WALL-CROSSING AND RECURSION FORMULAE FOR TROPICAL JUCYS COVERS

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Abstract. Hurwitz numbers enumerate branched genus g covers of the Riemann sphere with fixed ramification data or equivalently certain factorisations of permutations. Double Hurwitz numbers are an important class of Hurwitz numbers, obtained by considering ramification data with a specific structure. They exhibit many fascinating properties, such as a beautiful piecewise polynomial structure, which has been well-studied in the last 15 years. In particular, using methods from tropical geometry, it was possible to derive wall-crossing formulae for double Hurwitz numbers in arbitrary genus. Further, double Hurwitz numbers satisfy an explicit recursive formula. In recent years several related enumerations have appeared in the literature. In this work, we focus on two of those invariants, so-called monotone and strictly monotone double Hurwitz numbers. Monotone double Hurwitz numbers originate from random matrix theory, as they appear as the coefficients in the asymptotic expansion of the famous Harish-Chandra–Itzykson–Zuber integral. Strictly monotone double Hurwitz numbers are known to be equivalent to an enumeration of Grothendieck dessins d'enfants. These new invariants share many structural properties with double Hurwitz numbers, such as piecewise polynomiality. In this work, we enlarge upon this study and derive new explicit wall-crossing and recursive formulae for monotone and strictly monotone double Hurwitz numbers. The key ingredient is a new interpretation of monotone and strictly monotone double Hurwitz numbers in terms of tropical covers, which was recently derived by the authors. An interesting observation is the fact that monotone and strictly monotone double Hurwitz numbers satisfy wall-crossing formulae, which are almost identical to the classical double Hurwitz numbers.

1. Introduction

Hurwitz numbers [36] count branched genus g coverings of the projective line with fixed ramification data. These objects connect several areas of mathematics, such as algebraic geometry, representation theory, mathematical physics, and many more. In particular, they admit several equivalent definitions, among which is an interpretation due to Hurwitz in terms of factorisations in the symmetric group [37]. From this interpretation many variants of Hurwitz numbers arise by imposing additional conditions on the factorisations. In this paper, we focus on two such variants, namely *monotone* and *strictly monotone Hurwitz numbers*. Monotone Hurwitz numbers were introduced in [30] in the context of random matrix theory as

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they were proved to be the coefficients in the asymptotic expansion of the Harish-Chandra–Itzykson–Zuber integral, while strictly monotone Hurwitz numbers are equivalent to counting certain Grothendieck dessins d'enfants [2].

In studying Hurwitz numbers, one often restricts oneself to ramification with a certain structure. An important case is the one of *single Hurwitz numbers*, where one allows arbitrary ramification over ∞ but only simple ramification (i.e., ramification profile $(2, 1, \ldots, 1)$) over b other points, where b is determined by the Riemann–Hurwitz formula. These numbers admit a stunning connection to Gromov–Witten theory: the celebrated ELSV formula expresses single Hurwitz numbers in terms of intersection numbers on the moduli space of stable curves with marked points $\overline{\mathcal{M}}_{g,n}$ [24]. As a direct consequence single Hurwitz numbers are polynomial in the ramification profile over ∞ up to a combinatorial factor.

From the study of single Hurwitz numbers, it is natural to consider arbitrary ramification over two points and simple ramification else. More precisely, one allows fixed but arbitrary ramification profile μ (resp., ν) over 0 (resp., ∞) and simple ramification over b other points, where b is again determined by the Riemann–Hurwitz formula. In particular, μ and ν are partitions of the same size, i.e., $|\mu| = |\nu|$. The numbers one obtains this way are called *double Hurwitz numbers* and were first studied by Okounkov in [43]. It is an important open question and an active topic of research in Hurwitz theory whether double Hurwitz numbers satisfy an ELSV-type formula, i.e., an expression in terms of intersection numbers on some moduli space. In the literature there are two promising approaches.

ELSV-type formula via piecewise polynomiality. One approach towards deriving an ELSV-type formula for double Hurwitz numbers was introduced by Goulden, Jackson, and Vakil in [28]. Namely, one studies double Hurwitz numbers with a view towards polynomial behaviour–which in the single Hurwitz numbers case was a consequence of the inherent structure of the ELSV formula. This may give an indication for the shape of the desired ELSV-type formula. In their work Goulden, Jackson, and Vakil observe that double Hurwitz numbers are piecewise polynomial in the entries of the two arbitrary ramification profiles and determine the chambers of polynomiality. More precisely, considering the configuration space of pairs of partitions μ, ν of the same size, Goulden, Jackson, and Vakil proved that this space may be subdivided by a hyperplane arrangement W, such that double Hurwitz numbers are polynomial in each connected component of the complement of W. We note that this polynomiality is not up to a combinatorial factor. This leads them to a concrete conjecture for the shape of the ELSV-type formulae with the condition that all covers are fully ramified over ∞ which they prove for genus 0 and genus 1.

This piecewise polynomial behaviour was further examined in work of Shadrin, Shapiro, and Vainshtein. More precisely, the wall-crossing function was studied, i.e., the difference of the polynomials in two adjacent chambers. It was proved by Shadrin, Shapiro, and Vainshstein that in genus 0 the wall-crossing function may be expressed in terms of Hurwitz numbers with smaller input data [46]. This is called a *wall-crossing formula*. A succesful approach in arbitrary genus has been enabled by fruitful interactions between Hurwitz theory and tropical geometry [6, 12, 15]. Using an elegant description of double Hurwitz numbers in terms of tropical covers, wall-crossing formulae for $g \geq 0$ were derived by Cavalieri, Johnson, and Markwig in [13]. A different approach towards the wall-crossing function was carried out by Johnson in [38] using the fermionic Fock space formalism.

ELSV-type formula via CEO topological recursion. The second approach is via the so-called *Chekhov–Eynard–Orantin* (CEO) *topological recursion* [16,27], which is a powerful formalism originating from mathematical physics, that associates a family of multidifferentials on a Riemann surface to a spectral curve. It turns out for several enumerative problems one can find a spectral curve, such that the coefficients of the Taylor expansion of the associated multidifferentials yield the desired enumerative invariants. One says that such an enumerative problem *satisfies CEO topological recursion.* It was proved in [20,25] that any enumerative problem, which satisfies CEO topological recursion, also satisfies an ELSV-type formula although its explicit shape may be difficult to derive. Recently, there have been many fruitful interactions between Hurwitz theory and CEO topological recursion. For example, the now proved Bouchard-Mariño conjecture [7,8,26] states that single Hurwitz numbers in fact satisfy CEO topological recursion, which gives a new proof of the ELSV formula [21]. This result was the first instance of a problem in Hurwitz theory to be accessible by this formalism and has since been extended to other cases. It is an important open problem whether double Hurwitz numbers satisfy CEO topological recursion as well. In particular, a positive answer would imply an ELSV-type formula for double Hurwitz numbers.

1.1. **(Strictly) monotone double Hurwitz numbers.** As mentioned above, there are several variants of Hurwitz numbers, which are defined by counting similar factorisations of permutations in the symmetric group. The study of monotone and strictly monotone Hurwitz numbers has been an active field of research in recent years. It was shown in several instances that (strictly) monotone Hurwitz numbers share many features with their classical counterparts. For example, single monotone Hurwitz numbers satisfy CEO topological recursion [17] and thus an ELSV-type formula, which was derived in [2]. Morover, strictly monotone Hurwitz numbers satisfy CEO topological recursion for several cases of ramification data [19,20,23,40,42].

Similarly, many results translate to the (strictly) monotone double Hurwitz numbers case, e.g., it was proved in [31] that monotone double Hurwitz numbers are piecewise polynomial with the same chamber structure as in the classical case. This result was extended to strictly monotone double Hurwitz numbers in [32,33]. Moreover, it was proved in [18,32] that (strictly) monotone double Hurwitz numbers are related to tropical geometry. More precisely, there is an expression in terms of *combinatorial covers* which are graphs related to tropical covers but decorated with extra combinatorial data.

Using this tropical intepretation, the polynomial behaviour of (strictly) montone double Hurwitz numbers was further studied by the first author in [32], in terms of the aforementioned combinatorial covers. Using Ehrhart theory, algorithms were developed which compute the polynomials for monotone double Hurwitz numbers. We note that a priori these algorithms compute quasi-polynomials in a chamber structure much finer than necessary. In other words, the polynomial structure of (strictly) monotone double Hurwitz numbers is not fully visible from this tropical viewpoint. However, it was possible to derive wall-crossing formulae in genus 0.

Motivated by the work in [38], Kramer and the authors studied the piecewise polynomial behaviour of (strictly) monotone double Hurwitz numbers in the fermionic Fock space formalism in [33]. A formal power sum was introduced, whose coefficients are generating series of (strictly) monotone double Hurwitz numbers. It was proved that this large formal power sum admits a piecwise polynomial structure and wall-crossing formulae. These wall-crossing formulae encode as coefficients a generating series of evaluations of the wall-crossing function. A wall-crossing formula on the level of actual numbers remained an open question.

A common theme in studying (strictly) monotone double Hurwitz numbers is to consider some refinement of the enumeration and obtain results for this refinement. An important example is the study of recursive behaviour of monotone Hurwitz numbers. A recursion for single monotone Hurwitz numbers was proved in [22,29], while a recursion for (strictly) monotone double Hurwitz numbers remains an important open question. It is one of the main missing ingredients for CEO topological recursion for these invariants. However, it is possible to express monotone double Hurwitz numbers as a sum of enumerations for which one may derive recursions for each summand. This approach was taken in [18] for monotone orbifold Hurwitz numbers and in [35] for monotone double Hurwitz numbers, where each summand corresponds to certain decorations on the aforementioned combinatorial covers. However, this refinement has no natural interpretation in terms of the representation theory of the symmetric group.

1.2. **Results.** In this paper, we study the piecewise polynomiality of (strictly) monotone double Hurwitz numbers and study their recursive structure. The main tool is a new tropical interpretation derived by the authors in [34], which expresses (strictly) monotone double Hurwitz numbers in terms of tropical covers weighted by Gromov–Witten invariants.

1.3. **Piecewise polynomiality of (strictly) monotone double Hurwitz numbers.** Our study of the piecewise polynomiality of (strictly) monotone double Hurwitz numbers begins with the observation that using this new tropical interpretation, (strictly) monotone double Hurwitz numbers may naturally be written as a sum of smaller invariants, which we call λ-*invariants*. These $\lambda-$invariants correspond to (ordered) partitions of the number of intermediate simple branch points and can be expressed as vacuum expectations of certain operators in the bosonic Fock space formalism and are thus not just obtained by combinatorial data.

We begin by reformulating the tropical enumeration of this $\lambda-$invariant in terms of a weighted count of flows on abstract tropical curves. We observe that this enumeration mimicks the summation of a polynomial over the lattice points of an integral polytope and thus obtain the following result using Ehrhart theory.

Theorem. *The $\lambda-$invariants are piecewise polynomial with the same chamber structure as classical double Hurwitz numbers.*

Therefore, as monotone Hurwitz numbers are a finite sum of $\lambda-$invariants, we obtain a new proof of the following as an immediate corollary.

