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Perazzo n-folds and the weak Lefschetz property

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Abstract
In this paper, we determine the maximum h_{max} and the minimum h_{min} of the Hilbert vectors of Perazzo algebras A_F, where F is a Perazzo polynomial of degree d in $n+m+1$ variables. These algebras always fail the Strong Lefschetz Property. We determine the integers n, m, d such that h_{max} (resp. h_{min}) is unimodal, and we prove that A_F always fails the Weak Lefschetz Property if its Hilbert vector is maximum, while it satisfies the Weak Lefschetz Property if it is minimum, unimodal, and satisfies an additional mild condition. We determine the minimal free resolution of Perazzo algebras associated to Perazzo threefolds in $\mathbb{P}^4$ with minimum Hilbert vectors. Finally we pose some open problems in this context.

Keywords Perazzo hypersurface · Lefschetz properties · Gorenstein algebra · Hilbert function · Minimal free resolution

Mathematics Subject Classification 14J70 · 14M05 · 13E10

1 Introduction

A Perazzo form of degree d is by definition (see [7]) a homogeneous polynomial $F \in K[X_0, \dots, X_n, U_1, \dots, U_m]$

$$F = X_0 p_0 + X_1 p_1 + \cdots + X_n p_n + G,$$

with $n \geq m \geq 2$, $p_0, \dots, p_n \in K[U_1, \dots, U_m]_{d-1}$, $G \in K[U_1, \dots, U_m]_d$, where $p_0, \dots, p_n$ are algebraically dependent but linearly independent.

Dedicated to Enrique Arrondo on the occasion of his 60th birthday

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The interest of Perazzo forms comes form the fact that their sets of zeros are a class of examples of hypersurfaces with vanishing hessian which are not cones. The problem of characterizing the hypersurfaces not cones with vanishing hessian is an important open problem in projective algebraic geometry since the classical works of Hesse and of Gordan–Noether [6, 11, 12]. Hesse believed that the hessian of a homogenous polynomial vanishes if and only if its variety of zeros is a cone, as is indeed the case when the degree of the form is 2. However, P. Gordan and M. Noether proved that while Hesse's claim is true for forms in at most 4 variables, it is false for 5 or more variables and any degree ≥ 3. They also gave a complete description of the hypersurfaces in $\mathbb{P}^4$, non cones, having vanishing hessian: their equations are elements of $K[U_1, U_2][\Delta]$, where Δ is a Perazzo polynomial of the form $X_0 p_0 + X_1 p_1 + X_2 p_2$. Since then, many efforts have been made to find a characterization for arbitrary degree and number of variables. It turns out that all the counterexamples known so far can be built starting from Perazzo forms (see [7, Appendix A], [18, Chapter 7]). The original paper of Perazzo [17] deals with the case of cubic hypersurfaces, it was revisited in [8].

The study of hessians of homogeneous polynomials has gained new attention because of its connection to Lefschetz properties for graded Artinian Gorenstein algebras. Recall that a standard graded Artinian algebra A has the weak Lefschetz property (WLP) if multiplication by a generic linear form ℓ has maximal rank in each degree. Similarly A has the strong Lefschetz property (SLP) if multiplication by ℓ^s has maximal rank in each degree for every positive integer s. The study of the Lefschetz properties for graded Artinian algebras originates from the Hard Lefschetz Theorem, which implies that the cohomology ring A of any smooth complex projective variety has the SLP; moreover A is a graded Artinian Gorenstein algebra.

Although Lefschetz properties have been the subject of intense research in recent years, many natural problems are still open and the general picture is far from being understood. In the Gorenstein case every standard graded Artinian algebra can be written as A_F, the quotient of a ring of differential operators by the annihilator of a homogeneous polynomial F, called its Macaulay dual generator (see Sect. 2.1 for details). Due to work of Watanabe and Maeno–Watanabe [15, 20], a non-trivial characterization of Artinian Gorenstein algebras failing the SLP, in terms of the Macaulay dual generator, has been found. Indeed, A_F fails the SLP if and only if one of the non-trivial higher hessians of F vanishes. This result has been generalised to the WLP using the so called mixed hessians (see [9]).

It follows that the Artinian Gorenstein algebras A_F associated to Perazzo forms fail the SLP. It is therefore natural to pose the question if these algebras satisfy or fail the WLP. This question has been considered in some recent articles [1, 5, 16], where the case of Perazzo forms with $m = 2$ has been completely solved.

Before summarizing the results of those articles, we recall a few basic facts about the Hilbert functions of graded Artinian Gorenstein algebras. If A is such an algebra of socle degree d, then its Hilbert function is captured by its h-vector $(h_0, h_1, \dots, h_d)$, where $h_i = \dim_K A_i$. Since A is a Poincaré duality algebra, the h-vector results to be symmetric, i.e. $h_i = h_{d-i}$. On the set of h-vectors of the same length there is the natural componentwise partial order: given $h = (h_0, h_1, \dots, h_d)$ and $h' = (h'_0, h'_1, \dots, h'_d)$, we say that $h \leq h'$ if $h_i \leq h'_i$ for every i, $0 \leq i \leq d$.

In the quoted articles the following facts are proved. Let A_F be the Artinian Gorenstein algebra associated to a Perazzo form F with $n \geq m = 2$, $d \geq n + 1$. Let $(h_0, h_1, \dots, h_d)$ be its h-vector. Then:

(1) the Hilbert function of A_F is unimodal, i.e. $h_0 \leq h_1 \leq \cdots \leq h_k \geq h_{k+1} \geq \cdots \geq h_d$ for some k;

(2) $h_i \leq d+2$ for any index i and A_F has the WLP if and only if $\sharp\{i | h_i = d+2\} \leq 1$;
(3) the h-vectors of the algebras A_F for fixed n, m, d have a maximum and a minimum, that are completely described. In particular if the h-vector is maximum A_F fails the WLP, while if the h-vector is minimum A_F has the WLP if and only if $d \geq 2n$;
(4) the Perazzo forms in 5 variables such that the h-vector is minimum admit a precise description in terms of the position of the 2-plane generated by p_0, p_1, p_2 in $\mathbb{P}(K[U_1, U_2]_{d-1})$ with respect to the rational normal curve.

In this article we consider Perazzo forms in any number of variables with the aim to extend the results obtained for $m = 2$. Moreover we tackle the problem of describing the minimal free resolutions of Perazzo algebras. We are able to prove that, for fixed m, n, d, the h-vectors of Perazzo algebras have a maximum and a minimum as in the case $m = 2$, and we describe them in Proposition 3.2 and Theorem 4.2 respectively. But, diversely from the case $m = 2$, if $m \geq 3$ these h-vectors are not always unimodal. In Theorems 3.5, 3.6 and 4.4 we characterize the integers $n \geq m \geq 3$ such that h_{max} (resp. h_{min}) is not unimodal. We note that h_{max} is never unimodal for d large enough.

Regarding the WLP, we find that Perazzo algebras with maximal h-vector never have the WLP, while those with minimal h-vector have the WLP provided that h_{min} is unimodal and an additional mild condition is satisfied. The problem of characterizing when WLP holds for intermediate h-vectors remains open.

As for our second aim, we are able to compute in Theorem 5.4 the minimal free resolution for a class of Perazzo algebras, those in 5 variables with minimal h-vector. The proof is by induction on the degree d, the base of the induction being possible because of the explicit description of the algebras A_F with minimal h-vector.

Many questions remain open, and we devote the last section of this article to list a few open problems that we think deserve to be considered.

