



# On the slope inequalities for extremal curves

Valentina Beorchia<sup>1</sup> · Michela Brundu<sup>1</sup>

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## Abstract

The present paper concerns the question of the violation of the  $r$ -th inequality for extremal curves in  $\mathbb{P}^r$ , posed in [KM]. We show that the answer is negative in many cases (Theorem 4.13 and Corollary 4.14). The result is obtained by a detailed analysis of the geometry of extremal curves and their canonical model. As a consequence, we show that particular curves on a Hirzebruch surface do not violate the slope inequalities in a certain range (Theorem 6.4).

**Keywords** Gonality sequence · Gonal scroll

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## 1 Introduction

Let  $X$  be a smooth and connected projective curve of genus  $g \geq 3$  defined over an algebraically closed field with characteristic zero. For each integer  $r \geq 1$ , H. Lange and P. E. Newstead in (Lange and Newstead 2010, Section 4) introduced the notion of

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✉ Valentina Beorchia  
beorchia@units.it

Michela Brundu  
brundu@units.it

<sup>1</sup> Dipartimento di Matematica e Geoscienze, Università di Trieste, Via Valerio 12/b, 34127 Trieste, Italy

$r$ -gonality  $d_r(X)$  of  $X$ , which is the minimal integer  $d$  such that there exists a linear series  $g_d^r$  (hence there is a rational map  $X \dashrightarrow \mathbb{P}^r$  of degree  $d$ ). In particular, for  $r = 1$  we obtain the classical gonality  $\gamma(X)$  of the curve  $X$ . The sequence  $\{d_r(X)\}_{r \geq 1}$  is called the gonality sequence of  $X$ .

For any curve and any  $r \geq g$ , the numbers  $d_r$  are known by the Riemann-Roch Theorem. Hence there are only finitely many interesting numbers in a gonality sequence and in Lange and Newstead (2010) it is evident that these numbers are deeply related to each other. In particular, in many cases they satisfy the  $r$ -th slope inequality, that is

$$\frac{d_r(X)}{r} \geq \frac{d_{r+1}(X)}{r + 1} \tag{1.1}$$

and this has been widely studied also in Lange and Martens (2012).

Observe that if  $X$  does not satisfy some slope inequality, then the corresponding Brill–Noether number is negative (see Remark 3.5). Consequently, a Brill–Noether general curve must satisfy all slope inequalities. The same occurs also for very special curves like hyperelliptic, trigonal and bielliptic curves (see Lange and Newstead 2010, Remark 4.5).

The gonality sequence of a curve  $X$  is related to Brill–Noether theory of vector bundles on  $X$  (see Lange and Newstead 2010). Moreover, if a curve  $X$  satisfies the  $p$ -th slope inequality for any  $p < n$ , then semistable vector bundles of rank  $n$  on  $X$  satisfy Mercat’s Conjecture, which governs the dimension of their spaces of global sections (cf. Lange and Newstead 2010, Conjecture 3.1 and Corollary 4.16).

These results motivate the discussion of slope inequalities for specific classes of curves.

Some sporadic examples of curves violating some slope inequality can be found in Ballico (2012), Ballico (2013), Lange and Martens (2012), Kato and Martens (2015), Cools et al. (2019) and various families have been detected in Lange and Martens (2012). Among such examples there are extremal curves, that is curves attaining the Castelnuovo bound for the genus. In Lange and Martens (2012), Theorem 4.13 the authors prove that extremal curves of degree  $\geq 3r - 1$  in  $\mathbb{P}^r$  do not satisfy all slope inequalities.

Moreover, in Kato and Martens (2015), Theorem 4.4 and Corollary 4.5 the authors prove that an extremal curve in  $\mathbb{P}^3$  of degree  $d \geq 10$  satisfies

$$\frac{d_3}{3} < \frac{d_4}{4}. \tag{1.2}$$

In the same paper (see the final Questions), the authors propose a new investigation in this direction; more precisely, they pose the following

**Question** - Is it true that extremal curves of degree  $d \geq 3r + 1$  in  $\mathbb{P}^r$  satisfy

$$\frac{d_r}{r} < \frac{d_{r+1}}{r + 1} \tag{1.3}$$

for  $r \geq 4$ ?

In the present paper we show that the answer to the above question is, in many cases, negative. The main results are the following:

**Theorem** *Let  $X$  be a  $\gamma$ -gonal extremal curve of degree  $d \geq 3r - 1$  in  $\mathbb{P}^r$  ( $r \geq 3$ ) where  $\gamma \geq 4$ . Then  $d_r = d$ . If, in addition,  $X$  is not isomorphic to a plane curve, then  $X$  satisfies the following conditions:*

- (i)  $d_{r+1} \leq d + \gamma - 1$ ;
- (ii) *under some technical hypotheses on the congruence class of  $d$ , the  $r$ -th slope inequality holds, i.e.*

$$\frac{d_r}{r} \geq \frac{d_{r+1}}{r + 1}.$$

For more details on the assumptions, see Theorem 4.13. As a consequence, the following result holds (see Corollary 4.14):

**Corollary** *Let  $X$  be a  $\gamma$ -gonal extremal curve in  $\mathbb{P}^r$ , where  $\gamma \geq 4$ . If the degree  $d$  satisfies*

$$r(\gamma - 1) \leq d \leq \gamma(r - 1) + 1 \tag{1.4}$$

*then the  $r$ -th slope inequality holds.*

The technique involved in the proofs relies on the fact that extremal curves are either isomorphic to a smooth plane curve or they lie on a Hirzebruch surface (see Theorem 4.1). Then, in order to bound the requested gonality number, we consider the residual divisor to a point in a general hyperplane section. It turns out that the dimension of the linear system associated with such a residual divisor can be estimated by looking at the canonical model of the curve.

The organization of the paper is as follows.

Section 2 contains some preliminaries about extremal curves and curves on Hirzebruch surfaces.

Section 3 summarizes results on the gonal sequence and the slope inequality which can be found in the literature.

Section 4 investigates the geometry and gonality of extremal curves. They are particular curves on a Hirzebruch surface, but not so rare. Indeed, thanks to Theorem 4.6, any smooth irreducible curve on a Hirzebruch surface, under a suitable assumption on its class, can be embedded in a projective space as an extremal curve. Next we state and prove the main results, that is Theorem 4.13 and Corollary 4.14.

In Sect. 5, we recall a result of Lange and Martens (2012), where the authors show that in degree  $d = 3r - 1$  the  $r$ -th slope inequality is violated by extremal curves and we prove, by an *ad hoc* argument, that the same holds in degree  $d = 3r - 2$  if  $r \geq 5$ .

Finally, in Sect. 6, we focus on fourgonal extremal curves, and we show that certain foursecant curves on a Hirzebruch surface admit several embeddings as an extremal curve in some projective space. This allows us to exhibit specific classes of extremal curves whose  $r$ -th slope inequality holds for  $r$  in a suitable interval (see Theorem 6.4).

## 2 Notation and preliminary notions

If  $x$  is a positive real number, by  $[x]$  we denote the *integer part* of  $x$ , i.e. the largest integer not exceeding  $x$ .

In this paper  $\mathbb{P}^n$  denotes the projective space over an algebraically closed field of characteristic zero. We shall also use the following notation:

- given a projective scheme  $Z \subseteq \mathbb{P}^n$ ,  $\langle Z \rangle$  will denote the linear span of  $Z$  in  $\mathbb{P}^n$ ;
- by a *curve*  $X$  we shall mean a smooth irreducible curve, unless otherwise specified;
- given a linear system  $|D|$  on a curve  $X$ , we will denote by  $\varphi_D$  the morphism associated with  $|D|$  and by  $X_D = \varphi_D(X)$  the image of  $X$  under  $\varphi_D$ ; in particular, if  $X$  is a non-hyperelliptic curve of genus  $g$ ,  $X_K \subset \mathbb{P}^{g-1}$  will denote the *canonical model* of  $X$ ;
- we say that a linear series  $g_d^r$  is *simple* if for any divisor  $D \in g_d^r$  the induced rational map  $\varphi_D$  is birational onto its image.