Corollary ([31, Theorem 4],[33, Theorem 4.1])**.** *Monotone and strictly monotone double Hurwitz numbers are piecewise polynomial with the same chamber structure as their classical counterpart.*

As a next step, we revisit the methods developed in [13] for deriving wall-crossing formulae for classical double Hurwitz numbers in arbitrary genus which starts from a tropical interpretation. While the work in [13] was focused on the case of $3-$valent abstract tropical curves counted with a specific multiplicity, we show that the methods may be generalised to our case of abstract tropical curves of arbitrary valency and multiplicity given by Gromov–Witten invariants. In particular, we derive wall-crossing formulae for the $\lambda-$invariants. In particular, we prove the following surprising theorem.

Theorem. *Up to a combinatorial pre-factor, the $\lambda-$invariants satisfy the same wall-crossing formulae as classical double Hurwitz numbers.*

As the wall-crossing function of (strictly) monotone double Hurwitz numbers is the finite sum of wall-crossing functions of $\lambda-$invariants, this completely determines the wall-crossing structure of (strictly) monotone double Hurwitz numbers on the level of actual numbers.

1.3.1. *Comparison to previous approaches.* We see that in comparison to [32], the piecewise polynomial structure of (strictly) montone double Hurwitz numbers is completely visible in this new tropical picture. Moreover, the tropical combinatorics, which are strongly related to vacuum expectations in the bosonic Fock space, enable a complete description of the wall-crossing behaviour of these invariants on the level of actual numbers.

Moreover, in a sense, we take an opposite approach to [33]. As elaborated above, the generating series computing (strictly) monotone double Hurwitz numbers was enlarged [33] and wall-crossing formulae were derived for this enlarged series. In this paper, we observe that using this new tropical interpretation, (strictly) monotone double Hurwitz numbers may naturally be written as a finite sum of smaller invariants.

1.4. **A new recursion.** The derivation of a recursion for (strictly) monotone double Hurwitz numbers is an important open problem in Hurwitz theory and an active area of research (see, e.g., [18,22,35]). We propose a recursion for the previously defined $\lambda-$invariants as a substitute. As we have already observed that this enumerations are related to Gromov-Witten theory and admit a piecewise polynomial structure similar to classical double Hurwitz numbers, they may be candidates for an ELSV-type formula (which would imply an ELSV-type formula for (strictly) monotone double Hurwitz numbers). In the flavour of the CEO topological recursion approach towards ELSV-type formulae, we provide one of the main ingredients: An explicit recursive formula for $\lambda-$invariants. The combinatorial factors in this formula are given by geometric data, namely Gromov-Witten invariants.

1.5. **Structure of this paper.** In Section 2, we recall some of the basic facts around Hurwitz theory and tropical geometry. In Section 3, we introduce the necessary notation to state two of our main results. Mainly, we state a piecewise polynomiality result in Theorem 3.4 and wall-crossing formulae for the aforementioned $\lambda-$invariants. In Section 4, we prove those theorems. Finally, we derive a recursion for $\lambda-$invariants in Section 5.

2. Preliminaries

In this section, we recall the basic background needed for this work. In particular, we introduce several variants of Hurwitz numbers in subsection 2.1, review some basics of Gromov-Witten theory in subsection 2.2, and recall the tropical correspondence theorems expressing these variants in terms of tropical covers in subsection 2.3. We further fix the notation $\zeta(z) = 2\sinh(z/2) = e^{z/2} - e^{-z/2}$ and $\mathcal{S}(z) = \frac{\zeta(z)}{z}$.

2.1. **Hurwitz numbers.**

For a permutation $\sigma \in S_d$, we denote the partition corresponding to its conjugacy class by $C(\sigma)$.

Definition 2.1. Let g be a non-negative integer, $x \in (\mathbb{Z}\backslash\{0\})^n$ with $\sum x_i = 0$. Let x^+ (resp., x^-) be the tuple of positive entries of x (resp., $-x$) and denote $d = |x^+| = |x^-|$. Further, we set $b = 2g - 2 + n$. Then we define a factorisation of type (g, x) to be a tuple $(\sigma_1, \tau_1, \ldots, \tau_b, \sigma_2)$, such that

(1) $\sigma_i, \tau_j \in S_d$;
(2) $C(\sigma_1) = x^+$, $C(\sigma_2) = x^-$, $C(\tau_i) = (2, 1, \ldots, 1)$;
(3) $\sigma_2 = \tau_b \cdots \tau_1 \sigma_1$.

Further, we denote $\tau_i = (r_i\, s_i)$ with $r_i < s_i$. We call $(\sigma_1, \tau_1, \ldots, \tau_b, \sigma_2)$ a *monotone* factorisation if $s_i \leq s_{i+1}$ and a *strictly monotone* factorisation if $s_i < s_{i+1}$. We then define the *monotone double Hurwitz number* $h^{\leq,\bullet}_{g;x}$ to be the number of monotone factorisations times $\frac{1}{d!}$. Analogously, we define the *strictly monotone double Hurwitz number* by $h^{<,\bullet}_{g;x}$ to be the number of strictly monotone factorisations times $\frac{1}{d!}$.

Furthermore, we call a factorisation of type (g, x) transitive if

(4) $\langle \sigma_1, \sigma_2, \tau_1, \ldots, \tau_b \rangle$ is a transitive subgroup of S_d.

Then we define the *connected monotone double Hurwitz number* $h^{\leq,\circ}_{g;x}$ and the *connected strictly monotone double Hurwitz number* $h^{<,\circ}_{g;x}$ as before as the numbers of transitive (strictly) monotone factorisations of type (g, x) times $\frac{1}{d!}$.

Remark 2.2. By dropping the monotonicity condition on the transpositions in Definition 2.1, we obtain so-called *double Hurwitz numbers*. These numbers are equivalent to the enumeration of branched degree d morphisms $C \to \mathbb{P}^1_{\mathbb{C}}$ with ramification profile x^+ (x^-) over 0 (resp., ∞) and simple ramification over b fixed points of $\mathbb{P}^1_{\mathbb{C}}$.

2.2. **Gromov-Witten invariants with target $\mathbb{P}^1$.**

We now recall some of the notions of Gromov-Witten theory. A more detailed introduction in the context of tropical covers can be found in [14]. For a more general introduction to the topic, we recommend [48].

Let $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, d)$ denote the moduli stack of stable maps of degree d with a genus g curve with n marked points to $\mathbb{P}^1$. This stack is equipped with a virtual fundamental class of degree $2g - 2 + 2d + n$ and Gromov-Witten invariants are defined by integrating evaluation and descendant classes against the virtual class as recalled in the following. Let us first fix some notation.

- Let $(X, x_1, \ldots, x_n, f)$ denote an element of $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, d)$, that is, X is a connected projective curve of genus g with at worst nodal singularities, $x_1, \ldots, x_n$ are non-singular points on X, and $f : X \to \mathbb{P}^1$ is a map with $f_*([X]) = d[\mathbb{P}^1]$. The ith evaluation morphism is the map $ev_i : \overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, d) \to \mathbb{P}^1$ obtained by mapping the tuple $(X, x_1, \ldots, x_n, f)$ to x_i.

- The ith cotangent line bundle $\mathbb{L}_i \to \overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, d)$ is obtained by identifying the fiber of each point with the cotangent space $\mathbb{T}^*_{x_i}(X)$. The first Chern class of ith cotangent line bundle is called a psi class which we denote by $\psi_i = c_1(\mathbb{L}_i)$.

Definition 2.3. Fix g, n, d and let $k_1, \ldots, k_n$ be non-negative integers such that $k_1 + \cdots + k_n = 2g + 2d - 2$. The stationary Gromov-Witten invariant is defined by

$$\langle \tau_{k_1}(pt) \cdots \tau_{k_n}(pt) \rangle^{\mathbb{P}^1}_{g,n} = \int_{[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1)]^{\mathrm{vir}}} \prod ev_i^*(pt) \psi_i^{k_i},$$

where pt denotes the class of a point on $\mathbb{P}^1$.

Similarly, we consider the moduli space of relative stable maps $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, \nu, \mu, d)$ relative to two partitions μ, ν of d and define the relative Gromov-Witten invariants by

$$\langle \nu \mid \tau_{k_1}(pt) \cdots \tau_{k_n}(pt) \mid \mu \rangle^{\mathbb{P}^1}_{g,n} = \int_{[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, \nu, \mu, d)]^{\mathrm{vir}}} \prod ev_i^*(pt) \psi_i^{k_i}.$$

We note that in the following, we add subscripts "$\circ$" and "$\bullet$" which correspond to *connected* or *not necessarily connected* (for simplicity also called *disconnected*) Gromov-Witten invariants which in turn correspond to considering connected or disconnected stable maps.

2.3. **Tropical correspondence theorem.** We begin by defining *abstract tropical curves*, where we note that we use the notion introduced in [10, Section 1.2], [9, Definition 3.1.3], and [47, Definition 2.1]. This notion in turn originates from [41, Section 3.3]. See also [1, 11] for the relation between abstract algebraic and abstract tropical curves.

Definition 2.4. An *abstract tropical curve* is a connected metric graph Γ with unbounded edges called ends, together with a function associating a genus $g(v)$ to each vertex v. Let $V(\Gamma)$ be the set of its vertices. Let $E(\Gamma)$ and $E'(\Gamma)$ be the set of its internal (or bounded) edges and its set of all edges, respectively. The set of ends is therefore $E'(\Gamma) \setminus E(\Gamma)$, and all ends are considered to have infinite length. The genus of an abstract tropical curve Γ is $g(\Gamma) := h^1(\Gamma) + \sum_{v \in V(\Gamma)} g(v)$, where $h^1(\Gamma)$ is the first Betti number of the underlying graph. An *isomorphism* of a tropical curve is an automorphism of the underlying graph that respects edges' lengths and vertices' genera. The *combinatorial type* of a tropical curve is obtained by disregarding its metric structure.

As a next step, we consider maps between abstract tropical curves which in a sense–up to contraction of edges–mirrors the situation of covers between Riemann surfaces (for more details, see, e.g., [6]). These maps are called *tropical covers* or *harmonic morphisms*. The notion of harmonic morphisms between graphs were first introduced in [5, Section 2.1] and generalised to harmonic morphisms between weighted graphs in [10]. We use the notion of harmonic morphisms defined in [3, Section 2.9].

