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Research Article

Gianni Dal Maso and Rodica Toader*

Quasistatic crack growth in elasto-plastic materials with hardening: The antiplane case

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Abstract: We study a variational model for crack growth in elasto-plastic materials with hardening in the antiplane case. The main result is the existence of a solution to the initial value problem with prescribed time-dependent boundary conditions.

Keywords: Fracture mechanics, small strain associative elasto-plasticity with hardening, free-discontinuity problems, quasistatic evolution problems, rate independent processes, variational models

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1 Introduction

There are several models for elasto-plastic materials with hardening for which a complete mathematical theory is available. For this subject we refer to the classical books [22, 23, 25, 26], and papers [13, 14, 24, 28, 29], while for more recent results and for a review of the literature we refer to [6, 18, 19], and [30]. Recent numerical and optimization results have been obtained in [7] and [12]. In this paper we study a model for the quasistatic crack growth in elasto-plastic materials with hardening, where an energetic formulation for elasto-plasticity is combined with the variational approach to irreversible crack growth. More precisely, we adopt the model of plasticity with hardening in the small strain regime presented in [27, Section 4.3.1.1]. As for crack growth, we follow the variational formulation introduced in [17] (see also [4]) and use some tools developed in [9]. In order to avoid a lot of technical difficulties, We prefer to consider here only the case of antiplane shear. The general case would require to face a lot of other technical difficulties, using recent results in the literature concerning the GSBD space (see [8]), but we have not worked out the details.

The reference configuration is a bounded open set Ω in $\mathbb{R}^d$, $d \geq 2$, with Lipschitz boundary, and the crack is described by a subset Γ of $\overline{\Omega}$ of dimension $d-1$. The displacement is a function $u\colon \Omega \setminus \Gamma \to \mathbb{R}$ and the corresponding strain is determined by its gradient ∇u, which is additively decomposed into an elastic and a plastic part: $\nabla u = e + p$. As in [27, Section 4.3.1.1], we consider also the scalar isotropic-hardening parameter $\eta\colon \Omega \to \mathbb{R}$.

The energetic formulation of our problem is based on the energy used in linearized elasto-plasticity with hardening and on a dissipation distance depending also on the cracks. The energy is given by

$$\mathcal{E}(e, p, \eta) := \frac{\alpha}{2}\|e\|^2 + \frac{\beta}{2}\|p\|^2 + (\eta h|p) + \frac{\gamma}{2}\|\eta\|^2. \tag{1.1}$$

Here and in the rest of the paper the symbols $(\cdot|\cdot)$ and $\|\cdot\|$ denote the scalar product and the norm in $L^2(\Omega;\mathbb{R}^d)$ or $L^2(\Omega)$, according to the context. In the previous formula $\alpha > 0$ is the Hooke constant, $\beta > 0$ determines the kinematic hardening, $\gamma > 0$ determines the isotropic hardening, while $h \in \mathbb{R}^d$ is a vector reflecting possible coupling between kinematic and isotropic hardening. We assume that $|h|^2 < \beta\gamma$, so that the energy satisfies a suit-

***Corresponding author: Rodica Toader**, Dipartimento di Scienze Matematiche, Informatiche e Fisiche, Università di Udine, Via delle Scienze 208, 33100 Udine, Italy, e-mail: toader@uniud.it. https://orcid.org/0000-0002-2798-2556
Gianni Dal Maso, SISSA, Via Bonomea 265, 34136 Trieste, Italy, e-mail: dalmaso@sissa.it. https://orcid.org/0000-0002-1010-4968

able coerciveness condition, which is a standard assumption in the existence theory for this kind of problems (see [27, Proposition 4.3.1]).

If $\Gamma_1 \subset \Gamma_2$, the dissipation distance is given by

$$\mathcal{D}(p_2, \eta_2, \Gamma_2; p_1, \eta_1, \Gamma_1) := \int_\Omega R(p_2 - p_1, \eta_2 - \eta_1)\, dx + \mathcal{H}^{d-1}(\Gamma_2 \setminus \Gamma_1), \tag{1.2}$$

where R is a positively one-homogeneous dissipation potential satisfying the usual coerciveness and growth conditions (see (2.3) below), and $\mathcal{H}^{d-1}$ is the $(d-1)$-dimensional Hausdorff measure. To force the irreversibility of crack growth, we set $\mathcal{D}(p_2, \eta_2, \Gamma_2; p_1, \eta_1, \Gamma_1) := +\infty$ if $\Gamma_1 \not\subset \Gamma_2$.

The evolution $t \mapsto (u(t), e(t), p(t), \eta(t), \Gamma(t))$ of our system in the time interval $[0, T]$ is driven by a time-dependent Dirichlet boundary condition of the form $u(t) = w(t)$ on $\partial\Omega \setminus \Gamma(t)$, where $w\colon [0, T] \to H^1(\Omega)$ is a prescribed absolutely continuous function. More precisely, a quasistatic evolution $t \mapsto (u(t), e(t), p(t), \eta(t), \Gamma(t))$ with boundary condition w is a function that satisfies the following conditions:

Condition (GS) (Global stability). For every $t \in [0, T]$ we have $\nabla u(t) = e(t) + p(t)$ in $\Omega \setminus \Gamma(t)$, $u(t) = w(t)$ on $\partial\Omega \setminus \Gamma(t)$, and

$$\mathcal{E}(e(t), p(t), \eta(t)) \le \mathcal{E}(\hat{e}, \hat{p}, \hat{\eta}) + \mathcal{D}(\hat{p}, \hat{\eta}, \hat{\Gamma}; p(t), \eta(t), \Gamma(t))$$

for every crack $\hat{\Gamma}$, every hardening parameter $\hat{\eta}$, and every $(\hat{u}, \hat{e}, \hat{p})$ such that $\nabla\hat{u} = \hat{e} + \hat{p}$ in $\Omega \setminus \hat{\Gamma}$ and $\hat{u} = w(t)$ on $\partial\Omega \setminus \hat{\Gamma}$.

Condition (EDB) (Energy-dissipation balance). For every $t \in [0, T]$ we have

$$\mathcal{E}(e(t), p(t), \eta(t)) + \mathrm{Diss}(p(\cdot), \eta(\cdot), \Gamma(\cdot); 0, t) = \mathcal{E}(e(0), p(0), \eta(0)) + \alpha \int_0^t (e(s)|\nabla\dot{w}(s))\, ds,$$

where $\mathrm{Diss}(p(\cdot), \eta(\cdot), \Gamma(\cdot); 0, t)$ denotes the dissipation in the interval $[0, t]$ corresponding to the distance $\mathcal{D}$ introduced in (1.2) (see (2.5) below).

The main result of the paper is that, given $(u_0, e_0, p_0, \eta_0, \Gamma_0)$ satisfying condition (GS) at $t = 0$, there exists a quasistatic evolution with $u(0) = u_0$, $e(0) = e_0$, $p(0) = p_0$, $\eta(0) = \eta_0$, $\Gamma(0) = \Gamma_0$ (see Theorem 2.2). To obtain this result, we use the standard variational approach based on the construction of discrete-time approximate solutions obtained by solving incremental minimum problems. Then we prove the convergence of these approximate solutions to a continuous-time quasistatic evolution satisfying (GS) and (EDB). It is not difficult to prove that a similar result can be obtained if the Dirichlet boundary condition is imposed only on $\partial_D\Omega \setminus \Gamma(t)$, where $\partial_D\Omega$ is a prescribed subset of $\partial\Omega$.

In Section 2 we present a detailed description of our model and introduce the function spaces used for a precise formulation of the problem. In particular, the estimates available for the displacement u lead us to choose a subspace of the space $\mathrm{GSBV}(\Omega)$ of generalized special functions of bounded variation, for which we refer to [2, Section 4.5]. Unfortunately, we cannot choose the space $H^1(\Omega \setminus \Gamma)$ for the displacement, because there is no way to guarantee that the set Γ constructed in the proofs is closed, unless we impose additional unnatural topological assumptions.

As a consequence of the choice of $\mathrm{GSBV}(\Omega)$ for the displacement, the crack Γ belongs to the set $\mathcal{R}_{d-1}(\overline{\Omega})$ of $(\mathcal{H}^{d-1}, d-1)$-rectifiable subsets $\overline{\Omega}$ (see [15, Definition 3.2.14 (4)]).

In Section 3 we study the incremental minimum problems in detail. A nontrivial issue is the existence of a solution. This is due to the fact that, while an estimate of the L^2-norm of ∇u is easily available, there are no estimates on the L^2-norm of the displacement u (nor on any L^p-norm), due to the presence of the cracks. For this reason the compactness theorem in $\mathrm{GSBV}(\Omega)$ by Ambrosio [1] (see also [2, Theorem 4.36]) cannot be applied. To overcome this difficulty, we rely on a recent result proved in [20], which provides the convergence of a suitable modification of a minimizing sequence. In this way we obtain the existence of a solution to the incremental minimum problems. We conclude this section by showing that a solution $(u_1, e_1, p_1, \eta_1, \Gamma_1)$ of an incremental minimum problem satisfies also

$$\mathcal{E}(e_1, p_1, \eta_1) \le \mathcal{E}(\hat{e}, \hat{p}, \hat{\eta}) + \mathcal{D}(\hat{p}, \hat{\eta}, \hat{\Gamma}; p_1, \eta_1, \Gamma_1) \tag{1.3}$$

for every crack $\hat{\Gamma}$, every hardening parameter $\hat{\eta}$, and every $(\hat{u}, \hat{e}, \hat{p})$ such that $\nabla\hat{u} = \hat{e} + \hat{p}$ in $\Omega \setminus \hat{\Gamma}$ and $\hat{u} = u_1$ on $\partial\Omega \setminus \hat{\Gamma}$.

In Section 4 we recall a notion of convergence in $\mathcal{R}_{d-1}(\overline{\Omega})$, called σ-convergence, introduced in [21, Definition 6.1] and use its properties to prove the stability of the minimality condition (1.3) with respect to the natural convergences of e_1, p_1, η_1, and to the σ-convergence of Γ_1. The proof relies on the jump transfer lemma of Francfort and Larsen [16, Theorem 2.1], adapted to σ-convergence in [21, Theorem 7.4].

In Section 5 we use the results of the previous sections to prove the existence of a solution to the initial value problem for the quasistatic evolution described by conditions (GS) and (EDB). As usual, we first construct approximate solutions by solving incremental minimum problems and then prove the convergence of a suitable subsequence. To this aim we use two variants of Helly's Theorem, one for functions with bounded variation with values in separable Hilbert spaces, and one for increasing functions with values in $\mathcal{R}_{d-1}(\overline{\Omega})$ endowed with the σ-convergence.

Condition (GS) for the limit functions is obtained thanks to the results of Section 4. The upper energy-dissipation inequality can be easily obtained by semicontinuity, while the lower energy-dissipation inequality, which in other papers (starting from the proof of [9, Theorem 3.15]) is obtained by approximating a Lebesgue integral with suitable Riemann sums, is proved here through an easier argument (see Lemma 5.4).

