

Conditional stability up to the final time for backward-parabolic equations with Log-Lipschitz coefficients

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Abstract

We prove logarithmic conditional stability up to the final time for backward-parabolic operators whose coefficients are Log-Lipschitz continuous in t and Lipschitz continuous in x . The result complements previous achievements of Del Santo and Prizzi (2009) and Del Santo, Jäh and Prizzi (2015), concerning conditional stability (of a type intermediate between Hölder and logarithmic), arbitrarily closed, but not up to the final time.

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1. Introduction

In real world models, deterministic diffusion processes are often irreversible. Consider for example the heat equation

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$$\partial_t u = \Delta u$$

with Cauchy data $u(0, x) = u_0(x)$. The forward initial value problem is well posed in an appropriate space of physically meaningful configurations, but the evolution has a strong regularizing effect, so when one tries to reconstruct an initial configuration $u(0, x)$ from a final observation $u(T, x)$ at a positive time T , one needs to impose regularity conditions on $u(T, x)$, while in general the backward problem with Cauchy data at T has no solution. However, in a physical context an observation at a final time T records the configuration resulting from an *actual* evolution, so the problem of existence is less relevant than that of uniqueness and sensitiveness to errors in measurements. In [22] John introduced the notion of *well-behaved problem* for ill-posed problems. According to John a problem is *well-behaved* if “only a fixed percentage of the significant digits need be lost in determining the solution from the data” [22, p. 552]. More precisely, a problem is well-behaved if its solutions in a space \mathcal{H} depend Hölder continuously on the data belonging to a space \mathcal{K} , provided the solutions satisfy a prescribed *a priori* bound. According to the literature, we call *conditional stability* any continuous dependence (possibly weaker than Hölder) which is subordinated to a prescribed *a priori* bound.

In this paper we carry on the investigation about conditional stability of backward solutions for a general parabolic equation. For ease of notation we reformulate the problem inverting the sign of the time variable, so we deal with (forward) solutions of the backward-parabolic equation

$$\partial_t u + \sum_{i,j} \partial_{x_i} (a_{ij}(t, x) \partial_{x_j} u) = 0 \quad (1.1)$$

on the strip $[0, T] \times \mathbb{R}^n$. We assume throughout the paper that the matrix $(a_{ij})_{i,j=1}^n$ is symmetric and positive definite and that the coefficients a_{ij} ’s are at least Lipschitz continuous in x and Hölder continuous in t . These are the standard regularity assumptions which guarantee the (forward) well posedness for forward-parabolic equations in H^s , $0 \leq s \leq 2$ (see e.g. [2]). We denote by

$$\mathcal{H} := C^0([0, T], L^2(\mathbb{R}^n)) \cap C^0([0, T], H^1(\mathbb{R}^n)) \cap C^1([0, T], L^2(\mathbb{R}^n))$$

the space for admissible solutions of (1.1).

In [1] Agmon and Nirenberg proved, among other things, that the Cauchy problem for (1.1) on the interval $[0, T]$ is well-behaved in the space \mathcal{H} with data in $L^2(\mathbb{R}^n)$ on each subinterval $[0, T']$ with $T' < T$, provided the coefficients a_{ij} ’s are sufficiently smooth with respect to x and Lipschitz continuous with respect to t . In order to achieve their result they developed the so called *logarithmic convexity technique*. The main step consists in proving that the function $t \mapsto \log \|u(t, \cdot)\|_{L^2}$ is convex for every solution $u \in \mathcal{H}$ of (1.1). In the same year Glagoleva [17] obtained essentially the same result for a concrete operator like (1.1) with time independent coefficients. Her proof rests on energy estimates obtained through integration by parts. Some years later Hurd [19] developed the technique of Glagoleva to cover the case of a general equation of type (1.1), with coefficients depending Lipschitz continuously on time. The results of [1, 17, 19] can be summarized as follows:

Theorem A. *Assume the coefficients a_{ij} ’s are Lipschitz continuous with respect to t . For every $T' \in (0, T)$ and $D > 0$ there exist $\rho > 0$, $0 < \delta < 1$ and $K > 0$ such that, if $u \in \mathcal{H}$ is a solution of (1.1) on $[0, T]$ with $\|u(0, \cdot)\|_{L^2} \leq \rho$ and $\|u(t, \cdot)\|_{L^2} \leq D$ on $[0, T]$, then*

$$\sup_{t \in [0, T']} \|u(t, \cdot)\|_{L^2} \leq K \|u(0, \cdot)\|_{L^2}^\delta.$$

The constants ρ , K and δ depend only on T' and D , on the positivity constant of the matrix $(a_{ij})_{i,j=1}^n$, on the L^∞ norms of the coefficients a_{ij} 's and of their spatial derivatives, and on the Lipschitz constant of the coefficients a_{ij} 's with respect to time.

As T' approaches T , the constant K above blows up, while δ decays to 0, so one cannot expect that solutions are well behaved up to the final time T . From the physical point of view, going back to the forward parabolic equation, this means that the reconstruction of the past from observations at the final time $t = T$ worsens more and more as one gets closer to the initial time $t = 0$. Yet, as it was proved by various authors (e.g. Imanuvilov and Yamamoto [20], Yamamoto [27], Isakov [21]), some kind of conditional stability for the backward-parabolic equation (1.1) up to the final time T can be recovered if one settles for integral estimates rather than pointwise estimates. Moreover, pointwise estimates can be recovered by imposing stronger *a priori* bounds on the solutions. In any case, however, one doesn't get Hölder dependence but only logarithmic dependence on data. The results of [20,27,21] can be summarized as follows:

Theorem B. *Assume the coefficients a_{ij} 's are Lipschitz continuous with respect to t . For every $D > 0$ there exist $\rho > 0$, $0 < \delta \leq 1$ and $K > 0$ such that, if $u \in \mathcal{H}$ is a solution of (1.1) on $[0, T]$ with $\|u(0, \cdot)\|_{L^2} \leq \rho$ and $\|u(t, \cdot)\|_{L^2} \leq D$ on $[0, T]$, then*

$$\int_0^T \|u(t, \cdot)\|_{L^2}^2 dt \leq K \frac{1}{|\log \|u(0, \cdot)\|_{L^2}|^{2\delta}}.$$

Moreover, if $\|u(t, \cdot)\|_{H^1} \leq D$ on $[0, T]$, then

$$\sup_{t \in [0, T]} \|u(t, \cdot)\|_{L^2} \leq K \frac{1}{|\log \|u(0, \cdot)\|_{L^2}|^\delta}.$$

The constants ρ , K and δ depend only on D , on the positivity constant of the matrix $(a_{ij})_{i,j=1}^n$, on the L^∞ norms of the coefficients a_{ij} 's and of their spatial derivatives, and on the Lipschitz constant of the coefficients a_{ij} 's with respect to time.

In all the above mentioned results, Lipschitz continuity of the coefficients a_{ij} 's with respect to time plays an essential role. The possibility of replacing Lipschitz continuity by simple continuity was ruled out by Miller [26] and more recently by Mandache [23]. They constructed examples of operators of the form (1.1) which do not enjoy the uniqueness property in \mathcal{H} . In the example of Miller the coefficients a_{ij} 's are Hölder continuous in time, while in the more refined example of Mandache the modulus of continuity $\bar{\mu}$ of the coefficients a_{ij} 's with respect to time needs only to satisfy $\int_0^1 (1/\bar{\mu}(s)) ds < +\infty$. On the other hand, in [9,11,12] it was proved that if $\bar{\mu}$ satisfies the Osgood condition, i.e. $\int_0^1 (1/\bar{\mu}(s)) ds = +\infty$, then equation (1.1) enjoys the uniqueness property in \mathcal{H} . Therefore it would be natural to conjecture that if the Osgood condition is satisfied, then the Cauchy problem for (1.1) is well-behaved in \mathcal{H} with data in $L^2(\mathbb{R}^n)$. Unfortunately this is

not true, as shown by a counterexample in [10]. Nevertheless if the coefficients a_{ij} 's are Log-Lipschitz continuous in time, it was shown in [10,8] that a weaker conditional stability result holds:

Theorem C. *Assume the coefficients a_{ij} 's are Log-Lipschitz continuous with respect to t . For every $T' \in (0, T)$ and $D > 0$ there exist $\rho > 0$, $0 < \delta < 1$ and $K, N > 0$ such that, if $u \in \mathcal{H}$ is a solution of (1.1) on $[0, T]$ with $\|u(0, \cdot)\|_{L^2} \leq \rho$ and $\|u(t, \cdot)\|_{L^2} \leq D$ on $[0, T]$, then*

$$\sup_{t \in [0, T']} \|u(t, \cdot)\|_{L^2} \leq K e^{-N |\log \|u(0, \cdot)\|_{L^2}|^\delta}.$$

The constants ρ , K , N and δ depend only on T' and D , on the positivity constant of the matrix $(a_{ij})_{i,j=1}^n$, on the L^∞ norms of the coefficients a_{ij} 's and of their spatial derivatives, and on the Log-Lipschitz constant of the coefficients a_{ij} 's with respect to time.

Moreover, in [5] a (very feeble) conditional stability result was proved even when the coefficients a_{ij} 's are just Osgood continuous with respect to t , provided they depend only on time.

The proof of Theorem C relies on weighted energy estimates in the spirit of [17,19,20,27], but in order to overcome the obstructions created by the lack of time differentiability of the coefficients a_{ij} 's it is necessary to introduce a weight function taylored on the modulus of continuity of the a_{ij} 's (see Proposition 2.4), and a microlocal approximation procedure originally developed by Colombini and Lerner in [6] in the context of hyperbolic equations with Log-Lipschitz coefficients.

In this paper we shall exploit the same type of weighted energy estimates to extend Theorem B to the case of parabolic equations whose coefficients are Log-Lipschitz continuous in time (Theorems 5.1 and 5.3). Our results can be summarized as follows:

Theorem D. *Assume the coefficients a_{ij} 's are Log-Lipschitz continuous with respect to t . For every $D > 0$ there exist $\rho > 0$, $0 < \delta \leq 1$ and $K > 0$ such that, if $u \in \mathcal{H}$ is a solution of (1.1) on $[0, T]$ with $\|u(0, \cdot)\|_{L^2} \leq \rho$ and $\|u(t, \cdot)\|_{L^2} \leq D$ on $[0, T]$, then*

$$\int_0^T \|u(t, \cdot)\|_{L^2}^2 dt \leq K \frac{1}{|\log \|u(0, \cdot)\|_{L^2}|^{2\delta}}.$$

Moreover, if $\|u(t, \cdot)\|_{H^1} \leq D$ on $[0, T]$, then

$$\sup_{t \in [0, T]} \|u(t, \cdot)\|_{L^2} \leq K \frac{1}{|\log \|u(0, \cdot)\|_{L^2}|^\delta}.$$

The constants ρ , K and δ depend only on D , on the positivity constant of the matrix $(a_{ij})_{i,j=1}^n$, on the L^∞ norms of the coefficients a_{ij} 's and of their spatial derivatives, and on the Log-Lipschitz constant of the coefficients a_{ij} 's with respect to time.

Our results therefore complement the achievements of [10,8], and *en passant* improve them in some crucial technical points related to the regularity of the coefficients a_{ij} 's with respect to the x variable (see the discussion in the final part of section 2). Finally, in Section 6 we illustrate some applications of the main results.

2. The weighted energy estimate

We consider the backward-parabolic equation

$$\partial_t u + \sum_{j,k=1}^n \partial_{x_j} (a_{jk}(t, x) \partial_{x_k} u) = 0 \quad (2.1)$$

on the strip $[0, T] \times \mathbb{R}_x^n$.

Hypothesis 2.1. We assume throughout the paper that:

- for all $(t, x) \in [0, T] \times \mathbb{R}_x^n$ and for all $j, k = 1, \dots, n$,

$$a_{jk}(t, x) = a_{kj}(t, x);$$

- there exists $\kappa \in (0, 1)$ such that for all $(t, x, \xi) \in [0, T] \times \mathbb{R}_x^n \times \mathbb{R}_\xi^n$,

$$\kappa |\xi|^2 \leq \sum_{j,k=1}^n a_{jk}(t, x) \xi_j \xi_k \leq \frac{1}{\kappa} |\xi|^2; \quad (2.2)$$

- for all $j, k = 1, \dots, n$, $a_{jk} \in \text{Log Lip}([0, T], L^\infty(\mathbb{R}_x^n)) \cap L^\infty([0, T], \text{Lip}(\mathbb{R}_x^n))$.

We set

$$A_{LL} := \sup \left\{ \frac{|a_{jk}(t, x) - a_{jk}(s, x)|}{|t - s|(1 + |\log |t - s||)} \mid j, k = 1, \dots, n, \right. \\ \left. t, s \in [0, T], x \in \mathbb{R}_x^n, 0 < |s - t| \leq 1 \right\},$$

$$A := \sup \{ \|\partial_x^\alpha a_{jk}(t, \cdot)\|_{L^\infty} \mid |\alpha| \leq 1, t \in [0, T] \}.$$

Remark 2.2. By classical regularity theory for elliptic partial differential equations (see e.g. [16, Thms. 8.8 and 8.12]), for each $t \in [0, T]$ the operator

$$\mathcal{A}(t)u := - \sum_{j,k=1}^n \partial_{x_j} (a_{jk}(t, x) \partial_{x_k} u)$$

is self-adjoint and positive definite in $L^2(\mathbb{R}^n)$, with domain $H^2(\mathbb{R}^n)$. Moreover the dependence on t of the operator $\mathcal{A}(t)$ is better than Hölder continuous, so one can apply the abstract theory of linear parabolic equations (see e.g. [2, Thm. 4.4.1]) and obtain well posedness of the *forward* equation

$$\partial_t u - \sum_{j,k=1}^n \partial_{x_j} (a_{jk}(t, x) \partial_{x_k} u) = 0$$

in $H^\theta(\mathbb{R}^n)$ for every $0 \leq \theta \leq 2$.