Definition 2.5. A tropical cover is a surjective harmonic map $\pi : \Gamma_1 \to \Gamma_2$ between abstract tropical curves as in [3, Section 2.9], i.e.:

i). Let $V(\Gamma_i)$ denote the vertex set of Γ_i; then we require $\pi(V(\Gamma_1)) \subset V(\Gamma_2)$.
ii). Let $E'(\Gamma_i)$ denote the edge set of Γ_i; then we require $\pi^{-1}(E'(\Gamma_2)) \subset E'(\Gamma_1)$.

iii). For each edge $e \in E'(\Gamma_i)$, denote by $l(e)$ its length. We interpret $e \in E'(\Gamma_1), \pi(e) \in E'(\Gamma_2)$ as intervals $[0, l(e)]$ and $[0, l(\pi(e))]$; then we require π restricted to e to be a linear map of slope $\omega(e) \in \mathbb{Z}_{\geq 0}$, that is, $\pi : [0, l(e)] \to [0, l(\pi(e))]$ is given by $\pi(t) = \omega(e) \cdot t$. We call $\omega(e)$ the *weight* of e. If $\pi(e)$ is a vertex, we have $\omega(e) = 0$.

iv). For a vertex $v \in \Gamma_1$, let $v' = \pi(v)$. We choose an edge e' adjacent to v'. We define the local degree at v as

$$d_v = \sum_{\substack{e \in \Gamma_1 \\ \pi(e) = e'}} \omega_e.$$

We require d_v to be independent of the choice of edge e' adjacent to v'. We call this fact the *balancing* or *harmonicity condition*.

We furthermore introduce the following notions:

i). The *degree* of a tropical cover π is the sum over all local degrees of preimages of any point in Γ_2. Due to the harmonicity condition, this number is independent of the point in Γ_2 [10, Lemma 2.4].

ii). For any end e, we define a partion μ_e as the partition of weights of the ends of Γ_1 mapping to e. We call μ_e the *ramification profile* above e.

The following theorem expresses monotone and strictly monotone double Hurwitz numbers in terms of tropical covers weighted by Gromov-Witten invariants. Since one of the key ingredients for the proof of the following theorem is the classical Jucys Correspondence [39], we also call the covers involved in this theorem *tropical Jucys covers*.

Theorem 2.6 ([34, Theorem 4.1]). *Let g be a non-negative integer, and $x \in (\mathbb{Z} \backslash \{0\})^n$ with $|x^+| = |x^-| = d$. Then*

$$h^{\leq,\bullet}_{g;x} = \sum_{\lambda \vdash b} \sum_{\pi \in \Gamma(\mathbb{P}^1_{trop}, g; x, \lambda)} \frac{1}{|\mathrm{Aut}(\pi)|} \frac{1}{\ell(\lambda)!} \prod_{v \in V(\Gamma)} m_v \prod_{e \in E(\Gamma)} \omega_e,$$

$$h^{<,\bullet}_{g;x} = \sum_{\lambda \vdash b} \sum_{\pi \in \Gamma(\mathbb{P}^1_{trop}, g; x, \lambda)} \frac{1}{|\mathrm{Aut}(\pi)|} \frac{1}{\ell(\lambda)!} \prod_{v \in V(\Gamma)} (-1)^{1+\mathrm{val}(v)} m_v \prod_{e \in E(\Gamma)} \omega_e,$$

where $\Gamma(\mathbb{P}^1_{trop}, g; x, \lambda)$ is the set of tropical covers $\pi : \Gamma \longrightarrow \mathbb{P}^1_{trop} = \mathbb{R}$ with $b = 2g - 2 + n$ points $p_1, \ldots, p_b$ fixed on the codomain $\mathbb{P}^1_{trop}$ and λ an ordered partition of b, such that

i). *The unbounded left (resp., right) pointing ends of Γ have weights given by the partition x^+ (resp., x^-).*

ii). *The graph Γ has $l := \ell(\lambda) \leq b$ vertices. Let $V(\Gamma) = \{v_1, \ldots, v_l\}$ be the set of its vertices. Then we have $\pi(v_i) = p_i$ for $i = 1, \ldots, l$. Moreover, let $w_i = \mathrm{val}(v_i)$ be the corresponding valencies.*

iii). *We assign an integer $g(v_i)$ as the genus to v_i and the following condition holds true:*

$$h^1(\Gamma) + \sum_{i=1}^{l} g(v_i) = g.$$

iv). *We have $\lambda_i = \mathrm{val}(v_i) + 2g(v_i) - 2$.*

v). *For each vertex v_i, let y^+ (resp., y^-) be the tuple of weights of those edges adjacent to v_i which map to the right-hand (resp., left-hand) of p_i. The multiplicity m_{v_i} of v_i is defined to be*

$$\begin{aligned} m_{v_i} =&(\lambda_i - 1)!|\mathrm{Aut}(y^+)||\mathrm{Aut}(y^-)| \\ &\times \sum_{g_1^i + g_2^i = g(v_i)} \left\langle \tau_{2g_2^i - 2}(\omega) \right\rangle_{g_2^i}^{\mathbb{P}^1, \circ} \left\langle y^+, \tau_{2g_1^i - 2 + n}(\omega), y^- \right\rangle_{g_1^i}^{\mathbb{P}^1, \circ}. \end{aligned}$$

Furthermore, we obtain $h_{g;x}^{\leq,\circ}$ and $h_{g;x}^{<,\circ}$ by considering only connected source curves.

Remark 2.7. It is well known that

$$\langle \tau_{2l-2}(\omega) \rangle_{l,1}^{\mathbb{P}^1} = [z^{2l-1}] \frac{1}{\zeta(z)} = -\frac{2^{2l-1} - 1}{2^{2l-1}} \frac{B_{2l}}{(2l)!},$$

where B_{2l} is the $2l$th Bernoulli number. Furthermore, it was proved in [45, Theorem 2] that

$$\left\langle y^+, \tau_{2g-2+\ell(y^+)+\ell(y^-)}, y^- \right\rangle_g^{\mathbb{P}^1, \circ} = \frac{1}{|\mathrm{Aut}(y^+)||\mathrm{Aut}(y^-)|} [z^{2g}] \frac{\prod_{y_i^+} \mathcal{S}(y_i z) \prod_{x_i^-} \mathcal{S}(x_i z)}{\mathcal{S}(z)}.$$

3. Piecewise polynomiality and wall-crossings

We begin by defining a refinement of monotone and strictly monotone double Hurwitz numbers.

Definition 3.1. Let g be a non-negative integer $x \in \mathbb{Z}^n$, such that $|x^+| = |x^-|$. Furthermore, let λ' be an ordered partition of $2g - 2 + n$. Then we define

$$\vec{h}_{g;x,\lambda'}^{\leq,\circ} = \sum_{\pi \in \Gamma(\mathbb{P}^1_{\mathrm{trop}}, g; x, \lambda')} \frac{1}{|\mathrm{Aut}(\pi)|} \frac{1}{\ell(\lambda)!} \prod_{v \in V(\Gamma)} m_v \prod_{e \in E(\Gamma)} \omega_e.$$

Furthermore, let λ'' be an unordered partition of $2g - 2 + n$. Then we define

$$h_{g;x,\lambda''}^{\leq,\circ} = \sum_{\lambda} \vec{h}_{g;x,\lambda}^{\leq,\circ},$$

where the first sum is over all ordered partitions λ which are obtained by some ordering of λ''. Similarly, we define $\vec{h}_{g;x,\lambda'}^{<,\circ}$ and $h_{g;x,\lambda'}^{<,\circ}$. We further define their disconnected counterparts by considering disconnected tropical covers and decorate them with $\bullet$.

Remark 3.2. We observe that by definition

$$h_{g;x}^{\leq,\circ} = \sum_{\lambda'} \vec{h}_{g;x,\lambda'}^{\leq,\circ} = \sum_{\lambda''} h_{g;x,\lambda''}^{\leq,\circ}, \tag{1}$$

where the first sum is taken over all ordered partition λ' of $2g - 2 + n$ and the second sum is taken over all unordered partitions λ'' of $2g - 2 + n$.

We note that these numbers naturally appear as weighted sums of vacuum expectations of products of the $\mathcal{G}_l$ operators in the notation of [34, Lemma 3.4 and Section 3.4].

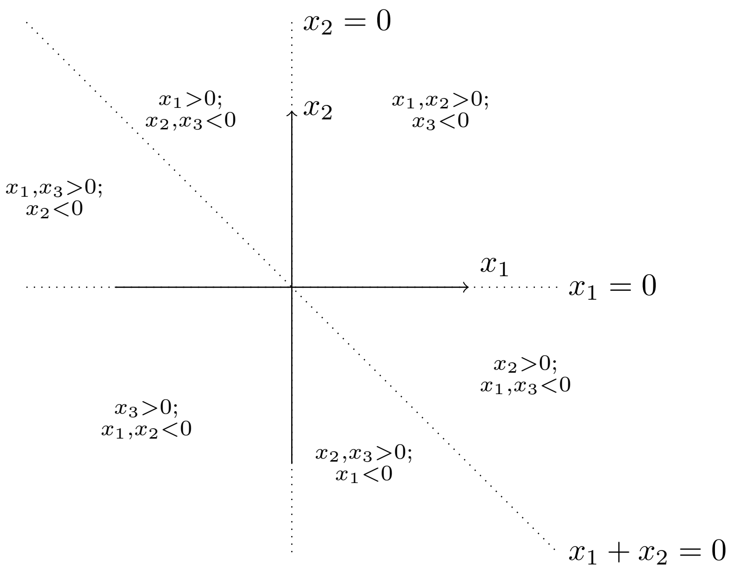

FIGURE 1. The resonance arrangement for $n = 3$, after projection onto the first two coordinates.

3.1. **Results.** In this section, we collect our results about the piecewise polynomial behaviour of $h^{\leq,\circ}_{g;x,\lambda}$ and $h^{<,\circ}_{g;x,\lambda}$. We first define the *resonance arrangement* which is the hyperplane arrangement in $\mathbb{R}^n$ given by

$$W_I = \left\{ x \in \mathbb{Z}^n \mid \sum_{i \in I} x_i = 0 \right\}$$

for all $I \subset \{1, \dots, n\}$. The connected components of the complement of the resonance arrangements are called *chambers*. We also refer to them by *$H-$chambers*.

Example 3.3. We illustrate the resonance arrangement for $n = 3$. The resonance arrangement lives in the hyperplane W in $\mathbb{R}^3$ given by $x_1 + x_2 + x_3 = 0$. There are 3 hyperplanes in $\mathbb{R}^3$ given by $(x_i = -x_j)_{i<j}$ dividing the surrounding space into 6 $H-$chambers. We consider the isomorphism $\mathbb{R}^2 \to W$ given by $(x_1, x_2) \mapsto (x_1, x_2, -x_1 - x_2)$. Note, that with $x_1 + x_2 + x_3 = 0$, the hyperplanes yield

(1) $x_1 = -x_2 = x_1 + x_3$, which is equivalent to $x_3 = 0$,
(2) $x_1 = -x_3 = x_1 + x_2$, which is equivalent to $x_2 = 0$,
(3) $x_2 = -x_3 = x_1 + x_2$, which is equivalent to $x_1 = 0$.

Thus, the hyperplane arrangement in Figure 1 is the resonance arrangement for $n = 3$. In each top-dimensional chamber, the inequalities $(x_i > -x_j)_{i<j}$ are determined.

The following is our first main theorem.

