem considered, that is, $\mathcal{U} = \Omega \setminus \overline{D}$, does not ensure the existence of a globally defined Airy function, since the hypothesis of simple connectedness is missing. For this reason, in Section 5 we shall make use of local Airy functions, defined either in interior disks (see the proof of Proposition 5.2) or in neighborhoods of the boundary of the cavity (see the proof of Proposition 5.3).

**Proposition 4.1.** *Under the above notation and assumptions, we have*

$$\frac{(1-|\nu|)^2}{E^2h^2}|\nabla^2\varphi|^2 \leq |\widehat{\nabla}a|^2 \leq \frac{(1+|\nu|)^2}{E^2h^2}|\nabla^2\varphi|^2. \tag{4.44}$$

*Proof.* By (4.6), we have $N_{11}=\varphi_{,22}$, $N_{22}=\varphi_{,11}$, $N_{12}=N_{21}=-\varphi_{,12}$, so that

$$|\nabla^2\varphi|^2 = \sum_{\alpha,\beta=1}^{2} N_{\alpha\beta}^2. \tag{4.45}$$

By (4.8), we have $\epsilon_{11}=\frac{1}{Eh}N_{11}-\frac{\nu}{Eh}N_{22}$, $\epsilon_{22}=\frac{1}{Eh}N_{22}-\frac{\nu}{Eh}N_{11}$, $\epsilon_{12}=\epsilon_{21}=\frac{1+\nu}{Eh}N_{12}$, so that

$$|\widehat{\nabla}a|^2 = \sum_{\alpha,\beta=1}^{2}\epsilon_{\alpha\beta}^2 = \frac{1}{(Eh)^2}\{(1+\nu^2)(N_{11}^2+N_{22}^2)+2(1+\nu)^2N_{12}^2-4\nu N_{11}N_{22}\}. \tag{4.46}$$

By the elementary inequality

$$|2N_{11}N_{22}| \leq N_{11}^2+N_{22}^2, \tag{4.47}$$

we see that, for every $\nu\in\mathbb{R}$,

$$|4\nu N_{11}N_{22}| \leq 2|\nu|(N_{11}^2+N_{22}^2), \tag{4.48}$$

that is,

$$-2|\nu|(N_{11}^2+N_{22}^2) \leq -4\nu N_{11}N_{22} \leq 2|\nu|(N_{11}^2+N_{22}^2). \tag{4.49}$$

By (4.46) and (4.49), the conclusion follows. ∎

## 5. Proof of the main result

**Proposition 5.1** (Lipschitz propagation of smallness). *Let $\Omega$ be a domain satisfying (3.2), (3.4). Let $D$ be an open subset of $\Omega$ satisfying (3.1), (3.3), (3.5). Let $a\in H^1(\Omega\setminus\overline{D},\mathbb{R}^2)$ be the solution to (1.1)–(1.2) satisfying (3.19). Let the elasticity tensor $\mathbb{C}=\mathbb{C}(x)\in L(\mathbb{M}^2,\mathbb{M}^2)$ be given by (3.12) with Lamé moduli $\lambda=\lambda(x)$, $\mu=\mu(x)$ satisfying (3.14) and (3.17). Let $\widehat{N}\in H^{-1/2}(\partial\Omega,\mathbb{R}^2)$, $\widehat{N}\not\equiv 0$, satisfy (3.9)–(3.11). Then there exists $s>1$, only depending on $\alpha_0$, $\gamma_0$, $\Lambda_0$ and $M_0$, such that for every $\rho>0$ and every $\bar{x}\in(\Omega\setminus\overline{D})_{s\rho}$, we have*

$$\int_{B_\rho(\bar{x})}|\widehat{\nabla}a|^2 \geq \frac{Cr_0^2}{\exp[A(r_0/\rho)^B]}\|\widehat{N}\|^2_{H^{-1/2}(\partial\Omega,\mathbb{R}^2)}, \tag{5.1}$$

*where $A,B,C>0$ are constants only depending on $\alpha_0$, $\gamma_0$, $\Lambda_0$, $M_0$, $M_1$ and $F$.*

*Proof.* The proof follows by merging the Lipschitz propagation of smallness estimate (3.5) contained in [17, Proposition 3.1], Korn inequalities (see, for instance, [12], [5]), trace inequalities [15] and equivalence relations for the $H^{-1/2}$ and $H^{-1}$ norms of the Neumann data $\widehat{N}$ in [17, Remark 3.4, (3.9)–(3.10)]. ∎