**Definition 2.1** If  $X$  and  $Y$  are two curves, a morphism  $\varphi : X \rightarrow Y$  is said *indecomposable* if it cannot be factorized as  $\varphi = \alpha \circ \beta$ , where  $\alpha$  and  $\beta$  are morphisms of degree bigger than one. In particular, if  $Y = \mathbb{P}^1$ , we say that a linear series  $|D| = g_d^1$  is *indecomposable* if the morphism  $\varphi_D$  associated with  $|D|$  is so.

**Definition 2.2** The *gonality* of a curve  $X$  is the minimum degree  $d$  of a linear series  $g_d^1$  on  $X$ ; if  $X$  has gonality  $\gamma$  then each series  $g_\gamma^1$  is called a *gonal series*. If  $\Gamma \in g_\gamma^1$  denotes a general *gonal divisor* of  $X$ , then the morphism  $\varphi_\Gamma : X \rightarrow \mathbb{P}^1$  is called a *gonal cover*.

Let  $X$  be a curve in  $\mathbb{P}^r$  of degree  $d \geq 2r + 1$  (the motivation of such a bound will be explained in the forthcoming Remark 4.2). Setting

$$m = m_{d,r} = \left\lceil \frac{d-1}{r-1} \right\rceil,$$

we can write, for a suitable integer  $\epsilon = \epsilon_{d,r}$ ,

$$d - 1 = m(r - 1) + \epsilon, \quad 0 \leq \epsilon \leq r - 2. \tag{2.1}$$

It is well-known that the genus  $g(X)$  of  $X$  satisfies the *Castelnuovo's bound* i.e.

$$g(X) \leq \pi(d, r) := m \left( \frac{m-1}{2}(r-1) + \epsilon \right). \tag{2.2}$$

Clearly, the values of  $m$  and  $\epsilon$  depend on  $d$  and  $r$ . So we introduce the following notation.

**Definition 2.3** Let  $X \subset \mathbb{P}^r$  a curve of degree  $d$ . The integer  $m = m_{d,r}$  in formula (2.1) will be called *m-ratio* of  $X$  in  $\mathbb{P}^r$ . Analogously, the integer  $\epsilon = \epsilon_{d,r}$  will be called *epsilon-remainder* of  $X$  in  $\mathbb{P}^r$ .

**Definition 2.4** A curve  $X$  is said an *extremal curve* in  $\mathbb{P}^r$  if it has a simple linear series  $g_d^r$  of degree  $d \geq 2r + 1$  and  $X$  has the maximal genus among all curves admitting such a linear series, i.e.

$$g(X) = \pi(d, r) = m_{d,r} \left( \frac{m_{d,r} - 1}{2} (r - 1) + \epsilon_{d,r} \right). \tag{2.3}$$

Observe that the notion above is *relative* to the space  $\mathbb{P}^r$  where the curve lies. Finally, let us recall a few notions about rational ruled surfaces.

**Definition 2.5** We denote by  $\mathbb{F}_n := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n))$  a *Hirzebruch surface* of invariant  $n$ , by  $C_0$  the unique (if  $n > 0$ ) unisecant curve with  $C_0^2 < 0$  and by  $L$  a line of its ruling.

It is well-known that  $\text{Pic}(\mathbb{F}_n) = \mathbb{Z}[C_0] \oplus \mathbb{Z}[L]$ , where  $C_0^2 = -n$ ,  $C_0 \cdot L = 1$  and  $L^2 = 0$ .

**Remark 2.6** If  $H = C_0 + \beta L$  is a very ample divisor on  $\mathbb{F}_n$ , then the associated morphism  $\varphi_H$  embeds  $\mathbb{F}_n$  in  $\mathbb{P}^r$  as a rational normal ruled surface  $R$  of degree  $r - 1$ , where  $r := H^2 + 1$ . With an easy computation, one can see that  $\beta = (r + n - 1)/2$ .

Finally, we recall that the canonical divisor of  $R$  is  $K \sim -2H + (r - 3)L$ .

In the sequel we will treat curves on (possibly embedded) Hirzebruch surfaces.

**Definition 2.7** If  $X \subset \mathbb{F}_n$  is a  $\gamma$ -gonal curve, we say that *the gonality of  $X$  is computed by the ruling of  $\mathbb{F}_n$*  if  $\gamma = X \cdot L$  (as far as  $n \geq 1$ ).

### 3 The slope inequalities

Let us recall a definition that generalizes the notion of the gonality of a curve.

**Definition 3.1** The  *$r$ -th gonality* of a curve  $X$  is defined to be

$$\begin{aligned} d_r &:= \min\{d \in \mathbb{N} \mid X \text{ admits a linear series } g_d^r\} \\ &= \min\{\text{deg } L \mid L \text{ line bundle on } X \text{ with } h^0(L) \geq r + 1\}. \end{aligned}$$

Moreover,  $(d_1, d_2, d_3, \dots, d_{g-1})$  is called the *gonality sequence* of  $X$ .

Recall that the gonality sequence is strictly increasing and weakly additive (see Lange and Martens 2012, Lemma 3.1).

**Lemma 3.2** *For any  $\gamma$ -gonal curve  $X$  of genus  $g$  and gonality sequence  $(d_1, d_2, \dots, d_{g-1})$ , the following properties hold:*

- $d_1 < d_2 < d_3 < \dots < d_{g-1}$ , where  $d_1 = \gamma$  and  $d_{g-1} = 2g - 2$ ;
- $d_{r+s} \leq d_r + d_s$  for all  $r, s$ ;
- $d_r \leq r\gamma$ , for all  $r$ .

**Remark 3.3** The gonality sequence is defined up to  $d_{g-1}$ , since  $d_r \geq 2g - 1$  for  $r \geq g$ . Therefore, by the Riemann-Roch Theorem,  $d_r = r + g$ , for all  $r \geq g$ , as observed in Lange and Newstead (2010), Remark 4.4 (b).

It is clear that, if the bound  $d_r \leq r\gamma$  is reached for all  $r$  up to a certain integer  $r_0$ , then the sequence

$$\left(\frac{d_r}{r}\right)_{r=1,\dots,r_0}$$

is constant and equal to the gonality  $\gamma$ . Otherwise, we can compare the above ratios: let us recall the following notion.

**Definition 3.4** The relation

$$\frac{d_r}{r} \geq \frac{d_{r+1}}{r+1} \tag{3.1}$$

is called the (*r*-th) *slope inequality*.

**Remark 3.5** Assume that, for some  $r$ , the slope inequality does not hold, i.e.

$$\frac{d_r}{r} < \frac{d_{r+1}}{r+1}$$

hence  $(r+1)d_r < rd_{r+1}$ . Using this inequality in the computation of the corresponding Brill–Noether number (see, for instance, Arbarello et al. 1985), we obtain that

$$\rho(d_r, r, g) = g - (r+1)(g - d_r + r) < r(d_{r+1} - g - r - 1) \leq 0.$$

Therefore  $\rho(d_r, r, g)$  is strictly negative.

**Definition 3.6** A curve  $X$  of genus  $g$  is called *Brill–Noether general* if  $\rho(d_r, r, g) \geq 0$ , for all  $1 \leq r \leq g - 1$ .

As a straightforward consequence, we have the following fact.

**Proposition 3.7** *Let  $X$  be a Brill–Noether general curve of genus  $g$  and gonality  $\gamma$ . Then*

$$\gamma = d_1 \geq \frac{d_2}{2} \geq \frac{d_3}{3} \geq \frac{d_4}{4} \geq \dots \geq \frac{d_r}{r} \geq \frac{d_{r+1}}{r+1} \dots \geq \frac{d_{g-1}}{g-1} = 2, \tag{3.2}$$

*i.e. all the slope inequalities hold.*

**Remark 3.8** Nevertheless, also “special” types of curves satisfy all the slope inequalities. For instance, in Lange and Newstead (2010), Remark 4.5, one can find the explicit values of the gonality sequence of a  $\gamma$ -gonal curve  $X$  in the following cases:

- if  $\gamma = 2$  ( $X$  hyperelliptic);
- if  $\gamma = 3$  ( $X$  trigonal);
- if  $\gamma = 4$  and  $X$  is bielliptic;
- if  $X$  is the general fourgonal curve.

In all the cases above, all the slope inequalities hold.

For this reason, from now on we will assume  $\gamma \geq 4$ .

### 4 Extremal curves and gonality

Let us first recall a result (see Arbarello et al. 1985, III, Theorem 2.5) that turns out to be important in the sequel since it describes the geometry of extremal curves. Let us keep the notation introduced in Section 2.

