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Superactivation of memory effects in a classical Markov environment

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Abstract

We investigate a phenomenon known as Superactivation of Backflow of Information (SBFI); namely, the fact that the tensor product of a non-Markovian dynamics with itself exhibits Backflow of Information (BFI) from environment to system even if the single dynamics does not. Such an effect is witnessed by the non-monotonic behaviour of the Helstrom norm and emerges in the open dynamics of two independent, but statistically coupled, parties. We physically interpret SBFI by means of the discrete-time non-Markovian dynamics of two open qubits collisionally coupled to an environment described by a classical Markov chain. In such a scenario, SBFI can be ascribed to the decrease of the qubit-qubit-environment correlations in favour of those of the two qubits, only. We further prove that the same mechanism at the roots of SBFI also holds in a suitable continuous-time limit. We also show that SBFI does not require entanglement to be witnessed, but only the quantumness of the Helstrom ensemble.

1. Introduction

Much study has recently been devoted to open quantum systems beyond the so-called Markovian regime [1], when memory effects, for instance due to strong coupling to the environment, cannot be neglected. In contrast to the classical case, many non-equivalent concepts of quantum non-Markovianity have been put forward, forming an intricate hierarchy [2]. In particular, for a one-parameter family $\{\Lambda_t\}_{t\geqslant 0}$ of completely positive and trace preserving (CPTP) maps, non-Markovianity can be characterized according to two major approaches, involving either the Divisibility of the dynamics or the notion of Information Flow. An open dynamics $\{\Lambda_t\}_{t\geqslant 0}$ is divisible if, for all $t\geqslant s\geqslant 0$, there exists an intertwiner map $\Lambda_{t,s}$ such that $\Lambda_t=\Lambda_{t,s}\Lambda_s$. Then, Λ_t is said to be (C)P-divisible if all $\Lambda_{t,s}$ are (C)PTP maps. Markovianity has often been identified with CP-divisibility [3, 4].

Instead, the so-called BLP approach [5] identifies Markovianity with the monotonic decrease in time of the distinguishability of generic states ρ and σ, namely $\partial_t||\Lambda_t[\Delta_\mu(\rho,\sigma)]||_1\leqslant 0$, where $||X||_1 = \mathrm{Tr}(\sqrt{X^\dagger X})$ denotes the trace-norm, while $\Delta_\mu(\rho,\sigma)=\mu\rho-(1-\mu)\sigma$ is the so-called Helstrom matrix, $\mu=1/2$ retrieving the trace distance $||\rho-\sigma||_1/2$. A revival of state distinguishability, signalled by $\partial_t||\Lambda_t[\Delta_\mu(\rho,\sigma)]||_1>0$, for some $t>0$, is then interpreted as Backflow of Information (BFI) from the environment to the system. Though a full-fledged microscopic characterization is still missing, BFI is usually associated with information stored in the form of system-environment correlations or with changes in the environmental state [6–10].

It is known that P-divisible families $\{\Lambda_t\}_{t\geqslant 0}$ cannot support BFI since the maps $\Lambda_{t,s}$ are contractive [11, 12]. On the other hand, if the maps Λ_t are invertible, P-divisible, but not CP-divisible, then the maps $\Lambda_t\otimes\Lambda_t$ cannot be P-divisible and thus show BFI at the level of a bipartite system even if the single system dynamics does not [13, 14]. We call such a phenomenon Superactivation of Backflow of Information (SBFI).

If one wants to access the actual flows of information, if any, between system and environment, possibly at the roots of BFI and SBFI, sticking to the reduced dynamics of the system only is useless. Rather, a certain degree of control over the compound system-environment dynamics is needed. As such, the microscopic physical mechanisms behind BFI are necessarily heavily model-dependent and still debated. For instance, when a classical environment acts as a control on the quantum open system, while in [15–17] the BFI seems not to be associated to an actual flow of information, it is instead so in [10]. Different points of view also appear regarding the discrimination of classical vs quantum effects behind the flows of information, in particular when trying to identify genuinely quantum memories [18, 19].

Collisional models, though providing a discrete-time description, appear a suitable tool for assessing emerging memory effects from the system-environment dynamics [20–24]. In the following, we therefore investigate the physics of SBFI in a discrete-time dissipative dynamics $\Lambda_n \otimes \Lambda_n$, $n \geqslant 0$, of two qubits that we obtain by means of a collisional model. An algebraic approach typical of quantum Markov Chains [25] will allow us to explain the emergence of SBFI in terms of the strength of the environment correlations and to describe it by means of the time-behaviour of the mutual information between the open system and the classical Markov chain as its environment. The results are as follows:

1. for two qubits collisionally and unitarily interacting with a Markov chain environment, SBFI appears if successive chain sites are sufficiently correlated, in which case, the mutual information decreases in discrete-time without changes in the Markov chain state. This shows that part of the information shared by the two-qubit and the collisional environment is released to the two-qubits. Such a result benefits from the algebraic approach to the collisional models developed in the following which provides a general context where to accommodate correlated environments as for instance those treated in [23, 24, 26, 27].
2. A suitable continuous-time limit of the discrete-time qubit dynamics is obtained by means of a non-unitary dynamical coupling between system and classical chain. Also in this case the mutual information shows a non-monotonic behaviour confirming the interpretation of the SBFI as a loss of correlations between system and environment to the advantage of the open system.
3. Finally, we show that the general non-classical resource needed for the emergence of SBFI is solely the quantumness of the Helstrom ensemble, with no need of entanglement.

2. Markov chain environment

As emphasized in the Introduction, we interpret the collisional scenario within an algebraic quantum spin chain approach. We choose the environment E to consist of an infinite spin chain, each site k supporting a same $d \times d$ matrix algebra: $\mathcal{A}_E^{(k)} = \mathcal{A}$. Local algebras $\mathcal{A}_{\boldsymbol{i}_{[-a,b]}}^{[-a,b]} = \otimes_{k=-a}^{b} \mathcal{A}_{i_k}^{(k)}$, supported by intervals $[-a, b]$ of integers $-a \leqslant j \leqslant b$, are generated by tensor products of the form $A_{\boldsymbol{i}_{[-a,b]}}^{[-a,b]} = \otimes_{j=-a}^{b} A_{i_j}^{(j)}$, where the upper index (j) indicates the site at which the operator A_{i_j} is located. These local operators can be embedded within the infinite chain as $\mathbb{1}_E^{a-1]} \otimes A_{\boldsymbol{i}_{[-a,b]}}^{[-a,b]} \otimes \mathbb{1}^{[b+1}$, where $\mathbb{1}_E^{a-1]} = \otimes_{k=-\infty}^{-a-1} \mathbb{1}_E^{(k)}$ and $\mathbb{1}_E^{[b+1} = \otimes_{k=b+1}^{+\infty} \mathbb{1}_E^{(k)}$. In the following, for sake of simplicity, we will omit the infinite tensor products $\mathbb{1}_E^{a-1]}$ and $\mathbb{1}_E^{[b+1}$. The collisional environment will then be described by the quasi-local (C^*) algebra $\mathcal{A}_E$ obtained by the so-called inductive limit of the local algebras [28].

Further, states over the chain are all positive, normalized linear expectations ω_E: $\mathcal{A}_E \to \mathbb{C}$, $\omega_E(\mathbb{1}) = 1$. When restricted to the local algebras, these expectations are represented by density matrices $\rho^{[-a,b]} \in \mathcal{A}_E^{[-a,b]}$ such that:

$$\omega_E(\underset{k=-a}{\overset{b}{\otimes}} A_{i_k}^{(k)}) = \mathrm{Tr}(\rho_E^{[-a,b]} \underset{k=-a}{\overset{b}{\otimes}} A_{i_k}^{(k)}), \tag{1}$$

for all $-a \leqslant b$ and $A_{i_k}^{(k)} \in \mathcal{A}$ at site k. Vice versa, a family of density matrices $\rho^{[-a,b]} \in \mathcal{A}_E^{[a,b]}$, gives rise to a state ω_E over the chain if, for all $-a \leqslant b$, $\mathrm{Tr}_b\, \rho^{[-a,b]} = \rho^{[-a,b-1]}$, where Tr_k defines the partial trace over the k-th site. Environment correlations are present over the subset $[-a, b]$ whenever the density matrix $\rho^{[-a,b]}$ does not factorize. Moreover, if $\mathrm{Tr}_{-a}\, \rho_E^{[-a,b]} = \rho_E^{[-a+1,b]} = \rho_E^{[-a,b-1]}$, then the state ω_E is invariant, $\omega_E \circ \Theta = \omega_E$, under the shift to the right,

$$\Theta(A_{i_{-a}}^{(-a)} \otimes \cdots \otimes A_{i_b}^{(b)}) = A_{i_{-a}}^{(-a+1)} \otimes \cdots \otimes A_{i_b}^{(b+1)}. \tag{2}$$

We couple such a chain to a system S described by a finite dimensional algebra $\mathcal{A}_S = M_\ell(\mathbb{C})$, with a state (expectation over $\mathcal{A}_S$) given by a density matrix ρ_S, $\omega_S(O_S) = \mathrm{Tr}(\rho_S O_S)$, $O_S \in \mathcal{A}_S$. As depicted in figure 1, the SE coupling is constructed as follows. Let Φ be a completely positive unital (CPU) map from $\mathcal{A}_S \otimes \mathcal{A}_E^{(0)}$ onto itself. Its action easily extends to the full algebra $\mathcal{A}_S \otimes \mathcal{A}_E$:

$$\Phi[O_S \otimes A_{\boldsymbol{i}_{[-a,-1]}}^{[-a,-1]} \otimes A_{i_0}^{(0)} \otimes A_{\boldsymbol{i}_{[1,b]}}^{[1,b]}] = A_{\boldsymbol{i}_{[-a,-1]}}^{[-a,-1]} \otimes \Phi[O_S \otimes A_{i_0}^{(0)}] \otimes A_{\boldsymbol{i}_{[1,b]}}^{[1,b]}. \tag{3}$$

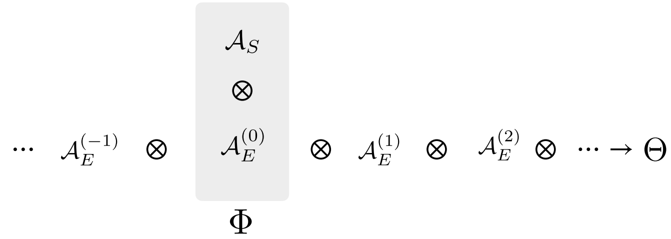

Figure 1. Scheme of the model for one party. The CPTP map Φ acts non-trivially only on the algebras of the system and of the 0th site of the chain, while Θ denotes the right shift on the chain.

The dynamics on the compound algebra $\mathcal{A}_S \otimes \mathcal{A}_E$ at discrete time n is then given by

$$\Phi_n \equiv (\Theta \circ \Phi)^n. \tag{4}$$

The maps Φ_n give the dynamics of operators in the Heisenberg picture; in the Schrödinger picture, an initial state ω_{SE} on $\mathcal{A}_S \otimes \mathcal{A}_E$ evolves at discrete time n into

$$\omega_{SE}^{(n)} = \omega_{SE} \circ \Phi_n. \tag{5}$$

Local restrictions to local algebras $\mathcal{A}_E^{[-a,b]}$ yield density matrices $\Omega_{S[-a,b]}^{(n)}$ such that

$$\mathrm{Tr}(\Omega_{S[-a,b]}^{(n)} O_S \otimes A_E^{[-a,b]}) = \omega_{SE} \circ \Phi_n(O_S \otimes A_E^{[-a,b]}), \tag{6}$$

with marginal states

$$\mathrm{Tr}(\Omega_S^{(n)} O_S) = \omega_{SE} \circ \Phi_n(O_S \otimes \mathbb{1}_E), \tag{7}$$

$$\mathrm{Tr}(\Omega_{[-a,b]}^{(n)} A_E^{[-a,b]}) = \omega_{SE} \circ \Phi_n(\mathbb{1}_S \otimes A_E^{[-a,b]}), \tag{8}$$

for all $O_S \in \mathcal{A}_S$ and $A_E^{[-a,b]} \in \mathcal{A}_E^{[-a,b]}$.

A factorized state $\omega_{SE} = \omega_S \otimes \omega_E$ on $\mathcal{A}_S \otimes \mathcal{A}_E$ is represented on $\mathcal{A}_S \otimes \mathcal{A}_E^{[-a,b]}$ by a factorized density matrix $\Omega_{S[-a,b]} = \rho_S \otimes \rho_E^{[-a,b]}$ and shows no correlations between system and collisional environment. Evidently, due to the dynamical coupling (3), correlations might develop between S and E under the action of Φ_n. Within the proposed algebraic setting, these correlations can be assessed by the mutual information. For a generic bipartite system $A + B$ with state ρ_{AB} and marginals $\rho_{A,B}$ the mutual information is given by:

$$\mathcal{I}_{AB} = S(\rho_A) + S(\rho_B) - S(\rho_{AB}) \geqslant 0, \tag{9}$$

where $S(\rho) = -\mathrm{Tr}\rho \log \rho$ is the von Neumann entropy. Indeed, $S(\rho_{AB}) \leqslant S(\rho_A) + S(\rho_B)$, while equality holds if only if $\rho_{AB} = \rho_A \otimes \rho_B$.

As we are interested in the discrete-time behaviour of correlations between open system S and sub-algebras $\mathcal{A}_E^{[-a,b]}$, we shall focus upon the following time-dependent mutual information

$$\mathcal{I}_{S[-a,b]}^{(n)} = S(\Omega_S^{(n)}) + S(\Omega_{[-a,b]}^{(n)}) - S(\Omega_{S[-a,b]}^{(n)}). \tag{10}$$

An increase/decrease with n of $\mathcal{I}_{S[-a,b]}^{(n)}$ would signal increasing/decreasing correlations between system and environment.

