Mathematical Social Sciences 134 (2025) 31–41

Contents lists available at ScienceDirect

Mathematical Social Sciences

journal homepage: www.elsevier.com/locate/mss

Upper semicontinuous utilities for all upper semicontinuous total preorders

Gianni Bosi, Gabriele Sbaiz *

Department of Economics, Business, Mathematics and Statistics, University of Trieste, Via Valerio 4/1, 34127 Trieste, Italy

ARTICLE INFO

JEL classification:
C60

MSC:
primary 54F05
secondary 91B16
06A05

Keywords:
Super-short topology
Strongly separable topology
Isolated set
Lindelöf topology

ABSTRACT

Let X be an arbitrary nonempty set. Then a topology t on X is said to be *completely useful* (or *upper useful*) if every upper semicontinuous *total* preorder $\precsim$ on the topological space (X, t) can be represented by an upper semicontinuous real-valued order-preserving function (i.e., utility function). In this paper the structures of completely useful topologies on X will be deeply studied and clarified. In particular, completely useful topologies will be characterized through the new notions of super-short and strongly separable topologies. Further, the incorporation of the *Souslin Hypothesis* and the relevance of these characterizations in mathematical utility theory will be discussed. Finally, various interrelations between the concepts of complete usefulness and other topological concepts that are of interest not only in mathematical utility theory are analyzed.

1. Introduction

Let X be an arbitrary nonempty set. *Completely useful* topologies t on X, i.e. topologies for which every *upper semicontinuous* total preorder admits an upper semicontinuous utility representation, have not been systematically studied in the literature up to now.[1]

Bosi and Herden (2002) introduced this important concept and characterized in various ways completely useful topologies (see also Bosi et al., 2024).

Using the previous concept of a completely useful topology, Rader's Theorem (Rader, 1963), which is the most famous result on the upper semicontinuous representability of upper semicontinuous total preorders, can be stated as follows: *Every second countable topology t on X is completely useful*.

Since it has been shown by Bosi and Herden (2002) that completely useful topologies are *useful* (i.e., every continuous total preorder has a continuous utility representation), the well known Debreu utility representation theorem (see Debreu, 1954, 1964) can be stated as follows: *Every second countable topology t on X is useful*.

The notion of completely useful topologies is deeply relevant in Mathematical Economics and Mathematical Utility Theory, since it is intrinsically linked to the notions of maximal choice and rational choice in optimization.

As stated by Robbins in Robbins (2007): « There is a sense in which the word rationality can be used which renders it legitimate to argue that at least some rationality is assumed before human behavior has an economic aspect — the sense, namely, in which it is equivalent to *purposive* ».

In this context, the optimization of a single underlying preference ordering or utility function represents a first step in giving sense to the notion of *purposive behavior*. An extension of this framework may require the existence of multiple preference orderings such that an alternative, chosen from an option set, is obtained through the maximization of all the possible underlying preference orderings. If these orderings are individual preferences, the set of chosen options is represented by the set of Pareto-efficient alternatives. Otherwise, the underlying preference orderings may be considered as future potential preferences for a decision-maker. In this case, the set of chosen options consists of those alternatives that will never be rejected independently to the potential preference ordering may materialize in future.

The previous discussion motivates the importance of the maximal elements according to an underlying preference relation (which is *not* necessarily complete). In particular, if we consider compact sets and we look specifically for maximal elements, it suffices to assume upper semicontinuity of a preference relation, and consequently of its real representation. This consideration is valid also in the case when, more generally, a *nontotal* preorder is considered. Indeed, an upper semicontinuous order-preserving representation obtains its maximum at some point, which is, obviously, a maximal element (point of maximum in

* Corresponding author.
E-mail addresses: gianni.bosi@deams.units.it (G. Bosi), gabriele.sbaiz@deams.units.it (G. Sbaiz).

[1] It is worth noting that if any upper semicontinuous total preorder has an upper semicontinuous representation, then actually we also have that any lower semicontinuous total preorder has a lower semicontinuous utility representation

https://doi.org/10.1016/j.mathsocsci.2025.01.002
Received 29 September 2024; Received in revised form 18 December 2024; Accepted 9 January 2025
Available online 15 January 2025

the total case) for the original preference (see for instance Remark 3.2.8 in Bridges and Mehta (1995), and the introduction in Bosi and Herden (2002)).

The importance of these spaces is also recognized in other contexts, such as the welfare economics (see Pivato, 2023), in which the notions of utility and multi-utility are of particular interest (see Pivato, 2013).

In this paper we, therefore, focus our attention on these topologies, in order to characterize them in several and, hopefully, simple ways.

Completely useful and useful topologies

Recently, completely useful topologies have been referred to as *upper useful* topologies by Bosi and Franzoi (2023). Nevertheless, in the present paper we prefer to go back to the original terminology inaugurated by Bosi and Herden (2002).

The reader may recall that a preorder $\precsim$ on X is said to be *upper semicontinuous* if, for every point $x \in X$, the set $l_{\prec}(x) = \{z \in X : z \prec x\}$ is an open subset of X. By duality a lower semicontinuous preorder $\precsim$ on X is defined.

The most famous result concerning the upper semicontinuous representability of upper semicontinuous total preorders is the famous Rader's theorem (**RT**) (see Rader, 1963), according to which an upper semicontinuous total preorder $\precsim$ on a topological space (X, t) is representable by an upper semicontinuous *utility function*, provided that (X, t) is a *second countable space* (i.e., there exists a countable basis $\mathcal{B} = \{B_n\}_{n\in\mathbb{N}}$ of t).

Using the concept of a completely useful topology t on X, Rader's theorem can be restated as follows (see also Richter, 1980; Isler, 1997; Mehta, 1997):

RT: Every second countable topology t on X is completely useful.

Recently, Bosi and Franzoi (2023, Theorem 2.14) have generalized Rader's theorem to the case of nontotal preorders, by introducing a suitable continuity condition of a nontotal preorder.

In addition, upper semicontinuous preference relations and their representability by a upper semicontinuous numerical function are frequently discussed in the literature (see e.g. Rader, 1963; Jaffray, 1975b; Richter, 1980; Sondermann, 1980; Bridges and Mehta, 1995; Subiza and Peris, 1997; Alcantud and Gutiérrez, 1999; Alcantud, 1999; Droste, 1999 and many others). The particular relevance of upper semicontinuous preference relations is mainly based upon three aspects. First of all, it often suffices to only assume upper semicontinuity, see for example Alcantud (1999), where spaces are considered to satisfy a weaker property than compactness. Secondly, continuity often cannot be reached without adding artificial conditions. For example, Arrow and Hahn in Arrow and Hahn (1971) have introduced the condition of a preference relation to be *locally nonsatiated*, which is extensively discussed in Chapter 2 of Bridges and Mehta (1995) or in the negative result of Alcantud (1999). Finally, upper semicontinuity often appears in a natural way and, thus, it can be applied in order to construct, by adding more or less appealing conditions, continuous utility representations. We recall, for instance, the Arrow-Hahn approach (Arrow and Hahn, 1971) or, more generally, the Euclidean distance approach, which is thoroughly discussed in Bridges and Mehta (1995), and the approach by Sondermann (1980), generalizing Neuefeind's construction of continuous utility representations (Neuefeind, 1972). We also note that, in the economic literature, situations have arisen, where assuming continuity instead of semicontinuity is a serious restriction (see, e.g., Dutta and Mitra, 1989).

The main advantage of the following considerations is the clarification of the topological structure of the general upper semicontinuous utility representation problem. This means that our approach is within the main stream of results that clarify the general structure of the utility representation problem (see Eilenberg, 1941; Wold, 1943-1944; Birkhoff, 1948; Debreu, 1954, 1964; Fleischer, 1960, 1963; Jaffray, 1975a,b; Mehta, 1986, 1988; Herden, 1989a,b, 1995; Estévez and Hervés, 1995; Candeal et al., 1998; Herden and Pallack, 2000 and many others). Indeed, in the following sections we shall discuss necessary and sufficient conditions for a topology t on X to be completely useful. These conditions can be applied in any concrete situation. In fact, in case that a given consumption set is endowed with some topological structure (in practice this is nearly always the case), then one has to check if the given structure satisfies any of the assumptions of the characterization theorems, which will be proved in this paper, in order to guarantee upper semicontinuous and, thus, also continuous representability of an arbitrary upper semicontinuous, respectively continuous, preference relation on the consumption set. If none of the assumptions of these theorems are satisfied, then a given upper semicontinuous preference relation may be not upper semicontinuously representable, and one has to look for additional conditions, like countably boundedness or convexity, in order to guarantee upper semicontinuous representability (see Monteiro, 1987; Estévez et al., 1999 and Candeal et al., 1995).

Much before the consideration of completely useful topologies, in Herden (1991) a topology t on X is said to be *useful* if every continuous total (linear) preorder $\precsim$ on X can be represented by a continuous real-valued order-preserving function, i.e. it has a continuous utility representation. Continuity of $\precsim$ means that the order topology $t_{\precsim}$, which is induced by $\precsim$, is coarser than t. Sufficient conditions for a topology t on X to be useful are, for instance, given by the classical Eilenberg–Debreu theorems **EDT** and **DT** (Eilenberg (1941), Debreu (1954, 1964) and see also below). Necessary and sufficient conditions for a topology t on X to be useful have been presented by Estévez and Hervés (1995) in case that t is a metrizable topology on X and have been generalized, more recently, by Herden and Pallack (2000) to arbitrary topologies on X. By using the concept of a useful topology t on X, the Eilenberg–Debreu theorems and the theorem of Estévez and Hervés (**EHT**) can be restated as follows:

EDT: Every connected and separable topology t on X is useful.
DT: Every second countable topology t on X is useful.
EHT: A metrizable topology t on X is useful if and only if t is second countable.

More recent results concerning useful topologies have been presented by Bosi and Herden (2019) and Bosi and Zuanon (2021, 2022). In particular, the conditions presented by Bosi and Herden (2002) are based on the concept of a *complete separable system* on a topological space, while Bosi and Zuanon (2021, 2022) have considered, in particular, the case of useful topologies which are in addition *completely regular*.

It has been shown in Proposition 4.4 of Bosi and Herden (2002) that completely useful topologies t on X are useful. This result justifies the concept of a completely useful topology. In order to illustrate this central fact, consider that, if a topology t is completely useful, and we take a continuous total preorder $\precsim$ on the topological space (X, t), then such total preorder is, in particular, upper semicontinuous, and therefore it admits an upper semicontinuous utility function. Since the preorder is representable and continuous, then it admits a continuous utility function by well known facts, which are substantially related to the famous *Debreu Open Gap Lemma* (see, e.g., Theorem 3.1.3 in Bridges and Mehta, 1995).