For $s > 0$, let $\mu(s) = s(1 + |\log(s)|)$. For $p \geq 1$, we define

$$\omega(p) := \int_{\frac{1}{p}}^1 \frac{1}{\mu(s)} ds = \log(1 + \log p).$$

The function $\omega : [1, +\infty) \rightarrow [0, +\infty)$ is bijective and strictly increasing. For $y \in (0, 1]$ and $\lambda > 1$, we set $\psi_\lambda(y) = \omega^{-1}(-\lambda \log(y)) = \exp(y^{-\lambda} - 1)$ and we define

$$\Phi_\lambda(y) := - \int_y^1 \psi_\lambda(z) dz.$$

The function $\Phi_\lambda : (0, 1] \rightarrow (-\infty, 0]$ is bijective and strictly increasing; moreover, it satisfies

$$y\Phi_\lambda''(y) = -\lambda(\Phi_\lambda'(y))^2 \mu\left(\frac{1}{\Phi_\lambda'(y)}\right) = -\lambda\Phi_\lambda'(y)\left(1 + \left|\log\left(\frac{1}{\Phi_\lambda'(y)}\right)\right|\right). \quad (2.3)$$

In the next lemma, we collect some properties of the functions ψ_λ and Φ_λ . The proof is left to the reader.

Lemma 2.3. *Let $\zeta > 1$. Then, for $0 < y \leq 1/\zeta$,*

$$\psi_\lambda(\zeta y) = \exp(\zeta^{-\lambda} - 1)(\psi_\lambda(y))^{\zeta^{-\lambda}}.$$

Define $\Lambda_\lambda(y) := y\Phi_\lambda(1/y)$. Then the function $\Lambda_\lambda : [1, +\infty) \rightarrow (-\infty, 0]$ is bijective and

$$\lim_{z \rightarrow -\infty} -\frac{1}{z} \psi_\lambda\left(\frac{1}{\Lambda_\lambda^{-1}(z)}\right) = +\infty. \quad \square$$

We denote by

$$\mathcal{H} := C^0([0, T], L^2(\mathbb{R}_x^n)) \cap C^0([0, T], H^1(\mathbb{R}_x^n)) \cap C^1([0, T], L^2(\mathbb{R}_x^n))$$

the space for admissible solutions of (2.1).

Proposition 2.4 (Weighted energy estimate). *Assume Hypothesis 2.1 is satisfied. There exists a constant $\alpha_1 > 0$ (depending only on A_{LL} , A and κ) and, setting $\alpha := \max\{\alpha_1, T^{-1}\}$, $\sigma := \frac{1}{\alpha}$ and $\tau := \frac{\sigma}{4}$, there exist constants $\bar{\lambda} > 1$, $\bar{\gamma} > 0$ and $M > 0$ (depending on A_{LL} , A , κ and α , and hence on T) such that, for all $\beta \geq \sigma + \tau$, $\lambda \geq \bar{\lambda}$ and $\gamma \geq \bar{\gamma}$ and whenever $u \in \mathcal{H}$ is a solution of equation (2.1), the estimate*

$$\begin{aligned}
& \int_0^s e^{2\gamma t} e^{-2\beta\Phi_\lambda\left(\frac{t+\tau}{\beta}\right)} \|u(t, \cdot)\|_{H^{1-\alpha t}}^2 dt \\
& \leq M\gamma \left((s + \tau) e^{2\gamma s} e^{-2\beta\Phi_\lambda\left(\frac{s+\tau}{\beta}\right)} \|u(s, \cdot)\|_{H^{1-\alpha s}}^2 \right. \\
& \quad \left. + \tau \Phi'_\lambda\left(\frac{\tau}{\beta}\right) e^{-2\beta\Phi_\lambda\left(\frac{\tau}{\beta}\right)} \|u(0, \cdot)\|_{L^2}^2 \right) \tag{2.4}
\end{aligned}$$

holds for all $s \in [0, \sigma]$.

Remark 2.5. If one would like to include lower order terms in (2.1), one has to suppose that the corresponding coefficients are L^∞ with respect to t and also Lip with respect to x . The constants in Proposition 2.4 then will depend also on the norms of the coefficients of the lower order terms.

In [10] estimate (2.4) was used to deduce the following local conditional stability result:

Theorem 2.6 ([10, Thm.1]). *Assume Hypothesis 2.1 is satisfied. Let α_1 , α and σ be as in Proposition 2.4. Then there exist constants ρ , δ , K and N , such that, whenever $u \in \mathcal{H}$ is a solution of (2.1) with $\|u(0, \cdot)\|_{L^2} \leq \rho$, the inequality*

$$\sup_{t \in [0, \sigma/8]} \|u(t, \cdot)\|_{L^2} \leq K(1 + \|u(\sigma, \cdot)\|_{L^2}) \exp(-N(|\log(\|u(0, \cdot)\|_{L^2})|^\delta))$$

holds true. The constants ρ , δ , K and N depend on A_{LL} , A , κ and α , and hence on T . \square

The fact that α_1 is independent of T and $\sigma = \min\{\alpha_1^{-1}, T\}$ allows one to iterate the local result of Theorem 2.6 a finite number of times, and to obtain conditional stability in the large.

Theorem 2.7 ([10, Thm. 2]). *Assume Hypothesis 2.1 is satisfied. Then for all $T' \in (0, T)$ and $D > 0$ there exist positive constants ρ' , δ' , K' and N' , depending only on A_{LL} , A , κ , T , T' and D , such that if $u \in \mathcal{H}$ is a solution of (2.1) satisfying $\sup_{t \in [0, T]} \|u(t, \cdot)\|_{L^2} \leq D$ and $\|u(0, \cdot)\|_{L^2} \leq \rho'$, the inequality*

$$\sup_{t \in [0, T']} \|u(t, \cdot)\|_{L^2} \leq K' \exp(-N' |\log(\|u(0, \cdot)\|_{L^2})|^\delta)$$

holds true. \square

Remark 2.8. Notice that, following Remark 2.2, it would be sufficient to impose an a-priori bound on $\|u(T, \cdot)\|_{L^2}$, which automatically implies the a-priori bound for $\|u(t, \cdot)\|_{L^2}$, $t \in [0, T]$.

Estimate (2.4) was proved in [10] when the coefficients $a_{ij}(t, x)$ are of class C^2 with respect to x (in this case the constant A contains also the L^∞ norm of the second order spatial derivatives of the a_{ij} 's). Actually, in [10] C^2 regularity was imposed to overcome a technical difficulty in managing a commutator term appearing in the dyadic decomposition of equation (2.1). However, once estimate (2.4) is achieved, Theorems 2.6 and 2.7 follow directly from it, and the additional regularity in x of the a_{ij} 's plays no role.

The C^2 requirement is somewhat “non natural”, since Lipschitz continuity in x of the a_{ij} ’s is sufficient in order that the domain of the operator $-\sum_{j,k=1}^n \partial_{x_j}(a_{jk}(t, x)\partial_{x_k})$ be $H^2(\mathbb{R}^n)$ (see [16, Thms. 8.8 and 8.12]).

In [8] a weaker version of estimate (2.4) was obtained by mean of Bony paraproducts (see [4]), when C^2 regularity in x is replaced by the more natural Lipschitz regularity. In this weaker version of (2.4) the spaces L^2 and $H^{1-\alpha t}$ were replaced by $H^{-\bar{\theta}}$ and $H^{1-\bar{\theta}-\alpha t}$ respectively, where $0 < \bar{\theta} < 1$, and the estimate hold for $s \in [0, \frac{7}{8}\sigma]$, where $\sigma = (1 - \bar{\theta})/\alpha$ ([8, Prop. 2.9]). Such weaker version of (2.4), together with some nontrivial modifications of the arguments in [10], led eventually to recover the continuity results of Theorems 2.6 and 2.7. However, the weaker weighted energy estimate of [8] turns out to be unfit for the purpose of reaching any kind of stability up to the final time T , especially because in that version of the estimate one can not integrate up to $s = \sigma$ in the left hand side of (2.4), but has to stop at $s = \sigma' < \sigma$. Therefore we shall go back to the strong weighted energy estimate (2.4) and demonstrate it in the Lipschitz continuous case, using some ideas contained in [8] and performing a more careful and precise analysis of some terms in the paramultiplication procedure.

3. Littlewood-Paley theory and Bony’s paraproduct

In this section, we review some elements of the Littlewood-Paley decomposition which we shall use throughout this paper to define Bony’s paraproduct. The proofs which are not contained in this section can be found in [10], [11] and [25].

Let $\chi \in C_0^\infty(\mathbb{R})$ with $0 \leq \chi(s) \leq 1$ be an even function and such that $\chi(s) = 1$ for $|s| \leq 11/10$ and $\chi(s) = 0$ for $|s| \geq 19/10$. We now define $\chi_k(\xi) = \chi(2^{-k}|\xi|)$ for $k \in \mathbb{Z}$ and $\xi \in \mathbb{R}_\xi^n$. Denoting by \mathcal{F} the Fourier-transform and by \mathcal{F}^{-1} its inverse, we define the operators

$$\begin{aligned} S_{-1}u &= 0 \text{ and } S_k u = \chi_k(D_x)u = \mathcal{F}^{-1}(\chi_k(\cdot)\mathcal{F}(u)(\cdot)), \quad k \geq 0, \\ \Delta_0 u &= S_0 u \text{ and } \Delta_k u = S_k u - S_{k-1}u, \quad k \geq 1. \end{aligned}$$

We define

$$\text{spec}(u) := \text{supp}(\mathcal{F}(u))$$

and we will use the abbreviation $\Delta_k u = u_k$. For $u \in \mathcal{S}'(\mathbb{R}_x^n)$, we have

$$u = \lim_{k \rightarrow +\infty} S_k u = \sum_{k \geq 0} \Delta_k u$$

in the sense of $\mathcal{S}'(\mathbb{R}_x^n)$.

We shall make use of the classical

Proposition 3.1 (*Bernstein’s inequalities*). *Let $u \in \mathcal{S}'(\mathbb{R}_x^n)$. Then, for $\nu \geq 1$,*

$$2^{\nu-1} \|u_\nu\|_{L^2} \leq \|\nabla_x u_\nu\|_{L^2} \leq 2^{\nu+1} \|u_\nu\|_{L^2}. \quad (3.1)$$

The right inequality of (3.1) holds also for $\nu = 0$. \square

In the following two propositions we recall the characterization of the classical Sobolev spaces and Lipschitz-continuous functions via Littlewood-Paley decomposition.

Proposition 3.2 ([7, Lemma 3.2]). *Let $\theta \in \mathbb{R}$. Then a tempered distribution $u \in \mathcal{S}'(\mathbb{R}_x^n)$ belongs to $H^\theta(\mathbb{R}_x^n)$ iff for all $k \geq 0$, $\Delta_k u \in L^2(\mathbb{R}_x^n)$ and*

$$\sum_{k=0}^{+\infty} 2^{2k\theta} \|\Delta_k u\|_{L^2}^2 < +\infty.$$

Moreover, there exists $C_\theta \geq 1$ such that for all $u \in H^\theta(\mathbb{R}_x^n)$, we have

$$\frac{1}{C_\theta} \|u\|_{H^\theta} \leq \left(\sum_{k=0}^{+\infty} 2^{2k\theta} \|\Delta_k u\|_{L^2}^2 \right)^{\frac{1}{2}} \leq C_\theta \|u\|_{H^\theta}.$$

The constant C_θ remains bounded for θ in compact subsets of \mathbb{R} . \square

Proposition 3.3 ([15, Lemma 3.2]). *A function $a \in L^\infty(\mathbb{R}_x^n)$ belongs to $\text{Lip}(\mathbb{R}_x^n)$ iff*

$$\sup_{k \in \mathbb{N}_0} \|\nabla_x(S_k a)\|_{L^\infty} < +\infty.$$

Moreover, there exists a positive constant C such that if $a \in \text{Lip}(\mathbb{R}_x^n)$, then

$$\|\Delta_k a\|_{L^\infty} \leq C 2^{-k} \|a\|_{\text{Lip}}, \quad \text{and} \quad \|\nabla_x(S_k a)\|_{L^\infty} \leq C \|a\|_{\text{Lip}},$$

where $\|a\|_{\text{Lip}} = \|a\|_{L^\infty} + \|\nabla\|_{L^\infty}$. \square

Let $a \in L^\infty(\mathbb{R}_x^n)$. Then, Bony's paraproduct of a and $u \in H^\theta(\mathbb{R}_x^n)$ is defined as

$$T_a u = \sum_{k \geq 3} S_{k-3} a \Delta_k u.$$

For the proof of our conditional stability result it is essential that T_a is a positive operator. Unfortunately, this is not implied by $a(x) \geq \kappa > 0$. Therefore, we have to modify the paraproduct a little bit. Following [7, Sect. 3.3.] we introduce the operator

$$T_a^m u = S_{m-1} a S_{m+2} u + \sum_{k \geq m+3} S_{k-3} a \Delta_k u, \quad (3.2)$$

where $m \in \mathbb{N}_0$; note $T_a^0 = T_a$. As it will be shown below, the operator T_a^m is a positive operator for positive a provided that m is sufficiently large. The next results were proved for T_a , but Lemma 3.10 in [7] guarantees that they hold also for T_a^m .