Let the environment E be a commutative chain with at each site a same commutative algebra $\mathcal{A} = D_d(\mathbb{C})$ spanned by 1-dimensional orthogonal projections $\{\Pi_i\}_{i=0}^{d-1}$, $\sum_{k=0}^{d-1}\Pi_k = \mathbb{1}$. It is turned into a Markov chain by endowing it with a state ω_E, identified by the local density matrices:

$$\rho_E^{[-a,b]} = \sum_{\boldsymbol{i}_{[-a,b]}} p_{\boldsymbol{i}_{[-a,b]}} \Pi_{\boldsymbol{i}_{[-a,b]}}^{[-a,b]} \in \mathcal{A}_E^{[-a,b]}, \tag{11}$$

where the projections $\Pi_{\boldsymbol{i}_{[-a,b]}}^{\otimes[-a,b]} = \otimes_{k=-a}^{b} \Pi_{i_k}^{(k)}$ generate the commutative sub-algebras $\mathcal{A}_E^{[-a,b]}$, and the probabilities $p_{\boldsymbol{i}_{[-a,b]}}$ satisfy:

$$p_{\boldsymbol{i}_{[-a,b]}} = T_{i_b i_{b-1}} T_{i_{b-1} i_{b-2}} \cdots T_{i_{-a+1} i_a} p_{i_{-a}} \tag{12}$$

where $p_i \geqslant 0$, $\sum_{i=1}^{d} p_i = 1$, while $T = [T_{ij}]$ satisfies $T_{ij} \geqslant 0$ and $\sum_{i=1}^{d} T_{ij} = 1$ so that $\mathrm{Tr}_b\, \rho_E^{[a,b]} = \rho_E^{[a,b-1]}$. Furthermore, the probability vector $\boldsymbol{p} = (p_1, \ldots, p_d)$ is chosen such that $T\boldsymbol{p} = \boldsymbol{p}$; then, $\mathrm{Tr}_a\, \rho_E^{[a,b]} = \rho_E^{[a+1,b]} = \rho_E^{[a,b-1]}$ and shift-invariance of the environment state is ensured, that is $\rho_E^{[a,b]} = \rho_E^{[a+n,b+n]}$ for all $n \in \mathbb{N}$.

Finally, let system and environment interact at site 0 through the map

$$\Phi[O_S \otimes A^{(0)}_{i_0}] = \sum_{i=0}^{d-1} \phi_i[O_S] \otimes \Pi_i A^{(0)}_{i_0} \Pi_i, \tag{13}$$

the maps ϕ_i being completely positive and unital, $\phi_i[\mathbb{1}] = \mathbb{1}$. Then, as proved in appendix A, its extension to the whole tensor product $\mathcal{A}_S \otimes \mathcal{A}_E$ gives the step-1 dynamics

$$\begin{aligned}\Phi_1[O_S \otimes A^{(0)}_{i_0}] &= \Theta\circ\Phi[O_S \otimes A^{(0)}_{i_0}] \\ &= \sum_{i=0}^{d-1} \phi_i[O_S] \otimes \Pi^{(1)}_i A^{(1)}_{i_0} \Pi^{(1)}_i.\end{aligned} \tag{14}$$

Furthermore, if the maps ϕ_i are invertible, Φ is an automorphism of the algebra $\mathcal{A}_S \otimes \mathcal{A}_E$; namely $\Phi[AB] = \Phi[A]\Phi[B]$ for all $A, B \in \mathcal{A}_S \otimes \mathcal{A}_E$.

In summary, the algebraic setting just presented accommodates a collisional model within a correlated multi-partite classical environment [23, 24], where system and ancilla at site $k = 0$ may either interact reversibly or be instantaneously immersed in the same dissipative environment before the shift is applied.

Notice that $\Phi[\mathbb{1}_S \otimes A^{(0)}_{i_0}] = \mathbb{1}_S \otimes A^{(0)}_{i_0}$. As a consequence, the environment is stationary,

$$\omega_{SE}(\Phi_n[\mathbb{1}_S \otimes A^{[a,b]}_E]) = \omega_{SE}(\mathbb{1}_S \otimes A^{[a,b]}_E). \tag{15}$$

Therefore, in the following, we focus upon the discrete-time reduced dynamics of the states of S, Λ_n: $\rho_S \mapsto \rho_{S_n}$. It is obtained from restricting to the system S,

$$\omega_{SE}(\Phi_n[O_S \otimes \mathbb{1}_E]) = \mathrm{Tr}(\Lambda_n[\rho_S] O_S). \tag{16}$$

We summarize the results concerning the reduced dynamics of system and environment in the following

Proposition 1. *The reduced dynamics arising by collisional coupling (14) of the system to a classical spin chain in a state specified by (11) and (12) consists of a discrete-time family of CPTP maps,*

$$\Lambda_n[\rho_S] := \sum_{\boldsymbol{i}_{[1,n]}} p_{\boldsymbol{i}_{[1,n]}} \phi^{\ddagger}_{\boldsymbol{i}_{[1,n]}}[\rho_S] = \Omega^{(n)}_S, \qquad \phi^{\ddagger}_{\boldsymbol{i}_{[1,n]}} = \phi^{\ddagger}_{i_n} \cdots \phi^{\ddagger}_{i_1}. \tag{17}$$

with $\phi^{\ddagger}_i$ the CPTP map dual to the CPU map ϕ_i in (13): $\mathrm{Tr}(\rho_S \phi_i[O_S]) = \mathrm{Tr}(\phi^{\ddagger}_i[\rho_S] O_S)$. On the other hand, the environment state is stationary,

$$\Omega^{(n)}_{[-a,b]} = \rho^{[-a,b]}_E. \tag{18}$$

The proof is reported in appendix A. The collision model naturally provides a discrete-time dynamics. In discrete-time, the notion of divisibility is naturally drawn from continuous-time: Λ_n is (C)P divisible if it can be written as $\Lambda_n = \Lambda_{n,m}\Lambda_m \forall\ n \geqslant m \in \mathbb{N}$, with $\Lambda_{n,m} = \Lambda_n \Lambda_m^{-1}$ a (C)PTP map.

2.1. Concrete collisional model

To investigate the physics behind the phenomenon of SBFI, we now consider two statistically coupled parties $S = S_1 + S_2$, each independently interacting with its own Markov-chain environment, with compound reduced dynamics $\Lambda_n \otimes \Lambda_n$. Thus $\mathcal{A}_S = M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$, while the Markov chain is chosen to consist of diagonal 4×4 matrices, $\mathcal{A}^{(k)}_E = D_4(\mathbb{C})$, and ϕ_i to be unital Pauli maps:

$$\phi_k[\sigma_j] = \mu^{(j)}_k \sigma_j, \qquad \mu^{(j)}_0 = \mu^{(0)}_k = 1, \qquad \mu^{(j)}_k = \varphi^{1-\delta_{jk}}, \tag{19}$$

for $j \neq 0$, $k \neq 0$ with φ a real parameter, where σ_j, $j = 1, 2, 3$, are the Pauli matrices, while $\sigma_0 = \mathbb{1}$. From (17), also Λ_n results a unital Pauli map; indeed,

$$\Lambda_n[\sigma_j] = \lambda^{(j)}_n \sigma_j, \qquad \lambda^{(j)}_n = \sum_{\boldsymbol{i}_{[1,n]}} p_{\boldsymbol{i}_{[1,n]}} \mu^{(j)}_{\boldsymbol{i}_{[1,n]}}, \tag{20}$$

where $\mu^{(j)}_{\boldsymbol{i}_{[1,n]}} \equiv \prod_{k=1}^n \mu^{(j)}_{i_k}$.

The maps Λ_n are invertible; then, $\Lambda_n = \Lambda_{n,n-1}\circ\Lambda_{n-1}$ with $\Lambda_{n,n-1} = \Lambda_n\circ\Lambda^{-1}_{n-1}$ and

$$\Lambda_{n,n-1}[\sigma_j] = \frac{\lambda^{(j)}_n}{\lambda^{(j)}_{n-1}} \sigma_j. \tag{21}$$

Let the Markov transition T in (12) be

$$T = \begin{pmatrix} p_0 & p_0 & p_0 & p_0 \\ p & p+\Delta & p-\Delta & p \\ p & p-\Delta & p+\Delta & p \\ r & r & r & r \end{pmatrix} \tag{22}$$

with positive parameters such that

$$0 \leqslant \Delta \leqslant p \leqslant \frac{1}{2}, \qquad p_0 + 2p + r = 1, \tag{23}$$

and with invariant probability vector $\boldsymbol{p} = (p_0, p, p, r)$. When $\Delta = 0$, it follows that $T_{ij} = p_i$, for all j so that the probabilities factorize, $p_{\boldsymbol{i}_{[-a,b]}} = \prod_{k=-a}^{b} p_{i_k}$, and

$$\rho_E^{[-a,b]} = \rho_E^{(-a)} \otimes \cdots \otimes \rho_E^{(b)}, \qquad \rho_E^{(j)} = \sum_{i=0}^{3} p_i \, \Pi_i^{(j)}.$$

Further, from (17), it follows that such an uncorrelated environment yields a reduced dynamics which is a CPTP discrete-time semigroup $\Lambda_n = \Lambda^n$, where $\Lambda[\rho_S] = \sum_{i=0}^{3} p_i \phi_i^{\ddagger}[\rho_S]$.

On the contrary, if $\Delta > 0$, the mutual information in (9) with $\rho_A = \rho_E^{(k)}$, $\rho_B = \rho_E^{(k+1)}$ and $\rho_{AB} = \rho_E^{[k,k+1]}$ yields

$$\mathcal{I}_{k,k+1} = 4p^2\left(\log 2 - h\left(\frac{1+Q}{2}\right)\right), \qquad Q \equiv \frac{\Delta}{p},$$

and $h(x) = -x\log x - (1-x)\log(1-x)$ decreases for $1/2 \leqslant x \leqslant 1$. Due to the stationarity of the Markov process, $\mathcal{I}_{k,k+1}$ is site independent and the correlations between any two successive environment sites increase with $0 \leqslant \Delta \leqslant p$. Furthermore, for $\Delta > 0$ the dynamical map Λ_n is no longer a semigroup and the evolution is governed by the following

Proposition 2. *Choosing the maps ϕ_k as in (19) and the transition matrix as in (22), the spectrum of the dynamics Λ_n,*

$$\Lambda_n[\sigma_j] = \lambda_n^{(j)}\sigma_j, \qquad \lambda_n^{(j)} = \sum_{\boldsymbol{i}_{[1,n]}} p_{\boldsymbol{i}_{[1,n]}} \, \mu_{\boldsymbol{i}_{[1,n]}}^{(j)}, \qquad j = 0, 1, 2, 3,$$

satisfies the following recurrences

$$\lambda_n^{(1,2)} =: \lambda_n = [1 - (p+r)(1-\varphi)]\,\lambda_{n-1} + p\,\Delta(1-\varphi)^2 \sum_{j=0}^{n-2} \lambda_j\,[(1+\varphi)\Delta]^{n-j-2}, \tag{24}$$

$$\lambda_n^{(3)} = [1 - 2p\,(1-\varphi)]\,\lambda_{n-1}^{(3)}. \tag{25}$$

Proof. Due to the form of the transition matrix,

$$T = \begin{pmatrix} p_0 & p_0 & p_0 & p_0 \\ p & p & p & p \\ p & p & p & p \\ r & r & r & r \end{pmatrix} + \Delta \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

summing over the index i_n in (20) yields

$$\lambda_n^{(j)} = A_{n-1}^{(j)}\,\lambda_{n-1}^{(j)} + \Delta(\mu_1^{(j)} - \mu_2^{(j)})\,B_{n-1}^{(j)} \qquad \forall\, j = 0, 1, 2, 3, \tag{26}$$

where, for $n \geqslant 1$

$$A_{n-1}^{(j)} = p_0 + p\,(\mu_1^{(j)} + \mu_2^{(j)}) + r\mu_3^{(j)}, \quad B_{n-1}^{(j)} = \sum_{\boldsymbol{i}_{[1,n-2]}} (T_{1 i_{n-2}}\mu_1^{(j)} - T_{2 i_{n-2}}\mu_2^{(j)})\,p_{\boldsymbol{i}_{[1,n-2]}}\,\mu_{\boldsymbol{i}_{[1,n-2]}}^{(j)}, \tag{27}$$

with $p_{\boldsymbol{i}^0} = 1$, $T_{i0} = p_i$ and $B_0^{(j)} = 0$. Then, summing over i_{n-2} in the expression for $B_{n-1}^{(j)}$, one gets

$$B_{n-1}^{(j)} = p\,(\mu_1^{(j)} - \mu_2^{(j)})\,\lambda_{n-2}^{(j)} + \Delta\,(\mu_1^{(j)} + \mu_2^{(j)})\,B_{n-2}^{(j)}, \tag{28}$$

and, iterating,

$$\begin{aligned} B_{n-1}^{(j)} &= p\,(\mu_1^{(j)} - \mu_2^{(j)})\,(\lambda_{n-2}^{(j)} + \Delta\,(\mu_1^{(j)} + \mu_2^{(j)})\,\lambda_{n-3}^{(j)}) + \Delta^2\,(\mu_1^{(j)} + \mu_2^{(j)})^2\,B_{n-3}^{(j)} \\ &= p\,(\mu_1^{(j)} - \mu_2^{(j)}) \sum_{k=0}^{n-2} \lambda_k^{(j)}\,\Delta^{n-k-2}\,(\mu_1^{(j)} + \mu_2^{(j)})^{n-k-2}, \end{aligned} \tag{29}$$

where we set $\lambda_0^{(j)} = 1$. Since $\mu_j^{(0)} = 1$ for $j = 0, 1, 2, 3$, from (23) it follows that $A_{n-1}^{(j)} = p_0 + 2p + r = 1$ and $B_{n-1}^{(j)} = 0$ so that $\lambda_n^{(0)} = 1$ for all $n \in \mathbb{N}$. On the other hand, the choice of the other coefficients $\mu_k^{(j)}$ in (19) gives

$$A_{n-1}^{(1,2)} = p_0 + p(1 + \varphi) + r\varphi, \qquad B_{n-1}^{(1,2)} = \pm p(1 - \varphi)\sum_{k=0}^{n-2} \lambda_k^{(1,2)} \Delta^{n-k-2}(1 + \varphi)^{n-k-2}, \tag{30}$$

$$A_{n-1}^{(3)} = p_0 + 2p\varphi + r \quad B_{n-1}^{(3)} = 0. \tag{31}$$

Since $p_0 + 2p + r = 1$ the expressions in (24) and (25) follow. □

We shall now study the model for two distinct choices of φ in (19), corresponding respectively to (1) a unitary coupling, discussed in Section 2.1.1, for which the solution of (24) can be analytically computed and (2) a dissipative coupling, presented in section 2.1.2, for which the natural stroboscopic limit of collisional models [20, 23, 29] is analytically available and allows one to compare the continuous-time scenario with the discrete-time one.