Clearly, the upper semicontinuous analogue of **EHT** exists (see Corollary 4.5 in Bosi and Herden, 2002). On the other hand, useful topologies t on X are not necessarily completely useful. Indeed, on every sufficiently large set, the cardinality of the set of all useful topologies t on X coincides with the cardinality of the power set of the set of all completely useful topologies t on X (see Theorem 5.1 in Bosi and Herden, 2002). This result demonstrates in a very strong way up to which degree the concept of a completely useful topology t on X strengthens the concept of a useful topology t on X.

Moreover, considering the theorems **EDT** and **DT** only the upper semicontinuous analogue of **DT** has been proved for the first time by Rader (1963).

It is worth highlighting that the concepts of useful and completely useful topologies play a significant role also in the general equilibrium theory within infinite-dimensional Banach spaces endowed with the weak topology or weak-$*$ topology (see e.g. Campión et al., 2006a,b, 2012).

More recently, Bosi and Zuanon (2020) and Bosi (2023) have introduced and have studied the concept of a *strongly useful* topology, in order to generalize the notion of a useful topology to the case of nontotal preorders. To this aim, these authors have referred to the definition of a *weakly continuous preorder*, which generalizes that of a continuous preorder. It is worth noting that strongly useful topologies are useful, since the concepts of a weakly continuous preorder and of a continuous preorder, respectively, coincide in the case when the preorder is total.

Motivated by the previous discussion, in this paper we are primarily concerned with the problem of characterizing in a simple way completely useful topologies.

Structure of the paper

The paper is structured as follows. Section 2 contains the main definitions and the preliminary results. In particular, the concept of a *super-short topology* is introduced, which indeed characterizes completely useful topologies. In Section 3 we present the characterization theorems, referring to the classical framework of set theory (i.e., **Z**ermelo-**F**raenkel + Axiom of **C**hoice axioms). In Section 4, we include the *Souslin Hypothesis* (**SH**), which have been already introduced by Vohra (1995) in a very interesting paper. However, the Souslin Hypothesis was already considered by Bosi and Herden (2002), in connection with completely useful topologies. Section 5 examines the relation between completely useful and strongly useful topologies. Indeed, it is shown that a completely useful topology is necessarily strongly useful. It is further proved that a topology is completely useful if and only if every upper semicontinuous and not necessarily total preorder admits an upper semicontinuous order-preserving function. Section 6 concludes summarizing the results obtained in the paper and highlighting some future directions.

2. Notation and preliminaries

We start this section by introducing the main notation and definitions that we shall employ throughout the paper. For further information on basic definitions in General Topology we address the interested reader to Dugundji (1978).

A preorder $\precsim$ on a set X is a *reflexive* and *transitive* binary relation on X. The *asymmetric part* (or *strict part*) $\prec$ and the *symmetric part* $\sim$ of a given preorder $\precsim$ on X are defined as usual, i.e., for all $x, y \in X$,

$x \prec y \Longleftrightarrow (x \precsim y)$ and $not(y \precsim x)$

and

$x \sim y \Longleftrightarrow (x \precsim y)$ and $(y \precsim x)$.

The relation $\sim$ is an equivalence, and we denote by $\precsim_{|\sim}$ the *quotient order* defined on the *quotient space* $X_{|\sim}$ as follows, for all $[x], [y] \in X_{|\sim}$:

$[x] \precsim_{|\sim} [y] \Longleftrightarrow x \precsim y.$

A preorder $\precsim$ on X is said to be *total* if, for all $x, y \in X$, either $x \precsim y$ or $y \precsim x$. A total preorder is said to be a *chain* if it is *antisymmetric* (i.e., for all $x, y \in X$, $x \precsim y$ and $y \precsim x$ imply that $x = y$). A *jump* in a preordered set $(X, \precsim)$ is a pair $(x, y) \in \prec$ such that for no $z \in X$ it happens that $x \prec z \prec y$ (i.e., the *order interval* $]x, y[$ is empty).

Given a totally preordered set $(X, \precsim)$, the *order topology* $t_{\precsim}$ is the topology on X whose subbasis consists of all sets of the form $l_{\prec}(x) = \{z \in X : z \prec x\}$ and $r_{\prec}(x) = \{z \in X : x \prec z\}$ where $x \in X$. The *upper order topology* $t^u_{\precsim}$ is the topology on X whose (sub)basis consists of all sets of the form $l_{\prec}(x) = \{z \in X : z \prec x\}$. Notice that the family $\{l_{\prec}(x) : x \in X\}$ is *linearly ordered by set inclusion* (i.e., a *chain*) when $\precsim$ is a total preorder on X (as well as the family $\{r_{\prec}(x) : x \in X\}$).

A subset D of a *preordered set* $(X, \precsim)$ is said to be *decreasing* if

$(x \in D)$ and $(z \precsim x) \Longrightarrow z \in D$, for all $z \in X$.

Moreover, a topology t on X is said to be:

1. *Hausdorff* if, given any two points $x, y \in X$ with $x \neq y$, there exist two open disjoint sets U, V such that $x \in U$ and $y \in V$;
2. T_1, if all singleton sets $\{x\}$ are closed;
3. *completely regular*, if it is T_1 and, for every $x \in X$, and every closed set $F \subseteq X$ not containing x, there exists a continuous function $f : (X, t) \to ([0, 1], t_{nat})$ such that $f(x) = 0$ and $f(y) = 1$ for $y \in F$;
4. *second countable*, if there exists a countable *basis* $\mathcal{B} = \{B_n\}_{n \in \mathbb{N}}$ for t;
5. *separable*, if there exists a countable subset $\mathcal{D} = \{d_n\}_{n \in \mathbb{N}}$ of X such that $O \cap \mathcal{D} \neq \emptyset$ for all $O \in t$ with $O \neq \emptyset$.

Given a preordered set $(X, \precsim)$, a function $f : (X, \precsim) \to (\mathbb{R}, \leq)$ is said to be *order-preserving* if it is *increasing* (i.e., for all $x, y \in X$, $x \precsim y$ implies that $f(x) \leq f(y)$) and, in addition, for all $x, y \in X$, $x \prec y$ implies that $f(x) < f(y)$. Clearly, in the particular case when the preorder $\precsim$ is total, a real-valued function f on $(X, \precsim)$ is order-preserving if and only if, for all $x, y \in X$, $x \precsim y$ is equivalent to $f(x) \leq f(y)$, and in this case it is said to be a *utility function*. A preorder $\precsim$ on a set X is said to be *Debreu separable* if there exists a countable subset $\mathcal{D}$ of X such that, for all $(x, y) \in \prec$, there exists $d \in \mathcal{D}$ such that $x \precsim d \precsim y$.

It is well known that, in the case of a total preorder, Debreu separability is a necessary and sufficient condition for the existence of a utility function.

The following *Representation Lemma*, which is an immediate consequence of Proposition 1.6.11 and Theorem 3.2.2 by Bridges and Mehta (1995) and Lemma 3.2 by Herden (1989a), characterizes the existence of an upper semicontinuous utility function for an upper semicontinuous total preorder. We shall denote by t_{nat} the *natural topology* (i.e., *interval topology*) on the real line $\mathbb{R}$.

Theorem 2.1 (*Representation Lemma*). *Let $(X, t, \precsim)$ be a topological totally preordered space. Then the following properties are equivalent:*

(1) There exists an upper semicontinuous utility function $f : (X, t, \precsim) \to (\mathbb{R}, t_{nat}, \leq)$.

(2) The following conditions are verified:

- *(2a) The total preorder $\precsim$ is upper semicontinuous;*
- *(2b) There exists a utility function $f' : (X, \precsim) \to (\mathbb{R}, \leq)$.*

(3) The following conditions hold:

- *(3a) The total preorder $\precsim$ is upper semicontinuous;*
- *(3b) The order topology $t_{\precsim}$ is second countable.*

(4) The following conditions are verified:

- *(4a) The total preorder $\precsim$ is upper semicontinuous;*
- *(4b) The order topology $t_{\precsim}$ is separable;*
- *(4c) There are only countable many jumps in $(X_{\precsim|\sim}, \precsim_{|\sim})$.*

(5) The following conditions hold:

- *(5a) The total preorder $\precsim$ is upper semicontinuous;*
- *(5b) The total preorder $\precsim$ is Debreu separable.*

A real valued function f from a preordered set $(X, \precsim)$ to a preordered set $(X', \precsim')$ is said to be an *order embedding* of $(X, \precsim)$ into $(X', \precsim')$ if, for all $x, y \in X$, $x \precsim y$ is equivalent to $f(x) \precsim' f(y)$. Further, a real-valued function f on X is said to be an *order isomorphism* of a preordered set $(X, \precsim)$ into a preordered set $(X', \precsim')$ if f is a bijective order embedding of $(X, \precsim)$ into $(X', \precsim')$. In this case, the preordered sets $(X, \precsim)$ and $(X', \precsim')$ are said to be *order isomorphic*.

A *well-order* $\precsim$ on a set X is a total preorder on X such that every nonempty set $X' \subset X$ has a least element (minimum). It is well known that every *well-ordered set* $(X, \precsim)$ is order-isomorphic to a uniquely determined *ordinal* α, which is its *order type*.

A *tree* is a preordered set $(T \precsim)$ such that, for every element $t \in T$, the set $l_{\prec}(t) = \{s \in T : s \prec t\}$ is well-ordered by $\precsim$. For every $t \in T$, the order type of $l_{\prec}(t) = \{s \in T : s \prec t\}$ is said to be the *level* of t.

A preorder $\precsim$ on a topological space (X, t) is said to be *upper semicontinuous* if, for every point $x \in X$, the set $l_{\prec}(x) = \{y \in X : y \prec x\}$ is an open subset of X. By duality, a *lower semicontinuous* preorder $\precsim$ on X is defined.

We recall that a real-valued function f on a topological space (X, t) is said to be *upper semicontinuous* if, for every number $\alpha \in \mathbb{R}$, $f^{-1}(]-\infty, \alpha[) = \{z \in X : f(z) < \alpha\}$ is open.

If (X, t) is a topological space, then denote by $\overline{A}$ the topological closure of any subset A of X. Further, $\mathcal{O}$ will stand for the collection of all subchains (i.e., subsets which are totally ordered by set inclusion) $(\mathbf{O}, \subset)$ of $(t, \subset)$. Without loss of generality, we shall assume that every subchain $(\mathbf{O}, \subset) \in \mathcal{O}$ contains X.

At this point, we introduce the following preorder $\precsim_{\mathcal{O}}$ on $\mathcal{O}$.