Theorem 3.4. *Let g be a non-negative integer, fix the length n of x, and let λ be an unordered partition of $2g-2+n$. The functions $h^{\leq,\circ}_{g;x,\lambda}$ and $h^{<,\circ}_{g;x,\lambda}$ are polynomials of degree at most $4g - 3 + n$ in each chamber of the resonance arrangement.*

Combining Theorem 3.4 and equation (1), we therefore obtain a new proof of the following result.

Corollary 3.5 ([31, Theorem 4], [33, Theorem 4.1])**.** *For a non-negative integer g and a fixed length n of x, the functions $h^{\leq,\circ}_{g;x}$ and $h^{<,\circ}_{g;x}$ are polynomial of degree $4g-3+n$ in each chamber of the resonance arrangement.*

This motivates the following definition.

Definition 3.6. Let $\mathfrak{c}_1, \mathfrak{c}_2$ be two $H-$*chambers* adjacent along the wall W_I, with $\mathfrak{c}_1$ being the chamber with $x_I = \sum_{i\in I} x_i < 0$. Let $P_i^\lambda(x)$ be the polynomial expressing $h_{g;x,\lambda}$ in $\mathfrak{c}_i$. We define the wall-crossing function by

$$WC_I^\lambda(x) = P_2^\lambda(x) - P_1^\lambda(x).$$

We derive the following expression of the wall-crossing function.

Theorem 3.7. *Let g be a non-negative integer, let n be the fixed length of x, and let λ be an unordered partition of $b = 2g-2+n$. Then we have*

$$WC_I^\lambda(x)$$
$$= \sum_{|y|=|z|=|x_I|} \sum_{\substack{\lambda^i\,unordered \\ \lambda^1\cup\lambda^2\cup\lambda^3=\lambda}} \left((-1)^{\ell(\lambda^2)} \frac{\prod y_i}{\ell(y)!}\frac{\prod z_i}{\ell(z)!} h^{\leq,\circ}_{g_1;(x_I,-y),\lambda^1} h^{\leq,\bullet}_{g_2;(y,-z),\lambda^2} h^{\leq,\circ}_{g_3;(z,x_{I^c}),\lambda^3}\right),$$

where y (resp. z) is an ordered tuple of length $\ell(y)$ (resp., $\ell(z)$) of positive integers with sum $|y|$ (resp., $|z|$) and g_1 is given by $|\lambda^1| = 2g_1 - 2 + \ell((x_I, -y))$ (and analogously for g_2, g_3)).

4. Proofs of chamber polynomiality and of wall-crossing formulae

In this section, we prove Theorems 3.4 and 3.7. We focus on the case of monotone Hurwitz numbers as the case of strictly monotone Hurwitz numbers is completely parallel. To begin with, we introduce a formal set-up for the proofs of both theorems in subsection 4.1. We continue in subsection 4.2 where we prove Theorem 3.4. Finally, we prove Theorem 3.7 in subsection 4.3. We follow the strategy of [13] which focuses on the case of trivalent graphs, however all results we cite hold for the graphs with higher valency considered in this paper with the same proofs. We also provide a running example for this case of higher valency throughout the proof, which is analogous to example 2.5 in [13] for the trivalent case.

4.1. **Formal set-up.** Instead of tropical covers, we work with combinatorial covers, where the information given by the cover is encoded as an orientation given on the graph.

Definition 4.1 (Combinatorial cover)**.** For fixed g, $x = (x_1, \dots, x_n) \in (\mathbb{Z}\backslash\{0\})^n$, $\lambda \vdash 2g-2+n$ unordered, a graph Γ is a *combinatorial cover* of type (g, x, λ), if

1. Γ is a connected graph with at most $2g-2+2n$ vertices.
2. Γ has n many $1-$valent vertices called *leaves*; the adjacent edges are called *ends* and are labeled by the weights $x_1, \dots, x_n$; further, all ends are oriented inwards. If $x_i > 0$, we say it is an *in-end*, otherwise it is an out-end.
3. We denote the set of edges which are *not* edges by $E^{in}(\Gamma)$.
4. There are $\ell(\lambda)$ inner vertices.
5. We denote the inner vertices by $v_1, \dots, v_{\ell(\lambda)}$ and assign a non-negative integer $g(v_i)$ to v_i which we call the *genus of* v_i; we further have $\lambda_i = \mathrm{val}(v_i) + 2g(v_i) - 2$.

(6) After reversing the orientation of the *out-ends*, Γ does not have sinks or sources.
(7) The internal vertices are ordered compatibly with the partial ordering induced by the directions of the edges.
(8) We have $g = b_1(\Gamma) + \sum g(v_i)$, where $b_1(\Gamma)$ is the first Betti number of Γ.
(9) Every internal edge e of the graph is equipped with a weight $\omega(e) \in \mathbb{N}$. The weights satisfy the balancing condition at each inner vertex: the sum of all weights of incoming edges equals the sum of the weights of outgoing edges.

The notation $\Gamma(x, \lambda, d, o)$ indicates that graph comes with directed edges (d) and with a compatible vertex ordering (o).

This definition encodes the data of the covers involved in Theorem 2.6 in graph theoretic language. More precisely, starting with a tropical cover $\pi \in \Gamma(\mathbb{P}^1_{trop}, g; x, \lambda)$, one obtains a combinatorial cover in the following way:

(1) let $\pi : \Gamma' \to \mathbb{P}^1_{trop}$ be the tropical cover; then we obtain the connected graph Γ of the combinatorial cover by placing vertices at the unbounded ends of Γ';
(2) the unbounded ends of Γ are labeled by $x = (x_1, \ldots, x_n)$;
(3) let v, w be adjacent inner vertices in Γ with $\pi(v) = p_i, \pi(w) = p_j$, and $i < j$; then the edge is oriented from v to w and all ends are directed inwards–this yields d;
(4) the directed edges induce a partial ordering o on the vertices;
(5) the rest of the data, such as weight of the edges, genus of the vertices, the partition λ translate immediately.

This process is invertible. Therefore Theorem 2.6 translates to

$$h^{\leq,\circ}_{g;x} = \sum_{\lambda \vdash b} \sum_{\Gamma} \frac{1}{|\mathrm{Aut}(\Gamma)|} \frac{1}{\ell(\lambda)!} \varphi_\Gamma,$$

where the second sum is over all combinatorial covers Γ of type (g, x, λ) and we have

$$\varphi_\Gamma = \prod_{i=1}^{\ell(\lambda)} m_{v_i} \prod_{e \in E^{in}(\Gamma)} \omega(e)$$

with

$$m_{v_i} = (\lambda_i - 1)! |\mathrm{Aut}(y^+)| |\mathrm{Aut}(y^-)| \\ \times \sum_{g_1^i + g_2^i = g(v_i)} \left\langle \tau_{2g_2^i - 2}(\omega) \right\rangle^{\mathbb{P}^1,\circ}_{g_2^i} \left\langle y^+, \tau_{2g_1^i - 2 + \ell(y^+) + \ell(\mathbf{y}^-)}(\omega), y^- \right\rangle^{\mathbb{P}^1,\circ}_{g_1^i},$$

where y^+ is the tuple of weights of incoming edges and y^- the tuple of weights of outgoing edges at v_i. Analogously, one obtains $h^{\leq,\circ}_{g;x,\lambda}$, $\vec{h}^{\leq,\circ}_{g;x,\lambda}$ and their disconnected counterparts.

Moreover, for an unordered partition λ, we have

$$h^{\leq,\circ}_{g;x,\lambda} = \sum_{\Gamma} \frac{1}{|\mathrm{Aut}(\Gamma)|} \frac{1}{\ell(\lambda)!} \varphi_\Gamma,$$

where the second sum is over all combinatorial covers Γ of type (g, x, λ).

Definition 4.2. Given g and x, an $x-$graph $\Gamma(x)$ (or simply Γ when there is no risk of confusion) is a connected, genus g combinatorial cover, where we forget the direction of the edges and the vertex ordering, such that the n ends are labeled $x_1, \ldots, x_n$.

4.1.1. *Hyperplane arrangements.* We view an $x-$graph Γ as a one-dimensional cell complex. The differential $d : \mathbb{R}E_\Gamma \to \mathbb{R}V_\Gamma$, sending a directed edge to the difference of its head and tail vertices, yields the following short exact sequence:

$$0 \to \ker(d) \to \mathbb{R}E_\Gamma \to \operatorname{im}(d) \to 0.$$

We decompose $\mathbb{R}E_\Gamma = \mathbb{R}^n \bigoplus \mathbb{R}^{|E^{in}(\Gamma)|}$ into ends and internal vertices. Then we have a vector of the form $(x, 0) \in \operatorname{im}(d)$ when $\sum x_i = 0$.

Definition 4.3. We define the *space of flows* to be

$$F_\Gamma(x) = d^{-1}(x, 0).$$

Inside the space of flows, we define a hyperplane arrangement

$$\mathcal{A}_\Gamma(x)$$

given by the restriction of the coordinate hyperplanes corresponding to the internal edges in $\mathbb{R}E_\Gamma$. The defining polynomial for this hyperplane arrangement is

$$\varphi_\mathcal{A} = \prod e_i,$$

where e_i are the coordinate functions on $\mathbb{R}E_\Gamma$ restricted to $F_\Gamma(x)$.

We note that often it is useful to fix a reference orientation on a given $x-$graph. The following lemma shows that this corresponds to fixing a bounded chamber in the hyperplane arrangement.

Lemma 4.4 ([13, Lemma 2.13, Corollary 2.14])**.** *The bounded chambers of $\mathcal{A}_\Gamma(x)$ correspond to orientations of Γ with no directed cycles. Moreover, given an $(x, \lambda)-$graph Γ, the bounded chambers of $\mathcal{A}_\Gamma(x)$ are in bijection with directed $(x, \lambda)-$graphs projecting to Γ after forgetting the orientations of the edges that come from a combinatorial cover (defined in Definition 4.1).* □

The following remark indicates an interesting structural result regarding the vertex contributions.

