



Braided Hopf Algebras and Gauge Transformations II: *-Structures and Examples

Paolo Aschieri^{1,2} · Giovanni Landi^{3,4,5} · Chiara Pagani³

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Abstract

We consider noncommutative principal bundles which are equivariant under a triangular Hopf algebra. We present explicit examples of infinite dimensional braided Lie and Hopf algebras of infinitesimal gauge transformations of bundles on noncommutative spheres. The braiding of these algebras is implemented by the triangular structure of the symmetry Hopf algebra. We present a systematic analysis of compatible *-structures, encompassing the quasitriangular case.

Keywords Noncommutative gauge transformations · Braided Lie algebras · Braided derivations · *-structures · Hopf–Galois extensions · Gauge transformations of the instanton bundle · Gauge transformations of the orthogonal bundle on theta 4-sphere

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✉ Paolo Aschieri
paolo.aschieri@uniupo.it

Giovanni Landi
landi@units.it

Chiara Pagani
cpagani@units.it

- 1 Dipartimento di Scienze e Innovazione Tecnologica, Università del Piemonte Orientale, viale T. Michel 11, 15121 Alessandria, Italy
- 2 INFN Torino, via P. Giuria 1, 10125 Turin, Italy
- 3 Università di Trieste, Via A. Valerio 12/1, 34127 Trieste, Italy
- 4 Institute for Geometry and Physics (IGAP) Trieste, Trieste, Italy
- 5 INFN, Sezione di Trieste, Trieste, Italy

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1 Introduction

The gauge group of a principal bundle can be given as bundle automorphisms (diffeomorphisms of the total space onto itself which respect the group action) covering the identity map on the base space. Elements of the gauge group act by pullback on the space of connection one-forms on the bundle, thus playing a central role for the definition of the moduli space of connections. In the dual algebraic language a principal bundle is given as an algebra extension $B \subseteq A$ which is H -Hopf–Galois, for H a Hopf algebra. A gauge transformation would then be given as an H -equivariant algebra morphism of A onto itself which restricts to the identity on B . This dual definition works well for the case of commutative algebras, and also for Hopf–Galois extensions with H coquasitriangular and B commutative [4]. It is however too restrictive in general, due to the scarcity of morphisms for a generic noncommutative algebra. Finding a good notion of bundle automorphisms and gauge transformations for noncommutative principal bundles is still an open problem.

In our previous paper [5] we looked at the problem from the infinitesimal view-point by considering algebra derivations, rather than algebra morphisms. We considered derivations of algebras which are commutative up to a braiding. The derivations are required to form a braided Lie algebra as well as a module over the algebra. Compatibility of these two structures is better understood in the context of triangular braidings.

We then studied the case of H -Hopf–Galois extensions which are equivariant under a triangular Hopf algebra (K, R) . Infinitesimal gauge transformations are now given by H -comodule maps that are vertical braided derivations, with the braiding implemented by the triangular structure R of the symmetry K . These maps form a braided Lie algebra $\text{aut}_B^R(A)$ and lead to a braided Hopf algebra $\mathcal{U}(\text{aut}_B^R(A))$ of infinitesimal gauge transformations. The construction is shown to be compatible with the theory of Drinfeld twists, and thus suitable for the study of noncommutative principal bundles that are obtained via twist deformation (quantization) of classical ones. We refer to [5] for details and for a discussion of different approaches and of the literature on the subject.

In the present paper we complement the general theory developed in [5], and briefly reviewed in § 2, with a systematic analysis of $*$ -structures on braided Hopf algebras associated with quasitriangular Hopf algebras. This is done in § 3 where we also study their compatibility with actions on $*$ -algebras. In the triangular case we further consider braided Lie $*$ -algebras and their representations on $*$ -algebras.

We then illustrate the general theory with the computation of the braided Lie $*$ -algebras of infinitesimal gauge transformations of two important examples of noncommutative principal bundles. These are given by two Hopf–Galois extensions of the algebra $\mathcal{O}(S_\theta^4)$ of the noncommutative 4-sphere S_θ^4 of [8] associated to an abelian twist. Additional examples obtained from cotriangular quantum groups, and from abelian as well as Jordanian twists are in [5, §7.1, §8.1]. In § 4.1 we determine the braided Lie $*$ -algebra $\text{aut}_{\mathcal{O}(S_\theta^4)}(\mathcal{O}(S_\theta^7))$ of infinitesimal gauge transformations of the $\mathcal{O}(SU(2))$ -Hopf–Galois extension $\mathcal{O}(S_\theta^4) \subset \mathcal{O}(S_\theta^7)$ of [12]. This bundle can also be obtained as a deformation by a twist on $\mathcal{O}(\mathbb{T}^2)$ of the Hopf–Galois extension $\mathcal{O}(S^4) \subset \mathcal{O}(S^7)$ of the classical $SU(2)$ -Hopf bundle [3]. This allows for the construction of $\text{aut}_{\mathcal{O}(S_\theta^4)}(\mathcal{O}(S_\theta^7))$ from its classical counterpart, following the general theory. The explicit description of the classical gauge Lie $*$ -algebra $\text{aut}_{\mathcal{O}(S^4)}(\mathcal{O}(S^7))$ relies on the $Spin(5)$ equivariance of the principal bundle $S^7 \rightarrow S^4$. This equivariance also implies that, as linear space, $\text{aut}_{\mathcal{O}(S^4)}(\mathcal{O}(S^7))$ splits as a direct sum over a class of representations of the Lie $*$ -algebra $\mathfrak{so}(5)$, of vertical $\mathcal{O}(SU(2))$ -equivariant derivations, this is done in § 4.1.4. Following a similar procedure, in § 4.2 we compute the braided Lie $*$ -algebra of infinitesimal gauge transformations of the $\mathcal{O}(SO_\theta(4, \mathbb{R}))$ -Hopf–Galois extension $\mathcal{O}(S_\theta^4) \subset \mathcal{O}(SO_\theta(5, \mathbb{R}))$ of the quantum homogeneous space $\mathcal{O}(S_\theta^4)$ [3, 14].

The braided Lie algebras of gauge transformations of these Hopf–Galois extensions of $\mathcal{O}(S_\theta^4)$ are re-obtained in [6] via an intrinsic construction, which does not use the twist procedure. They are further studied there in the context of Atiyah sequences of braided Lie algebras, generalising the Atiyah sequence of a principal bundle.

2 Braided Lie algebras of gauge transformations

The main objects investigated in this paper are K -equivariant Hopf–Galois extensions, for (K, R) a triangular Hopf algebra, and their braided Lie algebras of gauge symmetries. We briefly recall from [5] the main notions and results that are needed.

We work in the category of \mathbb{k} -modules for \mathbb{k} a commutative field, or the ring of formal power series in an indeterminate and coefficients in a field. All algebras are assumed to be unital and associative; morphisms of algebras preserve the unit. Dually for coalgebras. We use standard terminologies and notations in Hopf algebra theory. For H a bialgebra we also call H -equivariant a map of H -modules or H -comodules.

Recall that an algebra A is a right H -comodule algebra for a Hopf algebra H if it carries a right coaction $\delta : A \rightarrow A \otimes H$ which is a morphism of algebras. As usual we write $\delta(a) = a_{(0)} \otimes a_{(1)}$ in Sweedler notation with an implicit sum. Then the subspace of coinvariants $B := A^{coH} = \{b \in A \mid \delta(b) = b \otimes 1_H\}$ is a subalgebra of A . The algebra extension $B \subseteq A$ is called an H -Hopf–Galois extension if the canonical map

$$\chi := (m \otimes \text{id}) \circ (\text{id} \otimes_B \delta) : A \otimes_B A \longrightarrow A \otimes H, \quad a' \otimes_B a \longmapsto a' a_{(0)} \otimes a_{(1)} \tag{2.1}$$

is bijective. There may be additional requirements, such as faithful flatness of A as a right B -module, to be mentioned when needed.

In the present paper we deal with H -Hopf–Galois extensions which are K -equivariant for a Hopf algebra K . That is A carries also a left action $\triangleright : K \otimes A \rightarrow A$ that commutes with the right H -coaction, $\delta \circ \triangleright = (\triangleright \otimes \text{id}) \circ (\text{id} \otimes \delta)$ (the coaction δ is a K -module map where H has trivial K -action). On elements $k \in K, a \in A$,

$$(k \triangleright a)_{(0)} \otimes (k \triangleright a)_{(1)} = (k \triangleright a_{(0)}) \otimes a_{(1)}. \tag{2.2}$$

We further assume the Hopf algebra K to be quasitriangular. Recall that a bialgebra (or Hopf algebra) K is quasitriangular if there exists an invertible element $R \in K \otimes K$ (the universal R -matrix of K) with respect to which the coproduct Δ of K is quasico-commutative

$$\Delta^{cop}(k) = R\Delta(k)\bar{R} \tag{2.3}$$

for each $k \in K$, with $\Delta^{cop} := \tau \circ \Delta$, τ the flip map, and $\bar{R} \in K \otimes K$ the inverse of R , $R\bar{R} = \bar{R}R = 1 \otimes 1$. Moreover R is required to satisfy,

$$(\Delta \otimes \text{id})R = R_{13}R_{23} \quad \text{and} \quad (\text{id} \otimes \Delta)R = R_{13}R_{12}. \tag{2.4}$$

We write $R := R^\alpha \otimes R_\alpha$ with an implicit sum. Then $R_{12} = R^\alpha \otimes R_\alpha \otimes 1$, and similarly for R_{23} and R_{13} . From conditions (2.3) and (2.4) it follows that R satisfies the quantum Yang–Baxter equation $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$. The R -matrix of a quasitriangular bialgebra (K, R) is unital: $(\varepsilon \otimes \text{id})R = 1 = (\text{id} \otimes \varepsilon)R$, with ε the counit of K . When

K is a Hopf algebra, quasitriangularity implies that its antipode S is invertible and satisfies

$$(S \otimes \text{id})(R) = \bar{R}; \quad (\text{id} \otimes S)(\bar{R}) = R; \quad (S \otimes S)(R) = R. \tag{2.5}$$

The Hopf algebra K is said to be triangular when $\bar{R} = R_{21}$, with $R_{21} = \tau(R) = R_\alpha \otimes R^\alpha$.

2.1 Braided Hopf algebras

We recall that a braided bialgebra associated with a quasitriangular Hopf algebra (K, R) is a K -module (L, \triangleright_L) which is both a K -module algebra $(L, m_L, \eta_L, \triangleright_L)$ and a K -module coalgebra $(L, \Delta_L, \varepsilon_L, \triangleright_L)$ and is a bialgebra in the braided monoidal category of K -modules. That is, $\varepsilon_L : L \rightarrow \mathbb{k}$ and $\Delta_L : L \rightarrow L \boxtimes L$ are algebra maps with respect to the product in L and the product \blacksquare in $L \otimes L$ defined by

$$(x \otimes y) \blacksquare (x' \otimes y') := x \Psi_R(x' \otimes y) y' = x(R_\alpha \triangleright_L x') \otimes (R^\alpha \triangleright_L y) y' \tag{2.6}$$

for $x, y, x', y' \in L$ and

$$\Psi_R : L \otimes L \rightarrow L \otimes L, \quad \Psi_R(x \otimes y) = R_\alpha \triangleright_L y \otimes R^\alpha \triangleright_L x \tag{2.7}$$

the braiding. We denote $L \boxtimes L = (L \otimes L, \blacksquare)$. It is a K -module algebra with action

$$k \triangleright_{L \boxtimes L} (x \boxtimes y) := (k_{(1)} \triangleright_L x) \boxtimes (k_{(2)} \triangleright_L y). \tag{2.8}$$

Such an L is a braided Hopf algebra if there is a K -module map $S_L : L \rightarrow L$, the braided antipode, which satisfies

$$m_L \circ (\text{id}_L \otimes S_L) \circ \Delta_L = \eta_L \circ \varepsilon_L = m_L \circ (S_L \otimes \text{id}_L) \circ \Delta_L. \tag{2.9}$$

It turns out that S_L is a braided algebra map:

$$S_L(xy) = (R_\alpha \triangleright_L S_L(y))(R^\alpha \triangleright_L S_L(x)) \tag{2.10}$$

and a braided coalgebra map:

$$\begin{aligned} \Delta_L \circ S_L(x) &= S_L(R_\alpha \triangleright_L x_{(2)}) \boxtimes S_L(R^\alpha \triangleright_L x_{(1)}) \\ &= R_\alpha \triangleright_L S_L(x_{(2)}) \boxtimes R^\alpha \triangleright_L S_L(x_{(1)}). \end{aligned} \tag{2.11}$$

(For $\Psi_R = \tau$ the flip map, the previous conditions state that S_L is an antialgebra and anticoalgebra map.)

Due to the quasi-cocommutativity property (2.3), the action in (2.8) commutes with the braiding: $\triangleright_{L \boxtimes L} \circ \Psi_R = \Psi_R \circ \triangleright_{L \boxtimes L}$. More generally, given two K -modules V, W and braiding $\Psi_R : V \otimes W \rightarrow W \otimes V$, $\Psi_R(v \otimes w) = R_\alpha \triangleright_W w \otimes R^\alpha \triangleright_V v$, the actions

of the coalgebra K on their tensor products satisfy $k \triangleright_{W \otimes V} \circ \Psi_R = \Psi_R \circ k \triangleright_{V \otimes W}$ for $k \in K$.

2.2 Braided Lie algebras of derivations

We study derivations of quasi-commutative algebras. As mentioned in the introduction, compatibility of the braided Lie algebra and the module structures works well in the context of triangular braidings.

Here we take (K, R) to be triangular, an assumption which is enough for the purposes of the present paper. A braided Lie algebra associated with a triangular Hopf algebra (K, R) , or simply a K -braided Lie algebra, is a K -module \mathfrak{g} with a bilinear map

$$[\ , \] : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$$

that satisfies the following conditions:

- (i) K -equivariance: for $\Delta(k) = k_{(1)} \otimes k_{(2)}$ the coproduct of K ,

$$k \triangleright [u, v] = [k_{(1)} \triangleright u, k_{(2)} \triangleright v]$$

- (ii) braided antisymmetry:

$$[u, v] = -[R_\alpha \triangleright v, R^\alpha \triangleright u],$$

- (iii) braided Jacobi identity:

$$[u, [v, w]] = [[u, v], w] + [R_\alpha \triangleright v, [R^\alpha \triangleright u, w]],$$

for all $u, v, w \in \mathfrak{g}, k \in K$.

As shown in [5, §5.1], the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of a braided Lie algebra \mathfrak{g} associated with (K, R) is a braided Hopf algebra. The coproduct of $\mathcal{U}(\mathfrak{g})$ is determined requiring the elements of \mathfrak{g} to be primitive, $\Delta(u) = u \boxtimes 1 + 1 \boxtimes u$, for all $u \in \mathfrak{g}$.

Any K -module algebra A is a K -braided Lie algebra with bracket given by the braided commutator

$$[\ , \] : A \otimes A \rightarrow A, \quad a \otimes b \mapsto [a, b] = ab - (R_\alpha \triangleright b) (R^\alpha \triangleright a). \quad (2.12)$$

(See [5, Lemma 5.2].) In particular, if A is a K -module algebra, then also the K -module algebra $(\text{Hom}(A, A), \triangleright_{\text{Hom}(A, A)})$ of linear maps from A to A with action

$$\begin{aligned} \triangleright_{\text{Hom}(A, A)} : K \otimes \text{Hom}(A, A) &\rightarrow \text{Hom}(A, A) \\ k \otimes \psi &\mapsto k \triangleright_{\text{Hom}(A, A)} \psi : A \mapsto k_{(1)} \triangleright_A \psi(S(k_{(2)})) \triangleright A \end{aligned} \quad (2.13)$$

is a braided Lie algebra with the braided commutator; here S is the antipode of K . Elements ψ in $\text{Hom}(A, A)$ which satisfy

$$\psi(aa') = \psi(a)a' + (R_\alpha \triangleright a) (R^\alpha \triangleright_{\text{Hom}(A,A)} \psi)(a') \tag{2.14}$$

for any a, a' in A are called braided derivations. We denote $\text{Der}^R(A)$ the \mathbb{k} -module of braided derivations of A (to lighten notation we often drop the subscript R). It is a K -submodule of $\text{Hom}(A, A)$, with action given by the restriction of $\triangleright_{\text{Hom}(A,A)}$

$$\begin{aligned} \triangleright_{\text{Der}(A)}: K \otimes \text{Der}(A) &\rightarrow \text{Der}(A) \\ k \otimes \psi &\mapsto k \triangleright_{\text{Der}(A)} \psi : a \mapsto k_{(1)} \triangleright \psi(S(k_{(2)}) \triangleright a) \end{aligned} \tag{2.15}$$

and moreover, see [5, Prop. 5.7], a braided Lie subalgebra of $\text{Hom}(A, A)$ with

$$\begin{aligned} [,] : \text{Der}(A) \otimes \text{Der}(A) &\rightarrow \text{Der}(A) \\ \psi \otimes \lambda &\mapsto [\psi, \lambda] := \psi \circ \lambda - (R_\alpha \triangleright_{\text{Der}(A)} \lambda) \circ (R^\alpha \triangleright_{\text{Der}(A)} \psi). \end{aligned} \tag{2.16}$$

When the K -module algebra A is quasi-commutative, that is when

$$a a' = (R_\alpha \triangleright a') (R^\alpha \triangleright a), \tag{2.17}$$

for all $a, a' \in A$, the braided Lie algebra $\text{Der}(A)$ with

$$(a\psi)(a') := a \psi(a'), \tag{2.18}$$

for $\psi \in \text{Hom}(A, A)$, $a, a' \in A$, is also a left A -submodule of $\text{Hom}(A, A)$. The Lie bracket of $\text{Der}(A)$ satisfies ([5, Prop. 5.8])

$$\begin{aligned} [a\psi, a'\psi'] &= a\psi(a')\psi' + a(R_\alpha \triangleright a')[R^\alpha \triangleright_{\text{Der}(A)} \psi, \psi'] \\ &\quad - R_\beta R_\alpha \triangleright a' (R_\delta R_\gamma \triangleright_{\text{Der}(A)} \psi') (R^\delta R^\beta \triangleright a) R^\gamma R^\alpha \triangleright_{\text{Der}(A)} \psi \end{aligned} \tag{2.19}$$

for all $a, a' \in A$, $\psi, \psi' \in \text{Der}(A)$.

2.3 Infinitesimal gauge transformations

Let now $B = A^{coH} \subseteq A$ be a K -equivariant Hopf–Galois extension, for (K, R) a triangular Hopf algebra. Inside the braided Lie algebra $\text{Der}(A)$ we consider the subspace of braided derivations that are H -comodule maps,

$$\text{Der}^R_{\mathcal{M}^H}(A) = \{u \in \text{Der}(A) \mid \delta(u(a)) = u(a_{(0)}) \otimes a_{(1)}, a \in A\} \tag{2.20}$$

and then those derivations that are vertical,

$$\text{aut}_B^R(A) := \{u \in \text{Der}_{\mathcal{M}^H}^R(A) \mid u(b) = 0, b \in B\}. \tag{2.21}$$

The linear spaces $\text{Der}_{\mathcal{M}^H}^R(A)$ and $\text{aut}_B^R(A)$ are K -braided Lie subalgebras of $\text{Der}(A)$, [5, Prop. 7.2]. Elements of $\text{aut}_B^R(A)$ are regarded as infinitesimal gauge transformations of the K -equivariant Hopf–Galois extension $B = A^{coH} \subseteq A$, [5, Def. 7.1]. There is the corresponding braided Hopf algebra $\mathcal{U}(\text{aut}_B^R(A))$ of gauge transformations.

2.4 Twisting of braided Lie algebras

Important examples of noncommutative principal bundles come from twisting classical structures. Aiming at studying their braided Lie algebras of infinitesimal gauge transformations, we need to first consider twist deformations of braided Lie algebras.

We recall some basic results of the theory of Drinfeld twists [9].

Let K be a bialgebra (or Hopf algebra). A twist for K is an invertible element $F \in K \otimes K$ which is unital, $(\varepsilon \otimes \text{id})(F) = 1 = (\text{id} \otimes \varepsilon)(F)$, and satisfies the twist condition

$$(F \otimes 1)[(\Delta \otimes \text{id})(F)] = (1 \otimes F)[(\text{id} \otimes \Delta)(F)]. \tag{2.22}$$

For F and its inverse \bar{F} we write $F = F^\alpha \otimes F_\alpha$ and $\bar{F} =: \bar{F}^\alpha \otimes \bar{F}_\alpha$, with an implicit sum. The R -matrix R of a quasitriangular bialgebra K is a twist for K .

When K has a twist it can be endowed with a second bialgebra structure which is obtained by deforming its coproduct and leaving its counit and multiplication unchanged. Moreover if K is triangular, or more in general quasitriangular, so is the new bialgebra:

Proposition 2.1 *Let $F = F^\alpha \otimes F_\alpha$ be a twist on a bialgebra $(K, m, \eta, \Delta, \varepsilon)$. Then the algebra (K, m, η) with coproduct*

$$\Delta_F(k) := F\Delta(k)\bar{F} = F^\alpha k_{(1)}\bar{F}^\beta \otimes F_\alpha k_{(2)}\bar{F}_\beta, \quad k \in K \tag{2.23}$$

and counit ε is a bialgebra. If in addition K is a Hopf algebra, then the twisted bialgebra $K_F := (K, m, \eta, \Delta_F, \varepsilon)$ is a Hopf algebra with antipode $S_F(k) := u_F S(k) \bar{u}_F$, where u_F is the invertible element $u_F := F^\alpha S(F_\alpha)$ with $\bar{u}_F = S(\bar{F}^\alpha) \bar{F}_\alpha$ its inverse.

Finally, if (K, R) is a quasitriangular bialgebra (a Hopf algebra), such is the twisted bialgebra (Hopf algebra) K_F with R -matrix

$$R_F := F_{21} R \bar{F} = F_\alpha R^\beta \bar{F}^\gamma \otimes F^\alpha R_\beta \bar{F}_\gamma \tag{2.24}$$

and inverse $\bar{R}_F := F \bar{R} \bar{F}_{21} = F^\alpha \bar{R}^\beta \bar{F}_\gamma \otimes F_\alpha \bar{R}_\beta \bar{F}^\gamma$. If (K, R) is triangular, so is (K_F, R_F) : $R_F{}_{21} = F R_{21} \bar{F}_{21} = F \bar{R} \bar{F}_{21} = \bar{R}_F$.

Any K -module V with left action $\triangleright_V: K \otimes V \rightarrow V$, is also a K_F -module with the same linear map \triangleright_V , now thought as a map $\triangleright_V: K_F \otimes V \rightarrow V$. When thinking of V as a K_F -module we denote it by V_F , with action \triangleright_{V_F} . Moreover, any K -module morphism $\psi: V \rightarrow W$ can be thought of as a morphism $\psi_F: V_F \rightarrow W_F$.

If A is a K -module algebra, with multiplication m_A and unit η_A , in order for the action \triangleright_{A_F} to be an algebra map one has to endow the K_F -module A_F with a new algebra structure: the unit is unchanged, while the product is deformed to

$$m_{A_F}: A_F \otimes_F A_F \longrightarrow A_F, \quad a \otimes_F a' \longmapsto a \bullet_F a' := (\bar{F}^\alpha \triangleright_A a) (\bar{F}_\alpha \triangleright_A a'). \tag{2.25}$$

For any K -module algebra map $\psi: A \rightarrow A'$, the K_F -module map $\psi: A_F \rightarrow A'_F$ is an algebra map for the deformed products.

If C is a K -module coalgebra, the K_F -module C_F is a K_F -module coalgebra with counit $\varepsilon_F = \varepsilon$ as linear map and coproduct

$$\Delta_F: C_F \rightarrow C_F \otimes_F C_F, \quad c \mapsto \Delta_F(c) = F^\alpha \triangleright_{c_{(1)}} \otimes_F F_\alpha \triangleright_{c_{(2)}}. \tag{2.26}$$

The twist L_F of a braided Hopf algebra L is obtained twisting L as a K -module algebra and as a K -module coalgebra, cf. [5, Prop. 4.11].