The corresponding problem in linearly elastic-perfectly plastic materials (without hardening) is much more difficult. The only result (see [11]), concerns the planar case and is obtained under a constraint: the number of connected components of the cracks is bounded by a prescribed constant. No result has been proved so far in dimension $d > 2$, not even in the antiplane case. In our opinion the main technical obstruction is related to possible interactions between cracks and concentrated plastic shears.

2 The model and the main result

In this section we present a variational model of a quasistatic crack growth in an elasto-plastic material with hardening and state the main result of this paper. The model is based on the energetic formulation for rate-independent processes described in [27]. As explained in the introduction, we consider only the case of antiplane shear.

We assume that the elastic and plastic strains satisfy $e, p \in L^2(\Omega; \mathbb{R}^d)$, while the scalar isotropic-hardening parameter satisfies $\eta \in L^2(\Omega)$. The energy $\mathcal{E}(e, p, \eta)$ is defined by (1.1). Since $|h|^2 < \beta\gamma$, there exists $\nu_0 > 0$ such that $|h| < (\beta - 2\nu_0)^{\frac{1}{2}}(\gamma - 2\nu_0)^{\frac{1}{2}}$. This implies that for every $\pi \in \mathbb{R}^d$ and $\zeta \in \mathbb{R}$,

$$\frac{\beta}{2}|\pi|^2 + \zeta h\cdot\pi + \frac{\gamma}{2}|\zeta|^2 \geq \frac{\beta}{2}|\pi|^2 - (\beta - 2\nu_0)^{\frac{1}{2}}(\gamma - 2\nu_0)^{\frac{1}{2}}|\zeta|\,|\pi| + \frac{\gamma}{2}|\zeta|^2 \geq \nu_0(|\pi|^2 + |\zeta|^2). \tag{2.1}$$

Since $\mathcal{E}(e, p, \eta)$ is quadratic, by (2.1) we have that

$$\mathcal{E}\colon L^2(\Omega; \mathbb{R}^d) \times L^2(\Omega; \mathbb{R}^d) \times L^2(\Omega) \to [0, +\infty) \quad \text{is convex.} \tag{2.2}$$

To introduce the dissipation distance, we consider a bounded closed convex set $\Sigma \subset \mathbb{R}^d \times \mathbb{R}$ containing $(0, 0)$ in its interior. The dissipation potential $R\colon \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ is its support function:

$$R(\pi, \zeta) := \sup_{(\pi^*,\zeta^*)\in\Sigma} (\pi^*\cdot\pi + \zeta^*\zeta).$$

It is well known that R is convex and positively homogeneous of degree one. Moreover, it satisfies the inequalities

$$c_R(|\pi|^2 + |\zeta|)^{\frac{1}{2}} \leq R(\pi, \zeta) \leq C_R(|\pi|^2 + |\zeta|)^{\frac{1}{2}} \quad \text{for every } (\pi, \zeta) \in \mathbb{R}^d \times \mathbb{R} \tag{2.3}$$

for suitable constants $0 < c_R \leq C_R$. The corresponding functional $\mathcal{R}\colon L^2(\Omega; \mathbb{R}^d) \times L^2(\Omega) \to [0, +\infty)$ is

$$\mathcal{R}(p, \eta) := \int_\Omega R(p, \eta)\, dx.$$

To describe the contribution of the crack to the dissipation distance, we introduce the set $\mathcal{R}_{d-1}(\overline{\Omega})$ of $(\mathcal{H}^{d-1}, d-1)$-rectifiable subsets $\overline{\Omega}$ (see [15, Definition 3.2.14 (4)]) and consider the pseudo-distance $\mathcal{H}$ on $\mathcal{R}_{d-1}(\overline{\Omega})$

defined by

$$\mathcal{H}(\Gamma_2, \Gamma_1) := \begin{cases} \mathcal{H}^{d-1}(\Gamma_2 \setminus \Gamma_1) & \text{if } \Gamma_1 \subset \Gamma_2, \\ +\infty & \text{otherwise,} \end{cases}$$

where $\mathcal{H}^{d-1}$ is the $(d-1)$-dimensional Hausdorff measure. The complete dissipation distance in $L^2(\Omega; \mathbb{R}^d) \times L^2(\Omega) \times \mathcal{R}_{d-1}(\overline{\Omega})$ is then given by

$$\mathcal{D}(p_2, \eta_2, \Gamma_2; p_1, \eta_1, \Gamma_1) := \mathcal{R}(p_2 - p_1, \eta_2 - \eta_1) + \mathcal{H}(\Gamma_2, \Gamma_1). \tag{2.4}$$

Given a time interval $[s, t]$ and three functions $p\colon [s, t] \to L^2(\Omega; \mathbb{R}^d)$, $\eta\colon [s, t] \to L^2(\Omega)$, and $\Gamma\colon [s, t] \to \mathcal{R}_{d-1}(\overline{\Omega})$, the corresponding dissipation is defined by

$$\operatorname{Diss}(p(\cdot), \eta(\cdot), \Gamma(\cdot); s, t) := \sup \sum_{i=1}^{k} \mathcal{D}(p(t_i), \eta(t_i), \Gamma(t_i); p(t_{i-1}), \eta(t_{i-1}), \Gamma(t_{i-1})), \tag{2.5}$$

where the supremum is taken over all $k \in \mathbb{N}$ and over all subdivisions $s = t_0 < t_1 < \cdots < t_k = t$. In the same way we define

$$\operatorname{Diss}_R(p(\cdot), \eta(\cdot), \Gamma(\cdot); s, t) := \sup \sum_{i=1}^{k} \mathcal{R}(p(t_i) - p(t_{i-1}), \eta(t_i) - \eta(t_{i-1})). \tag{2.6}$$

If $\Gamma(\cdot)$ is increasing on $[s, t]$, i.e., $\Gamma(\tau_1) \subset \Gamma(\tau_2)$ for every $s \le \tau_1 < \tau_2 \le t$, it is clear from (2.4) that

$$\operatorname{Diss}(p(\cdot), \eta(\cdot), \Gamma(\cdot); s, t) = \operatorname{Diss}_R(p(\cdot), \eta(\cdot), \Gamma(\cdot); s, t) + \mathcal{H}^{d-1}(\Gamma(t) \setminus \Gamma(s)), \tag{2.7}$$

while

$$\operatorname{Diss}(p(\cdot), \eta(\cdot), \Gamma(\cdot); s, t) = +\infty$$

if $\Gamma(\cdot)$ is not increasing on $[s, t]$.

To describe the energetic formulation of our evolution problem, it is convenient to consider the displacement u as a function defined $\mathcal{L}^d$-a.e. in Ω, where $\mathcal{L}^d$ denotes the d-dimensional Lebesgue measure. Since u might have essential discontinuity points on Γ, it is natural to assume that it belongs to a suitable function space which allows for discontinuities along $(d-1)$-dimensional sets.

We recall that $\mathrm{BV}(\Omega)$ is the space of functions $u \in L^1(\Omega)$ whose distributional gradient Du is a bounded Radon measure on Ω with values in $\mathbb{R}^d$. The space $\mathrm{SBV}(\Omega)$ of special functions of bounded variation is composed of all functions $u \in \mathrm{BV}(\Omega)$ such that the singular part of Du is concentrated on a set of σ-finite $\mathcal{H}^{d-1}$-measure. The space $\mathrm{GSBV}(\Omega)$ of generalized special functions of bounded variation is the set of measurable functions u whose truncations belong to $\mathrm{SBV}_{\mathrm{loc}}(\Omega)$. We refer to [2, Section 4.5] for the details. In particular, we recall that for every $v \in \mathrm{GSBV}(\Omega)$ the approximate gradient ∇v is well defined $\mathcal{L}^d$-a.e. in Ω, the jump set J_v of v is a countably $(\mathcal{H}^{d-1}, d-1)$-rectifiable subset of Ω (according to [15, Definition 3.2.14 (3)]), and the trace of v on $\partial\Omega$ is well defined $\mathcal{H}^{d-1}$-a.e. on $\partial\Omega$ (see [2, Theorem 4.34]).

The estimates available for u lead us to formulate the problem in the space $\mathrm{GSBV}^2(\Omega)$ defined by

$$\mathrm{GSBV}^2(\Omega) := \{v \in \mathrm{GSBV}(\Omega) : \nabla v \in L^2(\Omega; \mathbb{R}^d),\ \mathcal{H}^{d-1}(J_v) < +\infty\}$$

and recall that $\mathrm{GSBV}^2(\Omega)$ is a vector space (see, e.g., [9, Proposition 2.3]). In our model we assume that $u \in \mathrm{GSBV}^2(\Omega)$, that the equality $\nabla u = e + p$ takes place $\mathcal{L}^d$-a.e. in Ω, and that $J_u \mathrel{\tilde{\subset}} \Gamma$, where $A \mathrel{\tilde{\subset}} B$ means $\mathcal{H}^{d-1}(A \setminus B) = 0$.

The Dirichlet boundary condition is prescribed through a function $w \in H^1(\Omega)$, imposing that the traces of u and w satisfy $u = w$ $\mathcal{H}^{d-1}$-a.e. on $\partial\Omega \setminus \Gamma$.

To simplify the exposition, given $\Gamma \in \mathcal{R}_{d-1}(\overline{\Omega})$ and $w \in H^1(\Omega)$, it is convenient to introduce the set $\mathcal{A}(\Gamma, w)$ of admissible pairs defined as the set of $(e, p) \in L^2(\Omega; \mathbb{R}^d) \times L^2(\Omega; \mathbb{R}^d)$ such that there exists $u \in \mathrm{GSBV}^2(\Omega)$ with the following properties:

$$\begin{aligned} &\nabla u = e + p \quad \mathcal{L}^d\text{-a.e. in } \Omega, \\ &J_u \mathrel{\tilde{\subset}} \Gamma, \\ &u = w \quad \mathcal{H}^{d-1}\text{-a.e. on } \partial\Omega \setminus \Gamma, \end{aligned}$$

where the last equality is intended in the sense of traces.

We study the evolution problem on the time interval $[0, T]$ with $T > 0$. The time-dependent boundary condition is given by means of a function $t \mapsto w(t)$. We assume that $w \in AC([0, T]; H^1(\Omega))$, the space of absolutely continuous functions from $[0, T]$ with values in $H^1(\Omega)$. We recall that $\nabla \dot{w}(t)$ is well defined for $\mathcal{L}^1$-a.e. $t \in [0, T]$ and that $\nabla \dot{w} \in L^1((0, T); L^2(\Omega; \mathbb{R}^d))$ (see [5, Appendix]).

We are now in a position to give the precise definition of a quasistatic evolution for our model in the framework of the notion of energetic solutions for rate-independent systems.