**Proposition 5.2** (Finite vanishing rate in the interior). *Under the hypotheses of Proposition 5.1, there exist $\widetilde{c}_0 < 1/2$ and $C > 0$, only depending on $\alpha_0$, $\gamma_0$ and $\Lambda_0$, such that, for every $\overline{r} \in (0, r_0)$ and every $\bar{x} \in \Omega \setminus \overline{D}$ such that $B_{\bar{r}}(\bar{x}) \subset \Omega \setminus \overline{D}$, and every $r_1 < \widetilde{c}_0 \bar{r}$, we have*

$$\int_{B_{r_1}(\bar{x})} |\widehat{\nabla} a|^2 \geq C \left(\frac{r_1}{\bar{r}}\right)^{\tau_0} \int_{B_{\bar{r}}(\bar{x})} |\widehat{\nabla} a|^2, \tag{5.2}$$

*where $\tau_0 \geq 1$ only depends on $\alpha_0$, $\gamma_0$, $\Lambda_0$, $M_0$, $M_1$, $r_0/\bar{r}$ and $F$.*

*Proof.* We can introduce in $B_{\bar{r}}(\bar{x})$ a locally defined Airy function $\varphi$ associated to the solution $a$. The proof follows by adapting the arguments in the proof of the analogous Proposition 3.5 in [19] which applies to the Kirchhoff–Love plate equation. The main difference consists in estimating the $L^2$ norms of $\varphi$ and $|\nabla \varphi|$ appearing in [19, (3.21)] in terms of the $L^2$ norm of $|\nabla^2 \varphi|$ and using (4.44), the stability estimate (3.20) and Proposition 5.1. ∎

**Proposition 5.3** (Finite vanishing rate at the boundary). *Under the hypotheses of Proposition 5.1, there exist $\bar{c}_0 < 1/2$ and $C > 0$, only depending on $\alpha_0$, $\gamma_0$, $\Lambda_0$, $M_0$, $\alpha$, such that, for every $\bar{x} \in \partial D$ and every $r_1 < \bar{c}_0 r_0$, we have*

$$\int_{B_{r_1}(\bar{x}) \cap (\Omega \setminus \overline{D})} |\widehat{\nabla} a|^2 \geq C \left(\frac{r_1}{r_0}\right)^{\tau} \int_{B_{r_0}(\bar{x}) \cap (\Omega \setminus \overline{D})} |\widehat{\nabla} a|^2, \tag{5.3}$$

*where $\tau \geq 1$ only depends on $\alpha_0$, $\gamma_0$, $\Lambda_0$, $M_0$, $\alpha$, $M_1$ and $F$.*

*Proof.* Following the arguments of Section 4, let us consider the Airy function $\varphi$ associated to the restriction of the solution $a$ to the simply connected domain $R_{r_0, 2M_0 r_0}(\bar{x}) \cap \Omega \setminus \overline{D}$, where the rectangle $R_{r_0, 2M_0 r_0}(\bar{x})$ has been defined in (2.1). We know that $\varphi$ satisfies the partial differential equation

$$\operatorname{div}(\operatorname{div}(\mathbb{L}\nabla^2 \varphi)) = 0 \quad \text{in } R_{r_0, 2M_0 r_0}(\bar{x}) \cap \Omega \setminus \overline{D}, \tag{5.4}$$

or, equivalently,

$$\begin{aligned} \Delta^2 \varphi + 2Eh\nabla\left(\frac{1}{Eh}\right) \cdot \nabla(\Delta\varphi) - Eh\Delta\left(\frac{\nu}{Eh}\right)\Delta\varphi \\ + Eh\nabla^2\left(\frac{1+\nu}{Eh}\right) \cdot \nabla^2\varphi = 0 \quad \text{in } R_{r_0, 2M_0 r_0}(\bar{x}) \cap (\Omega \setminus \overline{D}), \end{aligned} \tag{5.5}$$

and the homogeneous Dirichlet conditions

$$\varphi = \varphi_{,n} = 0 \quad \text{on } \partial D \cap R_{r_0, 2M_0 r_0}(\bar{x}). \tag{5.6}$$

Notice that, under our assumptions, the fourth order tensor $\mathbb{L}$ satisfies the strong convexity condition

$$\mathbb{L}A \cdot A \geq \frac{1}{5h\Lambda_0}|A|^2 \quad \text{in } \Omega \tag{5.7}$$

for every symmetric $2 \times 2$ matrix $A$. We also notice that the coefficients of the terms involving second and third order derivatives of $\varphi$ in (5.5) are of class $C^2$ and $C^3$ in $R_{r_0,2M_0r_0}(\bar{x}) \cap \Omega \setminus \overline{D}$, respectively, with corresponding $C^2$ and $C^3$ norm bounded by a constant only depending on $h$, $\alpha_0$, $\gamma_0$ and $\Lambda_0$. Therefore, we can apply the results obtained in [6]. More precisely, by applying [6, Corollary 2.3] to $v = \varphi$, there exists $c < 1$, only depending on $M_0$ and $\alpha$, such that, for every $r_1 < r_2 < cr_0$, we have