We next recall that the action of a braided Hopf algebra (or just bialgebra) L on a K -module algebra A is a K -equivariant action $\blacktriangleright_A: L \otimes A \rightarrow A$ which satisfies

$$x \blacktriangleright_A (aa') = (x_{(1)} \blacktriangleright_A (R_\alpha \triangleright_A a)) ((R^\alpha \triangleright_L x_{(2)}) \blacktriangleright_A a'), \tag{2.27}$$

for all $a, a' \in A$. When twisting this leads to an action

$$\blacktriangleright_{A_F}: L_F \otimes_F A_F \rightarrow A_F, \quad x \blacktriangleright_{A_F} a = (\bar{F}^\alpha \triangleright_L x) \blacktriangleright_A (\bar{F}_\alpha \triangleright_A a). \tag{2.28}$$

When \mathfrak{g} is a braided Lie algebra associated with a triangular Hopf algebra (K, R) , and F is a twist for K , the K_F -module \mathfrak{g}_F inherits from \mathfrak{g} a twisted bracket ([5, Prop. 5.14]):

Proposition 2.2 *The K_F -module \mathfrak{g}_F with bilinear map*

$$[\ , \]_F = \mathfrak{g}_F \otimes \mathfrak{g}_F \rightarrow \mathfrak{g}_F, \quad u \otimes v \rightarrow [u, v]_F := [\bar{F}^\alpha \triangleright_{\mathfrak{g}} u, \bar{F}_\alpha \triangleright_{\mathfrak{g}} v] \tag{2.29}$$

is a braided Lie algebra associated with (K_F, R_F) .

As a particular case of the above, consider the braided Lie algebra $(\text{Der}(A), [\ , \])$, for A a K -module algebra. It consists of the K -module of braided derivations of A to itself, with action $\triangleright_{\text{Hom}(A,A)}$ as in (2.15), and bracket the braided commutator (2.16). It is a braided Lie algebra associated with the triangular Hopf algebra (K, R) .

On the one hand, we obtain the braided Lie algebra $(\text{Der}(A)_F, [,]_F)$ associated with the triangular Hopf algebra (K_F, R_F) . The K_F -action $\triangleright_{\text{Der}(A)_F}$ coincides with $\triangleright_{\text{Der}(A)}$ as linear map. The Lie bracket is given by the braided commutator

$$[\psi, \lambda]_F = \psi \circ_F \lambda - (R_{F\alpha} \triangleright_{\text{Der}(A)} \lambda) \circ_F (R_F^\alpha \triangleright_{\text{Der}(A)} \psi),$$

with the composition (in fact in $(\text{Hom}(A, A), \circ)$) that is changed as in (2.25):

$$\psi \circ_F \phi = (\bar{F}^\alpha \triangleright_{\text{Der}(A)} \psi) \circ (\bar{F}_\alpha \triangleright_{\text{Der}(A)} \phi). \tag{2.30}$$

On the other hand, there is the braided Lie algebra $\text{Der}(A_F)$ of the K_F -module A_F associated with (K_F, R_F) . We use the notation $\Delta_F(k) =: k_{[1]} \otimes k_{[2]}$ for the coproduct in K_F to distinguish it from the original one $\Delta(k) = k_{(1)} \otimes k_{(2)}$ in K . The K_F -action is

$$\begin{aligned} \triangleright_{\text{Der}(A_F)}: K_F \otimes \text{Der}(A_F) &\rightarrow \text{Der}(A_F) \\ k \otimes \psi &\mapsto k \triangleright_{\text{Der}(A_F)} \psi : a \mapsto h_{[1]} \triangleright_{A_F} \psi(S_F(h_{[2]}) \triangleright_{A_F} a). \end{aligned} \tag{2.31}$$

with bracket

$$[\psi, \lambda]_{R_F} = \psi \circ \lambda - (R_{F\alpha} \triangleright_{\text{Der}(A_F)} \lambda) \circ (R_F^\alpha \triangleright_{\text{Der}(A_F)} \psi).$$

These two braided Lie algebras are isomorphic [5, Thm. 5.19]:

Theorem 2.3 *The braided Lie algebras $(\text{Der}(A)_F, [,]_F)$ and $(\text{Der}(A_F), [,]_{R_F})$ are isomorphic via the map*

$$\mathcal{D} : \text{Der}(A)_F \rightarrow \text{Der}(A_F), \quad \psi \mapsto \mathcal{D}(\psi) : a \mapsto (\bar{F}^\alpha \triangleright_{\text{Der}(A)_F} \psi)(\bar{F}_\alpha \triangleright_A a), \tag{2.32}$$

which satisfies $\mathcal{D}([\psi, \lambda]_F) = [\mathcal{D}(\psi), \mathcal{D}(\lambda)]_{R_F}$, for all $\psi, \lambda \in \text{Der}(A)_F$. It has inverse

$$\mathcal{D}^{-1} : \text{Der}(A_F) \rightarrow \text{Der}(A)_F, \quad \psi \mapsto \mathcal{D}^{-1}(\psi) : a \mapsto (F^\alpha \triangleright_{\text{Der}(A_F)} \psi)(F_\alpha \triangleright_{A_F} a). \tag{2.33}$$

This isomorphism extends as algebra map to the universal enveloping algebras

$$\mathcal{D} : \mathcal{U}(\text{Der}(A)_F) \rightarrow \mathcal{U}(\text{Der}(A_F))$$

resulting into a braided Hopf algebra isomorphism. We further have the braided Hopf algebras isomorphisms (see [5, Prop. 5.18]) $\mathcal{U}(\text{Der}(A))_F \simeq \mathcal{U}(\text{Der}(A)_F) \simeq \mathcal{U}(\text{Der}(A_F))$.

Remark 2.4 As shown in [5], the isomorphism $\mathcal{D} : \text{Der}(A)_F \rightarrow \text{Der}(A_F)$ is the restriction of a more general isomorphism $\mathcal{D} : ((\text{Hom}(A, A)_F, \circ_F), [,]_F) \rightarrow$

$(\text{Hom}(A_F, A_F), \circ), [\ , \]_{R_F}$). This result indeed holds in more generality for A just a K -module and not necessarily a K -module algebra.

As mentioned, when A is quasi-commutative the K -braided Lie algebra $\text{Der}(A)$ has an A -module structure defined in (2.18) that is compatible with the Lie bracket of $\text{Der}(A)$.

The K_F -braided Lie algebra $\text{Der}(A)_F$ has A_F -module structure

$$a \cdot_F \psi := (\bar{F}^\alpha \triangleright_A a) (\bar{F}_\alpha \triangleright_{\text{Der}(A)} \psi), \tag{2.34}$$

for all $\psi \in \text{Der}(A)_F$ and $a \in A_F$. The compatibility of the braided bracket with this module structure then, for all $\psi, \psi' \in \text{Der}(A)_F, a, a' \in A_F$, reads

$$\begin{aligned} [a \cdot_F \psi, a' \cdot_F \psi']_F &= a \cdot_F [\psi, a']_F \cdot_F \psi' + a \cdot_F (R_{F\alpha} \triangleright_{A_F} a') \cdot_F [R_F^\alpha \triangleright_{\text{Der}(A)_F} \psi, \psi']_F \\ &\quad - R_{F\beta} R_{F\alpha} \triangleright_{A_F} a' \cdot_F [R_{F\delta} R_{F\gamma} \triangleright_{\text{Der}(A)_F} \psi', R_F^\delta R_F^\beta \triangleright_{A_F} a]_F \cdot_F R_F^\gamma R_F^\alpha \triangleright_{\text{Der}(A)_F} \psi. \end{aligned} \tag{2.35}$$

Here an element in A is thought as a linear map $A \rightarrow A$ given by left multiplication. Then $[\psi, a]_F = [\bar{F}^\alpha \triangleright_{\text{Der}(A)} \psi, \bar{F}_\alpha \triangleright_A a] = (\bar{F}^\alpha \triangleright_{\text{Der}(A)} \psi)(\bar{F}_\alpha \triangleright_A a)$.

Also the K_F -braided Lie algebra $\text{Der}(A_F)$ has compatible A_F -module structure. With the product \bullet_F in (2.25) this is given as in (2.18) by

$$(a \bullet_F \psi)(a') = a \bullet_F \psi(a') \tag{2.36}$$

for any $a, a' \in A_F, \psi \in \text{Der}(A_F)$.

The isomorphism $\mathcal{D} : \text{Der}(A)_F \rightarrow \text{Der}(A_F)$ respects the A_F -module structures:

Corollary 2.5 *If the K -module algebra A is quasi-commutative, the braided Lie algebra isomorphism $\mathcal{D} : (\text{Der}(A)_F, [\ , \]_F) \rightarrow (\text{Der}(A_F), [\ , \]_{R_F})$ of Theorem 2.3 is also an isomorphism of the A_F -modules $\text{Der}(A)_F$ and $\text{Der}(A_F)$:*

$$\mathcal{D}(a \cdot_F \psi) = a \bullet_F \mathcal{D}(\psi),$$

for $a \in A_F$ and $\psi \in \text{Der}(A)_F$.

Next, let $B \subseteq A$ be a K -equivariant Hopf–Galois extension. We use the above isomorphisms for the K -braided Lie algebra of derivations $\text{Der}(A)$ and its braided subalgebras $\text{Der}_{\mathcal{M}^H}^R(A)$ and $\text{aut}_B^R(A)$ defined in (2.20) and in (2.21).

The K -braided Lie algebras $(\text{aut}_B^R(A), [\ , \]) \subseteq (\text{Der}_{\mathcal{M}^H}^R(A), [\ , \])$ are twisted to the K_F -braided Lie algebras $(\text{aut}_B^R(A)_F, [\ , \]_F) \subseteq (\text{Der}_{\mathcal{M}^H}^R(A)_F, [\ , \]_F)$ with bracket $[\ , \]_F$. These are braided Lie subalgebras of $(\text{Der}(A)_F, [\ , \]_F)$. We can equivalently consider the K_F -braided Lie algebras $(\text{aut}_{B_F}^{R_F}(A_F), [\ , \]_{R_F}) \subseteq (\text{Der}_{\mathcal{M}^{H_F}}^{R_F}(A_F), [\ , \]_{R_F})$ that are braided Lie subalgebras of $(\text{Der}(A_F), [\ , \]_{R_F})$. These are isomorphic, [5, Prop. 8.1]:

Proposition 2.6 *The isomorphism $\mathcal{D} : (\text{Der}(A)_F, [\ , \]_F) \rightarrow (\text{Der}(A_F), [\ , \]_{R_F})$ of braided Lie algebras in Theorem 2.3 restricts to isomorphisms $\mathcal{D} : \text{Der}_{\mathcal{M}^H}^R(A)_F \rightarrow \text{Der}_{\mathcal{M}^H}^{R_F}(A_F)$ and $\mathcal{D} : \text{aut}_B^R(A)_F \rightarrow \text{aut}_{B_F}^{R_F}(A_F)$ of (K_F, R_F) -braided Lie algebras.*

In § 4 we work out the braided Lie algebra of equivariant derivations and of infinitesimal gauge transformations for two important examples of principal bundles over the noncommutative 4-sphere S_θ^4 of [8]. We use the general theory developed in this section to obtain the braided Lie algebra of equivariant derivations and of infinitesimal gauge transformations of these noncommutative bundles from their classical counterparts.

These noncommutative principal bundles are $*$ -algebra extensions (which can be completed to C^* -algebras). The $*$ -structures canonically lift to the Lie algebras of braided derivations and of gauge transformations. Before considering these examples, in the next section we proceed with a systematic analysis of $*$ -structures for braided Hopf and Lie algebras. The results of § 4 are however presented in a self contained way so that § 3 might be skipped in a first reading.

3 Braided Hopf and Lie $*$ -algebras

In this section the ground field is $\mathbb{k} = \mathbb{C}$. We present a study of compatibility conditions for defining $*$ -structures on Hopf algebras and their representations. The study of braided Hopf $*$ -algebra actions on braided $*$ -algebras associated with quasitriangular Hopf algebras is new to the best of our knowledge.

A $*$ -structure on a vector space V is an antilinear involution $*$: $V \rightarrow V, v \mapsto v^*$, on an algebra A one also requires $*$: $A \rightarrow A$ to be antimultiplicative. A $*$ -structure on a Hopf algebra K is a $*$ -structure on the algebra K that satisfies $\Delta(k^*) = \Delta(k)^{* \otimes *}$ for all $k \in K$; it then follows that $\varepsilon(k^*) = \overline{\varepsilon(k)}$ and $(S \circ *)^2 = \text{id}$. In particular S is invertible, with $S^{-1} = * \circ S \circ *$. If V is a K -module with a $*$ -structure, one requires the compatibility condition

$$(k \triangleright_V v)^* = S^{-1}(k^*) \triangleright_V v^* \tag{3.1}$$

for all $k \in K, v \in V$. This condition is well defined: it is equivalent to require that $\succ_V : K \otimes V \rightarrow V$, defined by

$$k \succ_V v := (S^{-1}(k^*) \triangleright_V v^*)^*, \tag{3.2}$$

is an action of the Hopf algebra K on V that coincides with the starting one \triangleright_V . Indeed

$$k \succ (h \succ v) = k \succ (S^{-1}(h^*) \triangleright v^*)^* = (S^{-1}(k^*)S^{-1}(h^*) \triangleright v^*)^* = kh \succ v. \tag{3.3}$$

The condition (3.1) is also required for A a K -module $*$ -algebra. In this case (3.1) is also well defined with respect to the multiplication in A . Indeed, $k \triangleright_L 1_A = \varepsilon(k)1_A$

and

$$\begin{aligned} (k_{(1)} \triangleright a)(k_{(2)} \triangleright b) &= ((S^{-1}(k_{(2)}^*) \triangleright b^*)(S^{-1}(k_{(1)}^*) \triangleright a^*))^* \\ &= (S^{-1}(k^*) \triangleright (b^* a^*))^* = k \triangleright (ab) \end{aligned} \tag{3.4}$$

for all $k, h \in K, a, b \in A$. Here and in the following to lighten the notations we frequently omit the subscript on the actions.

Example 3.1 Any Hopf $*$ -algebra K with adjoint action $K \otimes K \rightarrow K, k \triangleright k' = k_{(1)}k'S(k_{(2)})$ is a K -module $*$ -algebra. This motivates the definition (3.1) following the conventions in [10] rather than those in [13].

Example 3.2 Condition (3.1) is dual to that for a comodule $*$ -algebra. Given a $*$ -algebra A which is a right comodule algebra for a Hopf $*$ -algebra \mathcal{U} , with coaction $\delta : A \rightarrow A \otimes \mathcal{U}, a \mapsto a_{(0)} \otimes a_{(1)}$, one requires $\delta(a^*) = \overline{(a_{(0)})^* \otimes (a_{(1)})^*}$. If K and \mathcal{U} are dually paired Hopf $*$ -algebras, one has $\langle k, u^* \rangle = \langle S^{-1}(k^*), u \rangle$. Then A is a module $*$ -algebra with K -action $k \triangleright a = a_{(0)} \langle k, a_{(1)} \rangle$ satisfying (3.1).

In the present paper we deal with braided Hopf and Lie algebras associated with a (quasi)triangular Hopf algebra (K, R) .

When K is quasitriangular we require its R -matrix to be antireal, that is, $R^{*\otimes*} = \bar{R}$. When K is triangular this condition coincides with the reality condition $R^{*\otimes*} = R_{21}$.

3.1 Braided Hopf $*$ -algebras

In this section we take (K, R) quasitriangular with R antireal. We use the braiding $\Psi_R : L \boxtimes L \rightarrow L \boxtimes L, \Psi_R(x \boxtimes y) = R_\alpha \triangleright_L y \boxtimes R^\alpha \triangleright_L x$, in (2.7) to induce the $*$ -structure from a K -module $*$ -algebra L to the K -module algebra $L \boxtimes L$ defined in §2.1 (and to more general tensor products of K -module $*$ -algebras).

Lemma 3.3 *Given a K -module $*$ -algebra L , the K -module algebra $L \boxtimes L$ with product \blacksquare in (2.6) and involution $(x \boxtimes y)^* = \Psi_R(y^* \boxtimes x^*)$, that is,*

$$(x \boxtimes y)^* = (R_\alpha \triangleright_L x^*) \boxtimes (R^\alpha \triangleright_L y^*), \tag{3.5}$$

for $x, y \in L$, is a K -module $*$ -algebra.

Proof The matrix R being antireal implies that (3.5) is an involution. It is antimultiplicative: using (3.1) for \triangleright_L together with antireality and properties (2.5) of the R -matrix we compute (omitting the subscript on the actions)

$$\begin{aligned} ((x \boxtimes y) \blacksquare (x' \boxtimes y'))^* &= (x(R_\alpha \triangleright x') \boxtimes (R^\alpha \triangleright y)y')^* \\ &= R_\beta \triangleright (x(R_\alpha \triangleright x'))^* \boxtimes R^\beta \triangleright ((R^\alpha \triangleright y)y')^* \\ &= R_\beta \triangleright ((R_\alpha \triangleright x')^* x^*) \boxtimes R^\beta \triangleright (y'^*(R^\alpha \triangleright y)^*) \end{aligned}$$

$$\begin{aligned}
 &= R_\beta \triangleright \left((\bar{R}_\alpha \triangleright x'^*) x^* \right) \boxtimes R^\beta \triangleright (y'^* \bar{R}^\alpha \triangleright y^*) \\
 &= (R_{\beta(1)} \bar{R}_\alpha \triangleright x'^*) (R_{\beta(2)} \triangleright x^*) \boxtimes (R^\beta_{(1)} \triangleright y'^*) (R^\beta_{(2)} \bar{R}^\alpha \triangleright y^*) \\
 &= (R_\beta R_\tau \bar{R}_\alpha \triangleright x'^*) (R_\gamma R_\mu \triangleright x^*) \boxtimes (R^\gamma R^\beta \triangleright y'^*) (R^\mu R^\tau \bar{R}^\alpha \triangleright y^*) \\
 &= (R_\beta \triangleright x'^*) (R_\gamma R_\mu \triangleright x^*) \boxtimes (R^\gamma R^\beta \triangleright y'^*) (R^\mu \triangleright y^*) \\
 &= (R_\beta \triangleright x'^* \boxtimes R^\beta \triangleright y'^*) \blacksquare (R_\mu \triangleright x^* \boxtimes R^\mu \triangleright y^*) \\
 &= (x' \boxtimes y')^* \blacksquare (x \boxtimes y)^* .
 \end{aligned}$$

Finally, we show the compatibility condition (3.1) between the $*$ -structures. Recalling that $L \boxtimes L$ has action (2.8), we compute

$$\begin{aligned}
 (k \triangleright (x \boxtimes y))^* &= (k_{(1)} \triangleright x \boxtimes k_{(2)} \triangleright y)^* \\
 &= R_\alpha \triangleright (k_{(1)} \triangleright x)^* \boxtimes R^\alpha \triangleright (k_{(2)} \triangleright y)^* \\
 &= R_\alpha S^{-1}(k^*_{(1)}) \triangleright x^* \boxtimes R^\alpha S^{-1}(k^*_{(2)}) \triangleright y^* \\
 &= R_\alpha (S^{-1}(k^*))_{(2)} \triangleright x^* \boxtimes R^\alpha (S^{-1}(k^*))_{(1)} \triangleright y^* \\
 &= (S^{-1}(k^*))_{(1)} R_\alpha \triangleright x^* \boxtimes (S^{-1}(k^*))_{(2)} R^\alpha \triangleright y^* \\
 &= S^{-1}(k^*) \triangleright (R_\alpha \triangleright x^* \boxtimes R^\alpha \triangleright y^*) \\
 &= S^{-1}(k^*) \triangleright (x \boxtimes y)^* \tag{3.6}
 \end{aligned}$$

where for the third last equality we used the quasi-cocommutative condition (2.3). \square

Considering the last and third expression in (3.6) one has

$$\left[S^{-1}(k^*) \triangleright (x \boxtimes y)^* \right]^* = \left[R_\alpha S^{-1}(k^*_{(1)}) \triangleright x^* \boxtimes R^\alpha S^{-1}(k^*_{(2)}) \triangleright y^* \right]^*$$

and, recalling the action (3.2), this reads $k \succ_{L \boxtimes L} (x \boxtimes y) = k_{(1)} \succ_L x \boxtimes k_{(2)} \succ_L y$. This proves that $\succ_{L \boxtimes L}$ is an action of the coalgebra K on $L \boxtimes L$.

Lemma 3.4 *The action $\succ: K \otimes L \rightarrow L$ commutes with the braiding isomorphism $\Psi_R: L \boxtimes L \rightarrow L \boxtimes L$, $\Psi_R(x \boxtimes y) = R_\alpha \triangleright y \boxtimes R^\alpha \triangleright x$ of the original K -action \triangleright ,*

$$k \succ (\Psi_R(x \boxtimes y)) = \Psi_R(k \succ (x \boxtimes y)) , \tag{3.7}$$

for all $k \in K, x, y \in L$.

Proof We first compute, using the antireality of the R -matrix,

$$(R^\alpha \triangleright x)^* \boxtimes (R_\alpha \triangleright y)^* = S^{-1}(R^\alpha \triangleright x) \triangleright x^* \boxtimes S^{-1}(R_\alpha \triangleright y) \triangleright y^* \tag{3.8}$$

$$= S^{-1}(\bar{R}^\alpha) \triangleright x^* \boxtimes S^{-1}(\bar{R}_\alpha) \triangleright y^* = \bar{R}^\alpha \triangleright x^* \boxtimes \bar{R}_\alpha \triangleright y^* , \tag{3.9}$$

for all $x, y \in \mathfrak{g}$. This and the $*$ -structure of $L \boxtimes L$ in (3.5) prove the lemma:

$$\begin{aligned}
 k \succ (\Psi_R(x \boxtimes y)) &= k \succ (R_\alpha \triangleright y \boxtimes R^\alpha \triangleright x) \\
 &= [S^{-1}(k^*) \triangleright (R_\alpha \triangleright y \boxtimes R^\alpha \triangleright x)]^* \\
 &= [S^{-1}(k^*) \triangleright (R_\beta \triangleright (R_\alpha \triangleright y)^* \boxtimes R^\beta \triangleright (R^\alpha \triangleright x)^*)]^* \\
 &= [S^{-1}(k_{(2)}^*) \triangleright y^* \boxtimes S^{-1}(k_{(1)}^*) \triangleright x^*]^* \\
 &= R_\alpha \triangleright (S^{-1}(k_{(2)}^*) \triangleright y^*)^* \boxtimes R^\alpha \triangleright (S^{-1}(k_{(1)}^*) \triangleright x^*)^* \\
 &= R_\alpha \triangleright (k_{(2)} \succ y) \boxtimes R^\alpha \triangleright (k_{(1)} \succ x) \\
 &= \Psi_R(k_{(1)} \succ x \boxtimes k_{(2)} \succ y). \quad \square
 \end{aligned}$$

Definition 3.5 A $*$ -structure on a braided Hopf algebra L associated with (K, R) is a $*$ -structure on the K -module algebra L such that the braided coproduct $\Delta_L : L \rightarrow L \boxtimes L$ is a $*$ -algebra map, $\Delta_L(x^*) = (\Delta_L(x))^*$ with $*$ -structure on $L \boxtimes L$ in (3.5).

The $*$ -algebra map condition for Δ_L is equivalent to $\Delta_L = * \circ \Delta_L \circ *$. This is well defined since $* \circ \Delta_L \circ * : L \rightarrow L \boxtimes L$ is a coassociative K -module map and an algebra map. The K -equivariance follows from that of Δ_L and the compatibility (3.1). The $*$ -algebra map property is straightforward and coassociativity is verified by direct computation.

The (braided) antipode S_L of a braided Hopf $*$ -algebra satisfies $S_L \circ * \circ S_L \circ * = \text{id}$ and so is invertible. Using (2.11), $\Delta_L \circ S_L = (S_L \otimes S_L) \circ \Psi_R \circ \Delta_L$, one gets:

$$\Delta_L \circ S_L^{-1} = (S_L^{-1} \otimes S_L^{-1}) \circ \bar{\Psi}_R \circ \Delta_L$$

with $\bar{\Psi}_R$ the inverse of Ψ_R : $\bar{\Psi}_R(x \otimes y) = \bar{R}^\alpha \triangleright_L y \otimes \bar{R}_\alpha \triangleright_L x$, for $x, y \in L$. Then, using (3.5), one gets

$$\Delta_L(S_L(x^*)) = R_\alpha R^\beta \triangleright S_L((x_{(2)})^*) \boxtimes R^\alpha R_\beta \triangleright S_L((x_{(1)})^*),$$

for each $x \in L$, together with

$$\Delta_L(S_L^{-1}(x^*)) = S_L^{-1}((x_{(2)})^*) \boxtimes S_L^{-1}((x_{(1)})^*). \tag{3.10}$$

A braided Hopf $*$ -algebra L acts on a K -module $*$ -algebra A , with action $\blacktriangleright_A : L \otimes A \rightarrow A$ satisfying (2.27), if the $*$ -structure of A satisfies the compatibility condition

$$(x \blacktriangleright_A a)^* = (\bar{R}^\alpha \triangleright_L S^{-1}(x^*)) \blacktriangleright_A (\bar{R}_\alpha \triangleright_A a^*) \tag{3.11}$$

that generalizes condition (3.1).

