Quadratic lifespan for the sublinear α-SQG sharp front problem

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Abstract

n this paper we consider the generalized surface quasi-geostrophic α-SQG equations, in the "sublinear regime" $\alpha \in (0,1)$ and we study the stability of vortex patches close to vortex discs. We shall prove that for regular, Sobolev initial vortex patches ε-close to a vortex disc, the solutions stay ε-close to a vortex disc for a time interval of order $O\left(\varepsilon^{-2}\right)$. The proof is based on a paradifferential Birkhoff normal form reduction, implemented in the case where the dispersion relation is sublinear.

Contents

1 Presentation of the problem

In this paper we consider the generalized surface quasi-geostrophic α-SQG equations

$$\begin{cases} \partial_t \theta(t,\zeta) + u(t,\zeta) \cdot \nabla\theta(t,\zeta) = 0, & (t,\zeta) \in \mathbb{R} \times \mathbb{R}^2 \\ \theta(0,\zeta) = \theta_0(\zeta) \end{cases}, \tag{1.1}$$

with velocity field

$$u := \nabla^{\perp} |D|^{-2+\alpha} \theta, \qquad |D| := (-\Delta)^{\frac{1}{2}}, \qquad \alpha \in [0,2). \tag{1.2}$$

These class of active scalar equations have been introduced in [27, 62] and, for $\alpha \to 0$, formally reduce to the 2D-Euler equation in vorticity formulation (in this case θ is the vorticity of the fluid). The case $\alpha = 1$ is the surface quasi-geostrophic (SQG) equation in [26] which models the evolution of the temperature θ for atmospheric and oceanic flows.

We are interested in a particular class of solutions in which the initial data θ_0 in (1.1) is a characteristic function of a sufficiently regular bounded subset $D(0) \subset \mathbb{R}^2$, i.e.

$$\theta_0(\zeta) = \mathbb{1}_{D(0)}(\zeta). \tag{1.3}$$

In the case $\alpha = 0$ the celebrated Yudovich theorem [68] guarantees us that there exists a unique global weak solution to (1.1) which admits a well-defined generalized Lagrangian flow associated to velocity u. Yudovich' theorem, though, says nothing on the persistence of regularity of the boundary of the deformed domain $D(t)$. The persistence of the regularity of the patch has been settled later on in the important works [25] and [18].

In the case $\alpha > 0$ there is no analogous of the Yudovich theorem, and the system composed of Eqs. (1.1) and (1.3) make sense only as a *Contour Dynamic Equation* (cf. [61, Section 8.3.1] for the case $\alpha = 0$), hence if $\partial D(t)$ is paramentrized by the curve $(t,x) \in [0,T] \times \mathbb{T} \mapsto X(t,x) = \begin{pmatrix} X_1(t,x) \\ X_2(t,x) \end{pmatrix} \in \mathbb{R}^2$ the evolution of the interface is given by evolutionary system

$$\begin{aligned} X_t(t,x) &= S[X,X](t,x) \\ S[p,q](x) &= \frac{c_\alpha}{2\pi} \text{ p.v.} \int \frac{p'(x) - q'(y)}{\left|p(x) - q(y)\right|^\alpha} \, \mathrm{d}y, \end{aligned} \qquad (t,x) \in \mathbb{R} \times \mathbb{T}, \ \alpha \in (0,1), \tag{1.4}$$

where, denoting Γ the Euler-Gamma function,

$$c_\alpha = \frac{\Gamma\left(\frac{\alpha}{2}\right)}{2^{1-\alpha}\Gamma\left(1 - \frac{\alpha}{2}\right)}. \tag{1.5}$$

Equation (1.4) describes the evolution of the α-SQG-sharp front problem. It is well-known (cf. [7, 43]) that for any $R > 0$ the profile

$$R\,\vec{\gamma}(x) := R \begin{pmatrix} \cos x \\ \sin x \end{pmatrix},$$

is a stationary solution of Eq. (1.4) known as the *Rankine vortex*. Analogously as to [7] we thus study the evolution of a patch whose profile is a radial perturbation of the Rankine vortex, thus transforming the vector-valued system Eq. (1.4) into the scalar evolution equation (see [7, Section 1] for more detailed computations)

$$\begin{aligned} -(1+h(x))\,h_t(x) &= S\left[(1+h)\gamma, (1+h)\gamma\right](x) \cdot \left(-(1+h(x))\,\gamma(x) + h'(x)\,\gamma'(x)\right) \\ &= \frac{c_\alpha}{2\pi} \int \frac{\cos(x-y)\left[(1+h(x))\,h'(y) - (1+h(y))\,h'(x)\right]}{\left[(1+h(x))^2 + (1+h(y))^2 - 2(1+h(x))(1+h(y))\cos(x-y)\right]^{\frac{\alpha}{2}}} \mathrm{d}y \\ &\quad + \frac{c_\alpha}{2\pi} \int \frac{\sin(x-y)\left[(1+h(x))(1+h(y)) + h'(x)\,h'(y)\right]}{\left[(1+h(x))^2 + (1+h(y))^2 - 2(1+h(x))(1+h(y))\cos(x-y)\right]^{\frac{\alpha}{2}}} \mathrm{d}y. \end{aligned} \tag{1.6}$$

1.1 Results of the manuscript

The local well-posedness issue for the Equation (1.4) has been solved in [38] and recently improved in [39] and assures us that initial data which are H^2 generate local-in-time solutions of Eq. (1.4). This indeed can be translated to Eq. (1.6) insuring that if $h_0 \in H^2$ than there exists a $T > 0$ and a unique solution $h \in \mathcal{C}\left([0,T]; H^2\right)$, of Eq. (1.6).

The aim of the present manuscript is to extend the local-existence result stated above via normal forms methods such as in [7, 8, 10, 14, 55], the result we prove is the following one:

Theorem 1.1 (Quadratic life-span)**.** *Let $\alpha \in (0,1)$. There exists $s_0 > 0$ such that for any $s \geq s_0$, there are $\varepsilon_0 > 0$, $c_{s,\alpha} > 0$, $C_{s,\alpha} > 0$ such that, for any h_0 in $H^s(\mathbb{T};\mathbb{R})$ satisfying $\|h_0\|_{H^s} \leq \varepsilon < \varepsilon_0$, the equation* (1.6) *with initial condition $h(0) = h_0$ has a unique classical solution*

$$h \in \mathcal{C}\left(\left[-T_{s,\alpha}, T_{s,\alpha}\right]; H^s(\mathbb{T};\mathbb{R})\right) \qquad \text{with} \qquad T_{s,\alpha} > c_{s,\alpha}\varepsilon^{-2}, \tag{1.7}$$

satisfying $\|h(t)\|_{H^s} \leq C_{s,\alpha}\,\varepsilon$, for any $t \in \left[-T_{s,\alpha}, T_{s,\alpha}\right]$.

Ideas of the proof Here we collect the main ingredients of the proof of Theorem 1.1.

Energy estimate The result in Theorem 1.1 is a consequence of an energy estimate of the form

$$\|h(t)\|_{H^s}^2 \lesssim_s \|h(0)\|_{H^s}^2 + \int_0^t \|h(\tau)\|_{H^{s_0}}^2 \|h(\tau)\|_{H^s}^2 \,\mathrm{d}\tau, \quad s \geq s_0 \gg 1 \tag{1.8}$$

which, if $\|h(0)\|_s \leq \varepsilon$, allows to control the norm $\|h(t)\|_s$ of the solution on the claimed time scale (1.7). A quartic energy estimates of the form (1.8) is possible only if there are no quadratic (in h) contributions in (1.6). The equation (1.6) indeed exhibits nontrivial quadratic contributions, in stark contrast to the scenarios involving perturbations of the half-space, as delineated in [28, Eq. (1.4)]. To remove the quadratic contribution in the energy estimate a classical approach is to implement a normal form transformation. However, since the equation (1.6) is quasi-linear, the direct normal form approach gives an unbounded transformation, rendering it ineffective for energy estimates due to the associated loss of derivative. Consequently, the main effort in the manuscript is the construction of a novel nonlinear transformation, denoted as Ψ_0, which maps the Sobolev spaces H^s onto itself for all $s \geq s_0 \gg 1$. The primary utility of this transformation is to redefine the variable of interest, h, as $g := \Psi_0 h$. Consequently, g satisfies a newly formulated nonlinear evolution equation, notable for its absence of quadratic contributions, a feature imperative for the derivation of (1.8). We construct Ψ_0 using a para-differential normal form approach for quasi-linear PDEs which we explain below.

The Hamiltonian unknown Defining $f := h + \frac{h^2}{2}$ allows us to transform (1.6) into an Hamiltonian quasi-linear PDE (cf. (2.5))

$$f_t = \partial_x \nabla H\big(f\big), \tag{1.9}$$

where the explicit for of the Hamiltonian is given in (2.6). Notice that the transformation $h \mapsto f$ is bounded and invertible in any Sobolev space and was first introduced in [13, 43]. It is important to notice that the equation (1.6) does not preserve the average of the unknown h, while such preservation is immediate for the evolution equation in the unknown f, given in (1.9). There is a strict connection between preservation of the average of f and incompressibility of the flow (1.2), we refer the interested reader to the introduction of [7].

The linearized equation To study small solutions we first linearize around $f = 0$, obtaining

$$f_t = \mathbb{V}_\alpha \partial_x f - \mathrm{i}\dot{\omega}(D) f, \quad \mathbb{V}_\alpha \in \mathbb{R}, \quad \dot{\omega}(\xi) \sim c_\alpha |\xi|^{\alpha-1}\xi. \tag{1.10}$$

Notice that since $f = h + h^2/2$ the linearization of Eqs. (1.6) and (1.9) around zero are the same. The constant transport operator $\mathbb{V}_\alpha \partial_x$ introduce a uniform translation (which correspond to a rotation of the patch). In view of the translation invariance of the problem (see Eq. (2.9)), it does not play any role in our stability analysis. On the contrary the Fourier multiplier $\mathrm{i}\dot{\omega}(D)$ is a dispersive operator whose symbol is the dispersion relation of the equation and play a crucial role in our paper. In other words, we can express the solutions of the linearized equation (1.10) in terms of the moving variable $v(t,x) := f(t, x - \mathbb{V}_\alpha t)$ using Fourier decomposition, as

$$v(t,x) = \sum_{j \in \mathbb{Z}\setminus\{0\}} v_j e^{\mathrm{i}jx - \mathrm{i}\dot{\omega}(j)t}, \tag{1.11}$$

thus suppressing the linear transport contribution in (1.10)

Property of the dispersion relation The explicit expression of $\dot{\omega}\big(j\big)$ is quite involved (see (2.13)). It is so remarkable that, to implement the normal form approach, only two basic property are needed:

1. **Absence of three wave interactions:** For any $j, k \in \mathbb{Z}$ we have

$$\left|\dot{\omega}\big(j+k\big) - \dot{\omega}\big(j\big) - \dot{\omega}\,(k)\right| \geq c > 0. \tag{1.12}$$

   The inequality in (1.12) was first proved in [7, Lemma 3.5] whose proof, notably, relies only on the convexity of the disperion relation. The lower bound in (1.12) is the most basic property in normal form theory and exclude possible resonant phenomena;

2. **Pseudo-differential property:** the function $\xi \in \mathbb{R} \mapsto \dot\omega(\xi)$ is a Fourier multiplier of order $\alpha \in (0,1)$ according to Definition 2.3, namely

$$\left|\partial_\xi^\beta \dot\omega(\xi)\right| \lesssim |\xi|^{\alpha-\beta} \quad \text{for any } \beta \in \mathbb{N}.$$

This property is proved in [7, Lemma 3.1] and is used in the para-differential normal form reduction (see the paragraph below) needed to compensate the loss of derivatives due to the quasi-linear nature of (1.6).

Para-differential normal form The construction of the normal form transformation Ψ_0 follows three conceptual steps:

1. **Paralinearization:** The first step in our analysis is the para-linearization of the Hamiltonian equation (1.9) obtaining an equation of the form

$$f_t + \mathrm{i}\,\mathrm{Op}^{BW}\left[V\left(f;x\right)\,\xi + \left(1+\nu\left(f;x\right)\right)\dot\omega_\alpha(\xi) + r\left(f;x,\xi\right)\right]\,f = R\left(f\right)f, \tag{1.13}$$

where V, ν and $\dot\omega$ are real, r is a symbol of order at most zero and $R\left(f\right)$ is an regularizing operator (cf. Definitions 3.1 and 3.9). This was first performed in [7, Theorem 4.1] and is here stated in Proposition 4.2;

2. **Reduction of quadratic paradifferential terms** up to smoothing remainder: this step is performed in Proposition 4.4 and is achieved via iterated conjugation by bounded flows generated by 1-homogeneous paradifferential operators. Since $\alpha \in (0,1)$ the para-differential reduction that we perform is completely different with respect to the one performed in [7] for the superlinear case $\alpha \in (1,2)$. Instead we perform a sub-linear reduction in the spirit of [10]. We describe roughly how the generic normal form step works. If one has to normalize a quadratic term of the form $\mathrm{Op}^{BW}\left[r(f;x,\xi)\right][u]$ where $r(f;x,\xi) = \sum_{j\neq 0} r_j(\xi) f_j e^{\mathrm{i}jx}$ is a simbol of order $m \leq 1$. Then one looks for a change of variables of the form $v = \Phi(f)[f]$ where $\Phi(f)$ is the time one flow map associated to the linear vector field $\mathrm{i}\mathrm{Op}^{BW}\left[g(f;x,\xi)\right]$ where $g(f;x,\xi) = \sum_{j\neq 0} g_j(\xi) f_j e^{\mathrm{i}jx}$ is a symbol of order m. Since $\dot\omega_\alpha(\xi)$ is a symbol of order $\alpha < 1$, the commutator $\left[\dot\omega_\alpha(D), \mathrm{Op}^{BW}\left[g\right]\right]$ is of order $m-(1-\alpha) < m$ and hence, in the new coordinates, the only quadratic term of order m is given by

$$\mathrm{Op}^{BW}\left[\partial_t g(f;x,\xi) + r(f;x,\xi)\right][v] = \mathrm{Op}^{BW}\left[g(\partial_t f;x,\xi) + r(f;x,\xi)\right][v].$$

If f solves (1.13), then $\partial_t f = -\mathrm{i}\dot\omega_\alpha(D)f + O(f^2)$, then we choose g in such a way that

$$g(-\mathrm{i}\omega_\alpha(D)f;x,\xi) + r(f;x,\xi) = 0$$

which can be done by defining

$$g(f;x,\xi) = \sum_{j\neq 0} \frac{r_j(\xi)}{\mathrm{i}\dot\omega_\alpha(j)} f_j e^{\mathrm{i}jx}.$$

Contrarily to [10] in the present manuscript we have to take as well into consideration the fact that the number of iterated conjugations is not fixed, but depends inversely on the quantity $1-\alpha$ thus invalidating the procedure in the endpoint case $\alpha = 1$. The conjugation procedure allows us to cancel the 1-homogeneous (in f) terms in the symbol $V\left(f;x\right)\,\xi + \left(1+\nu\left(f;x\right)\right)\dot\omega_\alpha(\xi) + r\left(f;x,\xi\right)$ in (1.13), as a result, the residual bilinear contribution emanates solely from $R\left(f\right)$;

3. **Birkhoff Normal Form on the smoothing term:** In this last step we need to cancel the only remaining quadratic interactions, the ones stemming from the smoothing term $R\left(f\right)$: this is the content of Proposition 5.1. Again, such result is achieved via a conjugation of a linear flow, $\Psi_{\mathtt{bir}}(f)$, but this time generated by a 1-homogeneous smoothing operator $Q(f)$ (cf. Definition 3.9). To cancel the quadratic contribution in $R(f)$ the operator $Q(f)$ have to solve the homological equation

$$Q\left(-\mathrm{i}\omega_\alpha(D)f\right) + \left[\!\left[Q\left(f\right), -\mathrm{i}\omega_\alpha(D)\right]\!\right] + R_1\left(f\right) = 0$$

which is solvable thanks to the non-resonance condition (1.12). The loss of derivatives introduced by the quasi-linear nature of the equation is compensated by the regularizing property of the generator.