The paper is organized as follows. We start by reviewing in Sect. 2 definitions and basic results concerning Artinian Gorenstein algebras associated to Perazzo hypersurfaces, minimal free resolutions and Lefschetz properties. In Sect. 3, we determine the maximal Hilbert function once the integers n, m, d are fixed. We study when this function is unimodal and we prove that Perazzo algebras with maximal Hilbert function do not have the WLP. In Sect. 4, similarly, we determine the minimal Hilbert function, study its unimodality and prove that Perazzo algebras with this Hilbert function, and satisfying an additional mild condition that implies the unimodality of the h-vector, have the WLP. We also characterize the integers n, m, d such that the maximum and the minimum Hilbert function coincide. In Sect. 5 we compute the minimal free resolution for the Perazzo algebras in 5 variables with minimal h-vector. Finally, in Sect. 6 we pose some relevant open problems in this circle of ideas.

2 Background

In this section we fix notations, we recall the basic facts on Hilbert functions, Lefschetz properties, minimal free resolutions as well as on Perazzo hypersurfaces needed later on.

2.1 Hilbert functions

Throughout this paper K will be an algebraically closed field of characteristic zero. Given a standard graded Artinian K-algebra $A = R/I$ where $R = K[x_0, x_1, \dots, x_N]$ and I is a homogeneous ideal of R, we denote by $\mathrm{HF}_A\colon \mathbb{Z} \longrightarrow \mathbb{Z}$ with $\mathrm{HF}_A(j) = \dim_K A_j =$

$\dim_K[R/I]_j$ its Hilbert function. Since A is Artinian, its Hilbert function is captured in its *h-vector* $h = (h_0, h_1, \dots, h_d)$ where $h_i = \mathrm{HF}_A(i) > 0$ and d is the last index with this property. The integer d is called the *socle degree of* A. We will use the terms "Hilbert function" and "h-vector" interchangeably along the paper.

We recall the construction of the Artinian Gorenstein algebra A_F with Macaulay dual generator a given form $F \in S = K[X_0, \dots, X_N]$; we denote by $R = K[x_0, \dots, x_N]$ the ring of differential operators acting on the polynomial ring S, i.e. $x_i = \frac{\partial}{\partial X_i}$. Therefore R acts on S by differentiation. Given polynomials $p \in R$ and $G \in S$ we will denote by $p \circ G$ the differential operator p applied to G. We define

$$\mathrm{Ann}_R\, F := \{p \in R \mid p \circ F = 0\} \subset R,$$

and $A_F = R/\mathrm{Ann}_R\, F$: it is a standard graded Artinian Gorenstein K-algebra and F is called its Macaulay dual generator. We remark that every standard graded Artinian Gorenstein K-algebra is of the form A_F for some form F, in view of the "Macaulay double annihilator Theorem" (see for instance [14, Lemma 2.12]). We may abbreviate and write $\mathrm{Ann}\, F$ when the ring R is understood.

As an important key tool to determine the unimodality of the Hilbert function of a Perazzo algebra or the minimal free resolution of Artinian Gorenstein algebras associated to Perazzo threefolds in $\mathbb{P}^4$, we state the following:

Proposition 2.1 *Let A_F be an Artinian Gorenstein graded K-algebra and set $I = \mathrm{Ann}\, F$. Then for every linear form $\ell \in A_1$ the sequence*

$$0 \longrightarrow \frac{R}{(I : \ell)}(-1) \longrightarrow A_F = \frac{R}{I} \longrightarrow \frac{R}{(I, \ell)} \longrightarrow 0 \tag{1}$$

is exact. Moreover $\frac{R}{(I:\ell)}$ is an Artinian Gorenstein graded algebra with $\ell \circ F$ as Macaulay dual generator.

Proof We get the result cutting the exact sequence

$$0 \longrightarrow \frac{(I : \ell)}{I}(-1) \longrightarrow \frac{R}{I}(-1) \xrightarrow{\times \ell} \frac{R}{I} \longrightarrow \frac{R}{(I, \ell)} \longrightarrow 0$$

into two short exact sequences. The second fact is a straightforward computation. □

2.2 Minimal free resolutions

Let $A = R/I$ be an Artinian graded K-algebra. It is well known that it has a minimal graded free R-resolution of the following type:

$$0 \longrightarrow F_{n+1} \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_i \longrightarrow \cdots \longrightarrow F_1 \longrightarrow R \longrightarrow A \longrightarrow 0$$

where

$$F_i = \oplus_j R(-j)^{\beta_{ij}^R(A)}$$

and the graded Betti numbers $\beta_{ij}^R(A)$ of A over R are defined as usual as the integers

$$\beta_{ij}^R(A) = \dim_K[\mathrm{Tor}_i^R(A, K)]_j.$$

These homological invariants are our main focus and indeed our goal in Sect. 5 is to determine the graded Betti numbers $\beta_{ij}^R(A_F)$ of an Artinian Gorenstein algebra A_F associated to a

Perazzo 3-fold F in $\mathbb{P}^4$ with termwise minimal Hilbert function. It is important to point out that the graded Betti numbers of an Artinian graded algebra A determine its graded Poincaré series. In fact, the graded Poincaré series of A over R is the generating function

$$P_A^R(t,s) = \sum_{i,j} \beta_{ij}^R(A) t^i s^j.$$

If R is regular, then the Poincaré series is in fact a polynomial.

2.3 Perazzo hypersurfaces

The simplest counterexample to Hesse's claim, i. e. a form with vanishing hessian which does not define a cone, is $XU^2 + YUV + ZV^2$. This example was extended to a class of cubic counterexamples in all dimensions by Perazzo in [17].

Definition 2.2 A *Perazzo hypersurface* $X = V(F) \subset \mathbb{P}^N$ is the hypersurface defined by a *Perazzo form*

$$F = X_0 p_0 + X_1 p_1 + \cdots + X_n p_n + G \in K[X_0, \ldots, X_n, U_1 \ldots, U_m]_d$$

where $n, m \geq 2$, $N = n + m$, $p_i \in K[U_1, \ldots, U_m]_{d-1}$ are algebraically dependent but linearly independent, and $G \in K[U_1, \ldots, U_m]_d$.

The Artinian Gorenstein algebra A_F associated to a Perazzo polynomial will be called Perazzo algebra.

The fact that the p_i's are algebraically dependent implies $\text{hess}_F = 0$, while the linear independence assures that $V(F)$ is not a cone.

We note that, to allow the linear independence of $p_0, \ldots, p_n$, we must assume

$$n + 1 \leq \binom{d+m-2}{m-1}. \tag{2}$$

If equality holds in (2) $p_0, \ldots, p_n$ form a basis of $K[U_1, \ldots, U_m]_{d-1}$; in this case A_F is called a *full Perazzo algebra*. Full Perazzo algebras were studied in [4] and [2]. On the other hand, to guarantee the algebraic dependence for a general choice of $p_0, \ldots, p_n$ we make the assumption that $n \geq m$.

The following lemma plays a key role in the induction step used in the proof of our main results (Theorem 3.8 and Theorem 5.4).

Lemma 2.3 *Let $F = X_0 p_0 + X_1 p_1 + \cdots + X_n p_n + G$ be a Perazzo form of degree d and let A_F be the associated Artinian Gorenstein algebra. Assume $n + 1 \leq \binom{d+m-3}{m-1}$. Then, for a general linear form $\ell \in A_F$, the polynomial $\ell \circ F$ defines a Perazzo form of degree $d - 1$.*

Proof We can write $\ell = a_0 X_0 + a_1 X_1 + \ldots a_n X_n + b_1 U_1 + \cdots + b_m U_m$ for some coefficients $a_i, b_j \in K$ not all zero. Then we can exhibit the action of ℓ on F as

$$\ell \circ F = X_0 \tilde{p_0} + \cdots + X_n \tilde{p_n} + \left(a_0 p_0 + \cdots + a_n p_n + b_1 \frac{\partial G}{\partial U_1} + \cdots + b_m \frac{\partial G}{\partial U_m} \right)$$

with

$$\tilde{p_0} = b_1 \frac{\partial p_0}{\partial U_1} + \cdots + b_m \frac{\partial p_0}{\partial U_m}, \ldots, \tilde{p_n} = b_1 \frac{\partial p_n}{\partial U_1} + \cdots + b_m \frac{\partial p_n}{\partial U_m}.$$

The form $\ell \circ F$ has degree $d-1$. It remains to prove that the polynomials $\tilde{p}_0, \dots, \tilde{p}_n$ are linearly independent, for a general choice of ℓ.