Remark 4.5. Recall that the contribution of each vertex is given by

$$m_{v_i} = (\lambda_i - 1)!|\operatorname{Aut}(y^+)||\operatorname{Aut}(y^-)| \\ \times \sum_{g_1^i + g_2^i = g(v_i)} \left\langle \tau_{2g_2^i - 2}(\omega) \right\rangle_{g_2^i}^{\mathbb{P}^1, \circ} \left\langle y^+, \tau_{2g_1^i - 2 + \ell(y^+) + \ell(y^-)}(\omega), y^- \right\rangle_{g_1^i}^{\mathbb{P}^1, \circ},$$

where y^+ are the incoming and y^- are the outgoing edge weights. Moreover, by [44, Theorem 2] the following identity holds:

$$\left\langle y^+, \tau_{2g_1^i - 2 + \ell(y^+) + \ell(y^-)}(\omega), y^- \right\rangle_{g_1^i}^{\mathbb{P}^1, \circ} \\ = \frac{1}{|\operatorname{Aut}(y^+)|} \frac{1}{|\operatorname{Aut}(y^-)|} [w^{g_1^i}] \frac{\prod_{y^+} \mathcal{S}(y_i^+ w) \prod_{y^-} \mathcal{S}(y_i^- w)}{\mathcal{S}(w)}.$$

Thus we obtain

$$m_{v_i} = (\lambda_i - 1)! \sum_{g_1^i + g_2^i = g(v_i)} \left\langle \tau_{2g_2^i - 2}(\omega) \right\rangle_{g_2^i}^{\mathbb{P}^1, \circ} [w^{g_1^i}] \frac{\prod_{y^+} \mathcal{S}(y_i^+ w) \prod_{y^-} \mathcal{S}(y_i^- w)}{\mathcal{S}(w)}.$$

We recall that $\mathcal{S}(w) = 1 + \frac{z^2}{24} + \frac{z^4}{1920} + O(z^6)$ and $\frac{1}{\mathcal{S}(w)} = 1 - \frac{z^2}{24} + \frac{7z^4}{5760} + O(z^6)$ are even power series. Therefore m_{v_i} is a polynomial in the adjacent edge weights and all appearing monomials are of even degree. We denote this polynomial by $M(v_i)$. This polynomial is independent of the flow of the respective branching graph.

Definition 4.6. Let Γ be an $x-$graph. We denote by $S_\Gamma(x)$ the contribution to $h_{g;x,\lambda}$ of all combinatorial covers having underlying $(x)-$graph Γ, where λ is obtained by

$$\lambda_i = \mathrm{val}(v_i) + 2g(v_i) - 2,$$

where v_i runs over all inner vertices, i.e., $\lambda = (\lambda_1, \ldots, \lambda_{\ell(\lambda)})$.

For a given $(x)-$graph Γ, we call $F-$chambers the chambers of $\mathcal{A}_\Gamma(x)$ in the flow space $F_\Gamma(x)$. Recall that all points in the same F-chamber A have the direction. Therefore, the edge weights all have the same sign with respect to the reference orientation. We start with two adjacent $F-$chambers A and B, which share the wall $e_i = 0$. Let A, such that $e_i > 0$ and B, such that $e_i < 0$. We start from a point in A, i.e., a weight distribution and orientation of edges on Γ. Note, that e_i represents the weight of some edge of Γ. Moving along the vector $-e_i$ the weight of the ith edge in Γ decreases. After passing through the wall defined by $e_i = 0$, we have for all $j \neq i$, that the weights e_j still have the same sign as in the chamber A. However, we also have $e_i < 0$ in chamber B. Each point in B therefore corresponds to an oriented graph in which the edge corresponding to e_i has opposite orientation with respect to the one in chamber A.

For an $F-$chamber A, let Γ_A denote the directed (x, λ)-graph with the edge directions corresponding to the chamber A. We use $m(A)$ (or $m(\Gamma_A)$), to denote the number of all possible orderings of the vertices of Γ_A from left to right (recall that the branch points are fixed over the base).

Lemma 4.7 ([13]). *For an F-chamber A, we have that $m(A)$ is zero if and only if A is unbounded.*

Roughly speaking, the reason for the above statement is that a chamber A can be unbounded if and only if the graphs contain an oriented loop which makes it impossible to order the vertices over the base. As $m(A)$ will appear as multiplicity in our formula, we can immediately discard all unbounded chambers, as their contribution vanishes completely.

We use $\mathrm{Ch}(\mathcal{A}_\Gamma(x))$ to denote the set of F-chambers of $\mathcal{A}_\Gamma(x)$. Clearly, the sign of

$$\varphi_{\mathcal{A}} = \prod_{i=1}^{n+|E(\Gamma)|} e_i$$

alternates on adjacent $F-$chambers (since we swap the direction of one edge, as explained above): we indicate with $\mathrm{sign}(A) = (-1)^{N(A)}$ the sign of $\varphi_{\mathcal{A}}$ on the chamber A, where $N(A)$ is the number of negative coordinates e_i in the chamber A.

For integer values of x, the space of flows $F_\Gamma(x)$ has an affine lattice, coming from the integral structure of $\mathbb{Z}E_\Gamma$. We denote this lattice by

$$\Lambda = F_\Gamma(x) \cap \mathbb{Z}E_\Gamma.$$

This notation allows a convenient interpretation of $S_\Gamma(x)$ in terms of the hyperplane arrangement $\mathcal{A}_\Gamma(x)$. Choices of the weights of the edges–i.e., the choice of a flow f on Γ–correspond to lattice points in Λ. We have that

$$\begin{aligned} S_\Gamma(x) &= \frac{1}{|\mathrm{Aut}(\Gamma)|} \sum_{A \in \mathrm{Ch}(\mathcal{A}_\Gamma(x))} m(A) \sum_{f \in A \cap \Lambda} \left(\prod_{e \in E'(\Gamma)} w(e) \prod_i M(v_i) \right) \\ &= \frac{1}{|\mathrm{Aut}(\Gamma)|} \sum_{A \in \mathrm{Ch}(\mathcal{A}_\Gamma(x))} \mathrm{sign}(A)\, m(A) \sum_{f \in A \cap \Lambda} \left(\varphi_{\mathcal{A}}(f) \prod_i M(v_i) \right), \end{aligned}$$

where $\prod_i M(v_i)$ is an even polynomial in the edge weights by Remark 4.5, and to pass from the first to the second line use that the product of all the edge weights of a flow f is the absolute value of $\varphi_{\mathcal{A}}$ computed at $f = (e_i)_i$ which if $f \in A$ is simply $\mathrm{sign}(A)\varphi_{\mathcal{A}}(f)$.

Example 4.8. We illustrate the introduced notions for the combinatorial cover $\Gamma(x, \lambda, d, o)$ in the top of Figure 2, where o is indicated in the left picture, d is indicated by the directed edges, $\lambda = (1, 1, 1, 1, 5, 2)$, and $x = (x_1, \dots, x_5)$.

In the middle of Figure 2, two flow spaces for Γ are given. On the left, we have $-(x_4+x_5) > x_2$ and $x_1+x_3 > 0$. On the right, we have crossed the wall $x_1+x_3 = 0$.

We further have $M(v_i) = 1$ for $i \neq 5$ and

$$M(v_5) = \frac{3a^4 + 10a^2(b^2 + c^2) + 3b^4 + 10b^2c^2 + 3c^4}{5760}$$

for $a = i, b = -j - (x_4 + x_5), c = -i - j - (x_4 + x_5)$.

4.2. **Polynomials and walls.** We begin with the proof of Theorem 3.4. We fix an $(x)-$graph Γ with reference orientation given by the flow f. We first observe that

$$\frac{1}{|\mathrm{Aut}(\Gamma)|} \left(\varphi_{\mathcal{A}}(f) \prod_i M(v_i) \right)$$

is a polynomial of degree $|E(\Gamma)| + 2\sum g_i$, as $\varphi_{\mathcal{A}(f)}$ is a polynomial of degree $|E(\Gamma)|$ and $M(v_i)$ is a polynomial of degree $2g_i$. Considering the Euler characteristic of Γ we obtain

$$|E(\Gamma)| = \ell(\lambda) + b_1(\Gamma) - 1 = \ell(\lambda) + g - \sum g_i - 1$$

and therefore

$$|E(\Gamma)| + 2\sum g_i = \ell(\lambda) + g + \sum g_i - 1.$$

Recalling $\lambda_i = \mathrm{val}(v_i) + 2g(v_i) - 2$ and the fact that $\mathrm{val}(v_i) \geq 2$, it is easily seen that the right-hand side maximizes for $\lambda = (1, \dots, 1)$. Thus, we have

$$|E(\Gamma)| + 2\sum g_i \leq 3g - 3 + n.$$

Similar to [13, Remark 2.11], we have that $F_\Gamma(x)$ is $b_1(\Gamma)-$dimensional.

Moreover, it is well known that summing a polynomial of degree d over the lattice points in a $b_1(\Gamma)-$dimensional integral polytope of fixed topology is a polynomial of degree $d + b_1(\Gamma)$ in the numbers defining the boundary of the polytope (see, e.g., [4, Theorem 4.2]. We further observe that each vertex is given by an integer vector because the incidence matrix of a directed graph is totally unimodular.

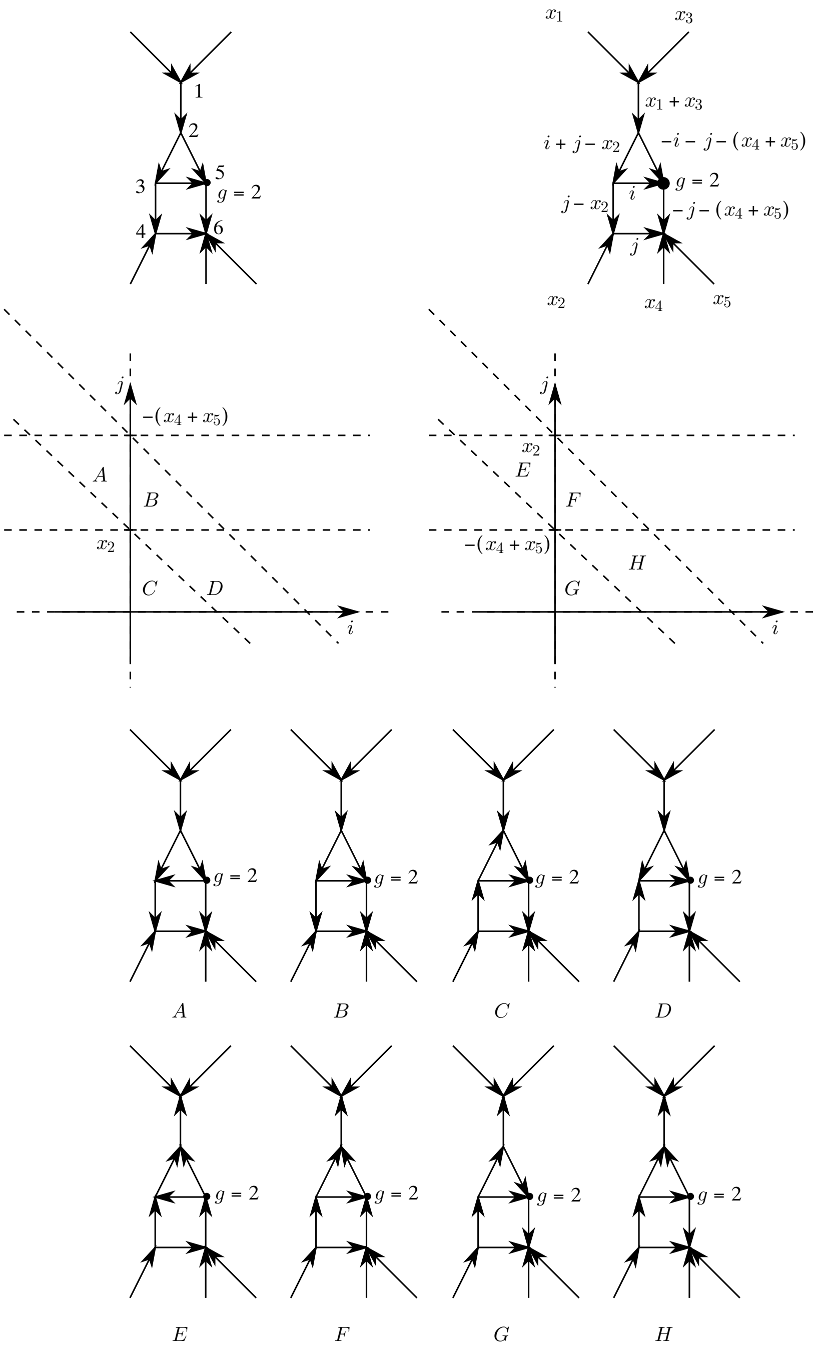

FIGURE 2. A combinatorial cover, its flow space in two adjacent chambers, and the corresponding orientations.