Proposition 3.6 *The compatibility condition (3.11) is well-defined.*

Proof The compatibility condition is equivalent to require that

$$x \succ_A a := \left[(\bar{R}^\alpha \triangleright_L S^{-1}(x^*)) \blacktriangleright_A (\bar{R}_\alpha \triangleright_A a^*) \right]^*$$

is an action of the braided Hopf algebra L on the K -module algebra A , which coincides with the starting action \blacktriangleright_A . We need to show that the map \succ defines a K -equivariant action that satisfies (2.27). To lighten the notation we omit the subscript on the actions. Firstly, \succ is an action. The condition $x \succ 1_A = \varepsilon_L(x)1_A$ follows from the unitality of R . Observing that the compatibility of the inverse antipode with the multiplication uses the inverse braiding, $S_L^{-1}(xy) = (\bar{R}^\alpha \triangleright_L S_L^{-1}(y))(\bar{R}_\alpha \triangleright_L S_L^{-1}(x))$, cf. (2.10), we compute

$$\begin{aligned} (xy) \succ v &= \left[(\bar{R}^\alpha \triangleright S^{-1}((xy)^*)) \blacktriangleright (\bar{R}_\alpha \triangleright v^*) \right]^* \\ &= \left[\left(\bar{R}^\alpha \triangleright ((\bar{R}^\gamma \triangleright S^{-1}(x^*))(\bar{R}_\gamma \triangleright S^{-1}(y^*))) \right) \blacktriangleright (\bar{R}_\alpha \triangleright v^*) \right]^* \\ &= \left[\left((\bar{R}^\alpha \triangleright_{(1)} \bar{R}^\gamma \triangleright S^{-1}(x^*))(\bar{R}^\alpha \triangleright_{(2)} \bar{R}_\gamma \triangleright S^{-1}(y^*)) \right) \blacktriangleright (\bar{R}_\alpha \triangleright v^*) \right]^* \\ &= \left[\left((\bar{R}^\beta \bar{R}^\gamma \triangleright S^{-1}(x^*))(\bar{R}^\alpha \bar{R}_\gamma \triangleright S^{-1}(y^*)) \right) \blacktriangleright (\bar{R}_\alpha \bar{R}_\beta \triangleright v^*) \right]^*. \end{aligned}$$

For the last equality we used the analogous of property (2.4) for \bar{R} . On the other hand

$$\begin{aligned} x \succ (y \succ v) &= x \succ \left[(\bar{R}^\alpha \triangleright S^{-1}(y^*)) \blacktriangleright (\bar{R}_\alpha \triangleright v^*) \right]^* \\ &= \left[(\bar{R}^\beta \triangleright S^{-1}(x^*)) \blacktriangleright \left(\bar{R}_\beta \triangleright \left[(\bar{R}^\alpha \triangleright S^{-1}(y^*)) \blacktriangleright (\bar{R}_\alpha \triangleright v^*) \right] \right) \right]^* \\ &= \left[(\bar{R}^\beta \triangleright S^{-1}(x^*))(\bar{R}_{\beta(1)} \bar{R}^\alpha \triangleright S^{-1}(y^*)) \blacktriangleright (\bar{R}_{\beta(2)} \bar{R}_\alpha \triangleright v^*) \right]^* \\ &= \left[(\bar{R}^\beta \bar{R}^\gamma \triangleright S^{-1}(x^*))(\bar{R}_\beta \bar{R}^\alpha \triangleright S^{-1}(y^*)) \blacktriangleright (\bar{R}_\gamma \bar{R}_\alpha \triangleright v^*) \right]^*, \end{aligned}$$

having used property (2.4) again. The two expressions then coincide due to the quantum Yang–Baxter equation $\bar{R}_{12}\bar{R}_{13}\bar{R}_{23} = \bar{R}_{23}\bar{R}_{13}\bar{R}_{12}$. Next we show K -equivariance:

$$\begin{aligned} k \triangleright (x \succ v) &= k \triangleright \left[(\bar{R}^\alpha \triangleright S^{-1}(x^*)) \blacktriangleright (\bar{R}_\alpha \triangleright v^*) \right]^* \\ &= \left[S^{-1}(k^*) \triangleright \left[(\bar{R}^\alpha \triangleright S^{-1}(x^*)) \blacktriangleright (\bar{R}_\alpha \triangleright v^*) \right] \right]^* \\ &= \left[\left((S^{-1}(k^*) \triangleright_{(1)} \bar{R}^\alpha) \triangleright S^{-1}(x^*) \right) \blacktriangleright \left((S^{-1}(k^*) \triangleright_{(2)} \bar{R}_\alpha) \triangleright v^* \right) \right]^* \\ &= \left[(\bar{R}^\alpha S^{-1}(k^*) \triangleright_{(2)} \triangleright S^{-1}(x^*)) \blacktriangleright (\bar{R}_\alpha S^{-1}(k^*) \triangleright_{(1)} \triangleright v^*) \right]^* \\ &= \left[(\bar{R}^\alpha S^{-1}(k^*) \triangleright S^{-1}(x^*)) \blacktriangleright (\bar{R}_\alpha S^{-1}(k^*) \triangleright v^*) \right]^* \\ &= \left[(\bar{R}^\alpha \triangleright S^{-1}(S^{-1}(k^*) \triangleright x^*)) \blacktriangleright (\bar{R}_\alpha S^{-1}(k^*) \triangleright v^*) \right]^* \end{aligned}$$

$$\begin{aligned}
 &= \left[(\bar{R}^\alpha \triangleright S^{-1}((k_{(1)} \triangleright x)^*)) \blacktriangleright (\bar{R}_\alpha \triangleright (k_{(2)} \triangleright v)^*) \right]^* \\
 &= (k_{(1)} \triangleright x) \succ (k_{(2)} \triangleright v)
 \end{aligned}$$

where for the fourth equality we used the co-commutativity (2.3) of the coproduct of K and for the sixth equality the K -equivariance of the braided antipode S of L .

We are left to show that (2.27) holds: using the same property for the action \blacktriangleright and property (3.10) for the inverse of the braided antipode, we compute

$$\begin{aligned}
 x \succ (aa') &= \left[(\bar{R}^\alpha \triangleright S^{-1}(x^*)) \blacktriangleright ((\bar{R}_{\alpha(1)} \triangleright a'^*)(\bar{R}_{\alpha(2)} \triangleright a^*)) \right]^* \\
 &= \left[(\bar{R}^\alpha \triangleright S^{-1}(x^*))_{(1)} \blacktriangleright (R_\gamma \bar{R}_{\alpha(1)} \triangleright a'^*) \right] \\
 &\quad \times \left[(R^\gamma \triangleright (\bar{R}^\alpha \triangleright S^{-1}(x^*))_{(2)}) \blacktriangleright (\bar{R}_{\alpha(2)} \triangleright a^*) \right]^* \\
 &= \left[(R^\gamma \triangleright (\bar{R}^\alpha \bar{R}^\mu \triangleright S^{-1}(x^*))_{(2)}) \blacktriangleright (\bar{R}_\mu \triangleright a^*) \right]^* \\
 &\quad \times \left[(\bar{R}^\alpha \bar{R}^\mu \triangleright S^{-1}(x^*))_{(1)} \blacktriangleright (R_\gamma \bar{R}_\alpha \triangleright a'^*) \right]^* \\
 &= \left[(R^\gamma \bar{R}^\alpha_{(2)} \bar{R}^\mu_{(2)} \triangleright S^{-1}(x_{(1)}^*)) \blacktriangleright (\bar{R}_\mu \triangleright a^*) \right]^* \\
 &\quad \times \left[(\bar{R}^\alpha_{(1)} \bar{R}^\mu_{(1)} \triangleright S^{-1}(x_{(2)}^*)) \blacktriangleright (R_\gamma \bar{R}_\alpha \triangleright a'^*) \right]^* \\
 &= \left[(R^\gamma \bar{R}^\alpha \bar{R}^\mu \triangleright S^{-1}(x_{(1)}^*)) \blacktriangleright (\bar{R}_\mu \bar{R}_\tau \triangleright a^*) \right]^* \\
 &\quad \times \left[(\bar{R}^\nu \bar{R}^\tau \triangleright S^{-1}(x_{(2)}^*)) \blacktriangleright (R_\gamma \bar{R}_\alpha \bar{R}_\nu \triangleright a'^*) \right]^* \\
 &= \left[(\bar{R}^\mu \triangleright S^{-1}(x_{(1)}^*)) \blacktriangleright (\bar{R}_\mu \bar{R}_\tau \triangleright a^*) \right]^* \\
 &\quad \times \left[(\bar{R}^\nu \bar{R}^\tau \triangleright S^{-1}(x_{(2)}^*)) \blacktriangleright (\bar{R}_\nu \triangleright a'^*) \right]^*
 \end{aligned}$$

On the other hand, also using the antireality of R ,

$$\begin{aligned}
 (x_{(1)} \succ (R_\alpha \triangleright a)) ((R^\alpha \triangleright x_{(2)}) \succ a') &= \left[(\bar{R}^\mu \triangleright S^{-1}(x_{(1)}^*)) \blacktriangleright (\bar{R}_\mu \triangleright (R_\alpha \triangleright a)^*) \right]^* \\
 &\quad \times \left[(\bar{R}^\nu \triangleright S^{-1}((R^\alpha \triangleright x_{(2)}^*)) \blacktriangleright (\bar{R}_\nu \triangleright a'^*) \right]^* \\
 &= \left[(\bar{R}^\mu \triangleright S^{-1}(x_{(1)}^*)) \blacktriangleright (\bar{R}_\mu \bar{R}_\alpha \triangleright a^*) \right]^* \left[(\bar{R}^\nu \triangleright S^{-1}(\bar{R}^\alpha \triangleright x_{(2)}^*)) \blacktriangleright (\bar{R}_\nu \triangleright a'^*) \right]^*
 \end{aligned}$$

and the two expressions coincide due to K -equivariance of the braided antipode. \square

3.2 Braided Lie \ast -algebras

In this section (K, R) is triangular, with $R^{\ast \otimes \ast} = R_{21} = \bar{R}$.

Definition 3.7 A \ast -structure on a K -braided Lie algebra \mathfrak{g} is an antilinear involution $\ast : \mathfrak{g} \rightarrow \mathfrak{g}$ which satisfies (3.1) and in addition $([u, v])^\ast = [v^\ast, u^\ast]$ for all $u, v \in \mathfrak{g}$.

The compatibility condition (3.1) is well-defined because it is equivalent to require that the K -action $\succ_{\mathfrak{g}}: K \otimes \mathfrak{g} \rightarrow \mathfrak{g}$ in (3.2), with (A, \cdot) replaced by $(\mathfrak{g}, [\ , \])$, is an action of K on the braided Lie algebra \mathfrak{g} , which coincides with the starting action $\triangleright_{\mathfrak{g}}$. The proof that $\succ_{\mathfrak{g}}: K \otimes \mathfrak{g} \rightarrow \mathfrak{g}$ is an action on the K -module \mathfrak{g} is as in (3.1). The K -equivariance property of the bracket, $k \succ_{\mathfrak{g}} [u, v] = [k_{(1)} \succ_{\mathfrak{g}} u, k_{(2)} \succ_{\mathfrak{g}} v]$ for all $k \in K$, that is, $k \succ_{\mathfrak{g}} \circ [\ , \] = [\ , \] \circ k \succ_{\mathfrak{g}}$ as maps $\mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$, is as in (3.4).

The compatibility of the action $\succ_{\mathfrak{g}}$ with the braided antisymmetry $[\ , \] = -[\ , \] \circ \Psi_R$ is due to K -equivariance of the braiding Ψ_R , see (3.7). Similarly, the compatibility of the action $\succ_{\mathfrak{g}}$ with the braided Jacobi identity (which is an equality between maps obtained from the bracket and the braiding) is due to K -equivariance of all the maps involved.

In the present paper the main example of K -braided Lie $*$ -algebra is that of braided derivations $\text{Der}(A)$ of a K -module $*$ -algebra A . Its $*$ -structure is defined by

$$\psi^*(a) := - \left((\bar{R}^\alpha \triangleright_{\text{Der}(A)} \psi) (\bar{R}_\alpha \triangleright_A a^*) \right)^*, \tag{3.12}$$

for all $\psi \in \text{Der}(A)$, $a \in A$. It lifts as an antilinear and antimultiplicative map to a $*$ -structure on the universal enveloping algebra $L = \mathcal{U}(\text{Der}(A))$. This $*$ -structure is compatible with the braided action $\blacktriangleright_A: \mathcal{U}(\text{Der}(A)) \otimes A \rightarrow A$ defined by $\psi \blacktriangleright_A a = \psi(a)$ for all $\psi \in \text{Der}(A) \subseteq \mathcal{U}(\text{Der}(A))$, $a \in A$. Since $\text{Der}(A)$ is the K -submodule of primitive elements, $S^{-1}(\psi) = -\psi$ and (3.12) also reads $\psi^*(a) = \left((\bar{R}^\alpha \triangleright_L S^{-1}(\psi)) (\bar{R}_\alpha \triangleright_A a^*) \right)^*$. This implies the compatibility:

$$x^* \blacktriangleright_A a = \left((\bar{R}^\alpha \triangleright_L S_L^{-1}(x)) \blacktriangleright_A (\bar{R}_\alpha \triangleright_A a^*) \right)^*. \tag{3.13}$$

This is the unique $*$ -structure compatible with \blacktriangleright_A , indeed (3.11) is equivalent to (3.13).

If the K -module $*$ -algebra A is quasi-commutative, see (2.17), the K -braided Lie $*$ -algebra $\text{Der}(A)$ is also a left A -module with action $\cdot : A \otimes \text{Der}(A) \rightarrow A$ defined in (2.18). This is compatible with the $*$ -structure: on $A \otimes \text{Der}(A)$ we have the $*$ -structure $(a \otimes \psi)^* = (R_\alpha \triangleright a^*) \otimes (R^\alpha \triangleright \psi^*)$ (cf. Lemma 3.3), then

$$(a \cdot \psi)^* = (R^\alpha \triangleright a^*) \cdot (R_\alpha \triangleright \psi^*)$$

for all $a \in A$, $\psi \in \text{Der}(A)$, that is, $* \circ \cdot = \cdot \circ *$ as maps $A \otimes \text{Der}(A) \rightarrow \text{Der}(A)$.

3.3 Twists and *-structures

3.3.1 Twist of Hopf *-algebras and their representations

A twist F of a Hopf *-algebra K is a twist of the Hopf algebra K that satisfies

$$F^{*\otimes*} = (S \otimes S)F_{21} . \tag{3.14}$$

Then K_F is a Hopf *-algebra with

$$*_F : K_F \rightarrow K_F , \quad k^{*F} := u k^* \bar{u} \tag{3.15}$$

where $u = F^\alpha S(F_\alpha)$ and $\bar{u} = S(\bar{F}^\alpha)\bar{F}_\alpha$ is its inverse. From (3.14) we have $u^* = u$, which implies that $*_F$ is involutive. The twist condition (2.22) implies the identity $F\Delta(u) = uS(F_{\beta(2)}) \otimes F^\beta S(F_{\beta(1)})$ or equivalently

$$F\Delta(u) = (u \otimes u)\bar{F}^{*\otimes*} . \tag{3.16}$$

Compatibility with the coproduct then follows: $\Delta_F(k^{*F}) = \Delta_F(k)^{*F\otimes*F}$ for all $k \in K$.

If A is a K -module *-algebra, A_F is a K_F -module *-algebra with $* : A_F \rightarrow A_F$ that is the same as the initial $*$ as antilinear map. Indeed this is antimultiplicative:

$$\begin{aligned} (a \cdot_F b)^* &= (\bar{F}_\alpha \triangleright b)^*(\bar{F}^\alpha \triangleright a)^* = S^{-1}(\bar{F}_\alpha^*) \triangleright b^* S^{-1}(\bar{F}^{\alpha*}) \triangleright a^* \\ &= \bar{F}^\alpha \triangleright b^* \bar{F}_\alpha \triangleright a^* = b^* \cdot_F a^* , \end{aligned} \tag{3.17}$$

where we used (3.1) and (3.14). Moreover it is compatible with the K_F -action,

$$(k \triangleright a)^* = (S(k))^* \triangleright a^* = (S_F(k))^{*F} \triangleright a^* = S_F^{-1}(k^{*F}) \triangleright a^* , \tag{3.18}$$

where we used $(S_F(k))^{*F} = (uS(k)\bar{u})^{*F} = (S(k))^*$ which holds since $u^* = u$.

3.3.2 Twist of quasitriangular and braided Hopf *-algebras, and of their representations

If $(K, R, *)$ is a quasitriangular Hopf *-algebra with R antireal, so is $(K_F, R_F, *_F)$ with R_F antireal. From (3.15), (3.16) and the equivalent expression $F_{21}^{*\otimes*}(\bar{u} \otimes \bar{u}) = \Delta^{cop}(\bar{u})\bar{F}_{21}$, we compute

$$\begin{aligned} R_F^{*F\otimes*F} &= (u \otimes u)(F_{21}R\bar{F})^{*\otimes*}(\bar{u} \otimes \bar{u}) = F\Delta(u)R^{*\otimes*}\Delta^{cop}(\bar{u})\bar{F}_{21} \\ &= F\Delta(u)\bar{R}\Delta^{cop}(\bar{u})\bar{F}_{21} = \bar{F}\bar{R}\bar{F}_{21} = \bar{R}_F \end{aligned}$$

where we used the quasi-cocommutativity property (2.3).

For a K -module algebra L one has the K_F -module algebra isomorphism

$$\varphi : L_F \boxtimes_F L_F \rightarrow (L \boxtimes L)_F, \quad x \boxtimes_F y \mapsto \varphi(x \boxtimes_F y) := \bar{F}^\alpha \triangleright x \boxtimes \bar{F}_\alpha \triangleright y \tag{3.19}$$

(leading to the monoidal equivalence of the categories of K -module algebras and K_F -module algebras). When L is a K -module $*$ -algebra so is $L \boxtimes L$ while $L_F, (L \boxtimes L)_F$ and $L_F \boxtimes_F L_F$ are K_F -module $*$ -algebras. This latter with $*$ -structure (cf. Lemma 3.3)

$$(x \boxtimes_F y)^{*F} = \Psi_{R_F}(y^* \boxtimes x^*) = (R_{F_\alpha} \triangleright x^*) \boxtimes (R_{F^\alpha} \triangleright y^*).$$

Lemma 3.8 *Let $(K, R, *)$ be a quasitriangular Hopf $*$ -algebra, with twist F and L a K -module $*$ -algebra. The isomorphism $\varphi : L_F \boxtimes_F L_F \rightarrow (L \boxtimes L)_F$ in (3.19) is a K_F -module $*$ -algebra isomorphism.*

Proof We show $\varphi^{-1} \circ * \circ \varphi = *_F$. Using the compatibility condition (3.1), (3.14) and $R_F = F_{21}R\bar{F}$ we have, for all $x, y \in L$,

$$\begin{aligned} \varphi^{-1}((\varphi(x \boxtimes_F y))^*) &= \varphi^{-1}((\bar{F}^\gamma \triangleright x \boxtimes \bar{F}_\gamma \triangleright y)^*) \\ &= F^\alpha R_\beta \triangleright (\bar{F}^\gamma \triangleright x)^* \boxtimes_F F_\alpha R^\beta \triangleright (\bar{F}_\gamma \triangleright y)^* \\ &= F^\alpha R_\beta S^{-1}(\bar{F}^{\gamma*}) \triangleright x^* \boxtimes_F F_\alpha R^\beta S^{-1}(\bar{F}_\gamma^*) \triangleright y^* \\ &= F^\alpha R_\beta \bar{F}_\gamma \triangleright x^* \boxtimes_F F_\alpha R^\beta \bar{F}^\gamma \triangleright y^* \\ &= R_{F_\beta} \triangleright x^* \boxtimes_F R_{F^\beta} \triangleright y^* = (x \boxtimes_F y)^{*F}. \end{aligned}$$

□

Theorem 3.9 *$(L_F, R_F, *)$ is a K_F -braided Hopf $*$ -algebra with $*$ -structure that, as an antilinear involution, is the same as that of $(L, R, *)$.*

Proof We already know that (L_F, R_F) is a K_F -braided Hopf algebra, we prove the compatibility with $*$. From (3.18) we see that $(L_F, *)$ is a K_F -module $*$ -algebra. Moreover the braided coproduct Δ_{L_F} is a $*$ -algebra map: $\Delta_{L_F}(x^*) = (\Delta_{L_F}(x))^*F$, for all $x \in L_F$. Indeed from (2.26) we have $\Delta_{L_F} \circ * = \varphi^{-1} \circ \Delta_L \circ * = \varphi^{-1} \circ * \circ \Delta_L = \varphi^{-1} \circ * \circ \varphi \circ \Delta_{L_F} = *_F \circ \Delta_{L_F}$. Thus $(L_F, R_F, *)$ is a K_F -braided Hopf $*$ -algebra according to Definition 3.5. □

If a K -braided Hopf $*$ -algebra L acts on a K -module $*$ -algebra A then the twisted K_F -braided Hopf $*$ -algebra L_F acts on the twisted K_F -module $*$ -algebra A_F with action

$$\blacktriangleright_{A_F}: L_F \otimes_F A_F \rightarrow A_F, \quad x \blacktriangleright_{A_F} a = (\bar{F}^\alpha \triangleright_{L_F} x) \blacktriangleright_A (\bar{F}_\alpha \triangleright_{A_F} a).$$

The compatibility of the $*$ -structure of A_F with the braided action $\blacktriangleright_{A_F}$ of L_F reads $(x \blacktriangleright_{A_F} a)^* = (\bar{R}_F^\beta \triangleright_{L_F} S_L^{-1}(x^*)) \blacktriangleright_{A_F} (\bar{R}_{F\beta} \triangleright_{A_F} a)$, cf. (3.11). This follows from $S_L^{-1}((k \triangleright x)^*) = S_L^{-1}(S^{-1}(k^*) \triangleright x^*) = S^{-1}(k^*) \triangleright S_L^{-1}(x^*)$, for $k \in K, x \in L$, owing to the K -equivariance of the braided antipode S_L .

3.3.3 Twist of braided Lie $*$ -algebras of derivations and gauge transformations

Let (K, R) be triangular. If \mathfrak{g} is a K -module Lie $*$ -algebra then \mathfrak{g}_F is a K_F -module Lie $*$ -algebra with the initial involution of \mathfrak{g} . The property $([u, v]_F)^* = [v^*, u^*]_F$ for all $u, v \in \mathfrak{g}_F$ is proven along the same lines of those in (3.17).

We now consider the K -braided Lie $*$ -algebra $\mathfrak{g} = \text{Der}(A)$ of derivations of the K -module $*$ -algebra A , with $*$ -structure defined in (3.12) and its universal enveloping $*$ -algebra $L = \mathcal{U}(\text{Der}(A))$. Similarly we have the K_F -braided Lie $*$ -algebra $\text{Der}(A_F)$, of derivations of the K_F -module $*$ -algebra A_F , with $*$ -structure $\tilde{*} := *_{\text{Der}(A_F)}$, defined as in (3.12),

$$\psi^{\tilde{*}}(a) := - \left((\bar{R}_F^\alpha \triangleright_{\text{Der}(A_F)} \psi) (\bar{R}_{F\alpha} \triangleright_{A_F} a^*) \right)^* \tag{3.20}$$

for all $\psi \in \text{Der}(A_F), a \in A_F$. The associated universal enveloping $*$ -algebra is $\mathcal{U}(\text{Der}(A_F))$.