Proposition 3.4 ([25, Prop. 5.2.1 and Thms. 5.2.8 and 5.2.9]). Let $m \in \mathbb{N} \setminus \{0\}$ and let $a \in L^\infty(\mathbb{R}_x^n)$. Let $\theta \in \mathbb{R}$.

Then T_a^m maps H^θ into H^θ and there exists $C_{m,\theta} > 0$ depending only on m and θ , such that, for all $u \in H^\theta$,

$$\|T_a^m u\|_{H^\theta} \leq C_{m,\theta} \|a\|_{L^\infty} \|u\|_{H^\theta}. \quad (3.3)$$

The constant $C_{m,\theta}$ can be chosen independent of θ when θ belongs to a compact subset of \mathbb{R} .

Let $m \in \mathbb{N} \setminus \{0\}$ and let $a \in \text{Lip}(\mathbb{R}_x^n)$. Then

- $a - T_a^m$ maps L^2 into H^1 and there exists $C_1 > 0$ depending only on m , such that, for all $u \in L^2$,

$$\|au - T_a^m u\|_{H^1} \leq C_1 \|a\|_{\text{Lip}} \|u\|_{L^2}; \quad (3.4)$$

- for every $i = 1, \dots, n$, the mapping $u \mapsto a \partial_{x_i} u - T_a^m \partial_{x_i} u$ extends from L^2 to L^2 , and there exists $C_0 > 0$ depending only on m , such that, for all $u \in L^2$,

$$\|a \partial_{x_i} u - T_a^m \partial_{x_i} u\|_{L^2} \leq C_0 \|a\|_{\text{Lip}} \|u\|_{L^2}. \quad (3.5)$$

Corollary 3.5. Let $\theta \in [0, 1]$. Then for every $i = 1, \dots, n$, the mapping $u \mapsto a \partial_{x_i} u - T_a^m \partial_{x_i} u$ extends from H^θ to H^θ , and for all $u \in H^\theta$,

$$\|a \partial_{x_i} u - T_a^m \partial_{x_i} u\|_{H^\theta} \leq C_0^{1-\theta} C_1^\theta \|a\|_{\text{Lip}} \|u\|_{H^\theta}. \quad (3.6)$$

Proof. By Proposition 3.4 the operator $(a - T_a^m) \partial_{x_j}$ is continuous from H^0 to H^0 and from H^1 to H^1 . The result follows by interpolation (see e.g. Theorems B.1, B.2 and B.7 in [24]). \square

Next we state a positivity result for T_a^m .

Proposition 3.6 ([7, Cor. 3.12]). Let $a \in L^\infty(\mathbb{R}_x^n) \cap \text{Lip}(\mathbb{R}_x^n)$ and suppose that $a(x) \geq \kappa > 0$ for all $x \in \mathbb{R}_x^n$. Then, there exists a constant $m_0 = m_0(\kappa, \|a\|_{\text{Lip}})$ such that

$$\langle T_a^m u | u \rangle_{L^2} \geq \frac{\kappa}{2} \|u\|_{L^2}^2,$$

for all $u \in L^2(\mathbb{R}_x^n)$ and $m \geq m_0$. A similar result is true for vector-valued functions if a is replaced by a positive symmetric matrix. \square

The next proposition is needed since T_a^m is not self-adjoint. However, the operator $(T_a^m - (T_a^m)^*) \partial_{x_j}$ is of order 0 and maps, if a is Lipschitz, L^2 continuously into L^2 .

Proposition 3.7 ([7, Prop. 3.8 and 3.11] and [11, Prop. 3.8]). Let $m \in \mathbb{N}$, $a \in L^\infty(\mathbb{R}_x^n) \cap \text{Lip}(\mathbb{R}_x^n)$. Then the mapping $u \mapsto (T_a^m - (T_a^m)^*) \partial_{x_j} u$ extends from L^2 to L^2 and there exists a constant $C_m > 0$ such that for all $u \in L^2(\mathbb{R}_x^n)$

$$\|(T_a^m - (T_a^m)^*) \partial_{x_j} u\|_{L^2} \leq C_m \|a\|_{\text{Lip}} \|u\|_{L^2}. \quad \square$$

We end this section with a property of the commutators $[\Delta_k, T_a^m]$ which will be crucial in the proof of the weighted energy estimate.

Proposition 3.8 ([11, Prop. 3.7]). *Let $m \in \mathbb{N} \setminus \{0\}$, let $\theta \in \mathbb{R}$ and let $a \in \text{Lip}$. Denote by $[\Delta_k, T_a^m]$ the commutator between Δ_k and T_a^m .*

Then there exists $C_{m,\theta}$ depending only on m and θ such that for all $u \in H^{1-\theta}$,

$$\left(\sum_{k=0}^{+\infty} 2^{-2k\theta} \|\partial_{x_j}([\Delta_k, T_a^m]\partial_{x_k}u)\|_{L^2}^2 \right)^{\frac{1}{2}} \leq C_{m,\theta} \|a\|_{\text{Lip}} \|u\|_{H^{1-\theta}}. \quad (3.7)$$

The constant $C_{m,\theta}$ can be chosen independent of θ when θ belongs to a compact subset of \mathbb{R} . \square

4. Proof of the weighted energy estimate

For ease of notation, we write the proof only in one space dimension. We divide the proof in several steps.

- *Microlocalization and approximation*

Let $u \in \mathcal{H}$ be a solution of (2.1) Let $\alpha \geq T^{-1}$, $\sigma = 1/\alpha$, $\tau = \sigma/4$, $\gamma > 0$, $\lambda > 1$, $\beta \geq \sigma + \tau$. For $t \in [0, \sigma]$ define $w(t, x) = e^{\gamma t} e^{-\beta \Phi_\lambda\left(\frac{t+\tau}{\beta}\right)} u(t, x)$. Then w satisfies

$$\partial_t w - \gamma w + \Phi'_\lambda\left(\frac{t+\tau}{\beta}\right) w + \partial_x(a(t, x)\partial_x w) = 0.$$

Now we add and subtract $\partial_x T_a^m \partial_x w$, where T_a^m is the paramultiplication operator defined in (3.2), with $m \geq m_0(\kappa, A)$, according to the positivity result of Proposition 3.6. We obtain

$$\partial_t w - \gamma w + \Phi'_\lambda\left(\frac{t+\tau}{\beta}\right) w + \partial_x(T_a^m \partial_x w) + \partial_x((a - T_a^m)\partial_x w) = 0. \quad (4.1)$$

We set $u_\nu = \Delta_\nu u$, $w_\nu = \Delta_\nu w$ and $v_\nu = 2^{-\alpha t \nu} w_\nu$. Then the function v_ν satisfies

$$\begin{aligned} \partial_t v_\nu &= \gamma v_\nu - \Phi'_\lambda\left(\frac{t+\tau}{\beta}\right) v_\nu - \partial_x(T_a^m \partial_x v_\nu) - \alpha \log 2 \nu v_\nu \\ &\quad - 2^{-\alpha t \nu} \partial_x([\Delta_\nu, T_a^m]\partial_x w) - 2^{-\alpha t \nu} \Delta_\nu \partial_x((a - T_a^m)\partial_x w). \end{aligned} \quad (4.2)$$

Now we make the scalar product of (4.2) with $(t + \tau)\partial_t v_\nu$ in $L^2(\mathbb{R}_x)$ and obtain

$$\begin{aligned}
& (t + \tau) \|\partial_t v_v(t)\|_{L^2}^2 = \gamma(t + \tau) \langle v_v | \partial_t v_v(t) \rangle_{L^2} \\
& - (t + \tau) \left\langle \Phi'_\lambda \left(\frac{t + \tau}{\beta} \right) v_v(t) | \partial_t v_v(t) \right\rangle_{L^2} \\
& - (t + \tau) \langle \partial_x (T_a^m \partial_x v_v(t)) | \partial_t v_v(t) \rangle_{L^2} \\
& - \alpha \log 2 (t + \tau) v \langle v_v(t) | \partial_t v_v(t) \rangle_{L^2} \\
& - (t + \tau) 2^{-\alpha t v} \langle \partial_x ([\Delta_v, T_a^m] \partial_x w(t)) | \partial_t v_v(t) \rangle_{L^2} \\
& - (t + \tau) 2^{-\alpha t v} \langle \Delta_v \partial_x ((a - T_a^m) \partial_x w(t)) | \partial_t v_v(t) \rangle_{L^2}.
\end{aligned} \tag{4.3}$$

To proceed further, we need to regularize the coefficient $a(t, x)$ with respect to t . We take a regular mollifier, i.e. an even, non-negative $\rho \in C_0^\infty(\mathbb{R})$ with $\text{supp}(\rho) \subseteq [-\frac{1}{2}, \frac{1}{2}]$ and $\int_{\mathbb{R}} \rho(s) ds = 1$. For $\varepsilon \in (0, 1]$, we set

$$a_\varepsilon(t, x) = \frac{1}{\varepsilon} \int_{\mathbb{R}} a(s, x) \rho\left(\frac{t-s}{\varepsilon}\right) ds.$$

A straightforward computation shows that for all $\varepsilon \in (0, 1]$, we have

$$a_\varepsilon(t, x) \geq \kappa > 0 \tag{4.4}$$

$$|a_\varepsilon(t, x) - a(t, x)| \leq A_{LL} \varepsilon (|\log \varepsilon| + 1) \tag{4.5}$$

as well as

$$|\partial_t a_\varepsilon(t, x)| \leq A_{LL} \|\rho'\|_{L^1(\mathbb{R})} (|\log \varepsilon| + 1)$$

for all $(t, x) \in [0, T] \times \mathbb{R}_x$. From these properties of $a_\varepsilon(t, x)$ and by Proposition 3.4, we immediately get

Lemma 4.1. *Let $m \in \mathbb{N}_0$ and $u \in L^2(\mathbb{R}_x^n)$. Then*

$$\|(T_a^m - T_{a_\varepsilon}^m)u\|_{L^2} \leq C_{m,0} A_{LL} \varepsilon (|\log \varepsilon| + 1) \|u\|_{L^2}$$

and

$$\|T_{\partial_t a_\varepsilon}^m u\|_{L^2} \leq C_{m,0} A_{LL} \|\rho'\|_{L^1(\mathbb{R})} (|\log \varepsilon| + 1) \|u\|_{L^2}. \quad \square$$

We set

$$a_v(t, x) := a_\varepsilon(t, x), \quad \text{with } \varepsilon = 2^{-2v}.$$

We replace T_a^m by $T_{a_v}^m + T_a^m - T_{a_v}^m$ in the third term of the right hand side of (4.3) and we obtain

$$\begin{aligned}
(t + \tau) \|\partial_t v_\nu(t)\|_{L^2}^2 &= \gamma(t + \tau) \langle v_\nu(t) \mid \partial_t v_\nu(t) \rangle_{L^2} \\
&\quad - (t + \tau) \left\langle \Phi'_\lambda \left(\frac{t + \tau}{\beta} \right) v_\nu(t) \mid \partial_t v_\nu(t) \right\rangle_{L^2} \\
&\quad - (t + \tau) \langle \partial_x (T_{a_\nu}^m \partial_x v_\nu(t)) \mid \partial_t v_\nu(t) \rangle_{L^2} \\
&\quad - (t + \tau) \langle \partial_x ((T_a^m - T_{a_\nu}^m) \partial_x v_\nu(t)) \mid \partial_t v_\nu(t) \rangle_{L^2} \\
&\quad - \alpha \log 2(t + \tau) \nu \langle v_\nu(t) \mid \partial_t v_\nu(t) \rangle_{L^2} \\
&\quad - (t + \tau) 2^{-\alpha t \nu} \langle \partial_x ([\Delta_\nu, T_a^m] \partial_x w(t)) \mid \partial_t v_\nu(t) \rangle_{L^2} \\
&\quad - (t + \tau) 2^{-\alpha t \nu} \langle \Delta_\nu \partial_x ((a - T_a^m) \partial_x w(t)) \mid \partial_t v_\nu(t) \rangle_{L^2}.
\end{aligned} \tag{4.6}$$

Now we replace $\partial_t v_\nu(t)$ in the term

$$-\alpha \log 2(t + \tau) \nu \langle v_\nu(t) \mid \partial_t v_\nu(t) \rangle_{L^2}$$

by the expression on the right hand side of (4.2) and we obtain

$$\begin{aligned}
-\alpha \log 2(t + \tau) \nu \langle v_\nu(t) \mid \partial_t v_\nu(t) \rangle_{L^2} &= \\
&\quad - \alpha \gamma \log 2(t + \tau) \nu \|v_\nu(t)\|_{L^2}^2 \\
&\quad + \alpha \log 2(t + \tau) \Phi'_\lambda \left(\frac{t + \tau}{\beta} \right) \nu \|v_\nu(t)\|_{L^2}^2 \\
&\quad + \alpha \log 2(t + \tau) \nu \langle v_\nu(t) \mid \partial_x T_a^m \partial_x v_\nu(t) \rangle_{L^2} \\
&\quad + \alpha^2 (\log 2)^2 (t + \tau) \nu^2 \|v_\nu(t)\|_{L^2}^2 \\
&\quad + \alpha \log 2(t + \tau) \nu 2^{-\alpha t \nu} \langle v_\nu(t) \mid \partial_x ([\Delta_\nu, T_a^m] \partial_x w(t)) \rangle_{L^2} \\
&\quad + \alpha \log 2(t + \tau) \nu 2^{-\alpha t \nu} \langle v_\nu(t) \mid \Delta_\nu \partial_x ((a - T_a^m) \partial_x w(t)) \rangle_{L^2}.
\end{aligned} \tag{4.7}$$