2.1.1. Unitary case

Set $\varphi = -1$; then, $\phi_k[X] = \sigma_k X \sigma_k$ and the map (13) becomes a 'controlled-unitary' typical of collisional models [26, 27]. In this scenario, the interaction between system S and the environment E is described by means of a unitary matrix $U_\tau = e^{-ig\tau \sum_k \sigma_k \otimes \Pi_k}$ for a duration $\tau = \pi/2g$: $\Phi[X] = U_{\pi/2g}^\dagger X\, U_{\pi/2g}$. Only $j = n - 2$ contributes to the sum in (24) and in Appendix B, the recurrence relations (24) and (25) are shown to yield

$$\lambda_n = \left(\frac{\beta + \alpha}{2\beta}\right)\left(\frac{\beta + \alpha}{2}\right)^n + \left(\frac{\beta - \alpha}{2\beta}\right)\left(\frac{\alpha - \beta}{2}\right)^n,$$
$$\lambda_n^{(3)} = (1 - 4p)^n, \tag{32}$$

where we set

$$\alpha \equiv 1 - 2(p + r)\,, \quad \beta \equiv \sqrt{\alpha^2 + 16\, p\, \Delta}\,. \tag{33}$$

Let $\alpha > 0$ so that $\lambda_n^{(1,2)} > 0$. The type of divisibility of the reduced dynamics depends on the environment correlations as follows.

Proposition 3.

(i) Λ_n *is P-divisible if and only if*

$$2p\Delta \;\leqslant\; r\alpha + p\alpha, \tag{34}$$

(ii) Λ_n *is CP-divisible if and only if*

$$2p\Delta \;\leqslant\; r\alpha, \tag{35}$$

(iii) $\Lambda_n \otimes \Lambda_n$ *is P-divisible if and only if*

$$2p\Delta \;\leqslant\; \alpha(r + p) - \frac{\alpha}{2}\left(1 + \sqrt{1 - 4p(1 - 2p)}\right). \tag{36}$$

For the proof, see Appendix B. Notice the strength of the environmental correlations Δ governs the divisibility degree of the reduced dynamics, in that (ii)⇒ (iii) ⇒ (i); on the other hand, (iii) ⇏ (ii) (see Remark 1 below).

To illustrate how the intensity of the environmental correlations relates to the emergence of SBFI, consider $r = 0$ so that $2\Delta \leqslant \alpha$ and, by (i), Λ_n is guaranteed to be P-divisible. Then, the discrete-time intertwiners $\Lambda_{n,m}$ are contractive and forbids BFI for a single qubit.

Then, we consider $p \ll 1$ and proceed with a perturbative analysis. Given any $X = X^\dagger \in M_2(\mathbb{C})$, one has that (see Appendix B for details) up to second order in p,

$$||\Lambda_{n,n-1}[X]||_1 - ||X||_1 = -K_1\, p + K_2(\Delta)\, p^2 + o(p^2),$$

with $K_1 \geqslant 0$ and $K_2(\Delta) > 0$ and no discrete-time dependence. Therefore, possible environment correlations ($\Delta \neq 0$) contribute with a positive second order term in the small parameter p; this latter cannot counteract the negative, correlation independent first order term which then makes the maps $\Lambda_{n,n-1}$ contractive for all time-steps n in the regime $0 \leqslant \Delta \leqslant p \ll 1$, thus concretely showing why there cannot be BFI for one qubit: the single qubit state distinguishability can never increase in time.

On the other hand, considering now two qubits, again setting $r = 0$, at leading order in $0 \leqslant p \ll 1$, the positivity condition (36) implies $\Delta/p \equiv Q \leqslant 1/2$. Therefore, if $Q > 1/2$, $\Lambda_{n,n-1} \otimes \Lambda_{n,n-1}$ cannot be positive and is thus not contractive. Moreover, being $\Lambda_n \otimes \Lambda_n$ invertible, the collisional dynamics of two qubits certainly exhibits SBFI, namely increasing distinguishability as witnessed by a suitably constructed two-qubit Helstrom statistical ensemble through the corresponding Helstrom matrix. Also, the lack of positivity of $\Lambda_{n,n-1} \otimes \Lambda_{n,n-1}$ for $Q > 1/2$ is easily seen by acting on totally symmetric projector P_2^+. Indeed, as shown in Appendix B,

$$||\Lambda_{n,n-1} \otimes \Lambda_{n,n-1}[P_2^+]||_1 - ||P_2^+||_1 = 4p^2 \quad (2\,Q - 1) > 0,$$

hence $\Lambda_{n,n-1} \otimes \Lambda_{n,n-1}$ is non-contractive, hence not Positive.

Remark 1. Unlike $\Lambda_t \otimes \Lambda_t$ in continuous time, in discrete time $\Lambda_n \otimes \Lambda_n$ can be P-divisible even if Λ_n is not CP-divisible. Indeed, the main result of [13] is based on the existence of time-local generators. Thus, even if Λ_n is not CP-divisible, $\Lambda_n \otimes \Lambda_n$ need not automatically display SBFI. However, as we saw above, in our case SBFI is triggered by sufficiently strong environment correlations that help to violate the inequality (36).

We now study the single and two qubit information flows from and into the collisional environment by means of the system-environment correlations as quantified by the mutual information. For that we restrict the system-environment state at discrete-time n, $\omega_{SE}^{(n)}$, on a local observable $O_S \otimes A_E^{[-a+1,b]}$, $a,\ b \in \mathbb{N}$. One thus retrieves the evolved local system-environment density matrix $\Omega^{(n)}_{S[-a+1,b]}$ through (6), given by (see Appendix C equation (C2))

$$\Omega^{(n)}_{S[-a,b]} = \sum_{\boldsymbol{\ell}_{[-n+1,b]}} p_{\boldsymbol{\ell}_{[-a,b]}}\, \phi^{\ddagger}_{\boldsymbol{\ell}_{[-n+1,0]}}[\rho_S] \otimes \Pi^{[-a,b]}_{\boldsymbol{\ell}_{[-n+1,b]}}. \tag{37}$$

We shall then consider the mutual information (10) relative to the compound state at discrete-time n (37) as a faithful quantifier of the system-chain correlations. In Appendix C, equation (C7), it is shown that the latter quantity takes the form

$$\mathcal{I}^{(n)}_{S[-a,b]} = S(\Lambda_n[\rho_S]) - \sum_{\boldsymbol{i}_{[1,n]}} p_{\boldsymbol{i}_{[1,n]}} \quad S(\phi^{\ddagger}_{\boldsymbol{i}_{[1,n]}}[\rho_S]).$$

Note that the previous expression depends only on n and not on the size of the portion of the chain considered. Taking into account, as above, two independent qubits coupled to identical chains, the maximal mutual information of their local density matrix reads

$$\mathcal{I}^{(n)}_{(S+S)E} = S(\Lambda_n \otimes \Lambda_n[\rho_{S+S}]) - \sum_{\ddagger \boldsymbol{i}_{[1,n]}, \boldsymbol{k}_{[1,n]}} p_{\boldsymbol{i}_{[1,n]}} p_{\boldsymbol{k}_{[1,n]}} S(\phi^{\ddagger}_{\boldsymbol{i}_{[1,n]}} \otimes \phi^{\ddagger}_{\boldsymbol{k}_{[1,n]}}[\rho_{S+S}]). \tag{38}$$

In the case under consideration, the unital maps ϕ_i are unitary; thus (38) yields

$$\mathcal{I}^{(n)}_{(S+S)E} = S(\Lambda_n \otimes \Lambda_n[\rho_{S+S}]) - S(\rho_{S+S}); \tag{39}$$

in particular, the variation of the mutual information between two discrete-times $n \geqslant m$ reduces to checking the behaviour of two-qubit entropy:

$$\Delta\mathcal{I}^{(n,m)}_{(S+S)E} \equiv S(\Lambda_n \otimes \Lambda_n[\rho_{S+S}]) - S(\Lambda_m \otimes \Lambda_m[\rho_{S+S}]). \tag{40}$$

Let us recall that the von Neumann entropy increases under PTP unital maps [30–32]; thus, when the unital single-qubit reduced dynamics is P-divisible, $\Delta\mathcal{I}^{(n,m)}_{SE} \geqslant 0$. On the other hand, moving to two qubits, choose as a concrete instance

$$r = 0, \quad \frac{1}{4} \leqslant p \leqslant \frac{1}{2}, \quad \Delta = \frac{1-2p}{2} \leqslant p \leqslant \frac{1}{2}, \tag{41}$$

so that Λ_n is P-divisible with (34) being saturated and the Pauli eigenvalues (32) at the first two successive discrete-time steps satisfy $\lambda_1 = \lambda_2 = \alpha = 1 - 2p$. Further, choosing $p = 1/4 + \epsilon$, $\epsilon \ll 1$, one can perform a perturbative study and show that the two-qubit completely symmetric projector $\rho_{S+S} = P_2^+$ witnesses a decrease of the two-qubit von Neumann entropy (details can be found in Appendix E),

$$\Delta\mathcal{I}^{(2,1)}_{(S+S)E} = -4\log(4/3)\,\epsilon^2 < 0, \tag{42}$$

hence a decrease of system environment correlations between the first and the second collision.

2.1.2. Dissipative case and stroboscopic limit

Let us now take $\varphi = e^{-2\gamma\tau}$, $\gamma, \tau > 0$ so that $\phi_0 = \mathrm{id}$ and for $k \neq 0$

$$\phi_k = e^{\tau \mathcal{L}_k}, \qquad \mathcal{L}_k[X] = \gamma\,(\sigma_k X \sigma_k - X). \tag{43}$$

In such case, our model is analogous to a collisional model in which the qubit $\mathcal{A}_S$ and the ancilla $\mathcal{A}_E^{(0)}$ undergo a joint dissipative evolution $O_S \otimes O_E^{(0)} \mapsto e^{\tau\mathbb{L}}[O_S \otimes O_E^{(0)}]$ for a time τ, before the shift on the chain is applied (the form of the the GKLS generator $\mathbb{L}$ is reported in Appendix D). The Markov chain correlations contribute with memory effects on top of this Markovian semigroup dynamics and, moreover, it allows one to retrieve a continuous-time dissipative dynamics and compare BFI and SBFI within such a continuous context. The technique employed is the so-called stroboscopic limit defined by $\tau \to 0$, $n \to \infty$, $n\tau \to t$. Choosing $\Delta = e^{-\kappa\tau}/2$, $p \to 1/2$ and, straightforwardly, $\lambda_t^{(3)} = e^{-2\gamma t}$, while the other two Pauli eigenvalues are both equal to the solution λ_t of the integro-differential equation

$$\dot{\lambda}_t = -\gamma\,\lambda_t + \gamma^2 \int_0^t \mathrm{d}s e^{-(\kappa+\gamma)(t-s)}\,\lambda_s, \tag{44}$$

which yields (see Appendix D)

$$\lambda_t = \quad e^{-(\gamma+\frac{\kappa}{2})t}\left[\cosh(Kt) + \frac{\kappa}{2K}\sinh(Kt)\right], \tag{45}$$

where $K \equiv \sqrt{\kappa^2 + 4\gamma^2}/2$. We thus obtain a family of P-divisible Pauli dynamical maps, with generator $\mathcal{L}_t[\rho] = \frac{1}{2}\sum_{i=1}^3 \gamma_t^{(i)}(\sigma_i\rho\sigma_i - \rho)$ and rates

$$\gamma_t^{(1)} = \gamma_t^{(2)} = \gamma, \tag{46}$$

$$\gamma_t^{(3)} = -\frac{2\gamma^2}{\sqrt{\kappa^2 + 4\gamma^2}\coth\left(\frac{1}{2}t\sqrt{\kappa^2 + 4\gamma^2}\right) + \kappa}, \tag{47}$$

with $\gamma_t^{(3)}$ being negative at all times.

System-environment correlations

Let us consider the case $\Delta = 1/2$ and $p = 1/2$. Notice that such case corresponds to $\kappa = 0$ and $\gamma_t^{(3)} = -\gamma\tanh(\gamma t)$, namely to the well known 'eternally' non-Markovian evolution firstly discussed in [33].

In such case, only two sequences $\boldsymbol{i}_{[1,n]}$ have non-vanishing probabilities and thus contribute to (17), namely $\boldsymbol{1} = 111\ldots$ and $\boldsymbol{2} = 222\ldots$ with probabilities $p_{\boldsymbol{1}} = p_{\boldsymbol{2}} = 1/2$. Accordingly, the continuous-time limit of (38) reads

$$\mathcal{I}^{(t)}_{(S+S)E} = S(\Lambda_t \otimes \Lambda_t[\rho_{S+S}]) - \frac{1}{4}\sum_{i,j=1,2} S(e^{t\mathcal{L}_i} \otimes e^{t\mathcal{L}_j}[\rho_{S+S}]). \tag{48}$$

Notice that, unlike in the unitary case, each of the entropies in the second term now grows in time due to the joint unital dissipative evolution of $\mathcal{A}_S$ and $\mathcal{A}_E^{(0)}$ that mixes them.

We study $\mathcal{I}^{(t)}_{(S+S)E}$ picking ρ_{S+S} of X-shape with respect to the eigenvectors of the matrix $\sigma_1 \otimes \sigma_1$:

$$\rho_X^{(1)} = \begin{pmatrix} \mu_1 & 0 & 0 & u \\ 0 & \nu & \nu & 0 \\ 0 & \bar{\nu} & 1 - (\mu_1 + \mu_2 + \nu) & 0 \\ \bar{u} & 0 & 0 & \mu_2 \end{pmatrix}. \tag{49}$$

In Appendix E, its decomposition in terms of the Pauli matrix tensor products $\{\sigma_i \otimes \sigma_j\}_{ij}$ is reported, from which the time-evolving states entering (48) can be easily inferred. In figure 2, we display the system-chain mutual information when the system is initialized in a state of the class (49), which displays a growth and collapse of correlations. We also compare such behaviour with that of $||\Lambda_t \otimes \Lambda_t[\Delta_\mu(\rho_X^{(1)}, \rho_X^{(3)})]||_1$, where $\rho_X^{(3)}$ has X shape in the computational basis.

Thus, the system-chain correlations can undergo a decrease for a certain time interval, despite the stationarity of the environment.