$$\int_{B_{r_1}(\bar{x})\cap(\Omega\setminus\overline{D})} \varphi^2 \geq \left(\frac{r_1}{r_0}\right)^{\frac{\log B}{\log(cr_0/r_2)}} \int_{B_{r_0}(\bar{x})\cap(\Omega\setminus\overline{D})} \varphi^2, \tag{5.8}$$

where $B > 1$ is given by

$$B = C\left(\frac{r_0}{r_2}\right)^C \frac{\int_{B_{r_0}(\bar{x})\cap(\Omega\setminus\overline{D})} \varphi^2}{\int_{B_{r_2}(\bar{x})\cap(\Omega\setminus\overline{D})} \varphi^2}, \tag{5.9}$$

and $C > 1$ only depends on $\alpha_0$, $\gamma_0$, $\Lambda_0$, $M_0$ and $\alpha$. Here $B = A^{-1}$, with $A$ the constant defined in [6, (2.17)]. Choose $r_2 = \overline{c}_0 r_0$, with $\overline{c}_0 = c/2$. We need to estimate the quantity $B$. By applying the Poincaré inequality (see, for instance, [5, Example 4.4]) and (4.44), we have

$$\int_{B_{r_0}(\bar{x})\cap(\Omega\setminus\overline{D})} \varphi^2 \leq Cr_0^4 \int_{B_{r_0}(\bar{x})\cap(\Omega\setminus\overline{D})} |\nabla^2\varphi|^2 = Cr_0^4 \int_{B_{r_0}(\bar{x})\cap(\Omega\setminus\overline{D})} |\widehat{\nabla} a|^2, \tag{5.10}$$

where $C > 0$ only depends on $\alpha_0$, $\gamma_0$, $\Lambda_0$, $M_0$ and $\alpha$. Moreover, by applying [6, Lemma 4.7] and (4.44), and recalling the choice of $r_2$, we have

$$\int_{B_{r_2}(\bar{x})\cap(\Omega\setminus\overline{D})} \varphi^2 \geq Cr_2^4 \int_{B_{r_2/2}(\bar{x})\cap(\Omega\setminus\overline{D})} |\nabla^2\varphi|^2 = Cr_0^4 \int_{B_{cr_0/4}(\bar{x})\cap(\Omega\setminus\overline{D})} |\widehat{\nabla} a|^2. \tag{5.11}$$

By the regularity of $\partial D$ and by (3.3), there exists $x_1 \in \Omega \setminus \overline{D}$ such that $B_{tcr_0/4}(x_1) \subset \Omega \setminus \overline{D}$ with $t = \frac{1}{1+\sqrt{1+M_0^2}}$. By applying Proposition 5.1 with $\rho = \frac{tcr_0}{4s}$ and by using the stability estimate for the direct problem (3.20), in view of (5.10)–(5.11), we can estimate $B \leq C$, with $C$ only depending on $\alpha_0$, $\gamma_0$, $\Lambda_0$, $M_0$, $\alpha$, $M_1$ and $F$. By using again the Poincaré inequality, [6, Lemma 4.7] and (4.44), we obtain the assertion. ■

From now on, we shall denote by $\mathcal{G}$ the connected component of $\Omega \setminus \overline{D_1 \cup D_2}$ such that $\Sigma \subset \partial\mathcal{G}$.

**Proposition 5.4** (Stability estimate of continuation from Cauchy data). *Under the hypotheses of Theorem 3.1, we have*

$$\int_{(\Omega\setminus\mathcal{G})\setminus\overline{D_1}} |\widehat{\nabla} a^{(1)}|^2 \leq r_0^2 \|\widehat{N}\|^2_{H^{-1/2}(\partial\Omega,\mathbb{R}^2)} \omega\left(\frac{\varepsilon}{\|\widehat{N}\|_{H^{-1/2}(\partial\Omega,\mathbb{R}^2)}}\right), \tag{5.12}$$

$$\int_{(\Omega\setminus\mathcal{G})\setminus\overline{D_2}} |\widehat{\nabla} a^{(2)}|^2 \leq r_0^2 \|\widehat{N}\|^2_{H^{-1/2}(\partial\Omega,\mathbb{R}^2)} \omega\left(\frac{\varepsilon}{\|\widehat{N}\|_{H^{-1/2}(\partial\Omega,\mathbb{R}^2)}}\right), \tag{5.13}$$

*where $\omega$ is an increasing continuous function on* $[0,\infty)$ *which satisfies*

$$\omega(t) \le C(\log|\log t|)^{-1/2} \quad \textit{for every } t < e^{-1}, \tag{5.14}$$