Definition 2.1. Given $w \in AC([0, T]; H^1(\Omega))$, a quasistatic evolution with boundary condition w is a function $t \mapsto (e(t), p(t), \eta(t), \Gamma(t))$ from $[0, T]$ into $L^2(\Omega; \mathbb{R}^d) \times L^2(\Omega; \mathbb{R}^d) \times L^2(\Omega) \times \mathcal{R}_{d-1}(\overline{\Omega})$ that satisfies the following conditions:

- Global Stability (GS): for every $t \in [0, T]$ we have $(e(t), p(t)) \in \mathcal{A}(\Gamma(t), w(t))$ and
$$\mathcal{E}(e(t), p(t), \eta(t)) \leq \mathcal{E}(\hat{e}, \hat{p}, \hat{\eta}) + \mathcal{D}(\hat{p}, \hat{\eta}, \hat{\Gamma}; p(t), \eta(t), \Gamma(t))$$
for every $\hat{\Gamma} \in \mathcal{R}_{d-1}(\overline{\Omega})$, $(\hat{e}, \hat{p}) \in \mathcal{A}(\hat{\Gamma}, w(t))$, and $\hat{\eta} \in L^2(\Omega)$,
- Energy-Dissipation Balance (EDB): the function $t \mapsto e(t)$ belongs to $L^\infty((0, T); L^2(\Omega; \mathbb{R}^d))$ and for every $t \in [0, T]$ we have
$$\mathcal{E}(e(t), p(t), \eta(t)) + \mathrm{Diss}(p(\cdot), \eta(\cdot), \Gamma(\cdot); 0, t) = \mathcal{E}(e(0), p(0), \eta(0)) + \alpha \int_0^t (e(s) | \nabla \dot{w}(s)) \, ds.$$

We are interested in the study of the existence of a quasistatic evolution with a prescribed initial condition $(e_0, p_0, \eta_0, \Gamma_0) \in L^2(\Omega; \mathbb{R}^d) \times L^2(\Omega; \mathbb{R}^d) \times L^2(\Omega) \times \mathcal{R}_{d-1}(\overline{\Omega})$. From the global stability condition it follows that the initial data must satisfy

$$(e_0, p_0) \in \mathcal{A}(\Gamma_0, w(0)), \tag{2.8}$$
$$\mathcal{E}(e_0, p_0, \eta_0) \leq \mathcal{E}(\hat{e}, \hat{p}, \hat{\eta}) + \mathcal{D}(\hat{p}, \hat{\eta}, \hat{\Gamma}; p_0, \eta_0, \Gamma_0), \tag{2.9}$$

for every $\hat{\Gamma} \in \mathcal{R}_{d-1}(\overline{\Omega})$, $(\hat{e}, \hat{p}) \in \mathcal{A}(\hat{\Gamma}, w(0))$, and $\hat{\eta} \in L^2(\Omega)$.

We are now in a position to state the main result of this paper.

Theorem 2.2. *Let* $w \in AC([0, T]; H^1(\Omega))$ *and let* $(e_0, p_0, \eta_0, \Gamma_0) \in L^2(\Omega; \mathbb{R}^d) \times L^2(\Omega; \mathbb{R}^d) \times L^2(\Omega) \times \mathcal{R}_{d-1}(\overline{\Omega})$ *be such that* (2.8) *and* (2.9) *hold. Then there exists a quasistatic evolution with boundary condition* w *such that* $(e(0), p(0), \eta(0), \Gamma(0)) = (e_0, p_0, \eta_0, \Gamma_0)$.

Theorem 2.2 will be proved through the usual variational approach. We first construct a discrete-time approximation by solving incremental minimum problems, then we prove the convergence of these approximate solutions to a solution according to Definition 2.1.

3 The incremental minimum problem

In this section we study the incremental minimum problems, which have the following general form. Given $p_0 \in L^2(\Omega; \mathbb{R}^d)$, $\eta_0 \in L^2(\Omega)$, $\Gamma_0 \in \mathcal{R}_{d-1}(\overline{\Omega})$, and $w_1 \in H^1(\Omega)$, the problem is to find

$$(e_1, p_1, \eta_1, \Gamma_1) \in L^2(\Omega; \mathbb{R}^d) \times L^2(\Omega; \mathbb{R}^d) \times L^2(\Omega) \times \mathcal{R}_{d-1}(\overline{\Omega})$$

such that

$$\begin{cases} (e_1, p_1) \in \mathcal{A}(\Gamma_1, w_1), \\ \mathcal{E}(e_1, p_1, \eta_1) + \mathcal{D}(p_1, \eta_1, \Gamma_1; p_0, \eta_0, \Gamma_0) \leq \mathcal{E}(\hat{e}, \hat{p}, \hat{\eta}) + \mathcal{D}(\hat{p}, \hat{\eta}, \hat{\Gamma}; p_0, \eta_0, \Gamma_0) \\ \quad \text{for every } \hat{\Gamma} \in \mathcal{R}_{d-1}(\overline{\Omega}), \ (\hat{e}, \hat{p}) \in \mathcal{A}(\hat{\Gamma}, w_1), \text{ and } \hat{\eta} \in L^2(\Omega). \end{cases} \tag{3.1}$$

To solve this problem it is convenient to express it in terms of the displacement u. In order to deal with the boundary condition we introduce a bounded open set $\Omega' \subset \mathbb{R}^d$ with Lipschitz boundary such that $\overline{\Omega} \subset \Omega'$ and we extend p_0, η_0, and w_1 to functions (denoted by the same symbols) belonging to the spaces $L^2(\Omega'; \mathbb{R}^d)$, $L^2(\Omega')$, and $H^1(\Omega')$, respectively. In this way the boundary condition $u = w_1$ on $\partial\Omega$ is rephrased as $u = w_1$ $\mathcal{L}^d$-a.e. in $\Omega' \setminus \overline{\Omega}$.

To express (3.1) in terms of ∇u and J_u, we introduce the function $f\colon \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \to [0, +\infty)$ defined by

$$f(\xi, \pi, \zeta) := \inf_{\hat\pi \in \mathbb{R}^d, \hat\zeta \in \mathbb{R}} g(\xi, \pi, \zeta, \hat\pi, \hat\zeta), \tag{3.2}$$

where $g(\xi, \pi, \zeta, \hat\pi, \hat\zeta) := \frac{\alpha}{2}|\xi - \hat\pi|^2 + \frac{\beta}{2}|\hat\pi|^2 + \hat\zeta\, h \cdot \hat\pi + \frac{\gamma}{2}|\hat\zeta|^2 + R(\hat\pi - \pi, \hat\zeta - \zeta)$. By (2.1) we have

$$g(\xi, \pi, \zeta, \hat\pi, \hat\zeta) \geq \frac{\alpha}{2}|\xi - \hat\pi|^2 + \nu_0|\hat\pi|^2 + \nu_0|\hat\zeta|^2. \tag{3.3}$$

Since $g(\xi, \pi, \zeta, \hat\pi, \hat\zeta)$ tends to $+\infty$ as $|(\hat\pi, \hat\zeta)| \to \infty$, the minimum in (3.2) is attained. Moreover, since g is strictly convex in $(\hat\pi, \hat\zeta)$, the minimum point is unique. We introduce two functions $\tilde\pi : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ and $\tilde\zeta : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ defined by

$$(\hat\pi, \hat\zeta) \text{ is a minimizer of } g(\xi, \pi, \zeta, \cdot, \cdot) \iff \begin{cases} \hat\pi = \tilde\pi(\xi, \pi, \zeta), \\ \hat\zeta = \tilde\zeta(\xi, \pi, \zeta). \end{cases}$$

By the uniqueness of the minimizer and the continuity of g , the functions $\tilde\pi$ and $\tilde\zeta$ depend continuously on (ξ, π, ζ), since the limit of minimizers is a minimizer. This implies that the function f is continuous.

Taking $\hat\pi = 0$ and $\hat\zeta = 0$ in (3.2), we obtain from (2.3) that

$$f(\xi, \pi, \zeta) \leq \frac{\alpha}{2}|\xi|^2 + R(-\pi, -\zeta) \leq \frac{\alpha}{2}|\xi|^2 + C_R(|\pi|^2 + |\zeta|^2)^{\frac{1}{2}}, \tag{3.4}$$

while by (3.3) we deduce that there exists $\mu_0 > 0$ such that

$$f(\xi, \pi, \zeta) \geq \mu_0|\xi|^2. \tag{3.5}$$

By (3.3) and (3.4) we have

$$\frac{\alpha}{2}|\xi - \tilde\pi(\xi, \pi, \zeta)|^2 + \nu_0|\tilde\pi(\xi, \pi, \zeta)|^2 + \nu_0|\tilde\zeta(\xi, \pi, \zeta)|^2 \leq \frac{\alpha}{2}|\xi|^2 + C_R(|\pi|^2 + |\zeta|^2)^{\frac{1}{2}},$$

from which we infer that there exists a constant $\Lambda_0 > 0$ such that

$$|\tilde\pi(\xi, \pi, \zeta)| + |\tilde\zeta(\xi, \pi, \zeta)| \leq \Lambda_0|\xi| + \Lambda_0(|\pi| + |\zeta|)^{\frac{1}{2}}. \tag{3.6}$$

By (2.1) the function $(\hat\pi, \hat\zeta) \mapsto \frac{\beta}{2}|\hat\pi|^2 + \hat\zeta\, h \cdot \hat\pi + \frac{\gamma}{2}|\hat\zeta|^2 + R(\hat\pi - \pi, \hat\zeta - \zeta)$ is convex. Hence we can conclude that

$$\xi \mapsto f(\xi, \pi, \zeta) \quad \text{is convex in } \mathbb{R}^d \text{ for every } (\pi, \zeta) \in \mathbb{R}^d \times \mathbb{R}. \tag{3.7}$$

We consider the auxiliary problem

$$\min_{\substack{u \in GSBV^2(\Omega') \\ u = w_1\ \mathcal{L}^d\text{-a.e. in } \Omega' \setminus \overline{\Omega}}} \int_{\Omega'} f(\nabla u, p_0, \eta_0)\, dx + \mathcal{H}^{d-1}(J_u \setminus \Gamma_0), \tag{3.8}$$

and study the existence of a solution.

Theorem 3.1. *Let $p_0 \in L^2(\Omega'; \mathbb{R}^d)$, $\eta_0 \in L^2(\Omega')$, $\Gamma_0 \in \mathcal{R}_{d-1}(\overline{\Omega})$, and $w_1 \in H^1(\Omega')$. Then there exists a solution to problem* (3.8).