Let $\ell' = b_1 U_1 + \cdots + b_m U_m$ and consider the map

$$\phi_b : K[U_1, \dots, U_m]_{d-1} \to K[U_1, \dots, U_m]_{d-2},$$

given by $p \mapsto \ell' \circ p$. The fact that the $(d-1)-$th powers of linear forms span $K[U_1, \dots, U_m]_{d-1}$ [13, Corollary 3.2], and similarly for $K[U_1, \dots, U_m]_{d-2}$, implies that ϕ_b is surjective. Therefore its kernel has dimension $\binom{m+d-3}{m-2}$. Now let $W = \langle p_0, \dots, p_n \rangle$. Since the p_i are linearly independent, W has dimension $n+1$. It does not fill the whole space $K[U_1, \dots, U_m]_{d-1}$, because $n+1 < \binom{m+d-2}{m-1}$. Using again that the $(d-1)-$th powers of linear forms span $K[U_1, \dots, U_m]_{d-1}$ and because of the assumption $n+1 \leq \binom{m+d-3}{m-1}$, we deduce that there is an open dense set B in K^m such that for any $(b_1, \dots, b_m) \in B$, the vector space $\ker \phi_b$ misses W. So, for any such m-tuple $(b_1, \dots, b_m)$, the map

$$\ell' : W \to \ell' \circ W$$

is an isomorphism, and therefore $\tilde{p}_0, \dots, \tilde{p}_n$ are linearly independent. □

2.4 Lefschetz properties

Definition 2.4 Let $A = R/I = \bigoplus_{i=0}^{d} A_i$ be a graded Artinian K-algebra. We say that A has the *weak Lefschetz property* (WLP, for short) if there is a linear form $\ell \in A_1$ such that, for all integers $i \geq 0$, the multiplication map

$$\times \ell \colon A_i \longrightarrow A_{i+1}$$

has maximal rank, i.e. it is injective or surjective. In this case, the linear form ℓ is called a *weak Lefschetz element* of A. We say that A fails the WLP in degree j if for a general form $\ell \in A_1$, the map $\times \ell \colon A_{j-1} \longrightarrow A_j$ does not have maximal rank.

We say that A has the *strong Lefschetz property* (SLP, for short) if there is a linear form $\ell \in A_1$ such that, for all integers $i \geq 0$ and $k \geq 1$, the multiplication map

$$\times \ell^k \colon A_i \longrightarrow A_{i+k}$$

has maximal rank. Such an element ℓ is called a *strong Lefschetz element* of A.

It is easy to prove that the h-vector $(h_0, h_1, \dots, h_d)$ of any graded Artinian K-algebra having the SLP or the WLP is *unimodal*, i.e. there exists an index k such that $h_0 \leq h_1 \leq \cdots \leq h_k \geq h_{k+1} \geq \cdots \geq h_d$.

Let A_F be an Artinian Gorenstein algebra associated to a Perazzo hypersurface of degree $d \geq 5$ in $\mathbb{P}^4$. Recall that by [5, Theorem 4.3] the algebra A_F has the weak Lefschetz property if the Hilbert function of A_F is the termwise minimal one, namely $(1, 5, 6, \dots, 6, 5, 1)$. Furthermore, in [5, Proposition 3.7] and [5, Theorem 4.1] it is proved that the maximal possible Hilbert function is

$$h_i = \begin{cases} 4i+1 \text{ for } 1 \leq i \leq \frac{d+1}{4} \\ d+2 \text{ for } \frac{d+1}{4} < i \leq \frac{d}{2} \\ \text{symmetry} \end{cases} \tag{3}$$

and that any algebra A_F with Hilbert function as in (3) fails the WLP. As a complete classification of Artinian Gorenstein algebras associated to Perazzo hypersurfaces of degree $d \geq 5$ in $\mathbb{P}^4$ with the weak Lefschetz property we have the following result.

Theorem 2.5 *Let A_F be an Artinian Gorenstein algebra associated to a Perazzo hypersurface $V(F) \subset \mathbb{P}^4$ of degree $d \geq 5$. Let $(h_0, h_1, \dots, h_d)$ be its h-vector. The algebra A_F has the WLP if and only if $\#\{i \mid h_i = d+2\} \leq 1$.*

Proof See [1, Theorem 3.11]. □

3 Maximal Hilbert function of a Perazzo algebra

In this section we determine the maximal h-vector $h_{max} = h_{max}(A_F)$ of a Perazzo algebra A_F for any given m, n, d, extending the results obtained in the case $n = m = 2$ in [5] and [1]; and $m = 2$ and $n \geq 2$ in [16]. Let d be the degree of the Macaulay dual generator F of A_F. We will see that, differently from the case $m = 2$, for $m \geq 3$ and $d \gg 0$ h_{max} is not always unimodal.

Let F be a Perazzo form as in Definition 2.2:

$$F = X_0 p_0 + X_1 p_1 + \cdots + X_n p_n + G \in S_d = K[X_0, \dots, X_n, U_1 \dots, U_m]_d,$$

with $p_0, \dots, p_n$ algebraically dependent but linearly independent.

We will use the following notations: for $i = 0, \dots, n$

$$p_i = \sum_{|\lambda|=d-1} \binom{d-1}{\lambda} p^i_\lambda U^\lambda \tag{4}$$

where $\lambda = (\lambda_1, \dots, \lambda_m)$ is a multi-index, $|\lambda| = \lambda_1 + \cdots + \lambda_m$, $\binom{d-1}{\lambda} = \frac{(d-1)!}{\lambda_1! \dots \lambda_m!}$ is the multinomial coefficient, and $U^\lambda = U_1^{\lambda_1} \dots U_m^{\lambda_m}$.

Then, for any multi-index $\gamma = (\gamma_1, \dots, \gamma_m)$ such that $|\gamma| \leq d-1$, the partial derivative $\frac{\partial^{|\gamma|} p_i}{\partial U^\gamma}$ of p_i with respect to γ is equal to

$$(d-1)(d-2)\dots(d-1-|\gamma|) \sum_{|\mu|=d-1-|\gamma|} \binom{d-1-|\gamma|}{\mu} p^i_{\mu+\gamma} U^\mu.$$

Similarly we put $G = \sum_{|\lambda|=d} \binom{d}{\lambda} G_\lambda U^\lambda$.

This notation will be useful to compute the h-vector of A_F, which is equivalent to computing the dimension of $\operatorname{Ann}_R(F)_i$ for $i = 0, \dots, [\frac{d}{2}]$.