**Theorem 4.1** *Let  $d$  and  $r$  be integers such that  $r \geq 3$ ,  $d \geq 2r + 1$ . Then extremal curves  $X \subset \mathbb{P}^r$  of degree  $d$  exist and any such a curve is one of the following:*

- (i) *The image of a smooth plane curve of degree  $k$  under the Veronese map  $\mathbb{P}^2 \rightarrow \mathbb{P}^5$ . In this case  $r = 5$ ,  $d = 2k$ .*
- (ii) *A non-singular member of the linear system  $|mH + L|$  on a rational normal ruled surface. In this case  $\epsilon = 0$ .*
- (iii) *A non-singular member of the linear system  $|(m + 1)H - (r - \epsilon - 2)L|$  on a rational normal ruled surface.*

In particular, any irreducible extremal curve is smooth.

**Remark 4.2** Observe that we assumed from the beginning that  $d \geq 2r + 1$ . Namely, if  $d < 2r$ , by Clifford Theorem the  $g_d^r$  is non-special and we obtain  $g = d - r$ . In particular, we have  $r > g$ ; by Remark 3.3 the gonality sequence is known and the  $r$ -th slope inequality holds.

Moreover, if  $d = 2r$  then  $m = 2$  and  $\epsilon = 1$ . Therefore  $\pi(d, r) = r + 1$ . Hence, if  $X$  is an extremal curve, then  $d_r = d_{g-1} = 2g - 2 = 2r$ . By Remark 3.3, we have  $d_{r+1} = d_g = 2g = 2r + 2$ , hence the  $r$ -th slope inequality holds.

**Remark 4.3** In the sequel, we shall not consider the case (i) in the Theorem 4.1 where the extremal curve is the image of smooth plane curves under the Veronese map, being the gonality sequence of smooth plane curves completely understood by Max Noether’s Theorem (see, for instance, Ciliberto (1982), Theorem 3.14).

More precisely, a plane curve of degree  $k \geq 5$  satisfies

$$d_r = \begin{cases} \alpha d - \beta, & \text{if } r < g = \frac{(d-1)(d-2)}{2} \\ r + g, & \text{if } r \geq g, \end{cases}$$

where  $\alpha$  and  $\beta$  are the uniquely determined integers with  $\alpha \geq 1$  and  $0 \leq \beta \leq \alpha$  such that  $r = \frac{\alpha(\alpha+3)}{2}$ .

In particular, as observed in Lange and Martens (2012), Proposition 4.3, whenever  $\beta \neq 0$ , the  $r$ -th slope inequality is satisfied, while if  $\beta = 0$  and  $\alpha \leq k - 4$ , such an inequality is violated.

In the case (i) of Theorem 4.1, we have  $r = 5$ , so  $\alpha = 2$  and  $\beta = 0$ . It follows that if  $k \geq 6$ , the 5-th slope inequality is violated.

The converse of the cases (ii) and (iii) of Theorem 4.1 holds: namely the above classes of curves on a ruled surface force the curve to be extremal, as the following result shows.

**Proposition 4.4** Let  $X \subset R \subset \mathbb{P}^r$ , where  $X$  is a smooth curve of degree  $d$  and  $R$  is a rational normal ruled surface. Setting  $m = m_{d,r}$  and  $\epsilon = \epsilon_{d,r}$ , if

$$X \in \begin{cases} |mH + L| \\ \text{or} \\ |(m + 1)H - (r - \epsilon - 2)L| \end{cases}$$

then  $X$  is an extremal curve in  $\mathbb{P}^r$ .

**Proof** The canonical divisor  $K$  of the surface  $R$  can be written as  $K \sim -2H + (r - 3)L$ . Therefore we can apply the Adjuncton Formula on  $R$  (where  $g$  denotes the genus of  $X$ ):

$$2g - 2 = (K + X) \cdot X.$$

In the first case  $X \sim mH + L$  we then obtain

$$2g - 2 = ((m - 2)H + (r - 2)L) \cdot (mH + L).$$

Taking into account that  $H^2 = r - 1$ , we finally obtain

$$2g = m(m - 1)(r - 1).$$

On the other hand,  $d = \text{deg}(X) = X \cdot H = mH^2 + L \cdot H = m(r - 1) + 1$ , hence  $d - 1 = m(r - 1)$ . Therefore, from (2.1), we have that  $\epsilon = 0$  and, so,  $\pi(d, r) = g$  as requested (see (2.2)).

In the second case  $X \sim (m + 1)H - (r - \epsilon - 2)L$  so we get

$$2g - 2 = ((m - 1)H + (\epsilon - 1)L) \cdot ((m + 1)H - (r - \epsilon - 2)L).$$

It is immediate to see that

$$2g = m(m - 1)(r - 1) + 2m\epsilon,$$

so again  $g = \pi(d, r)$ . □

The two results above characterize embedded extremal curves in terms of rational ruled surfaces. Now we show that any smooth irreducible curve on a Hirzebruch surface, under a certain assumption on its class, can be embedded in a projective space as an extremal curve.

To do this, the following known result will be useful (see, for instance, Hartshorne 1977, Ch.V, Corollary 2.18).

**Proposition 4.5** Let  $D$  be the divisor  $aC_0 + bL$  on the rational ruled surface  $\mathbb{F}_n$ . Then:

(a)  $D$  is very ample  $\iff D$  is ample  $\iff a > 0$  and  $b > an$ ;



(b) the linear system  $|D|$  contains an irreducible smooth curve  $\iff$  it contains an irreducible curve  $\iff a = 0, b = 1$  or  $a = 1, b = 0$  or  $a > 0, b > 0$  or  $a > 0, b = an, n > 0$ .

The Hirzebruch surface  $\mathbb{F}_0$  is isomorphic to  $\mathbb{P}^1 \times \mathbb{P}^1$ , so  $\text{Pic}(\mathbb{F}_0)$  is generated by two lines belonging to distinct rulings. But also in this case, we denote these generators by  $C_0$  and  $L$  (like in the case of  $\mathbb{F}_n$ , where  $n > 0$ ), even if their roles can be exchanged.

**Theorem 4.6** *Let  $X \sim \gamma C_0 + \lambda L$  be an irreducible smooth curve on  $\mathbb{F}_n$  with  $\gamma \geq 3$  and not isomorphic to a plane curve.*

*Dividing  $\lambda - n - 1$  by  $\gamma - 2$ , let us denote by  $\beta$  the quotient and  $\epsilon$  the remainder, i.e.*

$$\beta := \left\lceil \frac{\lambda - n - 1}{\gamma - 2} \right\rceil = \frac{\lambda - n - 1 - \epsilon}{\gamma - 2}, \quad 0 \leq \epsilon \leq \gamma - 3.$$

Moreover, set

$$r := 2\beta + 1 - n \quad \text{and} \quad d := \gamma(\beta - n) + \lambda.$$

Consider the complete linear system  $|H|$  on  $\mathbb{F}_n$  given by

$$H \sim C_0 + \beta L.$$

Then

- (i)  $\beta > n$ , for all  $n \geq 0$ ;
- (ii) the morphism  $\varphi_H$  embeds  $\mathbb{F}_n$  in  $\mathbb{P}^r$ ;
- (iii)  $\varphi_H(X)$  is a curve of degree  $d$ .

Assume in addition that

$$\lambda \geq \frac{\gamma(\gamma + n - 2)}{2}.$$

Then

- (iv)  $m_{d,r} = \gamma - 1$  and  $\epsilon_{d,r} = \epsilon$ ;
- (v)  $\varphi_H(X)$  is an extremal curve in  $\mathbb{P}^r$ .

**Proof** Note first that, by assumption,  $X$  is irreducible and smooth. Then, by Proposition 4.5 and the assumption  $\gamma \geq 3$ , we have

$$\lambda > 0, \quad \text{if } n = 0$$

and

$$\lambda \geq \gamma n, \quad \text{if } n > 0 \tag{4.1}$$

(i) If  $n = 0$ , it is clear that  $\beta \geq 1 \iff \lambda - 1 \geq \gamma - 2$ . If this is not the case, we observe that we can change the role of  $\gamma$  and  $\lambda$ , since on  $\mathbb{F}_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$  we can choose arbitrarily one of the two rulings.