Proposition 3.10 *The isomorphism $\mathcal{D} : \text{Der}(A)_F \rightarrow \text{Der}(A_F)$ of K_F -braided Lie algebras of Theorem 2.3 is a K_F -braided Lie $*$ -algebra isomorphism. It lifts to the isomorphism $\mathcal{D} : \mathcal{U}(\text{Der}(A)_F) \rightarrow \mathcal{U}(\text{Der}(A_F))$ of K_F -braided Hopf $*$ -algebras.*

Proof We have to show $\mathcal{D}(\psi^*) (a) = \mathcal{D}(\psi)^{\tilde{*}} (a)$ for all $\psi \in \text{Der}(A)_F, a \in A_F$. In analogy with (3.17) we have $(\bar{F}^\alpha \triangleright a)^* \otimes (\bar{F}_\alpha \triangleright \psi)^* = (\bar{F}_\alpha \triangleright a^*) \otimes (\bar{F}^\alpha \triangleright \psi^*)$ and therefore,

$$\begin{aligned} \mathcal{D}(\psi^*) (a) &= (\bar{F}^\alpha \triangleright_L \psi^*) (\bar{F}_\alpha \triangleright a) = (\bar{F}_\alpha \triangleright_L \psi)^* (\bar{F}^\alpha \triangleright a^*)^* \\ &= [(\bar{R}_\beta \triangleright_L S_L^{-1}(\bar{F}_\alpha \triangleright_L \psi)) (R^\beta \bar{F}^\alpha \triangleright a^*)]^* \\ &= [(\bar{R}_\beta \bar{F}_\alpha \triangleright_L S_L^{-1}(\psi)) (R^\beta \bar{F}^\alpha \triangleright a^*)]^* \end{aligned}$$

where we used (3.12) with $S(\psi) = -\psi$ (or (3.13)) and K -equivariance of the braided antipode S_L . On the other hand, by definition of the $*$ -structure on $\text{Der}(A_F)$ we have

$$\begin{aligned} \mathcal{D}(\psi)^{\tilde{*}} (a) &= [(\bar{R}_{F\beta} \triangleright_{L_F} S_L^{-1}(\mathcal{D}(\psi))) (R_F^\beta \triangleright a^*)]^* \\ &= [(\bar{R}_{F\beta} \triangleright_{L_F} \mathcal{D}(S_L^{-1}(\psi))) (R_F^\beta \triangleright a^*)]^* \\ &= [(\mathcal{D}(\bar{R}_{F\beta} \triangleright_L S_L^{-1}(\psi))) (R_F^\beta \triangleright a^*)]^* \\ &= [((\bar{F}^\alpha \bar{R}_{F\beta} \triangleright_L S_L^{-1}(\psi))) (\bar{F}_\alpha R_F^\beta \triangleright a^*)]^* . \end{aligned}$$

The proof then follows from $R_F = F_{21} R F^{-1}$.

The isomorphism $\mathcal{D} : \mathcal{U}(\text{Der}(A)_F) \rightarrow \mathcal{U}(\text{Der}(A_F))$ of K_F -braided Hopf algebras commutes with the $*$ -structures when restricted to the primitive elements $\text{Der}(A)_F$ and, since these are the generators, on all elements of $\mathcal{U}(\text{Der}(A)_F)$. The proof is by induction. If $x, y \in \mathcal{U}(\text{Der}(A)_F)$ satisfy $\mathcal{D}(x^*) = \mathcal{D}(x)^*$, $\mathcal{D}(y^*) = \mathcal{D}(y)^*$, then so does their product, $\mathcal{D}((x \cdot_F y)^*) = \mathcal{D}(y^* \cdot_F x^*) = \mathcal{D}(y^*) \bullet_F \mathcal{D}(x^*) = \mathcal{D}(y)^* \bullet_F \mathcal{D}(x)^* = (\mathcal{D}(x) \bullet_F \mathcal{D}(y))^* = (\mathcal{D}(x \cdot_F y))^*$. \square

Finally, for A a quasi-commutative K -module $*$ -algebra, $\text{Der}(A)$ is a K -braided Lie and A -module $*$ -algebra. The twisted algebra A_F is a K_F -braided quasi-commutative $*$ -algebra and $\text{Der}(A_F)$ a K_F -braided Lie and A_F -module $*$ -algebra. In particular,

$$([\psi, \eta]_{R_F})^{\tilde{*}} = [\eta^{\tilde{*}}, \psi^{\tilde{*}}]_{R_F}, \quad (a \bullet_F \psi)^{\tilde{*}} = (R_{F\alpha} \triangleright a^*) \bullet_F (R_F^\alpha \triangleright_{\text{Der}(A_F)} \psi^{\tilde{*}}). \tag{3.21}$$

From Theorem 2.3, Corollary 2.5 and Proposition 3.10, $\mathcal{D} : \text{Der}(A)_F \rightarrow \text{Der}(A_F)$ is an isomorphism of K_F -braided Lie and A_F -module $*$ -algebras. As in Theorem 2.6, for a Hopf–Galois extension this isomorphism restricts to equivariant derivations and to infinitesimal gauge transformations:

Corollary 3.11 *For $B = A^{coH} \subseteq A$ a (K, R) -equivariant Hopf–Galois extension with A a quasi-commutative H -comodule $*$ -algebra, the isomorphism $\mathcal{D} : \text{Der}(A)_F \rightarrow \text{Der}(A_F)$ of (K_F, r_F) -braided Lie and A_F -module $*$ -algebras restricts to isomorphisms*

$$\mathcal{D} : \text{Der}_{\mathcal{M}^H}^R(A)_F \rightarrow \text{Der}_{\mathcal{M}^H_F}^{R_F}(A_F), \quad \mathcal{D} : \text{aut}_B^R(A)_F \rightarrow \text{aut}_{B_F}^{R_F}(A_F)$$

of (K_F, R_F) -braided Lie and B_F -module $*$ -algebras.

4 Principal bundles over S_θ^4 and their gauge transformations

In this section we consider the twist deformation of the Hopf $SU(2)$ -bundle over the 4-sphere S_θ^4 , and then of the $SO(4)$ -bundle over S_θ^4 , seen as a homogeneous space.

4.1 The instanton bundle

The $H = \mathcal{O}(SU(2))$ Hopf–Galois extension $\mathcal{O}(S_\theta^4) \subset \mathcal{O}(S_\theta^7)$ of [12] can be obtained as a deformation by a twist on $K = \mathcal{O}(\mathbb{T}^2)$ of the Hopf–Galois extension $\mathcal{O}(S^4) \subset \mathcal{O}(S^7)$ of the classical $SU(2)$ Hopf bundle, [3]. We use that twist deformation in the framework of the theory developed in § 2.4, to obtain the braided Lie algebras $\text{Der}_{\mathcal{M}^H}(\mathcal{O}(S_\theta^7))$ and $\text{aut}_{\mathcal{O}(S_\theta^4)}(\mathcal{O}(S_\theta^7))$ from their classical counterparts $\text{Der}_{\mathcal{M}^H}(\mathcal{O}(S^7))$ and $\text{aut}_{\mathcal{O}(S^4)}(\mathcal{O}(S^7))$.

4.1.1 The classical Hopf bundle

Let us start with the Hopf–Galois extension $B \subset A$ of the classical $SU(2)$ -Hopf bundle $\pi : S^7 \rightarrow S^4$. The algebra $A := \mathcal{O}(S^7)$ is the commutative $*$ -algebra of coordinate functions on the 7-sphere S^7 with generators $\{z_a, z_a^*, a = 1, \dots, 4\}$, satisfying the sphere relation $\sum z_a^* z_a = 1$. It carries a right coaction of the Hopf algebra $\mathcal{O}(SU(2))$ of coordinate functions on $SU(2)$. This is the $*$ -algebra generated by commuting elements $\{w_j, w_j^*, j = 1, 2\}$, with $\sum w_j^* w_j = 1$, and standard Hopf algebra structure induced from the group structure of $SU(2)$. The right coaction $\delta : \mathcal{O}(S^7) \rightarrow \mathcal{O}(S^7) \otimes \mathcal{O}(SU(2))$ is defined on the algebra generators as

$$u \mapsto u \dot{\otimes} w, \quad u := \begin{pmatrix} z_1 & z_2 & z_3 & z_4 \\ -z_2^* & z_1^* & -z_4^* & z_3^* \end{pmatrix}^t, \quad w := \begin{pmatrix} w_1 & -w_2^* \\ w_2 & w_1^* \end{pmatrix}. \quad (4.1)$$

Here $\dot{\otimes}$ denotes the composition of the tensor product \otimes with matrix multiplication. As usual the coaction is extended to the whole $\mathcal{O}(S^7)$ as a $*$ -algebra morphism.

The $*$ -subalgebra $B = \mathcal{O}(S^7)^{co\mathcal{O}(SU(2))}$ of coinvariant elements for the coaction is identified with the $*$ -algebra $\mathcal{O}(S^4)$ of coordinate functions on the 4-sphere S^4 . As the algebraic counterpart of the principality of the Hopf bundle $\pi : S^7 \rightarrow S^4$, one has that the algebra $\mathcal{O}(S^7)$ is a (not trivial) faithfully flat Hopf–Galois extension of $\mathcal{O}(S^4)$.

A set of generators for the algebra B is given by the elements

$$\alpha := 2(z_1 z_3^* + z_2^* z_4), \quad \beta := 2(z_2 z_3^* - z_1^* z_4), \quad x := z_1 z_1^* + z_2 z_2^* - z_3 z_3^* - z_4 z_4^* \quad (4.2)$$

and their $*$ -conjugated α^*, β^* , with $x^* = x$. From the 7-sphere relation $\sum z_\mu^* z_\mu = 1$, it follows that they satisfy a 4-sphere relation $\alpha^* \alpha + \beta^* \beta + x^2 = 1$.

For future use we also note these generators satisfy the relations

$$\begin{aligned} (1-x)z_1 &= \alpha z_3 - \beta^* z_4 & (1-x)z_2 &= \alpha^* z_4 + \beta z_3 \\ (1+x)z_3 &= \alpha^* z_1 + \beta^* z_2 & (1+x)z_4 &= \alpha z_2 - \beta z_1 \end{aligned} \quad (4.3)$$

together with their $*$ -conjugated.

4.1.2 The equivariant derivations

Since the sphere S^7 and S^4 are the homogeneous spaces $S^7 = Spin(5)/SU(2)$ and $S^4 = Spin(5)/Spin(4) \simeq Spin(5)/SU(2) \times SU(2)$, the Hopf fibration $S^7 \rightarrow S^4$ is a $Spin(5)$ -equivariant $SU(2)$ -principal bundle. Then, the right-invariant vector fields $X \in so(5) \simeq spin(5)$ on $Spin(5)$ project to the right cosets S^7 and S^4 and generate the $\mathcal{O}(S^7)$ -module of vector fields on S^7 and the $\mathcal{O}(S^4)$ -module of those on S^4 . A

convenient generating set for the $\mathcal{O}(S^7)$ -module is given by the following right $SU(2)$ -invariant vector fields on S^7 (cf. [11]):

$$\begin{aligned}
 H_1 &= \frac{1}{2}(z_1\partial_1 - z_1^*\partial_1^* - z_2\partial_2 + z_2^*\partial_2^* - z_3\partial_3 + z_3^*\partial_3^* + z_4\partial_4 - z_4^*\partial_4^*) \\
 H_2 &= \frac{1}{2}(-z_1\partial_1 + z_1^*\partial_1^* + z_2\partial_2 - z_2^*\partial_2^* - z_3\partial_3 + z_3^*\partial_3^* + z_4\partial_4 - z_4^*\partial_4^*) \tag{4.4} \\
 E_{10} &= \frac{1}{\sqrt{2}}(z_1\partial_3 - z_3^*\partial_1^* - z_4\partial_2 + z_2^*\partial_4^*) & E_{-10} &= \frac{1}{\sqrt{2}}(z_3\partial_1 - z_1^*\partial_3^* - z_2\partial_4 + z_4^*\partial_2^*) \\
 E_{01} &= \frac{1}{\sqrt{2}}(z_2\partial_3 - z_3^*\partial_2^* + z_4\partial_1 - z_1^*\partial_4^*) & E_{0-1} &= \frac{1}{\sqrt{2}}(z_1\partial_4 - z_4^*\partial_1^* + z_3\partial_2 - z_2^*\partial_3^*) \\
 E_{11} &= -z_4\partial_3 + z_3^*\partial_4^* & E_{-1-1} &= z_4^*\partial_3^* - z_3\partial_4 \\
 E_{1-1} &= -z_1\partial_2 + z_2^*\partial_1^* & E_{-11} &= -z_2\partial_1 + z_1^*\partial_2^*. \tag{4.5}
 \end{aligned}$$

Here the partial derivatives ∂_a, ∂_a^* , are defined by $\partial_a(z_c) = \delta_{ac}$ and $\partial_a(z_c^*) = 0$ and similarly for $\partial_a^*, a, c = 1, 2, 3, 4$. The vector fields above are chosen so that their commutators close the Lie $*$ -algebra $so(5)$ in the form

$$\begin{aligned}
 [H_1, H_2] &= 0; & [H_j, E_r] &= r_j E_r; \\
 [E_r, E_{-r}] &= r_1 H_1 + r_2 H_2; & [E_r, E_s] &= N_{rs} E_{r+s}. \tag{4.6}
 \end{aligned}$$

The elements H_1, H_2 are the generators of the Cartan subalgebra, and E_r is labelled by

$$r = (r_1, r_2) \in \Gamma = \{(\pm 1, 0), (0, \pm 1), (\pm 1, \pm 1)\},$$

one of the eight roots. Also, $N_{rs} = 0$ if $r+s$ is not a root and $N_{rs} \in \{1, -1\}$ otherwise. The $*$ -structure is given by

$$H_j^* = H_j, \quad E_r^* = E_{-r}. \tag{4.7}$$

The $*$ -structure on vector fields X is defined by $X^*(f) = (S(X)(f^*))^* = -(X(f^*))^*$ for any function f , and one accordingly checks that for the vector fields in (4.4) and (4.5), $E_{-r}(z_a) = -(E_r(z_a^*))^*$ and $H_j(z_a) = -(H_j(z_a^*))^*$.

The vector fields (4.4) and (4.5), being invariant under the action of $SU(2)$, projects to a generating set for the $\mathcal{O}(S^4)$ -module of vector fields on S^4 . Explicitly one finds,

$$\begin{aligned}
 H_1^\pi &= \alpha\partial_\alpha - \alpha^*\partial_{\alpha^*} & H_2^\pi &= \beta\partial_\beta - \beta^*\partial_{\beta^*} \\
 E_{10}^\pi &= \frac{1}{\sqrt{2}}(2x\partial_{\alpha^*} - \alpha\partial_x) & E_{-10}^\pi &= \frac{1}{\sqrt{2}}(-2x\partial_\alpha + \alpha^*\partial_x) \\
 E_{11}^\pi &= \beta\partial_{\alpha^*} - \alpha\partial_{\beta^*} & E_{-1-1}^\pi &= -\beta^*\partial_\alpha + \alpha^*\partial_\beta \\
 E_{01}^\pi &= \frac{1}{\sqrt{2}}(2x\partial_{\beta^*} - \beta\partial_x) & E_{0-1}^\pi &= \frac{1}{\sqrt{2}}(-2x\partial_\beta + \beta^*\partial_x) \\
 E_{1-1}^\pi &= \beta^*\partial_{\alpha^*} - \alpha\partial_\beta & E_{-11}^\pi &= -\beta\partial_\alpha + \alpha^*\partial_{\beta^*} \tag{4.8}
 \end{aligned}$$

using analogous partial derivatives on $\mathcal{O}(S^4)$. Indeed the $\mathcal{O}(S^4)$ -module of vector fields on S^4 can be generated by the five elements $H_\mu = \partial_{\mu^*} - x_\mu D$, for $D = \sum_\mu x_\mu \partial_\mu$ the

Liouville vector field. The five weights μ are those of the representation [5] of $so(5)$ with

$$x_{00} = x, \quad x_{10} = \frac{1}{\sqrt{2}}\alpha, \quad x_{-10} = \frac{1}{\sqrt{2}}\alpha^*, \quad x_{01} = \frac{1}{\sqrt{2}}\beta, \quad x_{0-1} = \frac{1}{\sqrt{2}}\beta^*$$

and sphere relation $\sum_{\mu} x_{\mu}^* x_{\mu} = 1$. The commutators $[H_{\mu}, H_{\nu}]$ give the generators in (4.8).

Dually, the vector fields (4.4) and (4.5) are $H = \mathcal{O}(SU(2))$ -equivariant derivations and generate the $\mathcal{O}(S^4)$ -module of such derivations

$$\text{Der}_{\mathcal{M}^H}(\mathcal{O}(S^7)) = \{X \in \text{Der}(\mathcal{O}(S^7)) \mid \delta \circ X = (X \otimes \text{id}) \circ \delta\}. \tag{4.9}$$

The general H -equivariant derivation is then of the form

$$X = b_1 H_1 + b_2 H_2 + \sum_r b_r E_r \tag{4.10}$$

for generic elements $b_j, b_r \in \mathcal{O}(S^4)$. These derivations are real, that is $X^* = X$, if and only if $b_j^* = b_j$ and $b_r^* = b_{-r}$. On the generators of $\mathcal{O}(S^7)$ the derivation X is given as

$$X : \mathcal{O}(S^7) \rightarrow \mathcal{O}(S^7), \quad (z_1 \ z_2 \ z_3 \ z_4)^t \mapsto M \cdot (z_1 \ z_2 \ z_3 \ z_4)^t \tag{4.11}$$

where M is the 4×4 matrix with entries in $\mathcal{O}(S^4)$

$$M = \begin{pmatrix} a_1 & b_{1-1}^* & -b_{10}^* & b_{01} \\ -b_{1-1} & -a_1 & -b_{01}^* & -b_{10} \\ b_{10} & b_{01} & -a_2 & -b_{11} \\ -b_{01}^* & b_{10}^* & b_{11}^* & a_2 \end{pmatrix}, \quad a_1 = \frac{1}{2}(b_1 - b_2), \quad a_2 = \frac{1}{2}(b_1 + b_2). \tag{4.12}$$

The derivation (4.11) restricts to

$$X^{\pi} : \mathcal{O}(S^4) \rightarrow \mathcal{O}(S^4), \quad (\alpha \ \beta \ \alpha^* \ \beta^* \ x)^t \mapsto M^{\pi} (\alpha \ \beta \ \alpha^* \ \beta^* \ x)^t \tag{4.13}$$

with

$$M^{\pi} = \begin{pmatrix} b_1 & b_{1-1}^* & 0 & b_{11}^* & \sqrt{2}b_{10}^* \\ -b_{1-1} & b_2 & -b_{11}^* & 0 & \sqrt{2}b_{01}^* \\ 0 & b_{11} & -b_1 & b_{1-1} & \sqrt{2}b_{10} \\ -b_{11} & 0 & -b_{1-1}^* & -b_2 & \sqrt{2}b_{01} \\ -b_{10} & -b_{01} & -b_{10}^* & -b_{01}^* & 0 \end{pmatrix}. \tag{4.14}$$

4.1.3 The Lie \ast -algebra of gauge transformations

We next look for infinitesimal gauge transformations, that is H -equivariant derivations X as in (4.11) which are vertical: $X^\pi(b) = 0$, for $b \in \mathcal{O}(S^4)$. These are the kernel of the matrix M^π in (4.14). Their collection $\text{aut}_{\mathcal{O}(S^4)}(\mathcal{O}(S^7))$ is clearly an $\mathcal{O}(S^4)$ -module. It is also a Lie algebra with Lie bracket $[bX, b'X'] = bb'[X, X']$ for any $b, b' \in \mathcal{O}(S^4)$ and $X, X' \in \text{aut}_{\mathcal{O}(S^4)}(\mathcal{O}(S^7))$.

The $Spin(5)$ equivariance of the principal bundle $S^7 \rightarrow S^4$ implies that the Lie algebra $\text{aut}_{\mathcal{O}(S^4)}(\mathcal{O}(S^7))$ can be organised using the representation theory of the Lie algebra $so(5)$. Indeed, the $Spin(5)$ action on S^7 lifts to $\text{Der}_{\mathcal{M}H}(\mathcal{O}(S^7))$ via the adjoint action, $Ad_g X = L_g \circ X \circ L_g^{-1}$, where, as usual, $L_g(a)(p) = a(g^{-1}p)$ for $g \in Spin(5)$, $p \in S^7$ and $a \in \mathcal{O}(S^7)$. Since the $Spin(5)$ -action closes on the subalgebra $\mathcal{O}(S^4) \subseteq \mathcal{O}(S^7)$, it also closes on the Lie subalgebra $\text{aut}_{\mathcal{O}(S^4)}(\mathcal{O}(S^7))$ of vertical derivations, indeed $Ad_g X(b) = L_g(X(L_g^{-1}(b))) = 0$ for all $g \in Spin(5)$, $b \in \mathcal{O}(S^4)$. Infinitesimally, $[T, X](b) = 0$ for all $T \in so(5)$, $b \in \mathcal{O}(S^4)$.

It follows that $\text{aut}_{\mathcal{O}(S^4)}(\mathcal{O}(S^7)) = \bigoplus_\pi V_\pi$ as linear space, with the sum over a class of representations V_π of $so(5)$ of vertical $\mathcal{O}(SU(2))$ -equivariant derivations. This decomposition will be worked out in details in § 4.1.4.