*with* $C > 0$ *only depending on* $\alpha_0$, $\gamma_0$, $\Lambda_0$, $M_0$, $\alpha$ *and* $M_1$. *Moreover, there exists* $d_0 > 0$, *with* $d_0/r_0$ *only depending on* $M_0$ *and* $\alpha$, *such that if* $d_H(\overline{\Omega \setminus D_1}, \overline{\Omega \setminus D_2}) \le d_0$ *then* (5.12)–(5.13) *hold with* $\omega$ *given by*

$$\omega(t) \le C|\log t|^{-\sigma} \quad \textit{for every } t < 1, \tag{5.15}$$

*where* $\sigma, C > 0$ *only depend on* $\alpha_0$, $\gamma_0$, $\Lambda_0$, $M_0$, $\alpha$, $M_1$.

*Proof.* The proof can be easily obtained by adapting the proof of the analogous estimates in [17, Propositions 3.5 and 3.6]. The only difference consists in replacing the auxiliary function $w = a^{(1)} - a^{(2)}$ with $w = a^{(1)} - a^{(2)} - r$, where $r \in \mathcal{R}_2$ is the minimizer of problem (3.21), and noticing that $\widehat{\nabla} r = 0$. ■

*Proof of Theorem* 3.1. It is convenient to introduce the following auxiliary distances:

$$d = d_H(\overline{\Omega \setminus D_1}, \overline{\Omega \setminus D_2}), \tag{5.16}$$

$$d_m = \max\left\{\max_{x\in\partial D_1} \operatorname{dist}(x, \overline{\Omega \setminus D_2}), \max_{x\in\partial D_2} \operatorname{dist}(x, \overline{\Omega \setminus D_1})\right\}. \tag{5.17}$$

Let $\eta > 0$ be such that

$$\max_{i=1,2} \int_{(\Omega\setminus\overline{\mathcal{G}})\setminus\overline{D_i}} |\widehat{\nabla} a^{(i)}|^2 \le \eta. \tag{5.18}$$

**Step 1.** *Assume* $\eta \le r_0^2 \|\widehat{N}\|^2_{H^{-1/2}(\partial\Omega,\mathbb{R}^2)}$. *We have*

$$d_m \le C r_0 \left(\frac{\eta}{r_0^2 \|\widehat{N}\|^2_{H^{-1/2}(\partial\Omega,\mathbb{R}^2)}}\right)^{1/\tau}, \tag{5.19}$$

*where* $\tau$ *has been introduced in Proposition* 5.3 *and* $C$ *is a positive constant only depending on the a priori data.*

*Proof.* Without loss of generality, let $x_0 \in \partial D_1$ be such that

$$\operatorname{dist}(x_0, \overline{\Omega \setminus D_2}) = d_m > 0. \tag{5.20}$$

Since $B_{d_m}(x_0) \subset D_2 \subset \Omega \setminus \overline{\mathcal{G}}$, we have

$$B_{d_m}(x_0) \cap (\Omega \setminus \overline{D_1}) \subset (\Omega \setminus \overline{\mathcal{G}}) \setminus \overline{D_1} \tag{5.21}$$

and then, by (5.18),

$$\int_{B_{d_m}(x_0)\cap(\Omega\setminus\overline{D_1})} |\widehat{\nabla} a^{(1)}|^2 \le \eta. \tag{5.22}$$

Let us distinguish two cases. First, let

$$d_m < \overline{c}_0 r_0, \tag{5.23}$$

where $\overline{c}_0$ is the positive constant appearing in Proposition 5.3. By applying this proposition, we have

$$\eta \geq C\left(\frac{d_m}{r_0}\right)^\tau \int_{B_{r_0}(x_0)\cap(\Omega\setminus\overline{D_1})} |\widehat{\nabla} a^{(1)}|^2, \tag{5.24}$$

where $C > 0$ only depends on $\alpha_0$, $\gamma_0$, $\Lambda_0$, $\alpha$, $M_0$, $M_1$ and $F$.

By Proposition 5.1, we have

$$\eta \geq C\left(\frac{d_m}{r_0}\right)^\tau r_0^2 \|\widehat{N}\|^2_{H^{-1/2}(\partial\Omega,\mathbb{R}^2)}, \tag{5.25}$$

where $C > 0$ only depends on $\alpha_0$, $\gamma_0$, $\Lambda_0$, $\alpha$, $M_0$, $M_1$, $F$, from which we can estimate $d_m$, obtaining (5.19).