If  $n = 1$ , then  $\beta > 1$  if and only if  $\lambda - 2 > \gamma - 2$  and this holds by the assumption that  $X$  is not isomorphic to a plane curve.

Indeed, in general, any irreducible curve  $\gamma C_0 + \lambda L$  on  $\mathbb{F}_1$  satisfies  $\lambda \geq \gamma$  by (4.1). In particular, if  $\lambda = \gamma$ , we have  $\beta = 1$  and  $H = C_0 + L$ . On one hand, it is clear that the linear system  $|H|$  does not induce an embedding of  $\mathbb{F}_1$ , as it maps surjectively to  $\mathbb{P}^2$  and corresponds to the contraction of the exceptional curve  $C_0$ ; it is well known that this is the blowing up morphism  $\mathbb{F}_1 \rightarrow \mathbb{P}^2$  of the plane in a point.

On the other hand, under such a morphism, any smooth irreducible curve  $X \sim \gamma(C_0 + L)$  is mapped isomorphically to a smooth degree  $\gamma$  plane curve, which contradicts our assumption.

If  $n > 1$ , obviously  $\beta > n$  if and only if

$$\lambda - n - 1 > n(\gamma - 2).$$

i.e.  $\lambda > n(\gamma - 1) + 1$ . But this holds since  $n > 1$  implies  $n\gamma - n + 1 < n\gamma$  and, by (4.1),  $n\gamma \leq \lambda$ .

(ii) Again by Proposition 4.5, the linear system  $|H| = |C_0 + \beta L|$  is very ample if and only if  $\beta > 0$  if  $n = 0$  or  $\beta > n$  if  $n > 0$  and this holds by (i). Hence  $\varphi_H$  is an embedding.

Moreover, we have the well-known formula (see, for instance Brundu and Sacchiero (1998), Proposition 1.8 - (ii)):

$$r + 1 = h^0(\mathcal{O}_{\mathbb{F}_n}(C_0 + \beta L)) = 2\beta + 2 - n. \tag{4.2}$$

(iii) Therefore  $\varphi_H$  embeds  $X$  in  $\mathbb{P}^r$  as a curve of degree

$$(\gamma C_0 + \lambda L) \cdot (C_0 + \beta L) = \gamma(\beta - n) + \lambda = d,$$

as required.

(iv) In order to show that  $m_{d,r} = \gamma - 1$  and  $\epsilon_{d,r} = \epsilon$ , it is enough to prove that

$$d - 1 - (\gamma - 1)(r - 1) = \epsilon \quad \text{and} \quad 0 \leq \epsilon \leq r - 2.$$

Clearly

$$\begin{aligned} d - 1 - (\gamma - 1)(r - 1) &= \gamma(\beta - n) + \lambda - 1 - (\gamma - 1)(2\beta - n) \\ &= \lambda - n - 1 - \beta(\gamma - 2) \end{aligned}$$

and, substituting the value of  $\beta$ , we obtain the requested equality.

In order to show that  $\epsilon \leq r - 2$ , note first that, for any  $n \geq 0$ , we have

$$\epsilon \leq r - 2 \iff \lambda - n - 1 - \beta(\gamma - 2) \leq 2\beta - n - 1 \iff \lambda \leq \beta\gamma.$$

Since  $\epsilon \leq \gamma - 3$ , clearly

$$\beta = \frac{\lambda - n - 1 - \epsilon}{\gamma - 2} \geq \frac{\lambda - n + 2 - \gamma}{\gamma - 2},$$

so in order to show that  $\lambda \leq \beta\gamma$ , it is enough to prove that

$$\lambda \leq \gamma \frac{\lambda - n + 2 - \gamma}{\gamma - 2} \iff 2\lambda \geq \gamma(\gamma + n - 2)$$

and this holds by assumption.

(v) In order to prove that  $\varphi_H(X) \subset \mathbb{P}^r$  is an extremal curve, we compute the genus  $g$  of  $X$  using the Adjunction Formula obtaining

$$\begin{aligned} 2g - 2 &= (K_{\mathbb{P}^n} + X) \cdot X = (-2C_0 - (2 + n)L + \gamma C_0 + \lambda L) \cdot (\gamma C_0 + \lambda L) \\ &= 2(\lambda\gamma - \lambda - \gamma) - n\gamma(\gamma - 1), \end{aligned} \tag{4.3}$$

which yields

$$g = \lambda\gamma - \lambda - \gamma + 1 - \frac{n}{2}\gamma(\gamma - 1) = (\lambda - 1)(\gamma - 1) - \frac{n}{2}\gamma(\gamma - 1). \tag{4.4}$$

Now we can compute the Castelnuovo bound

$$\pi(d, r) = m_{d,r} \left( \frac{m_{d,r} - 1}{2}(r - 1) + \epsilon_{d,r} \right) = (\gamma - 1) \left( \frac{\gamma - 2}{2}(2\beta - n) + \epsilon \right).$$

Since

$$2\beta - n = 2 \frac{\lambda - n - 1 - \epsilon}{\gamma - 2} - n = \frac{2\lambda - 2 - 2\epsilon - n\gamma}{\gamma - 2}$$

we obtain

$$\pi(d, r) = (\gamma - 1) \left( \frac{1}{2}(2\lambda - 2 - 2\epsilon - n\gamma) + \epsilon \right) = (\gamma - 1)(\lambda - 1 - \epsilon - n\gamma/2 + \epsilon)$$

and, finally,

$$\pi(d, r) = (\gamma - 1)(\lambda - 1 - n\gamma/2) = (\gamma - 1)(\lambda - 1) - \frac{n}{2}\gamma(\gamma - 1).$$

Comparing this formula with (4.4), we see that  $\pi(d, r) = g$  and hence  $\varphi_H(X)$  is an extremal curve in  $\mathbb{P}^r$ . □

**Remark 4.7** By the irreducibility of  $X$ , we have  $\lambda \geq \gamma n$  from Proposition 4.5. As a consequence, the additional assumption in Theorem 4.6

$$\lambda \geq \frac{\gamma(\gamma + n - 2)}{2}.$$

holds if  $n \geq \gamma - 2$ .

In the sequel we will need to relate the gonality of  $X$  with its  $m$ -ratio. Since Theorem 4.1 claims that the extremal curves, not isomorphic to plane curves, lie on a rational normal ruled surface, we here recall the following result of Martens (see Martens (1996)) which describes such a relationship in the wider case of curves on ruled surfaces (see Definition 2.7).

**Theorem 4.8** *Let  $X$  be a reduced and irreducible curve on a Hirzebruch surface  $\mathbb{F}_n$  and assume that  $X$  is not a fibre. Then the gonality of  $X$  is computed by a ruling of the surface, unless  $n = 1$  and  $X \sim \alpha(C_0 + L)$  for some  $\alpha \geq 2$ , in which case  $X$  is isomorphic to a plane curve of degree  $\alpha$  and its gonality is  $\alpha - 1$ .*

**Remark 4.9** The exceptional case in Theorem 4.8 concerns curves of the type

$$X \sim \alpha(C_0 + L)$$

on  $\mathbb{F}_1$ . We observe that such a situation never occurs in the framework of extremal curves of type (ii) and (iii) in Theorem 4.1.

Note first that for a rational ruled surface  $R \subset \mathbb{P}^r$  of degree  $r - 1$  the hyperplane divisor  $H$  satisfies  $H \sim C_0 + \frac{r-1-C_0^2}{2}L$  by Remark 2.6. Hence  $R \cong \mathbb{F}_1$  if and only if

$$H \sim C_0 + \frac{r}{2}L.$$

It follows that, in case (ii), we have

$$X \sim mH + L = mC_0 + \left(\frac{mr}{2} + 1\right)L$$

and, so,  $X \sim \alpha C_0 + \alpha L$  if and only if  $\alpha = m = \frac{2}{2-r}$ , which is not possible for  $r \geq 2$ . In case (iii) we have

$$X \sim (m + 1)H - (r - 2 - \epsilon)L = (m + 1)C_0 + \left(\frac{(m + 1)r}{2} - r + 2 + \epsilon\right)L.$$

hence  $X \sim \alpha C_0 + \alpha L$  if and only if  $\alpha = m + 1$  and

$$m + 1 = \frac{(m + 1)r}{2} - r + 2 + \epsilon \Rightarrow \epsilon = (2 - r)(m - 1)/2.$$

But  $\epsilon \geq 0$  so we get a contradiction for  $r \geq 3$ .