Combining these facts, it follows that $S_\Gamma(x)$ is a polynomial in x of degree $\ell(\lambda)+g+\sum g_i-1+b_1(\Gamma)$ as long as varying x does not change the topology of $\mathcal{A}_\Gamma(x)$ which is maximal for $\ell(\lambda)=b$ and $g_i=0$.

Thus $h_{g;x,\lambda}$ is piecewise polynomial of maximal degree $4g-3+n$. We now determine the regions in which $h_{g;x,\lambda}$ is polynomial. More precisely, we prove that $h_{g;x,\lambda}$ is polynomial in each top-dimensional component in the complement of a hyperplane arrangement. We further compute those hyperplanes.

We note that the hyperplane arrangement given by $\mathcal{A}_\Gamma(x)$ is not always given by hyperplanes which only intersect transversally. Morally, the shape of the polynomial expressing $h_{g;x,\lambda}$ should only change when the topology of $\mathcal{A}_\Gamma(x)$ changes. When translating generic hyperplane arrangements the topology may change when one passes through a non-transversality. However, in our situation, there can be non-transversalities which appear for each value of x. Nonetheless, it is still true that the topology may change once one passes through additional non-transversalities. We call those non-transversalities which appear for any value x *good transversalities*. The following definition is a classification of these.

Definition 4.9. Suppose a set of k hyperplanes (equivalently, edges in Γ) in $\mathcal{A}_\Gamma(x)$ intersect in codimension $k-l$. We call this intersection *good* if there is a set L of l vertices in Γ so that I is precisely the set of edges incident to vertices in L.

Furthermore, we define the *discriminant locus* $\mathcal{D} \subset \mathbb{R}^n$ the set of values of x so that for some directed $(x)-$ graph Γ the hyperplane arrangement $\mathcal{A}_\Gamma(x)$ has a non-transverse intersection that is not good. The discriminant is a union of hyperplanes which we call the *discriminant arrangement*. We call these hyperplanes *walls* and the chambers defined by the arrangement *$H-$chambers*.

The $H-$chambers are the chambers of polynomiality of $h_{g;x,\lambda}$. Now, we establish that the walls correspond to the resonance arrangements

$$\sum_{i\in I} x_i = 0$$

for $I \subset \{1,\ldots,n\}$. We begin with the following definition.

Definition 4.10. A *simple cut* of a graph Γ is a minimal set C of edges that disconnects the ends of Γ: There are two ends of Γ such that every path between them contains an edge of C and this is true of no proper subset of C.

For an $(x,\lambda)-$graph, a flow in $F_\Gamma(x)$ is *disconnected* if for some simple cut C the flow on each edge of C is zero.

This yields the following lemma.

Lemma 4.11 ([13, Lemma 3.8])**.** *The discriminant arrangement $\mathcal{D}$ is given by the set of $x \in \mathbb{R}^n$ such that for some $x-$graph Γ, the space $F_\Gamma(x)$ admits a disconnected flow.*

Now, let Γ admit a disconnected flow and let C be the corresponding simple cut. Then it follows by the balancing condition that the sum $\sum_{i\in I} x_i$ of weights of ends belonging to a connected component of $\Gamma\backslash C$ is 0. Thus, the walls of the discriminant arrangement are a subset of the hyperplanes in the resonance arrangement. The arrangements are equal since it is easy to construct a graph Γ, with some edge e, such that $\Gamma\backslash e$ has two components, one containing the ends of I and the other containing the ends of I^c. Thus $h_{g;x,\lambda}$ is polynomial in each chamber of the resonance arrangement.

4.3. **Wall-crossing.** In this section, we prove Theorem 3.7. We first discuss the combinatorics of cutting an $x-$graph Γ into several smaller graphs.

Definition 4.12. Let Γ a directed graph and let E be a subset of the edges of Γ. We consider the graph whose edges are the connected components of $\Gamma\backslash E^c$ and whose vertices are E^C. We call this graph the *contraction* of Γ with respect to E and denote it by $\Gamma\!/_{E}$.

We fix a directed $x-$graph Γ_A and let $I \subset \{1, \ldots, n\}$ be some subset. Then the *set* $\mathrm{Cuts}_I(\Gamma_A)$ *of* $I-$*cuts of* Γ_A consists of those subsets C of the edges of Γ, such that $C = \emptyset$ or

(1) $\Gamma_A \backslash C$ is disconnected;
(2) the ends of Γ_A lie on exactly two components of $\Gamma_A \backslash C$, one containing all ends indexed by I, the other containing all ends indexed by I^c;
(3) the directed graph $\Gamma\!/_{C^c}$ is *acyclic* and has the component containing I as the initial vertex and the component containing I^c as the final vertex.

Let $v(\Gamma_A \backslash C)$ be the number of components of $\Gamma_A \backslash C$. Then we define the *rank of* C by

$$\mathrm{rk}(C) = v(\Gamma_A \backslash C) - 1.$$

By the discussion in [13, Section 6], we have

(2)
$$WC(x_2)=\sum_{\Gamma}\sum_{A\in\mathcal{BC}_\Gamma(x_2)}\sum_{C\in\mathrm{Cut}_I(\Gamma_A)}(-1)^{\mathrm{rk}(C)-1}\binom{\ell(\lambda)}{s,t_1,\ldots,t_N,u}\left(\sum_{\Lambda\cap A}\varphi_{\mathfrak{A}}\prod_i M(v_i)\right),$$

where $N = \mathrm{rk}(C) - 1$ and $t_1, \ldots, t_N$ are the numbers of inner vertices of the N inner components of $\Gamma_A \backslash C$.

Definition 4.13. Let Γ be an $x-$graph and $C \in \mathrm{Cuts}_I(\Gamma_A)$. We call C a thin cut if all edges in C are either adjacent to the inital component containing I or the component containg I^c. Furthermore, for a thin cut T, we denote by $P(T)$ the set of all cuts $C \in \mathrm{Cuts}_I(\Gamma_A)$ which contain T.

By [13, Lemma 8.2], we have

$$(-1)^t\binom{\ell(\lambda)}{s,t,u} = \sum_{C\in\mathrm{P}(T)}(-1)^{\mathrm{rk}(C)-1}\binom{\ell(\lambda)}{s,t_1,\ldots,t_N,u}. \tag{3}$$

Remark 4.14. We note that there is a sign mistake in the formulation of [13, Lemma 8.2] which occurs in the proof of [13, Lemma 8.4].

Combining equations (2) and (3), we obtain

$$WC(x_2) = \sum_{\Gamma}\sum_{A\in\mathcal{BC}_\Gamma(x_2)}\sum_{\substack{T\in\mathrm{Cut}_I(\Gamma_A)\\ \text{thin}}}(-1)^t\binom{\ell(\lambda)}{s,t,u}\left(\sum_{\Lambda\cap A}\varphi_{\mathcal{A}}\prod_i M(v_i)\right).$$

We now observe that each thin cut divides Γ_A into three parts: the initial component Γ_A^1, an intermediate part Γ_A^2, and a final component Γ_A^3. Moreover, the intermediate part may be disconnected. Thus, we observe that Γ_A^1 contributes to $h^{\le}_{g_1;(x_I,-y),\lambda_1}$, Γ_A^2 to $h^{\le,\bullet}_{g_2;(y,-z),\lambda_2}$ and Γ_A^3 to $h^{\le,\circ}_{g_3;(z,x_{I^c})}$, $\lambda_1 \cup \lambda_2 \cup \lambda_3 = \lambda$ and y, z are some partitions with $|y| = |x_I|$, $|z| = |x_{I^c}|$. Finally, we observe that

$$\phi_{\Gamma_A} = \frac{\ell(\lambda_1)!\ell(\lambda_2)!)\ell(\lambda_3)!}{\ell(\lambda)!}\frac{\prod y_i}{\ell(y)!}\frac{\prod z_i}{\ell(z)!}\phi_{\Gamma_A^1}\phi_{\Gamma_A^2}\phi_{\Gamma_A^3} \tag{4}$$

and

$$\binom{\ell(\lambda)}{s,t,u} = \binom{\ell(\lambda)}{\ell(\lambda_1),\ell(\lambda_2),\ell(\lambda_3)} = \frac{\ell(\lambda)!}{\ell(\lambda_1)!\ell(\lambda_2)!\ell(\lambda_3)!}$$

which cancels with the factor in equation (4). This completes the proof of Theorem 3.7.

5. A refined recursion for (strictly) monotone double Hurwitz numbers

In this section, we derive recursive formulae for $\vec{h}^{\leq}_{g;x,\lambda}$ and $\vec{h}^{<}_{g;x,\lambda}$. We then generalise these results for mixed usual/monotone/strictly monotone Hurwitz numbers.