Remark 2. The information lost by the system and subjected to BFI is generally thought to be stored either in system-environment correlations or in changes of the environmental state (notice that in our Example, the environment is stationary (15)) [9, 34]. In figure 2 the mutual information $\mathcal{I}_{SE}^{(t)}$ of (48) is plotted for X states with $\mu_{1,2} = \nu = 1/4$. As for the maximally entangled state state P_2^+ considered in (42), these states have maximally mixed marginals. For a state ρ_{S+S} with maximally mixed marginals, using trace preservation and factorization, one shows that

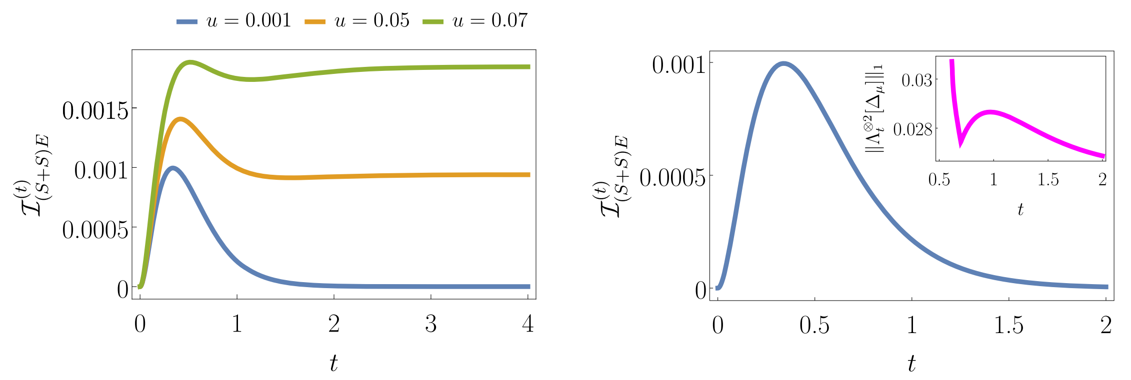

Figure 2. System-chain mutual information for the state $\rho_X^{(1)}$ with fixed $\mu_1 = \mu_2 = \nu = 1/4$, $v = i/8$ at different values of real u. For $u = 0.001$, the behaviour is compared with that of the trace norm of the Helstrom matrix between $\Lambda_t \otimes \Lambda_t[\rho_X^{(1)}]$ and $\Lambda_t \otimes \Lambda_t[\rho_X^{(3)}]$, with bias $\mu = 0.52$, $\rho_X^{(3)}$ being in the form (49) w.r.t.the computational basis and defined by parameters $\mu_1^{'} = \mu_2^{'} = 1/2,\ \nu^{'} = 0,\ u^{'} = 1/8,\ v^{'} = 0$. One easily sees that the marginals of $\rho_X^{(1)}$ are the maximally mixed state $\mathbb{1}_2/2$.

$$\mathrm{Tr}_{1(2)}(\Lambda_t \otimes \Lambda_t[\rho_{S+S}]) = \Lambda_t[\mathrm{Tr}_{1(2)}(\rho_{S+S})] = \frac{\mathbb{1}_2}{2}.$$

Similarly, one checks that the one-qubit local density matrix (37), obtained by tracing over one of the two open systems together with its own environment, reduces to

$$\Omega^{(t)}_{S[-a,b]} = \frac{\mathbb{1}_2}{2} \otimes \rho_E^{[-a,b]} \Rightarrow \mathcal{I}^{(t)}_{SE} = 0. \tag{50}$$

For such states, the bipartite correlations have a non-monotonic behaviour in time, while the qubit-chain marginals are uncorrelated at all times due to (50). Thus, in such case, the information is temporarily stored non-locally in the system-environment correlations.

3. Quantum signature of SBFI

SBFI is undoubtedly a memory effect with no classical counterpart, despite it might arise from the coupling to a classical collisional environment. The reason is that positivity and complete positivity coincide for mappings on commutative algebras. To illustrate this in more detail recall that, in a commutative setting, the Helstrom matrix takes the form

$$\Delta_\mu(\rho_S, \sigma_S) = \sum_i (\mu p_i - (1 - \mu) q_i) P_i, \tag{51}$$

p_i, q_i being, respectively, the eigenvalues of ρ_S and σ_S and P_i their common eigenprojectors. Thus, the Helstrom distinguishability reduces to the ℓ_1-norm of the vector $\|\mu|p\rangle - (1 - \mu)|q\rangle\|_{\ell_1}$, with $\||x\rangle\|_{\ell_1} = \sum_{i=1}^d |x_i|$. In the case of a classical bipartite system, consider a real vector $|x\rangle = \sum_{ij} x_{ij}|i\rangle \otimes |j\rangle \in \mathbb{R}^d \otimes \mathbb{R}^d$ evolving into $|x_t\rangle = T(t) \otimes T(t)|x\rangle$, under the action of a continuous-time P-divisible stochastic process $T(t)$, such that for all $t \geqslant s \geqslant 0$, $T(t) = T(t, s)T(s)$, with $T(t, s)$ a stochastic matrix, $T_{ik}(t, s) \geqslant 0$ and $\sum_i T_{ik}(t, s) = 1$. Under such dynamics, the ℓ_1-norm of a time-evolving vector $|x\rangle = \{x_{ij}\} \in \mathbb{R}^d \times \mathbb{R}^d$ cannot increase in time,

$$\partial_t \|T(t) \otimes T(t)|x\rangle\|_{\ell_1} \leqslant 0, \quad \forall\ |x\rangle \in \mathbb{R}^d \times \mathbb{R}^d. \tag{52}$$

Indeed,

$$\begin{aligned} \||x_t\rangle\|_{\ell_1} = \sum_{ij} \left| \sum_{kl} T_{ik}(t, s)\, T_{jl}(t, s) x_{kl}(s) \right| &\leqslant \sum_{kl}\sum_{ij} |T_{ik}(t, s)|\, |T_{jl}(t, s)|\, |x_{kl}(s)| \\ &\leqslant \sum_{kl} |x_{kl}(s)| = \||x_s\rangle\|_{\ell_1}, \end{aligned}$$

for all $t \geqslant s \geqslant 0$. For quantum systems, as we have seen, the phenomenon of SBFI is witnessed by the quantity

$$\begin{aligned} \Delta D_\mu(t + \tau, t) := &\|\Lambda_{t+\tau} \otimes \Lambda_{t+\tau}[\Delta_\mu(\rho_{S+S}, \sigma_{S+S})]\|_1 \\ &- \|\Lambda_t \otimes \Lambda_t[\Delta_\mu(\rho_{S+S}, \sigma_{S+S})]\|_1, \end{aligned} \tag{53}$$

assuming a strictly positive value at some $t, \tau > 0$, that is by revivals of the bipartite Helstrom distinguishability. The quantum character of such a memory effect can be assessed by the following measure of the quantum correlations present in the Helstrom ensemble $\mathcal{E}_H(t) = \{(\mu;\ \rho_{S+S}(t)),\ (1 - \mu;\ \sigma_{S+S}(t))\}$. The quantumness of a single-party ensemble $\mathcal{E} = \{(\mu_i,\ \rho_i)\}$ has been identified with the possibility of simultaneously diagonalizing it

[35, 36]; equivalently, if the ensemble is encoded into a quantum–classical state $\chi^{\mathcal{E}} = \sum_i \mu_i \rho_i \otimes \langle i|$, one can measure the ensemble quantumness in terms of the quantum correlations as left-sided quantum discord in $\chi^{\mathcal{E}}$ [36, 37]. Among the variety of available discord measures [38], we shall consider the so-called measurement induced geometric measure of quantum correlations defined in the trace norm by

$$\mathcal{Q}_{\{\mathbb{P}\}}(\rho) := \min_{\mathbb{P}} D(\rho,\ \mathbb{P} \otimes\ \mathrm{id}[\rho]),$$

where $\mathbb{P}[X] = \sum_i P_i X P_i$ is a projective measurement associated to $=\{P_i\}_i$, $P_i = \langle i|$ an orthonormal set of rank-1 projectors. If $\mathcal{E}$ is an ensemble of bipartite states, one rather focuses on finding a simultaneous diagonalization on a set of rank-1 projections of the type $\{P_i^1 \otimes P_j^2\}_{ij}$. Accordingly, in [39] the following measure of bipartite 'ensemble quantumness of correlations' was introduced:

$$\mathcal{Q}_{\{\mathbb{P}^1\otimes\mathbb{P}^2\}}(\chi^{\mathcal{E}}) = \min_{\mathbb{P}^1\otimes\mathbb{P}^2} \sum_i \mu_i\, D(\rho_i,\ \mathbb{P}^1 \otimes \mathbb{P}^2[\rho_i]), \tag{54}$$

with $\chi^{\mathcal{E}}$ now encoding the bipartite ensemble by means of an additional classical register.

The next result connects the bipartite quantumness of correlations of the Helstrom ensemble, as defined by (54), to the quantity (53) exposing SBFI.

Proposition 4. *Given a dynamics $\Lambda_t \otimes \Lambda_t$ with Λ_t P-divisible, the variation of the Helstrom distinguishability $\Delta D_\mu(t+\Delta t, t)$ can be bounded as follows*

$$\Delta D_\mu(t+\tau,\ t) \leqslant 2\ ||\Lambda_{t+\tau,t}||_\diamond^2 \quad \mathcal{Q}_{\{\mathbb{P}^1\otimes\mathbb{P}^2\}}(\chi^{\mathcal{E}_H}(t)), \tag{55}$$

where $\chi^{\mathcal{E}_H}(t) = \mu\rho_{S+S}(t) \otimes \langle 0| + (1-\mu)\sigma_{S+S}(t) \otimes \langle 1|$, *while* $||.||_\diamond$ denotes the diamond norm of a map.

Proof. Let us fix $\{|p_\alpha\rangle\}_\alpha$ with $|p_\alpha\rangle = |p_i^1\rangle \otimes |p_j^2\rangle$, $\{|p_i^1\rangle\}_i$, $\{|p_j^2\rangle\}_j$ being arbitrary local orthonormal basis, from which one has a corresponding orthonormal set of rank-1 projectors $\{P_i^1 \otimes P_j^2\}_{ij}$. Accordingly, a completely decohering map with respect to such basis is described by a (bi)local projective measurement:

$$\mathbb{P}_1 \otimes \mathbb{P}_2[X] = \sum_{ij} P_i^1 \otimes P_j^2\ X P_i^1 \otimes P_j^2.$$

Then, for $t > s > 0$ both in discrete and continuous time, considering the Helstrom matrix at time t, $\Lambda_t \otimes \Lambda_t[\Delta_\mu(\rho_{S+S}, \sigma_{S+S})]$, via the triangle inequality and the contractivity of $\Lambda_{t,s}$ and $\mathbb{P}_1 \otimes \mathbb{P}_2$, one estimates

$$\begin{aligned} ||(\Lambda_{t,s} \otimes\ \Lambda_{t,s}) \circ (\mathbb{P}_1 \otimes\ \mathbb{P}_2)[\Delta_\mu(s)]||_1 &\leqslant \sum_{ij} |\delta_\mu^{ij}(s)|\ ||\Lambda_{t,s}[P_i^1]||_1\ ||\Lambda_{t,s}[P_j^2]||_1 \\ &\leqslant ||\mathbb{P}_1 \otimes\ \mathbb{P}_2[\Delta_\mu(s)]||_1\ \leqslant\ ||\Delta_\mu(s)||_1, \end{aligned} \tag{56}$$

where $\delta_\mu^{ij}(s) := \langle p_i^1|\langle p_j^2|\Delta_\mu(s)|p_i^1\rangle|p_j^2\rangle$. Consider the induced trace norm and the diamond norm of $\Lambda\colon M_d(\mathbb{C}) \to M_{d'}(\mathbb{C})$ [40],

$$||\Lambda||_1 = \max\{||\Lambda[X]||_1\colon ||X||_1 \leqslant 1\}, \qquad ||\Lambda||_\diamond = ||\Lambda \otimes \mathrm{id}_d||_1.$$

Then, the variation of the Helstrom matrix,

$$\Delta D_\mu(t,\ s) := ||\Lambda_{t,s} \otimes \Lambda_{t,s}[\Delta_\mu(s)]||_1 - ||\Delta_\mu(s)||_1$$

can be upper-bounded as follows

$$\begin{aligned} \Delta D_\mu(t,\ s) &= ||\Lambda_{t,s}\otimes\Lambda_{t,s}[\Delta_\mu(s)]||_1 - ||\Delta_\mu(s)||_1 \\ &= ||\mu\ \Lambda_{t,s}\otimes\Lambda_{t,s}[\rho_{S+S}(s) - \mathbb{P}_1\otimes\mathbb{P}_2[\rho_{S+S}(s)]] \\ &\quad - (1-\mu)\Lambda_{t,s}\otimes\Lambda_{t,s}[\sigma_{S+S}(s) - \mathbb{P}_1\otimes\mathbb{P}_2[\sigma_{S+S}(s)]] + \Lambda_{t,s}\otimes\Lambda_{t,s}\circ\mathbb{P}_1\otimes\mathbb{P}_2[\Delta_\mu(s)]||_1 \\ &\quad - ||\Delta_\mu(s)||_1 \\ &\leqslant\ \mu\ ||\Lambda_{t,s}\otimes\Lambda_{t,s}[\rho_{S+S}(s) - \mathbb{P}_1\otimes\mathbb{P}_2[\rho_{S+S}(s)]]||_1 \\ &\quad + (1-\mu)\ ||\Lambda_{t,s}\otimes\Lambda_{t,s}[\sigma_{S+S}(s) - \mathbb{P}_1\otimes\mathbb{P}_2[\sigma_{S+S}(s)]]||_1 \\ &\quad + ||\Lambda_{t,s}\otimes\Lambda_{t,s}\circ\mathbb{P}_1\otimes\mathbb{P}_2[\Delta_\mu(s)]||_1 - ||\Delta_\mu(s)||_1. \end{aligned}$$

Using (56) and the fact that $||\Lambda \otimes \Lambda||_1 \leqslant ||\Lambda||_\diamond^2$, we have

$$\begin{aligned} \Delta D_\mu(t,\ s) \ &\leqslant\ ||\Lambda_{t,s}||_\diamond^2\ (\mu||\rho_{S+S}(s) - \mathbb{P}_1 \otimes\ \mathbb{P}_2[\rho_{S+S}(s)]||_1 \\ &+ (1-\mu)||\sigma_{S+S}(s) - \mathbb{P}_1 \otimes\ \mathbb{P}_2[\sigma_{S+S}(s)]||_1). \end{aligned}$$

Since $\mathbb{P}_{1,2}$ are arbitrary, one can tighten the latter inequality by minimizing over the projective measurements. One then finally obtains the following upper-bound for $\Delta D_\mu(t, s)$,

$$\Delta D_\mu(t,\ s)\ \leqslant\ 2\ ||\Lambda_{t,s}||_\diamond^2\ \ \mathcal{Q}_{\{\mathbb{P}^1\otimes\mathbb{P}^2\}}(\chi^{\mathcal{E}}(s)),$$

where the quantumness of the Helstrom ensemble $\mathcal{E}_H(s) = \{(\mu;\, \rho_{S+S}(s)),\, (1-\mu;\, \sigma_{S+S}(s))\}$, encoded in the quantum–classical state

$$\chi^{\mathcal{E}}(s) = \mu\ \rho_{S+S}(s)\langle 0| + (1-\mu)\ \sigma_{S+S}(s) \otimes \langle 1|,$$

is measured by the (left-sided) quantum correlations of $\chi^{\mathcal{E}_H}(s)$ [39]:

$$\mathcal{Q}_{\{\mathbb{P}^1\otimes\mathbb{P}^2\}}(\chi^{\mathcal{E}_H}(s)) = \frac{1}{2}\min_{\mathbb{P}^1\otimes\mathbb{P}^2}(\mu\|\rho_{S+S}(s) - \mathbb{P}^1\otimes\ \mathbb{P}^2[\rho_{S+S}(s)]\|_1 \\ + (1-\mu)\|\sigma_{S+S}(s) - \mathbb{P}^1\otimes\ \mathbb{P}^2[\sigma_{S+S}(s)]\|_1).$$

□

Remark 3. If SBFI triggers at time t, i.e. $\Delta D_\mu(t+\tau, t) > 0$, then the quantumness of correlations of the ensemble $\mathcal{E}_H(t) = \{(\mu,\, \rho_{S+S}(t));\, (1-\mu,\, \sigma_{S+S}(t))\}$ has to be strictly positive, that is, the state $\chi^{\mathcal{E}_H}(t)$ has to have a non zero quantum discord. In this sense, the Helstrom ensemble quantumness is a 'precursor' of non-Markovianity [41]. In particular, $\mathcal{Q}_{\{\mathbb{P}^1\otimes\mathbb{P}^2\}}(\chi^{\mathcal{E}_H}(t)) > 0$ does not imply that the states are entangled (as similarly noted in [42, 43]; a simple construction in the Pauli case for a quantum ensemble triggering SBFI but not involving entanglement is reported in Appendix F).