Proposition 3.1 *Let $i \leq [\frac{d}{2}]$ be an integer number. Let $h = (h_0, \dots, h_d)$ be the h-vector of the Perazzo algebra A_F. Then h_i is equal to the rank of the matrix containing in the columns the coefficients of the partial derivatives of F of order i.*

Proof We put $R = K[x_0, \dots, x_n u_1, \dots, u_m]$. We observe that $h_1 = n + m + 1$. From now on we assume $i \geq 2$. Since $h_i = \dim(A_F)_i = \dim[R/\operatorname{Ann}_R(F)]_i$, we need to determine the polynomials ϕ of R_i such that $\phi \circ F = 0$. Being F linear in $x_0, \dots, x_n$, all polynomials $\phi \in R_i$ of degree at least 2 in $x_0, \dots, x_n$ clearly belong to $\operatorname{Ann}_R(F)_i$. So assume that ϕ has degree ≤ 1 in $x_0, \dots, x_n$. We can write

$$\phi = x_0 \sum_{|\mu|=i-1} \alpha^0_\mu u^\mu + x_1 \sum_{|\mu|=i-1} \alpha^1_\mu u^\mu + \cdots + \sum_{|\nu|=i} \beta_\nu u^\nu.$$

Imposing $\phi \circ F = 0$ we get:

$$\sum_{|\mu|=i-1} \alpha^0_\mu \frac{\partial^{|\mu|} p_0}{\partial U^\mu} + \sum_{|\mu|=i-1} \alpha^1_\mu \frac{\partial^{|\mu|} p_1}{\partial U^\mu} + \cdots + X_0 \sum_{|\nu|=i} \beta_\nu \frac{\partial^{|\nu|} p_0}{\partial U^\mu} + \cdots + \sum_{|\nu|=i} \beta_\nu \frac{\partial^{|\nu|} G}{\partial U^\nu} = 0. \tag{5}$$

This gives rise to a homogeneous linear system of equations in the unknowns $\alpha^0_\mu, \alpha^1_\mu, \ldots, \beta_\nu$, with $|\mu| = i-1$, $|\nu| = i$. The equations of the system are obtained equaling to zero the coefficients of the monomials of degree $d-i$ in $U_1, \ldots, U_m$ and in $X_0, \ldots, X_n, U_1, \ldots, U_m$. Being char $K = 0$, in view of notation (4), it follows that, up to a non-zero constant, the coefficients of the unknowns are precisely the coefficients of the partial derivatives of F of order i. The theorem is proved. □

Proposition 3.2 *The maximal h-vector of a Perazzo algebra A_F, for fixed m, n, d satisfying (2), is $h_{max} = (h_0, \ldots, h_d)$ with*

$$h_i = \min\{\alpha_i + \beta_i, \alpha_i + \gamma_i\}$$

for any $0 \le i \le [\frac{d}{2}]$, where

$$\alpha_i = \binom{m+i-1}{m-1}, \ \beta_i = \binom{d+m-i-1}{m-1} \text{ and } \gamma_i = (n+1)\binom{m+i-2}{m-1}.$$

Proof It follows from a result of Iarrobino, that we recall in Lemma 3.3. Let F be a Perazzo polynomial as in Definition 2.2 and assume that $p_0, \ldots, p_n, G$ are *general*. Let us compute the h-vector of A_F. In view of Proposition 3.1, for any i, $1 \le i \le [\frac{d}{2}]$, we have

$$\begin{aligned} h_i &= \dim(A_F)_i \\ &= \dim\langle \frac{\partial^i F}{\partial X_0^{i_0} \ldots \partial X_n^{i_n} \partial U_1^{i_{n+1}} \ldots \partial U_m^{i_{n+m}}} \mid i_0 + \cdots + i_{n+m} = i \rangle \end{aligned}$$

where $i_j \ge 0$ for $j = 0, \ldots, n+m$. Being $p_0, \ldots, p_n$ general, this is equal to $\dim A + \dim B$ where

$$A = \langle \frac{\partial^{i-1} p_0}{\partial U_1^{j_1} \ldots \partial U_m^{j_m}}, \ldots, \frac{\partial^{i-1} p_n}{\partial U_1^{j_1} \ldots \partial U_m^{j_m}} \mid j_1 + \cdots + j_m = i-1, \, j_r \ge 0 \rangle,$$

$$\begin{aligned} B &= \langle \frac{\partial^i F}{\partial U_1^{j_1} \ldots \partial U_m^{j_m}} \mid j_1 + \cdots + j_m = i, \, j_r \ge 0 \rangle = \\ &= \langle \sum_{j=0}^n X_j \frac{\partial^i p_j}{\partial U_1^{j_1} \ldots \partial U_m^{j_m}} + \frac{\partial^i G}{\partial U_1^{j_1} \ldots \partial U_m^{j_m}} \mid j_1 + \cdots + j_m = i, \, j_r \ge 0 \rangle. \end{aligned}$$

From Lemma 3.3 we get

$$\begin{aligned} \dim A &= \min\left\{ (n+1)\binom{m+i-2}{m-1}, \binom{d+m-i-1}{m-1} \right\} = \min\{\beta_i, \gamma_i\}, \\ \dim B &= \binom{m+i-1}{m-1} = \alpha_i, \end{aligned}$$

which proves the thesis. □

Lemma 3.3 *Let $F_1, \ldots, F_r \in K[y_0, \ldots, y_s]$ be a set of r general forms of fixed degree d. Then, for any positive integer number $i \leq d$,*

$$\langle \frac{\partial^i F_1}{\partial y_0^{i_0} \ldots \partial y_s^{i_s}}, \ldots, \frac{\partial^i F_r}{\partial y_0^{i_0} \ldots \partial y_s^{i_s}} \mid i_0 + \ldots i_s = i, i_j \geq 0 \rangle$$

is a K-vector space of dimension

$$\min \left\{ r\binom{i+s}{s}, \binom{d+s-i}{s} \right\}. \tag{6}$$

Proof It follows from [13, Proposition 3.4]. □

We explicitly note that Lemma 3.3 implies that h_{max} is term-wise maximal. We also note that Proposition 3.2 extends Proposition 3.7 in [5] and Theorem 3.5 in [16] which refer to the case $m = 2$.

We observe that $\alpha_0 + \gamma_0 = 1$ and $\alpha_0 + \beta_0 = 1 + \binom{d+m-1}{m-1}$ so $h_0 = 1$, as expected. Moreover $\alpha_1 + \gamma_1 = m + n + 1$ and $\alpha_1 + \beta_1 = m + \binom{d+m-2}{m-1}$, so $h_1 = \alpha_1 + \gamma_1$, and $\beta_1 = \gamma_1$ if and only if $n + 1 = \binom{d+m-1}{m-1}$ which means that A_F is a full Perazzo algebra.

From now on we will use the notation $s := [\frac{d}{2}]$ so that $d = 2s$ if it is even, and $d = 2s + 1$ if it is odd.

We want to study the unimodality of h_{max}. Note that in the range $0 \leq i \leq s$, α_i, γ_i are strictly increasing functions of i, independent of d, while β_i is strictly decreasing and depends on d.

Lemma 3.4 *For any i, $0 \leq i \leq s$, $\alpha_i + \gamma_i$ is a strictly increasing function of i, while $\alpha_i + \beta_i$ is a strictly decreasing function of i.*

Proof The first assertion is clear because both α_i and γ_i are strictly increasing. To prove the second one, let $i \leq s - 1$. We have:

$$\begin{aligned}
&(\alpha_i + \beta_i) - (\alpha_{i+1} + \beta_{i+1}) \\
&= (\beta_i - \beta_{i+1}) - (\alpha_{i+1} - \alpha_i) \\
&= \tbinom{d+m-i-1}{m-1} - \tbinom{d+m-i-2}{m-1} - (\tbinom{m+i}{m-1} - \tbinom{m+i-1}{m-1}) \\
&= \tbinom{d+m-i-2}{m-2} - \tbinom{m+i-1}{m-2} > 0.
\end{aligned} \tag{7}$$

Indeed from the hypothesis $i \leq s - 1$ it follows $d - 2i - 1 > 0$, that is equivalent to $d + m - i - 2 > m + i - 1$. □

The maximal Hilbert vector of a Perazzo algebra with $m = 2, n \geq 2$ and $d \geq 3$ is unimodal (see [1, Theorem 3.6] for the case $n = 2$ and [16, Theorem 4.12] for the case $n \geq 2$). The result is no longer true for $m > 2$ and in next theorem we will determine when the maximal Hilbert vector of a Perazzo algebra with fixed m, n, d and $m \geq 3$ is unimodal.

Theorem 3.5 *Let h_{max} be the maximal Hilbert vector of a Perazzo algebra with fixed m, n, d, $m \geq 3$. Let $s = [\frac{d}{2}]$. Then h_{max} is unimodal if and only*

(1) $\gamma_{s-1} < \beta_{s-1}$, *and*
(2) $\alpha_{s-1} + \gamma_{s-1} \leq \alpha_s + \beta_s$.