Theorem 5.1. *Let μ and ν be partitions of some positive integer d. Moroever, let g be a non-negative integer. Furthermore, we fix an ordered partition $\lambda = (\lambda_1, \dots, \lambda_k)$ with $|\lambda| = 2g-2+\ell(\mu)+\ell(\nu)$ and denote $\lambda' = (\lambda_1, \dots, \lambda_{k-1})$. Then we have*

$$\vec{h}^{\leq,\circ}_{g;(\mu,-\nu),\lambda} = \frac{1}{k} \sum_{\substack{I,n,\mu^i,\\ \nu^i,\lambda^i,\gamma^i,g_i\\ \nu'}} \prod_{i=1}^{n} \vec{h}^{\leq,\circ}_{g_i;(\mu^i,(-\nu^i,-\gamma^i)),\lambda^i} \frac{1}{|\mathrm{Aut}(\nu_I)|} \cdot \left(\prod_{j=1}^{n} \prod_{l=1}^{\ell(\gamma^j)} (\gamma^j)_l \right)$$

$$\times \sum_{\substack{g_1^k+g_2^k=\\ \frac{\lambda_k+2-|I|+\ell(\gamma)}{2}}} \left\langle \tau_{2g_2^k-2} \right\rangle^{\mathbb{P}^1,\circ}_{g_2^k} \left\langle (\gamma^1,\dots,\gamma^n), \tau_{2g_1^k-2+\sum\ell(\gamma^i)+\ell(\nu')}, \nu' \right\rangle^{\mathbb{P}^1,\circ}_{g_1^k},$$

and

$$\vec{h}^{<,\circ}_{g;(\mu,-\nu),\lambda}$$

$$= \frac{1}{k} \sum_{\substack{I,n,\mu^i,\\ \nu^i,\lambda^i,\gamma^i,g_i\\ \nu'}} \prod_{i=1}^{n} \vec{h}^{<,\circ}_{g_i;(\mu^i,(-\nu^i,-\gamma^i)),\lambda^i} \cdot \frac{1}{|\mathrm{Aut}(\nu_I)|} \cdot \left(\prod_{i=1}^{n} \prod_{j=1}^{\ell(\gamma^i)} (\gamma^i)_j \right) (-1)^{\sum\ell(\gamma^j)+\ell(\nu')}$$

$$\times \sum_{\substack{g_1^k+g_2^k=\\ \frac{\lambda_k+2-|I|+\ell(\gamma)}{2}}} \left\langle \tau_{2g_2^k-2} \right\rangle^{\mathbb{P}^1,\circ}_{g_2^k} \left\langle (\gamma^1,\dots,\gamma^n), \tau_{2g_1^k-2+\sum\ell(\gamma^i)+\ell(\nu')}, \nu' \right\rangle^{\mathbb{P}^1,\circ}_{g_1^k},$$

where in both formulas, the first sum is over all

1. *subsets $I \subset \{1, \dots, \ell(\nu)\}$,*
2. *positive integers n,*
3. *decompositions of μ, ν, and λ into n partitions $\mu^1 \cup \dots \cup \mu^n = \mu$, $\nu^1 \cup \dots \cup \nu^n \cup \nu' = \nu$ and $\lambda^1 \cup \dots \cup \lambda^n = \lambda'$, where the μ^i must be non-empty,*
4. *partitions γ^i of $|\mu^i| - |\nu^i|$, where γ^i must be non-empty,*
5. *non-negative integers g_i with $\sum g_i = g - 1 + \frac{\lambda_k+2-n}{2} + \frac{3}{2}\sum\gamma^i$*

up to order.

Proof. This result is a consequence of Theorem 2.6. We focus on the case of monotone Hurwitz numbers, as the argument for strictly monotone Hurwitz numbers is the same up to a sign. The idea is to consider all covers contributing to $\vec{h}^{\leq}_{g;(\mu,-\nu),\lambda}$ and removing the last inner vertex which we denote by w. Let $\pi : \Gamma \to \mathbb{P}^1_{trop}$ be such a cover. When we remove the last inner vertex (and thus the adjacent ends which are indexed by I), the cover decomposes in possibly many disconnected components. Let n be their number. Each such component yields again a tropical cover $\pi^i : \Gamma^i \to \mathbb{P}^1_{trop}$ mapping to some subset $S^i \subset \{p_1, \dots, p_b\}$. Each cover π^i is contained in $\Gamma(\mathbb{P}^1_{trop}, g_i; (\mu^i, -\delta^i), \lambda^i)$ some non-negative integer g_i, a subpartition μ^i of μ, a partition δ^i of $|\mu^i|$, and a subpartition λ^i of λ. We note that δ^i can be decomposed into a subpartition of ν which we denote by ν^i and some partition γ^i given by the weights of the edges adjacent to the removed vertex and contained in the ith component, i.e., we have $\delta^i = (\nu^i, \gamma^i)$. This data satisfies conditions (1)–(5) stated in the theorem. The first four conditions are immediate. In order to observe the fifth condition, we consider the Euler characteristics of the graphs Γ and Γ_i. The Euler characteristic of Γ is given by

$$|V(\Gamma)| - |E(\Gamma)| = 1 - b_1(\Gamma) = 1 - g + \sum_{v \in V^{in}(\Gamma)} g(v) \tag{5}$$

and the Euler charcteristic of Γ_i is given by

$$|V(\Gamma_i)| - |E(\Gamma_i)| = 1 - b_1(\Gamma_i) = 1 - g + \sum_{v \in V^{in}(\Gamma_i)} g(v). \tag{6}$$

However, we see that

$$\Big(|V(\Gamma)| - 1 - |I| + \sum \ell(\gamma^i)\Big) - (|E(\Gamma) - |I|) = \sum_i |V(\Gamma_i)| - \sum |E(\Gamma_i)|, \tag{7}$$

since we remove a single vertex and $|I|$ many ends and leaves attached to it, i.e., $|I|$ vertices and $|I|$ edges. Moreover, all incoming edges of the removed vertices obtain an additional vertex which yields $\sum_i \ell(\gamma^i)$ many vertices. By combining equations (5), (6), and (7), we obtain

$$1 - g + \sum_{v \in V^{in}(\Gamma)} g(v) + \sum \ell(\gamma^i) = \sum_{i=1}^{n} \left(1 - g_i + \sum_{v \in V^{in}(\Gamma_i)} g(v)\right).$$

We observe that $\sum_i \sum_{v \in V^{in}(\Gamma_i)} g(v) = \sum_{v \in V(\Gamma)} g(v) - g(w)$ and therefore obtain

$$1 - g + g(w) + \sum \ell(\gamma^i) = n - \sum g_i.$$

However, we know that $\mathrm{val}(w) = n + \sum \ell(\gamma^i)$ and thus $g(w) = \frac{\lambda_k + 2 - n + \sum \ell(\gamma^i)}{2}$. Thus, we obtain

$$\sum g_i = g - 1 + \frac{\lambda_k + 2 - n}{2} + \frac{3}{2} \sum \ell(\gamma^i),$$

which is the last condition.

On the other hand, starting with data $I, n, \mu^i, \nu^i, \lambda^i, \gamma^i, g_i, \nu'$ satisfying these conditions, one can consider n tropical covers $\pi_i : \Gamma_i \to \mathbb{P}^1_{trop}$, where

$$\pi_i \in \Gamma(\mathbb{P}^1_{trop}, g_i, (\mu^i, (-\nu^i, \gamma^i), \lambda^i)).$$

We can then glue the π^is to a cover

$$\pi : \Gamma \to \mathbb{P}^1_{trop}$$

contributing to $\Gamma(\mathbb{P}^1_{trop}, g, (\mu, -\nu), \lambda)$: first, we choose subsets S^i of $\{p_1, \ldots, p_{k-1}\}$ with $|S^i| = \ell(\lambda^i)$. There are $\binom{\ell(\lambda)-1}{\ell(\lambda^1), \ldots, \ell(\lambda^n)}$ such choices. Then the vertices of π^i map to the points in S^i, while maintaining the order of the images of the vertices in π^i. We then join the edges with weights corresponding to the partitions γ^i to a single vertex w, such that these edges are incoming edges and w maps to p_k. Moreover, we attach $\ell(\nu')$ outgoing edges to w which are ends with weights in bijection to the entries of $\ell(\nu')$. This way, we obtain a cover $\pi \in \Gamma(\mathbb{P}^1_{trop}, g, (\mu, -\nu), \lambda)$. Let $\omega(\Gamma), \omega(\Gamma_i)$ be the weight of the graphs Γ and Γ_i. Then we observe that

$$\omega(\Gamma) = \frac{\prod \ell(\lambda^i)!}{\ell(\lambda)!} \cdot \frac{1}{|\mathrm{Aut}(\nu_I)|} \cdot \prod \omega(\Gamma_i) \cdot \left(\prod_{i=1}^{n} \prod_{j=1}^{\ell(\gamma^i)} (\gamma^i)_j \right)$$
$$\times \sum_{\substack{g_1^k + g_2^k = \\ \frac{\lambda_k + 2 - |I| + \ell(\gamma)}{2}}} \left\langle \tau_{2g_2^k - 2} \right\rangle_{g_2^k}^{\mathbb{P}^1, \circ} \left\langle (\gamma^i, \ldots, \gamma^n), \tau_{2g_1^k - 2 + + \sum \ell(\gamma^i) + \ell(\nu')}, \nu' \right\rangle_{g_1^k}^{\mathbb{P}^1, \circ}$$

where we note that $\frac{1}{|\mathrm{Aut}(\nu_I)|}$ contributes to $\frac{1}{|\mathrm{Aut}(\Gamma)|}$. This completes the proof. $\square$

We now want to generalise the statement above to mixed Hurwitz numbers. The following definition expresses mixed p-strictly monotone/ q-monotone/ $(b-(p+q))$-usual double Hurwitz numbers in terms of tropical covers weighted by Gromov-Witten invariants.

Definition 5.2. Let g be a non-negative integer, and $x \in (\mathbb{Z} \backslash \{0\})^n$ with $|x^+| = |x^-| = d$, $b = 2g - 2 + n$, let p and q be two integers such that $p + q \leq b$. Let $\lambda_{(1)}$ be a partition of p and let $\lambda_{(2)}$ be a partition of q, set $\tilde{\lambda}_i := 1$ for $i = 1, \ldots, b - (p+q)$, and finally set $\lambda := \lambda_{(1)} \cup \lambda_{(2)} \cup \tilde{\lambda}$. We are ready to define the λ-slice of the mixed p-strictly monotone/ q-monotone/ $(b-(p+q))$-usual double Hurwitz numbers

$$h^{\times,<,\leq,\bullet}_{g;x,p,q,\lambda} = \sum_{\pi \in \Gamma(\mathbb{P}^1_{\mathrm{trop}}, g; x, \lambda)} \frac{1}{|\mathrm{Aut}(\pi)|} \frac{1}{\ell(\lambda)!} \prod_{i=1}^{p} (-1)^{1+\mathrm{val}(v_i)} m_{v_i}$$
$$\times \prod_{j=p+1}^{p+q} m_{v_j} \prod_{k=p+q+1}^{b} m_{v_k} \prod_{e \in E(\Gamma)} \omega_e,$$

where $\Gamma(\mathbb{P}^1_{\mathrm{trop}}, g; x, \lambda)$ is the set of tropical covers $\pi : \Gamma \longrightarrow \mathbb{P}^1_{trop} = \mathbb{R}$ with $b = 2g - 2 + n$ points $p_1, \ldots, p_b$ fixed on the codomain $\mathbb{P}^1_{trop}$ and λ an ordered partition of b, such that

i). The unbounded left (resp., right) pointing ends of Γ have weights given by the partition x^+ (resp., x^-).
ii). The graph Γ has $l := \ell(\lambda) \leq b$ vertices. Let $V(\Gamma) = \{v_1, \ldots, v_l\}$ be the set of its vertices. Then we have $\pi(v_i) = p_i$ for $i = 1, \ldots, l$. Moreover, let $w_i = \mathrm{val}(v_i)$ be the corresponding valencies.

iii). We assign an integer $g(v_i)$ as the genus to v_i and the following condition holds true:

$$h^1(\Gamma) + \sum_{i=1}^{l} g(v_i) = g.$$

iv). We have $\lambda_i = \mathrm{val}(v_i) + 2g(v_i) - 2$.

v). For each vertex v_i, let y^+ (resp., y^-) be the tuple of weights of those edges adjacent to v_i which map to the right-hand (resp., left-hand) of p_i. The multiplicity m_{v_i} of v_i is defined to be

$$m_{v_i} = (\lambda_i - 1)! |\mathrm{Aut}(y^+)| |\mathrm{Aut}(y^-)| \\ \times \sum_{g_1^i + g_2^i = g(v_i)} \left\langle \tau_{2g_2^i - 2}(\omega) \right\rangle_{g_2^i}^{\mathbb{P}^1, \circ} \left\langle y^+, \tau_{2g_1^i - 2 + n}(\omega), y^- \right\rangle_{g_1^i}^{\mathbb{P}^1, \circ}.$$

Note that the m_{v_k} above always simplify to either one (in most of the cases) or two (only in case the two half-edges directed towards the same end have equal weights). Furthermore, we define $h^{\times,<,\leq,\circ}_{g;x,p,q,\lambda}$ by considering only connected source curves.