The two results recalled above (Theorems 4.1 and 4.8) lead to the following consequence, whose formulas immediately come from (2.1) and (2.3).

**Corollary 4.10** *Let  $X$  be a  $\gamma$ -gonal extremal curve in  $\mathbb{P}^r$  (where  $r \geq 3$ ) of degree  $d$ , genus  $g$  and  $m$ -ratio  $m$ . If  $X$  is not isomorphic to a plane curve, then there exists a rational normal ruled surface  $R$  such that  $X \subset R \subset \mathbb{P}^r$  and, setting  $\text{Pic}(R) = \mathbb{Z}[H] \oplus \mathbb{Z}[L]$ , either:*

(i) if  $X \in |mH + L|$  on  $R$  then  $\epsilon = 0$  and  $m = \gamma$ . Consequently,

$$d = \gamma(r - 1) + 1 \tag{4.5}$$

$$g = \frac{\gamma(\gamma - 1)(r - 1)}{2}. \tag{4.6}$$

or

(ii) if  $X \in |(m + 1)H - (r - \epsilon - 2)L|$  on  $R$  then  $m = \gamma - 1$ . Consequently,

$$d = (\gamma - 1)(r - 1) + \epsilon + 1 \tag{4.7}$$

$$g = (\gamma - 1) \left[ \frac{\gamma - 2}{2} (r - 1) + \epsilon \right]; \tag{4.8}$$

In particular, the gonial series  $g_\gamma^1$  on  $X$  comes from the restriction of the fibration  $\pi : R \rightarrow \mathbb{P}^1$  given by the ruling.

**Remark 4.11** Assume  $\gamma \geq 4$  in Corollary 4.10 and consider the case (ii):  $m = \gamma - 1$ .

If  $\epsilon = 0$  and  $d \geq 3r - 1$  then, from (4.7), we have

$$d = (\gamma - 1)(r - 1) + 1 \geq 3r - 1 \implies (\gamma - 4)r \geq \gamma - 3 \implies \gamma \geq 5.$$

We shall need the following result (see Lange and Martens 2012, Theorem 4.13).

**Theorem 4.12** Let  $X$  be an extremal curve of degree  $d \geq 3r - 1$  in  $\mathbb{P}^r$ . Then  $d_{r-1} = d - 1$  and  $X$  does not satisfy all slope inequalities.

Now we can state the main result of this section, which gives a negative answer to the question posed by Kato and Martens in Kato and Martens (2015), i.e. if the  $r$ -th slope inequality is violated for extremal curves in  $\mathbb{P}^r$  for any  $r \geq 4$ .

**Theorem 4.13** Let  $X$  be a  $\gamma$ -gonal extremal curve of degree  $d \geq 3r - 1$  in  $\mathbb{P}^r$  ( $r \geq 3$ ) where  $\gamma \geq 4$ . Then  $d_r = d$ .

If, in addition,  $X$  is not isomorphic to a plane curve, then  $X$  satisfies the following conditions:

- (i)  $d_{r+1} \leq d + \gamma - 1$ ;
- (ii) by assuming one of the following sets of hypotheses:

- (a) either  $\epsilon = 0$ ,  $m = \gamma$  and  $r \geq \gamma - 1$ ,
- (b) or  $\epsilon \geq \gamma - 2$ ,

then the  $r$ -th slope inequality holds, i.e.

$$\frac{d_r}{r} \geq \frac{d_{r+1}}{r + 1}.$$

**Proof** From Theorem 4.12, we obtain  $d_{r-1} = d - 1$ . Since  $d_r > d_{r-1} = d - 1$ , we get that  $d_r \geq d$ . On the other hand  $X$  possesses a  $g_d^r$  by assumption, so  $d_r \leq d$ , hence the first statement is proved.

(i) Since  $X \subset \mathbb{P}^r$  is a curve of degree  $d$ , denoting by  $|H|$  the hyperplane linear system in  $\mathbb{P}^r$ , the induced linear system  $|H_X|$  on  $X$  is a linear series  $g_d^r$ .

From Corollary 4.10,  $X$  is a  $\gamma$ -secant curve on a rational ruled surface  $R$  and the gonal series  $g_\gamma^1$  on  $X$  comes from the restriction of  $\pi : R \rightarrow \mathbb{P}^1$ . So, for any  $P \in X$ , we set  $\Gamma_P$  to be the gonal divisor containing  $P$ , i.e.  $\Gamma_P = \pi^{-1}(\pi(P)) \cap X$ .

Also observe that the general hyperplane  $H$  cuts on  $R$  an irreducible unisecant curve, say  $U_H$ . In particular, the general hyperplane  $H$  does not contain any line of the ruling. Therefore for a general  $H_X \in g_d^r$  and for any  $P \in H_X$ , we have

$$H_X \cap \Gamma_P = H \cap X \cap \Gamma_P = (H \cap R) \cap X \cap \Gamma_P = U_H \cap X \cap \Gamma_P = \{P\}.$$

Setting  $\Gamma_P = P + Q_1 + \dots + Q_{\gamma-1}$ , let us consider the divisor obtained by adding to  $H_X$  the  $\gamma - 1$  further points of the gonal divisor, i.e.  $D = H_X + Q_1 + \dots + Q_{\gamma-1}$ ; we have

$$\deg D = \deg H_X + \gamma - 1 = d + \gamma - 1.$$

Now let us consider the canonical model  $X_K \subset \mathbb{P}^{g-1}$ ; here we can apply the Geometric Riemann-Roch Theorem to all the divisors  $\Gamma_P$ ,  $H_X$  and  $D$ . First we obtain

$$\dim \langle \Gamma_P \rangle = \deg \Gamma_P - h^0(\mathcal{O}(\Gamma_P)) = \gamma - 2$$

and

$$\dim \langle H_X \rangle = \deg H_X - h^0(\mathcal{O}(H_X)) = d - r - 1.$$

Consequently, since the intersection  $\langle \Gamma_P \rangle \cap \langle H_X \rangle$  contains  $P$ , we have

$$\dim \langle D \rangle \leq \dim \langle H_X \rangle + \gamma - 2 = d - r + \gamma - 3.$$

Hence, again from the Geometric Riemann-Roch Theorem, we get that

$$t + 1 := h^0(\mathcal{O}(D)) = \deg D - \dim \langle D \rangle \geq d + \gamma - 1 - (d - r + \gamma - 3) = r + 2.$$

Therefore there exists an integer  $t \geq r + 1$  such that  $|D|$  is of the form  $g_{d+\gamma-1}^t$ . This implies that  $d_t \leq d + \gamma - 1$ . From Lemma 3.2, we finally obtain that  $d_{r+1} \leq d_t \leq d + \gamma - 1$ .

(ii) From the fact that  $d_r = d$  and (i), it is enough to show that

$$\frac{d}{r} \geq \frac{d + \gamma - 1}{r + 1},$$

or, equivalently,

$$d \geq r(\gamma - 1). \tag{4.9}$$

(a) Assume  $\epsilon = 0, m = \gamma$  and  $r + 1 \geq \gamma$ . Then, using (4.5),

$$d = \gamma(r - 1) + 1 = r\gamma - \gamma + 1 \geq r\gamma - (r + 1) + 1 = r(\gamma - 1)$$

i.e. (4.9), as required.

(b) Assume  $\epsilon \geq \gamma - 2 \geq 1$ . Then we express the degree using (4.7), obtaining

$$d = (\gamma - 1)(r - 1) + \epsilon + 1 \geq (\gamma - 1)(r - 1) + \gamma - 1 = r(\gamma - 1)$$

so we get (4.9), as required. □

Let us observe that the condition that  $X$  is not isomorphic to a plane curve is necessary since, otherwise, the inequality (i) does not hold (see Remark 4.3) and, even assuming (b), the  $r$ -th slope inequality fails.

**Corollary 4.14** *Let  $X$  be a  $\gamma$ -gonal extremal curve in  $\mathbb{P}^r$ , where  $\gamma \geq 4$ . If the degree  $d$  satisfies*

$$r(\gamma - 1) \leq d \leq \gamma(r - 1) + 1 \tag{4.10}$$

*then the  $r$ -th slope inequality holds.*

**Proof** Let us note first that, under the assumption (4.10), the curve  $X$  cannot be isomorphic to a plane curve. If so, by Theorem 4.1, (i), we would have  $r = 5, d = 2k$  and  $\gamma = k - 1$  by Max Noether Theorem, contradicting (4.10).