4. Conclusions

In this work we studied the SBFI in an open system of two qubits, each coupled to a classical Markov chain. The assumptions made in the treatment of the environment and the interaction allowed for a full analytical description of the system dynamics and system-environment correlations. Notice that there is a structural difficulty, both analytically and from the point of view of a microscopic derivation, to devise dynamics that are P-divisible but not CP-divisible. This reflects the lack of a general characterization of positive maps versus completely positive ones. Despite these general obstructions, the proposed model is sufficiently rich to provide a dynamics with a neat microscopic origin of its degree of divisibility, and able to display the SBFI effect.

Both in the discrete and continuous-time regimes, we investigated the emergence of bipartite memory effects by means of the system-chain mutual information of local density matrices obtained through an algebraic approach. Growths and collapses of correlations have been detected for both unitary and dissipative collisions: in the former case, the mutual information is simply the system's entropy up to a constant, while in the latter case it has the form of a Jensen-Shannon divergence. Despite the ongoing debate regarding the physical nature of Backflow of Information, especially in such kind of classical environments, the non-monotonicity of the aforementioned quantities provides a clear-cut physical interpretation in terms of system-environment correlations. Interestingly, despite information might be stored in and released through classical correlations, SBFI has no classical counterpart; however, the quantum resource needed to trigger it is only the quantumness of the Helstrom ensemble but not entanglement in its states.

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Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors.

Appendix A. Reduced dynamics

As seen in the main text, tensor product elements of the local algebra $\mathcal{A}_E^{[-a,b]}$ supported by the interval of integers $-a \leqslant j \leqslant b$ are denoted by means of the multi-indices $\boldsymbol{i}_{[-a,b]} = i_{-a}i_{-a+1}\cdots i_b$ as follows:

$$A^{\otimes[-a,b]}_{\boldsymbol{i}_{[-a,b]}} = A^{(-a)}_{i_{-a}} \otimes A^{(-a+1)}_{i_{-a+1}} \otimes \cdots \otimes A^{(b)}_{i_b} = \bigotimes_{k=-a}^{b} A^{(k)}_{i_k},$$

where the upper index in $A^{(k)}_{i_k}$ indicates the site k at which the operator A_{i_k} is located.

The collisional dynamics $\Phi_n = (\Theta\circ\Phi)^n$ comprises 1) the right shift Θ on $\mathcal{A}_E$ such that

$$\Theta[A^{[-a,b]}_{\boldsymbol{i}_{[-a,b]}}] = A^{[-a+1,b+1]}_{\boldsymbol{i}_{[-a,b]}},$$

and 2) the CPU map on the bipartite algebra $\mathcal{A}_S \otimes \mathcal{A}$ of system S and chain ancilla at site 0 defined by:

$$\Phi[O_S \otimes A^{(0)}_{i_0}] = \sum_{k=0}^{d-1} \phi_k[O_S] \otimes \Pi^{(0)}_k A^{(0)}_{i_0} \Pi^{(0)}_k, \quad A_{i_0} \in \mathcal{A} \quad atsite 0, \tag{A1}$$

with ϕ_k completely positive, unital maps on the system algebra $\mathcal{A}_S$. When extended to the whole algebra $\mathcal{A}_S \otimes \mathcal{A}_E$, Φ yields

$$\Phi_1[O_S \otimes A^{\otimes[-a,-1]}_{\boldsymbol{i}_{[-a,-1]}} \otimes A^{(0)}_{i_0} \otimes A^{\otimes[1,b]}_{\boldsymbol{i}_{[1,b]}}] = \sum_{k=0}^{d-1} \phi_k[O_S] \otimes A^{\otimes[-a+1,0]}_{\boldsymbol{i}_{[-a,-1]}} \otimes \Pi^{(1)}_k A^{(1)}_{i_0} \Pi^{(1)}_k \otimes A^{\otimes[2,b+1]}_{\boldsymbol{i}_{[1,b]}}. \tag{A2}$$

Iterating the action of $\Theta \circ \Phi$ one gets

$$\begin{aligned}\Phi_n[O_S \otimes \ A^{\otimes[-a,b]}_{\boldsymbol{i}_{[-a,b]}}] \ &= \ \sum_{\boldsymbol{k}_{[1,n]}} \phi_{\boldsymbol{k}_{[1,n]}}[O_S] \otimes \ A^{\otimes[-a+n,0]}_{\boldsymbol{i}_{[-a,-n]}} \otimes \ \Pi^{(1)}_{k_1} A^{(1)}_{i_{-n+1}} \Pi^{(1)}_{k_1} \otimes \cdots \\ &\otimes \Pi^{(n)}_{k_n} A^{(n)}_{i_0} \Pi^{(n)}_{k_n} \otimes \ A^{\otimes[n+1,b+n]}_{\boldsymbol{i}_{[1,b]}},\end{aligned} \tag{A3}$$

where $\boldsymbol{k}_{[a,b]}$ denotes the multi-index $k_a k_{a+1} \ldots k_b$ and $\phi_{\boldsymbol{k}_{[1,n]}} \equiv \phi_{k_1} \circ \phi_{k_2} \circ \cdots \circ \phi_{k_n}$.

A.1. System S reduced dynamics

The reduced dynamics Λ_n of the states of the open system S at discrete time n in (4) is obtained through (see (7)),

$$\mathrm{Tr}(\Omega^{(n)}_S O_S) = \omega_{SE} \circ \Phi_n(O_S \otimes \mathbb{1}_E);$$

namely by restricting the compound state $\omega_{SE} \circ \Phi_n$ to the system S algebra $\mathcal{A}_S \otimes \mathbb{1}_E$. Using (A3) one gets

$$\Phi_n[O_S \otimes \mathbb{1}_E] = \sum_{\boldsymbol{k}_{[1,n]}} \phi_{\boldsymbol{k}_{[1,n]}}[O_S] \otimes \otimes_{j=1}^{n} \Pi^{(j)}_{k_j}. \tag{A4}$$

Let us consider an initial factorized state $\omega_S \otimes \omega_E$ where the system state ω_S is represented by a density matrix ρ_S, while the restriction of the environment state ω_E to the algebra spanned by the orthogonal projections $\Pi_{\boldsymbol{k}_{[1,n]}} \equiv \otimes_{j=1}^{n} \Pi^{(j)}_{k_j}$ gives rise to the density matrix $\sum_{\boldsymbol{k}_{[1,n]}} p_{\boldsymbol{k}_{[1,n]}} \Pi_{\boldsymbol{k}_{[1,n]}}$. Then,

$$\omega_S \otimes \omega_E(\Phi_n[O_S \otimes \mathbb{1}_E]) = \mathrm{Tr}(\rho_S \sum_{\boldsymbol{k}_{[1,n]}} p_{\boldsymbol{k}_{[1,n]}} \phi_{\boldsymbol{k}_{[1,n]}}[O_S]) = \mathrm{Tr}(\sum_{\boldsymbol{k}_{[1,n]}} p_{\boldsymbol{k}_{[1,n]}} \phi^{\ddagger}_{\boldsymbol{k}_{[1,n]}}[\rho_S] O_S),$$

where $\Phi^{\ddagger}_{\boldsymbol{k}_{[1,n]}} = \phi^{\ddagger}_{k_n} \circ \cdots \circ \phi^{\ddagger}_{k_1}$ with $\phi^{\ddagger}_{k_i}$ the dual map of ϕ_{k_i}. Hence, in the Schrödinger picture, the dynamical map reads

$$\Lambda_n = \sum_{\boldsymbol{i}_{[1,n]}} p_{\boldsymbol{i}_{[1,n]}} \phi^{\ddagger}_{\boldsymbol{i}_{[1,n]}}. \tag{A5}$$

A.2. Environment E reduced dynamics

The single site operators A_i belong to the commutative algebra generated by the orthogonal projectors $\Pi^{(k)}_j$; then, $\sum_{k=0}^{d-1} \Pi_k A_i \Pi_k = A_i$. Therefore, due to the assumed unitality of the CP maps ϕ_k, from (A3) it follows that

$$\begin{aligned}\Phi_n[\mathbb{1}_S \otimes \ A^{\otimes[-a,b]}_{\boldsymbol{i}_{[-a,b]}}] &= \sum_{\boldsymbol{k}_{[1,n]}} \mathbb{1}_S \otimes \ A^{\otimes[-a+n,0]}_{\boldsymbol{i}_{[-a,-n]}} \otimes \ \Pi^{(1)}_{k_1} A^{(1)}_{i_{-n+1}} \Pi^{(1)}_{k_1} \otimes \cdots \\ &\qquad \otimes \Pi^{(n)}_{k_n} A^{(n)}_{i_0} \Pi^{(n)}_{k_n} \otimes \ A^{\otimes[n+1,b+n]}_{\boldsymbol{i}_{[1,b]}} \\ &= \mathbb{1}_S \otimes \ A^{\otimes[-a+n,0]}_{\boldsymbol{i}_{[-a,-n]}} \otimes \ A^{(1)}_{i_{-n+1}} \otimes \cdots \otimes A^{(n)}_{i_0} \otimes \ A^{\otimes[n+1,b+n]}_{\boldsymbol{i}_{[1,b]}} \\ &= \mathbb{1}_S \otimes \ \Theta^n[A^{\otimes[-a,b]}_{\boldsymbol{i}_{[-a,b]}}].\end{aligned} \tag{A6}$$

Since the environment state is shift-invariant by construction, it follows that the environment state is stationary:

$$\omega_{SE} \circ \Phi_n(\mathbb{1}_S \otimes \mathcal{A}_E) = \omega_{SE}(\mathbb{1}_S \otimes \mathcal{A}_E) = \omega_E(\mathcal{A}_E).$$

Appendix B. Λ_n in the unitary case

The unitary case correspond to choosing $\varphi = -1$ in (19). Then, only $j = n - 2$ contributes to the sum in (24) so that:

$$\lambda^{(\ell)}_n = (1 - 2(p + r))\, \lambda^{(\ell)}_{n-1} + 4\, p\, \Delta\, \lambda^{(\ell)}_{n-2}, \quad \ell = 1, 2. \tag{B1}$$

The general solutions of (B1) can be found with the ansatz $\lambda^{(\ell)}_n = x \lambda^{(\ell)}_{n-1}$ for all $n \geqslant 2$, by means of the roots $x^{\pm}$ of [44]

$$P(x) = x^2 - \alpha\, x + 4p\Delta, \qquad \alpha = 1 - 2(p + r). \tag{B2}$$

The general solution will thus have the form $\lambda_n^{(\ell)} = c_+ x_+^n + c_- x_-^n$, with the constants $c_\pm$ fixed by the initial conditions $\lambda_0^{(\ell)} = 1$ and $\lambda_1^{(\ell)} = \alpha$. The eigenvalues $\lambda_n^{(1,2,3)}$ then read

$$\lambda_n^{(\ell)} = \frac{\beta + \alpha}{2\beta}\left(\frac{\alpha + \beta}{2}\right)^n + \frac{\beta - \alpha}{2\beta}\left(\frac{\alpha - \beta}{2}\right)^n =: \lambda_n, \quad \beta = \sqrt{\alpha^2 + 16\, p\Delta}, \quad \ell = 1, 2, \tag{B3}$$

$$\lambda_n^{(3)} = (1 - 4p)^n. \tag{B4}$$

From the multiplicative action of the Pauli maps Λ_n on the Pauli matrices, one deduces that Λ_n is a convex combination of two discrete-time semigroups:

$$\Lambda_n = \frac{\beta + \alpha}{2\beta}\Psi_+^n + \frac{\beta - \alpha}{2\beta}\Psi_-^n, \qquad \Psi_\pm[X] = \sum_{i=0}^{3}\psi_\pm^{(i)}\,\mathrm{Tr}(\sigma_i X)\,\sigma_i, \qquad \textit{where} \tag{B5}$$

$$\psi_\pm^{(1,2)} = \frac{\alpha \pm \beta}{2}, \qquad \psi_\pm^{(0)} = 1, \quad \psi_\pm^{(3)} = 1 - 4p. \tag{B6}$$