Proof We will use repeatedly Lemma 3.4. Assume first that conditions (1) and (2) are satisfied. From (1) it follows that $h_{s-1} = \alpha_{s-1} + \gamma_{s-1}$. We consider now h_s: if $h_s = \alpha_s + \gamma_s$, then

h_{max} is unimodal by Lemma 3.4; if $h_s = \alpha_s + \beta_s$, the unimodality of h_{max} follows from Lemma 3.4 and condition (2).

Assume now that h_{max} is unimodal. We observe that, by Lemma 3.4, the existence of $i \le s-1$ such that $\beta_i \le \gamma_i$ is equivalent to $\beta_{s-1} \le \gamma_{s-1}$. Therefore if, by contradiction $\beta_{s-1} \le \gamma_{s-1}$, then there exists $i \le s-1$ such that $h_i = \alpha_i + \beta_i$, i.e. $\beta_i \le \gamma_i$. Being $\alpha_i + \beta_i \le \alpha_i + \gamma_i$, using Lemma 3.4 we get

$$\alpha_{i+1} + \beta_{i+1} < \alpha_i + \beta_i \le \alpha_i + \gamma_i < \alpha_{i+1} + \gamma_{i+1}.$$

Therefore $h_{i+1} = \alpha_{i+1} + \beta_{i+1}$ and $h_{i+1} < h_i = \alpha_i + \beta_i$ with $i+1 \le s$: this contradicts the unimodality of h_{max}. So condition (1) is satisfied. It implies that $h_{s-1} = \alpha_{s-1} + \gamma_{s-1}$. Finally, from $h_{s-1} \le h_s$ it is immediate to deduce condition (2). □

Now we want to translate the conditions (1) and (2) of Theorem 3.5 in inequalities involving n, m, d, with $m \ge 3$. We have to discute separately the cases d even and d odd.

Condition (1), $d = 2s$ **even.**

$\gamma_{s-1} < \beta_{s-1}$ is equivalent to

$$\binom{s+m}{m-1} > (n+1)\binom{s+m-3}{m-1}.$$

This reduces to the inequality

$$ns^3 - 3(m-1)s^2 - 3[(m-1)^2 + n]s - m(m-1)(m-2) < 0.$$

Looking at the signs of the coefficients of the powers of s, from Descartes' rule of signs we deduce that the associated equation of degree 3 in the unknown s has at most one real positive solution $\bar{s}_1$. Therefore Condition (1) is never satisfied for s large enough.

Condition (1), $d = 2s + 1$ **odd.**

$\gamma_{s-1} < \beta_{s-1}$ is equivalent to

$$\binom{s+m+1}{m-1} > (n+1)\binom{s+m-3}{m-1}.$$

This reduces to the inequality of degree 4

$$\begin{aligned} &ns^4 + 2(n-2m+2)s^3 - (n+1+6m^2-6m-1)s^2 - \\ &\quad (2n+2+4m^3-6m^2-2m+2)s - (m+1)m(m-1)(m-2) < 0. \end{aligned}$$

Again from Descartes' rule of signs we get that the associated equation has at most one positive solution $\bar{s}_2$, and we conclude as in the even case that Condition (1) is never satisfied for s large enough.

Condition (2), $d = 2s$ **even.** The condition $\alpha_{s-1} + \gamma_{s-1} \le \alpha_s + \beta_s$ translates in an inequality of degree 2 in s, of the form

$$ns^2 - (3m + n - 3)s - 2(m-1)(m-2) \le 0.$$

We conclude as in the previous cases.

Condition (2), $d = 2s + 1$ **odd.** This time we get an inequality of degree 3 in s with at most one positive solution:

$$ns^3 - 4(m-1)s^2 - (4m^2 - 8m + 4 + n)s - (m+1)(m-1)(m-2) \le 0$$

and we conclude as in the previous cases.

Note that, even if we do not get explicit bounds on s, we can summarize our computations in the following Theorem.

Theorem 3.6 *The maximal Hilbert vector of a Perazzo algebra with fixed $n \geq m \geq 3$ is not unimodal for d large enough.*

Proof It follows from the discussion after Theorem 3.5. □

Example 3.7 Case (2) in Theorem 3.5 can fail as the following examples show. For d even we take: $n = 7, m = 4, d = 6, s = 3, h_{max} = (1, 12, 42, 40, 42, 12, 1)$; $h_2 = \alpha_2 + \gamma_2$, $\alpha_2 + \beta_2 = 10 + \binom{9}{3}$, $h_3 = \alpha_3 + \beta_3, \alpha_3 + \gamma_3 = 100$. An example with d odd is the following: $n = 13, m = 3, d = 5, s = 2, h_{max} = (1, 17, 16, 17, 1)$; $h_2 = \alpha_2 + \beta_2$ and $\alpha_2 + \gamma_2 = 48$.

We want to study now if Perazzo algebras with maximal h-vector have the WLP. In the special case $m = n = 2$ this problem has been solved in the negative in [5], and in the case $m = 2$ and $n \geq 2$ in [16].

As recalled in Sect. 2.4, it is well known that the h-vector of an Artinian graded algebra with the WLP is unimodal. We want to prove that for arbitrary n, m, even if the h-vector is unimodal, the Perazzo algebras with maximal h-vector fail the Weak Lefschetz Property.

Theorem 3.8 *Let A_F be a Perazzo algebra with maximal h-vector for fixed m, n, d with $n \geq m \geq 2$ and $d \geq 6$. Then A_F fails the WLP.*

Proof For the case $m = 2$ the reader can look at [5, 16]. Let $m \geq 3$. Let $h = (h_0, \ldots, h_d)$ be the h-vector of A_F. If h is not unimodal, then the thesis trivially follows. So we assume that h is unimodal.

Take ℓ a general linear form and consider the exact sequence appearing in Proposition 2.1:

$$0 \to (A_{\ell \circ F})(-1) \to A_F \to A_F/(\ell) \to 0. \tag{8}$$

Since h is unimodal we know from Theorem 3.5 that $\gamma_{s-1} < \beta^d_{s-1}$ and $\alpha_s + \beta^d_s \geq \alpha_{s-1} + \gamma_{s-1}$. Since β_i depends on d, we keep track of it using the above notation.

We now discuss separately the cases d odd and d even.

Assume first that $d = 2s + 1$ is odd. We observe that, from the assumption $n \geq m$, it follows $\beta^d_s < \gamma_s$; indeed $\beta^d_s = \binom{m+s}{m-1}$ and $\gamma_s = (n+1)\binom{m+s-2}{m-1}$, so a simple computation shows that $\beta^d_s < \gamma_s$ is equivalent to $s > \frac{m-1}{n}$.

Therefore the h-vector of A_F is

$$(\alpha_0 + \gamma_0, \alpha_1 + \gamma_1, \ldots, \alpha_{s-1} + \gamma_{s-1}, \alpha_s + \beta^d_s, \alpha_s + \beta^d_s, \alpha_{s-1} + \gamma_{s-1}, \ldots, \alpha_0 + \gamma_0).$$

If A_F has the WLP then the h-vector of $A_{\ell \circ F}$ will be

$$(\alpha_0 + \gamma_0, \alpha_1 + \gamma_1, \ldots, \alpha_{s-1} + \gamma_{s-1}, \alpha_s + \beta^d_s, \alpha_{s-1} + \gamma_{s-1}, \ldots, \alpha_1 + \gamma_1, \alpha_0 + \gamma_0).$$

But $\alpha_s + \beta^d_s > \alpha_s + \beta^{d-1}_s$. This is a contradiction because $\ell \circ F$ is a Perazzo polynomial of degree $d - 1$ by Lemma 2.3: the condition $n + 1 \leq \binom{m+d-3}{m-1}$ is satisfied, otherwise a simple computation shows that $\beta^d_{s-1} \leq \gamma_{s-1}$, which implies that h is not unimodal in view of Theorem 3.5.