Now observe that the assumption implies

$$r(\gamma - 1) \leq \gamma(r - 1) + 1 \implies r \geq \gamma - 1.$$

Since  $X$  is an extremal curve and  $X$  is not isomorphic to a plane curve, by Corollary 4.10, we have two possible cases.

In the first one,  $m = \gamma, \epsilon = 0$  and  $d = \gamma(r - 1) + 1$  from (4.5). Since  $r \geq \gamma - 1$ , by Theorem 4.13 we obtain that the  $r$ -th slope inequality holds.

In the second case,  $m = \gamma - 1$  and  $d = (\gamma - 1)(r - 1) + \epsilon + 1$  from (4.7). The assumption (4.10) yields

$$(\gamma - 1)(r - 1) + \epsilon + 1 \geq r(\gamma - 1) \implies \epsilon \geq \gamma - 2.$$

Again by Theorem 4.13, we obtain that the  $r$ -th slope inequality holds. □

The study above and, in particular, Corollary 4.14, can be summarized in the following table where we shall consider only curves which are not isomorphic to plane curves.

The first column concerns the increasing degree and the last one the  $r$ -th slope inequality.

In particular, the case  $d = 2r$ , here omitted, has been described in Remark 4.2.

The first four lines of the table will be considered in detail in the next section.

**Table 1** The  $r$ -th slope inequality for extremal curves of low degree

$d$	$\gamma$	$m$	$\epsilon$	$r$ -th slope
$2r + 1 \leq d \leq 3r - 3$	3	2	$2 \leq \epsilon \leq r - 2$	Yes (trigonal)
$3r - 2$	3	3	0	Yes (trigonal)
$3r - 2$	4	3	0	★
$3r - 1$	4	3	1	No
$3r \leq d \leq 4r - 4$	4	3	$2 \leq \epsilon \leq r - 2$	Yes
$4r - 3$	4	4	0	Yes
$4r - 3$	5	4	0	
$4r - 2$	5	4	1	
$4r - 1$	5	4	2	
$4r \leq d \leq 5r - 5$	5	4	$2 \leq \epsilon \leq r - 2$	Yes
$5r - 4$	5	5	0	Yes
$5r - 4$	6	5	0	
$5r - 3$	6	5	1	
$5r - 2$	6	5	2	
$5r - 1$	6	5	3	
$5r \leq d \leq 6r - 6$	6	5	$2 \leq \epsilon \leq r - 2$	Yes
$6r - 5$	6	6	0	Yes
...	...	...	...	...

### 5 Violating cases

This section concerns the first four lines of the Table 1. More precisely, extremal curves of degree in the initial range are forced to have certain gonality (see Proposition 5.1). Moreover, the  $r$ -th slope inequality is violated by extremal curves of degree  $d = 3r - 1$  (see Proposition 5.2). Finally, we will treat the case of curves of degree  $d = 3r - 2$  in Proposition 5.5 and Theorem 5.6.

**Proposition 5.1** *Let  $X \subset \mathbb{P}^r$  (where  $r \geq 3$ ) be an extremal curve of degree  $d$  and genus  $g$ , not isomorphic to a plane curve and such that*

$$2r + 1 \leq d \leq 3r - 1.$$

*Then:*

- (i) *if  $2r + 1 \leq d \leq 3r - 3$  then  $X$  is trigonal ( $m = 2, \epsilon \geq 2$ );*
- (ii) *if  $d = 3r - 2$  then  $X$  is either trigonal or fourgonal (where  $m = 3, \epsilon = 0$ );*
- (iii) *if  $d = 3r - 1$  then  $X$  is fourgonal (where  $m = 3, \epsilon = 1$ ).*

**Proof** Let us recall that  $d = m(r - 1) + \epsilon + 1$  where  $m = [(d - 1)/(r - 1)]$ .

(i) In this case

$$\frac{d - 1}{r - 1} \leq \frac{3r - 4}{r - 1} < 3$$



hence  $m = 2$ . So  $d = 2(r - 1) + \epsilon + 1$  and the bound  $d \geq 2r + 1$  implies  $\epsilon \geq 2$ . Therefore we are in the case of Corollary 4.10 - (ii), hence  $X$  admits a  $g_3^1$ , i.e. it is a trigonal curve.

(ii) In this case

$$\frac{d - 1}{r - 1} = \frac{3r - 3}{r - 1} = 3$$

hence  $m = 3$ . Therefore  $d = 3(r - 1) + \epsilon + 1 = 3r - 2$  and, so,  $\epsilon = 0$ . By Corollary 4.10 there are two possibilities: in case (i) we have  $\gamma = m = 3$  so the curve  $X$  is trigonal.

Otherwise, in case (ii) of Corollary 4.10, we have that  $\gamma = m + 1 = 4$  so  $X$  is fourgonal.

(iii) In this case

$$\frac{d - 1}{r - 1} = \frac{3r - 2}{r - 1}$$

hence  $m = 3$ . Therefore  $d = 3(r - 1) + \epsilon + 1 = 3r - 1$  and, so,  $\epsilon = 1$ . So the situation is described by Corollary 4.10 - (ii) and, in particular,  $X$  possesses a  $g_4^1$ .  $\square$

Concerning the slope inequalities, the cases described above behave as follows: as far as  $X$  is trigonal, all the slope inequalities are fulfilled (see Remark 3.8). The case  $d = 3r - 1$  is described by Lange–Martens (see Lange and Martens 2012, Corollary 4.6) as follows.

**Proposition 5.2** *For any  $r \geq 2$  and any extremal curve  $X$  of degree  $d = 3r - 1$  in  $\mathbb{P}^r$ , we have*

$$\frac{d_r}{r} < \frac{d_{r+1}}{r + 1}.$$

Finally, the case  $d = 3r - 2$  is studied in the next Theorem 5.6. Its proof will involve the following two results (see, respectively, Lange and Martens 2012, Proposition 4.10 and Lemma 4.8).

**Proposition 5.3** *Let  $g_\delta^s$  be a very ample linear series on  $X$  with  $\delta \geq 3s - 1$  and  $\epsilon_{\delta,s} \neq 0$ . If  $g > \pi(\delta, s) - m + 2$ , then  $2\delta \leq g + 3s - 2$ .*

**Lemma 5.4** *Let  $X$  be a curve admitting a  $g_\delta^s$  with  $\delta \geq 2s - 1 \geq 3$  such that*

- (i)  $d_{s-1} = \delta - 1$ , and
- (ii)  $2\delta \leq g + 3s - 2$ .

*Then  $d_s = \delta$  and the linear series  $g_\delta^s$  is complete and very ample. Moreover, if  $g_{\delta'}^{s'} = |K_X - g_\delta^s|$  (hence  $\delta' = 2g - 2 - \delta$  and  $s' = g - 1 - \delta + s$ ), we have  $s' \geq s$ ,  $d_{s'+1} \geq \delta' + 3$  and, so,*

$$\frac{d_{s'}}{s'} < \frac{d_{s'+1}}{s' + 1}.$$

By Proposition 5.1 extremal curves in  $\mathbb{P}^r$  of degree  $d = 3r - 2$  can be either trigonal or fourgonal. Since we know that trigonal curves satisfy all slope inequalities, we shall focus the fourgonal case. We shall see that for  $r = 4$  extremal curves of degree  $3r - 2 = 10$  satisfy all slope inequalities, while for  $r \geq 5$ , such curves violate the  $r$ -th slope inequality.

**Proposition 5.5** *Let  $X \subset \mathbb{P}^4$  be a fourgonal extremal curve of degree 10. Then  $X$  satisfies all slope inequalities.*

**Proof** Since  $X$  is extremal, we have  $g(X) = \pi(10, 4) = 9$ . The curves of genus  $g \leq 13$  violating some slope inequality have been classified in Kato and Martens (2015), Theorem 3.5 (i). In particular, in genus  $g = 9$  the only examples are the extremal curves of degree 8 in  $\mathbb{P}^3$ . The gonality sequence of such curves has been determined in Lange and Martens (2012), Example 4.7, and in particular it satisfies  $d_4 = 11$ . But in our case  $X$  possesses a  $g_{10}^4$ , so  $d_4 \leq 10$ . It follows that no slope inequality is violated.  $\square$

Now we turn to the case  $r \geq 5$ .