Overview of the literature Theorem 1.1 builds upon several important results that have been proved in the past two decades in the context of quasi-linear hyperbolic PDEs. We divide such references thematically:

Foundational results The first local existence theory for Eq. (1.4) has been proved in [64] in a $\mathcal{C}^\infty$ setting and in [38] for finite regularity, which was lets extended in [24], we refer as well to [60]. The formation and avoidance of singularities is a very active research field which has seen several improvement in the recent past, cf. [39, 40, 47, 58, 59].

Traveling waves The existence of traveling waves by means of bifurcation methods has been extensively used in order to produce a large number of uniformly rotating structures, such as in [19–22, 29–31, 41, 44, 48–52, 63]. We refer to the introductions in [42, 43] for more references.

Quasi-periodic solutions Very recently quasi-periodic solutions for several patches configurations have been constructed in [13, 42, 43, 45, 46, 53, 65, 66] using techniques first developed in the context of the water-waves equations in [4, 5, 11, 12, 17, 33, 37]. We refer as well to [6, 35, 36] for KAM results for 2D and 3D hydro-dynamical systems.

Normal forms for quasi-linear systems The application of normal forms energy methods has been widely used in the past 10 years in order to extend lifespans for solutions of the water-waves and other hydro-dynamical systems in [3, 8–10, 14–16, 23, 32, 34, 55–57, 67, 69]. Recently it was proved in [7] the first normal-forms result for α-patches in the case $\alpha \in (1,2)$.

Further literature We mention that in the setting in which the patch is a small perturbation of the half plane it is possible to construct global-in-time solutions,such as in [1, 2, 28, 54], using dispersive techniques that *cannot* be applied in the periodic setting. We want to highlight that in the half-plane patch setting, contrary to Eq. (1.6), the nonlinear equation governing the perturbation of the half-plane lacks the quadratic component (cf. [28, Eq. (1.4)]).

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2 Background on the SQG patch problem as a contour equation

2.1 The Hamiltonian formalism

As explained in detail in [7, Section 3] the unknown h of Eq. (1.4) is *not* the good unknown in order to study the problem at hand. Setting in fact

$$f(t,x) := h(t,x) + \frac{h^2(t,x)}{2}, \tag{2.1}$$

we can recast Eq. (1.4) as a Hamiltonian evolution equation. Let us denote with $E_\alpha = E_\alpha\left(f\right)$, the *pseudo-energy* of the patch defined as

$$E_\alpha(t) := -\frac{1}{2}\iint \frac{\partial_{xy}\left[\sqrt{1+2f(t,x)}\,\sqrt{1+2f\left(t,y\right)}\cos\left(x-y\right)\right]}{\left[1+2f(x)+1+2f\left(y\right)-2\sqrt{1+2f(x)}\sqrt{1+2f\left(y\right)}\cos\left(x-y\right)\right]^{-\frac{\alpha}{2}}}\mathrm{d}y\,\mathrm{d}x. \tag{2.2}$$

The following result is proved in [43, Proposition 2.1]:

Proposition 2.1. *Let $\alpha \in (0,2)$. If h is a solution of Eq.* (1.6) *then the variable f defined in* (2.1) *solves the Hamiltonian equation*

$$f_t = \partial_x \nabla E_\alpha(f) \tag{2.3}$$

where $E_\alpha(f)$ is the pseudo-energy of the patch

$$E_\alpha(f) := -\frac{c_\alpha}{8\left(1-\frac{\alpha}{2}\right)^2} \iint \frac{\partial_{xy}\left[\sqrt{1+2f(x)}\sqrt{1+2f(y)}\cos(x-y)\right]}{\left[1+2f(x)+1+2f(y)-2\sqrt{1+2f(x)}\sqrt{1+2f(y)}\cos(x-y)\right]^{\frac{\alpha}{2}-1}} \mathrm{d}y\, \mathrm{d}x$$

with c_α defined in (1.5), *and its L^2-gradient $\nabla E_\alpha(f)$ is*

$$\nabla E_\alpha(f) = \frac{c_\alpha}{2\left(1-\frac{\alpha}{2}\right)} \int \frac{1+2f(y)+\sqrt{1+2f(x)}\,\partial_y\left[\sqrt{1+2f(y)}\sin(x-y)\right]}{\left[1+2f(x)+1+2f(y)-2\sqrt{1+2f(x)}\sqrt{1+2f(y)}\cos(x-y)\right]^{\frac{\alpha}{2}}} \mathrm{d}y. \tag{2.4}$$

Regarding the vortex patch equation in a rotating frame with angular velocity $\Omega \in \mathbb{R}$ amounts to look for solutions of (2.3) of the form $f(t,x) = \tilde{f}(t, x-\Omega t)$ namely solving, renaming $\tilde{f}$ simply as f,

$$f_t = \partial_x \nabla H_{\alpha,\Omega}(f) \tag{2.5}$$

where

$$H_{\alpha,\Omega}(f) := E_\alpha(f) + \Omega J(f) \tag{2.6}$$

and $J(f)$ is the *angular momentum*

$$J(f) := \frac{1}{2}\int_{-\pi}^{\pi} f(x)^2\, \mathrm{d}x. \tag{2.7}$$

The L^2-gradient of $J(f)$ is

$$\nabla J(f) = f. \tag{2.8}$$

Remark 2.2 (Symmetries and conservation laws). The Hamiltonian evolution equation (2.5) has the following symmetries:

- The invariance by **time translation** implies that the *energy* $E_\alpha(f)$ is a constant of motion;
- The invariance by **space rotation**, namely

  $$H_{\alpha,\Omega} \circ \mathrm{t}_\theta = H_{\alpha,\Omega}, \quad \forall \theta \in \mathbb{R}, \quad \mathrm{t}_\theta u(\cdot) := u(\cdot + \theta) \tag{2.9}$$

  implies that the angular momentum $J(f)$ in (2.7) is a constant of motion;
- The degeneracy of the Poisson structure ∂_x implies that the area

  $$A(f) := \int_{\mathbb{T}} f(x)\, \mathrm{d}x,$$

  is also a constant of motion.

These are the only known conservation laws and they do not provide any control of the H^2 norm required for propagate the available local Cauchy theory.

2.2 The linear problem

Definition 2.3. Let $m \in \mathbb{R}$, we define the space of Fourier multipliers of order m, $\tilde{\Gamma}_0^m$, as the space of smooth functions from $\mathbb{R}$ to $\mathbb{C}$ of the form $\xi \mapsto a(\xi)$ such that

$$\left|\partial_\xi^\beta a(\xi)\right| \le C_\beta \langle\xi\rangle^{m-\beta}, \qquad \forall \beta \in \mathbb{N}_0, |\xi| \ge 1/2.$$

We say that a is *real-to-real* if $a(D)\, u$ is a real-valued function for any u real valued.

Remark 2.4. Notice that the real-to-real condition in Definition 2.3 implies that $\overline{a(\xi)} = a(-\xi)$.

Remark 2.5. Notice that, since (2.5) preserves the average, we shall always work with functions with zero average. Consequently we implicitly substistute functions $\xi \mapsto a(\xi)$ that are not smooth in 0 with their smooth modification $\xi \mapsto \psi(\xi)\, a(\xi)$ where ψ is a smooth bump function which is identically zero close to zero and identically one for $|\xi| \ge 1/4$. This does not modify the action of the Fourier multiplier $\psi(D)\, a(D)$ once restricted onto spaces of zero-average functions.

The following result is a consequence of Lemma 3.1 and Remark 3.2 in [7]:

Lemma 2.6 (Linearization of $\nabla H_{\alpha,\Omega}$ around zero)**.** *For $\alpha \in (0,1)\cap(1,2)$, $\Omega \in \mathbb{R}$ and $H_{\alpha,\Omega}$ defined as in Eq.* (2.6)*, we have*

$$\mathrm{d}\nabla H_{\alpha,\Omega}(0) = -L_\alpha(|D|) - \Omega, \tag{2.10}$$

where

$$\begin{aligned} L_\alpha(\xi) &:= \mathbb{V}_\alpha - \frac{c_\alpha}{2\left(1-\frac{\alpha}{2}\right)} \frac{1}{1-\alpha} \frac{\Gamma(3-\alpha)}{\Gamma\left(1-\frac{\alpha}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)} \frac{\Gamma\left(\frac{\alpha}{2}+|\xi|\right)}{\Gamma\left(1-\frac{\alpha}{2}+|\xi|\right)}, \\ \mathbb{V}_\alpha &:= \frac{c_\alpha}{2\left(1-\frac{\alpha}{2}\right)} \frac{1}{1-\alpha} \frac{\Gamma(2-\alpha)}{\Gamma\left(1-\frac{\alpha}{2}\right)^2}, \end{aligned} \tag{2.11}$$

is a Fourier multiplier of order $\max\{0,\alpha-1\}$*. Additionally there exists a real Fourier multiplier $m^\star_{\alpha-3}$ of order $\alpha-3$ such that*

$$L_\alpha(\xi) = \mathbb{V}_\alpha - \frac{c_\alpha}{2\left(1-\frac{\alpha}{2}\right)} \frac{\Gamma(3-\alpha)}{\Gamma\left(1-\frac{\alpha}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)} \frac{1}{1-\alpha}\; |\xi|^{\alpha-1} + m^\star_{\alpha-3}(\xi). \tag{2.12}$$

Definition 2.7. We define the symbol of the linearized SQG equation as

$$\xi \mapsto \omega_\alpha(\xi) = \xi\, L_\alpha(\xi) \in \mathbb{R},$$

where L_α is defined in Equation (2.11). We define also the dispersive part of $\omega_\alpha(\xi)$ as

$$\dot\omega_\alpha(\xi) := \omega_\alpha(\xi) - \mathbb{V}_\alpha \xi = -\frac{c_\alpha}{2\left(1-\frac{\alpha}{2}\right)} \frac{\Gamma(3-\alpha)}{\Gamma\left(1-\frac{\alpha}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)} \frac{1}{1-\alpha}\; |\xi|^{\alpha-1}\xi + m^\star_{\alpha-3}(\xi)\,\xi. \tag{2.13}$$

Note that, since $\Gamma(r) > 0$ for any $r > 0$, one has

$$\dot\omega_\alpha(\xi) \neq 0 \quad \text{for any } \xi \neq 0. \tag{2.14}$$

We use as well the following result, which is proved in [7] (see Lemma 3.5).

Lemma 2.8 (Absence of three wave interactions)**.** *Let $\alpha \in (0,2)$. For any $n, j, k \in \mathbb{Z}\setminus\{0\}$ satisfying $k = j+n$, it results*

$$\left|\omega_\alpha(k) - \omega_\alpha(j) - \omega_\alpha(n)\right| \ge \omega_\alpha(2) > 0. \tag{2.15}$$

3 Functional setting

Along the paper we deal with real parameters

$$s \geq s_0 \gg \varrho \geq 4\frac{\alpha}{1-\alpha} \tag{3.1}$$

The values of s, s_0 and ϱ may vary from line to line while still being true the relation (3.1).

We expand a 2π-periodic function $u(x)$ in $L^2(\mathbb{T};\mathbb{C})$ in Fourier series as

$$u(x) = \sum_{j\in\mathbb{Z}} u_j e^{\mathrm{i}jx}, \qquad u_j := \frac{1}{2\pi}\int_{\mathbb{T}} u(x) e^{-\mathrm{i}jx}\,\mathrm{d}x. \tag{3.2}$$

A function $u(x)$ is *real* if and only if $\overline{u_j} = u_{-j}$, for any $j \in \mathbb{Z}$. For any $s \in \mathbb{R}$ we define the Sobolev space $H^s(\mathbb{T};\mathbb{R})$ as the completion of $\mathcal{C}^\infty(\mathbb{T};\mathbb{R})$ w.r.t. the

$$\|u\|_s := \|u\|_{H^s} = \left(\sum_{j\in\mathbb{Z}} \langle j\rangle^{2s} |u_j|^2\right)^{\frac{1}{2}}, \qquad\qquad \langle j\rangle := \max\left(1, |j|\right).$$

We shall denote with

$$H^s := H^s(\mathbb{T};\mathbb{R}),$$

We define

$$\Pi_0 u := u_0 = \frac{1}{2\pi}\int_{\mathbb{T}} u(x)\,\mathrm{d}x$$

the average of u and

$$\Pi_0^\perp u := (\mathrm{Id} - \Pi_0)\, u = \sum_{j\neq 0} u_j e^{\mathrm{i}jx}. \tag{3.3}$$

We define H_0^s the subspace of zero average functions of H^s. Clearly $H_0^0 = L_0^2$ with scalar product, for any $u, v \in L_0^2$,

$$\langle u \mid v\rangle := \int_{\mathbb{T}} u(x)\, v(x)\,\mathrm{d}x. \tag{3.4}$$

Given an interval $I \subset \mathbb{R}$ symmetric with respect to $t = 0$ and $s \in \mathbb{R}$, we denote with $C\left(I; H_0^s\right)$ the space of time continuous functions with values in H_0^s endowed with the norm

$$\sup_{t\in I} \|u(t,\cdot)\|_s \tag{3.5}$$

We denote $B_s(I;\epsilon_0)$,the ball of radius $\epsilon_0 > 0$ in $C(I, H_0^s)$.

Given a linear operator $R(u)[\cdot]$ acting on L_0^2 we associate the linear operator defined by the relation

$$\overline{R}(u)[v] := \overline{R(u)\left[\overline{v}\right]}, \quad \forall v : \mathbb{T} \to \mathbb{C}. \tag{3.6}$$

An operator $R(u)$ is *real* if $R(u) = \bar{R}(u)$ for any u real.

3.1 Paradifferential calculus

We introduce paradifferential operators (Definition 3.4) following [8], with minor modifications due to the fact that we deal with a scalar equation and not a system, and the fact that we consider operators acting on H_0^s and not on homogenous spaces $\dot{H}^s$. In this way we will mainly rely on results in [8, 14].