Let now $d = 2s$ even. The h-vector of A_F is

$$(\alpha_0 + \gamma_0, \alpha_1 + \gamma_1, \ldots, \alpha_{s-1} + \gamma_{s-1}, h_s, \alpha_{s-1} + \gamma_{s-1}, \ldots, \alpha_0 + \gamma_0).$$

If A_F has the WLP then the h-vector of $A_{\ell \circ F}$ will be

$$(\alpha_0 + \gamma_0, \alpha_1 + \gamma_1, \ldots, \alpha_{s-1} + \gamma_{s-1}, \alpha_{s-1} + \gamma_{s-1}, \ldots, \alpha_1 + \gamma_1, \alpha_0 + \gamma_0),$$

which implies that $\beta^{d-1}_{s-1} = \gamma_{s-1}$. But $\ell \circ F$ has odd degree $2s - 1$ and is a Perazzo polynomial as in the previous case, hence $\beta^{d-1}_{s-1} < \gamma_{s-1}$: a contradiction. □

4 Minimal Hilbert function of a Perazzo algebra

In this section we will compute the minimal h-vector of all Perazzo algebras with fixed m, n, d, extending the results obtained in [5, 16] for $m = 2$. We will use Proposition 3.1 saying that, in the h-vector of the Perazzo algebra A_F, h_i is equal to the rank of the matrix containing in the columns the coefficients of the partial derivatives of F of order i. So first of all we will give a precise description of this matrix.

Proposition 4.1 *The matrix of the linear system defined in* (5) *to compute h_i has the following form*

$$\begin{pmatrix} 0 & | & N_i \\ --- & | & --- \\ M_{i-1} & | & \Gamma_i \end{pmatrix} \tag{9}$$

where:

$$M_{i-1} = (C^0_{i-1}\ C^1_{i-1} \dots C^n_{i-1}),$$
$$N_i = \begin{pmatrix} C^0_i \\ C^1_i \\ \vdots \\ C^n_i, \end{pmatrix},$$

C^k_i, Γ_i are the catalecticant matrices for p_k, $k = 0, \dots, n$ and G defined as follows (see [14], Definition 1.3*):*

$$C^k_i = (p^k_{\delta+\eta})_{|\delta|=i,|\eta|=d-1-i} \text{ and } \Gamma_i = (G_{\delta+\eta})_{|\delta|=i,|\eta|=d-i}.$$

Proof It follows from the expression of F and the assumption that K has characteristic zero. □

We keep using the following notation introduced in Sect. 3:

$$\alpha_i = \binom{m+i-1}{m-1},\ \beta_i = \binom{d+m-i-1}{m-1}.$$

Theorem 4.2 *Let m, n, d be fixed with $n \geq m \geq 2$. Then the minimal h-vector of the Perazzo algebras A_F, with F polynomial of degree d as in Definition 12, is $h_{\min} = (h_0, \dots, h_d)$, where for $1 \leq i \leq \frac{d}{2}$*

$$h_i = \min\{2(n+1), \alpha_i + n + 1, \alpha_i + \beta_i\}. \tag{10}$$

Proof In view of Propositions 3.1 and 4.1, we have to look for the Perazzo polynomials F such that the rank of the matrix (9) is minimal for any index $i = 1, \dots, [\frac{d}{2}]$. Therefore we can assume $G = 0$, so that, for any i, $h_i = \operatorname{rank} M_{i-1} + \operatorname{rank} N_i$.

The minimal possible rank of each catalecticant matrix C^k_{i-1} or C^k_i is 1. Therefore the minimum between the number $n+1$ of catalecticant blocks of M_{i-1} and $\beta_i = \binom{m+d-i-1}{m-1} = \dim K[U_1, \dots, U_m]_{d-i}$ is a lower bound for the rank of M_{i-1} for any $i \geq 1$.

Similarly a lower bound for the rank of N_i is the minimum between the number of its columns, that is $\binom{m+i-1}{m-1} = \alpha_i$, and $n+1$ that is the number of its catalecticant blocks.

So we get the following lower bound:

$$h_i \geq \min\{\, n+1, \beta_i \,\} + \min\{\, n+1, \alpha_i \,\}.$$

Since $i \leq d-i$ in our range $i \leq \frac{d}{2}$, we get

$$h_i \geq \min\{\, 2(n+1), n+1+\alpha_i, \alpha_i+\beta_i \,\}.$$

To conclude the proof, we exhibit an example of a Perazzo algebra with h-vector as in (10). We observe that all the catalecticant matrices of the polynomial L^d, where L is a linear form, have rank 1. Therefore, in view of [13, Corollary 3.2] it is enough to take $F = X_0L_0^{d-1} + \cdots + X_nL_n^{d-1}$, where $L_0, \ldots, L_n \in K[U_1, \ldots, U_m]_1$ are general linear forms. □

For $m = 2$ formula (10) gives the expression of the minimum h-vector found in [5, Proposition 2.8] and [16, Theorem 3.4].

In [1, 5, 16] it was proved that in the case $m = 2$ the minimal h-vector of Perazzo algebras is always unimodal and that the Perazzo algebras with minimal h-vector have the Weak Lefschetz Property provided that $d \geq 2n$. This does not always happen in the general case, as next example shows that h_{min} can be non unimodal for some integers n, m, d.

Example 4.3 Let $m = 3$, $n = 9$, $d = 4$. These are the invariants of the famous Stanley's example [19], whose h-vector is $(1, 13, 12, 13, 1)$: it is clearly non unimodal and it is minimal for these invariants (see [4]); it corresponds to a full Perazzo algebra.

Theorem 4.4 *The minimal h-vector of Perazzo algebras with invariants $n \geq m > 2$ is unimodal if and only if $n+1 \leq \beta_{s-1}$, where $s = [\frac{d}{2}]$.*

Proof Theorem 4.2 implies that, given n, m, d, for any $1 \leq i \leq s$ with $s = [\frac{d}{2}]$:

(1) if $n+1 \leq \alpha_i \leq \beta_i$, then $h_i = 2(n+1)$;
(2) if $\alpha_i < n+1 \leq \beta_i$, then $h_i = n+1+\alpha_i$, which is an *increasing* function of i;
(3) if $n+1 > \beta_i$, then $h_i = \alpha_i + \beta_i$, which is a *decreasing* function of i depending also on d.

It follows that h_{min} is unimodal if and only if h_{s-1} is not of the form (3). This proves the Theorem. □

We are now able to prove that the Weak Lefschetz Property holds for the Perazzo algebras with minimal h-vector, provided it satisfies the condition that $n+1 \leq \beta_s$, which implies unimodality.

Theorem 4.5 *Let A_F be a Perazzo algebra with minimal h-vector for fixed m, n, d with $n \geq m \geq 3$. Assume that the h-vector of A_F is unimodal and that $n+1 \leq \beta_s$, where $s = [\frac{d}{2}]$. Then A_F has the WLP.*

Proof To prove that A_F has the WLP it is enough to check that, for a general linear form ℓ, the multiplication map $\times\ell : (A_F)_{s-1} \to (A_F)_s$ is injective. If by contradiction it is not injective, then using Proposition 2.1 and Lemma 2.3, we get that $\dim(A_{\ell\circ F})_{s-1} < \dim(A_F)_{s-1}$. But our assumption on n, m, d implies that the component of index $s-1$ of the minimal h-vector is the same for degrees d and $d-1$. This contradicts the minimality of the h-vector of A_F. □

Note that if d is odd, then the condition $n+1 \leq \beta_s$ means that the h-vector of $A_{\ell\circ F}$ is unimodal.

We characterize now the integers n, m, d such that the Perazzo algebras with these invariants have all the same Hilbert function, i.e. the maximal and the minimal h-vector coincide.