Proof. Let $(u_k)_k$ be a minimizing sequence for (3.8). By (3.5) we can apply [20, Theorem 3.1] with E_k and h_k therein given by

$$E_k(v) := \int_\Omega |\nabla v|^2\, dx + \mathcal{H}^{d-1}(J_v) \quad \text{and} \quad h_k = w_1$$

for every $k \in \mathbb{N}$. We obtain a subsequence (not relabelled), modifications $(y_k)_k$ of $(u_k)_k$, and a function u such that the following conditions hold:

$$y_k, u \in GSBV^2(\Omega') \quad \text{and} \quad y_k = u = w_1 \quad \mathcal{L}^d\text{-a.e. in } \Omega' \setminus \Omega, \tag{3.9}$$

$$y_k \to u \quad \mathcal{L}^d\text{-a.e. in } \Omega', \tag{3.10}$$

$$\nabla y_k \rightharpoonup \nabla u \quad \text{and} \quad \nabla u_k \rightharpoonup \nabla u \text{ weakly in } L^2(\Omega'; \mathbb{R}^d), \tag{3.11}$$

$$\mathcal{H}^{d-1}(J_u \cap A) \leq \liminf_{k\to\infty} \mathcal{H}^{d-1}(J_{y_k} \cap A) \quad \text{for every open set } A \subset \Omega', \tag{3.12}$$

$$\lim_{k\to\infty} \mathcal{H}^{d-1}(J_{y_k} \setminus J_{u_k}) = 0. \tag{3.13}$$

In [20, Theorem 3.1] inequality (3.12) is proved only for $A = \Omega'$. The result for an arbitrary A in (3.12) follows from Ambrosio's compactness theorem ([2, Theorem 4.36]) applied to $\mathrm{GSBV}^2(A)$. Inequality (3.13) can be obtained by a slight modification of the arguments used in [20]. Indeed, a careful inspection of the proof of (7) (i)–(ii) in [20, Theorem 3.2] allows us to replace (9) (ii) with the estimate $\mathcal{H}^{d-1}(J_v \setminus J_u) < C_M\theta$ for the functions u and v and the constants C_M and θ in [20, Corollary 3.3]. This leads to the inequality $\mathcal{H}^{d-1}(J_{y_k} \setminus J_{u_k}) \le \frac{1}{k}$ instead of (34) (i) in [20, Theorem 3.8].

By the convexity of $\xi \mapsto f(\xi, p_0(x), \eta_0(x))$ for $\mathcal{L}^d$-a.e. x in Ω' (see (3.7)) and by (3.11) we have

$$\int_{\Omega'} f(\nabla u, p_0, \eta_0)\,dx \le \liminf_{k\to\infty} \int_{\Omega'} f(\nabla u_k, p_0, \eta_0)\,dx. \tag{3.14}$$

Since $\mathcal{H}^{d-1}(\Gamma_0) < +\infty$, for every $\varepsilon > 0$ there exists a compact set $K \subset \Gamma_0$ such that $\mathcal{H}^{d-1}(\Gamma_0 \setminus K) < \varepsilon$. By (3.12) and (3.13) we have

$$\mathcal{H}^{d-1}(J_u \setminus \Gamma_0) \le \mathcal{H}^{d-1}(J_u \setminus K) \le \liminf_{k\to\infty} \mathcal{H}^{d-1}(J_{u_k} \setminus K) \le \liminf_{k\to\infty} \mathcal{H}^{d-1}(J_{u_k} \setminus \Gamma_0) + \varepsilon.$$

Passing to the limit as $\varepsilon \to 0$, we obtain

$$\mathcal{H}^{d-1}(J_u \setminus \Gamma_0) \le \liminf_{k\to\infty} \mathcal{H}^{d-1}(J_{u_k} \setminus \Gamma_0).$$

Recalling that $(u_k)_k$ is a minimizing sequence, this inequality together with (3.9) and (3.14) shows that u is a minimizer of (3.8). □

Given a solution u_1 of (3.8) we set

$$p_1 := \tilde{\pi}(\nabla u_1, p_0, \eta_0), \quad e_1 := \nabla u_1 - p_1, \quad \eta_1 := \tilde{\zeta}(\nabla u_1, p_0, \eta_0), \quad \Gamma_1 := J_{u_1} \cup \Gamma_0. \tag{3.15}$$

Then $p_1, e_1 \in L^2(\Omega'; \mathbb{R}^d)$ and $\eta_1 \in L^2(\Omega')$ by (3.6), while $\Gamma_1 \in \mathcal{R}_{d-1}(\overline{\Omega})$.

We now prove that $(e_1, p_1, \eta_1, \Gamma_1)$ solves the minimum problem (3.1).

Theorem 3.2. *Let u_1 be a solution of* (3.8) *and let $e_1, p_1, \eta_1, \Gamma_1$ be defined by* (3.15). *Then $(e_1, p_1, \eta_1, \Gamma_1)$ is a solution to the minimum problem* (3.1).

Proof. Condition $(p_1, \eta_1) \in \mathcal{A}(\Gamma_1, w_1)$ is satisfied by definition. To prove the minimality, we fix $\hat\Gamma \in \mathcal{R}_{d-1}(\overline{\Omega})$ with $\hat\Gamma \,\tilde{\supset}\, \Gamma_0$, $(\hat e, \hat p) \in \mathcal{A}(\hat\Gamma, w_1)$, and $\hat\eta \in L^2(\Omega)$. Let $\hat u \in \mathrm{GSBV}^2(\Omega)$ be such that $\nabla \hat u = \hat e + \hat p$ $\mathcal{L}^d$-a.e. in Ω, $J_{\hat u} \,\tilde{\subset}\, \hat\Gamma$, and $\hat u = w_1$ $\mathcal{H}^{d-1}$-a.e. on $\partial\Omega \setminus \hat\Gamma$. We now extend $\hat u, \hat e, \hat p, \hat\eta$ to Ω' by setting $\hat u := w_1$, $\hat e := \nabla w_1 - p_0$, $\hat p := p_0$, $\hat \eta := \eta_0$ in $\Omega' \setminus \Omega$. Then $\hat u \in \mathrm{GSBV}^2(\Omega')$, hence by (3.8) we obtain

$$\int_{\Omega'} f(\nabla u_1, p_0, \eta_0)\,dx + \mathcal{H}^{d-1}(J_{u_1} \setminus \Gamma_0) \le \int_{\Omega'} f(\nabla \hat u, p_0, \eta_0)\,dx + \mathcal{H}^{d-1}(J_{\hat u} \setminus \Gamma_0),$$

which gives

$$\int_{\Omega} f(\nabla u_1, p_0, \eta_0)\,dx + \mathcal{H}^{d-1}(J_{u_1} \setminus \Gamma_0) \le \int_{\Omega} f(\nabla \hat u, p_0, \eta_0)\,dx + \mathcal{H}^{d-1}(J_{\hat u} \setminus \Gamma_0).$$

By the definition (3.2) of f and (3.15) the previous inequality gives

$$\begin{aligned}
&\frac{\alpha}{2}\int_\Omega |e_1|^2\,dx + \frac{\beta}{2}\int_\Omega |p_1|^2\,dx + \int_\Omega \eta_1 h{\cdot}p_1\,dx + \frac{\gamma}{2}\int_\Omega |\eta_1|^2\,dx + \int_\Omega R(p_1 - p_0, \eta_1 - \eta_0)\,dx + \mathcal{H}^{d-1}(J_{u_1} \setminus \Gamma_0)\\
&\quad\le \frac{\alpha}{2}\int_\Omega |\hat e|^2\,dx + \frac{\beta}{2}\int_\Omega |\hat p|^2\,dx + \int_\Omega \hat\eta h{\cdot}\hat p\,dx + \frac{\gamma}{2}\int_\Omega |\hat\eta|^2\,dx + \int_\Omega R(\hat p - p_0, \hat\eta - \eta_0)\,dx + \mathcal{H}^{d-1}(J_{\hat u} \setminus \Gamma_0).
\end{aligned}$$

Since $\mathcal{H}^{d-1}(J_{u_1} \setminus \Gamma_0) = \mathcal{H}^{d-1}(\Gamma_1 \setminus \Gamma_0)$ and $\mathcal{H}^{d-1}(J_{\hat u} \setminus \Gamma_0) \le \mathcal{H}^{d-1}(\hat\Gamma \setminus \Gamma_0)$, the previous inequality gives

$$\mathcal{E}(e_1, p_1, \eta_1) + \mathcal{D}(p_1, \eta_1, \Gamma_1; p_0, \eta_0, \Gamma_0) \le \mathcal{E}(\hat e, \hat p, \hat \eta) + \mathcal{D}(\hat p, \hat\eta, \hat\Gamma; p_0, \eta_0, \Gamma_0),$$

thus proving that $(e_1, p_1, \eta_1, \Gamma_1)$ is a solution to the minimum problem (3.1). □

In the proof of Theorem 2.2 we shall use also the property of the solution of (3.1) provided by the following proposition.

Proposition 3.3. *Let* $(e_1, p_1, \eta_1, \Gamma_1)$ *be a solution to the minimum problem* (3.1). *Then*

$$\mathcal{E}(e_1, p_1, \eta_1) \leq \mathcal{E}(\hat{e}, \hat{p}, \hat{\eta}) + \mathcal{D}(\hat{p}, \hat{\eta}, \hat{\Gamma}; p_1, \eta_1, \Gamma_1) \tag{3.16}$$

for every $\hat{\Gamma} \in \mathcal{R}_{d-1}(\overline{\Omega})$, $(\hat{e}, \hat{p}) \in \mathcal{A}(\hat{\Gamma}, w_1)$, *and* $\hat{\eta} \in L^2(\Omega)$.

Proof. It is enough to observe that

$$\mathcal{D}(\hat{p}, \hat{\eta}, \hat{\Gamma}; p_0, \eta_0, \Gamma_0) - \mathcal{D}(p_1, \eta_1, \Gamma_1; p_0, \eta_0, \Gamma_0) \leq \mathcal{D}(\hat{p}, \hat{\eta}, \hat{\Gamma}; p_1, \eta_1, \Gamma_1)$$

by the triangular inequality. □

4 A notion of convergence for cracks and stability of minimizers

In this section we recall a notion of convergence for cracks introduced in [21] and use its properties to prove the stability of condition (3.16) (see Theorem 4.5).

Let $\mathcal{R}_{d-1}(\Omega')$ be the set of $(\mathcal{H}^{d-1}, d-1)$-rectifiable subsets of Ω', see [15, Definition 3.2.14 (4)]. Let $\mathcal{U}(\Omega')$ be the space of functions $u \in SBV(\Omega')$ with values in $\{0, 1\}$ and let $A(\Omega')$ be the collection of all open subsets of Ω'. Given a sequence $(\Gamma_k)_k$ in $\mathcal{R}_{d-1}(\Omega')$, let $\mathcal{H}_k\colon \mathcal{U}(\Omega) \times A(\Omega) \to [0, +\infty)$ be defined by

$$\mathcal{H}_k(u, A) := \mathcal{H}^{d-1}((J_u \setminus \Gamma_k) \cap A). \tag{4.1}$$

It is known that a subsequence, not relabelled, has the property that $\mathcal{H}_k(\,\cdot\,, A)$ Γ-converges with respect to the strong topology of $L^1(\Omega')$ to a functional $\mathcal{H}(\,\cdot\,, A)$ of the form

$$\mathcal{H}(u, A) = \int_{J_u \cap A} h(x, \nu)\, d\mathcal{H}^{d-1} \tag{4.2}$$

for some function $h\colon \Omega \times \mathbb{R}^d \to [0, +\infty)$.