3.1.1 Classes of symbols.

Roughly speaking the class $\tilde{\Gamma}^m_1$ contains symbols of order m which are linear with respect to u, whereas the class $\Gamma^m_{\geq 2}$ contains non-homogeneous symbols of order m that vanish at degree at least 2 in u. We denote $C_0^\infty(\mathbb{T};\mathbb{R}) := \bigcap_{s\in\mathbb{R}} H_0^s$ the space of smooth function with zero average.

Definition 3.1 (Symbols)**.** Let $m \in \mathbb{R}$ and $\epsilon_0 > 0$.

i) **Homogeneous symbols.** We denote by $\tilde{\Gamma}^m_1$ the space of linear maps from $C_0^\infty(\mathbb{T};\mathbb{R})$ to the space of C^∞ *complex* valued functions from $\mathbb{T}\times\mathbb{R}$ to $\mathbb{C}$, $(x,\xi)\mapsto a(u;x,\xi)$ of the form

$$a(u;x,\xi) := \sum_{j\in\mathbb{Z}\setminus\{0\}} a_j(\xi) u_j e^{\mathrm{i}jx}, \tag{3.7}$$

where $a_j(\xi)$ are complex valued Fourier multipliers satisfying, for some $\mu \geq 0$,

$$\left|\partial_\xi^\beta a_j(\xi)\right| \leq C_\beta \left|j\right|^\mu \langle\xi\rangle^{m-\beta}, \quad \forall j \in \mathbb{Z}\setminus\{0\}, \beta\in\mathbb{N}. \tag{3.8}$$

We say that an homogeneous symbol is *real-to-real* if

$$\overline{a_j(\xi)} = a_{-j}(-\xi) \tag{3.9}$$

We also denote by $\tilde{\Gamma}^m_0$ the space of constant coefficients symbols $\xi\mapsto a(\xi)$ which satisfy (3.8) with $j=1$ and $\mu = 0$.

ii) **Non-homogeneous symbols.** We denote by $\Gamma^m_{\geq 2}[\epsilon_0]$ the space of functions $(u;x,\xi)\mapsto a(u;x,\xi)$, defined for $u\in B_{s_0}(I;\epsilon_0)$ for some $s_0 > 0$ large enough, with complex values, such that for any $\sigma \geq s_0$, there are $C>0$, $0<\epsilon_0(\sigma)<\epsilon_0$ and for any $u\in B_{s_0}(I;\epsilon_0(\sigma))\cap C\left(I;H_0^\sigma\right)$ and any $\gamma,\beta\in\mathbb{N}_0$ with $\gamma\leq\sigma-s_0$, one has the estimate

$$\left\|\partial_\xi^\beta\partial_x^\gamma a(u;x,\xi)\right\|_{L_x^\infty(\mathbb{T})} \leq C\langle\xi\rangle^{m-\beta}\left\|u\right\|_{s_0}\left\|u\right\|_\sigma, \tag{3.10}$$

We say that a non-homogeneous symbol is *real-to-real* if

$$\overline{a(u;x,\xi)} = a(u;x,-\xi). \tag{3.11}$$

iii) **Symbols.** We denote by $\Gamma^m[\epsilon_0]$ the space of symbols

$$a(u;x,\xi) = a_1(u;x,\xi) + a_{\geq 2}(u;x,\xi)$$

where a_1 is a homogeneous symbol in $\tilde{\Gamma}^m_1$ and $a_{\geq 2}$ is a non-homogeneous symbol in $\Gamma^m_{\geq 2}[\epsilon_0]$. A symbol a is real-to-real if both a_1 and $a_{\geq 2}$ are real-to-real.

Remark 3.2. • The class of homogeneous symbols $\tilde{\Gamma}^m_1$ coincides with the the same class defined in [7,8,14] restricted onto real functions (see Remark 2.2 in [16], while the class of non-homogeneous symbols $\Gamma^m_{\geq 2}[\epsilon_0]$ coincides with the restriction onto real functions of the class $\Gamma^m_{0,0,2}[\epsilon_0]$ of non-homogeneous symbols in [7,8,14].

• If $a(u;x,\xi)$ is a homogeneous symbol in $\tilde{\Gamma}^m_1$ then

$$\left\|\partial_\xi^\beta\partial_x^\gamma a(u;x,\xi)\right\|_{L_x^\infty(\mathbb{T})} \leq C\langle\xi\rangle^{m-\beta}\left\|u\right\|_\sigma, \quad \sigma\geq s_0, \quad \gamma,\beta\in\mathbb{N}_0, \gamma\leq\sigma-s_0 \tag{3.12}$$

where $s_0 > \frac{1}{2}+\mu$.

• If $a(u;x,\xi)$ is a symbol in $\Gamma^m[\epsilon_0]$ then $\partial_x a(u;x,\xi)\in\Gamma^m[\epsilon_0]$ and $\partial_\xi a(u;x,\xi)\in\Gamma^{m-1}[\epsilon_0]$. If in addition $b(u;x,\xi)$ is a symbol in $\Gamma^{m'}[\epsilon_0]$ then $ab\in\Gamma^{m+m'}[\epsilon_0]$.

We also define classes of functions in analogy with our classes of symbols.

Definition 3.3 (Functions)**.** Let $\epsilon_0 > 0$. We denote by $\tilde{\mathcal{F}}_1$, resp. $\mathcal{F}_{\geq 2}[\epsilon_0]$, $\mathcal{F}[\epsilon_0]$, the subspace of $\tilde{\Gamma}^0_1$, resp. $\Gamma^0_{\geq 2}[\epsilon_0]$, $\Gamma^0[\epsilon_0]$, made of those real-to-real symbols which are independent of ξ.

Notice that the space of homogeneous and non-homogeneous functions are always real-valued.

3.1.2 Paradifferential quantization

Fix $0 < \delta \ll 1$ and consider a C^∞, even cut-off function $\chi\colon \mathbb{R} \to [0,1]$ such that

$$\chi(\xi) = \begin{cases} 1 & \text{if } |\xi| \le 1.1 \\ 0 & \text{if } |\xi| \ge 1.9\,, \end{cases} \qquad \chi_\delta(\xi) := \chi\left(\frac{\xi}{\delta}\right). \tag{3.13}$$

If $a(x,\xi)$ is a smooth symbol we define its Weyl quantization as the operator acting on a 2π-periodic function $u(x)$ (written as in (3.2)) as

$$\mathrm{Op}^W(a)u = \frac{1}{2\pi}\sum_{j\in\mathbb{Z}}\left(\sum_{k\in\mathbb{Z}} \hat{a}\left(j-k, \frac{j+k}{2}\right)\hat{u}_k\right)e^{\mathrm{i}jx} \tag{3.14}$$

where $\hat{a}(k,\xi)$ is the k-Fourier coefficient of the $2\pi-$periodic function $x \mapsto a(x,\xi)$.

Definition 3.4 (Bony-Weyl quantization)**.** If a is a symbol in $\Gamma^m[\epsilon_0]$, we set

$$a_\chi(u;x,\xi) := \sum_{j\in\mathbb{Z}} \chi_\delta\left(\frac{j}{\langle 2\xi\rangle}\right)\hat{a}_j(u;\xi)e^{\mathrm{i}jx}, \quad \hat{a}_j(u;\xi) := \frac{1}{2\pi}\int_{\mathbb{T}} a(u;x,\xi)e^{-\mathrm{i}jx}\,\mathrm{d}x.$$

and we define the *Bony-Weyl* quantization of a as

$$\mathrm{Op}^{BW}(a(u;x,\xi)) := \mathrm{Op}^W(a_\chi(u;x,\xi)). \tag{3.15}$$

In view of (3.14), one has

$$\mathrm{Op}^{BW}(a(u;x,\xi))v = \frac{1}{2\pi}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \chi_\delta\left(\frac{j-k}{\langle j+k\rangle}\right)\hat{a}_{j-k}\left(u;\frac{j+k}{2}\right)\hat{v}_k e^{\mathrm{i}jx} \tag{3.16}$$

Moreover if a is a linear symbol in $\widetilde{\Gamma}^m_1$ then, since $\hat{a}_j(u;\xi) = a_j(\xi)\hat{u}_j$ (see (3.7)), the expression (3.16) reduces in this case to

$$\mathrm{Op}^{BW}(a(u;x,\xi))v = \frac{1}{2\pi}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \chi_\delta\left(\frac{j-k}{\langle j+k\rangle}\right)a_{j-k}\left(\frac{j+k}{2}\right)\hat{u}_{j-k}\hat{v}_k e^{\mathrm{i}jx}$$

Remark 3.5. • The operator $\mathrm{Op}^{BW}(a)$ maps functions with zero average in functions with zero average and $\Pi_0^\perp \mathrm{Op}^{BW}[a] = \mathrm{Op}^{BW}[a]\Pi_0^\perp$.

• Definition 3.4 is independent of the cut-off functions χ, up to smoothing operators (Definition 3.9).

• The action of $\mathrm{Op}^{BW}(a)$ on the spaces H_0^s only depends on the values of the symbol $a = a(u;x,\xi)$ (or $a(u;t,x,\xi)$) for $|\xi| \ge 1$. Therefore, we may identify two symbols $a(u;t,x,\xi)$ and $b(u;t,x,\xi)$ if they agree for $|\xi| \ge 1/2$. In particular, whenever we encounter a symbol that is not smooth at $\xi = 0$, such as, for example, $a = g(x)|\xi|^{m-1}\xi$ for $m \in \mathbb{R}\setminus\{0\}$, or $\mathrm{sgn}(\xi)$, we will consider its smoothed out version $\psi(\xi)\,a$, where $\psi \in C^\infty(\mathbb{R};\mathbb{R})$ is an even and positive cut-off function satisfying

$$\psi(\xi) = 0 \text{ if } |\xi| \le \tfrac{1}{8}, \quad \psi(\xi) = 1 \text{ if } |\xi| > \tfrac{1}{4}, \quad \partial_\xi\psi(\xi) > 0 \quad \forall \xi \in \left(\tfrac{1}{8},\tfrac{1}{4}\right). \tag{3.17}$$

• Given a paradifferential operator $A = \mathrm{Op}^{BW}[a(x,\xi)]$ it results

$$\overline{A} = \mathrm{Op}^{BW}\left[\overline{a(x,-\xi)}\right], \quad A^\intercal = \mathrm{Op}^{BW}[a(x,-\xi)], \quad A^* = \mathrm{Op}^{BW}\left[\overline{a(x,\xi)}\right], \tag{3.18}$$

where $A^\intercal$ is the transposed operator with respect to the real scalar product $\langle u,v\rangle_r := \int_{\mathbb{T}} u(x)\,v(x)\,\mathrm{d}x$, and A^* denotes the adjoint operator with respect to the complex scalar product of L^2_0 in (3.4). It results $A^* = \overline{A}^\intercal$.

• A paradifferential is *symmetric* (i.e. $A = A^\intercal$) if $a(x,\xi) = a(x,-\xi)$. A operator $\partial_x\mathrm{Op}^{BW}(a(x,\xi))$ is Hamiltonian if and only if

$$a(x,\xi) \in \mathbb{R} \qquad \text{and} \qquad a(x,\xi) = a(x,-\xi) \quad \text{is even in } \xi. \tag{3.19}$$

We now provide the action of a paradifferential operator on Sobolev spaces, cf. [8, Prop. 3.8], with minor modifications due to the fact that the target spaces are Sobolev spaces of non-homogeneous type.

Proposition 3.6 (Action of a paradifferential operator)**.** *Let $m \in \mathbb{R}$.*

i) Let $a_1(u;x,\xi)\in\tilde\Gamma^m_1$ a real-to-real symbol. There is $s_0>0$ such that for any $s\in\mathbb{R}$, there is a constant $C>0$, depending only on s and on (3.8) *with $\beta=0$, such that, for any $u\in H^{s_0}_0$ and $v\in H^s$, one has*

$$\left\|\mathrm{Op}^{BW}(a_1(u;x,\xi))v\right\|_{s-m}\le C\|u\|_{s_0}\|v\|_s\,. \tag{3.20}$$

ii) Let $\epsilon_0>0$, $a_{\ge2}$ in $\Gamma^m_{\ge2}[\epsilon_0]$ a real-to-real symbol. There is $s_0>0$ such that for any $s\in\mathbb{R}$ there are $C>0$ and $\epsilon'\in(0,\epsilon_0]$ such that, for any u in $B_{s_0}(I;\epsilon')$,

$$\left\|\mathrm{Op}^{BW}(a_{\ge2}(u;\cdot))\right\|_{\mathcal{L}(H^s_0,H^{s-m}_0)}\le C\|u\|^2_{s_0}\,. \tag{3.21}$$

Class of m-Operators. We now define the class of m-Operators.

The class $\tilde{\mathcal{M}}^m_1$ contains linear operators that loose m derivatives and depends linearly with respect to u, while the class $\mathcal{M}^m_{\ge2}$ contains non-homogeneous operators which loose m derivatives, vanish at degree at least 2 in u. The constant μ in (3.23) takes into account possible loss of derivatives in the "low" frequencies which is traced by the parameter s_0 in the estimate (3.25) for non-homogeneous operators.

Definition 3.7 (Classes of m-operators)**.** Let $m\in\mathbb{R}$ and $\epsilon_0>0$.

i) **Homogeneous m- operators.** We denote by $\tilde{\mathcal{M}}^m_1$ the space of translation invariant, real, linear operators of the form

$$M(u)v=\sum_{j,k\in\mathbb{Z}\setminus\{0\}}M_{j,k}\,u_{j-k}v_ke^{\mathrm{i}jx}\,, \tag{3.22}$$

with coefficients $M_{j,k}$ satisfying the following: there are $\mu\ge0$, $C>0$ such that

$$\left|M_{j,k}\right|\le C\min\left\{\left|j-k\right|,|k|\right\}^{\mu}\max\left\{\left|j-k\right|,|k|\right\}^{-\varrho}\,. \tag{3.23}$$

and the reality condition

$$\overline{M_{j,k}}=M_{-j,-k}\,,\qquad\forall\, j,k\in\mathbb{Z}\setminus0\,. \tag{3.24}$$

ii) **Non-homogeneous m-operators.** We denote by $\mathcal{M}^m_{\ge2}[\epsilon_0]$ the space of operators $(u,v)\mapsto M(u)v$ defined on $B_{s_0}(I;\epsilon_0)\times C\left(I;H^{s_0}_0\right)$ for some $s_0>0$, which are linear in the variable v and such that the following holds true. For any $s\ge s_0$ there are $C>0$ and $\epsilon_0(s)\in]0,\epsilon_0[$ such that for any $u\in B_{s_0}(I;\epsilon_0(s))\cap C\left(I;H^s_0\right)$, any $v\in C\left(I;H^s_0\right)$, we have that

$$\|M(u)v\|_{s-m}\le C\left(\|u\|^2_{s_0}\|v\|_s+\|u\|_s\|u\|_{s_0}\|v\|_{s_0}\right)\,, \tag{3.25}$$

and we require additionally the *reality condition* $M(u)\,v=\bar M(u)\,v$ (cf. (3.6)).

iii) **m-Operators.** We denote by $\mathcal{M}^m[\epsilon_0]$, the space of operators

$$M(u)v=M_1(u)v+M_{\ge2}(u)v\,. \tag{3.26}$$

where M_1 is a homogeneous m-operator in $\tilde{\mathcal{M}}^m_1$, and $M_{\ge2}$ is a non–homogeneous m-operator in $\mathcal{M}^m_{\ge2}[\epsilon_0]$.