Definition 2.2. For any two chains $(\mathbf{O}, \subset) \in \mathcal{O}$, $(\mathbf{U}, \subset) \in \mathcal{O}$:

$$(\mathbf{O}, \subset) \precsim_{\mathcal{O}} (\mathbf{U}, \subset) \Longleftrightarrow \forall O \in \mathbf{O}, \exists \mathbf{T} \subset \mathbf{U} : \left(O = \bigcup_{U \in \mathbf{T}} U \ \vee \ O = \bigcap_{U \in \mathbf{T}} U \right).$$

Remark 2.3. Observe that $(\mathbf{O}, \subset) \precsim_{\mathcal{O}} (\mathbf{U}, \subset)$ means that the chain $(\mathbf{O}, \subset)$ can be *rebuilt* starting from the chain $(\mathbf{U}, \subset)$. If further $(\mathbf{O}, \subset) \prec_{\mathcal{O}} (\mathbf{U}, \subset)$, then there exists a set $U \in \mathbf{U}$ such that

$$\bigcup_{\mathbf{O} \ni O' \subsetneq U} O' \subsetneq U \subsetneq \bigcap_{U \subsetneq O'' \in \mathbf{O}} O''.$$

Let us define the total preorder naturally associated to a chain of open sets.

Definition 2.4. Let $(\mathbf{O}, \subset) \in \mathcal{O}$ be an arbitrary chain. Then the total preorder $\precsim_{\mathbf{O}}$ on X which is naturally associated to $\mathbf{O}$ is defined by

$$x \precsim_{\mathbf{O}} y \Longleftrightarrow \forall O \in \mathbf{O}, (y \in O \Longrightarrow x \in O).$$

Remark 2.5. It is immediate to check that, for every chain $(\mathbf{O}, \subset) \in \mathcal{O}$, the total preorder $\precsim_{\mathbf{O}}$ is upper semicontinuous on (X, t).

The following fundamental definition was first introduced by Bosi and Herden (2002).

Definition 2.6. Let $(\mathbf{O}, \subset) \in \mathcal{O}$ be an arbitrary chain. Then an open set $O \in \mathbf{O}$ is said to be an *isolated set* (of $\mathbf{O}$) if

$$\bigcup_{\mathbf{O} \ni O' \subsetneq O} O' \subsetneq O \subsetneq \bigcap_{O \subsetneq O'' \in \mathbf{O}} O''.$$

Remark 2.7. It is easy to see that, if $(\mathbf{O}, \subset) \in \mathcal{O}$ is an isolated set of $\mathbf{O}$, then every pair

$$(x, y) \in \left(O \setminus \bigcup_{\mathbf{O} \ni O' \subsetneq O} O' \right) \times \left(\bigcap_{O \subsetneq O'' \in \mathbf{O}} O'' \setminus O \right)$$

represents a jump in $(X_{|\sim_{\mathbf{O}}}, \precsim_{|\sim_{\mathbf{O}}})$.

Now, we are ready to present the new definition of a *weakly isolated set*.

Definition 2.8. Let $(\mathbf{O}, \subset) \in \mathcal{O}$ be an arbitrary chain. Then an isolated set O (of $\mathbf{O}$) is said to be a *weakly isolated* set (of $\mathbf{O}$) if

$$\overline{\bigcup_{\mathbf{O} \ni O' \subsetneq O} O'} = \overline{O} \text{ and } \overline{X \setminus \bigcap_{O \subsetneq O'' \in \mathbf{O}} O''} = X \setminus O.$$

Definition 2.9. A topological space (X, t) is said to be *strongly separable* if for every chain $(\mathbf{O}, \subset) \in \mathcal{O}$ there exists a countable subset $Y_{\mathbf{O}}$ of X such that the following two statements hold:

- $\left(Y_{\mathbf{O}} \cap O\right) \setminus O' \neq \emptyset$ for every pair $(O', O) \in \mathbf{O} \times \mathbf{O}$ such that there exists $O'' \in \mathbf{O}$ with $O' \subsetneq O'' \subsetneq O$;
- $\left(Y_{\mathbf{O}} \cap \bigcap_{O'' \subsetneq O \in \mathbf{O}} O'' \right) \setminus \bigcup_{\mathbf{O} \ni O' \subsetneq O} O' \neq \emptyset$ for every isolated set O (of $\mathbf{O}$).

The following definition was presented by Herden and Pallack (2001).

Definition 2.10. Let $(Z, \leq)$ be an arbitrary chain. Then the *length* $l(Z)$ of Z is the least upper bound of all cardinal numbers κ that can be order-embedded into $(Z, \leq)$ or $(Z, \geq)$. Moreover, a chain $(Z, \leq)$ is said to be *lengthy* if $l(Z) = |Z| = card(Z)$.

We recall at this moment the notions of short, hereditarily Lindelöf and hereditarily separable spaces.

Definition 2.11. A topological space (X, t) is said to be *short* if there exists no uncountable ordinal number α such that $(\alpha, \leq)$ can be order-embedded into $(t, \subset)$ or $(t, \supset)$.

Remark 2.12. In Proposition 4.2 of Bosi and Herden (2002), it has been shown that (X, t) is short if and only if (X, t) is a *hereditarily Lindelöf* and *hereditarily separable* space. The reader may recall that (X, t) is a hereditarily Lindelöf space if, for every subset A of X, the subspace $\left(A, t_{|A}\right)$ ($t_{|A}$ is the relativized topology on A) is a Lindelöf space, and that (X, t) is a hereditarily separable space if, for every subset A of X, the subspace $\left(A, t_{|A}\right)$ is a separable space.

Definition 2.13. A topological space (X, t) is said to be *strongly hereditarily Lindelöf* if, for every chain $(\mathbf{O}, \subset) \in \mathcal{O}$, there exists a countable subchain $(\mathbf{U}, \subset)$ of $(\mathbf{O}, \subset)$ such that for every $O \in \mathbf{O}$ there exists a subset $\mathbf{V}$ of $\mathbf{U}$ such that $O = \bigcup_{V \in \mathbf{V}} V$.

It is immediate to check that a topological space (X, t) is strongly hereditarily Lindelöf provided that the topology $t_{\mathbf{O}}$ generated by every chain $\mathbf{O} \in \mathcal{O}$ is second countable.

Moreover, it is worth recalling that two open sets $O, O' \in t$ are said to be *equivalent*, which will be abbreviated by writing $O \sim O'$, if $\overline{O} = \overline{O'}$. The corresponding equivalence classes are denoted as usual by $[O]$ and $\left[O'\right]$ respectively.

We now introduce a *key* definition of this paper.

Definition 2.14. Let X be a given nonempty set. Let t be an arbitrary topology on X.

(i) (X, t) is said to be *super-short* if every well-ordered subchain $\left(\mathcal{S}, \precsim_{\mathcal{O}}\right)$ of $\left(\mathcal{O}, \precsim_{\mathcal{O}}\right)$ is countable.
(ii) (X, t) is said to be *locally super-short* if for every open set $O \in t$, the induced topological space $\left([O], t_{\big|[O]} \right)$ is super-short.

Proposition 2.15. *If t is a super-short topology on X, then t is short.*

Proof. By contraposition, assume that the topology t on X is not short. Then there exists, for example, an uncountable ordinal number α which can be order embedded in $(t, \subset)$. Therefore, there exists an order embedding $\beta \to O_\beta$ where $O_\beta \in t$ and $\beta < \alpha$. Define, for every $\beta < \alpha$, the subchain $(\mathbf{O}_\beta, \subset) = (\{O_{\beta'} : \beta' \leq \beta\}, \subset)$ of $\mathcal{O}$. It is immediate to check that $(\mathbf{O}_\beta, \subset) \prec_{\mathcal{O}} (\mathbf{O}_{\beta''}, \subset)$, for all $\beta < \beta'' < \alpha$. Hence, we have that $(\{\mathbf{O}_\beta : \beta < \alpha\}, \precsim_{\mathcal{O}})$ is an uncountable well ordered subchain of $(\mathcal{O}, \precsim_{\mathcal{O}})$, and the proof is complete. □

The vice versa of Proposition 2.15 is not always true. Indeed, it will follow, from the results we will prove in Section 4, that there exist short topologies t on X that are not super-short. Therefore, these considerations justify the concept of a super-short topology t on X.

Definition 2.16. A topology t on a set X is said to satisfy the *countable chain condition* (*ccc*) if every family of disjoint open sets is countable. A chain $(X, \precsim)$ is said to satisfy the *countable chain condition* (*ccc*) if the corresponding order topology $t_{\precsim}$ satisfies *ccc*.

It is clear that a separable topology t satisfies *ccc*, as well as a hereditarily Lindelöf topology.

3. The structure of completely useful topologies

We start by recalling the definition of a completely useful topology.

Definition 3.1. Let X be an arbitrary nonempty set. Then a topology t on X is said to be *completely useful* (or *upper useful*) if every upper semicontinuous total preorder $\precsim$ on X can be represented by an upper semicontinuous real-valued order-preserving function (i.e., utility function).

We denote, for every chain $(\mathbf{O}, \subset) \in \mathcal{O}$, by $\mathbf{S}(\mathbf{O})$ the set of all pairs $(x, y) \in X \times X$ for which there exist open sets $O' \in \mathbf{O}$ and $O \in \mathbf{O}$ such that $O' \subsetneq O$ and $(x, y) \in O' \times (O \setminus O')$.

We present now a new characterization of completely useful topologies in classical set theory, which is primarily based on the concept of a super-short topology.

Theorem 3.2. *Let t be a arbitrary topology on a set X. The following assertions are equivalent:*

(i) *t is completely useful.*
(ii) *(X, t) is strongly separable.*
(iii) *(X, t) is super-short.*
(iv) *t is strongly hereditarily Lindelöf.*
(v) *The topology $t_{\mathbf{O}}$ generated by every chain $(\mathbf{O}, \subset) \in \mathcal{O}$ is second countable.*

Proof. (i) $\Longrightarrow$ (ii). Let t be a completely useful topology on X. Consider any chain $(\mathbf{O}, \subset) \in \mathcal{O}$ and its associated upper semicontinuous total preorder $\precsim_{\mathbf{O}}$, which is, therefore, representable by an upper semicontinuous utility function. From Theorem 2.1 (Representation Lemma), we have, for example, that the order topology $t_{\precsim_{\mathbf{O}}}$ on X has to be separable and that there are only countable many jumps in $(X_{|\sim_{\mathbf{O}}}, \precsim_{\mathbf{O}|\sim_{\mathbf{O}}})$. The former requirement implies the existence of a countable subset Y of X such that $(Y \cap O) \setminus O' \neq \emptyset$ for every triplet $O' \subsetneq O'' \subsetneq O$ of open sets of $\mathbf{O}$, while the latter implies the countability of the (pairwise disjoint) nonempty sets $\bigcap_{O'' \subsetneq O \in \mathbf{O}} O'' \setminus \bigcup_{\mathbf{O} \ni O' \subsetneq O} O' \neq \emptyset$ associated to the isolated sets O (of $\mathbf{O}$) (see Remark 2.7). So, we have that (X, t) has to be strongly separable.