Definition 4.1 (σ-convergence). Let $(\Gamma_k)_k$ be a sequence in $\mathcal{R}_{d-1}(\Omega')$ and let $\Gamma \in \mathcal{R}_{d-1}(\Omega')$. We say that Γ_k σ-converges to Γ in Ω' if for every $A \in A(\Omega')$ the functionals $\mathcal{H}_k(\,\cdot\,, A)$ defined by (4.1) Γ-converge with respect to the strong topology of $L^1(\Omega')$ to the functional $\mathcal{H}(\,\cdot\,, A)$ given by (4.2) and if Γ is the unique rectifiable set such that

$$h(x, \nu_\Gamma(x)) = 0 \quad \text{for } \mathcal{H}^{d-1}\text{-a.e. } x \in \Gamma$$

and such that for every $\Gamma' \in \mathcal{R}_{d-1}(\Omega')$ we have

$$h(x, \nu_{\Gamma'}(x)) = 0 \text{ for } \mathcal{H}^{d-1}\text{-a.e. } x \in \Gamma' \implies \Gamma' \,\tilde{\subset}\, \Gamma.$$

The following proposition summarizes the basic properties of σ-convergence.

Proposition 4.2. *Let* $(\Gamma_k)_k$ *be a sequence in* $\mathcal{R}_{d-1}(\Omega')$. *Then:*
(a) (Compactness) *If* $(\mathcal{H}^{d-1}(\Gamma_k))_k$ *is bounded, then there exist a subsequence, not relabelled, and a set* $\Gamma \in \mathcal{R}_{d-1}(\Omega')$ *such that*

$$\Gamma_k \quad \sigma\text{-converges in } \Omega' \text{ to } \Gamma.$$

(b) (Semicontinuity) *If* Γ_k σ*-converges in* Ω' *to* Γ, *then*

$$\mathcal{H}^{d-1}(\Gamma \cap A) \leq \liminf_{k\to\infty} \mathcal{H}^{d-1}(\Gamma_k \cap A) \tag{4.3}$$

for every open subset A *of* Ω'.
(c) (Stability) *If* Γ_k σ*-converges in* Ω' *to* Γ *and* $(\tilde{\Gamma}_k)_k$ *is a sequence in* $\mathcal{R}_{d-1}(\Omega')$ *such that* $\mathcal{H}^{d-1}(\tilde{\Gamma}_k \Delta \Gamma_k) \to 0$, *then* $\tilde{\Gamma}_k$ σ*-converges in* Ω' *to* Γ.

Proof. Property (a) is proved in [21, Proposition 6.3]. Under the assumption of (b) we observe that $\Gamma_k \cap A$ σ-converges in A to $\Gamma \cap A$ for every open set $A \subset \Omega'$. Therefore (4.3) follows from [21, Proposition 6.3] if $\liminf_{k\to\infty} \mathcal{H}^{d-1}(\Gamma_k \cap A) < +\infty$ and is trivial otherwise. Property (c) is proved in [21, Remark 6.2]. □

The following proposition presents the main property of σ-convergence. According to [9, Definition 2.6], we say that a sequence $(u_k)_k$ converges to u weakly in $\mathrm{GSBV}^2(\Omega')$ if $u_k, u \in \mathrm{GSBV}^2(\Omega')$, $u_k \to u$ $\mathcal{L}^d$-a.e. in Ω', $\nabla u_k \rightharpoonup \nabla u$ weakly in $L^2(\Omega'; \mathbb{R}^d)$, and $(\mathcal{H}^{d-1}(J_{u_k}))_k$ is bounded in $\mathbb{R}$.

Proposition 4.3. *Let $(\Gamma_k)_k$ be a sequence in $\mathcal{R}_{d-1}(\Omega')$ that σ-converges in Ω' to $\Gamma \in \mathcal{R}_{d-1}(\Omega')$. Let $(u_k)_k$ be a sequence in $\mathrm{GSBV}^2(\Omega')$ with $u_k \rightharpoonup u$ weakly in $\mathrm{GSBV}^2(\Omega')$ and $\mathcal{H}^{d-1}(J_{u_k} \setminus \Gamma_k) \to 0$. Then $J_u \,\tilde{\subset}\, \Gamma$.*

Proof. If, in addition, $u_k \in \mathrm{BV}(\Omega') \cap L^\infty(\Omega')$ and $(u_k)_k$ is bounded in $L^\infty(\Omega')$, then the result follows from [21, Proposition 6.8]. The general case can be obtained by truncation, arguing as in [9, Proposition 4.6]. □

In the proof of the stability result for (3.16) a crucial role is played by the following theorem which extends the jump transfer argument introduced in [16, Theorem 2.1].

Theorem 4.4 (Jump transfer in GSBV^2). *Let $(\Gamma_k)_k$ be a sequence in $\mathcal{R}_{d-1}(\overline{\Omega})$ σ-converging in Ω' to $\Gamma \in \mathcal{R}_{d-1}(\overline{\Omega})$ and let $v \in \mathrm{GSBV}^2(\Omega')$. Then there exists a sequence $(v_k)_k$ in $\mathrm{GSBV}^2(\Omega')$ such that*

$$v_k = v \quad \mathcal{L}^d\text{-a.e. in } \Omega' \setminus \Omega, \tag{4.4}$$

$$v_k \to v \quad \mathcal{L}^d\text{-a.e. in } \Omega', \tag{4.5}$$

$$\nabla v_k \to \nabla v \quad \text{strongly in } L^2(\Omega'; \mathbb{R}^d), \tag{4.6}$$

$$\limsup_{k\to\infty} \mathcal{H}^{d-1}(J_{v_k} \setminus \Gamma_k) \le \mathcal{H}^{d-1}(J_v \setminus \Gamma). \tag{4.7}$$

Proof. If, in addition, $v \in \mathrm{BV}(\Omega')$, then the result follows from [21, Theorem 7.4]. In the general case we conclude arguing as in [9, Theorem 5.3]. □

We are now ready to prove the stability result for (3.16).

Theorem 4.5 (Stability of minimizers). *Let $(e_k)_k, (p_k)_k, (\eta_k)_k, (\Gamma_k)_k, (w_k)_k$ be sequences in $L^2(\Omega; \mathbb{R}^d)$, $L^2(\Omega; \mathbb{R}^d)$, $L^2(\Omega)$, $\mathcal{R}_{d-1}(\overline{\Omega})$, and $H^1(\Omega)$, respectively. For every $k \in \mathbb{N}$ we suppose that $(e_k, p_k) \in \mathcal{A}(\Gamma_k, w_k)$ and*

$$\mathcal{E}(e_k, p_k, \eta_k) \le \mathcal{E}(\hat{e}, \hat{p}, \hat{\eta}) + \mathcal{D}(\hat{p}, \hat{\eta}, \hat{\Gamma}; p_k, \eta_k, \Gamma_k) \tag{4.8}$$

for every $\hat{\Gamma} \in \mathcal{R}_{d-1}(\overline{\Omega})$, $(\hat{e}, \hat{p}) \in \mathcal{A}(\hat{\Gamma}, w_k)$, and $\hat{\eta} \in L^2(\Omega)$. Assume that

$$e_k \rightharpoonup e \quad \text{weakly in } L^2(\Omega; \mathbb{R}^d), \tag{4.9}$$

$$p_k \rightharpoonup p \quad \text{weakly in } L^2(\Omega; \mathbb{R}^d), \tag{4.10}$$

$$\eta_k \rightharpoonup \eta \quad \text{weakly in } L^2(\Omega), \tag{4.11}$$

$$\Gamma_k \ \sigma\text{-converges in } \Omega' \text{ to } \Gamma, \tag{4.12}$$

$$w_k \to w \quad \text{strongly in } H^1(\Omega). \tag{4.13}$$

Then $(e, p) \in \mathcal{A}(\Gamma, w)$ and

$$\mathcal{E}(e, p, \eta) \le \mathcal{E}(\hat{e}, \hat{p}, \hat{\eta}) + \mathcal{D}(\hat{p}, \hat{\eta}, \hat{\Gamma}; p, \eta, \Gamma)$$

for every $\hat{\Gamma} \in \mathcal{R}_{d-1}(\overline{\Omega})$, $(\hat{e}, \hat{p}) \in \mathcal{A}(\hat{\Gamma}, w)$, and $\hat{\eta} \in L^2(\Omega)$.

To prove the theorem we use the following lemma.

Lemma 4.6. *Let $e \in L^2(\Omega; \mathbb{R}^d)$, $p \in L^2(\Omega; \mathbb{R}^d)$, $\eta \in L^2(\Omega)$, $\Gamma \in \mathcal{R}_{d-1}(\overline{\Omega})$, and $w \in H^1(\Omega)$. The following conditions are equivalent:*

(a) *$(e, p) \in \mathcal{A}(\Gamma, w)$ and*

$$\mathcal{E}(e, p, \eta) \le \mathcal{E}(\hat{e}, \hat{p}, \hat{\eta}) + \mathcal{D}(\hat{p}, \hat{\eta}, \hat{\Gamma}; p, \eta, \Gamma) \tag{4.14}$$

for every $\hat{\Gamma} \in \mathcal{R}_{d-1}(\overline{\Omega})$, $(\hat{e}, \hat{p}) \in \mathcal{A}(\hat{\Gamma}, w)$, and $\hat{\eta} \in L^2(\Omega)$.

(b) *$(e, p) \in \mathcal{A}(\Gamma, w)$ and*

$$0 \le \alpha(e|\tilde{e}) + \frac{\alpha}{2}\|\tilde{e}\| + \beta(p|\tilde{p}) + \frac{\beta}{2}\|\tilde{p}\|^2 + (\eta h|\tilde{p}) + (\tilde{\eta} h|p) + (\tilde{\eta} h|\tilde{p}) + \gamma(\eta|\tilde{\eta}) + \frac{\gamma}{2}\|\tilde{\eta}\|^2 + \mathcal{R}(\tilde{p}, \tilde{\eta}) + \mathcal{H}^{d-1}(\tilde{\Gamma} \setminus \Gamma) \tag{4.15}$$

for every $\tilde{\Gamma} \in \mathcal{R}_{d-1}(\overline{\Omega})$, with $\tilde{\Gamma} \,\tilde{\supset}\, \Gamma$, $(\tilde{e}, \tilde{p}) \in \mathcal{A}(\tilde{\Gamma}, 0)$, and $\tilde{\eta} \in L^2(\Omega)$.