Remark 3.8. Definition 3.7 of m- operator is an adaptation of the one in [14]. Indeed, since the SQG sharp front equation (2.3) is real, scalar and average preserving, the two definitions coincide. In particular, setting $\tilde M(u):=\Pi_0^\perp M(\Pi_0^\perp u)\Pi_0^\perp$, one can apply the theory developed in [14] without any changes.

Definition 3.9 (Smoothing operators)**.** Let $\varrho\ge0$. A $(-\varrho)$-operator $R(u)$ belonging to $\mathcal{M}^{-\rho}[\epsilon_0]$ is called a smoothing operator. We also denote

$$\tilde{\mathcal{R}}^{-\rho}_p:=\tilde{\mathcal{M}}^{-\rho}_p\,,\qquad\qquad \mathcal{R}^{-\rho}_{\ge2}[\epsilon_0]:=\mathcal{M}^{-\rho}_{\ge2}[\epsilon_0]\,,\qquad\qquad \mathcal{R}^{-\rho}[\epsilon_0]:=\mathcal{M}^{-\rho}[\epsilon_0]\,.$$

Remark 3.10. • The class of homogeneous smoothing operators $\tilde{\mathcal{R}}^{-\varrho}_1$ coincides with the class $\tilde{\mathcal{R}}^{-\varrho}_1$ defined in [7, 8, 14] (setting $p=0,1$), while the class of non-homogeneous smoothing operators $\mathcal{R}^{-\varrho}_{\ge2}[\epsilon_0]$ coincides with the class $\mathcal{R}^{-\varrho}_{0,0,2}[\epsilon_0]$ of non-homogeneous smoothing operators in [7, 8, 14].

• Proposition 3.6 implies that, if $a(u;x;\xi)$ is in $\Gamma^m[\epsilon_0]$ real-to-real, for some $m\in\mathbb{R}$, then $\mathrm{Op}^{BW}(a(u;x;\xi))$ defines a map in $\mathcal{M}^m[\epsilon_0]$.

• If $R(u)$ is a homogeneous smoothing remainder in $\tilde{\mathcal{R}}_1^{-\varrho}$ then

$$\|R(u)v\|_{s+\varrho}\lesssim_s \|u\|_s\|v\|_{s_0}+\|u\|_{s_0}\|v\|_s, \tag{3.27}$$

for any $u,v\in H_0^s$.

• The combination of (3.25) and (3.27) implies that the composition of smoothing operators $R_1\in\mathcal{R}^{-\varrho}[\epsilon_0]$ and $R_2\in\mathcal{R}^{-\varrho}[\epsilon_0]$ is a smoothing operator R_1R_2 in $\mathcal{R}^{-\varrho}_{\geq 2}[\epsilon_0]$.

3.1.3 Symbolic calculus.

The following result is proved in Proposition 3.12 in [8].

Proposition 3.11 (Composition of Bony-Weyl operators)**.** *Let $\varrho\geq 0$, $m,m'\in\mathbb{R}$, $\epsilon_0>0$. Consider a real-to-real symbols $a(u;x,\xi)\in\Gamma^m[\epsilon_0]$, $b(u;x,\xi)\in\Gamma^{m'}[\epsilon_0]$ and $\mathtt{m}(\xi)\in\tilde{\Gamma}_0^{m'}$. Then*

$$[\![\mathrm{Op}^{BW}[a(u;x,\xi)],\mathrm{Op}^{BW}[\mathtt{m}(\xi)]]\!]=\frac{1}{\mathrm{i}}\mathrm{Op}^{BW}\left[-\partial_\xi \mathtt{m}(\xi)\partial_x a(u;x,\xi)\right]+\mathrm{Op}^{BW}[r(u;x,\xi)]+R[u], \tag{3.28}$$

$$[\![\mathrm{Op}^{BW}[a(u;x,\xi)],\mathrm{Op}^{BW}[b(u;x,\xi)]]\!]=\frac{1}{\mathrm{i}}\mathrm{Op}^{BW}\left[\{a,\ b\}(u;x,\xi)\right]+\mathrm{Op}^{BW}[r_{\geq 2}(u;x,\xi)]+R_{\geq 2}[u], \tag{3.29}$$

where:

- *$\{a,\ b\}$ is the Poisson bracket defined as*

$$\{a,\ b\}(u;x,\xi):=\partial_\xi a(u;x,\xi)\,\partial_x b(u;x,\xi)-\partial_x a(u;x,\xi)\,\partial_\xi b(u;x,\xi)\in\Gamma^{m+m'-1}_{\geq 2}[\epsilon_0]; \tag{3.30}$$

- *$r(u;x,\xi)$ is a real-to-real symbol in $\Gamma^{m+m'-3}[\epsilon_0]$ while $r_{\geq 2}(u;x,\xi)$ is a real-to-real non-homogeneous symbol in $\Gamma^{m+m'-3}_{\geq 2}[\epsilon_0]$;*

- *$R[u]$ is a smoothing remainder in $\mathcal{R}^{-\varrho+m+m'}[\epsilon_0]$ while $R_{\geq 2}[u]$ is a non-homogeneous smoothing remainder in $\mathcal{R}^{-\varrho+m+m'}_{\geq 2}[\epsilon_0]$.*

Moreover if $a\in\Gamma^m_{\geq 2}[\epsilon_0]$ then the symbol $r(u;x,\xi)$ and the smoothing remainder $R[u]$ in (3.28) *are respectively in $\Gamma^{m+m'-3}_{\geq 2}[\epsilon_0]$ and $\mathcal{R}^{-\varrho+m+m'}_{\geq 2}[\epsilon_0]$.*

Remark 3.12. In our application the Fourier multiplier $\mathtt{m}(\xi)$ shall be the dispersion relation which is in in $\tilde{\Gamma}_0^\alpha$ with $\alpha\in(0,1)$. In this case the symbol $-\partial_\xi \mathtt{m}(\xi)\partial_x a(u;x,\xi)$ in (3.28), which belongs to $\Gamma^{m-(1-\alpha)}[\epsilon_0]$, becomes lower order with respect to $a(u;x,\xi)$.

The following lemma, which is a consequence of Proposition 2.15 (items (ii) and (iv)) in [14], shall be use below.

Lemma 3.13. *Let $m,m',m_0\in\mathbb{R}$, $\varrho\geq 0$, $\epsilon_0>0$, $M(u)$ be a m-operator in $\mathcal{M}^m[\epsilon_0]$ and $\mathrm{p}(\xi)$ a symbol in $\tilde{\Gamma}_0^{m_0}$. Then:*

1. *If $c(u)$ is a homogeneous symbol in $\tilde{\Gamma}_1^{m'}$,*

$$b_1(u):=c(-\mathrm{ip}(D)u;x,\xi)\quad\textit{and}\quad b_{\geq 2}(u;x,\xi):=c(M(u)u;x,\xi)$$

are symbols respectively in $\tilde{\Gamma}_1^{m'}$ and $\Gamma^{m'}_{\geq 2}[\epsilon_0]$;

2. *If $Q(u)$ is a homogeneous smoothing operator in $\tilde{\mathcal{R}}_1^{-\varrho}$,*

$$\tilde{R}_1(u):=Q(-\mathrm{ip}(D)u)\quad\textit{and}\quad R_{\geq 2}(u):=Q(M(u)u)$$

are smoothing operators respectively in $\tilde{\mathcal{R}}_1^{-\varrho+\max\{0,m_0\}}$ and $\mathcal{R}^{-\varrho+\max\{0,m\}}_{\geq 2}[\epsilon_0]$;

3. *If $R(u)\in\mathcal{R}^{-\varrho}_{\geq 2}[\epsilon_0]$ and $\mathsf{a}(u;x,\xi)\in\Gamma^m_{\geq 2}[\epsilon_0]$, $\varrho>m$ then*

$$R(u)\circ\mathrm{Op}^{BW}[\mathsf{a}(u;x,\xi)]\in\mathcal{R}^{-\varrho+m}_{\geq 2}[\epsilon_0],\qquad \mathrm{Op}^{BW}[\mathsf{a}(u;x,\xi)]\circ R(u)\in\mathcal{R}^{-\varrho+m}_{\geq 2}[\epsilon_0].$$

4 Para-differential reduction

Notation 4.1. From now on we denote with $r(f;x,\xi) = r_1(f;x,\xi) + r_{\geq 2}(f;x,\xi)$ any symbol in the space $\Gamma^0[\epsilon_0]$ with $r_1(f;x,\xi) \in \tilde{\Gamma}^0_1$ and $r_{\geq 2}(f;x,\xi) \in \Gamma^0_{\geq 2}[\epsilon_0]$ such that

$$\bar{r}(f;x,\xi) = -r(f;x,-\xi). \tag{4.1}$$

The explicit expression of $r(f;x,\xi)$ may vary from line to line.

The symbol $r(f;x,\xi)$ defined in Notation 4.1 is such that the operator $\mathrm{iOp}^{BW}[r(f;x,\xi)]$ is real-to-real.

In this section we para-linearize the equation of motion and we reduce it to a cubic equation up to a (already quadratic) smoothing remainder.

Proposition 4.2 (Paralinearization of the α-SQG patch equation)**.** *Let $\alpha \in (0,1) \cup (1,2)$, $\Omega \in \mathbb{R}$, $\varrho > 0$. There is $s_0 > 0$ and $\epsilon_0 > 0$ such that for $f \in B_{s_0}(I;\epsilon_0)$ the evolution equation* (2.5) *has the form*

$$f_t + \partial_x \circ \mathrm{Op}^{BW}\left[(1+\nu(f;x))L_\alpha(\xi) + \Omega + V(f;x) + P(f;x,\xi)\right] f = R(f)f. \tag{4.2}$$

where:

i) *$L_\alpha(\xi)$ is the real-to-real Fourier multiplier in $\tilde{\Gamma}^{\max\{0,\alpha-1\}}_0$ (cf. Definition 2.3), defined in Equation* (2.11)*;*

ii) *$\nu(f;x)$ and $V(f;x)$ are real functions in $\mathcal{F}[\epsilon_0]$;*

iii) *$P(f;x,\xi)$ is a real-to-real symbol in $\Gamma^{-1}[\epsilon_0]$ (see Definition 3.1),*

iv) *R is a smoothing operator in $\mathcal{R}^{-\varrho}[\epsilon_0]$ (see Definition 3.7).*

In particular , setting $\Omega = -\mathbb{V}_\alpha$ and noticing that $\nu(f;x) \in \mathcal{F}[\epsilon_0]$, Notation 4.1 and after a harmless relabeling of $V(f;x)$ we rewrite Eq. (4.2) *as*

$$f_t + \mathrm{i}\,\mathrm{Op}^{BW}\left[V(f;x)\,\xi + (1+\nu(f;x))\,\dot{\omega}_\alpha(\xi) + r(f;x,\xi)\right] f = R(f)f. \tag{4.3}$$

Remark 4.3. In view of the second bullet in Remark 3.10, the equation 4.3 can be written as

$$f_t = X(f) = -\mathrm{i}\dot{\omega}(D)f + M_{\mathtt{SQG}}(f)f, \tag{4.4}$$

where $M_{\mathtt{SQG}}(f)$ is a real 1-operator in $\mathcal{M}^1[\epsilon_0]$.

4.1 Paradifferential quadratic reduction

In the present section we suppress the quadratic components of the paradifferential term

$$\mathrm{Op}^{BW}\left[(1+\nu(f;x))\,\dot{\omega}_\alpha(\xi) + V(f;x)\,\xi + r(f;x,\xi)\right], \tag{4.5}$$

appearing in Equation (4.3). We will perform a reduction in decreasing order (up to order $-\varrho$) in the linear part ($O(f)$) of the equation (4.3). All said and done we prove the following result:

Proposition 4.4. *There is $\varrho_0 := \varrho_0(\alpha) > 0$ such that for any $\varrho > \varrho_0$ there are $s_0, \epsilon_0 > 0$ such that for any solution $f \in B_{s_0}(I;\epsilon_0)$ of* (4.3)*, there exists a real-to-real invertible linear map $\mathbf{B}(f)$ such that the following holds true:*
(i) **Boundedness:** *$\mathbf{B}(f)$ and $\mathbf{B}(f)^{-1}$ are bounded in Sobolev spaces, namely: for any $s \in \mathbb{R}$ there is $\epsilon'_0 := \epsilon'_0(s) \in (0,\epsilon_0]$ such that for any $f \in B_{s_0}(I;\epsilon'_0)$ and $\zeta \in H^s_0$*

$$\left\|\mathbf{B}(f)\zeta\right\|_s + \left\|\mathbf{B}(f)^{-1}\zeta\right\|_s \lesssim \|\zeta\|_s \tag{4.6}$$

(ii) **Conjugation:** *If f solves* (4.3) *then $z := \mathbf{B}(f)f$ solves*

$$\partial_t z = -\mathrm{iOp}^{BW}\left[\dot{\omega}_\alpha(\xi) + q_{\geq 2}(f;x,\xi) + r_{\geq 2}(f;x,\xi)\right] z + R(f)z \tag{4.7}$$

where:

- $\dot{\omega}_\alpha(\xi) \in \widetilde{\Gamma}_0^\alpha$ *is the Fourier multiplier defined in Equation* (2.13)*;*
- $q_{\geq 2}(f;x,\xi)$ *is a non-homogeneous real, odd in* ξ*, symbol in* $\Gamma^1_{\geq 2}[\epsilon_0]$*;*
- $r_{\geq 2}(f;x)$ *is a non-homogeneous symbol in* $\Gamma^0_{\geq 2}[\epsilon_0]$ *satisfying* (4.1)*;*
- $R(f)$ *is a real, smoothing operator in* $\mathcal{R}^{-\varrho+\varrho_0}[\epsilon_0]$.

Moreover the operator $\mathrm{i}\,\mathrm{Op}^{BW}\left[\dot{\omega}_\alpha(\xi) + q_{\geq 2}(f;x,\xi) + r_{\geq 2}(f;x,\xi)\right]$ *is real-to-real.*

Notation 4.5. We denote with $q_{\geq 2}(f;x,\xi)$ any real-valued, odd in ξ, symbol in the space $\Gamma^1_{\geq 2}[\epsilon_0]$. The explicit values of $q_{\geq 2}$ may implicitly vary from line to line.

The symbol $q_{\geq 2}(f;x,\xi)$ defined in Notation 4.1 is such that the operator $\mathrm{iOp}^{BW}\left[q_{\geq 2}(f;x,\xi)\right]$ is real-to-real and L^2-energy neutral.