By (4.6) and (4.7), we obtain

$$\begin{aligned}
(t + \tau) \|\partial_t v_\nu(t)\|_{L^2}^2 &= \gamma(t + \tau) \langle v_\nu(t) \mid \partial_t v_\nu(t) \rangle_{L^2} \\
&\quad - (t + \tau) \Phi'_\lambda \left(\frac{t + \tau}{\beta} \right) \langle v_\nu(t) \mid \partial_t v_\nu(t) \rangle_{L^2} \\
&\quad - (t + \tau) \langle \partial_x (T_{a_\nu}^m \partial_x v_\nu(t)) \mid \partial_t v_\nu(t) \rangle_{L^2} \\
&\quad - (t + \tau) \langle \partial_x ((T_a^m - T_{a_\nu}^m) \partial_x v_\nu(t)) \mid \partial_t v_\nu(t) \rangle_{L^2} \\
&\quad + \alpha \log 2(t + \tau) \Phi'_\lambda \left(\frac{t + \tau}{\beta} \right) \nu \|v_\nu(t)\|_{L^2}^2 \\
&\quad + \alpha \log 2(t + \tau) \nu \langle v_\nu(t) \mid \partial_x T_a^m \partial_x v_\nu(t) \rangle_{L^2} \\
&\quad + \alpha^2 (\log 2)^2 (t + \tau) \nu^2 \|v_\nu(t)\|_{L^2}^2 - \alpha \gamma \log 2(t + \tau) \nu \|v_\nu(t)\|_{L^2}^2 \\
&\quad + \alpha \log 2(t + \tau) \nu 2^{-\alpha t \nu} \langle v_\nu(t) \mid \partial_x ([\Delta_\nu, T_a^m] \partial_x w(t)) \rangle_{L^2} \\
&\quad + \alpha \log 2(t + \tau) \nu 2^{-\alpha t \nu} \langle v_\nu(t) \mid \Delta_\nu \partial_x ((a - T_a^m) \partial_x w(t)) \rangle_{L^2}
\end{aligned}$$

$$\begin{aligned}
& - (t + \tau) 2^{-\alpha t \nu} \langle \partial_x ([\Delta_\nu, T_a^m] \partial_x w(t)) \mid \partial_t v_\nu(t) \rangle_{L^2} \\
& - (t + \tau) 2^{-\alpha t \nu} \langle \Delta_\nu \partial_x ((a - T_a^m) \partial_x w(t)) \mid \partial_t v_\nu(t) \rangle_{L^2}.
\end{aligned}$$

A straightforward computation using Leibniz derivation rule with respect to t yields

$$\gamma(t + \tau) \langle v_\nu(t) \mid \partial_t v_\nu(t) \rangle_{L^2} = \frac{\gamma}{2} \frac{d}{dt} \left((t + \tau) \|v_\nu(t)\|_{L^2}^2 \right) - \frac{\gamma}{2} \|v_\nu(t)\|_{L^2}^2$$

and

$$\begin{aligned}
& - (t + \tau) \Phi'_\lambda \left(\frac{t + \tau}{\beta} \right) \langle v_\nu(t) \mid \partial_t v_\nu(t) \rangle_{L^2} = \\
& - \frac{1}{2} \frac{d}{dt} \left((t + \tau) \Phi'_\lambda \left(\frac{t + \tau}{\beta} \right) \|v_\nu(t)\|_{L^2}^2 \right) + \frac{1}{2} \frac{t + \tau}{\beta} \Phi''_\lambda \left(\frac{t + \tau}{\beta} \right) \|v_\nu(t)\|_{L^2}^2 \\
& + \frac{1}{2} \Phi'_\lambda \left(\frac{t + \tau}{\beta} \right) \|v_\nu(t)\|_{L^2(\mathbb{R}^n)}^2.
\end{aligned}$$

Next we consider the term $-(t + \tau) \langle \partial_x (T_{a_\nu}^m \partial_x v_\nu(t)) \mid \partial_t v_\nu(t) \rangle_{L^2}$. From (3.2) it can be seen that $\partial_t T_{a_\nu}^m = T_{\partial_t a_\nu}^m + T_{a_\nu}^m \partial_t$. A simple computation then shows that

$$\begin{aligned}
-(t + \tau) \langle \partial_x (T_{a_\nu}^m \partial_x v_\nu(t)) \mid \partial_t v_\nu(t) \rangle_{L^2} &= \frac{1}{2} \frac{d}{dt} \left((t + \tau) \langle T_{a_\nu}^m \partial_x v_\nu(t) \mid \partial_x v_\nu(t) \rangle_{L^2} \right) \\
& - \frac{1}{2} \langle T_{a_\nu}^m \partial_x v_\nu(t) \mid \partial_x v_\nu(t) \rangle_{L^2} \\
& - \frac{1}{2} (t + \tau) \langle T_{\partial_t a_\nu}^m \partial_x v_\nu(t) \mid \partial_x v_\nu(t) \rangle_{L^2} \\
& - \frac{1}{2} (t + \tau) \langle \partial_t \partial_x v_\nu(t) \mid ((T_{a_\nu}^m)^* - T_{a_\nu}^m) \partial_x v_\nu(t) \rangle_{L^2}.
\end{aligned}$$

At the end we obtain the identity

$$\begin{aligned}
(t + \tau) \|\partial_t v_\nu(t)\|_{L^2}^2 &= \frac{\gamma}{2} \frac{d}{dt} \left((t + \tau) \|v_\nu(t)\|_{L^2}^2 \right) - \frac{\gamma}{2} \|v_\nu(t)\|_{L^2}^2 \\
& - \frac{1}{2} \frac{d}{dt} \left((t + \tau) \Phi'_\lambda \left(\frac{t + \tau}{\beta} \right) \|v_\nu(t)\|_{L^2}^2 \right) \\
& + \frac{1}{2} \Phi'_\lambda \left(\frac{t + \tau}{\beta} \right) \|v_\nu(t)\|_{L^2}^2 + \frac{1}{2} \frac{t + \tau}{\beta} \Phi''_\lambda \left(\frac{t + \tau}{\beta} \right) \|v_\nu(t)\|_{L^2}^2 \\
& - (t + \tau) \langle \partial_x ((T_a^m - T_{a_\nu}^m) \partial_x v_\nu(t)) \mid \partial_t v_\nu(t) \rangle_{L^2} \\
& + \frac{1}{2} \frac{d}{dt} \left((t + \tau) \langle T_{a_\nu}^m \partial_x v_\nu(t) \mid \partial_x v_\nu(t) \rangle_{L^2} \right) \\
& - \frac{1}{2} \langle T_{a_\nu}^m \partial_x v_\nu(t) \mid \partial_x v_\nu(t) \rangle_{L^2} \\
& - \frac{1}{2} (t + \tau) \langle T_{\partial_t a_\nu}^m \partial_x v_\nu(t) \mid \partial_x v_\nu(t) \rangle_{L^2}
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2}(t+\tau)\langle\partial_t\partial_x v_\nu(t)\mid((T_{a_\nu}^m)^*-T_{a_\nu}^m)\partial_x v_\nu(t)\rangle_{L^2} \\
& -\alpha\gamma\log 2(t+\tau)v\|v_\nu(t)\|_{L^2}^2 \\
& +\alpha\log 2(t+\tau)\Phi'_\lambda\left(\frac{t+\tau}{\beta}\right)v\|v_\nu(t)\|_{L^2}^2 \\
& -\alpha\log 2(t+\tau)v\langle\partial_x v_\nu(t)\mid T_a^m\partial_x v_\nu(t)\rangle_{L^2} \\
& +\alpha^2(\log 2)^2(t+\tau)v^2\|v_\nu(t)\|_{L^2}^2 \\
& +\alpha\log 2(t+\tau)v2^{-\alpha tv}\langle v_\nu(t)\mid\mathcal{X}_\nu(t)\rangle_{L^2} \\
& -(t+\tau)2^{-\alpha tv}\langle\mathcal{X}_\nu(t)\mid\partial_t v_\nu(t)\rangle_{L^2}, \tag{4.8}
\end{aligned}$$

where we have set

$$\mathcal{X}_\nu(t):=(\partial_x([\Delta_\nu,T_a^m]\partial_x w(t))+\Delta_\nu(\partial_x((a-T_a^m)\partial_x w(t)))).$$

- *Estimates for $\nu = 0$*

In what follows, we denote by $C^{(1)}, C^{(2)}, C^{(3)}, \dots$ positive constants which depend only on A_{LL}, A and κ .

Setting $\nu = 0$, we get from (4.8)

$$\begin{aligned}
(t+\tau)\|\partial_t v_0(t)\|_{L^2}^2 &= \frac{\gamma}{2}\frac{d}{dt}\left((t+\tau)\|v_0(t)\|_{L^2}^2\right) - \frac{\gamma}{2}\|v_0(t)\|_{L^2}^2 \\
& -\frac{1}{2}\frac{d}{dt}\left((t+\tau)\Phi'_\lambda\left(\frac{t+\tau}{\beta}\right)\|v_0(t)\|_{L^2}^2\right) \\
& +\frac{1}{2}\Phi'_\lambda\left(\frac{t+\tau}{\beta}\right)\|v_0(t)\|_{L^2}^2 + \frac{1}{2}\frac{t+\tau}{\beta}\Phi''_\lambda\left(\frac{t+\tau}{\beta}\right)\|v_0(t)\|_{L^2}^2 \\
& -(t+\tau)\langle\partial_x((T_a^m-T_{a_0}^m)\partial_x v_0(t))\mid\partial_t v_0(t)\rangle_{L^2} \\
& +\frac{1}{2}\frac{d}{dt}\left((t+\tau)\langle T_{a_0}^m\partial_x v_0(t)\mid\partial_x v_0(t)\rangle_{L^2}\right) \\
& -\frac{1}{2}\langle T_{a_0}^m\partial_x v_0(t)\mid\partial_x v_0(t)\rangle_{L^2} - \frac{1}{2}(t+\tau)\langle\partial_x v_0(t)\mid T_{\partial_t a_0}^m\partial_x v_0(t)\rangle_{L^2} \\
& -\frac{1}{2}(t+\tau)\langle\partial_t\partial_x v_0(t)\mid((T_{a_0}^m)^*-T_{a_0}^m)\partial_x v_0(t)\rangle_{L^2} \\
& -(t+\tau)\langle\mathcal{X}_0(t)\mid\partial_t v_0(t)\rangle_{L^2}.
\end{aligned}$$

By Proposition 3.6 we have

$$-\frac{1}{2}\langle T_{a_0}^m\partial_x v_0(t)\mid\partial_x v_0(t)\rangle_{L^2}\leq-\frac{\kappa}{8}\|v_0(t)\|_{L^2}^2.$$

Using Propositions 3.1, 3.4 and Lemma 4.1, for $N_1, N_2 > 0$, we get

$$\left|\langle\partial_x v_0(t)\mid T_{\partial_t a_0}^m\partial_x v_0(t)\rangle_{L^2}\right|\leq C^{(1)}\|v_0\|_{L^2}^2,$$

$$|\langle T_{a-a_0}^m \partial_x v_0(t) \mid \partial_x \partial_t v_0(t) \rangle_{L^2}| \leq C^{(2)} N_1 \|v_0(t)\|_{L^2}^2 + \frac{1}{N_1} \|\partial_t v_0(t)\|_{L^2}^2,$$

and

$$|\langle (T_{a_0}^m)^* - T_{a_0}^m \partial_x v_0(t) \mid \partial_t \partial_x v_0(t) \rangle_{L^2}| \leq C^{(3)} N_2 \|v_0(t)\|_{L^2}^2 + \frac{1}{N_2} \|\partial_t v_0(t)\|_{L^2}^2.$$

Now, we choose N_1 and N_2 so large that

$$\frac{1}{N_1} + \frac{1}{N_2} - \frac{1}{2} < 0$$

and $\bar{\gamma}$ so large that

$$-\frac{\gamma}{4} + \left(C^{(1)} + C^{(2)} N_1 + C^{(3)} N_2 \right) (\sigma + \tau) < 0$$

for $\gamma \geq \bar{\gamma}$. With this choice, the term

$$C^{(1)}(t + \tau) \|v_0(t)\|_{L^2}^2 + C^{(2)} N_1 (t + \tau) \|v_0(t)\|_{L^2}^2 + C^{(3)} N_2 (t + \tau) \|v_0(t)\|_{L^2}^2$$

is absorbed by $-\frac{\gamma}{4} \|v_0(t)\|_{L^2}^2$, and the term

$$\frac{1}{N_1} (t + \tau) \|\partial_t v_0(t)\|_{L^2}^2 + \frac{1}{N_2} (t + \tau) \|\partial_t v_0(t)\|_{L^2}^2$$

is absorbed by $-\frac{1}{2} (t + \tau) \|\partial_t v_0(t)\|_{L^2}^2$. Hence, we get

$$\begin{aligned} & \frac{1}{2} (t + \tau) \|\partial_t v_0(t)\|_{L^2}^2 \\ & \leq \frac{\gamma}{2} \frac{d}{dt} \left((t + \tau) \|v_0(t)\|_{L^2}^2 \right) - \frac{\gamma}{4} \|v_0(t)\|_{L^2}^2 + \frac{1}{2} \Phi'_\lambda \left(\frac{t + \tau}{\beta} \right) \|v_0(t)\|_{L^2}^2 \\ & \quad - \frac{\kappa}{8} \|v_0(t)\|_{L^2}^2 - \frac{1}{2} \frac{d}{dt} \left((t + \tau) \Phi'_\lambda \left(\frac{t + \tau}{\beta} \right) \|v_0(t)\|_{L^2}^2 \right) \\ & \quad + \frac{1}{2} \frac{t + \tau}{\beta} \Phi''_\lambda \left(\frac{t + \tau}{\beta} \right) \|v_0(t)\|_{L^2}^2 \\ & \quad + \frac{1}{2} \frac{d}{dt} \left((t + \tau) \langle T_{a_0}^m \partial_x v_0(t) \mid \partial_x v_0(t) \rangle_{L^2} \right) - (t + \tau) \langle \mathcal{X}_0 \mid \partial_t v_0(t) \rangle_{L^2}. \end{aligned}$$