Proposition 4.6 *Let n, m, d be positive integers with $n \geq m \geq 2$, $n+1 \leq \binom{d+m-2}{m-1}$. Then the Perazzo algebras with these invariants have $h_{min} = h_{max}$ if and only if*

$$\binom{d+m-3}{m-1} \leq n+1. \tag{11}$$

Proof We will prove that Eq. 11 is equivalent to $h_{i,min} = h_{i,max} = \alpha_i + \beta_i$ for any index i. We will then exclude the possibility that $h_{2,min} = h_{2,max} \neq \alpha_2 + \beta_2$.

Since $\alpha_i + \beta_i$ is a decreasing function of i, the first assertion is equivalent to $h_{2,min} = h_{2,max} = \alpha_2 + \beta_2$. But $h_{2,min} = \alpha_2 + \beta_2$ if and only if $\binom{m+d-3}{m-1} \leq n+1$ and $h_{2,max} = \alpha_2 + \beta_2$ if and only if $\binom{m+d-3}{m-1} \leq (n+1)m$, which proves our claim.

We assume now by contradiction that $h_{2,min} = h_{2,max} \neq \alpha_2 + \beta_2$. This means that $h_{2,max} = \alpha_2 + \gamma_2 < \alpha_2 + \beta_2$, and $h_{2,min} = \min\{2n+2, n+1+\binom{m+1}{2}\} < \alpha_2 + \beta_2$. If $n+1 \leq \binom{m+1}{2}$, then $2n+2 = \binom{m+1}{2} + (n+1)m$ which is equivalent to $(n+1)(2-m) = \binom{m+1}{2}$, but this is impossible because $m \geq 2$. If $n+1 \geq \binom{m+1}{2}$, then we would have $n+1 = (n+1)m$: contradiction. □

Remark 4.7 Perazzo algebras with $h_{min} = h_{max}$ clearly include full Perazzo algebras. It has been conjectured in [2, Conjecture 2.6] that the h-vectors of full Perazzo algebras are minimal among the h-vectors of all Artinian Gorenstein algebras with the same degree and codimension. This conjecture has been proved for $d = 4$ and $m = 3, 4, 5$ in [4], and for any degree and $m = 3$ in [2].

5 Minimal free resolution of a Perazzo algebra

In this section, we determine the minimal free resolution of an Artinian Gorenstein algebra corresponding to a Perazzo threefold of $\mathbb{P}^4$ with termwise minimal Hilbert function, i. e. of the following type $(1, 5, 6, \ldots, 6, 5, 1)$.

When we deal with Perazzo hypersurfaces in $\mathbb{P}^4$, we use the notations $S = K[X, Y, Z, U, V]$ and $R = K[x, y, z, u, v]$. We have

$$F = Xp_0 + Yp_1 + Zp_2 + G \quad \text{where } p_0, p_1, p_2, G \in K[U, V] \tag{12}$$

and any choice of p_0, p_1, p_2 will be algebraically dependent.

An explicit classification of the possible dual generators F of degree $d \geq 5$ defining a Perazzo threefold with termwise minimal Hilbert function is given in [5, Theorem 5.4]. We state a slightly rephrased version of this result.

Lemma 5.1 *Let $F \in K[X, Y, Z, U, V]$ be a Perazzo form such that the algebra A_F has minimal Hilbert function $(1, 5, 6, 6, \ldots, 6, 5, 1)$. Then the dual generator F can be expressed as*

(i) $XU^{d-1} + YU^{d-2}V + ZU^{d-3}V^2$,
(ii) $XU^{d-1} + YU^{d-2}V + ZV^{d-1}$, *or*
(iii) $XU^{d-1} + Y(U + \lambda V)^{d-1} + ZV^{d-1}$ *with* $\lambda \in K^*$

after a linear change of variables.

Proof By [5, Theorem 5.4], there are three classes of forms up to a linear change of variables. In the first case, F can be written as

$$F = XU^{d-1} + YU^{d-2}V + ZU^{d-3}V^2 + aU^d + bU^{d-1}V + cU^{d-2}V$$

$$= (X + aU)U^{d-1} + (Y + bU)U^{d-2}V + (Z + cU)U^{d-3}V^2$$

for $a, b, c \in K$. A further linear change of variables gives the case (i). The other cases are obtained in the same manner. □

We know by [5, Theorem 4.3] that, in all the three cases of Lemma 5.1, A_F has the WLP which allows us to prove the following key lemma.

Lemma 5.2 *Let A_F be an Artinian Gorenstein algebra associated to a Perazzo threefold $X = V(F)$ of $\mathbb{P}^4$ of degree $d \geq 5$ and with termwise minimal Hilbert function. Let $\ell \in A_F$ be a general linear form and set $B = A_F/(\ell)$. We have:*

$$\beta_{ij}^R(B) := [Tor_i^R(B, K)]_j = 0 \textit{ for } j > i + 2.$$

Proof By [5, Theorem 4.3] the algebra A_F has the WLP which implies that the h-vector of B is $(1, 4, 1)$ and the socle degree of B is 2, i.e. $\operatorname{reg}(B) = 2$. Therefore, the minimal graded free resolution of B as R-module has the following shape:

$$0 \rightarrow \begin{array}{c} R(-6)^{\beta_{56}^R(B)} \\ \oplus \\ R(-7)^{\beta_{57}^R(B)} \end{array} \rightarrow \begin{array}{c} R(-5)^{\beta_{45}^R(B)} \\ \oplus \\ R(-6)^{\beta_{46}^R(B)} \end{array} \rightarrow \begin{array}{c} R(-4)^{\beta_{34}^R(B)} \\ \oplus \\ R(-5)^{\beta_{35}^R(B)} \end{array} \rightarrow$$

$$\begin{array}{c} R(-3)^{\beta_{23}^R(B)} \\ \oplus \\ R(-4)^{\beta_{24}^R(B)} \end{array} \rightarrow \begin{array}{c} R(-1) \\ \oplus \\ R(-2)^9 \\ \oplus \\ R(-3)^{\beta_{13}^R(B)} \end{array} \rightarrow R \rightarrow B \rightarrow 0$$

and we conclude that $\beta_{ij}^R := [Tor_i^R(B, K)]_j = 0$ for $j > i + 2$ which proves what we want. □

Example 5.3 Using Macaulay2 [10], we have computed the Betti table of Artinian Gorenstein algebras A_F associated to all 3 possible types of Perazzo threefolds F of $\mathbb{P}^4$ with termwise minimal Hilbert function and degree $5 \leq d \leq 8$ (see Lemma 5.1),

```
+-----------------------+
|       0  1  2  3  4  5|
|total: 1 14 35 35 14  1|
|    0: 1  .  .  .  .  .|
|    1: .  9 17 12  3  .|
|    2: .  1  3  3  1  .|
|    3: .  1  3  3  1  .|
|    4: .  3 12 17  9  .|
|    5: .  .  .  .  .  1|
+-----------------------+
```

```
+-----------------------+
|       0  1  2  3  4  5|
|total: 1 14 35 35 14  1|
|    0: 1  .  .  .  .  .|
|    1: .  9 17 12  3  .|
|    2: .  1  3  3  1  .|
|    3: .  .  .  .  .  .|
|    4: .  1  3  3  1  .|
|    5: .  3 12 17  9  .|
|    6: .  .  .  .  .  1|
+-----------------------+
```

```
+-----------------------+
|        0  1  2  3  4  5|
|total:  1 14 35 35 14  1|
|     0: 1  .  .  .  .  .|
|     1: .  9 17 12  3  .|
|     2: .  1  3  3  1  .|
|     3: .  .  .  .  .  .|
|     4: .  .  .  .  .  .|
|     5: .  1  3  3  1  .|
|     6: .  3 12 17  9  .|
|     7: .  .  .  .  .  1|
+-----------------------+
```

```
+-----------------------+
|        0  1  2  3  4  5|
|total:  1 14 35 35 14  1|
|     0: 1  .  .  .  .  .|
|     1: .  9 17 12  3  .|
|     2: .  1  3  3  1  .|
|     3: .  .  .  .  .  .|
|     4: .  .  .  .  .  .|
|     5: .  .  .  .  .  .|
|     6: .  1  3  3  1  .|
|     7: .  3 12 17  9  .|
|     8: .  .  .  .  .  1|
+-----------------------+
```