(ii) $\Longrightarrow$ (i). Assume that (X, t) is strongly separable, and consider any upper semicontinuous total preorder $\precsim$ on (X, t). Consider the chain $(\mathbf{O}, \subset) \in \mathcal{O}$, with $\mathbf{O} = \{l_{\prec}(x) = \{z \in X : z \prec x\} : x \in X\}$. Then, we have that the subchain given by

$$\mathbf{O}' = \{l_{\prec}(y) : (y \in X) \text{ and } y \text{ is the right endpoint of a jump } (x, y) \text{ in } (X_{|\sim}, \precsim_{|\sim})\}$$

is countable, which implies that there are only countable many jumps (x, y) in $(X_{|\sim}, \precsim_{|\sim})$. Further, since, for all $x, y, z \in X$, the condition $l_{\prec}(x) \subsetneq l_{\prec}(y) \subsetneq l_{\prec}(z)$ is equivalent to $x \prec y \prec z$, it is easy to see that the order topology $t_{\precsim}$ is separable. Therefore, t is completely useful by the aforementioned Representation Lemma.

(iii) $\Longrightarrow$ (ii). This implication follows by a straightforward indirect argument. Indeed, if (X, t) is not strongly separable then a routine transfinite induction argument allows us to construct an uncountable well-ordered subchain $(\mathcal{S}, \precsim)$ of $(\mathcal{O}, \precsim)$.

(i) $\Longrightarrow$ (iii). Let $(\mathbf{O}, \subset) \in \mathcal{O}$ be an arbitrarily chosen chain. Then the associated upper semicontinuous total preorder $\precsim_{\mathbf{O}}$ on X is representable by an upper semicontinuous function $f : X \to \mathbb{R}$, and we may conclude from the Representation Lemma that $(X_{|\sim_{\mathbf{O}}}, \precsim_{|\sim_{\mathbf{O}}})$ has only countably many jumps and that there exists a countable subset Y of X such that $Y \cap]u, v[\neq \emptyset$ for every nonempty open and bordered order interval $]u, v[$ of $\precsim_{\mathbf{O}}$. The jumps of $\precsim_{|\sim_{\mathbf{O}}}$ correspond to the isolated sets of $\mathbf{O}$ (see Remark 2.7). In addition, the definition of $\precsim_{\mathbf{O}}$ implies that the open and bordered order intervals $]u, v[$ of $\precsim_{\mathbf{O}}$ are defined by sets $O' \subsetneq O \in \mathbf{O}$ for which there exists an open set $O'' \in \mathbf{O}$ such that $O' \subsetneq O'' \subsetneq O$. Hence, assertion (iii) follows.

(i) $\Longrightarrow$ (iv). Let t be a completely useful topology on X, and consider a chain $(\mathbf{O}, \subset) \in \mathcal{O}$. Then, let $\precsim_{\mathbf{O}}$ be the upper semicontinuous total preorder associated to $\mathbf{O}$. Since $\precsim_{\mathbf{O}}$ is representable by an upper semicontinuous utility function, we may proceed as in the proof of the implication (i) $\Longrightarrow$ (ii). We may further observe that, for every $x \in X$, we have that

$$l_{\prec_{\mathbf{O}}}(x) = \{z \in X : z \prec_{\mathbf{O}} x\} = \bigcup_{O \not\ni x,\, O \in \mathbf{O}} O.$$

Hence, the property of (X, t) to be strongly hereditarily Lindelöf easily follows.

(iv) $\Longrightarrow$ (i). Let (X, t) be a strongly hereditarily Lindelöf topological space, and consider an upper semicontinuous total preorder $\precsim$ on (X, t). Consider the family $\mathbf{L} = \{l_{\prec}(x) : x \in X\}$. Due to the fact that $\mathbf{L}$ is a family of open subsets of X which is linearly ordered by set inclusion, by the property of (X, t) to be strongly hereditarily Lindelöf, there exists a countable subset X' of X such that, for every $x \in X$, one has that $l_{\prec}(x)$ is equal to the union of some sets $l_{\prec}(z)$ with $z \in X'$. Since we have already observed that, for all $x, y \in X$, $x \prec y$ is equivalent to $l_{\prec}(x) \subsetneq l_{\prec}(y)$, we may conclude that the preordered set $(X, \precsim)$ is Debreu separable, and therefore it is representable by an upper semicontinuous utility function. This consideration implies (i).

(i) $\Longleftrightarrow$ (v). This bi-implication is straightforward and hence omitted. □

Remark 3.3. The previous result implies, in particular, that a strongly separable topology t on X is always (hereditarily) separable. Indeed, the following chain of implications holds, as regards any topology t on X:

t strongly separable $\Rightarrow$ t completely useful $\Rightarrow$ t short $\Rightarrow$ t hereditarily separable.

The converse does not hold, as the following example shows. This consideration justifies the concept of a strongly separable topology t on X.

Therefore, let us present an example of a separable topology which is not strongly separable.

Example 3.4. Let c be the cardinality of the real line, and consider the closed and bounded real interval $[0, 1]$. The topological product $(X, t) := \left([0, 1]^c, t_{prod}\right)$ is a compact, connected and separable space and, thus, satisfies the assumptions of **EDT**. On the other hand, Corollary 4.13 in Bosi and Herden (2002) implies that, in case that κ is an ordinal number and t a completely useful topology on X that contains at least three open sets, the countability of κ is necessary for the topological product $\left(X^\kappa, t_{prod}\right)$ to be completely useful, which implies, in particular, that $\left([0, 1]^c, t_{prod}\right)$ is not completely useful. In particular, as a consequence of Theorem 3.2, (X, t) is an example of a separable topology which is not strongly separable. Further, this example illustrates the fact that **EDT** cannot be generalized to the semicontinuous case, which also shows that useful topologies t on X are not necessarily completely useful.

At this point, as an immediate consequence of the Alexandroff–Urysohn Metrization Theorem (see e.g. Herden, 1995), we get the following corollary, which applies to the case of completely regular topologies.

Corollary 3.5. *Let t be a completely regular topology on a set X. The following assertions are equivalent:*

(i) *t is completely useful.*
(ii) *The topology $t_{\mathbf{O}}$ generated by every chain $(\mathbf{O}, \subset) \in \mathcal{O}$ is separable.*

The famous Rader Theorem is also an easy consequence of the previous characterization of completely useful topologies (see Rader, 1963).

Corollary 3.6 (*Rader Theorem*). *Every second countable topology t on X is completely useful.*

Proof. From the equivalence of the assertions (i) and (iv) of Theorem 3.2, we only have to show that a second countable topological space (X, t) is strongly hereditarily Lindelöf. So, consider a chain $(\mathbf{O}, \subset) \in \mathcal{O}$, and denote by $t_{\mathbf{O}}$ the subtopology of t which is generated by the family $\mathbf{O}$. Since $t_{\mathbf{O}}$ is a subtopology of a second countable topological space and it is linearly ordered by set inclusion, we have that $t_{\mathbf{O}}$ is itself second countable (see considerations in Bosi and Herden, 2002) and therefore it is easily seen that (X, t) is strongly hereditarily Lindelöf. □

In order to deeply illustrate the concept of a super-short topology, let us present the following proposition.

Proposition 3.7. *The following assertions are equivalent:*

(i) *(X, t) is super-short.*
(ii) *Every chain $(\mathbf{O}, \subset) \in \mathcal{O}$ contains countable trees $(T, \supset)$ consisting, at each level α, of pairwise disjoint sets of the form $O \setminus O'$ such that $O', O \in \mathbf{O}$ and $O' \subsetneqq O'' \subsetneqq O$ for some set $O'' \in \mathbf{O}$ or O is an isolated set (of $\mathbf{O}$).*
(iii) *Every chain $(\mathbf{O}, \subset) \in \mathcal{O}$ contains a countable subchain $(\mathbf{U}, \subset)$ such that $\mathbf{S}(\mathbf{O}) = \mathbf{S}(\mathbf{U})$.*
(iv) *Every uncountable chain $(\mathbf{O}, \subset) \in \mathcal{O}$ contains at most countably many weakly isolated sets and at least one uncountable subchain $(\mathbf{U}, \subset)$ that can be order-embedded into $(\mathbb{R}, \leq)$.*

Proof. (i) $\Longrightarrow$ (ii). Thanks to Theorem 3.2, we have that if (X, t) is super-short then it is strongly separable. Hence, (ii) follows immediately from the definition of strongly separable topology.

(ii) $\Longrightarrow$ (i). Let $(T, \supset)$ be a tree of the type that is described in assertion (ii). Then each ordinal number α that represents a level of $(T, \supset)$ can be associated with the subchain $\left(\mathbf{O}_{\alpha}, \subset\right)$ of $\mathbf{O}$ which consists of all open sets $O \in \mathbf{O}$ that up to level α already have been used in the construction of $(T, \supset)$. This means, in particular, that for different levels $\alpha < \beta$ of $(T, \supset)$ the strict relation $\left(\mathbf{O}_{\alpha}, \subset\right) \prec \left(\mathbf{O}_{\beta}, \subset\right)$ holds. Hence, assertion (ii) implies assertion (i).

(ii) $\Longrightarrow$ (iii). Let us assume, in contrast, that there exists a chain $(\mathbf{O}, \subset) \in \mathcal{O}$ that does not contain a countable subchain $(\mathbf{U}, \subset)$ such that $\mathbf{S}(\mathbf{O}) = \mathbf{S}(\mathbf{U})$. Then we replace $(\mathbf{O}, \subset)$ by $(\mathbf{V}, \subset) := \left(\mathbf{O} \cup \{\emptyset, X\}, \subset\right)$. Since there exists no countable subchain $(\mathbf{U}, \subset)$ of $(\mathbf{O}, \subset)$ such that $\mathbf{S}(\mathbf{O}) = \mathbf{S}(\mathbf{U})$, then there exists, in particular, some set $O \in \mathbf{O}$ such that $\emptyset \subsetneqq O \subsetneqq X$ and we may start at level $\alpha = 0$ with the open sets $\emptyset, X \in \mathbf{V}$ in order to consider the set $X \setminus \emptyset = X$. Now a routine transfinite induction argument allows us to construct an uncountable tree $(T, \supset)$ that consists at each level α of pairwise disjoint sets of the form $O \setminus O'$ such that $O', O \in \mathbf{V}$ and $O' \subsetneqq O'' \subsetneqq O$ for some set $O'' \in \mathbf{V}$ or O is an isolated set (of $\mathbf{V}$).