The rest of the section is devoted to the proof of Proposition 4.4.

The map linear map $\mathbf{B}(f)$ has the form

$$\mathbf{B}(f) := \Phi(f)^{(N)} \circ \Phi(f)^{(N-1)} \circ \cdots \circ \Phi^{(0)}(f), \quad N := N(\varrho,\alpha) > 0$$

for suitable transformations $\Phi(f)^{(j)}$, $j = 0,\ldots,N$ obtained iteratively as flows of paradifferential PDEs as in Equation (A.4).

Step 1**.** (Reduction of the linear transport term)

We first construct $\Phi^{(0)}(f)$ to eliminate the linear component in the transport symbol

$$V(f;x)\xi = V_1(f;x)\xi + V_{\geq 2}(f;x)\xi \tag{4.8}$$

in Equation (4.3). To do so we look for a transformation of the form $\Phi^{(0)}(f) := \Phi^\tau_g(f)_{|\tau=1}$ where $\Phi^\tau_g(f)$ is the unique solution of Eq. (A.4) and generating symbol g is defined as in Eq. (A.1).

First of all, thanks to (A.5), the map $\Phi^{(0)}(f)$ satisfies the estimate (4.6). Then, the variable

$$w_0 := \Phi^{(0)}(f)f, \tag{4.9}$$

solves (cf. Eq. (4.2))

$$\begin{aligned}
\partial_t w_0 = &-\mathrm{i}\Phi^{(0)}(f)\ \mathrm{Op}^{BW}\left[V(f;x)\ \xi + \left(1+\nu(f;x)\right)\dot{\omega}_\alpha(\xi) + r(f;x,\xi)\right]\Phi^{(0)}(f)^{-1}\ w_0 && \text{(4.10a)}\\
&+\partial_t\Phi^{(0)}(f)\,\Phi^{(0)}(f)^{-1}\,w_0 && \text{(4.10b)}\\
&+\Phi^{(0)}(f)\,R(f)\,\Phi^{(0)}(f)^{-1}\ w_0. && \text{(4.10c)}
\end{aligned}$$

We now compute each term in Eqs. (4.10a) to (4.10c).

We apply Proposition A.2, Item i and obtain that

$$\text{(4.10a)} = -\mathrm{iOp}^{BW}\left[V_1(f;x)\ \xi + \left(1+\nu_1^{(0)}(f;x)\right)\dot{\omega}_\alpha(\xi) + q_{\geq 2}(f;x,\xi) + r(f;x,\xi)\right]w_0 + R(f)\,w_0, \tag{4.11}$$

where V_1 is the linear component of V (see (4.8)), the function $\nu_1^{(0)} \in \tilde{\mathcal{F}}_1^{\mathbb{R}}$ and $q_{\geq 2}$ is the quasilinear symbol of Notation 4.5. We apply Proposition A.2, (iii) and (ii) and obtain

$$\begin{aligned}
&\text{(4.10b)} = \mathrm{i}\ \mathrm{Op}^{BW}\left[\beta\left(-\mathrm{i}\,\dot{\omega}_\alpha(D)f;x\right)\ \xi +\ q_{\geq 2}(f,x,\xi)\right]w_0 + R(f)\,w_0,\\
&\text{(4.10c)} = R(f)\,w_0,
\end{aligned}$$

where we denote with $R(f)$ a smoothing remainder in $\mathcal{R}^{-\varrho+1}[\epsilon_0]$ which can change from line to line. Equation (4.10) is thus transformed into

$$\partial_t w_0 = -\mathrm{i}\ \mathrm{Op}^{BW}\left[\left(V_1(f;x) - \beta\left(-\mathrm{i}\dot{\omega}_\alpha(D)\,f;x\right)\right)\xi + \left(1+\nu_1^{(0)}(f;x)\right)\dot{\omega}_\alpha(\xi) + q_{\geq 2}(f;x,\xi) + r(f;x,\xi)\right]w_0$$

$$+R(f)\,w_0.$$

In Fourier, we have the homological equation

$$0 = V_1(f;x) - \beta\left(-\mathrm{i}\dot\omega_\alpha(D) f;x\right) = \sum_{j\in\mathbb{Z}\setminus\{0\}} \left((V_1)_j + \mathrm{i}\,\dot\omega_\alpha(j)\,\beta_j\right) f_j\, e^{\mathrm{i}jx},$$

so that, thanks to the non-degeneracy property (2.14), we can define

$$\beta_j := -\frac{(V_1)_j}{\mathrm{i}\,\dot\omega_\alpha(j)}. \tag{4.12}$$

Notice that since $\xi \mapsto \dot\omega_\alpha(\xi)$ is odd and V_1 is real valued the Fourier coefficients in Eq. (4.12) satisfy the reality condition $\overline{\beta_j} = \beta_{-j}$, hence we have that

$$\beta(f;x) := -\sum_{j\in\mathbb{Z}\setminus\{0\}} \frac{(V_1)_j}{\mathrm{i}\,\dot\omega_\alpha(j)}\, f_j\, e^{\mathrm{i}jx} \in \tilde{\mathcal{F}}_1. \tag{4.13}$$

Finally, with the choice (4.13), the equation for w_0 becomes

$$\partial_t w_0 = -\mathrm{i}\,\mathrm{Op}^{BW}\left[\left(1+v_1^{(0)}(f;x)\right)\dot\omega_\alpha(\xi) + q_{\geq 2}(f;x,\xi) + r(f;x,\xi)\right] w_0 + R(f)\,w_0. \tag{4.14}$$

Step 2**.** (Reduction of the linear symbols of positive order) Next we find a transformation in order to eliminate all the linear symbols of positive order in (4.14). To do so we shall prove the following inductive Lemma:

Lemma 4.6. *For any $n \geq 0$ there are $n+1$ real, bounded and invertible transformations*

$$\Phi^{(0)}(f),\dots,\Phi^{(n)}(f)$$

such that:

1. **Boundedness:** *For any $j = 0,\dots,n$ the map $\Phi^{(j)}(f)$ satisfies the bound* (4.6)*;*
2. **Conjugation:** *If f solves* (4.3) *then*

$$w_n := \Phi^{(n)}(f)\circ\cdots\circ\Phi^{(0)}(f)\,f \tag{4.15}$$

solves

$$\partial_t w_n = -\mathrm{i}\,\mathrm{Op}^{BW}\left[\dot\omega_\alpha(\xi) + b_1^{(n)}(f;x,\xi) + q_{\geq 2}^{(n)}(f;x,\xi) + r(f;x,\xi)\right] w_n + R(f)\,w_n. \tag{4.16}$$

where:

- $\dot\omega_\alpha(\xi) \in \widetilde{\Gamma}_0^\alpha$ *is the Fourier multiplier defined in Equation* (2.13)*;*
- $b_1^{(n)}$ *is a homogeneous real symbol in $\widetilde{\Gamma}_1^{\alpha - n\ell}$ odd in ξ;*
- $q_{\geq 2}^{(n)}$ *is a non-homogeneous real symbol as in Notation 4.5;*
- $r(f;x)$ *is a symbol as in Notation 4.1;*
- $R(f)$ *is a real, smoothing operator in $\mathcal{R}^{-\varrho+\varrho_0}[\epsilon_0]$ with $\varrho_0 = \varrho_0(n) := n+1$.*

Proof. **Initialization case:** $n = 0$**.** For $n = 0$ the thesis is exactly the conclusion of Step 1 choosing the map $\Phi(f)^{(0)}$ as in (4.9) and denoting the symbol

$$b_1^{(0)}(f;x,\xi) := v_1^{(0)}(f;x)\,\dot\omega(\xi) \in \tilde\Gamma_1^\alpha.$$

Notice that $b_1^{(0)}(f;x,\xi)$ is real valued and odd in ξ, so that $\mathrm{iOp}^{BW}\left[b_1^{(0)}(f;x,\xi)\right]$ is real-to-real.

Inductive step: We look for a transformation $\Phi^{(n+1)}(f)$ to eliminate the linear symbol $b_1^{(n)}$ in (4.16) up to lower order terms. We define $\Phi^{(n+1)}(f) := \Phi_g^\tau(f)|_{\tau=1}$ the time-1 flow of

$$\partial_\tau \Phi_g^\tau(f) = \mathrm{iOp}^{BW}\left[g(f;x,\xi)\right]\Phi_g^\tau(f), \quad \Phi_g^0(f) = \mathrm{Id},$$

where $g(f;x,\xi)$ is the real, odd in ξ, symbol in $\tilde{\Gamma}_1^{\alpha-n\ell}$ defined by

$$g(f;x,\xi) := b_1^{(n)}\left(\mathrm{i}\dot{\omega}_\alpha^{-1}(D)f;x,\xi\right). \tag{4.17}$$

First of all, thanks to (A.5), the map $\Phi^{(n+1)}(f)$ satisfies the estimate (4.6). Then, the variable

$$w_{n+1} := \Phi^{(n+1)}(f)\, w_n, \tag{4.18}$$

solves (cf. (4.16))

$$\partial_t w_{n+1} = -\mathrm{i}\Phi^{(n+1)}(f)\ \mathrm{Op}^{BW}\left[\dot{\omega}_\alpha(\xi) + b_1^{(n)}(f;x,\xi) + q_{\geq 2}^{(n)}(f;x,\xi) + r(f;x,\xi)\right]\Phi^{(n+1)}(f)^{-1}\ w_{n+1} \tag{4.19a}$$

$$+\partial_t\Phi^{(n+1)}(f)\,\Phi^{(n+1)}(f)^{-1}\, w_{n+1} \tag{4.19b}$$

$$+\Phi^{(n+1)}(f)\,R(f)\,\Phi^{(n+1)}(f)^{-1}\ w_{n+1}. \tag{4.19c}$$

We now compute each term in Eqs. (4.19a) to (4.19c).
We apply Proposition A.3, Item i and obtain that

$$(4.19a) = -\mathrm{i}\mathrm{Op}^{BW}\left[\dot{\omega}_\alpha(\xi) + b_1^{(n)}(f;x,\xi) + b_1^{(n+1)}(f;x,\xi) + \tilde{q}_{\geq 2}^{(n,1)}(f;x,\xi) + r(f;x,\xi)\right] w_{n+1} + R(f)\, w_{n+1}, \tag{4.20}$$

where $\tilde{q}_{\geq 2}^{(n,1)}(f;x,\xi) \in \Gamma^1_{\geq 2}[\epsilon_0]$, $b_1^{(n+1)}(f;x,\xi) \in \tilde{\Gamma}_1^{\alpha-\ell(n+1)}$ are real valued and odd in ξ, $R(f) \in \mathcal{R}^{-\varrho}[\epsilon_0]$. We apply Proposition A.3, (iii) and obtain

$$(4.19b) = \mathrm{i}\mathrm{Op}^{BW}\left[g\left(-\mathrm{i}\dot{\omega}_\alpha(D)f;x,\xi\right) + \tilde{q}_{\geq 2}^{(n,2)}(f;x,\xi)\right] w_{n+1} + R(f)\, w_{n+1} \tag{4.21}$$

where $\tilde{q}_{\geq 2}^{(n,2)} \in \Gamma^1_{\geq 2}[\epsilon_0]$, real and odd in ξ, and $R(f) \in \mathcal{R}^{-\varrho}[\epsilon_0]$. Finally, by Proposition A.3 (ii), we get

$$(4.19c) = R(f)\, w_0 \tag{4.22}$$

where $R(f) \in \mathcal{R}^{-\varrho+\varrho_0+1}[\epsilon_0]$. Note that the order of the smoothing operator above is $-\varrho+\varrho_0(n)+1 = -\varrho+\varrho_0(n+1)$. Defining

$$q_{\geq 2}^{(n+1)}(f;x,\xi) := \tilde{q}_{\geq 2}^{(n,1)}(f;x,\xi) + \tilde{q}_{\geq 2}^{(n,2)}(f;x,\xi)$$

and, summarizing all the contribution in (4.20), (4.21), (4.22) and noting that

$$b_1^{(n)}(f;x,\xi) - g\left(-\mathrm{i}\dot{\omega}_\alpha(D)f;x,\xi\right) = 0,$$

we get

$$\partial_t w_{n+1} = -\mathrm{i}\ \mathrm{Op}^{BW}\left[\dot{\omega}_\alpha(\xi) + b_1^{(n+1)}(f;x,\xi) + q_{\geq 2}^{(n+1)}(f;x,\xi) + r(f;x,\xi)\right] w_{n+1} + R(f)\, w_{n+1} \tag{4.23}$$

as claimed in (4.16). □

Choice of the number of iterative steps: We choose the number n_α of iterative steps such that

$$n_\alpha \geq \frac{\alpha}{1-\alpha}.$$

In this way we can include the symbol $b_1^{(n_\alpha)} \in \tilde{\Gamma}_1^{\alpha-\ell n_\alpha}$ in (4.16) in the symbol r since $\alpha - \ell n_\alpha \leq 0$. Relabeling

$$v := w_{n_\alpha} \tag{4.24}$$

the equation for v becomes

$$\partial_t v = -\mathrm{i}\ \mathrm{Op}^{BW}\left[\dot{\omega}_\alpha(\xi) + q_{\geq 2}(f;x,\xi) + r(f;x,\xi)\right] v + R(f)\, v, \tag{4.25}$$

with

$$q_{\geq 2} := q_{\geq 2}^{(n_\alpha)} \tag{4.26}$$

as in Notation 4.5, r as in Notation 4.1 and $R(f) \in \mathcal{R}^{-\varrho+\underline{\varrho}_0}[\epsilon_0]$ where

$$\underline{\varrho_0} := \varrho_0(n_\alpha) = n_\alpha + 1 \tag{4.27}$$

depends only on α. To prove Proposition 4.4 it is now sufficient to eliminate the linear component, $r_1(f;x,\xi)$, of the symbol

$$r(f;x,\xi) = r_1(f;x,\xi) + r_{\geq 2}(f;x,\xi).$$

This is the content of the following iterative Lemma:

Lemma 4.7 (Reduction of the linear symbol)**.** *For any $n \geq 0$ there are $n+1$ bounded and invertible transformations*

$$\Psi^{(0)}\left(f\right), \ldots, \Psi^{(n)}\left(f\right)$$

such that

1. **Boundedness:** *For any $j = 0, \ldots, n$ the map $\Psi^{(j)}\left(f\right)$ satisfies the bound* (4.6)*;*
2. **Conjugation:** *If f solves* (4.3) *and v is the variable in* (4.24) *then*

$$v_n := \Psi^{(n)}\left(f\right) \circ \cdots \circ \Psi^{(0)}\left(f\right) v \tag{4.28}$$

solves

$$\partial_t v_n = -\mathrm{i}\,\mathrm{Op}^{BW}\left[\dot{\omega}_\alpha\left(\xi\right) + q_{\geq 2}\left(f; x, \xi\right) + r_1^{(n)}\left(f; x, \xi\right) + r_{\geq 2}^{(n)}\left(f; x, \xi\right)\right] v_n + R\left(f\right) v_n \tag{4.29}$$

where:

- $\dot{\omega}_\alpha\left(\xi\right) \in \widetilde{\Gamma}_0^\alpha$ *is the Fourier multiplier defined in* (2.13)*;*
- $r_1^{(n)}$ *is a homogeneous symbol in* $\tilde{\Gamma}_1^{-n\ell}$ *satisfying* (4.1) *where we denote with*

$$\ell := 1 - \alpha > 0; \tag{4.30}$$

- $q_{\geq 2}$ *is a symbol as in Notation 4.5;*
- $r_{\geq 2}\left(f; x, \xi\right)$ *is a non-homogeneous symbol in* $\Gamma_{\geq 2}^0\left[\epsilon_0\right]$ *satisfying* (4.1)*;*
- $R\left(f\right)$ *is a smoothing operator in* $\mathcal{R}^{-\varrho+\underline{\varrho_0}}[\epsilon_0]$ *with* $\underline{\varrho_0}$ *given in* (4.27).