Proposition 4.1 *The Lie \ast -algebra $\text{aut}_{\mathcal{O}(S^4)}(\mathcal{O}(S^7))$ of infinitesimal gauge transformations of the $\mathcal{O}(SU(2))$ -Hopf–Galois extension $\mathcal{O}(S^4) \subset \mathcal{O}(S^7)$ is generated, as an $\mathcal{O}(S^4)$ -module, by the elements*

$$\begin{aligned}
 K_1 &:= 2xH_2 + \beta^*\sqrt{2}E_{01} + \beta\sqrt{2}E_{0-1} \\
 K_2 &:= 2xH_1 + \alpha^*\sqrt{2}E_{10} + \alpha\sqrt{2}E_{-10} \\
 W_{01} &:= \sqrt{2}(\beta H_1 + \alpha^*E_{11} + \alpha E_{-11}) \\
 W_{0-1} &:= \sqrt{2}(\beta^*H_1 + \alpha^*E_{1-1} + \alpha E_{-1-1}) \\
 W_{10} &:= \sqrt{2}(\alpha H_2 - \beta^*E_{11} + \beta E_{1-1}) \\
 W_{-10} &:= \sqrt{2}(\alpha^*H_2 + \beta^*E_{-11} - \beta E_{-1-1}) \\
 W_{11} &:= 2xE_{11} + \alpha\sqrt{2}E_{01} - \beta\sqrt{2}E_{10} \\
 W_{-1-1} &:= 2xE_{-1-1} + \alpha^*\sqrt{2}E_{0-1} - \beta^*\sqrt{2}E_{-10} \\
 W_{1-1} &:= -2xE_{1-1} + \beta^*\sqrt{2}E_{10} + \alpha\sqrt{2}E_{0-1} \\
 W_{-11} &:= -2xE_{-11} + \beta\sqrt{2}E_{-10} + \alpha^*\sqrt{2}E_{01}.
 \end{aligned} \tag{4.15}$$

The \ast -structure is given by

$$K_j^* = K_j, \quad W_r^* = W_{-r}. \tag{4.16}$$

Proof An H -equivariant real derivation $X = b_1H_1 + b_2H_2 + \sum_r b_rE_r$ vanishes on $\mathcal{O}(S^4)$ if $M^\pi (\alpha \alpha^* \beta \beta^* x)^t = 0$, for the associated matrix M^π in (4.14). This reads

$$\begin{aligned}
 b_1\alpha + b_{1-1}^*\beta + b_{11}^*\beta^* + \sqrt{2}b_{10}^*x &= 0 \\
 -b_1\alpha^* + b_{11}\beta + b_{1-1}\beta^* + \sqrt{2}b_{10}x &= 0 \\
 -b_{1-1}\alpha - b_{11}^*\alpha^* + b_2\beta + \sqrt{2}b_{01}^*x &= 0 \\
 -b_{11}\alpha - b_{1-1}^*\alpha^* - b_2\beta^* + \sqrt{2}b_{01}x &= 0 \\
 b_{10}\alpha + b_{10}^*\alpha^* + b_{01}\beta + b_{01}^*\beta^* &= 0.
 \end{aligned}
 \tag{4.17}$$

At the algebraic level of the present paper, it is enough to look for solutions with entries of the matrix M^π that are linear in the $\mathcal{O}(S^4)$ generators. An explicit computation leads to the derivations

$$\begin{aligned}
 U_1 &= i(2xH_1 + \alpha^*\sqrt{2}E_{10} + \alpha\sqrt{2}E_{-10}) \\
 U_2 &= i(2xH_2 + \beta^*\sqrt{2}E_{01} + \beta\sqrt{2}E_{0-1}) \\
 W_1 &= (\beta^* - \beta)H_1 + \alpha^*(E_{1-1} - E_{11}) + \alpha(-E_{-11} + E_{-1-1}) \\
 W_2 &= i((\beta^* + \beta)H_1 + \alpha^*(E_{1-1} + E_{11}) + \alpha(E_{-1-1} + E_{-11})) \\
 W_3 &= (\alpha^* - \alpha)H_2 + \beta^*(E_{-11} + E_{11}) - \beta(E_{-1-1} + E_{1-1}) \\
 W_4 &= i((\alpha^* + \alpha)H_2 + \beta^*(E_{-11} - E_{11}) + \beta(E_{1-1} - E_{-1-1})) \\
 T_1 &= 2x(E_{11} - E_{-1-1}) + \sqrt{2}(\alpha E_{01} - \alpha^*E_{0-1} - \beta E_{10} + \beta^*E_{-10}) \\
 T_2 &= i(2x(E_{11} + E_{-1-1}) + \sqrt{2}(\alpha E_{01} + \alpha^*E_{0-1} - \beta E_{10} - \beta^*E_{-10})) \\
 T_3 &= 2x(E_{1-1} - E_{-11}) + \sqrt{2}(\beta E_{-10} + \alpha^*E_{01} - \beta^*E_{10} - \alpha E_{0-1}) \\
 T_4 &= i(2x(E_{1-1} + E_{-11}) - \sqrt{2}(\beta E_{-10} + \alpha^*E_{01} + \beta^*E_{10} + \alpha E_{0-1})).
 \end{aligned}$$

The derivations in (4.15) are obtained as the linear combinations

$$\begin{aligned}
 K_1 &= -iU_2, \quad K_2 = -iU_1, \quad W_{01} = -\frac{\sqrt{2}}{2}(W_1 + iW_2), \quad W_{0-1} = \frac{\sqrt{2}}{2}(W_1 - iW_2), \\
 W_{10} &= -\frac{\sqrt{2}}{2}(W_3 + iW_4), \quad W_{-10} = \frac{\sqrt{2}}{2}(W_3 - iW_4), \quad W_{11} = \frac{1}{2}(T_1 - iT_2), \\
 W_{-1-1} &= -\frac{1}{2}(T_1 + iT_2), \quad W_{1-1} = -\frac{1}{2}(T_3 - iT_4), \quad W_{-11} = \frac{1}{2}(T_3 + iT_4).
 \end{aligned}$$

Each vertical derivation, $X = b_1H_1 + b_2H_2 + \sum_r b_rE_r$, with $b_j, b_r \in \mathcal{O}(S^4)$ which satisfy (4.17) is expressed as combination of the vertical derivations K_j, W_r in (4.15) as

$$X = c_1K_1 + c_2K_2 + \sum_r c_rW_r$$

with coefficients $c_1, c_2, c_r \in \mathcal{O}(S^4)$ given by

$$c_1 = \frac{1}{4}(2xb_2 + \sqrt{2}\beta b_{01} + \sqrt{2}\beta^*b_{0-1}) \quad c_2 = \frac{1}{4}(2xb_1 + \sqrt{2}\alpha b_{10} + \sqrt{2}\alpha^*b_{-10})$$

$$\begin{aligned}
 c_{01} &= \frac{\sqrt{2}}{4}(\beta^*b_1 + \alpha b_{11} + \alpha^*b_{-11}) & c_{0-1} &= \frac{\sqrt{2}}{4}(\beta b_1 + \alpha b_{-11} + \alpha^*b_{-1-1}) \\
 c_{10} &= \frac{\sqrt{2}}{4}(\alpha^*b_2 - \beta b_{11} + \beta^*b_{-1-1}) & c_{-10} &= \frac{\sqrt{2}}{4}(\alpha b_2 + \beta b_{-11} - \beta^*b_{-1-1}) \\
 c_{11} &= \frac{1}{4}(2xb_{11} + \sqrt{2}\alpha^*b_{01} - \sqrt{2}\beta^*b_{10}) & c_{-1-1} &= \frac{1}{4}(2xb_{-1-1} + \sqrt{2}\alpha b_{0-1} - \sqrt{2}\beta b_{-10}) \\
 c_{1-1} &= \frac{1}{4}(-2xb_{-1-1} + \sqrt{2}\beta b_{10} + \sqrt{2}\alpha^*b_{0-1}) & c_{-11} &= \frac{1}{4}(-2xb_{-11} + \sqrt{2}\beta^*b_{-10} + \sqrt{2}\alpha b_{01}).
 \end{aligned}
 \tag{4.18}$$

The proof uses the equation (4.17) for the kernel of M^π . Indeed, from (4.15) one computes:

$$\begin{aligned}
 X &= c_1K_1 + c_2K_2 + \sum_r c_rW_r \\
 &= (c_{01}\sqrt{2}\beta + c_22x + c_{0-1}\sqrt{2}\beta^*)H_1 + (c_{12}2x + c_{10}\sqrt{2}\alpha + c_{-10}\sqrt{2}\alpha^*)H_2 \\
 &\quad + (c_{11}\beta^*\sqrt{2} + c_{111}\alpha\sqrt{2} + c_{-111}\alpha^*\sqrt{2})E_{01} \\
 &\quad + (c_{11}\beta\sqrt{2} + c_{-1-1}\alpha\sqrt{2} + c_{-1-1}\alpha^*\sqrt{2})E_{0-1} \\
 &\quad + (c_2\alpha^*\sqrt{2} + c_{-1-1}\beta^*\sqrt{2} - c_{111}\beta\sqrt{2})E_{10} \\
 &\quad + (c_2\alpha\sqrt{2} - c_{-1-1}\beta^*\sqrt{2} + c_{-11}\beta\sqrt{2})E_{-10} \\
 &\quad + (c_{01}\sqrt{2}\alpha^* - c_{10}\sqrt{2}\beta^* + c_{11}2x)E_{11} \\
 &\quad + (c_{01}\sqrt{2}\alpha + c_{-10}\sqrt{2}\beta^* - 2xc_{-11})E_{-11} \\
 &\quad + (c_{10}\sqrt{2}\beta - c_{-1-1}2x + c_{0-1}\sqrt{2}\alpha^*)E_{1-1} \\
 &\quad + (c_{0-1}\sqrt{2}\alpha - c_{-10}\sqrt{2}\beta + c_{-1-1}2x)E_{-1-1} \\
 &= b_1H_1 + b_2H_2 + \sum_r b_rE_r
 \end{aligned}$$

where the last equality follows from equations (4.17) for the coefficients b_j, b_r .

The generators in (4.15) satisfy $K_j(f^*) = -(K_j(f))^*$ and $W_r(f^*) = -(W_{-r}(f))^*$ for $f \in \mathcal{O}(S^7)$, from which one gets the $*$ -structure in (4.16). This also follows from $H_j^* = H_j$ and $E_r^* = E_{-r}$, in (4.7) using $(bX)^* = b^*X^*$, for $b \in \mathcal{O}(S^4)$ and X a derivation. □

The action of the vertical derivations K_j, W_r on the algebra generators z_a of $\mathcal{O}(S^7)$ is listed in Table 2 in Appendix 1.

Proposition 4.2 *The generators in (4.15) transform under the adjoint representation of $so(5)$ with highest weight vector W_{11} :*

$$\begin{aligned}
 H_j \triangleright K_l &= [H_j, K_l] = 0, & H_j \triangleright W_r &= [H_j, W_r] = r_j W_r, \\
 E_r \triangleright K_j &= [E_r, K_j] = -r_j W_r, \\
 E_r \triangleright W_{-r} &= [E_r, W_{-r}] = r_1 K_1 + r_2 K_2, & E_r \triangleright W_s &= [E_r, W_s] = N_{rs} W_{r+s},
 \end{aligned}
 \tag{4.19}$$

with N_{rs} the structure constants of $so(5)$ as before, with $N_{rs} = 0$ if $r+s$ is not a root.

Proof By direct computation. □

Remark 4.3 The generators in (4.15) are not independent over the algebra $\mathcal{O}(S^4)$. Indeed one finds they satisfy the relations:

$$\begin{aligned}
 &\beta W_{0-1} - \beta^* W_{01} + \alpha W_{-10} - \alpha^* W_{10} = 0 \\
 &-\beta K_2 + \sqrt{2}x W_{01} - \alpha^* W_{11} + \alpha W_{-11} = 0 \\
 &-\beta^* K_2 + \sqrt{2}x W_{0-1} - \alpha W_{-1-1} + \alpha^* W_{1-1} = 0 \\
 &-\alpha K_1 + \sqrt{2}x W_{10} + \beta^* W_{11} + \beta W_{-1-1} = 0 \\
 &-\alpha^* K_1 + \sqrt{2}x W_{-10} + \beta W_{-1-1} + \beta^* W_{-11} = 0.
 \end{aligned}
 \tag{4.20}$$

These relations have a deep geometrical meaning. They are the vanishing ‘vertical’ components of five vector fields which are horizontal for a canonical connection on the principal bundle [6]. These horizontal vector fields carry the five dimensional representation of $so(5)$ the smallest not trivial vector representation of $so(5)$ with highest weight vector of weight $(1, 0)$. On the other hand, any d -dimensional representation of $so(5)$ as vertical vector fields on S^7 vanishes when $d < 10$. Indeed, the only vertical equivariant derivation which is linear in the generators of $\mathcal{O}(S^4)$ and with weight $(1, 0)$ is $X_{10} = \alpha^* H_2 + \beta^* E_{-11} - \beta E_{-1-1}$. This generator is annihilated by $E_{1,1}$, $E_{1,0}$ and $E_{1,-1}$, but it is not by $E_{0,1}$. Since $E_{0,1} \triangleright X_{10}$ has weight $(1, 1)$ which is not present in the five-dimensional representation, we conclude that the minimal space of (linear in the generators of $\mathcal{O}(S^4)$) derivations is ten-dimensional.

4.1.4 A representation theoretical decomposition of $\text{aut}_{\mathcal{O}(S^4)}(\mathcal{O}(S^7))$

The result of multiplying the generators of $\mathcal{O}(S^4)$ with the ten vector fields in (4.15) can be organised using the representation theory of $so(5)$ (cf. [1, 2]). An irreducible representation of $so(5)$ is characterised by two non negative integers (s, n) and we denote it $[d(s, n)]$. It has highest weight vector of weight $\frac{s}{2}(1, 1) + n(1, 0)$ and is of dimension $d(s, n) = \frac{1}{6}(1+s)(1+n)(2+s+n)(3+s+2n)$. The generic vector field in the $\mathcal{O}(S^4)$ -module $\text{aut}_{\mathcal{O}(S^4)}(\mathcal{O}(S^7))$ is a combination of the vector fields in (4.15) with coefficients given by polynomials in the generators of $\mathcal{O}(S^4)$. The algebra $\mathcal{O}(S^4)$ decomposes in the sum of irreducible representations of $so(5)$ (spherical harmonics on S^4) as

$$\mathcal{O}(S^4) = \bigoplus_{n \in \mathbb{N}_0} [d(0, n)]
 \tag{4.21}$$

with $[d(0, n)]$ the representation of highest weight vector α^n of weight $(n, 0)$ consisting of polynomials of homogeneous degree n in the generators of $\mathcal{O}(S^4)$ (see Appendix 1).

The 50 vector fields obtained by multiplying the vector fields in (4.15) with the generators of $\mathcal{O}(S^4)$ can be arranged according to the representations $[35] \oplus [10] \oplus [5]$. The highest weight vectors for these three representations are worked out to be given

respectively by:

$$\begin{aligned} Z_{21} &= \alpha W_{11}, \\ Y_{11} &= \sqrt{2}xW_{11} + \alpha W_{01} - \beta W_{10}, \\ X_{10} &= \beta^* W_{11} + \beta W_{1-1} - \alpha K_1 + \sqrt{2}xW_{10}, \end{aligned} \tag{4.22}$$

with the label denoting the value of the corresponding weight.

Lemma 4.4 *When represented as vector fields on the bundle, the representation [5] generated by the vector X_{10} above vanishes. Also, $Y_{11} = -\sqrt{2}W_{11}$ so that the representation [10] generated by Y_{11} is the one in Proposition 4.2. The vector Z_{21} makes up the representation [35], none of whose vectors do vanish.*

Proof The action of $so(5)$ on the vector X_{10} yields the additional four vectors

$$\begin{aligned} X_{00} &= \beta^* W_{01} - \beta W_{0-1} + \alpha^* W_{10} - \alpha W_{-10} \\ X_{01} &= \beta K_2 - \sqrt{2}xW_{01} + \alpha^* W_{11} - \alpha W_{-11} \\ X_{-10} &= -\alpha^* K_1 + \sqrt{2}xW_{-10} + \beta W_{-1-1} + \beta^* W_{-11} \\ X_{0-1} &= -\beta^* K_2 + \sqrt{2}xW_{0-1} - \alpha W_{-1-1} + \alpha^* W_{1-1}. \end{aligned} \tag{4.23}$$

These five derivations vanish on $\mathcal{O}(S^7)$; they are in fact the vanishing combinations of derivations in (4.20). (The relations (4.20) have then a representation-theoretical meaning.) Using the 4-sphere relation and the relations in (4.3) one shows that

$$Y_{11} = \sqrt{2}xW_{11} + \alpha W_{01} - \beta W_{10} = -\sqrt{2}W_{11}. \tag{4.24}$$

Then the vector Y_{11} generates the starting representation [10] in (4.15) as stated. \square

By construction $\text{aut}_{\mathcal{O}(S^4)}(\mathcal{O}(S^7))$ is closed under commutator. It turns out that the commutators of the derivations in (4.15) can be expressed again in terms of the derivations in (4.15) with coefficients which are linear in the generators of $\mathcal{O}(S^4)$ (all commutators are listed in Appendix 1. More specifically we have the following:

Lemma 4.5 *The commutators of the derivations in (4.15) can be organised according to the representation $[35] \oplus [10]$ of $so(5)$ already found and generated by αW_{11} and W_{11} .*

Proof There are 45 commutators. The non vanishing commutator with highest weight is $[W_{11}, W_{10}]$ with weight $(2, 1)$. A direct computation shows that

$$[W_{11}, W_{10}] = -\sqrt{2}\alpha W_{11}$$

and the corresponding representation is the [35] found in the previous lemma.

The commutator $[W_{11}, W_{1-1}]$, the only one of weight $(2, 0)$, belongs to the representation [35]. The latter comprises also two combinations of the three vectors

$$[K_1, W_{11}], [K_2, W_{11}], [W_{10}, W_{01}].$$

of weight $(1, 1)$. On the other hand, their combination

$$T_{11} = [K_1, W_{11}] + [K_2, W_{11}] + [W_{10}, W_{01}]$$

is annihilated by all positive element of $so(5)$ and generates a copy of the representation [10]. In fact this is just the starting representation in (4.15) of Proposition 4.2. An explicit computation gives

$$\begin{aligned} [K_1, W_{11}] &= 2x W_{11} - \sqrt{2}\beta W_{10} \\ [K_2, W_{11}] &= 2x W_{11} + \sqrt{2}\alpha W_{01} \\ [W_{10}, W_{01}] &= -\sqrt{2}\beta W_{10} + \sqrt{2}\alpha W_{01} \end{aligned}$$

so that using the relation (4.24) one obtains

$$T_{11} = 2\sqrt{2}Y_{11} = -4W_{11}. \tag{4.25}$$

Thus the representation [10] generated by T_{11} is the one in (4.15) as stated. □

Proposition 4.6 *The Lie algebra $\text{aut}_{\mathcal{O}(S^4)}(\mathcal{O}(S^7))$ decomposes as*

$$\text{aut}_{\mathcal{O}(S^4)}(\mathcal{O}(S^7)) = \bigoplus_{n \in \mathbb{N}_0} [d(2, n)].$$

Here $[d(2, n)]$ is the representation of $so(5)$ as derivations on $\mathcal{O}(S^7)$ of highest weight vector $\alpha^n W_{11}$ of weight $(n + 1, 1)$ and dimension $d(2, n) = \frac{1}{2}(n + 1)(n + 4)(2n + 5)$.

Proof From the splitting (4.21) of $\mathcal{O}(S^4)$ (and Appendix 1), we need to consider the $10 \cdot d(0, n)$ vector fields obtained by multiplying the 10 vector fields in (4.15) with the polynomials of degree n in the generators of $\mathcal{O}(S^4)$ that are in the representation $[d(0, n)]$ of highest weight vector α^n . Of these, $\alpha^n W_{11} = \alpha^{n-1} Z_{21}$ is a highest weight vector and generates the representation $[d(2, n)]$. For the remaining vectors fields, $\alpha^{n-1} Y_{11}$ is a highest weight vector generating the representation $[d(2, n - 1)]$. There is then the highest weight vector $\alpha^{n-1} X_{10}$ generating the representation $[d(0, n)]$. And finally there is the highest weight vector $\alpha^{n-3} \rho^2 Z_{21} = \alpha^{n-3} Z_{21}$ of the representation $[d(2, n - 2)]$ (here $\rho^2 := \alpha\alpha^* + \beta\beta^* + x^2 = 1$). There is no room for additional vectors since a direct computation shows that $d(2, n) + d(2, n - 1) + d(0, n) + d(2, n - 2) = 10 \cdot d(0, n)$. Since $Z_{21} = \alpha W_{11}$, $Y_{11} = -\sqrt{2}W_{11}$ and $X_{10} = 0$, the only representation which has not yet appeared in lower degree is $[d(2, n)]$, the one of highest weight $\alpha^n W_{11}$. □

4.1.5 Braided derivations and infinitesimal gauge transformations

The right invariant vector fields H_1 and H_2 of $Spin(5)$ are the vector fields of a maximal torus $\mathbb{T}^2 \subset Spin(5)$. They define the universal enveloping algebra K of the abelian Lie algebra $[H_1, H_2] = 0$. Their action (4.4) on $\mathcal{O}(S^7)$ commutes with the $\mathcal{O}(SU(2))$ right coaction on $\mathcal{O}(S^7)$. To the torus 2-cocycle of [3, Ex. 3.21] there corresponds then a twist

$$F := e^{\pi i \theta (H_1 \otimes H_2 - H_2 \otimes H_1)}, \quad \theta \in \mathbb{R}, \tag{4.26}$$

with universal R -matrix $R_F = \bar{F}^2$. In fact these elements belong to a topological completion of the algebraic tensor product $K \otimes K$. This fact does not play a role here since we diagonalise F (we systematically use it on eigen-functions of the generators H_1, H_2).

The twist F in (4.26) hence leads to the $\mathcal{O}(SU(2))$ -Hopf-Galois extension $\mathcal{O}(S_\theta^4) = \mathcal{O}(S_\theta^7)^{co\mathcal{O}(SU(2))} \subset \mathcal{O}(S_\theta^7)$ introduced in [12]. It satisfies equation (3.14) so that it is compatible with the $*$ -structure. To conform with the literature, in the following we use the subscript θ instead of F for twisted algebras and their multiplications, as well as for module structures. The $*$ -algebra $\mathcal{O}(S_\theta^7)$ is generated by coordinates $z_a, z_a^*, a = 1, 2, 3, 4$. Their commutation relations are obtained from (2.25) given that for the action of H_1 and H_2 the z_a have eigenvalues $\frac{1}{2}(1, -1), \frac{1}{2}(-1, 1), \frac{1}{2}(-1, -1), \frac{1}{2}(1, 1)$, for $a = 1, 2, 3, 4$. The only nontrivial relations among the z_a are:

$$\begin{aligned} z_{1\bullet\theta} z_3 &= e^{\pi i \theta} z_{3\bullet\theta} z_1, & z_{1\bullet\theta} z_4 &= e^{-\pi i \theta} z_{4\bullet\theta} z_1, \\ z_{2\bullet\theta} z_3 &= e^{-\pi i \theta} z_{3\bullet\theta} z_2, & z_{2\bullet\theta} z_4 &= e^{\pi i \theta} z_{4\bullet\theta} z_2. \end{aligned}$$

Those with the z_a^* are obtained using that they have eigenvalues opposite to the eigenvalues of the z_a . These coordinates satisfy the relation $z_{1\bullet\theta} z_1^* + z_{2\bullet\theta} z_2^* + z_{3\bullet\theta} z_3^* + z_{4\bullet\theta} z_4^* = 1$. The $*$ -subalgebra $\mathcal{O}(S_\theta^4)$ of $\mathcal{O}(SU(2))$ -coinvariants is generated by

$$\begin{aligned} \alpha &:= 2(z_{1\bullet\theta} z_3^* + z_{2\bullet\theta}^* z_4), & \beta &:= 2(z_{2\bullet\theta} z_3^* - z_{1\bullet\theta}^* z_4), \\ x &:= z_{1\bullet\theta} z_1^* + z_{2\bullet\theta} z_2^* - z_{3\bullet\theta} z_3^* - z_{4\bullet\theta} z_4^*. \end{aligned} \tag{4.27}$$

The only nontrivial commutation relations are

$$\alpha_{\bullet\theta} \beta = e^{-2\pi i \theta} \beta_{\bullet\theta} \alpha, \quad \alpha_{\bullet\theta} \beta^* = e^{2\pi i \theta} \beta^*_{\bullet\theta} \alpha \tag{4.28}$$

and their complex conjugates. They can be obtained from the twisted multiplication rule (2.25) by using that α, β, x are eigen-functions of H_1 and H_2 , with eigenvalues $(1, 0), (0, 1)$ and $(0, 0)$ respectively. They satisfy the relations $\alpha_{\bullet\theta} \alpha^* + \beta_{\bullet\theta} \beta^* + x_{\bullet\theta} x = 1$. From these one then establishes:

$$\begin{aligned} (1-x)_{\bullet\theta} z_1 &= \alpha_{\bullet\theta} z_3 - z_{4\bullet\theta} \beta^*, & (1-x)_{\bullet\theta} z_2 &= z_{4\bullet\theta} \alpha^* + \beta_{\bullet\theta} z_3, \\ (1+x)_{\bullet\theta} z_3 &= \alpha^*_{\bullet\theta} z_1 + \beta^*_{\bullet\theta} z_2, & (1+x)_{\bullet\theta} z_4 &= z_{2\bullet\theta} \alpha - z_{1\bullet\theta} \beta. \end{aligned} \tag{4.29}$$

Remark 4.7 The relations (4.29) are the analogues of the classical ones (4.3). However, in passing from the algebra $\mathcal{O}(S^4)$ to the algebra $\mathcal{O}(S_\theta^4)$ we rescaled by a phase the classical elements α, β . In the vector space $\mathcal{O}(S_\theta^4) = \mathcal{O}(S^4)$, one has $x = x$ and

$$\begin{aligned} \alpha &= 2(z_1 \bullet_\theta z_3^* + z_2^* \bullet_\theta z_4) = e^{-\frac{\pi i \theta}{2}} 2(z_1 z_3^* + z_2^* z_4) = e^{-\frac{\pi i \theta}{2}} \alpha \\ \beta &= 2(z_2 \bullet_\theta z_3^* - z_1^* \bullet_\theta z_4) = e^{\frac{\pi i \theta}{2}} 2(z_2 z_3^* - z_1^* z_4) = e^{\frac{\pi i \theta}{2}} \beta . \end{aligned}$$

Since the Lie algebra $so(5)$ is a braided Lie algebra associated with K with trivial R -matrix $R = 1 \otimes 1$, we can twist it to the braided Lie algebra $so_\theta(5)$ associated with $(K_F, R_F = \bar{F}^2)$. It has Lie brackets (see Proposition 2.2)

$$\begin{aligned} [H_1, H_2]_F &= [H_1, H_2] = 0 ; & [H_j, E_r]_F &= [H_j, E_r] = r_j E_r ; \\ [E_r, E_{-r}]_F &= [E_r, E_{-r}] = r_1 H_1 + r_2 H_2 ; \\ [E_r, E_s]_F &= e^{-i\pi\theta r \wedge s} [E_r, E_s] = e^{-i\pi\theta r \wedge s} N_{rs} E_{r+s} , \end{aligned} \tag{4.30}$$

with $r \wedge s := r_1 s_2 - r_2 s_1$. Here, as for the $so(5)$ -commutators in (4.6), $N_{rs} = 0$ if $r+s$ is not a root.