**Theorem 5.6** *Let  $X \subset \mathbb{P}^r$  with  $r \geq 5$  be a fourgonal extremal curve of degree  $d = 3r - 2$ . Then  $d_{r+1} = 3r + 1$ .*

*In particular,  $X$  violates the  $r$ -th slope inequality:*

$$\frac{d_r}{r} < \frac{d_{r+1}}{r + 1}.$$

**Proof** We note, first, that  $X$  is not isomorphic to a plane curve; indeed, in this case we would have  $r = 5$  and  $d = 3r - 2 = 13$ , while such curves lie on the Veronese surface and hence their degree is even.

Therefore  $X$  lies on a rational normal surface. As observed in Proposition 5.1,  $m_{d,r} = 3$ ,  $\epsilon_{d,r} = 0$ . Therefore by Theorem 4.1 and Corollary 4.10, the class of  $X$  on a ruled surface  $R \subset \mathbb{P}^r$  (of degree  $r - 1$ ) is given by

$$X \sim 4H - (r - 2)L \tag{5.1}$$

and the genus  $g$  of  $X$  turns out to be

$$g = \pi(3r - 2, r) = 3(r - 1). \tag{5.2}$$

It is not difficult to verify that the proof of Theorem 4.13-(i) applies also to our case, and since  $\gamma = 4$ , we have

$$d_{r+1} \leq d + 3 = 3r + 1.$$

To prove that  $d_{r+1} \geq 3r + 1$  for  $r \geq 5$ , we claim that  $X$  admits also an embedding in  $\mathbb{P}^{r-2}$  as an extremal curve. Keeping the notation of Remark 2.6, if  $H \sim C_0 + \beta L$ , then it is straightforward to see that  $\beta \geq -C_0^2$  and the equality holds if and only if  $R$  is a cone (see, for instance, Reid (1993), Theorem 2.5 and Remark (b)).

Since  $X$  is irreducible we necessarily have

$$0 \leq X \cdot C_0 = (4C_0 + (4\beta - r + 2)L) \cdot C_0 = 4C_0^2 + 4\beta - r + 2,$$

so that

$$\beta \geq -C_0^2 + \frac{r-2}{4} > -C_0^2,$$

hence

$$\beta \geq -C_0^2 + 1, \tag{5.3}$$

in particular  $R$  is not a cone. Therefore we can consider the projection of  $R$  from a line  $L$  of its ruling. Hence consider the divisor  $H' := H - L$  on  $R$ ; taking into account (5.3) we see that the linear system  $|H'|$  maps  $R$  in  $\mathbb{P}^{r-2}$  as a degree  $r - 3$  rational normal surface or a degree  $r - 3$  rational normal cone (see, for instance, Reid (1993), Theorem 2.5 and Remark (b)). Under such a morphism the image of  $X$  has degree

$$\begin{aligned} X \cdot H' &= (4H - (r - 2)L) \cdot (H - L) = 4H^2 - (r - 2) - 4 \\ &= 4(r - 1) - r - 2 = 3r - 6. \end{aligned}$$

As  $m_{3r-6,r-2} = \left\lceil \frac{3r-7}{r-3} \right\rceil = 3$  and  $\epsilon_{3r-6,r-2} = 2$ , the maximal genus is in this case

$$\pi(3r - 6, r - 2) = 3((r - 3) + 2) = 3(r - 1) = g(X),$$

which proves the claim.

The above construction provides a divisor  $H' \in g_\delta^s := g_{d-4}^{r-2}$ , so we can consider the birational morphism

$$\varphi_{H'} : X \rightarrow Y := \varphi_{H'}(X) \subset \mathbb{P}^s$$

where  $Y$  is an extremal curve of genus  $g = g(X) = 3r - 3$ ,

$$\deg(Y) = \delta = d - 4 = 3r - 6 \quad \text{and} \quad s = r - 2.$$

Consequently,  $Y \subset \mathbb{P}^s$  is smooth and  $g_\delta^s$  is very ample. Moreover, since  $r \geq 5$  then  $s \geq 3$ .

Furthermore, since  $Y$  is an extremal curve,  $\delta \geq 3s - 1$  and  $\epsilon_{\delta,s} = 2$ , we can apply both Theorem 4.12 and Proposition 5.3, obtaining, respectively, that  $d_{s-1} = \delta - 1$  and  $2\delta \leq g + 3s - 2$ .

Therefore all the assumptions of Lemma 5.4 are verified and from it we obtain that  $d_s = \delta$  and the Serre dual series  $g_{\delta'}^{s'} = |K_X - g_\delta^s|$  of  $g_\delta^s$  satisfies the following relation

$$d_{s'+1} \geq \delta' + 3,$$

where

$$\begin{aligned} \delta' &= 2g - 2 - \delta = 2(3r - 3) - 2 - (3r - 6) = 3r - 2, \\ s' &= g - 1 - \delta + s = 3r - 3 - 1 - (3r - 6) + r - 2 = r. \end{aligned}$$

Therefore the above inequality gives

$$d_{r+1} \geq 3r + 1,$$

as required.

Finally, since  $d_r \leq d$ , we have

$$\frac{d_r}{r} \leq \frac{d}{r} = \frac{3r - 2}{r} < \frac{3r + 1}{r + 1} = \frac{d_{r+1}}{r + 1}.$$

□

## 6 Fourgonal extremal curves

The setting of the current section is the following:  $X$  is a fourgonal extremal curve in  $\mathbb{P}^r$  (where  $r \geq 3$ ) of degree  $d$ , genus  $g$  and  $m$ -ratio  $m$ .

**Remark 6.1** Concerning space curves (i.e. the case  $r = 3$ ), let us consider the assumptions (a) and (b) of Theorem 4.13-(ii). Since  $0 \leq \epsilon \leq r - 2 = 1$ , then the only possibility is case (a), where  $\epsilon = 0$  and  $m = \gamma \leq r + 1 = 4$ . For this reason, regarding space curves, Theorem 4.13 describes only the fourgonal case. More precisely, if  $X$  is a fourgonal extremal curve of degree  $d$  in  $\mathbb{P}^3$  then it satisfies the following conditions:

- (i)  $d_3 = d$ ;
- (ii)  $d_4 \leq d + 3$ .
- (iii) If, in addition, we assume  $\epsilon = 0$  and  $m = 4$ , then the 3rd slope inequality holds, i.e.

$$\frac{d_3}{3} \geq \frac{d_4}{4}.$$

Note that this situation is quite specific. Namely, formulas (4.5) and (4.6) give  $d = 9$  and  $g = 12$ . So, comparing this fact with (1.2) (i.e. the violation of the third slope inequality), which holds for  $d \geq 10$ , we obtain that this bound on the degree given by Kato–Martens is sharp. Moreover, the above claims fit with the description of the gonal sequence given by the same authors in Kato and Martens (2015), Theorem 3.5.

It is easy to specialize the results of Section 4 to the following results in the fourgonal case.

**Corollary 6.2** *Let  $X$  be a fourgonal extremal curve of degree  $d \geq 3r - 1$  in  $\mathbb{P}^r$ , where  $r \geq 3$ . Then  $X$  satisfies the following conditions:*

- (i)  $d_r = d$ ;
- (ii)  $d_{r+1} \leq d + 3$ .
- (iii) If, in addition, we assume that  $d \geq 3r$ , then the  $r$ -th slope inequality holds, i.e.

$$\frac{d_r}{r} \geq \frac{d_{r+1}}{r+1}.$$

**Proof** Note first that  $X$  cannot be isomorphic to a plane curve (otherwise we would have  $r = 5$  and  $d = 2k \geq 14$ , so its gonality would be  $k - 1 \geq 6$ ). Therefore we can apply Theorem 4.13 and obtain (i) and (ii). Moreover,  $d \geq 3r$  and  $\gamma = 4$  imply that  $\epsilon \neq 1$  (see Table 1). Hence either assumption (a) or (b) in Theorem 4.13 hold.  $\square$

Moreover, a stronger form of Corollary 4.14 holds.