As the second case, let

$$d_m \geq \overline{c}_0 r_0. \tag{5.26}$$

By starting again from (5.22), applying Proposition 5.1 and recalling $d_m \leq M_1 r_0$, we have

$$d_m \leq C r_0 \left(\frac{\eta}{r_0^2 \|\widehat{N}\|^2_{H^{-1/2}(\partial\Omega,\mathbb{R}^2)}}\right), \tag{5.27}$$

where $C > 0$ only depends on $\alpha_0$, $\gamma_0$, $\Lambda_0$, $M_0$, $M_1$, $F$. Since we have assumed $\eta \leq r_0^2 \|\widehat{N}\|^2_{H^{-1/2}(\partial\Omega,\mathbb{R}^2)}$, also in this case we obtain (5.19). ∎

**Step 2.** *Assume* $\eta \leq r_0^2 \|\widehat{N}\|^2_{H^{-1/2}(\partial\Omega,\mathbb{R}^2)}$. *We have*

$$d \leq C r_0 \left(\frac{\eta}{r_0^2 \|\widehat{N}\|^2_{H^{-1/2}(\partial\Omega,\mathbb{R}^2)}}\right)^{1/\tau_1} \tag{5.28}$$

*with* $\tau_1 = \max\{\tau, \tau_0\}$ *and* $C > 0$ *only depends on* $\alpha_0$, $\gamma_0$, $\Lambda_0$, $\alpha$, $M_0$, $M_1$ *and* $F$.

*Proof.* We may assume that $d > 0$ and there exists $y_0 \in \overline{\Omega \setminus D_1}$ such that

$$\mathrm{dist}(y_0, \overline{\Omega \setminus D_2}) = d. \tag{5.29}$$

Since $d > 0$, we have $y_0 \in D_2 \setminus D_1$. Let

$$h = \mathrm{dist}(y_0, \partial D_1), \tag{5.30}$$

possibly $h = 0$.

There are three cases to consider:

(i) $h \leq d/2$;

(ii) $h > d/2$, $h \leq d_0/2$;

(iii) $h > d/2$, $h > d_0/2$.

Here the number $d_0$, with $0 < d_0 < r_0$, is such that $d_0/r_0$ only depends on $M_0$, and it is the same as in Proposition 5.4. In particular, [4, Proposition 3.6] shows that there exists an absolute constant $C > 0$ such that if $d \leq d_0$, then $d \leq Cd_m$.

*Case (i).* By definition, there exists $z_0 \in \partial D_1$ such that $|z_0 - y_0| = h$. By applying the triangle inequality, we get $\mathrm{dist}(z_0, \overline{\Omega \setminus D_2}) \geq d/2$. Since, by definition, $\mathrm{dist}(z_0, \overline{\Omega \setminus D_2}) \leq d_m$, we obtain $d \leq 2d_m$.

*Case (ii).* It turns out that $d < d_0$ and then, by the above recalled property, again we have $d \leq Cd_m$ for an absolute constant $C$.

*Case (iii).* Let $\widetilde{h} = \min\{h, r_0\}$. Obviously $B_{\widetilde{h}}(y_0) \subset \Omega \setminus \overline{D_1}$ and $B_d(y_0) \subset D_2$. Set

$$d_1 = \min\{d/2, \widetilde{c}_0 d_0/4\},$$

where $\widetilde{c}_0$ is the positive constant appearing in Proposition 5.2. Since $d_1 < d$ and $d_1 < \widetilde{h}$, we have $B_{d_1}(y_0) \subset D_2 \setminus \overline{D_1}$ and therefore $\eta \geq \int_{B_{d_1}(y_0)} |\widehat{\nabla} a^{(1)}|^2$.

Since $d_0/2 < \widetilde{h}$, we have $B_{d_0/2}(y_0) \subset \Omega \setminus \overline{D_1}$ so that we can apply Proposition 5.2 with $r_1 = d_1, \overline{r} = d_0/2$, obtaining $\eta \geq C(2d_1/d_0)^{\tau_0} \int_{B_{d_0/2}(y_0)} |\widehat{\nabla} a^{(1)}|^2$, with $C > 0$ only depending on $\alpha_0$, $\gamma_0$, $\Lambda_0$, $M_0$, $M_1$ and $F$. Next, by Proposition 5.1, recalling that $d_0/r_0$ only depends on $M_0$, we derive

$$d_1 \leq Cr_0 \left( \frac{\eta}{r_0^2 \|\widehat{M}\|^2_{H^{-1/2}(\partial\Omega, \mathbb{R}^2)}} \right)^{1/\tau_0},$$

where $C > 0$ only depends on $\alpha_0$, $\gamma_0$, $\Lambda_0$, $M_0$, $M_1$ and $F$. For $\eta$ small enough, we have $d_1 < \widetilde{c}_0 d_0/4$, so that $d_1 = d/2$ and

$$d \leq Cr_0 \left( \frac{\eta}{r_0^2 \|\widehat{M}\|^2_{H^{-1/2}(\partial\Omega, \mathbb{R}^2)}} \right)^{1/\tau_0},$$

where $C > 0$ only depends on $\alpha_0$, $\gamma_0$, $\Lambda_0$, $M_0$, $M_1$ and $F$. Collecting the three cases, the conclusion follows. ■