Similarly, the $\mathcal{O}(S^4)$ -module and Lie $*$ -algebra $\text{Der}_{\mathcal{M}^H}(\mathcal{O}(S^7))$ is deformed to the $\mathcal{O}(S_\theta^4)$ -module and braided Lie $*$ -algebra $(\text{Der}_{\mathcal{M}^H}(\mathcal{O}(S^7)))_F, [,]_F, \cdot_F, *$ associated with (K_F, R_F) . The module $(\text{Der}_{\mathcal{M}^H}(\mathcal{O}(S^7)))_F$ is generated by derivations H_j and E_r with module structure in (2.34):

$$a \cdot_F H_j = a H_j , \quad a_s \cdot_F E_r = e^{-\pi i \theta s \wedge r} a_s E_r ,$$

for all $a \in \mathcal{O}(S_\theta^7)$ and $a_s \in \mathcal{O}(S_\theta^7)$ eigen-functions of H_j with eigenvalues s_j (being E_r eigenvectors of H_j). The Lie brackets are determined by those of $so_\theta(5)$ in (4.30) using equation (2.35) for the module structure. The $*$ -structure is the same as that of $\text{Der}_{\mathcal{M}^H}(\mathcal{O}(S^7))$, as stated in the first sentence of § 3.3.3.

The Lie $*$ -algebra of infinitesimal gauge transformations $\text{aut}_{\mathcal{O}(S^4)}(\mathcal{O}(S^7))$ is generated, as an $\mathcal{O}(S^4)$ -module, by the operators in (4.15). Its twist deformation is the $\mathcal{O}(S_\theta^4)$ -module and braided Lie $*$ -algebra $(\text{aut}_{\mathcal{O}(S^4)}(\mathcal{O}(S^7)))_F, [,]_F, \cdot_F, *$ associated with $(K_F, R_F = \bar{F}^2)$. It has braided Lie bracket determined on generators:

$$\begin{aligned} [K_1, K_2]_F &= [K_1, K_2] ; & [K_j, W_r]_F &= [K_j, W_r] ; \\ [W_r, W_s]_F &= e^{-i\pi\theta r \wedge s} [W_r, W_s] , \end{aligned} \tag{4.31}$$

in parallel with the result in (4.30). On generic elements X, X' in the linear span of the generators in (4.15) and $b, b' \in \mathcal{O}(S^4)$, the equation (2.35) gives

$$[b \cdot_F X, b' \cdot_F X']_F = b \cdot_F (R_{F\alpha} \triangleright b') \cdot_F [R_F^\alpha \triangleright X, X']_F . \tag{4.32}$$

From Corollary 3.11 we have $\text{Der}_{\mathcal{M}^H}(\mathcal{O}(S_\theta^7)) = \mathcal{D}((\text{Der}_{\mathcal{M}^H}(\mathcal{O}(S^7)))_{\mathbb{F}})$ with \mathcal{D} an isomorphism of $K_{\mathbb{F}}$ -braided Lie and $A_{\mathbb{F}}$ -module $*$ -algebras. In particular $\mathcal{D}(H_j)^{\tilde{*}} = \mathcal{D}(H_j^*) = \mathcal{D}(H_j)$ and $\mathcal{D}(E_r)^{\tilde{*}} = \mathcal{D}(E_r^*) = \mathcal{D}(E_{-r})$, $j = 1, 2, r \in \Gamma$ (cf. (4.7)). Thus, recalling (4.30):

Proposition 4.8 *The braided Lie $*$ -algebra $\text{Der}_{\mathcal{M}^H}(\mathcal{O}(S_\theta^7))$ of equivariant derivations of the $\mathcal{O}(SU(2))$ -Hopf–Galois extension $\mathcal{O}(S_\theta^4) \subset \mathcal{O}(S_\theta^7)$ is generated, as an $\mathcal{O}(S_\theta^4)$ -module by elements*

$$\tilde{H}_j := \mathcal{D}(H_j), \quad \tilde{E}_r := \mathcal{D}(E_r), \quad j = 1, 2, \quad r \in \Gamma \tag{4.33}$$

with bracket closing the braided Lie algebra $\mathfrak{so}_\theta(5)$:

$$\begin{aligned} [\tilde{H}_1, \tilde{H}_2]_{\mathbb{R}^{\mathbb{F}}} &= \mathcal{D}([H_1, H_2]) = 0; & [\tilde{H}_j, \tilde{E}_r]_{\mathbb{R}^{\mathbb{F}}} &= \mathcal{D}([H_j, E_r]) = r_j \tilde{E}_r; \\ [\tilde{E}_r, \tilde{E}_{-r}]_{\mathbb{R}^{\mathbb{F}}} &= \mathcal{D}([E_r, E_{-r}]) = \sum_j r_j \tilde{H}_j; \\ [\tilde{E}_r, \tilde{E}_s]_{\mathbb{R}^{\mathbb{F}}} &= e^{-i\pi\theta r \wedge s} \mathcal{D}([E_r, E_s]) = e^{-i\pi\theta r \wedge s} N_{rs} \tilde{E}_{r+s} \end{aligned} \tag{4.34}$$

(and $N_{rs} = 0$ if $r+s$ is not a root). The $\mathcal{O}(S_\theta^4)$ -module structure is in (2.36) (with $\bullet_{\mathbb{F}} = \bullet_\theta$). The $*$ -structure on generators is given by $\tilde{H}_j^{\tilde{*}} = \tilde{H}_j$ and $\tilde{E}_r^{\tilde{*}} = \tilde{E}_{-r}$.

For $a_s \in \mathcal{O}(S_\theta^7)$ an eigen-function of H_j of eigenvalue s_j , the derivation \tilde{E}_r acts as

$$\tilde{E}_r(a_s) = (\bar{F}^\alpha \triangleright E_r)(\bar{F}_\alpha \triangleright a_s) = e^{-i\pi\theta r \wedge s} E_r(a_s). \tag{4.35}$$

On the product of two such eigen-functions $a_s, a_m \in \mathcal{O}(S_\theta^7)$, we can explicitly see that \tilde{E}_r acts as a braided derivation, with respect to the braiding $\mathbb{R}_{\mathbb{F}} = F_{21} \bar{F} = \bar{F}^2$:

$$\begin{aligned} \tilde{E}_r(a_s \bullet_\theta a_m) &= e^{-i\pi\theta r \wedge (s+m)} e^{-i\pi\theta s \wedge m} E_r(a_s a_m) \\ &= e^{-i\pi\theta (r \wedge (s+m) + s \wedge m)} [E_r(a_s) a_m + a_s E_r(a_m)] \\ &= e^{-i\pi\theta (r \wedge (s+m) + s \wedge m)} [e^{i\pi\theta (r+s) \wedge m} E_r(a_s) \bullet_\theta a_m + e^{i\pi\theta s \wedge (r+m)} a_s \bullet_\theta E_r(a_m)] \\ &= e^{-i\pi\theta r \wedge s} E_r(a_s) \bullet_\theta a_m + e^{-2i\pi\theta (r \wedge s + r \wedge m)} a_s \bullet_\theta E_r(a_m) \\ &= \tilde{E}_r(a_s) \bullet_\theta a_m + e^{-2i\pi\theta r \wedge s} a_s \bullet_\theta \tilde{E}_r(a_m). \end{aligned} \tag{4.36}$$

Using these results for the subalgebra $\text{aut}_{\mathcal{O}(S_\theta^4)}(\mathcal{O}(S_\theta^7)) = \mathcal{D}(\text{aut}_{\mathcal{O}(S^4)}(\mathcal{O}(S^7))_{\mathbb{F}})$ of vertical derivations, we have the following characterization of $\text{aut}_{\mathcal{O}(S_\theta^4)}(\mathcal{O}(S_\theta^7))$.

Proposition 4.9 *The braided Lie $*$ -algebra $\text{aut}_{\mathcal{O}(S_\theta^4)}(\mathcal{O}(S_\theta^7))$ of infinitesimal gauge transformations of the $\mathcal{O}(SU(2))$ -Hopf–Galois extension $\mathcal{O}(S_\theta^4) \subset \mathcal{O}(S_\theta^7)$ is generated, as an $\mathcal{O}(S_\theta^4)$ -module, by the elements*

$$\tilde{K}_j := \mathcal{D}(K_j), \quad \tilde{W}_r := \mathcal{D}(W_r), \quad j = 1, 2, \quad r \in \Gamma \tag{4.37}$$

with bracket given in Table 1. The braided Lie bracket of generic elements \tilde{X}, \tilde{X}' in $\text{aut}_{\mathcal{O}(S^4_\theta)}(\mathcal{O}(S^7_\theta))$ and $b, b' \in \mathcal{O}(S^4_\theta)$ is then given by

$$[b_{\bullet_\theta} \tilde{X}, b'_{\bullet_\theta} \tilde{X}']_{\text{Rf}} = b_{\bullet_\theta} (\text{Rf}_\alpha \triangleright b')_{\bullet_\theta} [\text{Rf}^\alpha \triangleright \tilde{X}, \tilde{X}']_{\text{Rf}}. \tag{4.38}$$

The $*$ -structure on generators is given by $\tilde{K}_j^* = \tilde{K}_j$ and $\tilde{W}_r^* = \tilde{W}_{-r}$. It is extended on the whole $\text{aut}_{\mathcal{O}(S^4_\theta)}(\mathcal{O}(S^7_\theta))$ via (3.21).

Proof For all $\tilde{X}, \tilde{X}' \in \text{aut}_{\mathcal{O}(S^4_\theta)}(\mathcal{O}(S^7_\theta))$ we have $[\tilde{X}, \tilde{X}']_{\text{Rf}} = \mathcal{D}([X, X']_{\text{F}})$ from Proposition 2.6 with the bracket on the right hand side given in (4.31). Using the classical Lie brackets listed in Table 3 of Appendix 1, we can compute the brackets of the generators of the braided gauge Lie algebra $\text{aut}_{\mathcal{O}(S^4_\theta)}(\mathcal{O}(S^7_\theta))$. For instance, for $[\tilde{W}_{-1-1}, \tilde{W}_{01}]_{\text{Rf}}$ we first compute

$$\begin{aligned} [W_{-1-1}, W_{01}]_{\text{F}} &= e^{\pi i \theta} [W_{-1-1}, W_{01}] = e^{\pi i \theta} (\sqrt{2} \beta W_{-1-1} - \sqrt{2} \alpha^* (K_1 + K_2)) \\ &= e^{\pi i \theta} (e^{\pi i \theta} \sqrt{2} \beta \cdot_{\text{F}} W_{-1-1} - \sqrt{2} \alpha^* \cdot_{\text{F}} (K_1 + K_2)). \end{aligned}$$

Here we used (4.31) to relate the brackets $[,]_{\text{F}}$ and $[,]$, the module structure of $\text{aut}_{\mathcal{O}(S^4)}(\mathcal{O}(S^7))_{\text{F}}$ in (2.34) and that the coordinates of the sphere S^4 are eigenfunctions of H_1 and H_2 . Next, applying the algebra map \mathcal{D} leads to

$$\begin{aligned} [\tilde{W}_{-1-1}, \tilde{W}_{01}]_{\text{Rf}} &= \mathcal{D}([W_{-1-1}, W_{01}]_{\text{F}}) \\ &= e^{2\pi i \theta} \sqrt{2} \varphi_\beta \beta_{\bullet_\theta} \tilde{W}_{-1-1} - e^{\pi i \theta} \sqrt{2} \varphi_\alpha^* \alpha^*_{\bullet_\theta} (\tilde{K}_1 + \tilde{K}_2). \end{aligned}$$

Here to pass from the generators of $\mathcal{O}(S^4)$ to those of $\mathcal{O}(S^4_\theta)$ we used (see Remark 4.7):

$$\alpha = \varphi_\alpha \alpha := e^{\frac{\pi i \theta}{2}} \alpha, \quad \beta = \varphi_\beta \beta := e^{-\frac{\pi i \theta}{2}} \beta. \tag{4.39}$$

Half of the brackets among the generators (4.37) of the braided Lie algebra $\text{aut}_{\mathcal{O}(S^4_\theta)}(\mathcal{O}(S^7_\theta))$ are listed in Table 1. The remaining ones can be obtained similarly, or more directly using the compatibility of the $*$ -structure with the braided Lie algebra and $\mathcal{O}(S^4)$ -module structures as in (3.21),

$$[\tilde{X}'^*, \tilde{X}^*]_{\text{Rf}} = ([\tilde{X}, \tilde{X}']_{\text{Rf}})^*, \quad (b_{\bullet_\theta} \tilde{X})^* = (\text{Rf}_\gamma \triangleright b^*)_{\bullet_\theta} (\text{Rf}' \triangleright \tilde{X}^*).$$

Recalling that \mathcal{D} is a $*$ -isomorphism we have

$$\tilde{K}_j^* = \mathcal{D}(K_j^*) = \mathcal{D}(K_j) = \tilde{K}_j, \quad \tilde{W}_r^* = \mathcal{D}(W_r^*) = \mathcal{D}(W_{-r}) = \tilde{W}_{-r}.$$

Therefore,

$$[\tilde{K}_j, \tilde{W}_{-r}]_{\text{Rf}} = [\tilde{K}_j^*, \tilde{W}_r^*]_{\text{Rf}} = ([\tilde{W}_r, \tilde{K}_j]_{\text{Rf}})^*,$$

$$[\tilde{W}_{-r}, \tilde{W}_{-s}]_{R_F} = [\tilde{W}_r^*, \tilde{W}_s^*]_{R_F} = ([\tilde{W}_s, \tilde{W}_r]_{R_F})^{\bar{*}},$$

with

$$[\tilde{W}_r, \tilde{K}_j]_{R_F} = -[\tilde{K}_j, \tilde{W}_r]_{R_F}, \quad [\tilde{W}_s, \tilde{W}_r]_{R_F} = -e^{-2i\pi\theta s \wedge r} [\tilde{W}_r, \tilde{W}_s]_{R_F}.$$

For instance,

$$\begin{aligned} [\tilde{K}_2, \tilde{W}_{0-1}]_{R_F} &= -([\tilde{K}_2, \tilde{W}_{01}]_{R_F})^{\bar{*}} = -\sqrt{2}(e^{-\pi i\theta} \varphi_\alpha^* \alpha^*_{\bullet\theta} \tilde{W}_{11} + e^{\pi i\theta} \varphi_\alpha \alpha_{\bullet\theta} \tilde{W}_{-11})^{\bar{*}} \\ &= -\sqrt{2}(e^{-\pi i\theta} \varphi_\alpha \alpha_{\bullet\theta} \tilde{W}_{-1-1} + e^{\pi i\theta} \varphi_\alpha^* \alpha^*_{\bullet\theta} \tilde{W}_{1-1}) \end{aligned}$$

while

$$\begin{aligned} [\tilde{W}_{10}, \tilde{W}_{0-1}]_{R_F} &= -(e^{-2\pi i\theta} [\tilde{W}_{-10}, \tilde{W}_{01}]_{R_F})^{\bar{*}} \\ &= -e^{2\pi i\theta} \sqrt{2}(e^{2\pi i\theta} \varphi_\beta \beta_{\bullet\theta} \tilde{W}_{-10} + \varphi_\alpha^* \alpha^*_{\bullet\theta} \tilde{W}_{01})^{\bar{*}} \\ &= -e^{2\pi i\theta} \sqrt{2}(\varphi_\beta^* \beta^*_{\bullet\theta} \tilde{W}_{10} + e^{-2\pi i\theta} \varphi_\alpha \alpha_{\bullet\theta} \tilde{W}_{0-1}) \\ &= -\sqrt{2}(e^{2\pi i\theta} \varphi_\beta^* \beta^*_{\bullet\theta} \tilde{W}_{10} + \varphi_\alpha \alpha_{\bullet\theta} \tilde{W}_{0-1}). \end{aligned}$$

□

The action of any element \tilde{W}_r on an algebra element $a_s \in \mathcal{O}(S_\theta^7)$ is as in (4.35),

$$\tilde{W}_r(a_s) = e^{-i\pi\theta r \wedge s} W_r(a_s), \tag{4.40}$$

with a braided derivation property as in (4.36),

$$\tilde{W}_r(a_s \bullet_\theta a_m) = \tilde{W}_r(a_s) \bullet_\theta a_m + e^{-2i\pi\theta r \wedge s} a_s \bullet_\theta \tilde{W}_r(a_m). \tag{4.41}$$

4.2 The orthogonal bundle on the homogeneous space S_θ^4

The 4-sphere of the previous example is the prototype of more general noncommutative θ -spheres S_θ^{2n} . These are quantum homogeneous spaces of quantum groups $SO_\theta(2n + 1, \mathbb{R})$ [7, 14]. The Hopf–Galois extension $\mathcal{O}(S_\theta^{2n}) = \mathcal{O}(SO_\theta(2n + 1, \mathbb{R}))^{co\mathcal{O}(SO_\theta(2n, \mathbb{R}))} \subset \mathcal{O}(SO_\theta(2n + 1, \mathbb{R}))$ was obtained in [3] from the extension of the classical $SO(2n)$ -bundle $SO(2n + 1) \rightarrow S^{2n}$ via a twist deformation process for quantum homogeneous spaces.

As in the previous section, the modules of infinitesimal gauge transformations of these noncommutative Hopf–Galois extensions are obtained by deforming those of the corresponding classical bundles. We study here the case $n = 2$ of the Hopf–Galois extension $\mathcal{O}(S_\theta^4) = \mathcal{O}(SO_\theta(5, \mathbb{R}))^{co\mathcal{O}(SO_\theta(4, \mathbb{R}))} \subset \mathcal{O}(SO_\theta(5, \mathbb{R}))$. We address the general case in [6].

Table 1 The braided brackets of vertical derivations

$$\begin{aligned}
 [\tilde{K}_1, \tilde{K}_2]_{R_F} &= \sqrt{2}(\varphi_\alpha^* \alpha^*_{\bullet\theta} \tilde{W}_{10} - \varphi_\alpha \alpha_{\bullet\theta} \tilde{W}_{-10}) \\
 [\tilde{K}_1, \tilde{W}_{01}]_{R_F} &= -\sqrt{2} \varphi_\beta \beta_{\bullet\theta} \tilde{K}_2 + 2x_{\bullet\theta} \tilde{W}_{01} \\
 [\tilde{K}_1, \tilde{W}_{-1}]_{R_F} &= -2x_{\bullet\theta} \tilde{W}_{-1} + \sqrt{2} e^{\pi i \theta} \varphi_\beta^* \beta^*_{\bullet\theta} \tilde{W}_{10} \\
 [\tilde{K}_1, \tilde{W}_{10}]_{R_F} &= \sqrt{2} e^{-\pi i \theta} \varphi_\beta \beta_{\bullet\theta} \tilde{W}_{-1} - \sqrt{2} e^{\pi i \theta} \varphi_\beta^* \beta^*_{\bullet\theta} \tilde{W}_{11} \\
 [\tilde{K}_1, \tilde{W}_{11}]_{R_F} &= 2x_{\bullet\theta} \tilde{W}_{11} - \sqrt{2} e^{-\pi i \theta} \varphi_\beta \beta_{\bullet\theta} \tilde{W}_{10} \\
 [\tilde{K}_2, \tilde{W}_{01}]_{R_F} &= \sqrt{2} e^{-\pi i \theta} \varphi_\alpha^* \alpha^*_{\bullet\theta} \tilde{W}_{11} + \sqrt{2} e^{\pi i \theta} \varphi_\alpha \alpha_{\bullet\theta} \tilde{W}_{-11} \\
 [\tilde{K}_2, \tilde{W}_{-1}]_{R_F} &= 2x_{\bullet\theta} \tilde{W}_{-1} - \sqrt{2} e^{-\pi i \theta} \varphi_\alpha \alpha_{\bullet\theta} \tilde{W}_{0-1} \\
 [\tilde{K}_2, \tilde{W}_{10}]_{R_F} &= 2x_{\bullet\theta} \tilde{W}_{10} - \sqrt{2} \varphi_\alpha \alpha_{\bullet\theta} \tilde{K}_1 \\
 [\tilde{K}_2, \tilde{W}_{11}]_{R_F} &= 2x_{\bullet\theta} \tilde{W}_{11} + \sqrt{2} e^{\pi i \theta} \varphi_\alpha \alpha_{\bullet\theta} \tilde{W}_{01} \\
 [\tilde{W}_{01}, \tilde{W}_{-1}]_{R_F} &= \sqrt{2} \varphi_\beta \beta_{\bullet\theta} \tilde{W}_{-1} + \sqrt{2} e^{\pi i \theta} \varphi_\alpha \alpha_{\bullet\theta} (\tilde{K}_2 - \tilde{K}_1) \\
 [\tilde{W}_{01}, \tilde{W}_{10}]_{R_F} &= \sqrt{2} \varphi_\beta \beta_{\bullet\theta} \tilde{W}_{10} - \sqrt{2} e^{\pi i \theta} \varphi_\alpha \alpha_{\bullet\theta} \tilde{W}_{01} \\
 [\tilde{W}_{01}, \tilde{W}_{11}]_{R_F} &= \sqrt{2} \varphi_\beta \beta_{\bullet\theta} \tilde{W}_{11} \\
 [\tilde{W}_{-1}, \tilde{W}_{10}]_{R_F} &= \sqrt{2} \varphi_\alpha \alpha_{\bullet\theta} \tilde{W}_{-1} \\
 [\tilde{W}_{-1}, \tilde{W}_{11}]_{R_F} &= -\sqrt{2} e^{-\pi i \theta} \varphi_\alpha \alpha_{\bullet\theta} \tilde{W}_{10} \\
 [\tilde{W}_{10}, \tilde{W}_{11}]_{R_F} &= \sqrt{2} \varphi_\alpha \alpha_{\bullet\theta} \tilde{W}_{11} \\
 [\tilde{W}_{-1-1}, \tilde{W}_{01}]_{R_F} &= \sqrt{2} e^{2\pi i \theta} \varphi_\beta \beta_{\bullet\theta} \tilde{W}_{-1-1} - \sqrt{2} e^{\pi i \theta} \varphi_\alpha^* \alpha^*_{\bullet\theta} (\tilde{K}_1 + \tilde{K}_2) \\
 [\tilde{W}_{-1-1}, \tilde{W}_{-1}]_{R_F} &= \sqrt{2} e^{-2\pi i \theta} \varphi_\beta^* \beta^*_{\bullet\theta} \tilde{W}_{0-1} \\
 [\tilde{W}_{-1-1}, \tilde{W}_{10}]_{R_F} &= \sqrt{2} \varphi_\alpha \alpha_{\bullet\theta} \tilde{W}_{-1-1} + \sqrt{2} e^{\pi i \theta} \varphi_\beta^* \beta^*_{\bullet\theta} (\tilde{K}_1 + \tilde{K}_2) \\
 [\tilde{W}_{-1-1}, \tilde{W}_{11}]_{R_F} &= -2x_{\bullet\theta} (\tilde{K}_1 + \tilde{K}_2) - \sqrt{2} \varphi_\alpha \alpha_{\bullet\theta} \tilde{W}_{-10} - \sqrt{2} \varphi_\beta \beta_{\bullet\theta} \tilde{W}_{0-1} \\
 [\tilde{W}_{-10}, \tilde{W}_{01}]_{R_F} &= \sqrt{2} e^{2\pi i \theta} \varphi_\beta \beta_{\bullet\theta} \tilde{W}_{-10} + \sqrt{2} \varphi_\alpha^* \alpha^*_{\bullet\theta} \tilde{W}_{01} \\
 [\tilde{W}_{-10}, \tilde{W}_{-1}]_{R_F} &= -\sqrt{2} \varphi_\alpha^* \alpha^*_{\bullet\theta} \tilde{W}_{-1-1} + \sqrt{2} e^{-\pi i \theta} \varphi_\beta^* \beta^*_{\bullet\theta} (\tilde{K}_2 - \tilde{K}_1) \\
 [\tilde{W}_{10}, \tilde{W}_{-10}]_{R_F} &= \sqrt{2} (\varphi_\beta^* \beta^*_{\bullet\theta} \tilde{W}_{01} + \varphi_\beta \beta_{\bullet\theta} \tilde{W}_{0-1}) \\
 [\tilde{W}_{-11}, \tilde{W}_{01}]_{R_F} &= \sqrt{2} e^{2\pi i \theta} \varphi_\beta \beta_{\bullet\theta} \tilde{W}_{-11} \\
 [\tilde{W}_{0-1}, \tilde{W}_{01}]_{R_F} &= \sqrt{2} (\varphi_\alpha^* \alpha^*_{\bullet\theta} \tilde{W}_{10} - \varphi_\alpha \alpha_{\bullet\theta} \tilde{W}_{-10}) \\
 [\tilde{W}_{-11}, \tilde{W}_{-1}]_{R_F} &= 2x_{\bullet\theta} (\tilde{K}_1 - \tilde{K}_2) - \sqrt{2} \varphi_\beta^* \beta^*_{\bullet\theta} \tilde{W}_{01} + \sqrt{2} \varphi_\alpha \alpha_{\bullet\theta} \tilde{W}_{-10}
 \end{aligned}$$

Let $\mathcal{O}(M(4, \mathbb{R}))$ be the commutative complex $*$ -algebra with generators m_{JL} , and capital indices J, L running from 1 to 4. It has the standard bialgebra structures

$$\Delta(M) = M \dot{\otimes} M \quad , \quad \varepsilon(M) = \mathbb{1}_4, \quad \text{for } M = (m_{JL}), \quad (4.42)$$

in matrix notation, where $\dot{\otimes}$ denotes the combination of tensor product and matrix multiplication. The Hopf algebra $\mathcal{O}(SO(4, \mathbb{R}))$ of coordinate functions on $SO(4, \mathbb{R})$ is the quotient of $\mathcal{O}(M(4, \mathbb{R}))$ by the bialgebra ideal

$$I_Q := \langle M^t Q M - Q \ ; \ M Q M^t - Q \rangle, \quad Q := \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix} = Q^t = Q^{-1} \quad (4.43)$$

and the further assumption $\det(M) = 1$. Indeed, this is a $*$ -ideal for the $*$ -structure $*(M) = Q M Q^t$ in $\mathcal{O}(M(4, \mathbb{R}))$. If we introduce on the set of indices $\{1, \dots, 4\}$ the involution defined by $1' = 3$ and $2' = 4$, the $*$ -structure can be given as

$$(m_{JL})^* = m_{J'L'}, \quad J, L = 1, \dots, 4. \quad (4.44)$$

The $*$ -bialgebra $\mathcal{O}(SO(4, \mathbb{R}))$ is a Hopf $*$ -algebra with antipode $S(M) := Q M^t Q^{-1}$.