Proof. **Initialization case: $n = 0$.** It is sufficient to choose $\Psi\left(f\right)^{(0)} = \mathrm{Id}$ since equation (4.25) already has the claimed form (4.37).

Inductive step: We look for a transformation $\Psi^{(n+1)}\left(f\right)$ to eliminate the linear symbol $r_1^{(n)}$ in (4.37) up to lower order terms. We define $\Psi^{(n+1)}\left(f\right) := \Phi_g^\tau\left(f\right)|_{\tau=1}$ the time-1 flow of

$$\partial_\tau \Phi_g^\tau\left(f\right) = \mathrm{Op}^{BW}\left[g\left(f; x, \xi\right)\right] \Phi_g^\tau\left(f\right), \quad \Phi_g^0\left(f\right) = \mathrm{Id},$$

where

$$g\left(f; x, \xi\right) := r_1^{(n)}\left(\mathrm{i}\dot{\omega}_\alpha^{-1}(D) f; x, \xi\right) \tag{4.31}$$

is a symbol in $\tilde{\Gamma}_1^{-n\ell}$ thanks to Lemma 3.13. Then, by (A.5), the map $\Psi^{(n+1)}\left(f\right)$ satisfies the estimate (4.6). Then, the variable

$$v_{n+1} := \Psi^{(n+1)}\left(f\right) v_n, \tag{4.32}$$

solves (cf. (4.37))

$$\partial_t v_{n+1} = -\mathrm{i}\Phi^{(n+1)}\left(f\right)\ \mathrm{Op}^{BW}\left[\dot{\omega}_\alpha\left(\xi\right) + q_{\geq 2}\left(f; x, \xi\right) + r_1^{(n)}\left(f; x, \xi\right) + r_{\geq 2}^{(n)}\left(f; x, \xi\right)\right] \Phi^{(n+1)}\left(f\right)^{-1}\ v_{n+1} \tag{4.33a}$$

$$+ \partial_t \Phi^{(n+1)}\left(f\right) \Phi^{(n+1)}\left(f\right)^{-1} v_{n+1} \tag{4.33b}$$

$$+ \Phi^{(n+1)}\left(f\right) R\left(f\right) \Phi^{(n+1)}\left(f\right)^{-1}\ v_{n+1}. \tag{4.33c}$$

We now compute each term in Eqs. (4.33a) to (4.33c).

We apply Proposition A.3, Item i and obtain that

$$(4.19\text{a}) = -\mathrm{i}\mathrm{Op}^{BW}\left[\dot{\omega}_\alpha\left(\xi\right) + q_{\geq 2}\left(f; x, \xi\right) + r_1^{(n)}\left(f; x, \xi\right) + r_1^{(n+1)}\left(f; x, \xi\right) + \tilde{r}_{\geq 2}^{(n,1)}\left(f; x, \xi\right)\right] v_{n+1} + R\left(f\right) v_{n+1}, \tag{4.34}$$

where $\tilde{r}_{\geq 2}^{(n,1)}\left(f; x, \xi\right) \in \Gamma_{\geq 2}^0[\epsilon_0]$, $r_1^{(n+1)}\left(f; x, \xi\right) \in \tilde{\Gamma}_1^{-\ell(n+1)}$ satisfying (4.1) and $R\left(f\right) \in \mathcal{R}^{-\varrho}[\epsilon_0]$. We apply Proposition A.3, (iii) and obtain

$$(4.33\text{b}) = \mathrm{i}\mathrm{Op}^{BW}\left[g\left(-\mathrm{i}\dot{\omega}_\alpha(D) f; x, \xi\right) + \tilde{r}_{\geq 2}^{(n,2)}\left(f; x, \xi\right)\right] v_{n+1} + R\left(f\right) v_{n+1} \tag{4.35}$$

where $\tilde{r}^{(n,2)}_{\geq 2} \in \Gamma^0_{\geq 2}[\epsilon_0]$ and $R\left(f\right) \in \mathcal{R}^{-\varrho}[\epsilon_0]$. Finally, by Proposition A.3, (ii), we get

$$\text{(4.19c)} = R\left(f\right) w_0 \tag{4.36}$$

where $R\left(f\right) \in \mathcal{R}^{-\varrho+\underline{\varrho_0}}[\epsilon_0]$.

Note that the order of the smoothing operators remains unchanged.

Defining

$$r^{(n+1)}_{\geq 2}\left(f;x,\xi\right) := \tilde{r}^{(n,1)}_{\geq 2}\left(f;x,\xi\right) + \tilde{r}^{(n,2)}_{\geq 2}\left(f;x,\xi\right)$$

and, summarizing all the contribution in (4.34), (4.35), (4.36) and noting that

$$r^{(n)}_1\left(f;x,\xi\right) - g\left(-\mathrm{i}\dot{\omega}_\alpha(D)f;x,\xi\right) = 0,$$

we get

$$\partial_t v_{n+1} = -\mathrm{i}\,\mathrm{Op}^{BW}\left[\dot{\omega}_\alpha\left(\xi\right) + b^{(n+1)}_1\left(f;x,\xi\right) + q^{(n+1)}_{\geq 2}\left(f;x,\xi\right) + r\left(f;x,\xi\right)\right] v_{n+1} + R\left(f\right) v_{n+1} \tag{4.37}$$

as claimed in (4.16). □

Choice of the number of iterative steps: We choose the number $\bar{n}_{\alpha,\varrho}$ of iterative steps such that

$$\bar{n}_{\alpha,\varrho} \geq \frac{\varrho}{1-\alpha}.$$

In this way we can include the operator $\mathrm{Op}^{BW}\left[r^{(\bar{n}_{\alpha,\varrho})}_1\right] \in \tilde{\mathcal{R}}^{-\ell\bar{n}_{\alpha,\varrho}}_1$ in (4.37) in the smoothing remainder r since $-\ell\bar{n}_{\alpha,\varrho} \leq -\varrho$. Relabeling

$$z := v_{\bar{n}_{\alpha,\varrho}} \tag{4.38}$$

the equation for z becomes

$$\partial_t z = -\mathrm{i}\,\mathrm{Op}^{BW}\left[\dot{\omega}_\alpha\left(\xi\right) + q_{\geq 2}\left(f;x,\xi\right) + r_{\geq 2}\left(f;x,\xi\right)\right] z + R\left(f\right) z, \tag{4.39}$$

which has the claimed form (4.7).

5 The normal form reduction for the smoothing remainder

In this section we eliminate, from the smoothing remainder

$$R\left(f\right) = R_1\left(f\right) + R_{\geq 2}\left(f\right), \tag{5.1}$$

appearing in Equation (4.7) its linear (with respect to f) component up to a higher homogeneity smoothing remainder.

Proposition 5.1. *Let $\alpha \in (0,1)$. There exists $\underline{\varrho} := \underline{\varrho}(\alpha)$, such that for any $\varrho \geq \underline{\varrho}$ there is $\underline{s_0} > 0$ such that for any $s \geq \underline{s_0}$, there is $\underline{\epsilon_0}(s) > 0$ such that for any $0 < \epsilon_0 \leq \underline{\epsilon_0}(s)$ and any real solution $f \in B_{\underline{s_0}}(I;\epsilon_0) \cap C\left(I;H^s_0\right)$ of the equation* (4.2) *there exists a real invertible operator $\Psi_{\mathtt{bir}}\left(f;t\right)$ on H^s_0 satisfying the following:*

- **Boundedness:** *for any $s \geq s_0$ there are $C := C_s > 0$ and $\epsilon'_0(s) \in (0,\epsilon_0)$, such that for any $f,\ v \in B_{\underline{s_0}}\left(I;\epsilon'_0(s)\right) \cap C\left(I;H^s_0\right)$ and for any $t \in I$,*

$$\left\|\Psi_{\mathtt{bir}}\left(f;t\right) v\right\|_s + \left\|\Psi_{\mathtt{bir}}\left(f;t\right)^{-1} v\right\|_s \leq C\left(\|v\|_s + \|f\|_s \|v\|_{s_0}\right); \tag{5.2}$$

- **Conjugation:** *Let $z = \mathbf{B}\left(f\right) f$ the auxiliary variable which solves* (4.7)*, then the variable*

$$g := \Psi_{\mathtt{bir}}\left(f;t\right) z = \Psi_{\mathtt{bir}}\left(f;t\right) \mathbf{B}\left(f;t\right) f \tag{5.3}$$

solves the equation

$$\partial_t g + \mathrm{i}\,\mathrm{Op}^{BW}\left[\dot{\omega}_\alpha\left(\xi\right) + q_{\geq 2}\left(f;x,\xi\right) + r_{\geq 2}\left(f;x,\xi\right)\right] g = R_{\geq 2}\left(f\right) g \tag{5.4}$$

where

– $\dot{\omega}_\alpha(\xi) \in \widetilde{\Gamma}_0^\alpha$ *is the Fourier multiplier defined in Equation* (2.13)*;*

– $q_{\geq 2}(f;x,\xi)$ *is a symbol as in Notation 4.5;*

– $r_{\geq 2}(f;x,\xi)$ *is a non-homogeneous symbol satisfying* (4.1)*;*

– $R_{\geq 2}(f)$ *is a real smoothing operator in* $\mathcal{R}_{\geq 2}^{-\varrho+\underline{\varrho}}[\epsilon_0]$.

In view of (5.3), the bounds in (5.2) and (4.6) imply in particular that for any $s \geq s_0$, there exists $C := C_{s,\alpha} > 0$ such that

$$C^{-1}\left\|f(t)\right\|_s \leq \left\|g(t)\right\|_s \leq C\left\|f(t)\right\|_s, \qquad \forall t \in I. \tag{5.5}$$

Proof. Notice that $R_1(f)$ in Eq. (5.1) is a real homogenous smoothing operator in $\tilde{\mathcal{R}}_1^{-\varrho+\varrho_0}$, that we expand (cf. (3.22)) as

$$R_1(f)\,v = \sum_{\substack{n,k,j\in\mathbb{Z}\setminus\{0\},\\ n+j=k}} R_{j,k} f_{j-k} v_k e^{\mathrm{i}jx}, \qquad R_{j,k} \in \mathbb{C}, \tag{5.6}$$

In order to remove $R_1(f)$ we conjugate (4.39) with the flow (cf. Lemma A.4)

$$\partial_\tau \Phi_Q^\tau(f) = Q(f)\,\Phi_Q^\tau(f), \qquad \Phi_Q^0(f) = \mathrm{Id}, \tag{5.7}$$

generated by the 1-homogenous smoothing operator

$$Q(f)\,v = \sum_{\substack{n,k,j\in\mathbb{Z}\setminus\{0\},\\ n+j=k}} Q_{n,j,k} f_n v_j e^{\mathrm{i}kx}, \qquad Q_{n,j,k} := \frac{-(R_1)_{n,j,k}}{\mathrm{i}\big(\dot{\omega}_\alpha(j) - \dot{\omega}_\alpha(k) - \dot{\omega}_\alpha(j-k)\big)}, \tag{5.8}$$

which is well-defined by Lemma 2.8. Notice that, since R_1 satisfies (3.24) (i.e. it is a real-to-real operator) and $\dot{\omega}_\alpha(\xi)$ is odd, the coefficients $Q_{n,j,k}$ satisfy (3.24) and thus $Q(f)$ is real-to-real. We thus apply Proposition A.5, Items i and ii and obtain that

$$\begin{aligned}\Phi_Q(u)\circ\big(\mathrm{Op}^{BW}\left[\dot{\omega}_\alpha(\xi) + q_{\geq 2}(f;x,\xi) + r_{\geq 2}(f;x,\xi)\right] + R_1(f) + R_{\geq 2}(f)\big)\circ\Phi_Q(u)^{-1}\\ = \mathrm{Op}^{BW}\left[\dot{\omega}_\alpha(\xi) + q_{\geq 2}(f;x,\xi) + r_{\geq 2}(f;x,\xi)\right] + \left[\!\left[Q(f), \dot{\omega}_\alpha(D)\right]\!\right] + R_1(f) + R^1_{\geq 2}(f),\end{aligned}$$

where $R^1_{\geq 2}(f) \in \mathcal{R}_{\geq 2}^{-\varrho+\varrho_0+1}[\epsilon_0]$. Next, by Proposition A.5-iii we obtain that

$$\partial_t \Phi_Q(f)\circ\Phi_Q(f)^{-1} = -\mathrm{i}\,Q\left(\dot{\omega}_\alpha(D)\,f\right) + R^2_{\geq 2}(f),$$

where $R^2_{\geq 2}(f) \in \mathcal{R}_{\geq 2}^{-\varrho+\varrho_0+1}[\epsilon_0]$. We now prove that $Q(f)$ solves the homological equation

$$Q\left(-\mathrm{i}\dot{\omega}_\alpha(D)f\right) + \left[\!\left[Q(f), -\mathrm{i}\dot{\omega}_\alpha(D)\right]\!\right] + R_1(f) = 0. \tag{5.9}$$

Writing (5.8) as

$$Q(f)\,v = \sum_{k,j\in\mathbb{Z}\setminus\{0\}} \left[Q(f)\right]_k^j v_j e^{\mathrm{i}kx} \quad \text{with}\left[Q(f)\right]_k^j := q_{n,j,k} f_n,$$

we see that the homological equation (5.9) amounts to

$$\left[Q(-\mathrm{i}\dot{\omega}_\alpha(D)f)\right]_k^j + \left[Q(f)\right]_k^j\left(\mathrm{i}\dot{\omega}_\alpha(k) - \mathrm{i}\dot{\omega}_\alpha(j)\right) + \left[R_1(f)\right]_k^j = 0, \quad \text{for any } j,k \in \mathbb{Z}\setminus\{0\},$$

and then, recalling (5.6), to $q_{n,j,k}\,\mathrm{i}\big(\dot{\omega}_\alpha(k) - \dot{\omega}_\alpha(j) - \dot{\omega}_\alpha(n)\big) + (r_1)_{n,j,k} = 0$. This proves (5.9), and, as a consequence, we conclude the proof of Proposition 5.1 setting

$$\underline{\varrho} := \varrho_0 + 1, \quad \Psi_{\mathtt{bir}}(f;t) := \Phi_Q^1(f;t).$$

□

5.1 Energy estimates and proof of Theorem 1.1

The first step is to choose the parameters in Proposition 5.1. In the statement of Proposition 5.1 we fix $\varrho := \underline{\varrho}(\alpha)$. Then Proposition 5.1 gives us $\underline{s_0} > 0$. For any $s \geq \underline{s_0}$ there is $0 < \underline{\epsilon_0}(s)$ where $\underline{\epsilon_0}(s)$ is defined in Proposition 5.1.