**Step 3.** *We have*

$$d_{\mathcal{H}}(\overline{D_1}, \overline{D_2}) \leq \sqrt{1 + M_0^2}\, d. \tag{5.31}$$

*Proof.* The proof is based on purely geometrical arguments; we refer to [19, proof of Theorem 3.1, Step 3]. ■

*Conclusion.* By Proposition 5.4,

$$d \leq Cr_0 \left( \log \left| \log \left( \frac{\varepsilon}{\|\widehat{N}\|^2_{H^{-1/2}(\partial\Omega, \mathbb{R}^2)}} \right) \right| \right)^{-1/(2\tau_1)} \tag{5.32}$$

with $\tau_1 \geq 1$ and $C > 0$ only depending on $\alpha_0$, $\gamma_0$, $\Lambda_0$, $\alpha$, $M_0$, $M_1$ and $F$. By this first rough estimate, there exists $\varepsilon_0 > 0$, only depending on $\alpha_0$, $\gamma_0$, $\Lambda_0$, $\alpha$, $M_0$, $M_1$ and $F$, such that if $\varepsilon \leq \varepsilon_0$, then $d \leq d_0$. Therefore, we can apply the second statement of Proposition 5.4, obtaining the conclusion. ∎

## 6. Generalized Plane Stress problem

In this section we derive the Generalized Plane Stress (GPS) problem for the statical equilibrium of a thin elastic plate under in-plane boundary loads. Our analysis follows the classical approach of the theory of structures, according to the original idea introduced by Filon [11]. Alternative, more formal derivations have been proposed to justify the GPS problem. The interested reader can refer, among others, to the contributions [8], [7] and [21].

Let $\mathcal{U}$ be a bounded domain in $\mathbb{R}^2$, and consider the cylinder $\mathcal{C} = \mathcal{U} \times (-h/2, h/2)$ with middle plane $\mathcal{U} \times \{x_3 = 0\}$ (which we will simply denote by $\mathcal{U}$ in what follows) and thickness $h$. Here, $\{O, x_1, x_2, x_3\}$ is a Cartesian coordinate system, with origin $O$ belonging to the plane $x_3 = 0$ and the $x_3$ axis orthogonal to $\mathcal{U}$. The cylinder is called a *plate* if $h$ is small with respect to the linear dimensions of $\mathcal{U}$, e.g., $h \ll \mathrm{diam}(\mathcal{U})$.

Suppose that the faces $\mathcal{U} \times \{x_3 = \pm h/2\}$ of the plate are free of applied loads, and all external surface forces acting on the lateral surface $\partial\mathcal{U} \times (-h/2, h/2)$ lie in planes parallel to the middle plane $\mathcal{U}$, and are independent of $x_3$. We shall further assume that body forces vanish in $\mathcal{C}$. The plate is assumed to be made of linearly elastic isotropic material, with Lamé moduli independent of the $x_3$-coordinate, e.g., $\lambda = \lambda(x_1, x_2)$, $\mu = \mu(x_1, x_2)$ for every $(x_1, x_2, 0) \in \overline{\mathcal{U}}$. Moreover, let $\lambda, \mu \in C^{0,1}(\overline{\mathcal{U}})$ be such that $\mu \geq \alpha_0$, $2\mu + 3\lambda \geq \gamma_0$ in $\overline{\mathcal{U}}$, with $\alpha_0$, $\gamma_0$ positive constants.

Under the above assumptions, the problem of elastostatics consists in finding a displacement $u$ solving

$$\begin{cases} T_{ij,j} = 0 & \text{in } \mathcal{C}, & (6.1)\\ T_{i3} = 0 & \text{on } \mathcal{U} \times \{x_3 = \pm h/2\}, & (6.2)\\ T_{\alpha\beta} n_\beta = \widehat{t}_\alpha & \text{on } \partial\mathcal{U} \times (-h/2, h/2), & (6.3)\\ T_{3\beta} n_\beta = 0 & \text{on } \partial\mathcal{U} \times (-h/2, h/2), & (6.4)\\ T_{ij} = 2\mu E_{ij} + \lambda(E_{kk})\delta_{ij} & \text{in } \mathcal{C}, & (6.5)\\ E_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) & \text{in } \mathcal{C}, & (6.6) \end{cases}$$

where the force field $\widehat{t} = (\widehat{t}_1, \widehat{t}_2, 0)$, with $\widehat{t}_\alpha = \widehat{t}_\alpha(x_1, x_2)$, $\alpha = 1, 2$, assigned on $\partial\mathcal{U} \times (-h/2, h/2)$ satisfies the compatibility conditions

$$\int_{\partial\mathcal{U}\times(-h/2,h/2)} \widehat{t} = 0, \quad \int_{\partial\mathcal{U}\times(-h/2,h/2)} x \times \widehat{t} = 0 \tag{6.7}$$