Similarly, one has the algebra $\mathcal{O}(M(5, \mathbb{R}))$, the commutative $*$ -bialgebra with generators n_{JL} (capital indices J, L now run from 1 to 5). The coproduct and counit are

$$\Delta(N) = N \dot{\otimes} N, \quad \varepsilon(N) = \mathbb{1}_5, \quad \text{for } N := (n_{JL}). \tag{4.45}$$

The algebra of coordinate functions on $SO(5, \mathbb{R})$ is the quotient of $\mathcal{O}(M(5, \mathbb{R}))$ by the bialgebra $*$ -ideal

$$J_Q = \langle N^t Q N - Q ; N Q N^t - Q \rangle, \quad Q := \begin{pmatrix} 0 & \mathbb{1}_2 & 0 \\ \mathbb{1}_2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{4.46}$$

and the additional requirement that $\det(N) = 1$. The $*$ -structure is now

$$(n_{JL})^* = n_{J'L'}, \quad J, L = 1, \dots, 5 \tag{4.47}$$

with $S' = 5$. Then $\mathcal{O}(SO(5, \mathbb{R}))$ is a Hopf $*$ -algebra with antipode $S(N) := Q N^t Q^{-1}$.

We shall select the last column of N by writing $n_{j5} = u_j$, for $j = 1, \dots, 4$ and $n_{55} = x$.

The surjective Hopf $*$ -algebra morphism

$$\pi : \mathcal{O}(SO(5, \mathbb{R})) \longrightarrow \mathcal{O}(SO(4, \mathbb{R})), \quad N \longmapsto \begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix} \tag{4.48}$$

induces a right coaction of $\mathcal{O}(SO(4, \mathbb{R}))$ on $\mathcal{O}(SO(5, \mathbb{R}))$:

$$\begin{aligned} \delta := (\text{id} \otimes \pi) \Delta : \mathcal{O}(SO(5, \mathbb{R})) &\longrightarrow \mathcal{O}(SO(5, \mathbb{R})) \otimes \mathcal{O}(SO(4, \mathbb{R})) \\ N &\longmapsto N \dot{\otimes} \begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned} \tag{4.49}$$

The $*$ -subalgebra of $\mathcal{O}(SO(5, \mathbb{R}))$ made of coinvariant elements is isomorphic to the $*$ -algebra of coordinate functions $\mathcal{O}(S^4)$ on the 4-sphere S^4 . It is indeed generated by the elements u_i, u_i^* and x in the last column of the defining matrix $N = (n_{JK})$ of $\mathcal{O}(SO(5, \mathbb{R}))$, which satisfy the sphere equation $(S(N)N)_{55} = 2(u_1^* u_1 + u_2^* u_2) + x^2 = 1$. The algebra extension $\mathcal{O}(S^4) = \mathcal{O}(SO(5, \mathbb{R}))^{co\mathcal{O}(SO(4, \mathbb{R}))} \subset \mathcal{O}(SO(5, \mathbb{R}))$ is Hopf–Galois.

The coordinate algebra of this orthogonal 4-sphere is isomorphic to the one of the 4-sphere of the previous section (the base space algebra of the $SU(2)$ -fibration) with generators in (4.2) via the identification

$$u_1 \rightarrow \frac{1}{\sqrt{2}}\alpha, \quad u_2 \rightarrow \frac{1}{\sqrt{2}}\beta, \quad x \rightarrow x. \tag{4.50}$$

We now determine infinitesimal gauge transformations of the Hopf–Galois extension $\mathcal{O}(S^4) \subset \mathcal{O}(SO(5, \mathbb{R}))$ using, as done for the previous example, the representation theory of $\mathfrak{so}(5)$ as vector fields on the bundle. Like it was the case for the instanton bundle, the $\mathcal{O}(S^4)$ -module of infinitesimal gauge transformations is generated by the

ten generators of $so(5)$. The crucial difference (see Proposition 4.13) is that in the present example, vector fields of given degree n in the generators of $\mathcal{O}(S^4)$ split in the sum of two irreducible representations with distinct highest weight vectors with same weight.

We then start with the $\mathcal{O}(S^4)$ -module of equivariant derivations for $H := \mathcal{O}(SO(4, \mathbb{R}))$.

$$\text{Der}_{\mathcal{M}^H}(\mathcal{O}(SO(5, \mathbb{R}))) = \{X \in \text{Der}(\mathcal{O}(SO(5, \mathbb{R}))) \mid \delta \circ X = (X \otimes \text{id}) \circ \delta\}. \tag{4.51}$$

Let ∂_{IJ} denote the derivation on $\mathcal{O}(M(5, \mathbb{R}))$ given on the generators by $\partial_{IJ}(n_{KL}) = \delta_{IK}\delta_{JL}$, for $I, J, K, L = 1, \dots, 5$. Then, recalling the coaction (4.49), $\text{Der}_{\mathcal{M}^H}(\mathcal{O}(SO(5, \mathbb{R})))$ is generated by the derivations

$$\begin{aligned} H_1 &:= n_{1K}\partial_{1K} - n_{3K}\partial_{3K} & H_2 &:= n_{2K}\partial_{2K} - n_{4K}\partial_{4K} \\ E_{10} &:= n_{5K}\partial_{3K} - n_{1K}\partial_{5K} & E_{-10} &:= n_{3K}\partial_{5K} - n_{5K}\partial_{1K} \\ E_{01} &:= n_{5K}\partial_{4K} - n_{2K}\partial_{5K} & E_{0-1} &:= n_{4K}\partial_{5K} - n_{5K}\partial_{2K} \\ E_{11} &:= n_{2K}\partial_{3K} - n_{1K}\partial_{4K} & E_{-1-1} &:= n_{3K}\partial_{2K} - n_{4K}\partial_{1K} \\ E_{1-1} &:= n_{4K}\partial_{3K} - n_{1K}\partial_{2K} & E_{-11} &:= n_{3K}\partial_{4K} - n_{2K}\partial_{1K} \end{aligned} \tag{4.52}$$

with summation on $K = 1, \dots, 5$ understood. They satisfy $H_j^* = H_j$ and $E_r^* = E_{-r}$. These ten generators close the Lie $*$ -algebra of $so(5)$ in (4.6), from which the labels used. It is important to notice that due to the equivariance for the right coaction of $\mathcal{O}(SO(4, \mathbb{R}))$, when applied to a generator n_{JK} , these derivations do not move the second index. This fact will play a role later on. Being equivariant, they restrict to derivations on the $*$ -subalgebra $\mathcal{O}(S^4)$ of coinvariants:

$$\begin{aligned} H_1^\pi &= u_1\partial_{u_1} - u_1^*\partial_{u_1^*} & H_2^\pi &= u_2\partial_{u_2} - u_2^*\partial_{u_2^*} \\ E_{10}^\pi &= x\partial_{u_1^*} - u_1\partial_x & E_{-10}^\pi &= u_1^*\partial_x - x\partial_{u_1} \\ E_{01}^\pi &= x\partial_{u_2^*} - u_2\partial_x & E_{0-1}^\pi &= u_2^*\partial_x - x\partial_{u_2} \\ E_{11}^\pi &= u_2\partial_{u_1^*} - u_1\partial_{u_2^*} & E_{-1-1}^\pi &= u_1^*\partial_{u_2} - u_2^*\partial_{u_1} \\ E_{1-1}^\pi &= u_2^*\partial_{u_1^*} - u_1\partial_{u_2} & E_{-11}^\pi &= u_1^*\partial_{u_2^*} - u_2\partial_{u_1}, \end{aligned} \tag{4.53}$$

using partial derivations for the generators n_{j5} of $\mathcal{O}(S^4)$. With the isomorphism in (4.50) these derivations coincide with the ones in (4.8).

The generic equivariant derivation is of the form $X = b_1H_1 + b_2H_2 + \sum_r b_rE_r$, with $b_j, b_r \in \mathcal{O}(S^4)$ and H_j, E_r in (4.52). The condition for X to be vertical only uses its restriction to $\mathcal{O}(S^4)$, that is the derivations in (4.53) and thus coincides with the conditions (4.17) under the isomorphism (4.50). Then, in parallel with Proposition 4.1 for the generators (4.15), and noticing that its proof only uses the algebra structure of $\mathcal{O}(S^4)$, we have the following result (we are dropping an overall factor of 2).

Proposition 4.10 *The Lie \ast -algebra $\text{aut}_{\mathcal{O}(S^4)}(\mathcal{O}(SO(5, \mathbb{R})))$ of infinitesimal gauge transformations of the $\mathcal{O}(SO(4, \mathbb{R}))$ -Hopf–Galois extension $\mathcal{O}(S^4) \subset \mathcal{O}(SO(5, \mathbb{R}))$ is generated, as an $\mathcal{O}(S^4)$ -module, by the derivations*

$$\begin{aligned}
 K_1 &:= xH_2 + u_2^\ast E_{01} + u_2 E_{0-1} & K_2 &:= xH_1 + u_1^\ast E_{10} + u_1 E_{-10} \\
 W_{01} &:= u_2 H_1 + u_1^\ast E_{11} + u_1 E_{-11} & W_{0-1} &:= u_2^\ast H_1 + u_1^\ast E_{1-1} + u_1 E_{-1-1} \\
 W_{10} &:= u_1 H_2 - u_2^\ast E_{11} + u_2 E_{1-1} & W_{-10} &:= u_1^\ast H_2 + u_2^\ast E_{-11} - u_2 E_{-1-1} \\
 W_{11} &:= xE_{11} + u_1 E_{01} - u_2 E_{10} & W_{-1-1} &:= xE_{-1-1} + u_1^\ast E_{0-1} - u_2^\ast E_{-10} \\
 W_{1-1} &:= -xE_{1-1} + u_2^\ast E_{10} + u_1 E_{0-1} & W_{-11} &:= -xE_{-11} + u_2 E_{-10} + u_1^\ast E_{01}
 \end{aligned}
 \tag{4.54}$$

with $K_j^\ast = K_j$ and $W_r^\ast = W_{-r}$. They are eigen-operators for H_1 and H_2 and transform under the adjoint representation [10] of $so(5)$ (that is (4.19) hold), with W_{11} the highest weight vector.

In particular we have that $H_j \triangleright K_l = 0$ and $H_j \triangleright W_r = r_j W_r$, and this induces a (left, see later) \mathbb{Z}^2 -grading on the derivations. They satisfy analogue relation of those in (4.20):

$$\begin{aligned}
 u_2 W_{0-1} - u_2^\ast W_{01} + u_1 W_{-10} - u_1^\ast W_{10} &= 0 \\
 -u_2 K_2 + x W_{01} - u_1^\ast W_{11} + u_1 W_{-11} &= 0 \\
 u_2^\ast K_2 - x W_{0-1} + u_1 W_{-1-1} - u_1^\ast W_{1-1} &= 0 \\
 -u_1 K_1 + x W_{10} + u_2^\ast W_{11} + u_2 W_{1-1} &= 0 \\
 u_1^\ast K_1 - x W_{-10} - u_2 W_{-1-1} - u_2^\ast W_{-11} &= 0.
 \end{aligned}
 \tag{4.55}$$

Remark 4.11 The Lie bracket of the generators H_j and W_r are those in Table 3 of Appendix 1 (with the identification (4.50) and up to a rescaling). While the Lie \ast -algebras $\text{aut}_{\mathcal{O}(S^4)}(\mathcal{O}(S^7))$ and $\text{aut}_{\mathcal{O}(S^4)}(\mathcal{O}(SO(5, \mathbb{R})))$ have the same Lie bracket on generators, they differ as $\mathcal{O}(S^4)$ -modules, and hence as Lie \ast -algebras, since in the orthogonal case there is no analogue of the constraint (4.24) occurring in the instanton case.

As in the case of the instanton bundle (cf. Lemma 4.4), the fifty vector fields obtained by multiplying the vector fields in (4.54) with the generators of $\mathcal{O}(S^4)$ can be arranged according to the representations $[35] \oplus [10] \oplus [5]$ of $so(5)$. The highest weight vectors for these three representations are given respectively by:

$$\begin{aligned}
 Z_{21} &= u_1 W_{11}, \\
 Y_{11} &= x W_{11} + u_1 W_{01} - u_2 W_{10}, \\
 X_{10} &= u_2^\ast W_{11} + u_2 W_{1-1} - u_1 K_1 + x W_{10},
 \end{aligned}
 \tag{4.56}$$

with the label denoting the value of the corresponding weight. These are the analogous of the vector fields found in (4.22) for the $SU(2)$ Hopf bundle.

When represented as vector fields on the bundle, the vector X_{10} is zero (because of relations analogous to (4.23)) and the representation [5] it generates vanishes. On the other hand, the vector field Y_{11} is no longer proportional to W_{11} , as it was for the Hopf bundle case due to the constraint (4.24). An explicit computation, using also the condition $(N^\dagger N)_{5K} = \delta_{5K}$ yields:

$$\begin{aligned} W_{11} &= (xn_{2K} - u_2n_{5K})\partial_{3K} + (-xn_{1K} + u_1n_{5K})\partial_{4K} + (u_2n_{1K} - u_1n_{2K})\partial_{5K} \\ Y_{11} &= u_1(u_2n_{1K} - u_1n_{2K})\partial_{1K} + u_2(u_2n_{1K} - u_1n_{2K})\partial_{2K} \\ &\quad + (n_{2K} - u_2\delta_{5K} + u_1^*(u_2n_{1K} - u_1n_{2K}))\partial_{3K} \\ &\quad + (-n_{1K} + u_1\delta_{5K} + u_2^*(u_2n_{1K} - u_1n_{2K}))\partial_{4K} + x(u_2n_{1K} - u_1n_{2K})\partial_{5K}. \end{aligned}$$

The vector field Y_{11} generates a different copy of the ten-dimensional representation of $so(5)$ that we denote by $\widehat{[10]}$ to distinguish it from the ten-dimensional representation [10] of highest weight vector W_{11} . Notice that while [10] consists of vector fields which are combinations of those in (4.54) with coefficients of degree zero in the generators of $\mathcal{O}(S^4)$, elements of $\widehat{[10]}$ are combinations with coefficients of degree one.

Next, in parallel with Lemma 4.5, the representation $[35] \oplus \widehat{[10]}$ of $so(5)$ just found are the ones that occur in the decomposition of the commutators.

Lemma 4.12 *The commutators of the derivations in (4.54) can be organised according to the representations $[35] \oplus \widehat{[10]}$ of $so(5)$ of highest weight vectors αW_{11} and Y_{11} respectively.*

Proof There are 45 non vanishing commutators. The non vanishing commutator with highest weight is $[W_{11}, W_{10}]$ with weight $(2, 1)$. A direct computation shows that

$$[W_{11}, W_{10}] = -u_1 W_{11}$$

and the corresponding representation is the [35] found above.

The other highest weight vector, of weight $(1, 1)$, is

$$T_{11} = [K_1, W_{11}] + [K_2, W_{11}] + [W_{10}, W_{01}] = 4(xW_{11} - u_2W_{10} + u_1W_{01}) = 4Y_{11}.$$

Thus the representation generated by T_{11} is the ten dimensional $\widehat{[10]}$. □

By using the decomposition of $\mathcal{O}(S^4)$ in (4.21), in parallel to Proposition 4.6, we have:

Proposition 4.13 *There is a decomposition*

$$\text{aut}_{\mathcal{O}(S^4)}(\mathcal{O}(SO(5, \mathbb{R}))) = \bigoplus_{n \in \mathbb{N}_0} [d(2, n)] \oplus [d(\widehat{2, n-1})].$$

Here $[d(2, n)]$, respectively $[d(2, \widehat{n - 1})]$, is the representation of $so(5)$ with highest weight vector $\alpha^n W_{11}$ of weight $(n + 1, 1)$, respectively $\alpha^{n-1} Y_{11}$ of weight $(n, 1)$; they consist of derivations on $\mathcal{O}(SO(5, \mathbb{R}))$ which are combinations of the derivations in (4.54) with polynomial coefficients of degree n in the generators of $\mathcal{O}(S^4)$.

4.2.1 Braided derivations and infinitesimal gauge transformations

Let us now pass to the twisted Hopf–Galois extension

$$\mathcal{O}(S_\theta^4) = \mathcal{O}(SO_\theta(5, \mathbb{R}))^{co\mathcal{O}(SO_\theta(4, \mathbb{R}))} \subset \mathcal{O}(SO_\theta(5, \mathbb{R})).$$

We briefly recall its construction from twist deformation (see [3, §4.1] for details). Consider the 2-cocycle $\gamma : \mathcal{O}(\mathbb{T}^2) \otimes \mathcal{O}(\mathbb{T}^2) \rightarrow \mathbb{C}$ on $\mathcal{O}(\mathbb{T}^2) \subset \mathcal{O}(SO(4, \mathbb{R}))$, given on the generators by $\gamma(t_1 \otimes t_2) = e^{-\pi i \theta}$, $\gamma(t_2 \otimes t_1) = e^{\pi i \theta}$ and $\gamma(t_1 \otimes t_1) = \gamma(t_2 \otimes t_2) = 1$. Notice that here $\gamma = \sigma^2$, where σ is the cocycle used in §4.1.5. We use it to deform the Hopf $*$ -algebra $\mathcal{O}(SO(4, \mathbb{R}))$ into the noncommutative Hopf $*$ -algebra $\mathcal{O}(SO_\theta(4, \mathbb{R}))$. This latter has same coalgebra structure as the original one but twisted algebra multiplication,

$$m_{IJ \bullet_\theta} m_{KL} = \gamma(T_I \otimes T_K) m_{IJ} m_{KL} \bar{\gamma}(T_J \otimes T_L), \quad I, J, K, L = 1, \dots, 4,$$

where $T := \text{diag}(t_1, t_2, t_1^*, t_2^*)$. (Here again, to conform with the literature we use the subscript θ instead of γ for twisted algebras and their multiplications.) We set λ

$$\lambda_{IJ} := (\gamma(T_I \otimes T_J))^2$$

so that $\lambda_{IJ} = \exp(-2i\pi\theta_{IJ})$. Since $\bar{\gamma}(T_J \otimes T_L) = \gamma(T_L \otimes T_J)$, and $\gamma(T_L \otimes T_J^*) = \bar{\gamma}(T_L \otimes T_J)$ we have $\lambda_{JI} = \lambda_{IJ}^{-1} = \lambda_{IJ'}$. It follows that the generators in $\mathcal{O}(SO_\theta(4, \mathbb{R}))$ satisfy the commutation relations

$$m_{IJ \bullet_\theta} m_{KL} = \lambda_{IK} \lambda_{LJ} m_{KL \bullet_\theta} m_{IJ}, \quad I, J, K, L = 1, \dots, 4. \tag{4.57}$$

The twisted antipode turns out to be equal to the starting one, $S_\theta(m_{IJ}) = S(m_{IJ})$. The relations (4.43) become

$$M^t \bullet_\theta Q \bullet_\theta M = Q, \quad M \bullet_\theta Q \bullet_\theta M^t = Q, \tag{4.58}$$

together with $\det_\theta(M) = 1$. The $*$ -structure is as in (4.44).

Next, using the projection π in (4.48) we lift the 2-cocycle from $\mathcal{O}(SO(4, \mathbb{R}))$ to $\mathcal{O}(SO(5, \mathbb{R}))$ (or equivalently we consider the same torus \mathbb{T}^2 embedded in $SO(5)$). The resulting Hopf $*$ -algebra is denoted by $\mathcal{O}(SO_\theta(5, \mathbb{R}))$ and has generators n_{IJ} with relations

$$n_{IJ \bullet_\theta} n_{KL} = \lambda_{IK} \lambda_{LJ} n_{KL \bullet_\theta} n_{IJ}, \quad I, J, K, L = 1, \dots, 5, \tag{4.59}$$

where now $T := \text{diag}(t_1, t_2, t_1^*, t_2^*, 1)$, and orthogonality conditions $N^t \bullet_\theta Q \bullet_\theta N = Q$ and $N \bullet_\theta Q \bullet_\theta N^t = Q$, with $\det_\theta(N) = 1$. The $*$ -structure is as in (4.47).

The quantum homogeneous space $\mathcal{O}(S^4)$ is deformed into the quantum homogeneous space $\mathcal{O}(S_\theta^4) \subset \mathcal{O}(SO_\theta(5, \mathbb{R}))$, consisting of coinvariants of $\mathcal{O}(SO_\theta(5, \mathbb{R}))$ under the $\mathcal{O}(SO_\theta(4, \mathbb{R}))$ -coaction. This noncommutative $*$ -subalgebra $\mathcal{O}(S_\theta^4)$ is generated by five elements $\{u_I = n_{I5}\}_{I=1,\dots,5} = \{u_i, u_{i'} = u_i^*, x\}_{i=1,2}$ with commutation relations, obtained from (4.59),

$$u_I \bullet_\theta u_J = \lambda_{IJ} u_J \bullet_\theta u_I . \tag{4.60}$$

The orthogonality conditions imply the sphere relation $2 \sum_{i=1}^2 u_i^* \bullet_\theta u_i + x^2 = 1$. From the general theory in [3], the algebra extension $\mathcal{O}(S_\theta^4) \subset \mathcal{O}(SO_\theta(5, \mathbb{R}))$ of the quantum homogeneous space $\mathcal{O}(S_\theta^4) = \mathcal{O}(SO_\theta(5, \mathbb{R}))^{co\mathcal{O}(SO_\theta(4, \mathbb{R}))}$ is still Hopf–Galois.