(iii) $\Longrightarrow$ (iv). Let $(\mathbf{O}, \subset) \in \mathcal{O}$ be an uncountable chain. Then assertion (iii) implies immediately that $(\mathbf{O}, \subset)$ contains at most countably many isolated sets. This means, in particular, that $(\mathbf{O}, \subset)$ contains at most countably many weakly isolated sets. Hence, it suffices to show that $(\mathbf{O}, \subset)$ contains an uncountable subchain $(\mathbf{W}, \subset)$ that can be order-embedded into $(\mathbb{R}, \leq)$. The existence of a countable subchain $(\mathbf{U}, \subset)$ of $(\mathbf{O}, \subset)$ such that $\mathbf{S}(\mathbf{O}) = \mathbf{S}(\mathbf{U})$ implies, in particular, that there exists a countable subchain $(\mathbf{U}, \subset)$ of $(\mathbf{O}, \subset)$ such that $(\mathbf{O}, \subset) \precsim (\mathbf{U}, \subset)$. Let now $\mathbf{F}$ be the collection of all sets $O \in \mathbf{O}$ that are not contained in $\mathbf{U}$ but for which there exists a subset $\mathbf{S}$ of $\mathbf{U}$ such that $O = \bigcup_{U \in \mathbf{S}} U$ and let, in addition, $\mathbf{G}$ be the collection of all sets $O \in \mathbf{O}$ that are not contained in $\mathbf{U}$ but for which there exists a subset $\mathbf{T}$ of $\mathbf{U}$ such that $O = \bigcap_{U \in \mathbf{T}} U$. Since $\mathbf{O}$ is an uncountable set we may assume, without loss of generality, that $\mathbf{F}$ is an uncountable set. The case that $\mathbf{G}$ is an uncountable set then follows by a dual argument. We, therefore, set $(\mathbf{W}, \subset) := (\mathbf{U} \cup \mathbf{F}, \subset)$. Then $(\mathbf{W}, \subset)$ is an uncountable subchain of $(\mathbf{O}, \subset)$. The countability of $\mathbf{U}$ implies that $(\mathbf{U}, \subset)$ is order-embeddable into $(\mathbb{R}, \leq)$. Since every set $O \in \mathbf{F}$ is the least upper bound of a subset $\mathbf{P}$ of $\mathbf{U}$ it follows, furthermore, that $(\mathbf{W}, \subset)$ can be order-embedded into $(\mathbb{R}, \leq)$.

(iv) $\Longrightarrow$ (ii). We assume, in contrast, that there exists a chain $(\mathbf{O}, \subset) \in \mathcal{O}$ for which there exists an uncountable tree $(T, \supset)$ that consists at each level α of pairwise disjoint sets of the form $O \setminus O'$ such that $O', O \in \mathbf{O}$, $O' \subsetneqq O'' \subsetneqq O$ for some set $O'' \in \mathbf{O}$ or O is an isolated set (of $\mathbf{O}$). Since uncountable well-ordered chains $(\mathbf{C}, \subset) \in \mathcal{O}$ and $(\mathbf{D}, \supset) \in \mathcal{O}$ respectively do not contain uncountable subchains that can be order-embedded into $(\mathbb{R}, \leq)$, it follows that the given topology t on X has to be short. Therefore, we may conclude that (X, t) is a hereditarily Lindelöf space, which implies that (X, t) satisfies *ccc*. With the help of *ccc*, it follows that $(\mathbf{O}, \subset)$ contains at most countably many isolated sets. Indeed, let $O \in \mathbf{O}$ be an isolated set that is not weakly isolated. Then at least one of the sets $O \setminus \overline{\bigcup_{\mathbf{O} \ni O' \subsetneqq O} O'}$ or $\left(X \setminus \overline{O}\right) \setminus \left(\overline{X \setminus \bigcap_{O \subsetneqq O'' \in \mathbf{O}} O''}\right)$ is a nonempty open subset of X. Since open subsets of X of the form $O \setminus \overline{\bigcup_{\mathbf{O} \ni O' \subsetneqq O} O'}$ or $\left(X \setminus \overline{O}\right) \setminus \left(\overline{X \setminus \bigcap_{O \subsetneqq O'' \in \mathbf{O}} O''}\right)$ are pairwise disjoint, thus, ccc implies that $(\mathbf{O}, \subset)$ only contains countably many isolated sets that are not weakly isolated. Let us now assume that $(\mathbf{O}, \subset)$ contains uncountably many pairs of triplets $O' \subsetneqq O'' \subsetneqq O$ and $N' \subsetneqq N'' \subsetneqq N$ such that $\left(O \setminus O'\right) \cap \left(N \setminus N'\right) = \emptyset$. Then the subchain $(\mathbf{K}, \subset)$ of $(\mathbf{O}, \subset)$, that is constructed by eliminating for all these triplets every set $O^{llthesetripletsevery} \in \mathbf{O}$ and $N''' \in \mathbf{O}$ such that $O' \subsetneqq O''' \subsetneqq O''$ or $O'' \subsetneqq O''' \subsetneqq O$ and $N' \subsetneqq N''' \subsetneqq N''$ or $N'' \subsetneqq N''' \subsetneqq N$, contains uncountably many isolated sets. This contradicts the already proved result that every chain $(\mathbf{L}, \subset) \in \mathcal{O}$ only contains countably many isolated sets. We, thus, may summarize our considerations in order to conclude with the help of the shortness of (X, t) and our assumption that $(T, \supset)$ is an uncountable tree that consists at each level α of at most countably many pairwise disjoint sets of the form $O \setminus O'$ such that $O', O \in \mathbf{O}$, $O' \subsetneqq O'' \subsetneqq O$ for some set $O'' \in \mathbf{O}$ or O is an isolated set (of $\mathbf{O}$) and that, moreover, the least upper bound of the lengths of all branches of $(T, \supset)$ have to be $\aleph_1$ (the cardinality of the set of all countable ordinal numbers). Hence, $(T, \supset)$ is an *Aronszajn tree* (see, for instance, Jech, 1978). Therefore, we may conclude that the subchain $(\mathbf{E}, \subset)$ of $(\mathbf{O}, \subset)$ that corresponds to $(T, \supset)$ has to be an *Aronszajn chain*, i.e. an uncountable short chain that does not contain any uncountable subchain $(\mathbf{H}, \subset)$ that is order-embeddable into $(\mathbb{R}, \leq)$. This last conclusion contradicts assertion (iv) and, thus, proves the validity of point (ii). □

Remark 3.8. Interested reader in *Aronszajn chains* may consult Baumgartner (1981) or Beardon et al. (2002) for details.

4. The structure of completely useful topologies under the souslin hypothesis

Let X be an arbitrary nonempty set. In Theorem 4.8 and Corollary 4.9 by Bosi and Herden (2002) it has been shown that in **ZFC** + **SH** (**Z**ermelo-**F**raenkel + Axiom of **C**hoice + **S**ouslin **H**ypothesis) completely useful topologies t on X can be characterized in a very elegant way. Therefore it is the first aim of this section to still complete the results of Bosi and Herden (2002) on completely useful topologies that have been proved in **ZFC** + **SH**.

The reader may recall that **SH** states that every order-dense and (almost) complete unbordered chain that satisfies *ccc* is order-isomorphic to the real line. **SH** has been posed by M. Souslin Souslin (1923) in the only paper that he has published during his life. Since the late sixties it is known that **SH** is independent of **ZFC**. In mathematical utility theory **SH** has been applied by Vohra (1995) in order to prove in **ZFC** + **SH** a general continuous representation theorem. It is easy to see that **SH** is equivalent to the assertion that there exists no *Souslin chain*, which means that there exists no chain that contains a non-representable subchain that satisfies *ccc* and only has countably many jumps (see Section 6 of Beardon et al., 2002).

We recall the equivalent version of **SH** which has been presented by Vohra (1995):

Definition 4.1 (*Souslin Hypothesis*)**.** For every totally preordered set $(X \precsim)$, if the order topology $t^{\precsim}$ satisfies ccc, then it is separable.

The following theorem combines **SH** with the structure of completely useful topologies t on X.

Theorem 4.2. *The following assertions are equivalent:*

(i) ***SH*** *holds.*
(ii) *For every set X and every topology t on X, the concept t to be completely useful is equivalent to any of the following equivalent conditions:*
   (1) *Every chain $(\mathbf{O}, \subset) \in \mathcal{O}$ only contains countably many isolated sets.*
   (2) *Every chain $(\mathbf{O}, \subset) \in \mathcal{O}$ that only consists of isolated sets is countable.*
   (3) *t satisfies ccc and every chain $(\mathbf{O}, \subset) \in \mathcal{O}$ only contains countably many weakly isolated sets.*
   (4) *(X, t) is a hereditarily Lindelöf and a hereditarily separable topological space, and every chain $(\mathbf{O}, \subset) \in \mathcal{O}$ that only consists of isolated sets is lengthy.*
   (5) *(X, t) is a hereditarily Lindelöf and a hereditarily separable topological space, and for every chain $(\mathbf{O}, \subset) \in \mathcal{O}$ its subchain $(\mathbf{W}, \subset)$ that consists of the weakly isolated sets of $\mathbf{O}$ is lengthy.*
   (6) *(X, t) is a hereditarily Lindelöf and a hereditarily separable topological and for every chain $(\mathbf{O}, \subset) \in \mathcal{O}$, that only consists of isolated sets, its subchain $(\mathbf{W}, \subset)$ which consists of the weakly isolated sets of $\mathbf{O}$ is lengthy.*

Proof. In Theorem 4.8 of Bosi and Herden (2002) it has been proved that **SH** is equivalent to the assertion that for every (nonempty) set X a topology t on X is completely useful if and only if every chain $(\mathbf{O}, \subset) \in \mathcal{O}$ only contains countably many isolated sets. Then, in order to prove the theorem, it is enough to verify the equivalence of the conditions (1), (2), (3), (4), (5), (6).

The bi-implication (1) $\Longleftrightarrow$ (2) is straightforward.

(1) $\Longrightarrow$ (3). Clearly, condition (1) implies that t is short (see Lemma 4.1 in Bosi and Herden, 2002). This means, in particular, that t satisfies *ccc*. Therefore, condition (3) is an easy consequence of condition (1).

(3) $\Longrightarrow$ (1). The proof of this implication is implicitly in the proof of the implication (iv) $\Longrightarrow$ (ii) of Proposition 3.7.

(1) $\Longrightarrow$ (4). The conditions (X, t) to be a hereditarily Lindelöf and a hereditarily separable topological space are equivalent to the condition (X, t) to be a short space. Hence, with the help of Lemma 4.1 in Bosi and Herden (2002), the validity of the implication (1) $\Longrightarrow$ (4) follows immediately from the definition of a lengthy chain.

(4) $\Longrightarrow$ (5) $\Longrightarrow$ (6). The verification of these implications is an easy task and, therefore, omitted.