We perform a H^s_0 energy estimate on Equation (5.4). First, using the equation (5.4) for the variable g and the reality of the symbols $\dot\omega_\alpha(\xi)$ and $q_{\geq 2}(f;x,\xi)$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|g\right\|_s^2 = \left\langle \mathrm{i}\left[\!\left[\mathrm{Op}^{BW}\left[q_{\geq 2}(f;x,\xi)\right], |D|^s\right]\!\right] g \,\middle|\, |D|^s g\right\rangle + \left\langle |D|^s g \,\middle|\, \mathrm{i}\left[\!\left[\mathrm{Op}^{BW}\left[q_{\geq 2}(f;x,\xi)\right], |D|^s\right]\!\right] g\right\rangle \tag{5.10}$$
$$+ \left\langle |D|^s B_{\geq 2}(f)\, g \,\middle|\, |D|^s g\right\rangle + \left\langle |D|^s g \,\middle|\, |D|^s B_{\geq 2}(f)\, g\right\rangle \tag{5.11}$$

where the bounded operator $B_{\geq 2}(f) := -\mathrm{i}\,\mathrm{Op}^{BW}\left[r_{\geq 2}(f;x,\xi)\right] + R_{\geq 2}(f)$ satisfies (cf. Notation 4.1 and Eqs. (3.21) and (3.25))

$$\left\|B_{\geq 2}(f)\, g\right\|_s \lesssim_{s,\alpha} \left\|f\right\|_{s_0}^2 \left\|g\right\|_s + \left\|f\right\|_{s_0} \left\|g\right\|_{s_0} \left\|f\right\|_s .$$

Next applying (3.28) to the Fourier multiplier $\mathtt{m}(\xi) := |\xi|^{2s} \in \tilde\Gamma^{2s}_0$ and the symbol $a := q_{\geq 2} \in \Gamma^1_{\geq 2}[\epsilon_0]$, together with Proposition 3.6, we get

$$|(5.11)| + |(5.10)| \lesssim_{s,\alpha} \left\|f\right\|_{s_0}^2 \left\|g\right\|_s^2 + \left\|f\right\|_{s_0} \left\|f\right\|_s \left\|g\right\|_s \left\|g\right\|_{s_0} .$$

hence, after an integration in time, we obtain the quartic energy estimate

$$\left\|g(t)\right\|_s^2 \leq \left\|g(0)\right\|_s^2 + \bar C_1(s,\alpha) \int_0^t \left\|f(t')\right\|_{s_0}^2 \left\|g(t')\right\|_s^2 + \left\|f(t')\right\|_{s_0} \left\|f(t')\right\|_s \left\|g(t')\right\|_s \left\|g(t')\right\|_{s_0} \mathrm{d}t', \quad \forall 0 < t < T,$$

and, by (5.5), we deduce

$$\left\|f(t)\right\|_s^2 \leq \bar C_2(s,\alpha) \left(\left\|f(0)\right\|_s^2 + \int_0^t \left\|f(t')\right\|_s^4 \mathrm{d}t' \right), \quad \forall 0 < t < T. \tag{5.12}$$

The energy estimate (5.12), the equivalence $\left\|f\right\|_s \sim \left\|h\right\|_s$ stemming from (2.1) and the local existence result in [38] (which amounts to a local existence result for the equation (4.2)), imply, by a standard bootstrap argument, Theorem 1.1. ☐

A Flows and conjugations

In this section we prove some abstract results about the conjugation of paradifferential operators and smoothing remainders under flows, we shall refer continuously the reader to [8, 10, 14].

Let $u \in B_{s_0}(I;\epsilon_0)$, $\beta \in \tilde{\mathcal{F}}_1$ and $g := g(u,\tau;x,\xi)$ be a symbol of the form

$$g(u,\tau;x,\xi) := \frac{\beta(u;x)}{1+\tau\beta_x(u;x)}\,\xi := b(u,\tau;x)\,\xi, \qquad \in \Gamma^1[\epsilon_0], \qquad \delta = 1, \tag{A.1}$$
$$g(u;x,\xi) = \mathrm{Re}\, g(u;x,\xi) \qquad \in \tilde\Gamma^\delta_1, \qquad \delta \in (0,\alpha], \tag{A.2}$$
$$g(u;x,\xi) \qquad \in \tilde\Gamma^\delta_1, \qquad \delta \leq 0. \tag{A.3}$$

Notice that the symbols in Eqs. (A.2) and (A.3) are considered to be constant in τ and linear with respect to u.

Consider the flow $\Phi^\tau_g(u)$, $\tau \in [-1,1]$ defined by

$$\begin{cases} \partial_\tau \Phi^\tau_g(u) = \mathrm{i}G(u,\tau)\,\Phi^\tau_g(u) \\ \Phi^0(u) = \mathrm{Id} \end{cases}, \qquad G(u) := \mathrm{Op}^{BW}\left[g(u,\tau;x,\xi)\right]. \tag{A.4}$$

We have the following.

Lemma A.1 (**Linear flows**). *There are $s_0, \epsilon_0 > 0$ such that, for any $u \in B_{s_0}(I;\epsilon_0)$, the problem* (A.4) *admits a unique solution $\Phi_g^\tau(u)$. Moreover, for any $s \in \mathbb{R}$ we have that $\Phi_g^\tau(u) \in \mathcal{L}\left(H_0^s\right)$ and there is a constant $C(s) > 0$ such that*

$$\begin{aligned}
&\left\|\Phi_g^\tau(u)\, w\right\|_s + \left\|\Phi_g^\tau(u)^{-1}\, w\right\|_s \le C(s)\, \|w\|_s\,, \\
&\qquad\qquad \left\|\left[\Phi_g^\tau(u) - \mathrm{Id}\right] w\right\|_{s-\delta} + \left\|\left[\Phi_g^\tau(u)^{-1} - \mathrm{Id}\right] w\right\|_{s-\delta} \le C(s)\, \|u\|_{s_0}\, \|w\|_s
\end{aligned} \tag{A.5}$$

for any $w \in H_0^s$.

Proof. The result is classical and follows by standard energy estimate arguments. See, for example, Lemma 3.22 in [8]. Moreover, since the operator $G(u) = \mathrm{Op}^{BW}\left[g(u,\tau;x,\xi)\right]$ satisfies $\Pi_0^\perp G(u) = G(u)\Pi_0^\perp$, it is a standard fact that also its flow satisfies $\Pi_0^\perp \Phi_g^\tau(u) = \Phi_g^\tau(u)\Pi_0^\perp$. As a consequence $\Phi_g^\tau(u)$ leaves invariant the space H_0^s. □

We set $\Phi_g(u) := \Phi_g^1(u)$ and its inverse $\Phi_g(u)^{-1} := \Phi_g^\tau(u)\Big|_{\tau=-1}$.

A.1 Conjugation by a flow generated by a real symbol of order one

We study the case in which g is as in Eq. (A.1). In such setting we recall (see [8]) that, given $\beta \in \tilde{\mathcal{F}}_1^{\mathbb{R}}$ we can define a path of diffeomorphism of $\mathbb{T}$ via the transformation

$$\Psi(u,\tau;x) := x + \tau\beta(u;x) \quad \text{with inverse} \quad \Psi^{-1}\left(u,\tau;y\right) := y + \breve{\beta}\left(u,\tau;y\right) \quad \text{where } \breve{\beta} \in \mathcal{F}[\epsilon_0]\,. \tag{A.6}$$

We set also $\Psi(u;x) := \Psi(u,1;x)$.

The following are Egorov-type theorems for whose proof we refer the reader to [10, Lemmas A.4 and A.5].

Proposition A.2 (Conjugations for a transport flow)**.** *Let $m \le 1$, $\varrho > 0$,and let $\Phi_g(u)$ be the flow generated by g as per Lemma A.1.*

i Space conjugation of a para-differential operator: *Let $a \in \Gamma^m[\epsilon_0]$ and*

$$a^{(m)}(u;x,\xi) := a\left(u;y,\xi\,\partial_y\Psi^{-1}\left(u;y\right)\right)\big|_{y=\Psi(u;x)} \in \Gamma^m[\epsilon_0]\,. \tag{A.7}$$

Then

$$\begin{aligned}
\Phi_g(u)\circ \mathrm{Op}^{BW}[a(u;x,\xi)]\circ\Phi_g(u)^{-1} &= \mathrm{Op}^{BW}\left[a^{(m)}(u;x,\xi) + a_{\ge 2}^{(m-2)}(u;x,\xi)\right] + R[u] \\
&= \mathrm{Op}^{BW}\left[a(u;x,\xi) + a_{\ge 2}^{(m)}(u;x,\xi)\right] + R[u]\,,
\end{aligned} \tag{A.8}$$

where:

- *$a_{\ge 2}^{(m-2)}(u;x,\xi)$ and $a_{\ge 2}^{(m)}(u;x,\xi)$ are non-homogeneous symbols respectively in $\Gamma_{\ge 2}^{m-2}[\epsilon_0]$ and $\Gamma_{\ge 2}^m[\epsilon_0]$;*
- *$R[u]$ is a smoothing operator in $\mathcal{R}_{\ge 2}^{-\varrho}[\epsilon_0]$.*

In addition:

(a) If a is real then $a_{\ge 2}^{(m)}$ is real;

(b) If $a(u;x,\xi) = V(u;x)\xi$ for some $V \in \mathcal{F}[\epsilon_0]$ (hence $m = 1$) then in Eq. (A.8) *$a_{\ge 2}^{(m-2)} \equiv 0$ is nil and $a^{(m)}(u;x,\xi) = \tilde{V}(u;x)\,\xi$ for a suitable function $\tilde{V} \in \mathcal{F}[\epsilon_0]$.*

ii Space conjugation of a smoothing remainder: *If $R(u)$ is a smoothing operator in $\mathcal{R}^{-\varrho}[\epsilon_0]$ then*

$$\Phi_g(u)\circ R(u)\circ\Phi_g(u)^{-1} = R(u) + R_{\ge 2}(u)\,,$$

where $R \in \mathcal{R}_{\ge 2}^{-\varrho+1}[\epsilon_0]$;

iii Conjugation of ∂_t: *If $u = f$ solution of Equation* (4.4) *then*

$$\partial_t \Phi_g\left(f\right)\Phi_g\left(f\right)^{-1} = \mathrm{i}\,\mathrm{Op}^{BW}\left[\beta\left(-\mathrm{i}\,\dot\omega_\alpha(D)f;x\right)\,\xi + \mathrm{i}\,\mathsf{V}_{\geq 2}\left(f;x\right)\xi\right] + R\left(f\right), \tag{A.9}$$

where

- $\dot\omega_\alpha(D)$ *is the real Fourier multiplier defined in* (2.13)*;*
- $\mathsf{V}_{\geq 2}\left(f;x\right)$ *is a real function in* $\mathcal{F}_{\geq 2}\left[\epsilon_0\right]$*;*
- $R\left(f\right)$ *is a smoothing operator in* $\mathcal{R}^{-\varrho}\left[\epsilon_0\right]$.

Proof. **i** Follows by Lemmas A4 and A5 in [10].

ii We first define the operator

$$P^\tau(f) := \Phi_g^\tau(u)\circ R(u)\circ\Phi_g^\tau(u)^{-1}. \tag{A.10}$$

Then $\Phi_g(u)\circ R(u)\circ\Phi_g(u)^{-1} = P^1(f)$ The operator $P^\tau(f)$ solves the Heisenberg equation

$$\partial_\tau P^\tau(f) = \llbracket G(f,\tau)\,,\,P^\tau(f)\rrbracket\,.$$

Moreover, by (A.5), (3.27) and (3.20), (3.21) we have

$$\|P^\tau(f)w\|_{s+\varrho} \lesssim_s \|f\|_s\|w\|_{s_0} + \|f\|_{s_0}\|w\|_s, \quad \|G(f,\tau)w\|_{s-1} \lesssim_s \|f\|_{s_0}\|w\|_s, \quad \forall s\geq s_0.$$

Then

$$\left\|\left(P^1(f)-R(f)\right)w\right\|_{s+\varrho-1} \lesssim \int_0^1 \left\|\llbracket G(f,\tau)\,,\,P^\tau(f)\rrbracket\,w\right\|_{s+\varrho-1}\mathrm{d}\tau \lesssim_s \|f\|_{s_0}^2\|w\|_s + \|f\|_{s_0}\|f\|_s\|w\|_{s_0}$$

iii Differentiating (A.4) with respect to time, we get

$$\partial_t\Phi_g\left(f\right)\Phi_g\left(f\right)^{-1} = \int_0^1 \Phi_g\left(f\right)\left[\Phi_g^\tau\left(f\right)\right]^{-1}\mathrm{Op}^{BW}\left[\mathrm{i}\partial_t g(f,\tau;x,\xi)\right]\Phi_g^\tau\left(f\right)\Phi_g\left(f\right)^{-1}\,\mathrm{d}\tau. \tag{A.11}$$

We claim that that

$$\partial_t b(f,\tau;x) = \beta(-\mathrm{i}\dot\omega(D)f;x) + \beta_{\geq 2}(f,\tau;x), \quad \beta_{\geq 2}(f,\tau;x)\in\mathcal{F}^{\mathbb{R}}_{\geq 2}[\epsilon_0]. \tag{A.12}$$

Indeed, differentiating $b(f(t),\tau;x)$ with respect to t and using the equation for f (4.4), we get

$$\partial_t b(f,\tau;x) = \beta\left(-\mathrm{i}\dot\omega(D)f;x\right) + \beta\left(M_{\mathsf{SQG}}\left(f\right)f;x\right) - \tau\left[\frac{\beta\left(f;x\right)\beta_x\left(X\left(f\right);x\right)}{(1+\tau\beta_x\left(f;x\right))^2} + \frac{\beta_x\left(f;x\right)\beta\left(X\left(f\right);x\right)}{(1+\tau\beta_x\left(f;x\right))}\right].$$

then (A.12) follows from the fact that, using Lemma 3.13 for each internal compositions,

$$\beta_{\geq 2}(f,\tau;x) := \beta\left(M_{\mathsf{SQG}}\left(f\right)f;x\right) - \tau\left[\frac{\beta\left(f;x\right)\beta_x\left(X\left(f\right);x\right)}{(1+\tau\beta_x\left(f;x\right))^2} + \frac{\beta_x\left(f;x\right)\beta\left(X\left(f\right);x\right)}{(1+\tau\beta_x\left(f;x\right))}\right]$$

is a function in $\mathcal{F}^{\mathbb{R}}_{\geq 2}[\epsilon_0]$.