Remark 5.3. It is a straightforward generalisation of theorem 2.6 in [34] the fact that these numbers $h^{\times,<,\leq,\bullet}_{g;x,p,q,\lambda}$ are the λ-slices of mixed usual/monotone/strictly-monotone Hurwitz numbers, meaning that if we define

$$h^{\times,<,\leq,\bullet}_{g;x,p,q} := \sum_{\substack{\lambda = (\lambda_{(1)}, \lambda_{(2)}, \tilde{\lambda}) \vdash b \\ \lambda_i = 1,\ i = p+q+1, \dots, b \\ \lambda_{(1)} \vdash p,\ \lambda_{(2)} \vdash q}} h^{\times,<,\leq,\bullet}_{g;x,p,q,\lambda},$$

then $h^{\times,<,\leq,\bullet}_{g;x,p,q}$ enumerates all weighted ramified covers of degree $d = |x^+| = |x^-|$ of the Riemann sphere by genus g compact surfaces where the ramification profiles over zero and infinity are given by x^+ and x^-, respectively, and all other ramifications are simple (and therefore can be represented as transpositions $(a_i, b_i)_{i=1,\dots,b}$ with $1 \leq a_i < b_i \leq d$), in such a way that the first p simple ramifications satisfy the strictly monotone condition, the following q satisfy the weakly monotone condition, and the remaining $b-(p+q)$ are usual simple ramifications (and hence do not satisfy any additional requirement):

(1) $b_i < b_{i+1}$ for $i = 1, \dots, p-1$,
(2) $b_i \leq b_{i+1}$ for $i = p+1, \dots, p+q-1$.

With the notation above, we are going to generalise Theorem 5.1 by cutting one vertex of the tropical covers. However, there are now three different types of vertices, as opposed to one in Theorem 5.1: the strictly monotone vertices, the weakly monotone vertices, and the usual vertices. We therefore obtain three different recursions, depending on which type of vertex we are cutting. Note that the first and the second type of vertex differ just by a sign factor in their weights, whereas the third type is extremely simple as its genus is zero and its cardinality must be equal to three. It is, moreover, possible to have the first and second type of vertices which happen to be usual vertices (this happens if and only if they come from parts of λ equal to one in the first $p+q$ parts): we still treat them according to their general nature, as the formula for their weight in that case naturally specialises to the weight of usual vertices.

Corollary 5.4. *Let μ and ν be partitions of some positive integer d, let g, p, q be non-negative integers, let λ be a partition $\lambda = \lambda_{(1)} \cup \lambda_{(2)} \cup \tilde{\lambda} = (\lambda_1, \ldots, \lambda_k)$ with $|\lambda| = b = 2g - 2 + \ell(\mu) + \ell(\nu)$, $\lambda_{(1)} \vdash p$, $\lambda_{(2)} \vdash q$, $p + q \leq b$, $\tilde{\lambda}_i = 1$ for all i, and for a partition σ denote $\sigma' = \sigma \setminus \{\sigma_{\ell(\sigma)}\}$. Then we have the following three recursions:*

i). ***Cutting along a strictly monotone vertex.***

$$\begin{aligned}
&\vec{h}^{\times,<,\leq,\circ}_{g;(\mu,-\nu),p,q,\lambda} \\
&= \frac{1}{k} \sum_{\substack{I,n,\mu^i,\nu^i,\nu' \\ \lambda^i_{(1)},\lambda^i_{(2)},\tilde{\lambda}^i,\gamma^i,g_i}} \prod_{i=1}^{n} \vec{h}^{<,\circ}_{g_i;(\mu^i,(-\nu^i,-\gamma^i)),\lambda^i} \cdot (-1)^{\sum \ell(\gamma^j)+\ell(\nu')} \cdot \frac{1}{|\mathrm{Aut}(\nu_I)|} \cdot \left(\prod_{i=1}^{n} \prod_{j=1}^{\ell(\gamma^i)} (\gamma^i)_j \right) \\
&\times \sum_{\substack{g_1^k+g_2^k= \\ \frac{\lambda_k+2-|I|+\ell(\gamma)}{2}}} \left\langle \tau_{2g_2^k-2} \right\rangle^{\mathbb{P}^1,\circ}_{g_2^k} \left\langle (\gamma^i, \ldots, \gamma^n), \tau_{2g_1^k-2+\sum \ell(\gamma^i)+\ell(\nu')}, \nu' \right\rangle^{\mathbb{P}^1,\circ}_{g_1^k},
\end{aligned}$$

ii). ***Cutting along a weakly monotone vertex.***

$$\begin{aligned}
\vec{h}^{\times,<,\leq,\circ}_{g;(\mu,-\nu),p,q,\lambda} &= \frac{1}{k} \sum_{\substack{I,n,\mu^i,\nu^i,\nu' \\ \lambda^i_{(1)},\lambda^i_{(2)},\tilde{\lambda}^i,\gamma^i,g_i}} \prod_{i=1}^{n} \vec{h}^{<,\circ}_{g_i;(\mu^i,(-\nu^i,-\gamma^i)),\lambda^i} \cdot \frac{1}{|\mathrm{Aut}(\nu_I)|} \cdot \left(\prod_{i=1}^{n} \prod_{j=1}^{\ell(\gamma^i)} (\gamma^i)_j \right) \\
&\times \sum_{\substack{g_1^k+g_2^k= \\ \frac{\lambda_k+2-|I|+\ell(\gamma)}{2}}} \left\langle \tau_{2g_2^k-2} \right\rangle^{\mathbb{P}^1,\circ}_{g_2^k} \left\langle (\gamma^i, \ldots, \gamma^n), \tau_{2g_1^k-2+\sum \ell(\gamma^i)+\ell(\nu')}, \nu' \right\rangle^{\mathbb{P}^1,\circ}_{g_1^k}.
\end{aligned}$$

iii). ***Cutting along a usual vertex.***

$$\vec{h}^{\times,<,\leq,\circ}_{g;(\mu,-\nu),p,q,\lambda} = \frac{1}{k} \sum_{\substack{I,n\leq 2,\mu^i,\nu^i,\nu' \\ \lambda^i_{(1)},\lambda^i_{(2)},\tilde{\lambda}^i,\gamma^i,g_i}} \prod_{i=1}^{n} \vec{h}^{<,\circ}_{g_i;(\mu^i,(-\nu^i,-\gamma^i)),\lambda^i} \cdot \frac{1}{|\mathrm{Aut}(\nu_I)|} \cdot \left(\prod_{i=1}^{n} \prod_{j=1}^{\ell(\gamma^i)} (\gamma^i)_j \right),$$

where in all three formulas, the first sum is over all

(1) *subsets $I \subset \{1, \ldots, \ell(\nu)\}$,*
(2) *positive integers n (smaller than or equal to 2 in the third recursion),*
(3) *decompositions of μ, ν, and λ into n partitions $\mu^1 \cup \cdots \cup \mu^n = \mu, \nu^1 \cup \cdots \cup \nu^n \cup \nu' = \nu$, where the μ^i must be non-empty,*
(4) *partitions γ^i of $|\mu^i| - |\nu^i|$, where γ^i must be non-empty,*
(5) *non-negative integers g_i with $\sum g_i = g - 1 + \frac{\lambda_k+2-n}{2} + \frac{3}{2} \sum \gamma^i$,*
(6) *in the third case we require $|\nu'| = 3 - \sum_i \ell(\gamma^i)$,*

up to order, and moreover

i). *when cutting over a strictly monotone vertex we have*

$$\lambda^1_{(1)} \cup \cdots \cup \lambda^n_{(1)} = \lambda'_{(1)}, \qquad \lambda^1_{(2)} \cup \cdots \cup \lambda^n_{(2)} = \lambda_{(2)}, \qquad \tilde{\lambda}^1 \cup \cdots \cup \tilde{\lambda}^n = \tilde{\lambda};$$

ii). *when cutting over a weakly monotone vertex we have*

$$\lambda^1_{(1)} \cup \cdots \cup \lambda^n_{(1)} = \lambda_{(1)}, \qquad \lambda^1_{(2)} \cup \cdots \cup \lambda^n_{(2)} = \lambda'_{(2)}; \qquad \tilde{\lambda}^1 \cup \cdots \cup \tilde{\lambda}^n = \tilde{\lambda};$$

iii). *when cutting over a usual vertex we have*

$$\lambda^1_{(1)} \cup \cdots \cup \lambda^n_{(1)} = \lambda_{(1)}, \qquad \lambda^1_{(2)} \cup \cdots \cup \lambda^n_{(2)} = \lambda_{(2)}, \qquad \tilde{\lambda}^1 \cup \cdots \cup \tilde{\lambda}^n = \tilde{\lambda}'.$$

Proof. The proof is a straightforward generalisation of the one of Theorem 5.1. The only difference is that we need to keep track of the partitions of p and q when cutting, and eliminate the right cut vertex from the summations over $\lambda^i_{(1)}, \lambda^i_{(2)}, \tilde{\lambda}^i$. Let us point out explicitly in the three cases what the differences are in the computation.

- When cutting over a strictly monotone vertex, the length of the partition $\lambda_{(1)}$ descreases by one as one vertex has been cut, whereas the partitions corresponding to weakly monotone vertices and usual vertices remain the same. Let n be the number of connected components: the vertices of the three types distribute in all possible ways in the n components. This explains condition i). in the statement. The reasoning is completely analogue when cutting over weakly monotone vertex or a usual vertex, which gives conditions ii). and iii).
- The conditions (1)–(5) have exactly the same geometric meaning as in Theorem 5.1, with the exception of condition (2) when cutting out a usual vertex: in fact a usual vertex has valency equal to three, and therefore the graph after the cut can have only either one or two connected components. Therefore n is bounded by 2. This also explains condition (6).
- The cut vertex in the third recursion has genus zero, therefore the recursion has trivial residue Gromov-Witten invariant.
- The extra minus signs appear only in the first recursion, as we cut along a vertex of strict monotone-type, and therefore we need to multiply by the minus signs in the weight of the cut vertex. The other recursions also carry these minus signs, but they remain hidden in the definition of the weigths of the strict monotone vertices throughout the whole recursive procedure.

This concludes the proof of the corollary. □

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