Further, we recall that Φ satisfies equation (2.3), i.e.

$$y \Phi''_\lambda(y) = -\lambda (\Phi'_\lambda(y))^2 \mu \left(\frac{1}{\Phi'_\lambda(y)} \right) = -\lambda \Phi'_\lambda(y) \left(1 + \left| \log \left(\frac{1}{\Phi'_\lambda(y)} \right) \right| \right)$$

for $\lambda > 1$. From this, we see that

$$\frac{1}{2}\Phi'_\lambda\left(\frac{t+\tau}{\beta}\right)\|v_0(t)\|_{L^2}^2 + \frac{1}{2}\frac{t+\tau}{\beta}\Phi''_\lambda\left(\frac{t+\tau}{\beta}\right)\|v_0(t)\|_{L^2}^2 < 0,$$

and thus we get

$$\begin{aligned} \frac{\gamma}{8}\|v_0(t)\|_{L^2}^2 &\leq -\frac{1}{2}(t+\tau)\|\partial_t v_0(t)\|_{L^2}^2 + \frac{\gamma}{2}\frac{d}{dt}\left((t+\tau)\|v_0(t)\|_{L^2}^2\right) - \frac{\gamma}{8}\|v_0(t)\|_{L^2}^2 \\ &\quad - \frac{\kappa}{8}\|v_0(t)\|_{L^2}^2 + \frac{1}{2}\frac{d}{dt}\left((t+\tau)\langle T_{a_0}^m \partial_x v_0(t) \mid \partial_x v_0 \rangle_{L^2}\right) \\ &\quad - (t+\tau)\langle \mathcal{X}_0 \mid \partial_t v_0(t) \rangle_{L^2} - \frac{1}{2}\frac{d}{dt}\left((t+\tau)\Phi'_\lambda\left(\frac{t+\tau}{\beta}\right)\|v_0(t)\|_{L^2}^2\right). \end{aligned}$$

Integrating in t over $[0, s] \subseteq [0, \sigma]$, we obtain

$$\begin{aligned} \frac{\gamma}{8}\int_0^s \|v_0(t)\|_{L^2}^2 dt &\leq \left(\frac{\gamma}{2} + C^{(4)}\right)(s+\tau)\|v_0(s)\|_{L^2}^2 + \frac{1}{2}\tau\Phi'_\lambda\left(\frac{\tau}{\beta}\right)\|v_0(0)\|_{L^2}^2 \\ &\quad - \frac{\gamma}{8}\int_0^s \|v_0(t)\|_{L^2}^2 dt - \frac{\kappa}{8}\int_0^s \|v_0(t)\|_{L^2}^2 dt \\ &\quad - \frac{1}{2}\int_0^s (t+\tau)\|\partial_t v_0(t)\|_{L^2}^2 dt - \int_0^s (t+\tau)\langle \mathcal{X}_0(t) \mid \partial_t v_0(t) \rangle_{L^2} dt, \end{aligned}$$

where we have used the estimates

$$|\langle \partial_x v_0(s) \mid T_{a_0}^m \partial_x v_0(s) \rangle_{L^2}| \leq 2C^{(4)}\|v_0(s)\|_{L^2}^2$$

and

$$\langle \partial_x v_0(0) \mid T_{a_0}^m \partial_x v_0(0) \rangle_{L^2} \geq \frac{\kappa}{2}\|\partial_x v_0(0)\|_{L^2}^2,$$

which follow from Propositions 3.4 and 3.6 respectively.

- *Estimates for $\nu \geq 1$*

Now, we consider (4.8) for $\nu \geq 1$. From Lemma 4.1 and Proposition 3.7, for N_3 and $N_4 > 0$, we obtain

$$\begin{aligned} &|\langle (T_a^m - T_{a_\nu}^m)\partial_x v_\nu(t) \mid \partial_x \partial_t v_\nu(t) \rangle_{L^2}| \\ &\leq C_{a,m}^{(5)}N_3\nu^2\|v_\nu(t)\|_{L^2}^2 + \frac{1}{N_3}\|\partial_t v_\nu(t)\|_{L^2}^2 \\ &\leq C_{a,m}^{(5)}N_3\nu 2^{2\nu}\|v_\nu(t)\|_{L^2}^2 + \frac{1}{N_3}\|\partial_t v_\nu(t)\|_{L^2}^2 \end{aligned} \tag{4.9}$$

and

$$|\langle \partial_x v_\nu(t) | T_{\partial_t a_\nu}^m \partial_x v_\nu(t) \rangle_{L^2}| \leq C_{a,m}^{(6)} \nu 2^{2\nu} \|v_\nu(t)\|_{L^2}^2, \quad (4.10)$$

as well as

$$|\langle ((T_{a_\nu}^m)^* - T_{a_\nu}^m) \partial_x v_\nu(t) | \partial_t \partial_x v_\nu(t) \rangle_{L^2}| \leq C_{a,m}^{(7)} N_4 2^{2\nu} \|v_\nu(t)\|_{L^2}^2 + \frac{1}{N_4} \|\partial_t v_\nu(t)\|_{L^2}^2 \quad (4.11)$$

Using again the positivity estimate in Proposition 3.6 as well as Proposition 3.1, we obtain

$$\begin{aligned} & -\alpha \log 2(t + \tau) \nu \langle \partial_x v_\nu(t) | T_a^m \partial_x v_\nu(t) \rangle_{L^2} \\ & \leq -\alpha \frac{\kappa \log 2}{4} (t + \tau) \nu 2^{2\nu} \|v_\nu(t)\|_{L^2}^2. \end{aligned} \quad (4.12)$$

Now, we choose N_3 and N_4 so large that

$$\frac{1}{N_3} + \frac{1}{N_4} - \frac{1}{2} < 0,$$

and α_1 large enough such that

$$-\frac{\alpha_1 \kappa \log 2}{4} + N_3 C_{a,m}^{(5)} + C_{a,m}^{(6)} + C_{a,m}^{(7)} N_4 < 0,$$

and we set $\alpha := \max\{T^{-1}, \alpha_1\}$. With this choice, we get

$$\begin{aligned} & \frac{\gamma}{4} \|v_\nu(t)\|_{L^2}^2 + \frac{1}{2} (t + \tau) \|\partial_t v_\nu(t)\|_{L^2}^2 \\ & \leq \frac{\gamma}{2} \frac{d}{dt} \left((t + \tau) \|v_\nu(t)\|_{L^2}^2 \right) - \frac{\gamma}{4} \|v_\nu(t)\|_{L^2}^2 \\ & \quad - \frac{1}{2} \frac{d}{dt} \left((t + \tau) \Phi'_\lambda \left(\frac{t + \tau}{\beta} \right) \|v_\nu(t)\|_{L^2}^2 \right) \\ & \quad + \frac{1}{2} \Phi'_\lambda \left(\frac{t + \tau}{\beta} \right) \|v_\nu(t)\|_{L^2}^2 + \frac{1}{2} \frac{t + \tau}{\beta} \Phi''_\lambda \left(\frac{t + \tau}{\beta} \right) \|v_\nu(t)\|_{L^2}^2 \\ & \quad + \frac{1}{2} \frac{d}{dt} \left((t + \tau) \langle T_{a_\nu}^m \partial_x v_\nu(t) | \partial_x v_\nu(t) \rangle_{L^2} \right) \\ & \quad - \alpha \gamma \log 2(t + \tau) \nu \|v_\nu(t)\|_{L^2}^2 - \frac{1}{2} \langle T_{a_\nu}^m \partial_x v_\nu(t) | \partial_x v_\nu(t) \rangle_{L^2} \\ & \quad + \alpha \log 2(t + \tau) \Phi'_\lambda \left(\frac{t + \tau}{\beta} \right) \nu \|v_\nu(t)\|_{L^2}^2 \\ & \quad + \alpha^2 (\log 2)^2 \nu^2 (t + \tau) \|v_\nu(t)\|_{L^2}^2 - \frac{3\alpha \kappa \log 2}{4} (t + \tau) \nu 2^{2\nu} \|v_\nu(t)\|_{L^2}^2 \\ & \quad + \alpha \log 2 \nu 2^{-\alpha t \nu} (t + \tau) \langle v_\nu(t) | \mathcal{X}_\nu(t) \rangle_{L^2} \\ & \quad - (t + \tau) 2^{-\alpha t \nu} \langle \mathcal{X}_\nu(t) | \partial_t v_\nu(t) \rangle_{L^2}. \end{aligned} \quad (4.13)$$

Since $y \Phi''_\lambda(y) = -\lambda \Phi'_\lambda(y) (1 + |\log(\Phi'_\lambda(y))|)$, if we take $\lambda \geq \bar{\lambda} > 2$, we have

$$\frac{1}{4} \frac{t + \tau}{\beta} \Phi''_{\lambda} \left(\frac{t + \tau}{\beta} \right) \leq -\frac{1}{2} \Phi'_{\lambda} \left(\frac{t + \tau}{\beta} \right),$$

and hence, the term $\frac{1}{2} \Phi'_{\lambda} \left(\frac{t + \tau}{\beta} \right) \|v_{\nu}(t)\|_{L^2}^2$ in (4.13) is absorbed by the term $\frac{1}{4} \frac{t + \tau}{\beta} \Phi''_{\lambda} \left(\frac{t + \tau}{\beta} \right) \|v_{\nu}(t)\|_{L^2}^2$. Now we need to absorb

$$\alpha \log 2(t + \tau) \Phi'_{\lambda} \left(\frac{t + \tau}{\beta} \right) v \|v_{\nu}(t)\|_{L^2}^2. \quad (4.14)$$

There are two terms in (4.13) that will help to achieve this. One is

$$-\frac{\alpha}{4} \frac{\kappa \log 2}{4} (t + \tau) v 2^{2\nu} \|v_{\nu}(t)\|_{L^2}^2 \quad (4.15)$$

and the other one is

$$\frac{1}{4} \frac{t + \tau}{\beta} \Phi''_{\lambda} \left(\frac{t + \tau}{\beta} \right) \|v_{\nu}(t)\|_{L^2}^2. \quad (4.16)$$

Let $\kappa' = \min\{4 \log 2, \frac{\kappa \log 2}{4}\}$. If $\nu \geq \frac{1}{2 \log 2} \log \left(\frac{4 \log 2}{\kappa'} \Phi'_{\lambda} \left(\frac{t + \tau}{\beta} \right) \right)$, then

$$-\frac{\alpha}{4} \frac{\kappa \log 2}{4} v 2^{2\nu} \leq -\alpha \log 2 \Phi'_{\lambda} \left(\frac{t + \tau}{\beta} \right) v.$$

On the contrary, if $\nu < \frac{1}{2 \log 2} \log \left(\frac{4 \log 2}{\kappa'} \Phi'_{\lambda} \left(\frac{t + \tau}{\beta} \right) \right)$ then

$$\frac{4 \log 2}{\kappa'} \Phi'_{\lambda} \left(\frac{t + \tau}{\beta} \right) > 2^{2\nu}$$

and, hence, by (2.3), we obtain

$$\begin{aligned} \frac{1}{4} \frac{t + \tau}{\beta} \Phi''_{\lambda} \left(\frac{t + \tau}{\beta} \right) &= -\frac{1}{4} \lambda \left(\Phi'_{\lambda} \left(\frac{t + \tau}{\beta} \right) \right)^2 \mu \left(\frac{1}{\Phi'_{\lambda} \left(\frac{t + \tau}{\beta} \right)} \right) \\ &\leq -\frac{1}{4} \lambda \left(\Phi'_{\lambda} \left(\frac{t + \tau}{\beta} \right) \right)^2 \mu \left(\frac{1}{\frac{4 \log 2}{\kappa'} \Phi'_{\lambda} \left(\frac{t + \tau}{\beta} \right)} \right) \\ &\leq -\frac{1}{4} \lambda \frac{\kappa'}{4 \log 2} \Phi'_{\lambda} \left(\frac{t + \tau}{\beta} \right) \left(1 + \log \left(\frac{4 \log 2}{\kappa'} \Phi'_{\lambda} \left(\frac{t + \tau}{\beta} \right) \right) \right) \\ &\leq -\frac{1}{4} \lambda \frac{\kappa'}{4 \log 2} \Phi'_{\lambda} \left(\frac{t + \tau}{\beta} \right) (1 + 2\nu \log 2) \\ &\leq -\lambda \frac{\kappa' (1 + \log 2)}{16 \log 2} \Phi'_{\lambda} \left(\frac{t + \tau}{\beta} \right) v, \end{aligned}$$

where we have used the fact that the function $\varepsilon \mapsto \varepsilon(|\log \varepsilon| + 1)$ is increasing. Consequently, if we choose $\lambda \geq \bar{\lambda}$ with

$$\bar{\lambda} \geq \frac{16\alpha(\log 2)^2(\sigma + \tau)}{\kappa'(1 + \log 2)},$$

we have

$$\frac{1}{4} \frac{t + \tau}{\beta} \Phi_\lambda'' \left(\frac{t + \tau}{\beta} \right) \leq -\alpha \log 2 (t + \tau) \Phi_\lambda' \left(\frac{t + \tau}{\beta} \right) \nu$$

and hence, the term (4.14) is compensated by (4.15) and (4.16).