□

A.2 Conjugation by a flow generated by a real symbol of order $\delta \leq \alpha$

We study the case in which g is as in Eqs. (A.2) and (A.3).

Proposition A.3 (Conjugation by flows generated by symbols of order smaller than one)**.** *With the same hypothesis of Proposition A.2, let $a\in\Gamma^m\left[\epsilon_0\right]$ and $\Phi_g^\tau(u)$ be the flow generated by g as in Eqs.* (A.2) *and* (A.3) *per Lemma A.1. One has the following conjugation rules:*

i Space conjugation: *[10, Lemma A.6]*

$$\Phi_g(u)\circ \mathrm{Op}^{BW}[\mathrm{i}a(u;x,\xi)]\circ\Phi_g(u)^{-1} = \mathrm{i}\mathrm{Op}^{BW}[a(u;x,\xi)+r_{\geq 2}(u;x,\xi)]+R_{\geq 2}(u), \tag{A.13}$$
$$\Phi_g(u)\circ \mathrm{Op}^{BW}[-\mathrm{i}\omega(\xi)]\circ\Phi_g(u)^{-1} = \mathrm{Op}^{BW}[-\mathrm{i}\omega(\xi)+\mathrm{i}\tilde r(u;x,\xi)]+R(u)$$

where:

- $r_{\geq 2}(u;x,\xi)$ *is a non-homogeneous symbol in* $\Gamma^{m-\ell}_{\geq 2}[\epsilon_0]$, *with* $\ell := 1-\alpha>0$ *(cfr.* (4.30)*);*
- $R_{\geq 2}(u)$ *is a non-homogeneous smoothing remainder in* $\mathcal{R}^{-\varrho}_{\geq 2}[\epsilon_0]$*;*
- $\tilde r(u;x,\xi)$ *is a symbol in* $\Gamma^{\delta-\ell}[\epsilon_0]$*;*
- $R(u)$ *is a non-homogeneous smoothing remainder in* $\mathcal{R}^{-\varrho}[\epsilon_0]$*;*

Moreover:

- *if* $a(u;x,\xi)$ *and* $g(u;x,\xi)$ *are real valued (and odd in* ξ*) then* $r_{\geq 2}(u;x,\xi)$ *and* $\tilde r(u;x,\xi)$ *are real valued (and odd in* ξ*) as well;*
- *if* $\mathrm{i}a(u;x,\xi)$ *and* $\mathrm{i}g(u;x,\xi)$ *are real-to real (cfr. Eqs.* (3.9) *and* (3.11)*) then* $\mathrm{i}r_{\geq 2}(u;x,\xi)$ *and* $\mathrm{i}\tilde r(u;x,\xi)$ *are real-to-real as well.*

ii Conjugation of smoothing operators: *Given* $R\in\mathcal{R}^{-\varrho}[\epsilon_0]$ *we have that*

$$\Phi_g(u)\circ R(u)\circ\Phi_g(u)^{-1} = R(u)+R_{\geq 2}(u)$$

where $R_{\geq 2}(u)\in\mathcal{R}^{-\varrho+\max\{0,\delta\}}_{\geq 2}[\epsilon_0]$

iii Conjugation of ∂_t**:** *If* $u=f$ *solution of Equation* (4.4) *then*

$$\partial_t\Phi_g(f)\circ\Phi_g(f)^{-1} = \mathrm{Op}^{BW}\left[\mathrm{i}g\left(-\mathrm{i}\dot\omega_\alpha(D)f;x,\xi\right)+r_{\geq 2}\right]+R_{\geq 2}(f).$$

where $r_{\geq 2}$ *is a symbol in* $\Gamma^{\delta-\ell}_{\geq 2}[\epsilon_0]$ *and* $R_{\geq 2}$ *is a smoothing remainder in* $\mathcal{R}^{-\varrho}_{\geq 2}[\epsilon_0]$. *Moreover if* g *is real then* $r_{\geq 2}$ *is real;*

Proof. Item **i** is proved in [10, Lemma A.6]. We only need to verify the reality conditions Eqs. (3.9) and (3.11). We prove such statement for (A.13) only, the proof for the latter being the same. Let $M(u) := \mathrm{Op}^{BW}[\mathrm{i}r_{\geq 2}(u;x,\xi)]+R_{\geq 2}(u)$, $M(u)$ is indeed real-to-real, hence $M(u)=\frac{M(u)+\bar M(u)}{2} = \mathrm{Op}^{BW}\left[\mathrm{i}\,\frac{r_{\geq 2}(u;x,\xi)-\overline{r_{\geq 2}}(u;x,-\xi)}{2}\right]+\frac{R_{\geq 2}(u)+\overline{R_{\geq 2}}(u)}{2}$, hence with the relabeling $\mathrm{i}r_{\geq 2}(u;x,\xi)\rightsquigarrow \mathrm{i}\,\frac{r_{\geq 2}(u;x,\xi)-\overline{r_{\geq 2}}(u;x,-\xi)}{2}$ and $R_{\geq 2}(u)\rightsquigarrow\frac{R_{\geq 2}(u)+\overline{R_{\geq 2}}(u)}{2}$ we prove the desired claim.

The proof of item **ii** is the same of the analogous item **ii** in Proposition A.2. To prove item **iii** we start from (A.11) and we set

$$\tilde g_{\geq 2}(f;x,\xi) := \partial_t g(f;x,\xi)-g(-\mathrm{i}\dot\omega(D)f;x,\xi).$$

By (4.4), one has $\tilde g_{\geq 2}(f;x,\xi)=g\left(M_{\mathtt{SQG}}\left(f\right)f;x,\xi\right)$, which belongs to $\Gamma^{\delta}_{\geq 2}[\epsilon_0]$ thanks to Lemma 3.13.

□

A.3 Conjugation by flows generated by linear smoothing operators

In this section we study the conjugation rules for the flow generated by a linear smoothing operator of the form $\Phi_Q(u) := \Phi^1_Q(u)$ where $\Phi^\tau_Q(u)$, $\tau\in[-1,1]$ are given by

$$\partial_\tau\Phi^\tau_Q(u) = Q(u)\circ\Phi^\tau_Q(u), \qquad \Phi^0_Q(u)=\mathrm{Id}, \qquad Q(u)\in\tilde{\mathcal{R}}^{-\varrho}_1. \tag{A.14}$$

We only state the properties with the flow $\Phi_Q(u)$. The proof follows from standard theory for linear ODEs in Banach spaces.

Lemma A.4. *There exists $s_0>0$ such that for any $u\in C(I;H^{s_0}_0)$ the problem* (A.14) *admit unique solution $\Phi^\tau_Q(u)$. Moreover $\Phi^\tau_Q(u)-\mathrm{Id}-\tau Q(u)$ is a ϱ-smoothing operator in $\mathcal{R}^{-\varrho}_{\geq 2}[r]$ for any $r>0$. In particular $\Phi^{\pm\tau}_Q(u)-\mathrm{Id}$ is in $\mathcal{R}^{-\varrho}[r]$ and for any $s\geq s_0$ there is $\epsilon_0>0$ such that*

$$\left\|\Phi^\tau_Q(u)v\right\|_s+\left\|\Phi^{-\tau}_Q(u)v\right\|_s\lesssim_s \|v\|_s+\|v\|_{s_0}\|u\|_s, \tag{A.15}$$

for any $u\in B_{s_0}(I;\epsilon_0)$, and $v\in C(I;H^s(\mathbb{T};\mathbb{R}))$ uniformly in $\tau\in[-1,1]$.

Proof. First of all there exist $s_0>0$ such that $Q(u)\in\mathcal{L}(H^s_0,H^s_0)$ for any $s\geq s_0$ and $u\in H^s_0$ (see (3.27)). Then the well-posedness of $\Phi^\tau_Q(u)$, follows by standard ODE's theory in Banach spaces (see also Lemma A.3 in [10]), and $\Phi^\tau_Q(u)^{-1}=\Phi^{-\tau}_Q(u)$ since the ODE is autonomous. Moreover one has the Taylor's expansion

$$\Phi^\tau_Q(u)=\mathrm{Id}+\tau Q(u)+\tau^2\sum_{k=0}^{\infty}\frac{\tau^k}{(k+2)!}Q^{k+2}(u).$$

Applying iteratively (3.27) to the smoothing operator $Q(u)$, we get that there is $L_s>0$ such that

$$\left\|Q^{k+2}(u)v\right\|_{s+\varrho}\leq L_s^{k+2}\left(\|u\|^{k+2}_{s_0}\|v\|_s+\|u\|^{k+1}_{s_0}\|u\|_s\|v\|_{s_0}\right).$$

Then, for any $r>0$ and $u\in B_{s_0}(I;r)$, we deduce that

$$\begin{aligned}\left\|\Phi^\tau_Q(u)-\mathrm{Id}-\tau Q(u)\right\|_{s+\varrho}&\leq\tau^2\sum_{k=0}^{\infty}\frac{(L_s r)^k}{(k+2)!}\left(\|u\|^2_{s_0}\|v\|_s+\|u\|_{s_0}\|u\|_s\|v\|_{s_0}\right)\\&\leq e^{L_s r}\left(\|u\|^2_{s_0}\|v\|_s+\|u\|_{s_0}\|u\|_s\|v\|_{s_0}\right)\end{aligned}\tag{A.16}$$

which, in view of (3.25), proves that $\Phi^\tau_Q(u)-\mathrm{Id}-\tau Q(u)$ is a ϱ-smoothing operator in $\mathcal{R}^{-\varrho}_{\geq 2}[\epsilon_0]$. Then the smoothing estimates in (A.16) and (3.27) for $Q(u)$ imply (A.15) for any $u\in B_{s_0}(I;\epsilon_0)$ for a sufficiently small $\epsilon_0:=\epsilon_0(s)>0$. ☐

We set $\Phi_Q(u):=\Phi^1_Q(u)$. Its inverse is given by $\Phi_Q(u)^{-1}=\Phi^\tau_Q(u)\Big|_{\tau=-1}$.

Proposition A.5 (Conjugation by flows generated by smoothing operators)**.** *Let $m\in\mathbb{R}$, $\varrho,\varrho'>0$, $Q(u)\in\tilde{\mathcal{R}}^{-\varrho}_1$ and let $\Phi_Q(u)$ be the flow generated by Q as per Lemma A.4. Then the following holds:*

i Space conjugation: *If $a\in\Gamma^m[\epsilon_0]$, then*

$$\Phi_Q(u)\circ\mathrm{Op}^{BW}[a(u;x,\xi)]\circ\Phi_Q(u)^{-1}-\mathrm{Op}^{BW}[a(u;x,\xi)]\in\mathcal{R}^{-\varrho+\max\{m,0\}}_{\geq 2}[\epsilon_0],\tag{A.17}$$

$$\Phi_Q(u)\circ\omega(D)\circ\Phi_Q(u)^{-1}-\left(\omega(D)+\left[\!\left[Q\left(f\right),\omega(D)\right]\!\right]\right)\in\mathcal{R}^{-\varrho+\alpha}_{\geq 2}[\epsilon_0]\tag{A.18}$$

ii Conjugation of smoothing operators: *If $R\in\mathcal{R}^{-\varrho'}[\epsilon_0]$, then*

$$\Phi_Q(u)\circ R(u)\circ\Phi_Q(u)^{-1}-R(u)\in\mathcal{R}^{-\min\{\varrho,\varrho'\}}_{\geq 2}[\epsilon_0].\tag{A.19}$$

iii Conjugation of ∂_t: *If $u=f$ solution of Equation* (4.4) *then*

$$\partial_t\Phi_Q\left(f\right)\circ\Phi_Q\left(f\right)^{-1}-\ Q\left(-\mathrm{i}\omega_\alpha(D)f\right)\in\mathcal{R}^{-\varrho+1}_{\geq 2}[\epsilon_0]\ .\tag{A.20}$$

Proof. i We prove (A.17) first. Denoting $A(u):=\mathrm{Op}^{BW}[a(u;x,\xi)]$ we have indeed that

$$\begin{aligned}&\Phi_Q(u)\circ A(u)\circ\Phi_Q(u)^{-1}=A(u)+R_A(u)\\&\qquad R_A(u):=A(u)\circ\left(\Phi^{-1}_Q(u)-\mathrm{Id}\right)+\left(\Phi_Q(u)-\mathrm{Id}\right)\circ A(u)\circ\Phi_Q(u)^{-1},\end{aligned}$$

hence the fact that $R_A(u)\in\mathcal{R}^{-\varrho+\max\{m,0\}}_{\geq 2}[\epsilon_0]$ follows from a repeated application of Lemma 3.13, Item 3 and Remark 3.10, the last bullet.
We prove now (A.18). By a Taylor expansion we have that

$$\Phi_Q(u) \circ \omega(D) \circ \Phi_Q(u)^{-1}$$
$$= \omega(D) + [\![Q(u) , \omega(D)]\!] + \int_0^1 (1-\tau)\, \Phi_Q^{\tau}(u) [\![Q(u) , [\![Q(u) , \omega(D)]\!]]\!] \Phi_Q^{-\tau}(u)\, \mathrm{d}\tau. \quad \text{(A.21)}$$

It is immediate that $[\![Q(u) , [\![Q(u) , \omega(D)]\!]]\!] \in \mathcal{R}_{\geq 2}^{-\varrho+\alpha}[\epsilon_0]$, next being $\Phi_Q^{-\tau}(u)$ and $\Phi_Q^{\tau}(u)$ bounded operators in Sobolev spaces uniformly in $|\tau| \leq 1$ (cf. Eq. (A.15)) we obtain that the integral term in Eq. (A.21) is a smoothing operator in $\mathcal{R}_{\geq 2}^{-\varrho}[\epsilon_0]$, this concludes the proof of (A.18).

ii Follows the same lines as in the proof of (A.17), with the exception that we have to use repeatedly Remark 3.10, the last bullet, instead of Lemma 3.13, Item 3.

iii Differentiating with respect to t the equation (A.14) and using that f solves (4.4), we get

$$\partial_t \Phi_Q(f) \circ \Phi_Q(f)^{-1} = \int_0^1 \Phi_Q^{\tau}(f)\, Q\big(X(f)\big)\, \Phi_Q^{-\tau}(f)\, \mathrm{d}\tau.$$

Next, thanks to Lemma 3.13, we have that $Q\big(X(f)\big)$ is a smoothing remainder in $\mathcal{R}^{-\varrho+1}[\epsilon_0]$. Moreover, by (A.19) and (4.4) we have

$$\int_0^1 \Phi_Q^{\tau}(f)\, Q\big(X(f)\big)\, \Phi_Q^{-\tau}(f)\, \mathrm{d}\tau = Q(X(f)) + R_{\geq 2}(f) = Q(-\mathrm{i}\dot{\omega}(D) f) + Q(M_{\mathtt{SQG}}(f)) + R_{\geq 2}(f)$$

where $R_{\geq 2}$ is in $\mathcal{R}_{\geq 2}^{-\varrho}[\epsilon_0]$ and applying again Lemma 3.13 to $Q(M_{\mathtt{SQG}}(f))$ we get the claimed expansion (A.20).

□

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