When considering the braided Lie $*$ -algebra of infinitesimal gauge transformations of this Hopf–Galois extension, it is useful to think of the latter as the result of a double deformation done with commuting left coaction of $\mathcal{O}(\mathbb{T}^2)$ and right coaction of $\mathcal{O}(\mathbb{T}^2) \subset \mathcal{O}(SO(4, \mathbb{R}))$. This second $\mathcal{O}(\mathbb{T}^2)$ disappears when considering $\mathcal{O}(SO(4, \mathbb{R}))$ equivariant quantities. This is the case for the algebra $\mathcal{O}(S^4)$ of coinvariant elements. It is also the case for the equivariant derivations in Proposition 4.10 and it is in this sense that those derivations can be thought of as having trivial right \mathbb{Z}^2 -grading (they do not move the second index in a generator n_{JK} as already mentioned).

Thus for the braided Lie $*$ -algebra of infinitesimal gauge transformations of the Hopf–Galois extension $\mathcal{O}(S_\theta^4) = \mathcal{O}(SO_\theta(5, \mathbb{R}))^{co\mathcal{O}(SO_\theta(4, \mathbb{R}))}$ we just need to consider the left torus action and the construction goes exactly as for the $SU(2)$ instanton case of the previous section. In particular we can repeat the construction in §4.1.5 verbatim by considering the maximal torus $\mathbb{T}^2 \subset SO(5)$, generated by the right invariant vector fields H_1 and H_2 of $SO(5)$, and use the twist

$$F := e^{\pi i \theta (H_1 \otimes H_2 - H_2 \otimes H_1)} \in K \otimes K \subset U(\mathfrak{so}(5))^{op} \otimes U(\mathfrak{so}(5))^{op} , \quad \theta \in \mathbb{R}, \tag{4.61}$$

of K , where K is generated by the right invariant vector fields H_1 and H_2 . Hence K is the universal enveloping algebra of the Cartan subalgebra of $\mathfrak{so}(5)^{op}$, the Lie algebra $\mathfrak{so}(5)$ being that of left invariant vector fields on $SO(5)$, cf. [5, §7.1]. The braided Lie $*$ -algebras of braided derivations and of infinitesimal gauge transformations of the Hopf–Galois extension $\mathcal{O}(S_\theta^4) \subset \mathcal{O}(SO_\theta(5, \mathbb{R}))$ are the twist (left) deformations of $\text{Der}_{\mathcal{M}^H}(\mathcal{O}(SO(5, \mathbb{R})))$ and of $\text{aut}_{\mathcal{O}(S^4)}(\mathcal{O}(SO(5, \mathbb{R})))$ respectively (with the right torus action playing no role).

As $\mathcal{O}(S^4)$ -modules, the Lie algebra $\text{Der}_{\mathcal{M}^H}(\mathcal{O}(SO(5, \mathbb{R})))$, $H = \mathcal{O}(SO(4, \mathbb{R}))$, is generated by the operators in (4.52) while the Lie algebra of infinitesimal gauge transformations $\text{aut}_{\mathcal{O}(S^4)}(\mathcal{O}(SO(5, \mathbb{R})))$ is generated by the operators in (4.54).

From Corollary 3.11 $\text{Der}_{\mathcal{M}^H}(\mathcal{O}(SO_\theta(5, \mathbb{R}))) = \mathcal{D}(\text{Der}_{\mathcal{M}^H}(\mathcal{O}(SO(5, \mathbb{R}))_{\mathbb{F}}))$, where now $H = \mathcal{O}(SO_\theta(4, \mathbb{R}))$, and \mathcal{D} is an isomorphism of $\mathcal{O}(SO_\theta(5, \mathbb{R}))$ -braided Lie and $\mathcal{O}(SO_\theta(5, \mathbb{R}))$ -module $*$ -algebras. In parallel with Proposition 4.8, we then have:

Proposition 4.14 *The braided Lie $*$ -algebra $\text{Der}_{\mathcal{M}^H}(SO_\theta(5, \mathbb{R}))$ of equivariant derivations of the $\mathcal{O}(SO_\theta(4, \mathbb{R}))$ -Hopf–Galois extension $\mathcal{O}(S_\theta^4) \subset \mathcal{O}(SO_\theta(5, \mathbb{R}))$ is generated, as an $\mathcal{O}(S_\theta^4)$ -module, by elements*

$$\tilde{H}_j := \mathcal{D}(H_j), \quad \tilde{E}_r := \mathcal{D}(E_r), \quad j = 1, 2, \quad r \in \Gamma. \tag{4.62}$$

As in (4.34) the bracket closes the braided Lie algebra $so_\theta(5)$,

$$\begin{aligned} [\tilde{H}_1, \tilde{H}_2]_{\mathbb{R}\mathbb{F}} &= 0; & [\tilde{H}_j, \tilde{E}_r]_{\mathbb{R}\mathbb{F}} &= r_j \tilde{E}_r; \\ [\tilde{E}_r, \tilde{E}_{-r}]_{\mathbb{R}\mathbb{F}} &= \sum_j r_j \tilde{H}_j; & [\tilde{E}_r, \tilde{E}_s]_{\mathbb{R}\mathbb{F}} &= e^{-i\pi\theta r \wedge s} N_{rs} \tilde{E}_{r+s} \end{aligned}$$

with $N_{rs} = 0$ if $r+s$ is not a root. The $*$ -structure is $\tilde{H}_j^* = \tilde{H}_j$ and $\tilde{E}_r^* = \tilde{E}_{-r}$.

Corollary 3.11 also yields $\text{aut}_{\mathcal{O}(S_\theta^4)}(\mathcal{O}(SO_\theta(5, \mathbb{R}))) = \mathcal{D}(\text{aut}_{\mathcal{O}(S_\theta^4)}(\mathcal{O}(SO(5, \mathbb{R}))_\theta))$. From Remark 4.11 the brackets among the generators $\mathcal{D}(K_j)$ and $\mathcal{D}(W_r)$ of $\text{aut}_{\mathcal{O}(S_\theta^4)}(\mathcal{O}(S_\theta^7))$ and among those of $\text{aut}_{\mathcal{O}(S_\theta^4)}(\mathcal{O}(SO_\theta(5, \mathbb{R})))$ have the same expression. This follows comparing the twist expressions (4.26) and (4.61) that define the isomorphisms \mathcal{D} . Nevertheless the braided Lie $*$ -algebras $\text{aut}_{\mathcal{O}(S_\theta^4)}(\mathcal{O}(S_\theta^7))$ and of $\text{aut}_{\mathcal{O}(S_\theta^4)}(\mathcal{O}(SO_\theta(5, \mathbb{R})))$ are different because of the different $\mathcal{O}(S_\theta^4)$ -module structures. Thus as in Proposition 4.9:

Proposition 4.15 *The braided Lie $*$ -algebra $\text{aut}_{\mathcal{O}(S_\theta^4)}(\mathcal{O}(SO_\theta(5, \mathbb{R})))$ of infinitesimal gauge transformations of the $\mathcal{O}(SO_\theta(4, \mathbb{R}))$ -Hopf–Galois extension $\mathcal{O}(S_\theta^4) \subset \mathcal{O}(SO_\theta(5, \mathbb{R}))$ is generated, as an $\mathcal{O}(S_\theta^4)$ -module, by the elements*

$$\tilde{K}_j := \mathcal{D}(K_j), \quad \tilde{W}_r := \mathcal{D}(W_r), \quad j = 1, 2, \quad r \in \Gamma \tag{4.63}$$

with brackets in Table 1 with the identifications $\varphi_\alpha \alpha \rightarrow \sqrt{2}u_1$, $\varphi_\beta \beta \rightarrow \sqrt{2}u_2$, $x \rightarrow x$ following from (4.39) and (4.50) (and up to a rescaling). The braided Lie bracket of generic elements in $\text{aut}_{\mathcal{O}(S_\theta^4)}(\mathcal{O}(SO_\theta(5, \mathbb{R})))$ is given by

$$[b \bullet_\theta \tilde{X}, b' \bullet_\theta \tilde{X}']_{\mathbb{R}\mathbb{F}} = b \bullet_\theta (\mathbb{R}\mathbb{F}_\alpha \triangleright b') \bullet_\theta [\mathbb{R}\mathbb{F}^\alpha \triangleright \tilde{X}, \tilde{X}']_{\mathbb{R}\mathbb{F}} \tag{4.64}$$

for $b, b' \in \mathcal{O}(S_\theta^4)$ and \tilde{X}, \tilde{X}' in the linear span of the generators in (4.63). The $*$ -structure on the generators $\tilde{K}_j^* = \tilde{K}_j$, $\tilde{W}_r^* = \tilde{W}_{-r}$ is extended to the whole $\text{aut}_{\mathcal{O}(S_\theta^4)}(\mathcal{O}(SO_\theta(5, \mathbb{R})))$ via the $\mathcal{O}(S_\theta^4)$ -module compatibility (3.21), $(b \bullet_\theta \tilde{X})^* = (\mathbb{R}\mathbb{F}_\gamma \triangleright b^*) \bullet_\theta (\mathbb{R}\mathbb{F}^\gamma \triangleright \tilde{X}^*)$. It is compatible with the bracket, $[\tilde{X}'^*, \tilde{X}^*]_{\mathbb{R}\mathbb{F}} = ([\tilde{X}, \tilde{X}']_{\mathbb{R}\mathbb{F}})^*$.

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Appendix A. Decomposition of $\mathcal{O}(S^4)$

We study how $\mathcal{O}(S^4)$ decomposes in the sum of irreducible representations $[d(s, n)]$ of $so(5)$. In the algebra $\mathcal{O}(\mathbb{R}^5)$, both α and $\rho^2 = \alpha\alpha^* + \beta\beta^* + x^2$ are annihilated by all raising operators W_r (the ones for positive roots), and thus their powers and products. They are of weight $(1, 0)$ and $(0, 0)$ respectively.

Let $V^{(r)}$ be the $\binom{4+r}{r}$ -dimensional vector space of monomials of degree r in the indeterminates $\alpha, \alpha^*, \beta, \beta^*, x$. The vectors $\alpha^{r-2k} \rho^{2k}$ are highest weight vectors of $V^{(r)}$ and

$$V^{(r)} = \bigoplus_{k=0}^{\lfloor \frac{r}{2} \rfloor} [d(0, r - 2k)] \tag{4.65}$$

where $[d(0, r - 2k)]$ is the irreducible representation with highest weight vector $\alpha^{r-2k} \rho^{2k}$ of weight $(r - 2k, 0)$, and of dimension $d(0, r - 2k) = \frac{1}{6}(1 + r - 2k)(2 + r - 2k)(3 + 2r - 4k)$. Indeed, for $r = 2m$,

$$\begin{aligned} \sum_{k=0}^m d(0, 2m - 2k) &= \sum_{k=0}^m d(0, 2m) = \frac{1}{3} \sum_{k=0}^m (3 + 13k + 18k^2 + 8k^3) \\ &= \frac{1}{3} \left(3 + 13 \frac{m(m+1)}{2} + 18 \frac{m(m+1)(2m+1)}{6} + 8 \frac{m^2(m+1)^2}{4} \right) \\ &= \frac{1}{6} (m + 1)(6 + 19m + 16m^2 + 4m^3), \end{aligned}$$

which coincides with the dimension $\binom{4+2m}{2m}$ of $V^{(2m)}$. Similar computations go for r odd.

For $\alpha, \alpha^*, \beta, \beta^*, x$ coordinate functions on $\mathcal{O}(S^4)$, $\rho^2 = 1$ and with fixed r , all representations $[d]$ in the decomposition (4.65) already appeared in $V^{(r')}$ for some $r' < r$, but for $[d(0, r)]$. We hence conclude that

$$\mathcal{O}(S^4) = \bigoplus_{n \in \mathbb{N}_0} [d(0, n)]$$

where $[d(0, n)]$ has highest weight vector α^n of weight $(n, 0)$.

Appendix B. Matrix representation of the braided Lie algebra $so_\theta(5)$

We give a matrix representation of the braided Lie algebra $so_\theta(5)$ as defined in (4.34).

Consider weights $\mu, \nu = (0, 0), (\pm 1, 0), (0, \pm 1)$ with order

$$(1, 0) = 1, (0, 1) = 2, (-1, 0) = 3, (0, -1) = 4, (0, 0) = 5.$$

Using this order for an identification between weights and row/column indices, define matrices $\mathcal{E}_{\mu\nu}$ of components

$$(\mathcal{E}_{\mu\nu})_{\sigma\tau} := \lambda^{\mu \wedge \nu} \delta_{\mu\sigma} \delta_{\nu\tau}.$$

The product of two such matrices is found to be

$$\mathcal{E}_{\mu\nu} \mathcal{E}_{\tau\sigma} = \lambda^{(\mu-\nu) \wedge (\tau-\sigma)} \delta_{\nu\tau} \mathcal{E}_{\mu\sigma}.$$

The minus signs in the exponents are due to $\mathcal{E}_{\mu\nu}$ having weight $\mu - \nu$ (cf. (4.57)).

A direct computation shows the following:

Lemma 1 *The matrices*

$$\begin{aligned} K_1 &:= \mathcal{E}_{11} - \mathcal{E}_{33} & K_2 &:= \mathcal{E}_{22} - \mathcal{E}_{44} \\ K_{10} &:= \mathcal{E}_{15} - \mathcal{E}_{53} & K_{-10} &:= \mathcal{E}_{51} - \mathcal{E}_{35} \\ K_{11} &:= \mathcal{E}_{14} - \mathcal{E}_{23} & K_{-1-1} &:= \mathcal{E}_{41} - \mathcal{E}_{32} \\ K_{01} &:= \mathcal{E}_{25} - \mathcal{E}_{54} & K_{0-1} &:= \mathcal{E}_{52} - \mathcal{E}_{45} \\ K_{1-1} &:= \mathcal{E}_{12} - \mathcal{E}_{43} & K_{-11} &:= \mathcal{E}_{21} - \mathcal{E}_{44} \end{aligned}$$

give a matrix representation of the algebra $so_\theta(5)$ (see (4.34)) with the identification $K_r \leftrightarrow \tilde{E}_r$ and setting $[K_r, K_s]_{R_F} := K_r K_s - \lambda^{2r \wedge s} K_s K_r$ for the braided commutator of matrices.

Table 2 Vertical vector fields on $\mathcal{O}(S^7)$

	z_1	z_2	z_3	z_4
K_1	$-xz_1 + \beta^*z_4$	$xz_2 + \beta z_3$	$-xz_3 + \beta^*z_2$	$xz_4 + \beta z_1$
K_2	$xz_1 + \alpha z_3$	$-xz_2 - \alpha^*z_4$	$-xz_3 + \alpha^*z_1$	$xz_4 - \alpha z_2$
W_{01}	$\frac{1}{\sqrt{2}}\beta z_1 - \sqrt{2}\alpha z_2$	$-\frac{1}{\sqrt{2}}\beta z_2$	$-\frac{1}{\sqrt{2}}\beta z_3 - \sqrt{2}\alpha^*z_4$	$\frac{1}{\sqrt{2}}\beta z_4$
W_{0-1}	$\frac{1}{\sqrt{2}}\beta^*z_1$	$-\frac{1}{\sqrt{2}}\beta^*z_2 - \sqrt{2}\alpha^*z_1$	$-\frac{1}{\sqrt{2}}\beta^*z_3$	$\frac{1}{\sqrt{2}}\beta^*z_4 - \sqrt{2}\alpha z_3$
W_{10}	$-\frac{1}{\sqrt{2}}\alpha z_1$	$\frac{1}{\sqrt{2}}\alpha z_2 - \sqrt{2}\beta z_1$	$-\frac{1}{\sqrt{2}}\alpha z_3 + \sqrt{2}\beta^*z_4$	$\frac{1}{\sqrt{2}}\alpha z_4$
W_{-10}	$-\frac{1}{\sqrt{2}}\alpha^*z_1 - \sqrt{2}\beta^*z_2$	$\frac{1}{\sqrt{2}}\alpha^*z_2$	$-\frac{1}{\sqrt{2}}\alpha^*z_3$	$\frac{1}{\sqrt{2}}\alpha^*z_4 + \sqrt{2}\beta z_3$
W_{11}	αz_4	βz_4	$-2xz_4 + \alpha z_2 - \beta z_1 = 0$ $(1-x)z_4$	
W_{-1-1}	$-\beta^*z_3$	α^*z_3	0	$-2xz_3 + \alpha^*z_1 + \beta^*z_2 = (1-x)z_3$
W_{1-1}	0	$2xz_1 - \beta^*z_4 + \alpha z_3 = \beta^*z_1$ $(1+x)z_1$		αz_1
W_{-11}	$2xz_2 + \beta z_3 + \alpha^*z_4 = 0$ $(1+x)z_2$		α^*z_2	$-\beta z_2$

The matrices K_r in the lemma are of the form

$$K_{\mu,v} := \mathcal{E}_{\mu v^*} - \mathcal{E}_{v\mu^*}, \quad \text{for } \mu + v = r. \tag{4.66}$$

The braided commutator $[K_{\mu,v}, K_{\tau,\sigma}]_{R_F} = K_{\mu,v}K_{\tau,\sigma} - \lambda^{2(\mu+v)\wedge(\tau+\sigma)}K_{\tau,\sigma}K_{\mu,v}$ is found to be

$$[K_{\mu,v}, K_{\tau,\sigma}]_{R_F} = \lambda^{(\mu+v)\wedge(\tau+\sigma)}(\delta_{v^*\tau}K_{\mu,\sigma} - \delta_{\mu^*\tau}K_{v,\sigma} - \delta_{v^*\sigma}K_{\mu,\tau} + \delta_{\sigma^*\mu}K_{v,\tau}). \tag{4.67}$$

In the classical limit, $\lambda = 1$, the matrices $\mathcal{E}_{\mu v}$ reduce to the usual elementary matrices and those in Lemma 1 give the defining matrix representation of the Lie algebra $so(5)$.

The Lie $*$ -algebra $\text{aut}_{\mathcal{O}(S^4)}(\mathcal{O}(S^7))$

In Table 2 we list the action of the generators (4.15) of the Lie algebra of infinitesimal gauge transformations on the generators of the algebra $\mathcal{O}(S^7)$.

We list in Table 3 half of the brackets of the generators (4.15) of the Lie algebra of infinitesimal gauge transformations $\text{aut}_{\mathcal{O}(S^4)}(\mathcal{O}(S^7))$, obtained by direct computation. The remaining brackets are obtained using the $*$ -structure:

$$\begin{aligned} [K_j, W_{-r}] &= [K_j^*, W_r^*] = -([K_j, W_r])^*, \\ [W_{-r}, W_{-s}] &= [W_r^*, W_s^*] = -([W_r, W_s])^*, \end{aligned}$$

Table 3 The brackets of vertical derivations

$[K_1, K_2] = \sqrt{2}(\alpha^*W_{10} - \alpha W_{-10})$	
$[K_1, W_{01}] = -\sqrt{2}\beta K_2 + 2xW_{01}$	$[K_2, W_{01}] = \sqrt{2}(\alpha^*W_{11} + \alpha W_{-11})$
$[K_1, W_{1-1}] = -2xW_{1-1} + \sqrt{2}\beta^*W_{10}$	$[K_2, W_{1-1}] = 2xW_{1-1} - \sqrt{2}\alpha W_{0-1}$
$[K_1, W_{10}] = \sqrt{2}(-\beta^*W_{11} + \beta W_{1-1})$	$[K_2, W_{10}] = 2xW_{10} - \sqrt{2}\alpha K_1$
$[K_1, W_{11}] = 2xW_{11} - \sqrt{2}\beta W_{10}$	$[K_2, W_{11}] = 2xW_{11} + \sqrt{2}\alpha W_{01}$
$[W_{01}, W_{1-1}] = \sqrt{2}\beta W_{1-1} + \sqrt{2}\alpha(-K_1 + K_2)$	$[W_{-1-1}, W_{01}] = \sqrt{2}\beta W_{-1-1} - \sqrt{2}\alpha^*(K_1 + K_2)$
$[W_{01}, W_{10}] = \sqrt{2}(\beta W_{10} - \alpha W_{01})$	$[W_{-1-1}, W_{1-1}] = \sqrt{2}\beta^*W_{0-1}$
$[W_{01}, W_{11}] = \sqrt{2}\beta W_{11}$	$[W_{-1-1}, W_{10}] = \sqrt{2}\alpha W_{-1-1} + \sqrt{2}\beta^*(K_1 + K_2)$
$[W_{1-1}, W_{10}] = \sqrt{2}\alpha W_{1-1}$	
	$[W_{-1-1}, W_{11}] = -2x(K_1 + K_2) - \sqrt{2}(\alpha W_{-10} + \beta W_{0-1})$
$[W_{1-1}, W_{11}] = -\sqrt{2}\alpha W_{10}$	$[W_{-10}, W_{01}] = \sqrt{2}(\beta W_{-10} + \alpha^*W_{01})$
$[W_{10}, W_{11}] = \sqrt{2}\alpha W_{11}$	$[W_{-10}, W_{1-1}] = -\sqrt{2}\alpha^*W_{1-1} + \sqrt{2}\beta^*(-K_1 + K_2)$
$[W_{10}, W_{-10}] = \sqrt{2}(\beta^*W_{01} + \beta W_{0-1})$	$[W_{0-1}, W_{01}] = \sqrt{2}(\alpha^*W_{10} + \alpha W_{-10})$
$[W_{-11}, W_{01}] = \sqrt{2}\beta W_{-11}$	$[W_{-11}, W_{1-1}] = 2x(K_1 - K_2) - \sqrt{2}(\beta^*W_{01} - \alpha W_{-10})$

with $K_j^* = K_j$ and $W_r^* = W_{-r}$ (see (4.16)) and $(bX)^* = b^*X^*$ for each $b \in \mathcal{O}(S^4)$ and X a derivation. For example, one computes

$$[K_2, W_{0-1}] = -([K_2, W_{01}])^* = -\sqrt{2}(\alpha^*W_{1-1} + \alpha W_{-1-1})$$

$$[W_{10}, W_{0-1}] = -([W_{-10}, W_{01}])^* = -\sqrt{2}(\beta^*W_{10} + \alpha W_{0-1}).$$

References

1. Antoine, J.-P., Speiser, D.: Characters of irreducible representations of the simple groups. I. General theory. *J. Math. Phys.* **5**, 1226–1234 (1964)
2. Antoine, J.-P., Speiser, D.: Characters of irreducible representations of the simple groups. II. Application to classical groups. *J. Math. Phys.* **5**, 1560–1572 (1964)
3. Aschieri, P., Bieliavsky, P., Pagani, C., Schenkel, A.: Noncommutative principal bundles through twist deformation. *Commun. Math. Phys.* **352**, 287–344 (2017)
4. Aschieri, P., Landi, G., Pagani, C.: The gauge group of a noncommutative principal bundle and twist deformations. *J. Noncommun. Geom.* **14**, 1501–1559 (2020)
5. Aschieri, P., Landi, G., Pagani, C.: Braided Hopf algebras and gauge transformations. [arXiv:2203.13811](https://arxiv.org/abs/2203.13811)
6. Aschieri, P., Landi, G., Pagani, C.: Atiyah sequences of braided Lie algebras and their splittings (in preparation)
7. Connes, A., Dubois-Violette, M.: Noncommutative finite-dimensional manifolds. I. Spherical manifolds and related examples. *Commun. Math. Phys.* **230**, 539–579 (2002)
8. Connes, A., Landi, G.: Noncommutative manifolds: the instanton algebra and isospectral deformations. *Commun. Math. Phys.* **221**, 141–159 (2001)
9. Drinfel'd, V.G.: Quasi-Hopf algebras. *Leningrad Math. J.* **1**, 1419–1457 (1990)
10. Klimyk, A., Schmüdgen, K.: *Quantum Groups and Their Representations*. Texts and Monographs in Physics, Springer-Verlag, Berlin (1997)
11. Landi, G.: Spin-Hall effect with quantum group symmetry. *Lett. Math. Phys.* **75**, 187–200 (2006)
12. Landi, G., Van Suijlekom, W.D.: Principal fibrations from noncommutative spheres. *Commun. Math. Phys.* **260**, 203–225 (2005)
13. Majid, S.: *Foundations of Quantum Group Theory*. Cambridge University Press, Cambridge (1995)

14. Varilly, J.: Quantum symmetry groups of noncommutative spheres. *Commun. Math. Phys.* **221**, 511–523 (2001)

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