It will be sufficient to consider the case $\alpha > 0$, namely $r < 1/2 - p$. If $p \neq 1/4$, then $\lambda_n^{(j)}$ and Λ_n. We can thus compute the intertwining maps $\Lambda_{n,n-1} = \Lambda_n \circ \Lambda_{n-1}^{-1}$ between two subsequent collisions. Setting $\gamma \coloneqq \frac{\beta+\alpha}{2} > \frac{\beta-\alpha}{2} =: \delta > 0$, these maps are of Pauli type with eigenvalues

$$\lambda_{n,n-1} \coloneqq \lambda_{n,n-1}^{(1)} = \lambda_{n,n-1}^{(2)} = \frac{\lambda_n}{\lambda_{n-1}} = \frac{\gamma^{n+1} + (-1)^n\,\delta^{n+1}}{\gamma^n + (-1)^{n-1}\,\delta^n}, \qquad \lambda_{n,n-1}^{(3)} = 1 - 4p. \tag{B7}$$

The P-divisibility of the discrete family of Pauli maps Λ_n, that is the contractivity of the intertwining maps $\Lambda_{n.n-1}$, is equivalent to asking that $|\lambda_{n,n-1}^{(i)}| \leqslant 1, i = 1, 2, 3$. In order to show this, we first prove that

$$\lambda_{2,1} > \lambda_{n,n-1} \qquad \forall\ n > 2. \tag{B8}$$

To see this, let $[0, 1] \ni x \equiv \delta/\gamma$. For even $n = 2k > 2$,

$$\lambda_{n,n-1} = \gamma\,\frac{1 + x^{n+1}}{1 - x^n}, \tag{B9}$$

monotonically decreases with n. Instead, for odd $n = 2k + 1 > 2$,

$$\lambda_{n,n-1} = \gamma\,\frac{1 - x^{n+1}}{1 + x^n}, \tag{B10}$$

increases with n; nevertheless,

$$\frac{\lambda_{n,n-1}}{\lambda_{2,1}} = \frac{1 - x^{n+1}}{1 + x^n}\frac{1 - x^2}{1 + x^3} \leqslant 1. \tag{B11}$$

Notice that $p_0 = 1 - r - 2p \geqslant 0$ and $r \geqslant 0$ imply $0 \leqslant p \leqslant 1/2$, so that $|\lambda_{n,n-1}^{(3)}| = |1 - 4p| \leqslant 1$. From (B7) and the previous discussion, one checks when $|\lambda_{2,1}| \leqslant 1$:

$$0 \leqslant \lambda_{2,1} = 1 - 2(p + r) + \frac{4\,\Delta\, p}{1 - 2(p + r)} \leqslant 1 \;\Leftrightarrow\; \frac{\Delta}{\alpha} \leqslant \frac{r}{2p} + \frac{1}{2}, \tag{B12}$$

where α has been defined in (B2).

Instead, the conditions for the complete positivity of $\Lambda_{n,n-1}$ are obtained by asking for the positivity of the eigenvalues of the 4×4 Choi matrix $\Lambda_{n,n-1} \otimes \mathrm{id}[P_+^2]$, where P_+^2 projects onto the totally symmetric vector $|\Psi_+^2\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. The eigenvalues are easily computed to be: $E_1(n) = p$, twice degenerate and

$$E_2(n) = \frac{1}{4}(1 + \lambda_{n,n-1}^{(3)} + 2\lambda_{n,n-1}) = \frac{1 - 2p}{2} + \frac{\lambda_{n,n-1}}{2}, \tag{B13}$$

$$E_3(n) = \frac{1}{4}(1 + \lambda_{n,n-1}^{(3)} - 2\lambda_{n,n-1}) = \frac{1 - 2p}{2} - \frac{\lambda_{n,n-1}}{2}. \tag{B14}$$

From $0 \leqslant p \leqslant 1/2$ and $\lambda_{n,n-1} \geqslant 0$ it follows that $E_1(n) \geqslant 0$. Further, (B8) implies $E_3(n) \geqslant E_3(2)$; then the positivity of $E_3(n)$ is ensured by

$$1 + \lambda_{2,1}^{(3)} - 2\lambda_{2,1} = 4r - \frac{8\,p\,\Delta}{\alpha} \geqslant 0 \;\Leftrightarrow\; \frac{\Delta}{\alpha} \leqslant \frac{r}{2p}. \tag{B15}$$

We now consider the positivity of $\Lambda_{n,n-1} \otimes \Lambda_{n,n-1}$. Since $\Lambda_{n,n-1}$ are Pauli maps, then $\Lambda_{n,n-1} \otimes \Lambda_{n,n-1}$ is positive if and only if $\Lambda_{n,n-1}^2$ is completely positive [45], that is if and only if the Choi matrix $\Lambda_{n,n-1}^2 \otimes \mathbb{1}[P_+^2] \geqslant 0$. Recasting

$$P_+^{(2)} = \frac{1}{4}(\mathbb{1}\otimes\mathbb{1} + \sigma_1\otimes\sigma_1 - \sigma_2\otimes\sigma_2 + \sigma_3\otimes\sigma_3) = \frac{1}{4}\begin{pmatrix} \mathbb{1}+\sigma_3 & \sigma_1 + i\sigma_2 \\ \sigma_1 - i\sigma_2 & \mathbb{1}-\sigma_3\end{pmatrix}$$

yields

$$\Lambda_{n,n-1}\otimes\Lambda_{n,n-1}[P_+^{(2)}] = \Lambda_{n,n-1}^2\otimes\mathbb{1}[P_+^{(2)}] = \frac{1}{4}\begin{pmatrix} 1+(\lambda_{n,n-1}^{(3)})^2 & 0 & 0 & 2\lambda_{n,n-1}^2 \\ 0 & 1-(\lambda_{n,n-1}^{(3)})^2 & 0 & 0 \\ 0 & 0 & 1-(\lambda_{n,n-1}^3)^2 & 0 \\ 2\lambda_{n,n-1}^2 & 0 & 0 & 1+(\lambda_{n,n-1}^{(3)})^2 \end{pmatrix}. \tag{B16}$$

Then, from $0 \leqslant p \leqslant 1/2$ it follows that $\Lambda_{n,n-1}\otimes\Lambda_{n,n-1}$ is completely positive iff

$$1 + (1-4p)^2 - 2(\lambda_{n,n-1})^2 \geqslant 0. \tag{B17}$$

Moreover, since $\lambda_{n,n-1} \geqslant \lambda_{2,1} \geqslant 0$, we get the inequality

$$4\,p^2\left(\frac{\Delta}{\alpha}\right)^2 + 2\,p\,\alpha\left(\frac{\Delta}{\alpha}\right) - (p^2 + r\,p_0) \leqslant 0, \tag{B18}$$

where we recall that $p_0 = 1 - 2p - r$. equation (B18) then gives the condition for P-divisibility of $\Lambda_n\otimes\Lambda_n$,

$$\frac{\Delta}{\alpha} \leqslant \frac{r}{2p} + \frac{1}{2} - \frac{1-\sqrt{1-4p(1-2p)}}{4p}. \tag{B19}$$

Clearly, (B15)$\Rightarrow$(B19)$\Rightarrow$(B12). On the other hand, (B19)$\not\Rightarrow$(B15). This is in contrast to the case of continuous-time dynamics with a time-local generator. Indeed, as proved in theorem 1 of [13], positive $\Lambda_{t,s}\otimes\Lambda_{t,s}$ for all $t \geqslant s \geqslant 0$ are possible if and only if $\Lambda_{t,s}$ are completely positive.

To see explicitly how the environmental correlations relate to the lack of BFI for one qubit and super-activation of BFI for two qubits, let us consider $r = 0$ and $p \ll 1$ and let $\Delta = Q\,p$. From (B3) and (B4), one sees that

$$\lambda_{n,n-1} = 1 - 2p + 4Qp^2 + O(p^3), \qquad \lambda_{n,n-1}^{(3)} = 1 - 4p, \tag{B20}$$

for all $n \geqslant 2$. Notice that the eigenvalues of $X = X^\dagger = x_0 + \sum_{i=1}^3 x_i\sigma_i$ in $M_2(\mathbb{C})$ are $x_0 \pm ||\boldsymbol{x}||$. Then, $||X||_1 = 2x_0 = \mathrm{Tr}X$ if $x_0 \geqslant ||\boldsymbol{x}||$, otherwise $||X||_1 = 2||\boldsymbol{x}||$. Let us assume $\mathrm{Tr}(X) = 2x_0 \geqslant 0$ and set $Y = \Lambda_{n,n-1}[X]$, its eigenvalues being $x_0 \pm ||\boldsymbol{y}||$, with $\boldsymbol{y} = (\lambda_{n,n-1}x_1,\ \lambda_{n,n-1}x_2,\ \lambda_{n,n-1}^{(3)}x_3)$. Thus, $||Y||_1 = ||X||_1 = 2x_0$ if $x_0 > ||\boldsymbol{y}||$, otherwise $||Y||_1 = 2||\boldsymbol{y}||$. Then, expanding up to the second order in p one finds

$$||\boldsymbol{y}||^2 = ||\boldsymbol{x}||^2 - 2p\,(2(x_1^2 + x_2^2) + x_3^2)) + 4p^2(Q(x_1^2 + x_2^2) + x_3^2) + O(p^3). \tag{B21}$$

Therefore, for $x_0 \leqslant ||\boldsymbol{y}||$, $x_0 \leqslant ||\boldsymbol{x}||$ so that $||Y||_1 \leqslant ||\boldsymbol{x}|| = ||X||_1$ and contractivity ensues. Indeed that a Q-dependent, positive contribution in (B21) only appears at second order in p and is dominated by a strictly negative contribution, thus preventing BFI for one qubit. Also, notice that up to second order in p there is no dependence on the successive discrete-time steps n and $n-1$.

Instead, let us consider the case of two qubits and consider the trace norm of $Z := \Lambda_{n,n-1}\otimes\Lambda_{n,n-1}[P_2^+]$ in the same small p regime. From (B16) one sees that the eigenvalues of Z are $1 - (\lambda^{(3)})^2 \geqslant 0$ twice degenerate and

$$1 + (\lambda^{(3)})^2 + 2\lambda_{n,n-1}^2 \geqslant 0, \qquad 1 + (\lambda^{(3)})^2 - 2\lambda_{n,n-1}^2.$$

If the latter is positive it follows that $||Z||_1 = 1 = ||P_2^+||_1$; otherwise, if $2\lambda_{n,n-1}^2 > 1 + (\lambda_{n,n-1}^{(3)})^2$, which for small p occurs whenever $Q > 1/2$,

$$||Z||_1 = 2(1 - (\lambda^{(3)})^2) + 4\lambda_{n.n-1}^2 \simeq 1 + 4p^2(2Q - 1)$$

becomes larger than 1 for $Q > 1/2$. Therefore, unlike for a single qubit, for two qubits the leading correction is a term of order 2 in p; this becomes positive for sufficiently correlated sites in the Markov chain environment in which case $\Lambda_{n,n-1}\otimes\Lambda_{n,n-1}$ ceases to be contractive.

Appendix C. Local system-chain density matrices and mutual information

Let us consider again the local algebra $\mathcal{A}_E^{[-a,b]}$ supported by the integers $0 \leqslant a \leqslant j \leqslant b$ whose elements are linear combinations of tensor products $A_{\boldsymbol{i}_{[-a,b]}}^{\otimes[-a,b]}$. Each single-site operator belongs to the commutative algebra $\mathcal{A} = D_d(\mathbb{C})$ generated by the orthogonal projections Π_j, $0 \leqslant j \leqslant d-1$ and is thus of the form $A_{i_k}^{(k)} = \sum_{\ell_k=0}^{d-1} a_{i_k}^{\ell_k}\Pi_{\ell_k}^{(k)}$. Then,

$$A_{\boldsymbol{i}_{[-a,b]}}^{\otimes[-a,b]} = \sum_{\boldsymbol{\ell}_{[-a,b]}} a_{\boldsymbol{i}_{[-a,b]}}^{\boldsymbol{\ell}_{[-a,b]}} \Pi_{\boldsymbol{\ell}_{[-a,b]}}^{[-a,b]}, \quad \Pi_{\boldsymbol{\ell}_{[-a,b]}}^{[-a,b]} \equiv \bigotimes_{k=-a}^{b} \Pi_{\ell_k}^{(k)}, \quad a_{\boldsymbol{i}_{[1-a,b]}}^{\boldsymbol{\ell}_{[-a,b]}} \equiv \prod_{k=-a}^{b} a_{i_k}^{\ell_k}.$$

The dynamics (A3) thus gives

$$\Phi_n[O_S \otimes A_{\boldsymbol{i}_{[-a,b]}}^{\otimes[-a,b]}] = \begin{cases} \sum_{\boldsymbol{\ell}_{[-a,b]}} a_{\boldsymbol{i}_{[-a,b]}}^{\boldsymbol{\ell}_{[-a,b]}} \phi_{\boldsymbol{\ell}_{[-n+1,0]}}[O_S] \otimes \Pi_{\boldsymbol{\ell}_{[-a,b]}}^{[-a+n,n+b]} & \dots\ 0 \leqslant n \leqslant a \\ \sum_{\boldsymbol{\ell}_{[-n+1,b]}} a_{\boldsymbol{i}_{[-a,b]}}^{\boldsymbol{\ell}_{[-a,b]}} \phi_{\boldsymbol{\ell}_{[-n+1,0]}}[O_S] \otimes \Pi_{\boldsymbol{\ell}_{[-n+1,b]}}^{[1,n+b]} & \dots\ n > a \end{cases}, \tag{C1}$$

where $\phi_{\boldsymbol{\ell}_{[-n+1,0]}} \equiv \phi_{\ell_{-n+1}} \circ \cdots \circ \phi_{\ell_0}$.