(6) $\Longrightarrow$ (3). Since (X, t) is short it follows that t satisfies *ccc*. In addition the shortness of (X, t) implies that for every chain $(\mathbf{O}, \subset) \in \mathcal{O}$, that only consists of isolated sets, its subchain $(\mathbf{W}, \subset)$ that consists of the weakly isolated sets of $\mathbf{O}$ is countable. Hence, we may conclude that every chain $(\mathbf{O}, \subset) \in \mathcal{O}$ only consists of countably many weakly isolated sets. □

Corollary 4.3. *Let (X, t) be an arbitrary topological space. Then in **ZFC** + **SH** the following conditions are equivalent:*

(i) *(X, t) is completely useful.*
(ii) *Every chain $(\mathbf{O} \subset) \in \mathcal{O}$ only contains countably many isolated sets.*
(iii) *Every chain $(\mathbf{O}, \subset) \in \mathcal{O}$ that only consists of isolated sets is countable.*
(iv) *t satisfies ccc and every chain $(\mathbf{O}, \subset) \in \mathcal{O}$ only contains countably many weakly isolated sets.*
(v) *(X, t) is a hereditarily Lindelöf and a hereditarily separable topological space, and every chain $(\mathbf{O}, \subset) \in \mathcal{O}$ that only consists of isolated sets is lengthy.*
(vi) *(X, t) is a hereditarily Lindelöf and a hereditarily separable topological space, and for every chain $(\mathbf{O}, \subset) \in \mathcal{O}$ its subchain $(\mathbf{W}, \subset)$ that consists of the weakly isolated sets of $\mathbf{O}$ is lengthy.*
(vii) *(X, t) is a hereditarily Lindelöf and a hereditarily separable topological space, and for every chain $(\mathbf{O}, \subset) \in \mathcal{O}$, which only consists of isolated sets, its subchain $(\mathbf{W}, \subset)$ that consists of the weakly isolated sets of $\mathbf{O}$ is lengthy.*

At this point the goal is to describe the structure of completely useful topologies t on X in **ZFC**. In particular, the following theorem completes Theorem 4.11 of Bosi and Herden (2002) by clarifying the interrelations between the characterizations of completely useful topologies t on X in **ZFC** + **SH** and **ZFC** respectively.

Theorem 4.4. *Let t be an arbitrary topology on X. Then the following assertions are equivalent:*

(i) *t is completely useful.*
(ii) *(X, t) is super-short.*
(iii) *(X, t) satisfies the equivalent conditions of Corollary 4.3 and every uncountable chain $(\mathbf{O}, \subset) \in \mathcal{O}$ contains at least one uncountable subchain $(\mathbf{U}, \subset)$ that can be order-embedded into $(\mathbb{R}, \leq)$.*
(iv) *(X, t) satisfies the equivalent conditions of Corollary 4.3 and it is locally super-short.*

Proof. (i) $\Longleftrightarrow$ (ii). This bi-implication has already proved in Theorem 3.2.

(ii) $\Longleftrightarrow$ (iii). Because of Proposition 3.7 and Corollary 4.3 the equivalence of the assertions (ii) and (iii) immediately follows.

(ii) $\Longrightarrow$ (iv). Since a super-short topology obviously is locally super-short also this implication is a consequence of Proposition 3.7 and Corollary 4.3.

(iv) $\Longrightarrow$ (ii). We verify that assertion (iv) implies the point (ii) of Theorem 3.2. Let, therefore, $(\mathbf{O}, \subset) \in \mathcal{O}$ be arbitrarily chosen. Since (X, t) satisfies the equivalent conditions of Corollary 4.3 and since (X, t) is locally super-short it suffices to prove that there exist at most countably many equivalence classes $[O] \cap \mathbf{O}$ of open sets $O \in \mathbf{O}$ that satisfy one of the following conditions:

C1: There exist sets $O', O \in [O] \cap \mathbf{O}$ such that $O' \subsetneq O$ and $O \subsetneq \bigcap_{O \subsetneq O'' \in \mathbf{O}} O''$.

C2: There exist sets $O', O'', O \in [O] \cap \mathbf{O}$ such that $O' \subsetneq O'' \subsetneq O$.

Let us now assume, in contrast, that there exist uncountably many equivalence classes $[O] \cap \mathbf{O}$ of open sets $O \in \mathbf{O}$ that satisfy at least one of the conditions **C1** or **C2**. Then we choose in each equivalence class $[O] \cap \mathbf{O}$, which satisfies condition **C1**, open sets $O' \subsetneqq O$ and in each equivalence class $[O] \cap \mathbf{O}$, that *merely* satisfies condition **C2**, open sets $O' \subsetneqq O'' \subsetneqq O$. In this way a subchain $(\mathbf{U}, \subset)$ of $(\mathbf{O}, \subset)$ is defined such that contains uncountably many isolated sets. This contradiction proves assertion (ii). □

Remark 4.5. Assertion (iii) of the previous theorem implies that for every topological space (X, t), that satisfies the equivalent conditions of Corollary 4.3, the set $\mathcal{O}$ cannot contain any Aronszajn chain which is not a Souslin chain. This observation is remarkable since Aronszajn chains can be constructed in **ZFC** while Souslin chains cannot be constructed in **ZFC** (see also Remark 3.8 in this respect).

5. Topologies associated to continuous and upper semicontinuous representability

Let X be an arbitrarily chosen set. Let us present two definitions concerning suitable concepts of (upper semi-) continuity of nontotal preorders and, respectively, topologies associated to the existence of continuous representations.

Definition 5.1. A preorder $\precsim$ on a topological space (X, t) is said to be:

(i) *weakly upper semicontinuous* if, for every pair $(x, y) \in \prec$, there exists some open decreasing subset O_{xy} of X such that $x \in O_{xy}$ and $y \in X \setminus O_{xy}$;
(ii) *weakly continuous* if, for every pair $(x, y) \in \prec$, there exists a continuous isotone (or equivalently *increasing*) function $f : (X, \precsim, t) \to (\mathbb{R}, \leq, t_{nat})$ such that $f(x) < f(y)$.

Definition 5.2. A topology t on a nonempty set X is said to be:

(i) *useful* if for every continuous total preorder $\precsim$ on (X, t) there exists a continuous utility function $f : (X, \precsim, t) \to (\mathbb{R}, \leq, t_{nat})$;
(ii) *strongly useful* if for every weakly continuous (*not* necessarily total) preorder $\precsim$ on (X, t) there exists a continuous utility function $f : (X, \precsim, t) \to (\mathbb{R}, \leq, t_{nat})$.

Remark 5.3. It is known that a completely useful topology is useful (see Bosi and Herden, 2002), and also that a strongly useful topology is useful (see Bosi, 2023).

In order to prove an important result concerning the existence of upper semicontinuous order-preserving functions for all upper semicontinuous preorders, we present an extension theorem concerning upper semicontinuous nontotal preorders.

Theorem 5.4. *Let t be an arbitrary topology on X and $\precsim$ an arbitrary upper semicontinuous preorder on (X, t). Then there exists an upper semicontinuous total preorder $\lesssim$ on (X, t) such that $\precsim \subset \lesssim$ and $\prec \subset <$.*

Proof. Let $\precsim$ be an arbitrary upper semicontinuous preorder on (X, t). Let us first notice that $\precsim$ is also weakly upper semicontinuous. Indeed, for every pair $(x, y) \in \prec$, one has that $O_{xy} = l_{\prec}(y)$ is an open decreasing subset of X such that $x \in O_{xy}$ and $y \in X \setminus O_{xy}$. Then we denote by $\mathcal{R}_{\precsim}$ the set of all weakly upper semicontinuous preorders $\trianglelefteq$ on (X, t) for which the inclusions $\precsim \subset \trianglelefteq$ and $\prec \subset \triangleleft$ hold. We observe that $\mathcal{R}_{\precsim}$ is nonempty since $\precsim \in \mathcal{R}_{\precsim}$. Moreover, the relation $\preceq$ on $\mathcal{R}_{\precsim}$ defined as

$$\preccurlyeq \preceq \trianglelefteq \Longleftrightarrow (\preccurlyeq \subset \trianglelefteq) \text{ and } (\prec \subset \triangleleft)$$

is a partial order $\preceq$ on $\mathcal{R}_{\precsim}$. At this point, we may apply Zorn's Lemma in order to prove that $\left(\mathcal{R}_{\precsim}, \preceq\right)$ contains a maximal element $\lesssim$. Since $\precsim \in \left(\mathcal{R}_{\precsim}, \preceq\right)$, it follows that $\left(\mathcal{R}_{\precsim}, \preceq\right)$ is nonempty. Now we consider an arbitrary subchain $(\mathcal{S}, \preceq)$ of $\left(\mathcal{R}_{\precsim}, \preceq\right)$. We claim that this chain has an upper bound $\leqslant$ that is defined in two steps. At first we define a relation $\trianglelefteq$ on X by setting

$$x \trianglelefteq y \Longleftrightarrow \exists \preccurlyeq \in \mathcal{S}, (x \preccurlyeq y).$$

Then we set

$$\leqslant := \trianglelefteq \setminus \{(y, x) \in \trianglelefteq : \exists \preccurlyeq \in \mathcal{S}, (x \prec y)\}.$$

Since $(\mathcal{S}, \preceq)$ is a chain we may conclude that $\leqslant$ is a preorder on X. In addition, the definition of $\leqslant$ implies that for all $\preccurlyeq \in \mathcal{S}$ and all $x, y \in X$ the implications $x \preccurlyeq y \Longrightarrow x \leqslant y$ and $x \prec y \Longrightarrow x < y$ hold, whence it follows that $\leqslant$ is an upper bound of $(\mathcal{S}, \preceq)$. Next, we assert that the preorder $\leqslant$ is weakly upper semicontinuous. To verify the previous claim, let some pair $(x, y) \in <$ be arbitrarily chosen. First we consider the set $R_{(x,y)}$ of all $\preccurlyeq \in \mathcal{S}$ such that $x \prec y$. Then for every $\preccurlyeq \in R_{(x,y)}$ we choose the set $\mathbf{O}_{(\preccurlyeq)}$ of all open $\preccurlyeq$-decreasing subsets O of X such that $x \in O$ and $y \in X \setminus O$. The definition of $\leqslant$ implies that $R_{(x,y)}$ is not empty. Hence, $U := \bigcup_{\preccurlyeq \in R_{(x,y)}} \bigcup_{O \in \mathbf{O}_{(\preccurlyeq)}} O$ is a nonempty open subset of X. With the help of the definition of $\leqslant$, the definition of U allows us to conclude, moreover, that U is an open $\leqslant$-decreasing subset of X such that $x \in U$ and $y \in X \setminus U$, which implies that $\leqslant$ is weakly upper semicontinuous. Since the definition of $\leqslant$ implies that $\leqslant \in \mathcal{R}_{\precsim}$, we therefore have proved that each chain $(\mathcal{S}, \preceq)$ in the partially ordered set $\left(\mathcal{R}_{\precsim}, \preceq\right)$ has an upper bound. We, thus, may apply Zorn's Lemma in order to conclude that $\left(\mathcal{R}_{\precsim}, \preceq\right)$ contains a maximal element $\lesssim$.