(see, for example, [13, §45]). The above boundary value problem is called the *plane problem of elastostatics*. It is known that, under our assumptions and for $\widehat{t}_\alpha \in H^{-1/2}(\partial\mathcal{U}, \mathbb{R}^2)$,

$\alpha = 1, 2$, there exists a solution $u \in H^1(\mathcal{C}, \mathbb{R}^3)$ which is unique up to an infinitesimal rigid displacement $r(x) = a + b \times x$, with $a, b \in \mathbb{R}^3$ constant vectors.

We now formulate the Generalized Plane Stress (GPS) problem associated to (6.1)–(6.6). The GPS problem is a two-dimensional boundary value problem formulated in terms of the thickness averages of $u$, $E$ and $T$, under the a priori assumption

$$T_{33} = 0 \quad \text{in } \mathcal{C}. \tag{6.8}$$

For a physically plausible justification of the above assumption under the hypothesis of small $h$, we refer to [22, §67] and to the paper [11] by Filon, who first derived the GPS problem.

Given a function $f : \mathcal{C} \to \mathbb{R}^3$, $f \in H^1(\mathcal{C})$, let us define the function $\tilde{f} : \mathcal{C} \to \mathbb{R}^3$ as follows:

$$\begin{cases} \tilde{f}_1(x_1, x_2, x_3) = f_1(x_1, x_2, -x_3), & (6.9)\\ \tilde{f}_2(x_1, x_2, x_3) = f_2(x_1, x_2, -x_3), & (6.10)\\ \tilde{f}_3(x_1, x_2, x_3) = -f_3(x_1, x_2, -x_3). & (6.11) \end{cases}$$

By definition of the plane problem, if $u$ is a solution to (6.1)–(6.6), then so is $\tilde{u}$. Moreover, $u - \tilde{u}$ is a solution to (6.1)–(6.6) with $\hat{t} = 0$, and therefore $u - \tilde{u} \in \mathcal{R}_3$. Noticing that $(u_1 - \tilde{u}_1)|_{x_3=0} = (u_2 - \tilde{u}_2)|_{x_3=0} = 0$, we have $u - \tilde{u} = a_3 e_3 + (b_1 e_1 + b_2 e_2) \times \sum_{i=1}^3 x_i e_i$ with $a_3, b_1, b_2 \in \mathbb{R}$. Now, it is easy to see that, choosing $r' \in \mathcal{R}_3$ as $r' = \sum_{i=1}^3 a_i' e_i + \sum_{i=1}^3 b_i' e_i \times \sum_{i=1}^3 x_i e_i$ with $a_3' = -a_3/2$, $b_1' = -b_1/2$, $b_2' = -b_2/2$, the solution $u + r'$ to (6.1)–(6.6) satisfies the condition $u + r' = \widetilde{u + r'}$ for all $a_1'$, $a_2'$, $b_3' \in \mathbb{R}$.

We next introduce the *thickness average* $\bar{f}$ of a function $f : \mathcal{C} \to \mathbb{R}^3$, $\bar{f} : U \to \mathbb{R}$, defined as

$$\bar{f}(x_1, x_2) = \frac{1}{h} \int_{-h/2}^{h/2} f(x_1, x_2, x_3)\, dx_3. \tag{6.12}$$

Taking into account that the thickness average of an $x_3$-odd function is zero, and the $x_3$-derivative of an $x_3$-even function is $x_3$-odd, for every point $(x_1, x_2) \in \mathcal{U}$ we have

$$\begin{cases} \overline{u}_3 = \overline{E}_{\alpha 3} = \overline{T}_{\alpha 3} = 0, \quad \alpha = 1, 2, & (6.13)\\ \overline{E}_{\alpha\beta} = \frac{1}{2}(\overline{u}_{\alpha,\beta} + \overline{u}_{\beta,\alpha}), \quad \alpha, \beta = 1, 2, & (6.14)\\ \overline{T}_{\alpha\beta} = 2\mu \overline{E}_{\alpha\beta} + \lambda(\overline{E}_{\gamma\gamma} + \overline{E}_{33})\delta_{\alpha\beta}, \quad \alpha, \beta = 1, 2, & (6.15)\\ \overline{T}_{33} = 2\mu \overline{E}_{33} + \lambda(\overline{E}_{\gamma\gamma} + \overline{E}_{33}), & (6.16) \end{cases}$$

where the solution $u + r'$ is denoted by $u$. Using the a priori assumption (6.8) in (6.16), the function $\overline{E}_{33}$ can be expressed in terms of $\overline{E}_{\gamma\gamma}$, and the *two-dimensional* constitutive equation can be written as