Let us now consider the discrete-time evolution of local density matrices that is obtained by duality:

$$\omega_S \otimes \omega_E \circ \Phi_n[O_S \otimes A_E^{[-a,b]}] = \mathrm{Tr}(\Omega_{S[-a,b]}^{(n)}\ O_S \otimes A_E^{[-a,b]}).$$

Using the shift invariance of the environment state ω_E one gets:

$$\omega_S \otimes \omega_E \circ \Phi_n[O_S \otimes A_{\boldsymbol{i}_{[-a,b]}}^{\otimes[-a,b]}] = \begin{cases} \sum_{\boldsymbol{\ell}_{[-a,b]}} p_{\boldsymbol{\ell}_{[-a,b]}} a_{\boldsymbol{i}_{[-a,b]}}^{\boldsymbol{\ell}_{[-a,b]}} \mathrm{Tr}(\phi_{\boldsymbol{\ell}_{[-n+1,0]}}^{\ddagger}[\rho_S]\ O_S) & \dots\ n \leqslant a \\ \sum_{\boldsymbol{\ell}_{[-n+1,b]}} p_{\boldsymbol{\ell}_{[-n+1,b]}} a_{\boldsymbol{i}_{[-a,b]}}^{\boldsymbol{\ell}_{[-a,b]}} \mathrm{Tr}(\phi_{\boldsymbol{\ell}_{[-n+1,0]}}^{\ddagger}[\rho_S]\ O_S) & \dots\ n > a \end{cases}.$$

Therefore, the local density matrices at discrete time-step n, $\Omega_{S[-a,b]}^{(n)}$, read

$$\Omega_{S[-a,b]}^{(n)} = \begin{cases} \sum_{\boldsymbol{\ell}_{[-a,b]}} p_{\boldsymbol{\ell}_{[-a,b]}} \phi_{\boldsymbol{\ell}_{[-n+1,0]}}^{\ddagger}[\rho_S] \otimes \Pi_{\boldsymbol{\ell}_{[-a,b]}}^{[-a,b]} & \dots\ n \leqslant a \\ \sum_{\boldsymbol{\ell}_{[-n+1,b]}} p_{\boldsymbol{\ell}_{[-n+1,b]}} \phi_{\boldsymbol{\ell}_{[-n+1,0]}}^{\ddagger}[\rho_S] \otimes \Pi_{\boldsymbol{\ell}_{[-n+1,b]}}^{[-n+1,b]} & \dots\ n > a \end{cases}. \tag{C2}$$

To quantify the system-chain correlations, we compute the mutual information

$$\mathcal{I}(\Omega_{S[-a,b]}^{(n)}) = S(\Omega_S^{(n)}) + S(\Omega_{[-a,b]}^{(n)}) - S(\Omega_{S[-a,b]}^{(n)}); \tag{C3}$$

relative to the evolved local density matrices $\Omega_{S[-a,b]}^{(n)}$ (C2) and their marginals $\Omega_S^{(n)}$, respectively $\Omega_{[-a,b]}^{(n)}$ that are obtained by performing the trace over $\mathcal{A}_S$, respectively $\mathcal{A}_E$. Using the notation in (11), they read

$$\Omega_S^{(n)} = \sum_{\boldsymbol{\ell}_{[-n+1,0]}} p_{\boldsymbol{\ell}_{[-n+1,0]}} \phi_{\boldsymbol{\ell}_{[-n+1,0]}}^{\ddagger}[\rho_S], \tag{C4}$$

$$\Omega_{[-a,b]}^{(n)} = \begin{cases} \sum_{\boldsymbol{\ell}_{[-a,b]}} p_{\boldsymbol{\ell}_{[-a,b]}} \Pi_{\boldsymbol{\ell}_{[-a,b]}}^{[-a,b]} = \rho_E^{[-a,b]} & \dots\ n \geqslant a \\ \sum_{\boldsymbol{\ell}_{[-n+1,b]}} p_{\boldsymbol{\ell}_{[-n+1,b]}} \Pi_{\boldsymbol{\ell}_{[-n+1,b]}}^{[1-n,b]} = \rho_E^{[-n+1,b]} & \dots\ n \geqslant a \end{cases}. \tag{C5}$$

Notice that (C4) follows since

$$a \geqslant n \Rightarrow \sum_{\boldsymbol{\ell}_{[-a,-n]};\boldsymbol{\ell}_{[1,b]}} p_{\boldsymbol{\ell}_{[-a,b]}} = p_{\boldsymbol{\ell}_{[-n+1,0]}}.$$

Furthermore, by relabelling the indices in the right-hand side of (C4), one obtains $\Omega_S^{(n)} = \Lambda_n[\rho_S]$ with Λ_n as in (A5).

Since the contributing operators in (C2) are all orthogonal, one gets

$$S(\Omega_{S[-a,b]}^{(n)}) = \begin{cases} S(\rho_E^{[-a,b]}) + \sum_{\boldsymbol{\ell}_{[-n+1,0]}} p_{\boldsymbol{\ell}_{[-n+1,0]}} S(\phi_{\boldsymbol{\ell}_{[-n+1,0]}}^{\ddagger}) & \dots\ n \leqslant a \\ S(\rho_E^{[-n+1,b]}) + \sum_{\boldsymbol{\ell}_{[-n+1,0]}} p_{\boldsymbol{\ell}_{[-n+1,0]}} S(\phi_{\boldsymbol{\ell}_{[-n+1,0]}}^{\ddagger}[\rho_S]) & \dots\ n > a \end{cases}. \tag{C6}$$

Therefore, by relabeling the summation indices, the mutual information simplifies to

$$\mathcal{I}(\Omega_{S[-a+1,b]}^{(n)}) = S(\Lambda_n[\rho_S]) - \sum_{\boldsymbol{\ell}_{[1,n]}} p_{\boldsymbol{\ell}^{[1,n]}}\ S(\Phi_{\boldsymbol{\ell}_{[1,n]}}^{\ddagger}[\rho_S]) \equiv \mathcal{I}_{SE}^{(n)}. \tag{C7}$$

Notice that the right-hand side of (C7) only depends on n and not on the specific local sub-algebra $\mathcal{A}_E^{[-a,b]}$ Analogously, for two qubits evolving in the same collisional environment, the mutual information relative to an initial state $\omega_{S+S} \otimes \omega_E$ where ω_{S+S} is an expectation corresponding to a two-qubit state ρ_{S+S}, one similarly derive

$$\mathcal{I}_{SE}^{(n)} := S(\Lambda_n \otimes \Lambda_n[\rho_{S+S}]) - \sum_{\boldsymbol{\ell}_{[1,n]}, \boldsymbol{k}_{[1,n]}} p_{\boldsymbol{\ell}_{[1,n]}} p_{\boldsymbol{k}_{[1,n]}} \quad S(\phi^{\ddagger}_{\boldsymbol{\ell}_{[1,n]}} \otimes \phi^{\ddagger}_{\boldsymbol{k}_{[1,n]}}[\rho_{S+S}]).$$

Appendix D. Stroboscopic limit

Let us consider Pauli maps as in (43) of the form $\phi_k = e^{\tau \mathcal{L}_k}$ and $\varphi = e^{-2\gamma\tau}$. This choice corresponds to the case in which the system, identified by $\mathcal{A}_S$, and the chain ancilla at site (0), described by $\mathcal{A}_E^{(0)}$, are dissipatively coupled for a time τ through the following GKLS generator,

$$\mathbb{L}[O_S \otimes \; O_E^{(0)}] = \gamma \sum_{i=0}^{3} \left((\sigma_i \otimes \; \Pi_i)\, O_S \otimes \; O_E^{(0)}\, (\sigma_i \otimes \; \Pi_i) - \frac{1}{2}\{\mathbb{1}_2 \otimes \; \Pi_i,\, O_S \otimes \; O_E^{(0)}\} \right), \tag{D1}$$

which satisfies $\mathbb{L}[O_S \otimes \Pi_j] = \mathcal{L}_j[O_S] \otimes \Pi_j$. The reduced dynamics will be of the Pauli type, with spectrum $\lambda_n^{(i)}$ obeying the recurrences (24) and (25). In the so-called stroboscopic limit typical of collision models, one takes

$$\tau \to 0, \qquad n \to \infty, \qquad n\tau \to t, \tag{D2}$$

and expands (25) at first order in τ obtaining

$$\frac{\lambda_n^{(3)} - \lambda_{n-1}^{(3)}}{\tau} = -2\gamma\lambda_{n-1}^{(3)} \Rightarrow \dot{\lambda}_t^{(3)} = -2\gamma\lambda_t^{(3)} \Rightarrow \lambda_t^{(3)} = e^{-2\gamma t}. \tag{D3}$$

On the other hand, denoting by λ_n the other two equal Pauli eigenvalues and expanding (24) up to order τ yield the following finite-difference equation:

$$\frac{\lambda_n - \lambda_{n-1}}{\tau} = -2(p + r)\gamma\lambda_{n-1} + 2p\,\gamma^2 \sum_{j=0}^{n-2} \tau\,(2\Delta)^{n-j-1}(1 - \gamma\tau)^{n-j-2}\lambda_j\,. \tag{D4}$$

Choosing $\Delta = \frac{e^{-\kappa\tau}}{2}$, the stroboscopic limit (D2) and the constraints (23) yield $p \to 1/2$, $r \to 0$ and turn (D4) into the integro-differential equation

$$\dot{\lambda}_t = -\gamma\,\lambda_t + \gamma^2 \int_0^t \mathrm{d}s e^{-(\kappa+\gamma)(t-s)}\lambda_s\,. \tag{D5}$$

The latter is readily solvable through its Laplace transform $\widetilde{\lambda}_z = \int_0^{+\infty} \mathrm{d}t e^{-zt}\lambda_t$, with the initial condition $\lambda_{t=0} = 1$, yielding:

$$\widetilde{\lambda}_z = \frac{z + \kappa + \gamma}{z^2 + z(\kappa + 2\gamma) + \kappa\gamma} \quad \text{with simple poles at} \quad z_{\pm} = \frac{-(\kappa + 2\gamma) \pm \sqrt{\kappa^2 + 4\gamma^2}}{2} \leqslant 0. \tag{D6}$$

Therefore, for $a \geqslant z_+$, one gets

$$\lambda_t = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \mathrm{d}z e^{zt}\widetilde{\lambda}_z = \quad e^{-(\gamma+\frac{\kappa}{2})t}\left[\cosh\left(\frac{1}{2}t\sqrt{\kappa^2 + 4\gamma^2}\right) + \frac{\kappa \sinh\left(\frac{1}{2}t\sqrt{\kappa^2 + 4\gamma^2}\right)}{\sqrt{\kappa^2 + 4\gamma^2}}\right]. \tag{D7}$$

By inspection of the Choi matrix of Λ_t, namely $\Lambda_t \otimes \mathrm{id}_2[P_2^+]$, one realizes that complete positivity requires

$$1 + \lambda_t^{(3)} - 2\lambda_t \geqslant 0, \tag{D8}$$

which is checked to be always satisfied.

Remark 4. From (D3) and (D7), one derives that the dynamical map Λ_t can be written as a convex composition of two-semigroups,

$$\Lambda_t = a e^{t\mathcal{L}_-} + (1 - a)e^{t\mathcal{L}_+}, \qquad 0 \leqslant a = \frac{1}{2} + \frac{\kappa}{2\sqrt{\kappa^2 + 4\gamma^2}} \leqslant 1. \tag{D9}$$

Notice that only $e^{t\mathcal{L}_-}$ is completely positive, while $e^{t\mathcal{L}_+}$ is only positive. Nevertheless, Λ_t is always completely positive and P-divisible. Indeed, the contractivity of the the Pauli intertwiners $\Lambda_{t,s}$, as discussed in Appendix B for the discrete-time case, is equivalent to requiring that the Pauli eigenvalues are monotonically decreasing functions of time,

$$\dot{\lambda}_t \leqslant 0, \qquad \dot{\lambda}_t^{(3)} \leqslant 0\,. \tag{D10}$$

This is verified since the Pauli spectrum evolves according to

$$\dot{\lambda}_t = -\Gamma_t\,\lambda_t, \qquad \dot{\lambda}_t^{(3)} = -\Gamma_t^{(3)}\,\lambda_t^{(3)}, \tag{D11}$$

where $\Gamma_t^{(3)} = 2\gamma \geqslant 0$ and

$$\Gamma_t = \gamma - \frac{2\gamma^2}{\sqrt{\kappa^2 + 4\gamma^2}\coth\left(\frac{1}{2}t\sqrt{\kappa^2 + 4\gamma^2}\right) + \kappa}.$$

The case $\kappa = 0$ has already been discussed in the main text, while the positivity of Γ_t for $\kappa > 0$ is equivalent to

$$1 + \sqrt{1 + \left(\frac{2\gamma}{\kappa}\right)^2}\coth\left(\frac{t}{2}\sqrt{\kappa^2 + 4\gamma^2}\right) \geqslant \frac{2\gamma}{\kappa}, \tag{D12}$$

which is clearly verified since $\coth(x) \geqslant 1$ for $x \geqslant 0$.

Appendix E. Details about the Mutual information

In the case of a unitary coupling between system and collisional environment, the variation of the system-chain mutual information reduces to the variation of the von Neumann entropy in discrete time as in (40). As considered in the main text, the choice (41) together with $p = 1/4 + \epsilon$ yield the following Pauli eigenvalues at discrete-time steps 1, respectively 2: $\lambda_1 = \lambda_2 = \frac{1}{2} - 2\epsilon$, respectively $\lambda_1^{(3)} = -4\epsilon$, $\lambda_2^{(3)} = 16\epsilon^2$.