Proof. (a) ⇒ (b) Let $\tilde\Gamma \in \mathcal{R}_{d-1}(\overline{\Omega})$, with $\tilde\Gamma \,\tilde\supset\, \Gamma$, $(\tilde e, \tilde p) \in \mathcal{A}(\tilde\Gamma, 0)$, and $\tilde\eta \in L^2(\Omega)$. Define $\hat\Gamma = \tilde\Gamma$, $\hat e = e + \tilde e$, $\hat p = p + \tilde e$, $\hat\eta = \eta + \tilde\eta$, and observe that $(\hat e, \hat p) \in \mathcal{A}(\hat\Gamma, w)$. Substituting in (4.14) and using the definition of $\mathcal{E}$ and $\mathcal{D}$, we obtain (4.15). The proof of the other implication is similar. □

Proof of Theorem 4.5. Since $(e_k, p_k) \in \mathcal{A}(\Gamma_k, w_k)$ for every $k \in \mathbb{N}$ there exists $u_k \in GSBV^2(\Omega)$ such that

$$\nabla u_k = e_k + p_k \quad \mathcal{L}^d\text{-a.e. in } \Omega, \tag{4.16}$$

$$J_{u_k} \,\tilde\subset\, \Gamma_k, \tag{4.17}$$

$$u_k = w_k \quad \mathcal{H}^{d-1}\text{-a.e. on } \partial\Omega \setminus \Gamma_k. \tag{4.18}$$

We extend w_k and w to Ω' in such a way that $w_k \in H^1(\Omega')$, $w \in H^1(\Omega')$, and $w_k \to w$ strongly in $H^1(\Omega')$. Moreover, we extend u_k to Ω' by setting $u_k := w_k$ in $\Omega' \setminus \Omega$ and observe that $u_k \in GSBV^2(\Omega')$ and that by (4.17) and (4.18) the jump of the extension in Ω' satisfies $J_{u_k} \,\tilde\subset\, \Gamma_k$.

By (4.9) and (4.16) we deduce that $(\nabla u_k)_k$ is bounded in $L^2(\Omega'; \mathbb{R}^d)$. Since also $\mathcal{H}^{d-1}(J_{u_k})$ is bounded, we can apply [20, Theorem 3.1] and obtain a subsequence (not relabelled), modifications $(y_k)_k$ of $(u_k)_k$, and a function u satisfying (3.9)–(3.13), with w_1 replaced by w. Applying Proposition 4.3, we conclude that $J_u \,\tilde\subset\, \Gamma$, hence $J_u \cap \Omega \,\tilde\subset\, \Gamma$ and $u = w$ on $\partial\Omega \setminus \Gamma$ in the sense of traces. By (3.11), (4.10), and (4.11) we obtain that $\nabla u = e + p$ $\mathcal{L}^d$-a.e. in Ω, which together with the previous remarks gives $(e, p) \in \mathcal{A}(\Gamma, w)$.

Let us fix $\tilde\Gamma \in \mathcal{R}_{d-1}(\overline{\Omega})$ with $\tilde\Gamma \,\tilde\supset\, \Gamma$, $(\tilde e, \tilde p) \in \mathcal{A}(\tilde\Gamma, 0)$, and $\tilde\eta \in L^2(\Omega)$. Then there exists a function $\tilde v \in GSBV^2(\Omega)$ such that

$$\nabla \tilde v = \tilde e + \tilde p \quad \mathcal{L}^d\text{-a.e. in } \Omega,$$

$$J_{\tilde v} \,\tilde\subset\, \tilde\Gamma,$$

$$\tilde v = 0 \quad \text{on } \partial\Omega \setminus \tilde\Gamma.$$

We extend $\tilde v$ to $\Omega' \setminus \Omega$ by setting $\tilde v := 0$ on $\Omega' \setminus \Omega$ and observe that $\tilde v \in GSBV^2(\Omega')$. By the Jump Transfer Theorem 4.4 there exists a sequence $(\tilde v_k)_k$ in $GSBV^2(\Omega')$ which satisfies (4.4)–(4.7). We define $\tilde e_k := \nabla \tilde v_k - \tilde p$ and $\tilde\Gamma_k := \Gamma_k \cup J_{\tilde v_k}$. Then $\tilde e_k \to \tilde e$ strongly in $L^2(\Omega; \mathbb{R}^d)$ by (4.6), and $(\tilde e_k, \tilde p) \in \mathcal{A}(\tilde\Gamma_k, 0)$ for every k.

Using the implication (a) ⇒ (b) in Lemma 4.6, from (4.8) we deduce that

$$\begin{aligned} 0 \le \alpha(e_k|\tilde e_k) + \frac{\alpha}{2}\|\tilde e_k\|^2 + \beta(p_k|\tilde p) + \frac{\beta}{2}\|\tilde p\|^2 + (\eta_k h|\tilde p) + (\tilde\eta h|p_k) + (\tilde\eta h|\tilde p) \\ + \gamma(\eta_k|\tilde\eta) + \frac{\gamma}{2}\|\tilde\eta\|^2 + \mathcal{R}(\tilde p, \tilde\eta) + \mathcal{H}^{d-1}(J_{\tilde v_k} \setminus \Gamma_k) \end{aligned}$$

for every k. Passing to the limit as $k \to \infty$ and using (4.7) we obtain

$$\begin{aligned} 0 \le \alpha(e|\tilde e) + \frac{\alpha}{2}\|\tilde e\|^2 + \beta(p|\tilde p) + \frac{\beta}{2}\|\tilde p\|^2 + (\eta h|\tilde p) + (\tilde\eta h|p) + (\tilde\eta h|\tilde p) \\ + \gamma(\eta|\tilde\eta) + \frac{\gamma}{2}\|\tilde\eta\|^2 + \mathcal{R}(\tilde p, \tilde\eta) + \mathcal{H}^{d-1}(J_{\tilde v} \setminus \Gamma). \end{aligned}$$

Since $J_{\tilde v} \,\tilde\subset\, \tilde\Gamma$, from this inequality we obtain (4.15). The conclusion follows from the implication (b) ⇒ (a) in Lemma 4.6. □

5 Proof of Theorem 2.2

In this section we prove Theorem 2.2. We use the standard procedure based on the construction of discrete-time approximate solutions obtained by solving incremental minimum problems. To this end for every $k \in \mathbb{N}$ we consider a subdivision $0 = t_k^0 < t_k^1 < \cdots < t_k^k = T$ of $[0, T]$ such that

$$\max_{1\le i\le k}(t_k^i - t_k^{i-1}) \to 0 \tag{5.1}$$

as $k \to \infty$.

Given $w \in AC([0, T]; H^1(\Omega))$ and $(e_0, p_0, \eta_0, \Gamma_0) \in L^2(\Omega; \mathbb{R}^d) \times L^2(\Omega; \mathbb{R}^d) \times L^2(\Omega) \times \mathcal{R}_{d-1}(\overline{\Omega})$ satisfying (2.8) and (2.9), the values of the approximate solutions at times t_k^i are defined in the following way. For every $k \in \mathbb{N}$

we set $e_k^0 := e_0$, $p_k^0 := p_0$, $\eta_k^0 := \eta_0$, and $\Gamma_k^0 := \Gamma_0$. Then, for $i = 1, \dots, k$ we define e_k^i, p_k^i, η_k^i, and Γ_k^i inductively as a solution to (3.1) with e_0, p_0, η_0, Γ_0, and w_1 replaced by e_k^{i-1}, p_k^{i-1}, η_k^{i-1}, Γ_k^{i-1}, and $w(t_k^i)$, respectively.

A crucial role in the proof of Theorem 2.2 is played by the following energy estimate for the discrete-time approximate solutions.

Lemma 5.1 (Discrete energy estimate). *There exists a sequence $a_k \to 0+$ such that*

$$\mathcal{E}(e_k^i, p_k^i, \eta_k^i) + \sum_{j=1}^{i} \mathcal{D}(p_k^j, \eta_k^j, \Gamma_k^j; p_k^{j-1}, \eta_k^{j-1}, \Gamma_k^{j-1}) \le \mathcal{E}(e_0, p_0, \eta_0) + \alpha \sum_{j=1}^{i} (e_k^j | \nabla w(t_k^j) - \nabla w(t_k^{j-1})) + a_k \tag{5.2}$$

for every $k \in \mathbb{N}$ and every $1 \le i \le k$.

Proof. Let us fix k and i. For every $1 \le j \le i$ we take $\hat{\Gamma} := \Gamma_k^{j-1}$, $\hat{e} := e_k^{j-1} + \nabla w(t_k^j) - \nabla w(t_k^{j-1})$, $\hat{p} := p_k^{j-1}$, and $\hat{\eta} := \eta_k^{j-1}$ as competitor in the minimum problem satisfied by $(e_k^j, p_k^j, \eta_k^j, \Gamma_k^j)$ and we obtain the inequality

$$\begin{aligned}&\mathcal{E}(e_k^j, p_k^j, \eta_k^j) + \mathcal{D}(p_k^j, \eta_k^j, \Gamma_k^j; p_k^{j-1}, \eta_k^{j-1}, \Gamma_k^{j-1}) \\ &\quad \le \mathcal{E}(\eta_k^{j-1}, p_k^{j-1}, \eta_k^{j-1}) + \alpha(e_k^j | \nabla w(t_k^j) - \nabla w(t_k^{j-1})) + \frac{\alpha}{2} \|\nabla w(t_k^j) - \nabla w(t_k^{j-1})\|^2.\end{aligned}$$

Summing this inequalities for $1 \le j \le i$, we obtain (5.2) with

$$a_k := \frac{\alpha}{2} \sum_{j=1}^{k} \|\nabla w(t_k^j) - \nabla w(t_k^{j-1})\|^2 \le M_k \sup_{1 \le j \le k} \|\nabla w(t_k^j) - \nabla w(t_k^{j-1})\|,$$

where $M_k := \frac{\alpha}{2} \sum_{j=1}^{k} \|\nabla w(t_k^j) - \nabla w(t_k^{j-1})\|$. Since $w \in AC([0,T]; H^1(\Omega))$, by (5.1) we conclude that $a_k \to 0+$. □

The discrete energy estimate proved in the previous lemma leads to the following a priori estimate for the discrete-time approximate solutions.

Lemma 5.2 (A priori estimate). *There exists $C > 0$ such that*

$$\|e_k^i\|^2 + \|p_k^i\|^2 + \|\eta_k^i\|^2 + \mathcal{H}^{d-1}(\Gamma_k^i) \le C \tag{5.3}$$

for every $k \in \mathbb{N}$ and $1 \le i \le k$.

Proof. Let $M_k := \max_{1 \le i \le k}(\|e_k^i\|^2 + \|p_k^i\|^2 + \|\eta_k^i\|^2)$. From (5.2) and the coerciveness (2.1) we obtain that there exists a constant $C_0 > 0$ such that

$$\|e_k^i\|^2 + \|p_k^i\|^2 + \|\eta_k^i\|^2 + \mathcal{H}^{d-1}(\Gamma_k^i \setminus \Gamma_0) \le C_0 \sum_{j=1}^{k} \|e_k^j\| \, \|\nabla w(t_k^j) - \nabla w(t_k^{j-1})\|. \tag{5.4}$$

Since $w \in AC([0,T]; H^1(\Omega))$, there exists a constant $C_1 > 0$ such that

$$\sum_{j=1}^{k} \|\nabla w(t_k^j) - \nabla w(t_k^{j-1})\|_{L^2(\Omega;\mathbb{R}^d)} \le C_1$$

for every $k \in \mathbb{N}$, hence

$$\|e_k^i\|^2 + \|p_k^i\|^2 + \|\eta_k^i\|^2 \le C_0 C_1 M_k^{\frac{1}{2}}$$

for every $1 \le i \le k$. Taking the supremum with respect to i, we obtain that $M_k \le C_0^2 C_1^2$. Taking into account (5.4), we now deduce that $\mathcal{H}^{d-1}(\Gamma_k^i) \le C_0^2 C_1^2 + \mathcal{H}^{d-1}(\Gamma_0)$, which concludes the proof. □

To prove the global stability condition (GS) in Definition 2.1 we observe that the approximate solutions satisfy a discrete-time version of the same property.