Now we consider the term

$$(t + \tau) \alpha^2 \log^2(2) \nu^2 \|v_\nu(t)\|_{L^2}. \quad (4.17)$$

If $\nu \geq \frac{1}{\log 2} \log \left(\frac{16\alpha \log 2}{\kappa} \right) =: \bar{\nu}_1$, then

$$-\frac{\alpha}{4} \frac{\kappa \log 2}{4} \nu 2^{2\nu} + \alpha^2 \log^2(2) \nu^2 \leq 0.$$

If $\nu \leq \bar{\nu}_1$, then we choose a possibly larger $\bar{\gamma}$ such that

$$\frac{\gamma}{4} \geq \alpha^2 \log^2(2) \bar{\nu}_1^2 (\sigma + \tau)$$

for all $\gamma \geq \bar{\gamma}$. We obtain

$$-\frac{\gamma}{4} + \alpha^2 \log^2(2) \nu^2 (t + \tau) \leq 0,$$

and, consequently, (4.17) is absorbed by

$$-\frac{\alpha}{4} \frac{\kappa \log 2}{4} (t + \tau) \nu 2^{2\nu} \|v_\nu(t)\|_{L^2}^2 - \frac{\gamma}{4} \|v_\nu(t)\|_{L^2}^2.$$

The term $-\alpha \gamma \log 2 (t + \tau) \nu \|v_\nu(t)\|_{L^2}^2$ can be neglected since it is negative. However, we stress here that it is a crucial term in order to achieve our energy estimate for an equation including also lower order terms. Recalling also Propositions 3.1 and 3.6, we obtain

$$\begin{aligned} & \frac{1}{2} (t + \tau) \|\partial_t v_\nu(t)\|_{L^2}^2 + \frac{\gamma}{8} \|v_\nu(t)\|_{L^2}^2 \\ & \geq \frac{\gamma}{2} \frac{d}{dt} \left((t + \tau) \|v_\nu(t)\|_{L^2}^2 \right) - \frac{1}{2} \frac{d}{dt} \left((t + \tau) \Phi_\lambda' \left(\frac{t + \tau}{\beta} \right) \|v_\nu(t)\|_{L^2}^2 \right) \\ & \quad + \frac{1}{2} \frac{d}{dt} \left((t + \tau) \langle T_{a_\nu}^m \partial_x v_\nu(t) | \partial_x v_\nu(t) \rangle_{L^2} \right) \\ & \quad - \frac{\kappa}{8} 2^{2\nu} \|v_\nu(t)\|_{L^2}^2 \end{aligned}$$

$$\begin{aligned}
& - \frac{\alpha \kappa \log 2}{4} \frac{2^{2\nu}}{4} (t + \tau) v 2^{2\nu} \|v_\nu(t)\|_{L^2}^2 \\
& + \alpha \log 2 v 2^{-\alpha t \nu} (t + \tau) \langle v_\nu(t) | \mathcal{X}_\nu(t) \rangle_{L^2} \\
& - (t + \tau) 2^{-\alpha t \nu} \langle \mathcal{X}_\nu(t) | \partial_t v_\nu(t) \rangle_{L^2} - \frac{\gamma}{8} \|v_\nu(t)\|_{L^2}^2.
\end{aligned}$$

Integrating over $[0, s] \subseteq [0, \sigma]$, we get

$$\begin{aligned}
& \frac{\kappa}{8} \int_0^s 2^{2\nu} \|v_\nu(t)\|_{L^2}^2 dt + \frac{\gamma}{8} \int_0^s \|v_\nu(t)\|_{L^2}^2 dt \\
& \leq \frac{1}{2} \tau \Phi'_\lambda \left(\frac{\tau}{\beta} \right) \|v_\nu(0)\|_{L^2}^2 + \left(\frac{\gamma}{2} + C^{(4)} 2^{2\nu} \right) (s + \tau) \|v_\nu(s)\|_{L^2}^2 \\
& \quad - \frac{\alpha \kappa \log 2}{4} \frac{2^{2\nu}}{4} \int_0^s (t + \tau) v 2^{2\nu} \|v_\nu(t)\|_{L^2}^2 dt - \frac{\gamma}{8} \int_0^s \|v_\nu(t)\|_{L^2}^2 dt \\
& \quad - \frac{1}{2} \int_0^s (t + \tau) \|\partial_t v_\nu(t)\|_{L^2}^2 dt \\
& \quad - \int_0^s (t + \tau) 2^{-\alpha t \nu} \langle \mathcal{X}_\nu(t) | \partial_t v_\nu(t) \rangle_{L^2} dt \\
& \quad + \alpha \log 2 \int_0^s v 2^{-\alpha t \nu} (t + \tau) \langle v_\nu(t) | \mathcal{X}_\nu(t) \rangle_{L^2} dt,
\end{aligned}$$

where we have used the estimate

$$|\langle \partial_x v_\nu(s) | T_{a_\nu}^m \partial_x v_\nu(s) \rangle_{L^2}| \leq C^{(4)} 2^{2\nu} \|v_\nu(s)\|_{L^2}^2.$$

- End of the proof

Now we sum over ν and we obtain

$$\begin{aligned}
& \frac{\kappa}{8} \int_0^s \sum_{\nu \geq 0} 2^{2\nu} \|v_\nu(t)\|_{L^2}^2 dt + \frac{\gamma}{8} \int_0^s \sum_{\nu \geq 0} \|v_\nu(t)\|_{L^2}^2 dt \\
& \leq \frac{1}{2} \tau \Phi'_\lambda \left(\frac{\tau}{\beta} \right) \sum_{\nu \geq 0} \|v_\nu(0)\|_{L^2}^2 - \frac{\gamma}{8} \int_0^s \sum_{\nu \geq 0} \|v_\nu(t)\|_{L^2}^2 dt \\
& \quad - \frac{1}{2} \int_0^s (t + \tau) \sum_{\nu \geq 0} \|\partial_t v_\nu(t)\|_{L^2}^2 dt
\end{aligned}$$

$$\begin{aligned}
& + \frac{\gamma}{2}(s + \tau) \sum_{v \geq 0} \|v_v(s)\|_{L^2}^2 + C^{(4)}(s + \tau) \sum_{v \geq 0} 2^{2v} \|v_v(s)\|_{L^2}^2 \\
& - \frac{\alpha}{4} \frac{\kappa \log 2}{4} \int_0^s (t + \tau) \sum_{v \geq 0} v 2^{2v} \|v_v(t)\|_{L^2}^2 dt \\
& - \int_0^s (t + \tau) \sum_{v \geq 0} 2^{-\alpha t v} \langle \mathcal{X}_v(t) \mid \partial_t v_v(t) \rangle_{L^2} dt \\
& + \alpha \log 2 \int_0^s (t + \tau) \sum_{v \geq 0} v 2^{-\alpha t v} \langle v_v(t) \mid \mathcal{X}_v(t) \rangle_{L^2} dt.
\end{aligned}$$

Now we have

$$\begin{aligned}
& \left| \sum_{v \geq 0} 2^{-\alpha t v} \langle \mathcal{X}_v(t) \mid \partial_t v_v(t) \rangle_{L^2} \right| \\
& \leq \sum_{v \geq 0} 2^{-\alpha t v} \left| \langle \partial_x([\Delta_v, T_a^m] \partial_x w(t)) \mid \partial_t v_v(t) \rangle_{L^2} \right| \\
& + \sum_{v \geq 0} 2^{-\alpha t v} \left| \langle \Delta_v(\partial_x((a - T_a^m) \partial_x w(t))) \mid \partial_t v_v(t) \rangle_{L^2} \right| \\
& \leq \sum_{v \geq 0} 2^{-\alpha t v} \|\partial_x([\Delta_v, T_a^m] \partial_x w(t))\|_{L^2} \|\partial_t v_v(t)\|_{L^2} \\
& + \sum_{v \geq 0} 2^{-\alpha t v} \|\Delta_v(\partial_x((a - T_a^m) \partial_x w(t)))\|_{L^2} \|\partial_t v_v(t)\|_{L^2} \\
& \leq \left(\sum_{v \geq 0} 2^{-2\alpha t v} \|\partial_x([\Delta_v, T_a^m] \partial_x w(t))\|_{L^2}^2 \right)^{\frac{1}{2}} \left(\sum_{v \geq 0} \|\partial_t v_v(t)\|_{L^2}^2 \right)^{\frac{1}{2}} \\
& + \left(\sum_{v \geq 0} 2^{2(1-\alpha)t v} \|\Delta_v((a - T_a^m) \partial_x w(t))\|_{L^2}^2 \right)^{\frac{1}{2}} \left(\sum_{v \geq 0} \|\partial_t v_v(t)\|_{L^2}^2 \right)^{\frac{1}{2}}
\end{aligned}$$

By Corollary 3.5, Proposition 3.8 and Proposition 3.2 we get

$$\begin{aligned}
& \left| \sum_{v \geq 0} 2^{-\alpha t v} \langle \mathcal{X}_v(t) \mid \partial_t v_v(t) \rangle_{L^2} \right| \leq C^{(5)} \|w(t)\|_{H^{1-\alpha t}} \left(\sum_{v \geq 0} \|\partial_t v_v(t)\|_{L^2}^2 \right)^{\frac{1}{2}} \\
& + C^{(6)} \|(a - T_a^m) \partial_x w(t)\|_{H^{1-\alpha t}} \left(\sum_{v \geq 0} \|\partial_t v_v(t)\|_{L^2}^2 \right)^{\frac{1}{2}} \\
& \leq C^{(7)} \|w(t)\|_{H^{1-\alpha t}} \left(\sum_{v \geq 0} \|\partial_t v_v(t)\|_{L^2}^2 \right)^{\frac{1}{2}}
\end{aligned}$$

$$\begin{aligned}
&\leq C^{(8)} \left(\sum_{\nu \geq 0} 2^{2(1-\alpha t)\nu} \|w_\nu(t)\|_{L^2}^2 \right)^{\frac{1}{2}} \left(\sum_{\nu \geq 0} \|\partial_t v_\nu(t)\|_{L^2}^2 \right)^{\frac{1}{2}} \\
&\leq C^{(9)} \left(\sum_{\nu \geq 0} 2^{2\nu} \|v_\nu(t)\|_{L^2}^2 \right)^{\frac{1}{2}} \left(\sum_{\nu \geq 0} \|\partial_t v_\nu(t)\|_{L^2}^2 \right)^{\frac{1}{2}} \\
&\leq C^{(10)} \sum_{\nu \geq 0} 2^{2\nu} \|v_\nu(t)\|_{L^2}^2 + \frac{1}{2} \sum_{\nu \geq 0} \|\partial_t v_\nu(t)\|_{L^2}^2.
\end{aligned}$$

In the same way one can prove that

$$\left| \sum_{\nu \geq 0} \nu 2^{-\alpha t \nu} \langle \mathcal{X}_\nu(t) | v_\nu(t) \rangle_{L^2} \right| \leq C^{(11)} \sum_{\nu \geq 0} 2^{2\nu} \|v_\nu(t)\|_{L^2}^2.$$

We thus obtain

$$\begin{aligned}
&\frac{\kappa}{8} \int_0^s \sum_{\nu \geq 0} 2^{2\nu} \|v_\nu(t)\|_{L^2}^2 dt + \frac{\gamma}{8} \int_0^s \sum_{\nu \geq 0} \|v_\nu(t)\|_{L^2}^2 dt \\
&\leq \frac{1}{2} \tau \Phi'_\lambda \left(\frac{\tau}{\beta} \right) \sum_{\nu \geq 0} \|v_\nu(0)\|_{L^2}^2 - \frac{\gamma}{8} \int_0^s \sum_{\nu \geq 0} \|v_\nu(t)\|_{L^2}^2 dt \\
&\quad + \frac{\gamma}{2} (s + \tau) \sum_{\nu \geq 0} \|v_\nu(s)\|_{L^2}^2 + C^{(4)} (s + \tau) \sum_{\nu \geq 0} 2^{2\nu} \|v_\nu(s)\|_{L^2}^2 \\
&\quad - \frac{\alpha \kappa \log 2}{4} \int_0^s (t + \tau) \sum_{\nu \geq 0} \nu 2^{2\nu} \|v_\nu(t)\|_{L^2}^2 dt \\
&\quad + C^{(12)} \int_0^s (t + \tau) \sum_{\nu \geq 0} 2^{2\nu} \|v_\nu(t)\|_{L^2}^2 dt.
\end{aligned}$$

Now the term

$$C^{(12)} \int_0^s (t + \tau) \sum_{\nu \geq 0} 2^{2\nu} \|v_\nu(t)\|_{L^2}^2 dt$$

can be absorbed by

$$-\frac{\alpha \kappa \log 2}{4} \int_0^s (t + \tau) \sum_{\nu \geq 0} \nu 2^{2\nu} \|v_\nu(t)\|_{L^2}^2 dt$$

for high frequencies, and by

$$-\frac{\gamma}{8} \int_0^s \sum_{v \geq 0} \|v_v(t)\|_{L^2}^2 dt$$

for low frequencies by choosing $\bar{\gamma}$ larger if necessary.