Since ϵ is taken as a small perturbative parameter, it follows that the intertwiner $\Lambda_{2,1}$ is a positive map. Indeed, the corresponding Pauli eigenvalues satisfy

$$\lambda_{2,1} = \frac{\lambda_2}{\lambda_1} = 1, \quad |\lambda_{2,1}^{(3)}| = |\frac{\lambda_2^{(3)}}{\lambda_1^{(3)}}| = 4\epsilon < 1.$$

Then, consider the first two time-step dynamics of two-qubit totally symmetric state P_2^+:

$$\Lambda_1 \otimes \Lambda_1[P_2^+] = \begin{pmatrix} \frac{1}{4} + 4\epsilon^2 & \cdot & \cdot & \frac{1}{8}(1 - 4\epsilon)^2 \\ \cdot & \frac{1}{4} - 4\epsilon^2 & \cdot & \cdot \\ \cdot & \cdot & \frac{1}{4} - 4\epsilon^2 & \cdot \\ \frac{1}{8}(1 - 4\epsilon)^2 & \cdot & \cdot & \frac{1}{4} + 4\epsilon^2 \end{pmatrix} \tag{E1}$$

$$\Lambda_2 \otimes \Lambda_2[P_2^+] = \begin{pmatrix} \frac{1}{4} + 64\epsilon^4 & \cdot & \cdot & \frac{1}{8}(1 - 4\epsilon)^2 \\ \cdot & \frac{1}{4} - 64\epsilon^4 & \cdot & \cdot \\ \cdot & \cdot & \frac{1}{4} - 64\epsilon^4 & \cdot \\ \frac{1}{8}(1 - 4\epsilon)^2 & \cdot & \cdot & \frac{1}{4} + 64\epsilon^4 \end{pmatrix}. \tag{E2}$$

By evaluating the spectrum of the two states and expanding the von Neumann entropies of the two above states (E1) up to second order in ϵ, one gets

$$S(\Lambda_1 \otimes \Lambda_1[P_2^+]) = \frac{20\log(2) - 3\log(3)}{8} + \log(3)\,\epsilon + \left(8\log(2) - 6\log(3) - \frac{16}{3}\right)\epsilon^2 + O(\epsilon^3),$$
$$S(\Lambda_2 \otimes \Lambda_2[P_2^+]) = \frac{20\log(2) - 3\log(3)}{8} + \log(3)\,\epsilon - \left(2\log(3) + \frac{16}{3}\right)\epsilon^2 + O(\epsilon^3).$$

Their difference coincides with the variation of the system-chain correlations and is given, up to order ϵ^2, by

$$\Delta\mathcal{I}_{SE}^{(2,1)}(P_2^+) = S(\Lambda_2 \otimes \Lambda_2[P_2^+]) - S(\Lambda_1 \otimes \Lambda_1[P_2^+]) = -4\log(\frac{4}{3})\epsilon^2 + O(\epsilon^3) < 0\,.$$

Let us now compute the quantum mutual information for the case $p = 1/2$, $\Delta = 1/2$, corresponding to master equation rates $\gamma_t = 1$, $\gamma_t^{(3)} = -\tanh(t)$ and Pauli eigenvalues $\lambda_t = e^{-t}\cosh(t)$, $\lambda_t^{(3)} = e^{-2t}$. Notice that, with these choices, the stochastic matrix T in (22) takes a particularly simple form:

$$T = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{E3}$$

so that the only non-zero probabilities $p_{\boldsymbol{i}_{[1,n]}}$ correspond to the sequences $\boldsymbol{i}_{[1,n]} = 111\cdots \equiv \boldsymbol{1}$ and $\boldsymbol{i}_{[1,n]} = 222\cdots \equiv \boldsymbol{2}$. Accordingly, only two CPTP unital semigroups $\phi_{\boldsymbol{i}_{[1,n]}}$ contribute in (17),

$$\Lambda_t = \frac{\phi_1 + \phi_2}{2}, \tag{E4}$$

with equal, time-independent weights $p_1 = p_2 = 1/2$. In the continuous-time limit, ϕ_1 and ϕ_2 are the Pauli maps defined by

$$\begin{aligned} \phi_1[\sigma_1] &= 1, & \phi_1[\sigma_2] &= e^{-2t}, & \phi_1[\sigma_3] &= e^{-2t}, \\ \phi_2[\sigma_1] &= e^{-2t}, & \phi_2[\sigma_2] &= 1, & \phi_2[\sigma_3] &= e^{-2t}. \end{aligned} \quad \text{(E5)} \tag{E5}$$

Notice that for other choices of T, in the stroboscopic limit, the weights $p_{i_{[1,n]}}$ would generally become functions of time as well. In the special case of (E3), the mutual information as function of t reads

$$\mathcal{I}^{(t)}_{SE}(\rho_S) = S(\Lambda_t \otimes \Lambda_t)[\rho_{S+S}]) - \frac{1}{4}\sum_{i,j=1,2} S(\phi_i \otimes \phi_j[\rho_{S+S}]). \tag{E6}$$

The non-monotonic behaviour of the system-chain mutual information (E6) has been inspected numerically by means of the following family of X states,

$$\begin{aligned} \rho_X^{(1)} = \frac{1}{4}\{&\mathbb{1}_4 - (1 - 2(\mu_1 + \nu))\sigma_1 \otimes \mathbb{1}_2 + (1 - 2(\mu_2 + \nu))\mathbb{1}_2 \otimes \sigma_1 \\ &- (1 - 2(\mu_1 + \mu_2))\sigma_1 \otimes \sigma_1 - 2\,\mathrm{Re}(u - v)\sigma_2 \otimes \sigma_2 \\ &+ 2\,\mathrm{Re}(u + v)\sigma_3 \otimes \sigma_3 + 2\mathrm{Im}(u + v)\sigma_2 \otimes \sigma_3 + 2\mathrm{Im}(u - v)\sigma_3 \otimes \sigma_2\}, \end{aligned} \tag{E7}$$

having the X shape when written in the basis of $\sigma_1 \otimes \sigma_1$, which can be obtained from the standard one by applying the matrix $V \otimes V$, $V = \frac{\sigma_1 + \sigma_3}{\sqrt{2}}$. The positivity condition are then readily obtained and read

$$0 \leqslant \mu_1, \mu_2 \leqslant 1 \qquad 0 \leqslant \nu \leqslant 1 - (\mu_1 + \mu_2), \qquad |u| \leqslant \sqrt{\mu_1\mu_2}, \qquad |v| \leqslant \sqrt{\nu(1 - \mu_1 - \mu_2 - \nu)}.$$

Setting $\alpha_t = e^{-t}\cosh(t)$ and $\beta_t = e^{-2t}$, the states in (E6) read

$$\begin{aligned} \Lambda_t \otimes \Lambda_t[\rho_X^{(1)}] = \frac{1}{4}\{&\mathbb{1}_4 - (1 - 2(\mu_1 + \nu))\alpha_t\sigma_1 \otimes \mathbb{1}_2 + (1 - 2(\mu_2 + \nu))\alpha_t\mathbb{1}_2 \otimes \sigma_1 \\ &- (1 - 2(\mu_1 + \mu_2))\alpha_t^2\sigma_1 \otimes \sigma_1 - 2\mathrm{Re}(u - v)\alpha_t^2\sigma_2 \otimes \sigma_2 \\ &+ 2\mathrm{Re}(u + v)\beta_t^2\sigma_3 \otimes \sigma_3 + 2\mathrm{Im}(u + v)\alpha_t\beta_t\sigma_2 \otimes \sigma_3 + 2\mathrm{Im}(u - v)\alpha_t\beta_t\sigma_3 \otimes \sigma_2\}, \\ \Phi_1 \otimes \Phi_1[\rho_X^{(1)}] = \frac{1}{4}\{&\mathbb{1}_4 - (1 - 2(\mu_1 + \nu))\sigma_1 \otimes \mathbb{1}_2 + (1 - 2(\mu_2 + \nu))\mathbb{1}_2 \otimes \sigma_1 \\ &- (1 - 2(\mu_1 + \mu_2))\sigma_1 \otimes \sigma_1 - 2\mathrm{Re}(u - v)\beta_t^2\sigma_2 \otimes \sigma_2 \\ &+ 2\mathrm{Re}(u + v)\beta_t^2\sigma_3 \otimes \sigma_3 + 2\mathrm{Im}(u + v)\beta_t^2\sigma_2 \otimes \sigma_3 + 2\mathrm{Im}(u - v)\beta_t^2\sigma_3 \otimes \sigma_2\}, \\ \Phi_2 \otimes \Phi_2[\rho_X^{(1)}] = \frac{1}{4}\{&\mathbb{1}_4 - (1 - 2(\mu_1 + \nu))\beta_t\sigma_1 \otimes \mathbb{1}_2 + (1 - 2(\mu_2 + \nu))\beta_t\mathbb{1}_2 \otimes \sigma_1 \\ &- (1 - 2(\mu_1 + \mu_2))\beta_t^2\sigma_1 \otimes \sigma_1 - 2\mathrm{Re}(u - v)\sigma_2 \otimes \sigma_2 \\ &+ 2\,\mathrm{Re}(u + v)\beta_t^2\sigma_3 \otimes \sigma_3 + 2\mathrm{Im}(u + v)\beta_t\sigma_2 \otimes \sigma_3 + 2\mathrm{Im}(u - v)\beta_t\sigma_3 \otimes \sigma_2\}, \\ \Phi_1 \otimes \Phi_2[\rho_X^{(1)}] = \frac{1}{4}\{&\mathbb{1}_4 - (1 - 2(\mu_1 + \nu))\sigma_1 \otimes \mathbb{1}_2 + (1 - 2(\mu_2 + \nu))\beta_t\mathbb{1}_2 \otimes \sigma_1 \\ &(1 - 2(\mu_1 + \mu_2))\beta_t^2\sigma_1 \otimes \sigma_1 - 2\mathrm{Re}(u - v)\beta_t\sigma_2 \otimes \sigma_2 \\ &+ 2\mathrm{Re}(u + v)\beta_t^2\sigma_3 \otimes \sigma_3 + 2\mathrm{Im}(u + v)\beta_t^2\sigma_2 \otimes \sigma_3 + 2\mathrm{Im}(u - v)\beta_t\sigma_3 \otimes \sigma_2\}, \\ \Phi_2 \otimes \Phi_1[\rho_X^{(1)}] = \frac{1}{4}\{&\mathbb{1}_4 - (1 - 2(\mu_1 + \nu))\beta_t\sigma_1 \otimes \mathbb{1}_2 + (1 - 2(\mu_2 + \nu))\mathbb{1}_2 \otimes \sigma_1 \\ &- (1 - 2(\mu_1 + \mu_2))\beta_t\sigma_1 \otimes \sigma_1 - 2\mathrm{Re}(u - v)\beta_t\sigma_2 \otimes \sigma_2 \\ &+ 2\mathrm{Re}(u + v)\beta_t^2\sigma_3 \otimes \sigma_3 + 2\mathrm{Im}(u + v)\beta_t\sigma_2 \otimes \sigma_3 + 2\mathrm{Im}(u - v)\beta_t^2\sigma_3 \otimes \sigma_2\}. \end{aligned}$$

Appendix F. Quantum Helstrom ensemble without entanglement

Consider the eternally non-Markovian evolution Λ_t generated by Pauli rates $\gamma_t(t) = 1$, $\gamma_t^{(3)} = -\tanh(t)$. The symmetric projector P_+^2 always detects SBFI for Pauli second tensor powers and, for small $\epsilon > 0$,

$$||\Lambda_{s+\epsilon,s} \otimes \Lambda_{s+\epsilon,s}[P_+^2]||_1 \simeq 1 + 2\epsilon\tanh(s) > ||P_+^2||_1. \tag{F1}$$

for sufficiently small $\epsilon \ll s$. Now, we argue that there exists a Helstrom matrix

$$\Delta_\mu = \mu\rho_1 - (1 - \mu)\rho_2, \tag{F2}$$

with ρ_1, ρ_2 separable, such that $\Lambda_s \otimes \Lambda_s[\Delta_\mu] = \alpha P_+^2$ so that

$$||\Lambda_t \otimes \Lambda_t[\Delta_\mu]||_1 = \alpha||\Lambda_{t,s} \otimes \Lambda_{t,s}[P_+^2]||_1 > \alpha||P_+^2||_1 = ||\Lambda_s \otimes \Lambda_s[\Delta_\mu]||_1. \tag{F3}$$

Consider the isotropic state

$$\rho_a = (1-a)\frac{\mathbb{1}_4}{4} + aP_2^+, \qquad 0 \leqslant a \leqslant 1, \tag{F4}$$

which is separable for $a \leqslant 1/3$. The preimage of P_+^2 is

$$\Lambda_s^{-1} \otimes \Lambda_s^{-1}[P_2^+] = \frac{1}{a}\Lambda_s^{-1} \otimes \Lambda_s^{-1}[\rho_a] - \frac{1-a}{a}\frac{\mathbb{1}_4}{4} \tag{F5}$$

Recall that $\Lambda_s^{-1} \otimes \Lambda_s^{-1}$ only guarantees hermiticity, but not, in general, positivity preservation. Nevertheless, for sufficiently small a, $\Lambda_s^{-1} \otimes \Lambda_s^{-1}[\rho_a]$ is separable by being sufficiently close to the separable state $\frac{\mathbb{1}_4}{4}$. Explicitly, in Fano form,

$$\rho_a = \frac{1}{4}[\mathbb{1}_4 + a(\sigma_1 \otimes \sigma_1 - \sigma_2 \otimes \sigma_2 + \sigma_3 \otimes \sigma_3)]. \tag{F6}$$

Given the Pauli eigenvalues $\lambda_t = e^{-t}\cosh(t)$, $\lambda_t^{(3)} = e^{-2t}$, the algebraic inverse of ρ_a is

$$\Lambda_a^{-1} \otimes \Lambda_a^{-1}[\rho_a] = \frac{1}{4}\left[\mathbb{1}_4 + a\frac{e^{2s}}{\cosh^2(s)}(\sigma_1 \otimes \sigma_1 - \sigma_2 \otimes \sigma_2) + e^{4s}\sigma_3 \otimes \sigma_3)\right]. \tag{F7}$$

The matrix in (F7) is positive provided that $0 \leqslant a \leqslant e^{-4s}$. Fix for instance $s = \operatorname{arctanh}(\frac{1}{2}) \approx 0.55$. Then,

$$\rho_a^0 = \Lambda_s^{-1} \otimes \Lambda_s^{-1}[\rho_a] = \frac{1}{4}\begin{pmatrix} 1+9a & 0 & 0 & \frac{9a}{2} \\ 0 & 1-9a & 0 & 0 \\ 0 & 0 & 1-9a & 0 \\ \frac{9a}{2} & 0 & 0 & 1+9a \end{pmatrix} \tag{F8}$$

is a physical state if $a < 1/9$. Its partial transpose is

$$T \otimes \mathrm{id}[\rho_a^0] = \frac{1}{4}\begin{pmatrix} 1+9a & 0 & 0 & 0 \\ 0 & 1-9a & \frac{9a}{2} & 0 \\ 0 & \frac{9a}{2} & 1-9a & 0 \\ 0 & 0 & 0 & 1+9a \end{pmatrix}, \tag{F9}$$

which is positive for $a \leqslant 2/27 \equiv a^*$. Hence, for $a \leqslant a^*$, one has a well defined separable state ρ_a^0, such that

$$||\Lambda_t \otimes \Lambda_t[\Delta_{\mu(a)}]||_1 - ||\Lambda_s \otimes \Lambda_s[\Delta_{\mu(a)}]||_1 > 0 \tag{F10}$$

for some $t > s$, where $\Delta_{\mu(a)} = \mu(a)\rho_a^0 - (1-\mu(a))\frac{\mathbb{1}_4}{4}$, with $\mu(a) = \frac{a}{1-a}$, ρ_r^0 separable. Since $\mathbb{1}_4/4$ is a fully incoherent state with respect to every basis, the ensemble quantumness of correlations reduces to the geometric measure of quantum discord of the isotropic state,

$$\mathcal{Q}_{\{\mathbb{P}^1\otimes\mathbb{P}^2\}}(\chi^{\mathcal{E}}(t)) = \mu(a)\min_{\mathbb{P}^1\otimes\mathbb{P}^2}||\rho_a - \mathbb{P}^1 \otimes \mathbb{P}^2[\rho_a]||_1 > 0.$$

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