Remark 5.3. By Proposition 3.3 the quadruple $(e_k^i, p_k^i, \eta_k^i, \Gamma_k^i)$ satisfies

$$\mathcal{E}(e_k^i, p_k^i, \eta_k^i) \le \mathcal{E}(\hat{e}, \hat{p}, \hat{\eta}) + \mathcal{D}(\hat{p}, \hat{\eta}, \hat{\Gamma}; p_k^i, \eta_k^i, \Gamma_k^i)$$

for every $\hat{\Gamma} \in \mathcal{R}_{d-1}(\overline{\Omega})$, $(\hat{e}, \hat{p}) \in \mathcal{A}(\hat{\Gamma}, w(t_k^i))$, and $\hat{\eta} \in L^2(\Omega)$.

Proof of Theorem 2.2. Let e_k^i, p_k^i, η_k^i, and Γ_k^i be defined as at the beginning of this section. For every $k \in \mathbb{N}$ and every $t \in [0, T]$ we consider their piecewise constant interpolations defined by

$$e_k(t) := e_k^{i-1}, \quad p_k(t) := p_k^{i-1}, \quad \eta_k(t) := \eta_k^{i-1}, \quad \Gamma_k(t) := \Gamma_k^{i-1}, \quad w_k(t) := w(t_k^{i-1})$$

for every $t \in [t_k^{i-1}, t_k^i)$ for $1 \le i < k$, and $t \in [t_k^{i-1}, t_k^i]$ for $i = k$.

Note that $t \mapsto \Gamma_k(t)$ is increasing, $(e_k(t), p_k(t)) \in \mathcal{A}(\Gamma_k(t), w_k(t))$, and by Remark 5.3 we have

$$\mathcal{E}(e_k(t), p_k(t), \eta_k(t)) \le \mathcal{E}(\hat{e}, \hat{p}, \hat{\eta}) + \mathcal{D}(\hat{p}, \hat{\eta}, \hat{\Gamma}; p_k(t), \eta_k(t), \Gamma_k(t))$$

for every $\hat{\Gamma} \in \mathcal{R}_{d-1}(\overline{\Omega})$, $(\hat{e}, \hat{p}) \in \mathcal{A}(\hat{\Gamma}, w_k(t))$, and $\hat{\eta} \in L^2(\Omega)$.

By (5.3) we obtain that there exists $C > 0$ such that

$$\|e_k(t)\|^2 + \|p_k(t)\|^2 + \|\eta_k(t)\|^2 + \mathcal{H}^{d-1}(\Gamma_k(t)) \le C \tag{5.5}$$

for every $k \in \mathbb{N}$ and every $t \in [0, T]$.

Recalling (2.5)–(2.7), by (5.2) it follows that there exists a sequence $b_k \to 0+$ such that

$$\mathcal{E}(e_k(t), p_k(t), \eta_k(t)) + \mathrm{Diss}_R(p_k, \eta_k; 0, t) + \mathcal{H}^{d-1}(\Gamma_k(t) \setminus \Gamma_0) \le \mathcal{E}(e_0, p_0, \eta_0) + \alpha \int_0^t (e_k(\tau)|\nabla \dot{w}(\tau))\, d\tau + b_k \tag{5.6}$$

for every $k \in \mathbb{N}$ and every $t \in [0, T]$.

Since $\nabla \dot{w} \in L^1((0, T); L^2(\Omega; \mathbb{R}^d))$, by (5.5) and (5.6), we obtain that there exists a constant $M_R > 0$ such that $\mathrm{Diss}_R(p_k, \eta_k; 0, T) \le M_R$ for every $k \in \mathbb{N}$. By (2.3) this implies that the functions $t \mapsto p_k(t)$ and $t \mapsto \eta_k(t)$ from $[0, T]$ into $L^2(\Omega; \mathbb{R}^d)$ and $L^2(\Omega)$, respectively, have equibounded variation. By Helly's Theorem for functions of bounded variation with values in a separable Hilbert space (see, for instance, [3, Theorem 1.126]) there exist a subsequence, not relabelled, and two functions $p\colon [0, T] \to L^2(\Omega; \mathbb{R}^d)$ and $\eta\colon [0, T] \to L^2(\Omega)$ with $\mathrm{Diss}_R(p, \eta; 0, T) \le M_R$, such that for every $t \in [0, T]$,

$$p_k(t) \rightharpoonup p(t) \quad \text{weakly in } L^2(\Omega; \mathbb{R}^d), \tag{5.7}$$

$$\eta_k(t) \rightharpoonup \eta(t) \quad \text{weakly in } L^2(\Omega). \tag{5.8}$$

The arguments used in [10, Theorem 6.3] and [9, Theorem 4.8] lead to a variant of Helly's theorem for increasing functions with values in $\mathcal{R}_{d-1}(\overline{\Omega})$ endowed with σ-convergence. More precisely, the bound on $\mathcal{H}^{d-1}(\Gamma_k(t))$ in (5.5) implies that there exist a subsequence, not relabelled, and an increasing function $\Gamma\colon [0, T] \to \mathcal{R}_{d-1}(\overline{\Omega})$ such that

$$\Gamma_k(t)\ \sigma\text{-converges to } \Gamma(t) \quad \text{for every } t \in [0, T].$$

Let us fix $t \in [0, T]$. By (5.5) there exist a subsequence $(e_{k_j}(t))_j$, depending on t, and a function $e^* \in L^2(\Omega; \mathbb{R}^d)$ such that

$$e_{k_j}(t) \rightharpoonup e^* \quad \text{weakly in } L^2(\Omega; \mathbb{R}^d).$$

By Theorem 4.5 we obtain that $(e^*, p(t)) \in \mathcal{A}(\Gamma(t), w(t))$ and

$$\mathcal{E}(e^*, p(t), \eta(t)) \le \mathcal{E}(\hat{e}, \hat{p}, \hat{\eta}) + \mathcal{D}(\hat{p}, \hat{\eta}, \hat{\Gamma}; p(t), \eta(t), \Gamma(t)) \tag{5.9}$$

for every $\hat{\Gamma} \in \mathcal{R}_{d-1}(\overline{\Omega})$, $(\hat{e}, \hat{p}) \in \mathcal{A}(\hat{\Gamma}, w(t))$, and $\hat{\eta} \in L^2(\Omega)$.

By taking $\hat{p} = p(t)$, $\hat{\eta} = \eta(t)$, and $\hat{\Gamma} = \Gamma(t)$ in (5.9) we obtain

$$\|e^*\|^2 \le \|\hat{e}\|^2 \tag{5.10}$$

for every $\hat{e} \in L^2(\Omega; \mathbb{R}^d)$, such that $(\hat{e}, p(t)) \in \mathcal{A}(\Gamma(t), w(t))$.

We claim that there exists a unique function $e^* \in L^2(\Omega; \mathbb{R}^d)$ such that $(e^*, p(t)) \in \mathcal{A}(\Gamma(t), w(t))$ and (5.10) holds. Indeed, if e_* satisfies the same properties, then $\|e^*\| = \|e_*\|$, and if e^* and e_* do not coincide $\mathcal{L}^d$-a.e. in Ω, then the function $\tilde{e} := \frac{1}{2}e^* + \frac{1}{2}e_*$ satisfies $(\tilde{e}, p(t)) \in \mathcal{A}(\Gamma(t), w(t))$ and, by strict convexity, $\|\tilde{e}\|^2 < \|e^*\|^2$, which contradicts the minimality of e^* and proves the claim.

This uniqueness property implies that for every $t \in [0, T]$ there exists $e(t) \in L^2(\Omega; \mathbb{R}^d)$ such that the whole sequence satisfies

$$e_k(t) \rightharpoonup e(t) \quad \text{weakly in } L^2(\Omega; \mathbb{R}^d). \tag{5.11}$$

This shows that the function $t \mapsto e(t)$ from $[0, T]$ into $L^2(\Omega; \mathbb{R}^d)$ is weakly measurable, i.e., $t \mapsto (e(t), g)$ is measurable for every $g \in L^2(\Omega; \mathbb{R}^d)$. Consequently, $t \mapsto e(t)$ from $[0, T]$ into $L^2(\Omega; \mathbb{R}^d)$ is strongly measurable. Recalling (5.5), this proves that $e \in L^\infty(0, T; L^2(\Omega; \mathbb{R}^d))$.

Moreover, by (5.9) we have $(e(t), p(t)) \in \mathcal{A}(\Gamma(t), w(t))$ and

$$\mathcal{E}(e(t), p(t), \eta(t)) \leq \mathcal{E}(\hat{e}, \hat{p}, \hat{\eta}) + \mathcal{D}(\hat{p}, \hat{\eta}, \hat{\Gamma}; p(t), \eta(t), \Gamma(t)) \tag{5.12}$$

for every $\hat{\Gamma} \in \mathcal{R}_{d-1}(\overline{\Omega})$, $(\hat{e}, \hat{p}) \in \mathcal{A}(\hat{\Gamma}, w(t))$, and $\hat{\eta} \in L^2(\Omega)$. This proves condition (GS) in Definition 2.1.