Theorem 5.4 *Let A_F be an Artinian Gorenstein algebra associated to a Perazzo threefold $X = V(F)$ of $\mathbb{P}^4$ of degree $d \geq 5$ and with minimal Hilbert function. The Betti diagram of A_F looks like*

```
+-----------------------+
|        0  1  2  3  4  5|
|total:  1 14 35 35 14  1|
|     0: 1  .  .  .  .  .|
|     1: .  9 17 12  3  .|
|     2: .  1  3  3  1  .|
|     3: .  .  .  .  .  .|
|     .:.   .  .  .  .  .|
|   d-3: .  .  .  .  .  .|
|   d-2: .  1  3  3  1  .|
|   d-1: .  3 12 17  9  .|
|     d: .  .  .  .  .  1|
+-----------------------+
```

Proof We proceed by induction on d. For $5 \leq d \leq 8$ the result is true (see Example 5.3). Assume $\deg(F) = d + 1 \geq 9$. Let $\ell \in A_F$ be a general linear form and consider the exact sequence:

$$0 \longrightarrow A_{\ell \circ F} \longrightarrow A_F \longrightarrow B = A_F/(\ell) \longrightarrow 0 \tag{13}$$

which gives us the long exact sequence

$$\cdots \longrightarrow [Tor_{i+1}^R(B, K)]_j \longrightarrow [Tor_i^R(A_{\ell \circ F}(-1), K)]_j \longrightarrow [Tor_i^R(A_F, K)]_j \longrightarrow [Tor_i^R(B, K)]_j \longrightarrow \cdots .$$

Using Lemma 5.2, we get that for any $j > i + 3$ we have:

$$\begin{aligned} \beta^R_{ij-1}(A_{\ell \circ F}) &= [Tor^R_i(A_{\ell \circ F}, K)]_{j-1} \\ &= [Tor^R_i(A_{\ell \circ F}(-1), K)]_j \\ &= [Tor^R_i(A_F, K)]_j \\ &= \beta^R_{ij}(A_F). \end{aligned} \tag{14}$$

By Lemma 2.3, $A_{\ell \circ F}$ is an Artinian Gorenstein algebra associated to a Perazzo form $\ell \circ F$ of degree d. Using the exact sequence (13) and the fact that A_F has the WLP ([5, Theorem 4.3]) we obtain that the Hilbert function of $A_{\ell \circ F}$ is termwise minimal, i.e. $A_{\ell \circ F}$ has Hilbert function $(1, 5, 6, 6, \cdots, 6, 6, 5, 1)$. Therefore, by hypothesis of induction we know all graded Betti numbers of $A_{\ell \circ F}$ and the equalities (14) give us:

$$\begin{aligned} 1 &= \beta_{1,d-1}(A_{\ell \circ F}) = \beta_{1,d}(A_F) \\ 3 &= \beta_{1,d}(A_{\ell \circ F}) \quad = \beta_{1,d+1}(A_F) \\ 0 &= \beta_{1,j-1}(A_{\ell \circ F}) = \beta_{1,j}(A_F) \text{ for } d + j \geq 5; \end{aligned}$$

$$\begin{aligned} 3 &= \beta_{2,d}(A_{\ell \circ F}) \quad = \beta_{2,d+1}(A_F) \\ 12 &= \beta_{2,d+1}(A_{\ell \circ F}) = \beta_{2,d+2}(A_F) \\ 0 &= \beta_{2,j-1}(A_{\ell \circ F}) = \beta_{2,j}(A_F) \text{ for } d + j \geq 6; \end{aligned}$$

$$\begin{aligned} 3 &= \beta_{3,d+1}(A_{\ell \circ F}) = \beta_{3,d+2}(A_F) \\ 17 &= \beta_{3,d+2}(A_{\ell \circ F}) = \beta_{3,d+3}(A_F) \\ 0 &= \beta_{3,j-1}(A_{\ell \circ F}) = \beta_{3,j}(A_F) \text{ for } d + j \geq 7; \end{aligned}$$

$$\begin{aligned} 1 &= \beta_{4,d+2}(A_{\ell \circ F}) = \beta_{4,d+3}(A_F) \\ 9 &= \beta_{4,d+3}(A_{\ell \circ F}) = \beta_{4,d+4}(A_F) \\ 0 &= \beta_{4,j-1}(A_{\ell \circ F}) = \beta_{4,j}(A_F) \text{ for } d + j \geq 8. \end{aligned}$$

Therefore, the Betti diagram of A_F being $\deg(F) = d + 1$ has the following shape ($*$ means not yet determined and . means zero):

```
+------------------------+
|        0  1  2  3  4  5|
|total:  1 14 35 35 14  1|
|     0: 1  .  .  .  .  .|
|     1: .  *  *  *  *  .|
|     2: .  *  *  *  *  .|
|     3: .  *  *  *  *  .|
|     4:.   *  *  *  *  .|
|     5:.   .  .  .  .  .|
|     .:.   .  .  .  .  .|
|   d-2: .  .  .  .  .  .|
|   d-1: .  1  3  3  1  .|
|     d: .  3 12 17  9  .|
|   d+1: .  .  .  .  .  1|
+------------------------+
```

Using now the fact that the minimal graded free R-resolution of an Artinian Gorenstein algebra A_F is self dual we conclude what we want. □

6 Final remarks and open problems

We end this paper with a couple of concrete problems which naturally arise from our results and we believe they deserve further consideration.

In Theorem 3.8 we prove that any Perazzo algebra A_F with maximal h-vector for fixed $n \geq m \geq 2$ and $d \geq 6$ fails WLP while in Theorem 3.5 we determine all Perazzo algebras with unimodal maximal h-vector for fixed n, m and d with $m \geq 3$. The analogous results for Perazzo algebras A_F with minimal h-vector for fixed n, m and d with $n \geq m \geq 3$ are obtained in Theorems 4.5 and 4.4.

For Perazzo algebras A_F with intermediate h-vector both possibilities occur: there are examples failing WLP and examples satisfying WLP as well as examples with unimodal h-vector and examples with non-unimodal h-vector (see [5]). Therefore, the major questions/problems left open are the following two:

Problem 1 (i) To classify all Perazzo algebras A_F with unimodal Hilbert function.
(ii) To classify all Perazzo algebras A_F with WLP.

For a complete answer to Problem 1 for $n = m = 2$ the reader can look at [1] and [5] and for $n \geq 2$ and $m = 2$ at [16]. To our knowledge for all other values $n \geq m \geq 3$ no answer is known.

In the last decades big effort has been made in understanding the minimal free resolution (MFR, for short) of any artinian Gorenstein algebra. In 1977, Buchsbaum and Eisenbud proved that any Gorenstein codimension 3 ideal is generated by the $2t \times 2t$ pfaffians of a skew symmetric matrix of size $(2t+1)\times(2t+1)$ and this fact completely determines the MFR of any Artinian Gorenstein algebra of codimension 3 (see [3, Theorem 2.1]). Nevertheless for codimension ≥ 4 little is known apart from the selfduality (up to twist) of the MFR of any Artinian Gorenstein algebra. Using our knowledge of Perazzo algebras we propose as an intermediate step the following problem:

Problem 2 (i) To determine the MFR of any Perazzo algebra with minimal (resp. maximal) Hilbert function.
(ii) To determine the MFR of any full Perazzo algebra.
(iii) To determine the MFR of *any* Perazzo algebra.

The above problem is interesting *per se* but also because we believe that the WLP of Perazzo algebras of any codimension c could be determined by their MFR.

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