All in all, we finally obtain

$$\begin{aligned} & \frac{\kappa}{8} \int_0^s \sum_{v \geq 0} 2^{2v} \|v_v(t)\|_{L^2}^2 dt + \frac{\gamma}{8} \int_0^s \sum_{v \geq 0} \|v_v(t)\|_{L^2}^2 dt \\ & \leq \frac{1}{2} \tau \Phi'_\lambda \left(\frac{\tau}{\beta} \right) \sum_{v \geq 0} \|v_v(0)\|_{L^2}^2 + \frac{\gamma}{2} (s + \tau) \sum_{v \geq 0} \|v_v(s)\|_{L^2}^2 \\ & \quad + C^{(4)}(s + \tau) \sum_{v \geq 0} 2^{2v} \|v_v(s)\|_{L^2}^2. \end{aligned}$$

From this, going back to u_v and using Proposition 3.2, the weighted energy estimate (2.4) follows. \square

5. Conditional stability up to the final time

In this section we state and prove two global stability theorems for solutions of (2.1) up to the final time T . The first result gives a logarithmic type control of $\|u\|_{L^2((0,T),L^2)}$ in terms of $\|u(0)\|_{L^2}$.

Theorem 5.1. *Assume Hypothesis 2.1 is satisfied. Then for all $D_0 > 0$ there exist positive constants ρ'' , δ'' and K'' , depending only on A_{LL} , A , κ , T and D_0 , such that if $u \in \mathcal{H}$ is a solution of (2.1) satisfying $\sup_{t \in [0, T]} \|u(t, \cdot)\|_{L^2} \leq D_0$ and $\|u(0, \cdot)\|_{L^2} \leq \rho''$, the inequality*

$$\|u\|_{L^2((0,T),L^2)} \leq K'' \frac{1}{|\log \|u(0)\|_{L^2}|^{\delta''}}$$

holds true.

Remark 5.2. Notice that, following Remark 2.2, it would be sufficient to impose an a-priori bound on $\|u(T, \cdot)\|_{L^2}$, which automatically implies the a-priori bound for $\|u(t, \cdot)\|_{L^2}$, $t \in [0, T]$.

Proof of Theorem 5.1. First we observe that, due to Theorem 2.7, it is not restrictive to assume that $\alpha_1 \leq T^{-1}$. Indeed, if this is not the case we can take T' , $0 < T' < T$, such that $T - T' < \alpha_1^{-1}$, and then in $[0, T']$ we apply the pointwise estimate given by Theorem 2.7, so we just need to estimate $\int_{T'}^T \|u(t)\|_{L^2}^2 dt$ in terms of $\|u(T')\|_{L^2}$.

With such assumption we can apply Proposition 2.4 with $\alpha = 1/T$, $\sigma = T$ and $\tau = T/4$ and we can find $\lambda > 1$, $\gamma > 0$ and $M > 0$ such that for all $\beta \geq T + \tau = \frac{5}{4}\tau$ and whenever $u \in \mathcal{H}$ is a solution of equation (2.1), then

$$\begin{aligned} & \int_0^T e^{2\gamma t} e^{-2\beta\Phi_\lambda(\frac{t+\tau}{\beta})} \|u(t, \cdot)\|_{H^{1-\alpha t}}^2 dt \\ & \leq M\gamma((T+\tau)e^{2\gamma T} e^{-2\beta\Phi_\lambda(\frac{T+\tau}{\beta})} \|u(T, \cdot)\|_{L^2}^2 + \tau \Phi'_\lambda(\frac{\tau}{\beta}) e^{-2\beta\Phi_\lambda(\frac{\tau}{\beta})} \|u(0, \cdot)\|_{L^2}^2). \end{aligned}$$

Now for any $r \in (0, T)$ we have

$$\begin{aligned} & \int_0^{T-r} e^{2\gamma t} e^{-2\beta\Phi_\lambda(\frac{t+\tau}{\beta})} \|u(t, \cdot)\|_{L^2}^2 dt \\ & \leq M\gamma((T+\tau)e^{2\gamma T} e^{-2\beta\Phi_\lambda(\frac{T+\tau}{\beta})} \|u(T, \cdot)\|_{L^2}^2 + \tau \Phi'_\lambda(\frac{\tau}{\beta}) e^{-2\beta\Phi_\lambda(\frac{\tau}{\beta})} \|u(0, \cdot)\|_{L^2}^2), \end{aligned}$$

where we have used the fact that $\|u(t, \cdot)\|_{L^2} \leq \|u(t, \cdot)\|_{H^{1-\alpha t}}$. Now, the function Φ_λ is increasing and consequently the function $t \mapsto e^{-2\beta\Phi_\lambda((t+\tau)/\beta)}$ is decreasing. We deduce that

$$\begin{aligned} & e^{-2\beta\Phi_\lambda(\frac{T-r+\tau}{\beta})} \int_0^{T-r} \|u(t, \cdot)\|_{L^2}^2 dt \\ & \leq M' \left(e^{-2\beta\Phi_\lambda(\frac{T+\tau}{\beta})} \|u(T, \cdot)\|_{L^2}^2 + \Phi'_\lambda(\frac{\tau}{\beta}) e^{-2\beta\Phi_\lambda(\frac{\tau}{\beta})} \|u(0, \cdot)\|_{L^2}^2 \right), \end{aligned}$$

where $M' = M\gamma 2Te^{2\gamma T}$. Then

$$\begin{aligned} & \int_0^{T-r} \|u(t, \cdot)\|_{L^2}^2 dt \leq M' \Phi'_\lambda(\frac{\tau}{\beta}) \left(e^{2\beta(\Phi_\lambda(\frac{T-r+\tau}{\beta}) - \Phi_\lambda(\frac{T+\tau}{\beta}))} \|u(T, \cdot)\|_{L^2}^2 \right. \\ & \quad \left. + e^{2\beta(\Phi_\lambda(\frac{T-r+\tau}{\beta}) - \Phi_\lambda(\frac{\tau}{\beta}))} \|u(0, \cdot)\|_{L^2}^2 \right) \\ & \leq M' \Phi'_\lambda(\frac{\tau}{\beta}) e^{2\beta(\Phi_\lambda(\frac{T-r+\tau}{\beta}) - \Phi_\lambda(\frac{T+\tau}{\beta}))} \left(\|u(T, \cdot)\|_{L^2}^2 \right. \\ & \quad \left. + e^{-2\beta\Phi_\lambda(\frac{\tau}{\beta})} \|u(0, \cdot)\|_{L^2}^2 \right), \end{aligned}$$

where we used the fact that $\Phi'_\lambda(\frac{\tau}{\beta}) \geq 1$ and $\Phi_\lambda(\frac{T+\tau}{\beta}) \leq 0$. We recall that the function Φ_λ is concave, so

$$\Phi_\lambda\left(\frac{T-r+\tau}{\beta}\right) - \Phi_\lambda\left(\frac{T+\tau}{\beta}\right) \leq \Phi'_\lambda\left(\frac{T+\tau}{\beta}\right) \left(\frac{T-r+\tau}{\beta} - \frac{T+\tau}{\beta}\right) = -\Phi'_\lambda\left(\frac{T+\tau}{\beta}\right) \frac{r}{\beta},$$

and then

$$\int_0^{T-r} \|u(t, \cdot)\|_{L^2}^2 dt \leq M' \Phi'_\lambda\left(\frac{\tau}{\beta}\right) e^{-2r\Phi'_\lambda\left(\frac{T+\tau}{\beta}\right)} \left(\|u(T, \cdot)\|_{L^2}^2 + e^{-2\beta\Phi_\lambda\left(\frac{\tau}{\beta}\right)} \|u(0, \cdot)\|_{L^2}^2 \right).$$

By Lemma 2.3 we have that

$$\Phi'_\lambda\left(\frac{T+\tau}{\beta}\right) = \psi_\lambda\left(\frac{T+\tau}{\tau} \frac{\tau}{\beta}\right) = \exp\left(\left(\frac{T+\tau}{\tau}\right)^{-\lambda} - 1\right) \left(\psi_\lambda\left(\frac{\tau}{\beta}\right)\right)^{\left(\frac{T+\tau}{\tau}\right)^{-\lambda}}.$$

We remind that $\tau = T/4$, so $\frac{T+\tau}{\tau} = 5$, and

$$\Phi'_\lambda\left(\frac{T+\tau}{\beta}\right) = \bar{N} \psi_\lambda\left(\frac{\tau}{\beta}\right)^{\bar{\delta}},$$

where $\bar{\delta} = 5^{-\lambda}$ and $\bar{N} = e^{\bar{\delta}-1}$. It follows that

$$\int_0^{T-r} \|u(t, \cdot)\|_{L^2}^2 dt \leq M' \psi_\lambda\left(\frac{\tau}{\beta}\right) e^{-2r\bar{N}\psi_\lambda\left(\frac{\tau}{\beta}\right)^{\bar{\delta}}} \left(\|u(T, \cdot)\|_{L^2}^2 + e^{-2\beta\Phi_\lambda\left(\frac{\tau}{\beta}\right)} \|u(0, \cdot)\|_{L^2}^2 \right).$$

Now we observe that

$$\psi_\lambda\left(\frac{\tau}{\beta}\right) e^{-r\bar{N}\psi_\lambda\left(\frac{\tau}{\beta}\right)^{\bar{\delta}}} = r^{-1/\bar{\delta}} r^{1/\bar{\delta}} \psi_\lambda\left(\frac{\tau}{\beta}\right) e^{-r\bar{N}\psi_\lambda\left(\frac{\tau}{\beta}\right)^{\bar{\delta}}} \leq C_{\bar{N}, \bar{\delta}} r^{-1/\bar{\delta}}$$

where

$$C_{\bar{N}, \bar{\delta}} := \sup_{z \geq 0} z e^{-\bar{N}z^{\bar{\delta}}}.$$

Then

$$\int_0^{T-r} \|u(t, \cdot)\|_{L^2}^2 dt \leq M' C_{\bar{N}, \bar{\delta}} r^{-1/\bar{\delta}} e^{-r\bar{N}\psi_\lambda\left(\frac{\tau}{\beta}\right)^{\bar{\delta}}} \left(\|u(T, \cdot)\|_{L^2}^2 + e^{-2\beta\Phi_\lambda\left(\frac{\tau}{\beta}\right)} \|u(0, \cdot)\|_{L^2}^2 \right).$$

We choose now β in such a way that $e^{-\beta\Phi_\lambda\left(\frac{\tau}{\beta}\right)} = \|u(0, \cdot)\|_{L^2}^{-1}$ i.e.

$$\frac{\beta}{\tau} \Phi_\lambda\left(\frac{\tau}{\beta}\right) = \frac{1}{\tau} \log \|u(0, \cdot)\|_{L^2}.$$

We obtain $\beta = \tau \Lambda_\lambda^{-1}\left(\frac{1}{\tau} \log \|u(0, \cdot)\|_{L^2}\right)$, where $\Lambda_\lambda(y) = y\Phi_\lambda(1/y)$. If $\|u(0, \cdot)\|_{L^2} \leq \bar{\rho} := e^{\tau\Lambda_\lambda(5)}$, then $\beta \geq T + \tau$. We have then

$$\int_0^{T-r} \|u(t, \cdot)\|_{L^2}^2 dt \leq M' C_{\bar{N}, \bar{\delta}} r^{-1/\bar{\delta}} e^{-r\bar{N}\psi_\lambda\left(\frac{1}{\Lambda_\lambda^{-1}\left(\frac{1}{\tau} \log \|u(0, \cdot)\|_{L^2}\right)}\right)^{\bar{\delta}}} \left(\|u(T, \cdot)\|_{L^2}^2 + 1 \right).$$

By Lemma 2.3 we have that

$$\lim_{z \rightarrow -\infty} -\frac{1}{z} \psi_\lambda \left(\frac{1}{\Lambda_\lambda^{-1}(z)} \right) = +\infty,$$

so

$$\psi_\lambda \left(\frac{1}{\Lambda_\lambda^{-1}(z)} \right) \geq |z|$$

if $z < 0$ and $|z|$ is sufficiently large. It follows that there exists $\tilde{\rho} \leq \bar{\rho}$ such that, if $\|u(0)\|_{L^2} \leq \tilde{\rho}$, then

$$\int_0^{T-r} \|u(t, \cdot)\|_{L^2}^2 dt \leq M' C_{\tilde{N}, \tilde{\delta}} r^{-1/\tilde{\delta}} e^{-r\tilde{N}(\frac{1}{r} |\log \|u(0, \cdot)\|_{L^2})|^{\tilde{\delta}}} \left(\|u(T, \cdot)\|_{L^2}^2 + 1 \right).$$

On the other hand,

$$\int_{T-r}^T \|u(t, \cdot)\|_{L^2}^2 dt \leq D_0 r.$$

It follows that for all $r > 0$

$$\int_0^T \|u(t, \cdot)\|_{L^2}^2 dt \leq \bar{M} \left(r + r^{-1/\tilde{\delta}} e^{-r\tilde{N}(|\log \|u(0, \cdot)\|_{L^2})|^{\tilde{\delta}}} \right),$$

where

$$\bar{M} = (M' C_{\tilde{N}, \tilde{\delta}} + 1)(D_0 + 1) \quad \text{and} \quad \tilde{N} = \frac{\tilde{N}}{\tau^{\tilde{\delta}}}.$$

Finally we choose

$$r = |\log \|u(0)\|_{L^2}|^{-\tilde{\delta}/2},$$

so we get

$$\begin{aligned} \int_0^T \|u(t, \cdot)\|_{L^2}^2 dt &\leq \bar{M} \left(\frac{1}{|\log \|u(0)\|_{L^2}|^{\tilde{\delta}/2}} + |\log \|u(0)\|_{L^2}|^{1/2} e^{-\tilde{N}(|\log \|u(0, \cdot)\|_{L^2})|^{\tilde{\delta}/2}} \right) \\ &\leq \bar{M} (1 + E_{\tilde{N}, \tilde{\delta}}) \frac{1}{|\log \|u(0)\|_{L^2}|^{\tilde{\delta}/2}}, \end{aligned}$$

where

$$E_{\tilde{N}, \tilde{\delta}} = \sup_{z \geq 0} z^{(1+\tilde{\delta})/2} e^{-\tilde{N}z^{\tilde{\delta}/2}}.$$

The proof is complete. \square

Under a stronger a-priori bound on admissible solutions in $[0, T]$, namely assuming an a-priori bound in H^1 rather than in L^2 , we can prove a pointwise stability estimate of logarithmic type up to the final time T .