We now prove condition (EDB). By the convexity of $\mathcal{E}$ (see (2.2)) and the weak convergence of the functions (see (5.7), (5.8), and (5.11)) for every $t \in [0, T]$ we have

$$\mathcal{E}(e(t), p(t), \eta(t)) \leq \liminf_{k\to\infty} \mathcal{E}(e_k(t), p_k(t), \eta_k(t)). \tag{5.13}$$

For every subdivision $0 = t_0 < t_1 < \cdots < t_m = t$ we have

$$\begin{aligned}\sum_{i=1}^m \mathcal{R}(p(t_i) - p(t_{i-1}), \eta(t_i) - \eta(t_{i-1})) &\leq \liminf_{k\to\infty} \sum_{i=1}^m \mathcal{R}(p_k(t_i) - p_k(t_{i-1}), \eta_k(t_i) - \eta_k(t_{i-1}))\\ &\leq \liminf_{k\to\infty} \mathrm{Diss}_R(p_k(\cdot), \eta_k(\cdot); 0, t).\end{aligned}$$

Passing to the supremum over all subdivisions, we obtain

$$\mathrm{Diss}_R(p(\cdot), \eta(\cdot); 0, t) \leq \liminf_{k\to\infty} \mathrm{Diss}_R(p_k(\cdot), \eta_k(\cdot); 0, t) \quad \text{for every } t \in [0, T]. \tag{5.14}$$

By Proposition 4.3 we have

$$\mathcal{H}^{d-1}(\Gamma(t) \setminus \Gamma_0) \leq \liminf_{k\to\infty} \mathcal{H}^{d-1}(\Gamma_k(t) \setminus \Gamma_0). \tag{5.15}$$

By (5.11) for every $\tau \in [0, T]$ we have

$$(e_k(\tau)|\nabla \dot{w}(\tau)) \to (e(\tau)|\nabla \dot{w}(\tau)).$$

By (5.5) we can apply the Dominated Convergence Theorem and we obtain

$$\int_0^t (e_k(\tau)|\nabla \dot{w}(\tau))\, d\tau \to \int_0^t (e(\tau)|\nabla \dot{w}(\tau))\, d\tau. \tag{5.16}$$

Passing to the limit in (5.6), by (5.13)–(5.16) we obtain

$$\mathcal{E}(e(t), p(t), \eta(t)) + \mathrm{Diss}_R(p(\cdot), \eta(\cdot); 0, t) + \mathcal{H}^{d-1}(\Gamma(t) \setminus \Gamma_0) \leq \mathcal{E}(e_0, p_0, \eta_0) + \alpha \int_0^t (e(\tau)|\nabla \dot{w}(\tau))\, d\tau \tag{5.17}$$

for every $t \in [0, T]$.

To prove the opposite inequality, we use the following lemma, which can be interpreted as a weaker form of the approximation of a Bochner integrable function f by means of suitable Riemann sums. Given $t > 0$, let

$$s_k^i := \frac{i}{k} t \quad \text{for every } k \in \mathbb{N} \text{ and } i = 0, 1, \ldots, k. \tag{5.18}$$

Rather than subdividing the interval $[0, t]$ by means of these points, for every $s \in (0, s_k^1)$ we subdivide it by means of the points

$$\tau_{k,s}^0 := 0, \qquad \tau_{k,s}^i := s_k^i - s \quad \text{for } i = 1, \ldots, k, \qquad \tau_{k,s}^{k+1} = t, \tag{5.19}$$

and for every s we consider the piecewise constant functions f^k_s obtained from f on this subdivision. It turns out that the mean value f^k of f^k_s with respect to s approximates f strongly in L^1.

Lemma 5.4. *Let X be a Banach space, let $t > 0$, let s^i_k and $\tau^i_{k,s}$ be defined by* (5.18) *and* (5.19)*, let $f\colon [0,t] \to X$ be a Bochner integrable function, let $f^k_s\colon (0,t] \to X$ be the piecewise constant function defined by*

$$f^k_s(\tau) := f(\tau^i_{k,s}) \quad \text{for } \tau \in (\tau^{i-1}_{k,s}, \tau^i_{k,s}],\ i = 1, \dots, k+1$$

and let $f^k\colon [0,t] \to X$ be the Bochner integrable function defined by

$$f^k(\tau) := \frac{1}{s^1_k} \int_0^{s^1_k} f^k_s(\tau)\, ds \quad \text{for every } \tau \in [0,t].$$

Then

$$\lim_{k\to\infty} \int_0^t \|f^k(\tau) - f(\tau)\|_X \, d\tau = 0. \tag{5.20}$$

Proof. If $s^{i-1}_k < \tau \le s^i_k$ for some $i \in \{1, \dots, k-1\}$, an elementary change of variables gives

$$s^1_k f^k(\tau) = \int_{s^{i-1}_k}^{\tau} f(\sigma + s^1_k)\, d\sigma + \int_{\tau}^{s^i_k} f(\sigma)\, d\sigma = \int_{s^{i-1}_k}^{\tau} \big(f(\sigma + s^1_k) - f(\sigma)\big)\, d\sigma + \int_{s^{i-1}_k}^{s^i_k} f(\sigma)\, d\sigma. \tag{5.21}$$

If $s^{k-1}_k < \tau \le s^k_k = t$, the same argument gives

$$s^1_k f^k(\tau) = \int_{s^{k-1}_k}^{\tau} f(t)\, d\sigma + \int_{\tau}^{s^k_k} f(\sigma)\, d\sigma = \int_{s^{k-1}_k}^{\tau} \big(f(t) - f(\sigma)\big)\, d\sigma + \int_{s^{k-1}_k}^{s^k_k} f(\sigma)\, d\sigma. \tag{5.22}$$

Let $\overline{f}^k\colon [0,t] \to X$ be the piecewise constant function defined by

$$\overline{f}^k(\tau) := \frac{1}{s^1_k} \int_{s^{i-1}_k}^{s^i_k} f(\sigma)\, d\sigma$$

for $i \in \{1, \dots, k\}$ and $s^{i-1}_k < \tau \le s^i_k$. It is known (see, for instance, [5, Appendix]) that

$$\lim_{k\to\infty} \int_0^t \|\overline{f}^k(\tau) - f(\tau)\|_X \, d\tau = 0. \tag{5.23}$$

From (5.21) and (5.22) we deduce that

$$\|f^k(\tau) - \overline{f}^k(\tau)\|_X \le \frac{1}{s^1_k} \int_{s^{i-1}_k}^{s^i_k} \|f(\sigma + s^1_k) - f(\sigma)\|_X \, d\sigma$$

if $s^{i-1}_k < \tau \le s^i_k$ for some $i \in \{1, \dots, k-1\}$, while

$$\|f^k(\tau) - \overline{f}^k(\tau)\|_X \le \frac{1}{s^1_k} \int_{s^{k-1}_k}^{s^k_k} \|f(t) - f(\sigma)\|_X \, d\sigma$$

if $s^{k-1}_k < \tau \le s^k_k = t$. Integrating these inequalities with respect to τ and adding with respect to i, we obtain

$$\int_0^t \|f^k(\tau) - \overline{f}^k(\tau)\|_X \, d\tau \le \int_0^{s^{k-1}_k} \|f(\sigma + s^1_k) - f(\sigma)\|_X \, d\sigma + \int_{s^{k-1}_k}^{s^k_k} \|f(t) - f(\sigma)\|_X \, d\sigma.$$

By the continuity in L^1 of translations and by the absolute continuity of the integral, from this inequality we obtain

$$\lim_{k\to\infty}\int_0^t \|f^k(\tau)-\overline{f}^k(\tau)\|_X\,d\tau = 0,$$

which, together with (5.23), gives (5.20). □

Proof of Theorem 2.2 (continuation). We fix $t \in (0,T]$. For every $k\in\mathbb{N}$ let s_k^i and $\tau_{k,s}^i$ be defined by (5.18) and (5.19). For every $s\in(0,s_k^1)$ let $e_s^k\colon [0,t]\to L^2(\Omega;\mathbb{R}^d)$ be the piecewise constant function defined by

$$e_s^k(\tau) := e(\tau_{k,s}^i)\quad \text{for } \tau\in(\tau_{k,s}^{i-1},\tau_{k,s}^i],\ i=1,\dots,k+1$$

and let $e^k\colon [0,t]\to L^2(\Omega;\mathbb{R}^d)$ be the Bochner integrable function defined by

$$e^k(\tau) := \frac{1}{s_k^1}\int_0^{s_k^1} e_s^k(\tau)\,ds\quad \text{for every } \tau\in[0,t].$$

By Lemma 5.4 we have

$$\lim_{k\to\infty}\int_0^t \|e^k(\tau)-e(\tau)\|\,d\tau = 0.$$

Given $k\in\mathbb{N}$, $s\in(0,s_k^1)$, and $i\in\{1,\dots,k+1\}$, we use (5.12), with $t=\tau_{k,s}^{i-1}$, $\hat e = e(\tau_{k,s}^i)-\nabla w(\tau_{k,s}^i)+\nabla w(\tau_{k,s}^{i-1})$, $\hat p = p(\tau_{k,s}^i)$, $\hat\eta=\eta(\tau_{k,s}^i)$, and $\hat\Gamma=\Gamma(\tau_{k,s}^i)$, obtaining

$$\begin{aligned}\mathcal{E}(e(\tau_{k,s}^{i-1}),p(\tau_{k,s}^{i-1}),\eta(\tau_{k,s}^{i-1})) \le{}& \mathcal{E}(e(\tau_{k,s}^{i}),p(\tau_{k,s}^{i}),\eta(\tau_{k,s}^{i})) + \mathcal{R}(p(\tau_{k,s}^{i})-p(\tau_{k,s}^{i-1}),\eta(\tau_{k,s}^{i})-\eta(\tau_{k,s}^{i-1}))\\ &+\mathcal{H}^{d-1}(\Gamma(\tau_{k,s}^i)\setminus\Gamma(\tau_{k,s}^{i-1})) - \alpha\big(e(\tau_{k,s}^i)|\nabla w(\tau_{k,s}^i)-\nabla w(\tau_{k,s}^{i-1})\big)\\ &+\frac{\alpha}{2}\|\nabla w(\tau_{k,s}^i)-\nabla w(\tau_{k,s}^{i-1})\|^2.\end{aligned}$$

Summing for $i=1,\dots,k+1$, we obtain

$$\begin{aligned}\mathcal{E}(e(0),p(0),\eta(0)) \le{}& \mathcal{E}(e(t),p(t),\eta(t)) + \mathrm{Diss}_R(p(\cdot),\eta(\cdot);0,t)+\mathcal{H}^{d-1}(\Gamma(t)\setminus\Gamma_0)\\ &-\alpha\int_0^t \big(e_s^k(\tau)|\nabla\dot w(\tau)\big)\,d\tau + \varepsilon_k(w,t),\end{aligned}\tag{5.24}$$

where

$$\varepsilon_k(w,t) := \frac{\alpha}{2}\sup_{s\in(0,s_k^1)}\|\nabla w(\tau_{k,s}^i)-\nabla w(\tau_{k,s}^{i-1})\|\int_0^t\|\nabla\dot w(\tau)\|\,d\tau.$$

Since $w\in \mathrm{AC}([0,T];H^1(\Omega))$, we have

$$\lim_{k\to\infty}\varepsilon_k(w,t)=0.\tag{5.25}$$

Taking the mean value for $s\in(0,s_k^1)$, from (5.24) we obtain

$$\begin{aligned}\mathcal{E}(e(0),p(0),\eta(0)) \le{}& \mathcal{E}(e(t),p(t),\eta(t)) + \mathrm{Diss}_R(p(\cdot),\eta(\cdot);0,t)+\mathcal{H}^{d-1}(\Gamma(t)\setminus\Gamma_0)\\ &-\alpha\int_0^t \big(e^k(\tau)|\nabla\dot w(\tau)\big)\,d\tau + \varepsilon_k(w,t).\end{aligned}$$

Passing to the limit as $k\to\infty$, from Lemma 5.4 and from (5.25) we obtain the inequality opposite to (5.17), thus concluding the proof of condition (EDB) in Definition 2.1. □

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