It remains to show that $\lesssim$ is a total preorder on X. Therefore we choose, for every pair $(x, y) \in <$ the union $O_{(x,y)}$ of all $\lesssim$-decreasing open subsets O of X such that $x \in O$ and $y \in X \setminus O$. We denote by $\mathbf{O}$ the set of all open $\lesssim$-decreasing subsets O of X obtained in this way. Now we verify that $(\mathbf{O}, \subset)$ is a chain. Let us assume, in contrast, that there exist sets $O_{(s,t)}, O_{(u,v)} \in \mathbf{O}$ such that $O_{(s,t)} \cap O_{(u,v)} \subsetneqq O_{(s,t)}$ and $O_{(s,t)} \cap O_{(u,v)} \subsetneqq O_{(u,v)}$. Then the maximality of $O_{(s,t)}$ and $O_{(u,v)}$ implies that $t \in O_{(u,v)}$ and $v \in O_{(s,t)}$ respectively. To prove the previous statement, we suppose, without loss of generality, that $t \notin O_{(u,v)}$.

Then, we have the following observation which we abbreviate by $(*)$:

$O_{(s,t)} \cup O_{(u,v)}$ is a $\lesssim$-decreasing open subset of X that is strictly greater than $O_{(s,t)}$ and also separates the points s and t.

With the help of the previous observation, we may conclude that:

$O_{(s,t)} \cap O_{(u,v)}$ is the greatest $\lesssim$-decreasing open subset of X that contains all points $y \in X$ and $y' \in X$ such that $y < t$ and $y' < v$ respectively.

Let us abbreviate this last conclusion by $(**)$.

Then we proceed by defining a relation $\lesssim^e$ on X by setting

$$\lesssim^e := \lesssim \cup \{(y, v) \in X \times X : y \lesssim t\} \cup \{(y', t) \in X \times X : y' \lesssim v\} \cup \{(v, z) \in X \times X : t \lesssim z\} \cup \{(t, z') \in X \times X : v \lesssim z'\}.$$

Since $\lesssim$ is a preorder on X the definition of $\lesssim^e$ implies that $\lesssim^e$ is a preorder on X. With the help of $(*)$ and $(**)$ it follows, moreover, that for every pair $(x, y) \in <^e$ there exists some open $\lesssim^e$-decreasing subset O of the form $O_{(x,y)}$ or $O_{(s,t)} \cap O_{(u,v)}$ of X, according to the two distinct cases $(*)$ and $(**)$, such that $x \in O$ and $y \in X \setminus O$. Hence, $\lesssim^e$ is a weakly upper semicontinuous preorder on (X, t). Since $\lesssim \prec \lesssim^e$, this conclusion contradicts the maximality of $\lesssim$ and, therefore, proves that $(\mathbf{O}, \subset)$ is a chain. As in the proof of Lemma 3.1 in Bosi and Herden (2002), the chain $(\mathbf{O}, \subset)$ induces the upper semicontinuous total preorder $\lesssim_{\mathbf{O}}$ on X. The definition of $(\mathbf{O}, \subset)$ implies that $\lesssim \preceq \lesssim_{\mathbf{O}}$. Hence, it follows from the maximality of $\lesssim$ that $\lesssim = \lesssim_{\mathbf{O}}$ which, finally, implies that $\lesssim$ is a total preorder on X. Then, the proof of the theorem is finished. □

The following theorem appears as an immediate consequence of the previous Theorem 5.4, and proves that the problem of characterizing completely useful topologies is equivalent to that of characterizing all the topologies such that every upper semicontinuous and not necessarily total preorder admits an upper semicontinuous order-preserving function.

Theorem 5.5. *Let t be an arbitrary topology on a set X. Then the following conditions are equivalent:*

(i) *t is completely useful.*
(ii) *Every upper semicontinuous and not necessarily total preorder $\precsim$ on (X,t) admits an upper semicontinuous order-preserving function $f : (X,t,\precsim) \to (\mathbb{R}, t_{nat}, \leq)$.*

Now, in order to show that a completely useful topological space is necessarily strongly useful we need some further definitions and results. At first, we recall that, from Nachbin (1965), an ordered topological space $(X,t,\leq)$ is said to be a *completely regular ordered topological space* if (X,t) is a Hausdorff space, and if, for every point $x \in X$ and every decreasing (open) neighborhood O of x, there exists a continuous isotone function $f : (X,t,\leq) \to ([0,1], t_{nat}, \leq)$ such that $f(x) = 0$ and $f(X \setminus O) = \{1\}$.

Let (X,t) be an arbitrary topological space, and consider any set $F(X,t,\mathbb{R}) := F$ of continuous real-valued functions on (X,t). Further, let $\tau_F(X,\mathbb{R})$ be the coarsest topology on X satisfying the property that every function $f \in F$ remains being continuous. As in Bosi and Zuanon (2021), two points $x,y \in X$ are considered as being *equivalent* if $f(x) = f(y)$ for all functions $f \in F$. For two equivalent points $x,y \in X$, we write $x \sim y$. Consider that F induces a weakly continuous preorder $\precsim_F$ on (X,t) by setting

$$x \precsim_F y \Longleftrightarrow \forall f \in F, (f(x) \leq f(y)).$$

Then consider the quotient space $\left(X_{|\sim}, \tau_F(X,\mathbb{R})_{|\sim}\right)$ that is associated to the equivalence relation $\sim$.

Finally, denote by $\precsim_{F|\sim}$ the quotient order that is induced by $\precsim_F$.

Now the following proposition holds.

Proposition 5.6. $\left(X_{|\sim}, \precsim_{F|\sim}, \tau_F(X,\mathbb{R})_{|\sim}\right)$ *is a completely regular ordered topological space.*

Proof. First we recall that a Hausdorff topology t on a set X is completely regular if and only if t coincides with its weak topology, i.e. the coarsest topology on X for which all continuous real-valued functions f on X are continuous. This is a well known result in general topology which may be found for instance in Cigler and Reichel (1978). Let now F be any (nonempty) set of continuous real-valued functions on X. Then the afore-quoted result implies that the Hausdorff topology, that is associated with the coarsest topology on X for which all functions $f \in F$ are continuous, has to be completely regular. Because of the particular definition of $\precsim_F$, the arguments that prove that a Hausdorff space is completely regular if and only if t coincides with its weak topology can be generalized and, thus, applied in our situation. This means that in case that x is an interior point of some (open) decreasing subset O of $X_{|\sim}$ there exists a continuous isotone function $f : \left(X_{|\sim}, \precsim_{F|\sim}, \tau_F(X,\mathbb{R})_{|\sim}\right) \to \left([0,1], \leq, t_{nat}\right)$ such that $f(x) = 0$ and $f\left(X_{|\sim} \setminus O\right) = \{1\}$. Hence, we may omit the details of the proof for the sake of brevity. □

The following proposition is an immediate consequence of Theorem 3.2, and therefore its proof is omitted.

Proposition 5.7. *If t is a completely useful topology on a set X, then every topology t' on X, which is contained in t, is also completely useful.*

Finally, the next theorem underlines the relevance of the concepts that have been studied in this paper also if one is mainly interested in continuous preorders and their continuous order-preserving representations.

Theorem 5.8. *Let (X,t) be a completely useful topological space. Then (X,t) is a strongly useful topological space.*

Proof. Let (X,t) be a completely useful topological space. Then, from Proposition 5.7, we have that:

(X,t') is a completely useful topological space for every topology t' on X that is coarser than t.

Let us abbreviate the previous observation by $(*)$.

We proceed by considering an arbitrarily weakly continuous preorder $\precsim$ on (X,t). If we are able to verify that there exists a continuous order-preserving function $f : (X,\precsim,t) \to \left(\mathbb{R},\leq,t_{nat}\right)$, then the result is proved. Therefore, we choose the set F of all continuous isotone functions $f : (X,\precsim,t) \to \left(\mathbb{R},\leq,t_{nat}\right)$ in order to consider the topology $\tau_F(X,\mathbb{R})$ on X. The definition of $\precsim_F$ implies that $\precsim \subset \precsim_F$ and $\prec \subset \prec_F$. Hence, we may assume, without loss of generality, that $\precsim = \precsim_F$. With the help of $(*)$ we may conclude that $\left(X, \tau_F(X,\mathbb{R})\right)$ is a completely useful topological space. Because of the definition of $\sim$, it follows that the quotient space $\left(X_{|\sim}, \tau_F(X,\mathbb{R})_{|\sim}\right)$ is a completely useful topological space. Since $\left(X_{|\sim}, \tau_F(X,\mathbb{R})_{|\sim}\right)$ is completely useful and therefore, thanks to Theorem 5.5, is such that every upper semicontinuous preorder admits an upper semicontinuous order-preserving function, there exists an upper semicontinuous order-preserving function $g : \left(X_{|\sim}, \precsim_{|\sim}, \tau_F(X,\mathbb{R})_{|\sim}\right) \to \left(\mathbb{R},\leq,t_{nat}\right)$. By considering, for every rational number q, the open decreasing subsets $g^{-1}(]-\infty,q[)$ of $X_{|\sim}$ we, therefore, may conclude that there exists a countable family $\left\{O_n\right\}_{n\in\mathbb{R}}$ of open decreasing subsets O_n of $X_{|\sim}$ such that for every pair $(x,y) \in \prec_{|\sim}$ there exists a natural number n such that $x \in O_n$ and $y \in X_{|\sim} \setminus O_n$. Let now some $n \in \mathbb{N}$ be arbitrarily chosen. Then, Proposition 5.6 implies that for every point $z \in O_n$ there exists a continuous isotone function $f_z : \left(X_{|\sim}, \precsim_{|\sim}, \tau_F(X,\mathbb{R})_{|\sim}\right) \to \left([0,1],\leq,t_{nat}\right)$ such that $f_z(z) = 0$ and $f_z\left(X_{|\sim} \setminus O_n\right) = \{1\}$. Since $\left(X_{|\sim}, \tau_F(X,\mathbb{R})_{|\sim}\right)$ is completely useful, therefore it is hereditarily Lindelöf. In addition, it follows that there exists a countable subset $P_n = \left\{z_1, z_2, \ldots, z_n, \ldots\right\}$ of O_n, which, without loss of generality, may be assumed to be infinite, such that $O_n = \bigcup_{z\in O_n} f_z^{-1}([0,1[) = \bigcup_{z_k\in P_n} f_{z_k}^{-1}([0,1[)$ or, equivalently, $X_{|\sim} \setminus O_n = \bigcap_{z_k\in P_n} f_{z_k}^{-1}(\{1\})$. Thanks to the previous considerations, we set $f_n := \sum_{z_k\in P_n} \frac{1}{2^k} f_{z_k}$ and $f := \sum_{n\in\mathbb{R}} \frac{1}{2^n} f_n$. Then, the definition of f implies immediately that $f : \left(X_{|\sim}, \precsim_{|\sim}, \tau_F(X,\mathbb{R})_{|\sim}\right) \to \left(\mathbb{R},\leq,t_{nat}\right)$ is a continuous order-preserving function. Let, finally, $\pi : \left(X, \tau_F(X,\mathbb{R})\right) \to \left(X_{|\sim}, \tau_F(X,\mathbb{R})_{|\sim}\right)$ be the quotient map. Then $f \circ \pi : (X,\precsim,t) \to \left(\mathbb{R},\leq,t_{nat}\right)$ is the desired continuous order-preserving function and the proof is complete. □

Remark 5.9. Theorem 5.8 allows us to establish the following chain of implications concerning topologies associated to continuous or upper semicontinuous representability of (not necessarily total) preorders. Indeed, as regards any topology of t on a set X, we have that:

t completely useful $\Rightarrow$ t strongly useful $\Rightarrow$ t useful.