**Corollary 6.3** *Let  $X$  be an extremal curve of degree  $d$  in  $\mathbb{P}^r$ , where  $r \geq 3$ . Then the following conditions are equivalent:*

- (i) the degree satisfies  $3r \leq d \leq 4r - 4$ ;
- (ii)  $X$  is fourgonal and the  $r$ -th slope inequality holds.

**Proof** Immediate, from Table 1.  $\square$

As a consequence of the results of the previous sections, we are able to determine the gonal sequence of some extremal curve in a certain interval and to show that all the slope inequalities hold there. Indeed, on one hand, in Theorem 5.6 we have seen that, by projecting an extremal curve lying in  $\mathbb{P}^r$  from a gonal divisor, we may obtain an extremal curve in  $\mathbb{P}^{r-2}$ .

On the other hand, for an extremal curve in  $\mathbb{P}^r$ , the values of  $d_r$  and  $d_{r-1}$  are determined by Theorems 4.12 and 4.13.

**Theorem 6.4** *Let*

$$X \sim 4(C_0 + nL) \subset \mathbb{F}_n$$

*be a smooth irreducible curve, where  $n \geq 3$ . For any integer  $a$  such that*

$$0 \leq a \leq \left\lceil \frac{n-3}{2} \right\rceil, \tag{6.1}$$

*the gonal sequence of  $X$  satisfies*

$$d_{n+2a} = 4(n+a) - 1, \quad d_{n+2a+1} = 4(n+a). \tag{6.2}$$

*Moreover, the following bound holds*

$$d_{n+2\left\lceil \frac{n-3}{2} \right\rceil+2} \leq 4n + 4 \left\lceil \frac{n-3}{2} \right\rceil + 3. \tag{6.3}$$

*Consequently, for any  $n \leq r \leq n + 2 \left\lceil \frac{n-3}{2} \right\rceil + 1$ , the  $r$ -th slope inequality holds.*

**Proof** For any integer  $a$  as above, consider the divisor  $H_a$  on  $\mathbb{F}_n$  defined by

$$H_a = C_0 + (n + a)L$$

and set

$$r_a := n + 2a + 1.$$

By Proposition 4.5-(a) and (4.2), if  $a \neq 0$ , then the linear system  $|H_a|$  embeds  $\mathbb{F}_n$  isomorphically in  $\mathbb{P}^{r_a}$ .

If  $a = 0$ , then  $\varphi_{H_0}(\mathbb{F}_n)$  is a ruled surface of degree  $H_0^2 = n$  in  $\mathbb{P}^{n+1}$ . Moreover

$$\text{deg}(\varphi_{H_0}(C_0)) = C_0 \cdot H_0 = 0$$

so the unisecant  $C_0$  is contracted to a point. Hence  $\varphi_{H_0}(\mathbb{F}_n)$  is a rational normal cone and  $\varphi_{H_0}$  is an isomorphism between  $\mathbb{F}_n \setminus C_0$  and its image.

In both cases, the corresponding morphism  $\varphi_{H_a}$  is an isomorphism on  $X$ . Namely, if  $a \neq 0$  it is clear. If  $a = 0$ , it follows from  $X \cdot C_0 = 4(C_0 + nL) \cdot C_0 = 0$ , so  $X \subset (\mathbb{F}_n \setminus C_0)$ .

Therefore, for any  $a \geq 0$ , the curve  $X_a := \varphi_{H_a}(X) \subset \mathbb{P}^{r_a}$  has degree

$$\delta_a := X \cdot H_a = 4(C_0 + nL) \cdot (C_0 + (n + a)L) = 4(n + a).$$

We claim that  $X_a$  is an extremal curve in  $\mathbb{P}^{r_a}$ . In order to show this, we compute first the genus of  $X$ . As usual, the Adjunction formula gives

$$\begin{aligned} 2g - 2 &= (K_{\mathbb{F}_n} + X) \cdot X = (-2C_0 - (2 + n)L + 4C_0 + 4nL) \cdot (4C_0 + 4nL) \\ &= 12n - 8. \end{aligned}$$

Therefore we obtain

$$g = 6n - 3. \tag{6.4}$$

Now we compute the Castelnuovo bound of the genus. Taking into account the bound (6.1) of the integer  $a$ , we obtain that

$$m_{\delta_a, r_a} = \left\lceil \frac{4n + 4a - 1}{n + 2a} \right\rceil = 3, \quad \epsilon_{\delta_a, r_a} = n - 2a - 1$$

and

$$2 \leq \epsilon_{\delta_a, r_a} \leq n - 1.$$

An immediate computation of the Castelnuovo bound given in (2.2) shows that

$$\pi(\delta_a, r_a) = 3(n + 2a + \epsilon_{\delta_a, r_a}) = 3(2n - 1) = 6n - 3,$$

which coincides with the value of  $g$  determined in (6.4), so  $X_a \subset \mathbb{P}^{r_a}$  is an extremal curve.

In order to prove (6.2), note first that  $\epsilon_{\delta_a, r_a} \geq 2$  implies that  $n - 2a \geq 3$ ; therefore  $\delta_a \geq 3r_a$ . So we can apply the cited result of Lange–Martens (see Theorem 4.12) to the extremal curve  $X_a \subset \mathbb{P}^{r_a}$ , obtaining that

$$d_{n+2a} = \delta_a - 1 = 4(n + a) - 1.$$

Second, since  $X_a$  is a fourgonal extremal curve and  $r_a \geq 3$  (again from (6.1)), we can apply also Corollary 6.2-(i), obtaining that

$$d_{n+2a+1} = \delta_a = 4(n + a).$$

Therefore (6.2) is proved. As a consequence, for any  $0 \leq a \leq \lfloor \frac{n-3}{2} \rfloor$ , both the  $(r_a - 1)$ -th and  $r_a$ -th slope inequalities are satisfied.

Now consider the highest value  $\bar{a} = \lfloor \frac{n-3}{2} \rfloor$ . Also in this case  $X_{\bar{a}}$  is an extremal fourgonal curve of degree  $\delta_{\bar{a}}$  in  $\mathbb{P}^{r_{\bar{a}}}$ . Hence, by Corollary 6.2-(ii),

$$d_{n+2\bar{a}+2} \leq 4(n + \bar{a}) + 3$$

hence also (6.3) is proved.

Finally, as shown before,  $\delta_a \geq 3r_a$  and so Corollary 6.2-(iii) yields the  $r$ -th slope inequality in the considered range of  $r$ . □

**Remark 6.5** The gonial subsequence (6.2) and the bound (6.3) in Theorem 6.4 can be explicitly written in both the following cases, according to the parity of  $n$ .

If  $n$  is even, i.e.  $n = 2k$  for some  $k$ , then  $2 \lfloor \frac{n-3}{2} \rfloor = 2(k - 2) = n - 4$  so

$$\begin{aligned} d_n &= 4n - 1, \quad d_{n+1} = 4n, \quad d_{n+2} = 4n + 3, \quad d_{n+3} = 4n + 4, \quad \dots, \quad d_{2n-4} \\ &= 6n - 9, \quad d_{2n-3} = 6n - 8 \end{aligned}$$

and  $d_{2n-2} \leq 6n - 5$ .

If  $n$  is odd, i.e.  $n = 2k + 1$  for some  $k$ , then  $2 \lfloor \frac{n-3}{2} \rfloor = 2(k - 1) = n - 3$  so

$$\begin{aligned} d_n &= 4n - 1, \quad d_{n+1} = 4n, \quad d_{n+2} = 4n + 3, \quad d_{n+3} = 4n + 4, \quad \dots, \quad d_{2n-3} \\ &= 6n - 7, \quad d_{2n-2} = 6n - 6 \end{aligned}$$

and  $d_{2n-1} \leq 6n - 3$ .

Theorem 6.4 shows that, as far as  $n \geq 5$ , there are classes of extremal curves in  $\mathbb{P}^n$  which gonial sequence does not follow the pattern given in Kato and Martens (2015), Proposition 4.2, for extremal curves in  $\mathbb{P}^3$ .

Moreover, using the results in Brundu and Sacchiero (2014) and Beorchia and Sacchiero (2016), it is possible to verify that the curves of Theorem 6.4 are very special curves in the fourgonal moduli space. Therefore the problem of describing the locus of curves violating the slope inequalities is intriguing and deserves further investigations.

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