$$\overline{T}_{\alpha\beta} = 2\mu \overline{E}_{\alpha\beta} + \lambda^* \overline{E}_{\gamma\gamma}\delta_{\alpha\beta} \tag{6.17}$$

with

$$\lambda^* = \frac{2\mu\lambda}{\lambda + 2\mu}. \tag{6.18}$$

Integrating over the thickness in (6.1)–(6.6), and neglecting those equations which yield identities, we obtain the averaged equations of equilibrium and the corresponding boundary conditions, and $\overline{u} \in H^1(\mathcal{U}, \mathbb{R}^2)$ is a solution to

$$\begin{cases} \overline{T}_{\alpha\beta,\beta} = 0 & \text{in } \mathcal{U}, \quad (6.19)\\ \overline{T}_{\alpha\beta} n_\beta = \widehat{t}_\alpha & \text{on } \partial\mathcal{U}, \quad (6.20)\\ \overline{T}_{\alpha\beta} = 2\mu \overline{E}_{\alpha\beta} + \lambda^*(\overline{E}_{\gamma\gamma})\delta_{\alpha\beta} & \text{in } \mathcal{U}, \quad (6.21)\\ \overline{E}_{\alpha\beta} = \frac{1}{2}(\overline{u}_{\alpha,\beta} + \overline{u}_{\beta,\alpha}) & \text{in } \mathcal{U}, \quad (6.22) \end{cases}$$

where the force field $\widehat{t} = \widehat{t}_1 e_1 + \widehat{t}_2 e_2$ applied on $\partial\mathcal{U}$ satisfies the compatibility conditions

$$\int_{\partial\mathcal{U}} \widehat{t} = 0, \qquad \int_{\partial\mathcal{U}} x \times \widehat{t} = 0. \tag{6.23}$$

Notice that the constitutive equation (6.21) can be written as

$$\overline{T}_{\alpha\beta} = \frac{E}{1-\nu^2}\big((1-\nu)\overline{E}_{\alpha\beta} + \nu(\overline{E}_{\gamma\gamma})\delta_{\alpha\beta}\big) \tag{6.24}$$

with

$$\mu = \frac{E}{2(1+\nu)}, \qquad \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \tag{6.25}$$

where $E$, $\nu$ are the Young modulus and the Poisson coefficient of the material, respectively. Finally, by defining

$$a_\alpha = \overline{u}_\alpha, \quad \epsilon_{\alpha\beta} = \overline{E}_{\alpha\beta} = \widehat{\nabla} a, \quad N_{\alpha\beta} = h\overline{T}_{\alpha\beta}, \quad \widehat{N}_\alpha = h\widehat{t}_\alpha, \quad \alpha, \beta = 1, 2, \tag{6.26}$$

we obtain the GPS problem

$$\begin{cases} N_{\alpha\beta,\beta} = 0 & \text{in } \mathcal{U}, \quad (6.27)\\ N_{\alpha\beta} n_\beta = \widehat{N}_\alpha & \text{on } \partial\mathcal{U}, \quad (6.28)\\ N_{\alpha\beta} = \frac{Eh}{1-\nu^2}((1-\nu)\epsilon_{\alpha\beta} + \nu(\epsilon_{\gamma\gamma})\delta_{\alpha\beta}) & \text{in } \overline{\mathcal{U}}, \quad (6.29)\\ \epsilon_{\alpha\beta} = \frac{1}{2}(a_{\alpha,\beta} + a_{\beta,\alpha}) & \text{in } \mathcal{U}, \quad (6.30) \end{cases}$$

with

$$\int_{\partial\mathcal{U}} \widehat{N} \cdot r = 0 \quad \text{for every } r \in \mathcal{R}_2. \tag{6.31}$$

Now, the constitutive equation (6.29) can be written as

$$N_{\alpha\beta}(x) = C_{\alpha\beta\gamma\delta}(x)\epsilon_{\gamma\delta}, \tag{6.32}$$

where the fourth order tensor $\mathbb{C} = (C_{\alpha\beta\gamma\delta})$ is defined as

$$\mathbb{C}(x)A = \frac{Eh}{1-\nu^2(x)}\big((1-\nu(x))\widehat{A} + \nu(\mathrm{tr}(A))I_2\big) \tag{6.33}$$

for every $2 \times 2$ matrix $A$. With this notation, the GPS problem (6.27)–(6.30) can be rewritten in the form

$$\begin{cases} \mathrm{div}(\mathbb{C}\nabla a) = 0 & \text{in } \mathcal{U}, \quad (6.34)\\ (\mathbb{C}\nabla a)n = \widehat{N} & \text{on } \partial\mathcal{U}. \quad (6.35) \end{cases}$$

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