At this point, in order to conclude this section, we present an example of a strongly useful topology which is not completely useful.

Example 5.10. Let X be any uncountable set, and let t be the *cofinite topology* on X (i.e., t consists of the empty set and all subsets Y of X such that their complement is finite). Then, (X,t) is an *irreducible topological space*, in the sense that the intersection of any two disjoint nonempty open subsets of X is nonempty. Since the constant functions are the only continuous real-valued functions on X, and the only weakly continuous total preorders on X are either the preorder $\precsim := \sim$, or the preorder $\precsim = \bowtie$, where $\bowtie$ is the *incomparability relation* on X (i.e., for all $x,y \in X$, $x \bowtie y$ if and only if neither $x \precsim y$ nor $y \precsim x$), we have that t is strongly useful. On the other hand, the set $\{X \setminus \{x\} : x \in X\}$ is an open cover of X, which does not admit a countable subcover. Therefore, (X,t) is not a Lindelöf space, and therefore it cannot be a completely useful space, since it is not short (see the bi-implication (i) $\Longleftrightarrow$ (iii) of Theorem 3.2).

6. Conclusion

The structures of *completely useful topologies* (i.e., topologies such that every upper semicontinuous total preorder is representable by an upper semicontinuous utility function) have been determined, and the relevance of these topologies in mathematical utility theory has been discussed. The *Souslin Hypothesis* has been also incorporated, in order to prove interesting and simple characterizations. Finally, various interrelations between completely useful topologies and other types of topologies, which are of interest in mathematical utility theory, have been described. This paper contributes to a general theory, which studies topological structures on a given set X, in order to guarantee the existence of continuous or semicontinuous order-preserving representations for (all) binary relations on X of a certain type.

The clarification of these topological structures is of interest not only in Economics, but also in any theory which studies the interrelations between order and topology. In a future paper, we shall consider other kinds of binary relations, like *interval orders* and *semiorders*, which are of particular interest in Economics, with respect to the identification of general topological conditions guaranteeing their continuous or at least upper semicontinuous representability.

CRediT authorship contribution statement

Gianni Bosi: Writing – review & editing, Writing – original draft, Supervision, Methodology, Formal analysis, Conceptualization. **Gabriele Sbaiz:** Writing – review & editing, Writing – original draft, Methodology, Formal analysis, Conceptualization.

Funding

This research received no external funding.

Acknowledgments

G. Sbaiz is member of the INdAM (Italian Institute for Advanced Mathematics) group.

Data availability

No data was used for the research described in the article.

References

Alcantud, J.C.R., 1999. Weak utilities from acyclicity. Theory and Decision 47 (2), 179–190.
Alcantud, J.C.R., Gutiérrez, J.M., 1999. Topological properties of spaces ordered by preferences. Int. J. Math. Math. Sci. 22 (1), 17–27.
Arrow, K., Hahn, F., 1971. General Competitive Analysis. Oliver and Boyd, Edinburgh.
Baumgartner, J.E., 1981. Order types of real numbers and other uncountable orderings. In: NATO Adv. Study Inst. Ser. C: Math. Phys. Sci.. In: Ordered sets, vol. 83, Reidel, Dordrecht-Boston, Mass, pp. 239–277.
Beardon, A.F., Candeal, J.C., Herden, G., Induráin, E., Mehta, G.B., 2002. The non-existence of a utility function and the structure of non-representable preference relations. J. Math. Econom. 37 (1), 17–38.
Birkhoff, G., 1948. Lattice Theory. vol. 25, American Mathematical Society Colloquium Publication, Providence, R.I.
Bosi, G., 2023. Continuous order-preserving functions for all kind of preorders. Order 40 (1), 87–97.
Bosi, G., Daris, R., Sbaiz, G., 2024. New characterizations of completely useful topologies in mathematical utility theory. Rend. Istit. Mat. Univ. Trieste 56, 1–12.
Bosi, G., Franzoi, L., 2023. A simple characterization of the existence of upper semicontinuous order-preserving functions. Econ. Theory Bull. 11, 203–210.
Bosi, G., Herden, G., 2002. On the structure of completely useful topologies. Appl. Gen. Topol. 3 (2), 145–167.
Bosi, G., Herden, G., 2019. The structure of useful topologies. J. Math. Econom. 82, 69–73.
Bosi, G., Zuanon, M., 2020. Topologies for the continuous representability of every nontotal weakly continuous preorder. Econ. Theory Bull. 8 (2), 369–378.
Bosi, G., Zuanon, M., 2021. Topologies for the continuous representability of all continuous total preorders. J. Optim. Theory Appl. 188 (2), 420–431.
Bosi, G., Zuanon, M., 2022. Characterization of useful topologies in mathematical utility theory by countable chain conditions. Axioms 11, 195.
Bridges, D.S., Mehta, G.B., 1995. Representation of Preference Orderings. Springer Verlag, Berlin-Heidelberg.
Campión, M.J., Candeal, J.C., Granero, A.S., Induráin, E., 2006a. Ordinal representability in Banach spaces. Lond. Math. Soc. Lect. Notes 337, 183.
Campión, M.J., Candeal, J.C., Induráin, E., 2006b. The existence of utility functions for weakly continuous preferences on a Banach space. Math. Social Sci. 51 (2), 227–237.
Campión, M.J., Candeal, J.C., Induráin, E., Mehta, G.B., 2012. Continuous order representability properties of topological spaces and algebraic structures. J. Korean Math. Soc. 49 (3), 449–473.
Candeal, J.C., Hervés, C., Induráin, E., 1998. Some results on representation and extension of preferences. J. Math. Econom. 29 (1), 75–81.
Candeal, J.C., Induráin, E., Mehta, G.B., 1995. Some utility theorems on inductive limits of preordered topological spaces. Bull. Aust. Math. Soc. 52 (2), 235–246.
Cigler, J., Reichel, H.C., 1978. Topologie. Bibliographisches Institut, Mannheim-Wien-Zürich.
Debreu, G., 1954. Representation of a preference ordering by a numerical function. In: Thrall, R., Coombs, C., Davies, R. (Eds.), Decision Processes. Wiley, New York, pp. 159–166.
Debreu, G., 1964. Continuity properties of a paretian utility. Internat. Econom. Rev. 5, 285–293.
Droste, M., 1999. Uniqueness of semicontinuous ordinal utility functions. J. Econ. Suppl. 8, 23–38.
Dugundji, J., 1978. Topology. Allyn and Bacon Series in Advanced Mathematics, Boston, Reprinting of the 1966 original.
Dutta, P.K., Mitra, T., 1989. On continuity of the utility function in intemporal allocation models: an example. Internat. Econom. Rev. 30, 527–536.
Eilenberg, S., 1941. Ordered topological spaces. Amer. J. Math. 63, 39–45.
Estévez, M., Hervés, C., 1995. On the existence of continuous preference orderings without utility representations. J. Math. Econom. 24 (4), 305–309.
Estévez, M., Hervés, C., Verdejo, A., 1999. Numerical representation of countably bounded preferences. J. Econ. Suppl. 8, 39–51.
Fleischer, I., 1960. Numerical representation of utility. J. Soc. Indust. Appl. Math. 9, 48–50.
Fleischer, I., 1963. Corrections to embeddings linearly ordered sets in real lexicographic products. Nor. Vid. Selsk. Forh. 36, 34–35.
Herden, G., 1989a. On the existence of utility functions. Math. Social Sci. 17 (3), 297–313.
Herden, G., 1989b. On the existence of utility functions II. Math. Social Sci. 18 (2), 37–44.
Herden, G., 1991. Topological spaces for which every continuous total preorder can be represented by a continuous utility function. Math. Social Sci. 22 (2), 123–136.
Herden, G., 1995. On some equivalent approaches to mathematical utility theory. Math. Social Sci. 29 (1), 19–31.
Herden, G., Pallack, A., 2000. Useful topologies and separable systems. Appl. Gen. Topol. 1 (1), 61–82.
Herden, G., Pallack, A., 2001. Interrelations between the length, the structure and the cardinality of a chain. Order 18 (2), 191–200.
Isler, R., 1997. Semicontinuous utility functions in topological spaces. Riv. Mat. Sci. Econ. Soc. 20 (1), 111–116.
Jaffray, J., 1975a. Existence of a continuous utility function: an elementary proof. Econometrica 43 (5–6), 981–983.
Jaffray, J., 1975b. Semicontinuous extension of a partial order. J. Math. Econom. 2 (3), 395–406.
Jech, T.J., 1978. Set Theory. Academic Press, New York.
Mehta, G.B., 1986. Existence of an order-preserving function on a normally preordered space. Bull. Aust. Math. Soc. 34, 141–147.
Mehta, G.B., 1988. Some general theorems on the existence of order-preserving functions. Math. Social Sci. 15 (2), 135–143.
Mehta, G.B., 1997. A remark on a utility representation theorem of rader. Econom. Theory 9 (2), 367–370.
Monteiro, P.K., 1987. Some results on the existence of utility functions on path connected spaces. J. Math. Econom. 16 (2), 147–156.
Nachbin, L., 1965. Topology and Order. Van Nostrand, Princeton.
Neuefeind, W., 1972. On continuous utility. J. Econom. Theory 5 (1), 174–176.
Pivato, M., 2013. Multiutility representations for incomplete difference preorders. Math. Social Sci. 66 (3), 196–220.
Pivato, M., 2023. Compact spaces of continuous preferences. Available at SSRN 4588051.
Rader, T., 1963. The existence of a utility function to present preferences. Rev. Econ. Stud. 30, 229–232.
Richter, M., 1980. Continuous and semicontinuous utility. Internat. Econom. Rev. 21 (2), 293–299.
Robbins, L., 2007. An Essay on the Nature and Significance of Economic Science. Ludwig von Mises Institute.

Sondermann, D., 1980. Utility representations for partial orders. J. Econom. Theory 23 (2), 183–188.
Souslin, M., 1923. Sur un corps dénomberable de nombres réels. Fund. Math. 4, 311–315.
Subiza, B., Peris, J.E., 1997. Numerical representation for lower quasi-continuous preferences. Math. Social Sci. 33 (2), 149–156.
Vohra, R., 1995. The Souslin Hypothesis and continuous utility functions: a remark. Econom. Theory 5 (3), 537–540.
Wold, H., 1943-1944. A synthesis of pure demand analysis (Parts I, II, III). Skand. Aktuarietidskr 26, 85–118, 220–263, 69–120.