Theorem 5.3. *Assume Hypothesis 2.1 is satisfied. Then for all $D_1 > 0$ there exist positive constants ρ''' , δ''' and K''' , depending only on A_{LL} , A , κ , T and D_1 , such that if $u \in \mathcal{H}$ is a solution of (2.1) satisfying $\sup_{t \in [0, T]} \|u(t, \cdot)\|_{H^1} \leq D_1$ and $\|u(0, \cdot)\|_{L^2} \leq \rho'''$, the inequality*

$$\sup_{t \in [0, T]} \|u(t, \cdot)\|_{L^2} \leq K''' \frac{1}{|\log \|u(0, \cdot)\|_{L^2}|^{\delta'''}}$$

holds true.

Remark 5.4. Notice that, following Remark 2.2, it would be sufficient to impose an a-priori bound on $\|u(T, \cdot)\|_{H^1}$, which automatically implies the a-priori bound for $\|u(t, \cdot)\|_{H^1}$, $t \in [0, T]$.

Proof of Theorem 5.3. We begin by noticing that, since u solves (2.1), then

$$\|\partial_t u(t, \cdot)\|_{H^{-1}} \leq \frac{1}{\kappa} D_1$$

It follows from Morrey's inequality that

$$\sup_{t \in [0, T]} \|u(t, \cdot)\|_{H^{-1}} \leq C_T \|u\|_{L^2((0, T), H^{-1})}^{1/2} \|u\|_{H^1((0, T), H^{-1})}^{1/2}$$

(for a direct simple proof see [3, proof of Thm. 8.8]). Then by Theorem 5.1 for $\|u(0)\|_{L^2} \leq \rho''$ we get

$$\sup_{t \in [0, T]} \|u(t, \cdot)\|_{H^{-1}} \leq C_T \left(\frac{1}{\kappa} D_1\right)^{1/2} \left(K'' \frac{1}{|\log \|u(0)\|_{L^2}|^{\delta''}}\right)^{1/2}.$$

The conclusion follows observing that for each fixed $t \in [0, T]$ we have

$$\begin{aligned} \|u(t, \cdot)\|_{L^2} &\leq \|u(t, \cdot)\|_{H^1}^{1/2} \|u(t, \cdot)\|_{H^{-1}}^{1/2} \\ &\leq D_1^{1/2} C_T^{1/2} \left(\frac{1}{\kappa} D_1\right)^{1/4} \left(K'' \frac{1}{|\log \|u(0)\|_{L^2}|^{\delta''}}\right)^{1/4}. \quad \square \end{aligned}$$

6. Reconstruction of the initial condition for parabolic equations

In view of applications it is convenient to rephrase Theorem 5.3. Consider the (forward) parabolic equation

$$\partial_t u - \sum_{j,k=1}^n \partial_{x_j} (a_{jk}(t, x) \partial_{x_k} u) = 0 \quad (6.1)$$

on the strip $[0, T] \times \mathbb{R}_x^n$ and assume Hypothesis 2.1 is satisfied. Then we have:

Corollary 6.1. *Let $D > 0$. There exist positive constants ρ_D , δ_D and K_D , depending only on A_{LL} , A , κ , T and D , such that if $u, v \in C^0([0, T], H^1) \cap C^1([0, T], L^2)$ are solutions of (6.1) satisfying $\|u(0, \cdot)\|_{H^1} \leq D$, $\|v(0, \cdot)\|_{H^1} \leq D$ and $\|u(T, \cdot) - v(T, \cdot)\|_{L^2} \leq \rho_D$, then the inequality*

$$\|u(0, \cdot) - v(0, \cdot)\|_{L^2} \leq K_D \frac{1}{|\log \|u(T, \cdot) - v(T, \cdot)\|_{L^2}|^{\delta_D}}$$

holds true.

Corollary 6.1 can be exploited to reconstruct the initial condition of an unknown solution $u(t)$ of (6.1), provided we can measure with arbitrary accuracy its final configuration $u_T := u(T)$. More precisely, suppose that for every $\theta > 0$ we can perform a measurement $v_{\theta, T}$ of u_T such that

$$\|v_{\theta, T} - u_T\|_{L^2} \leq \theta.$$

Moreover, suppose that we know *a priori* that $\|u(0)\|_{H^1} \leq D$ for some $D > 0$. We are interested in finding a computable approximation of $u(0)$. If it were possible to solve equation (6.1) backward in time with final condition $v(T) = v_{\theta, T}$, then by Corollary 6.1 we would get that $v(0)$ is close to $u(0)$, provided $\|v(0)\|_{H^1} \leq D$ and $v_{\theta, T}$ is sufficiently close to u_T . However, equation (6.1) with final condition $v(T) = v_{\theta, T}$ in general has no solution, due to the regularizing effect of equation (6.1) forward in time, and to the fact that $v_{\theta, T}$ does not possess any regularity, since it is the output of a measurement. There are various strategies to overcome this major obstruction. We mention the technique of quasi reversibility (see e.g. [13]), which consists in perturbing the equation to make it solvable backward in time, and the technique of Fourier truncation, which consists in approximating $v_{\theta, T}$ with a very regular function obtained truncating its Fourier transform. We illustrate the second technique through an example inspired by [14] (see also [18]).

We consider the equation

$$\partial_t u - \sum_{j,k=1}^n a_{jk}(t) \partial_{x_j} \partial_{x_k} u = 0 \quad (6.2)$$

on the strip $[0, T] \times \mathbb{R}_x^n$ and assume that the coefficients $a_{jk}(t)$ are Log-Lipschitz continuous. Moreover, setting

$$a(t, \xi) := \sum_{j,k=1}^n a_{jk}(t) \xi_j \xi_k,$$

we assume that

$$\frac{1}{2} |\xi|^2 \leq a(t, \xi) \leq 2 |\xi|^2, \quad (t, \xi) \in [0, T] \times \mathbb{R}_\xi^n.$$

Denote by \mathcal{F} the Fourier transform with respect to the x variable, and by \mathcal{F}^{-1} its inverse. Let $u \in C^0([0, T], H^1) \cap C^1([0, T], L^2)$ be a solution of (6.2) and let $\hat{u}(t, \xi) := (\mathcal{F}u)(t, \xi)$. Then

$$\partial_t \hat{u}(t, \xi) = -a(t, \xi) \hat{u}(t, \xi).$$

We set

$$A(t, \xi) := \int_0^t a(s, \xi) ds,$$

and we observe that $A(t, \xi)$ is increasing in t . Since $u(0, \cdot) \in L^2(\mathbb{R}_x^n)$, we have the following explicit representation of $\hat{u}(t, \xi)$ and hence of $u(t, x)$:

$$\hat{u}(t, \xi) = e^{-A(t, \xi)} \hat{u}(0, \xi), \quad (t, \xi) \in [0, T] \times \mathbb{R}_\xi^n.$$

On the other hand, if $\phi_T(\xi)$ is such that

$$e^{A(T, \xi)} \phi_T(\xi) \in L^2(\mathbb{R}_\xi^n), \quad (6.3)$$

then (6.2) can be solved backward in time with final condition $w(T) = w_T := \mathcal{F}^{-1} \phi_T$ and the explicit solution is $w(t, x) = (\mathcal{F}^{-1} \phi)(t, x)$, where

$$\phi(t, \xi) = e^{A(T, \xi) - A(t, \xi)} \phi_T(\xi).$$

As above, suppose we know *a priori* that $\|u(0)\|_{H^1} \leq D$. Moreover, suppose that for every $\theta > 0$ we can perform a measurement $v_{\theta, T}$ of u_T such that

$$\|v_{\theta, T} - u_T\|_{L^2} \leq \theta.$$

Let \hat{u}_T and $\hat{v}_{\theta, T}$ be the Fourier transform of u_T and $v_{\theta, T}$. For $R > 0$ define:

$$\hat{u}_{T, R}(\xi) := \chi_R(\xi) \hat{u}_T(\xi) \quad \text{and} \quad \hat{v}_{\theta, T, R}(\xi) := \chi_R(\xi) \hat{v}_{\theta, T}(\xi)$$

where $\chi_R(\xi)$ is the characteristic function of the ball of radius R in \mathbb{R}_ξ^n . Both $\hat{u}_{T, R}$ and $\hat{v}_{\theta, T, R}$ satisfy (6.3) so we can solve (6.2) backward in time with data at T given by $u_{T, R} = \mathcal{F}^{-1} \hat{u}_{T, R}$ and $v_{\theta, T, R} = \mathcal{F}^{-1} \hat{v}_{\theta, T, R}$. The explicit representations of the corresponding solutions are $u_R(t, x) := \mathcal{F}^{-1}(\hat{u}_R)(t, x)$ and $v_{\theta, R}(t, x) := \mathcal{F}^{-1}(\hat{v}_{\theta, R})(t, x)$, where

$$\hat{u}_R(t, \xi) = e^{A(T, \xi) - A(t, \xi)} \hat{u}_{T, R}(\xi)$$

and

$$\hat{v}_{\theta, R}(t, \xi) = e^{A(T, \xi) - A(t, \xi)} \hat{v}_{\theta, T, R}(\xi).$$

It is straightforward to check that $\|u_R(0)\|_{H^1} \leq D$. Now we have

$$\begin{aligned} \|v_{\theta, R}(0)\|_{H^1} &\leq \|u_R(0)\|_{H^1} + \|v_{\theta, R}(0) - u_R(0)\|_{H^1} \\ &\leq D + \left(\int_{|\xi| \leq R} (1 + |\xi|^2) e^{2A(T, \xi)} |\hat{v}_{\theta, T, R}(\xi) - \hat{u}_{T, R}(\xi)|^2 d\xi \right)^{\frac{1}{2}} \\ &\leq D + (1 + R^2)^{\frac{1}{2}} e^{2TR^2} \|v_{\theta, T} - u_T\|_{L^2} \leq D + e^{(2T+1)R^2} \theta. \end{aligned}$$

Moreover, we have

$$\begin{aligned} \|u_T - v_{\theta, T, R}\|_{L^2} &\leq \|\hat{u}_T - \hat{u}_{T, R}\|_{L^2} + \|\hat{u}_{T, R} - \hat{v}_{\theta, T, R}\|_{L^2} \\ &\leq \left(\int_{|\xi| \geq R} |\hat{u}_T(\xi)|^2 d\xi \right)^{\frac{1}{2}} + \left(\int_{|\xi| \leq R} |\hat{u}_T(\xi) - \hat{v}_{\theta, T, R}(\xi)|^2 d\xi \right)^{\frac{1}{2}} \\ &\leq \left(\int_{|\xi| \geq R} (1 + |\xi|^2)(1 + |\xi|^2)^{-1} e^{-2A(T, \xi)} |\hat{u}(0, \xi)|^2 d\xi \right)^{\frac{1}{2}} + \theta \\ &\leq (1 + R^2)^{-\frac{1}{2}} e^{-(T/2)R^2} \|u(0)\|_{H^1} + \theta \leq e^{-(T/2)R^2} D + \theta \end{aligned}$$

Now, assuming without loss of generality that $\theta < 1$, we choose $R(\theta) := (2T + 1)^{-1/2} |\log \theta|^{1/2}$ and we notice that $R(\theta)$ tends to $+\infty$ as $\theta \rightarrow 0$. With this choice we have

$$\|v_{\theta, R}(0)\|_{H^1} \leq D + 1$$

and

$$\|u_T - v_{\theta, T, R}\|_{L^2} \leq D\theta^{T/(4T+2)} + \theta \leq (D + 1)\theta^{T/(4T+2)}.$$

Now let $\rho = \rho_{D+1}$, $K = K_{D+1}$ and $\delta = \delta_{D+1}$ be the constants given by Corollary 6.1. Then for sufficiently small θ we have that

$$\|u_T - v_{\theta, T, R}\|_{L^2} \leq \rho.$$

Finally, by Corollary 6.1, we get

$$\|u(0) - v_{\theta, R(\theta)}(0)\|_{L^2} \leq \tilde{K} \frac{1}{|\log \theta|^\delta},$$

where \tilde{K} can be explicitly expressed in terms of T , D , K and δ . Therefore $v_{\theta, R(\theta)}(0)$ is the desired approximation of $u(0)$ in L^2 .

Data availability

No data was used for the research described in the article.

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