On Selection of Standing Wave at Small Energy in the 1D Cubic Schrödinger Equation with a Trapping Potential

Scipio Cuccagna[1], **Masaya Maeda**[2]

[1] Department of Mathematics and Geosciences, University of Trieste, via Valerio 12/1, 34127 Trieste, Italy. E-mail: scuccagna@units.it
[2] Department of Mathematics and Informatics, Graduate School of Science, Chiba University, Chiba 263-8522, Japan. E-mail: maeda@math.s.chiba-u.ac.jp

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Abstract: Combining virial inequalities by Kowalczyk, Martel and Munoz and Kowalczyk, Martel, Munoz and Van Den Bosch with our theory on how to derive nonlinear induced dissipation on discrete modes, and in particular the notion of Refined Profile, we show how to extend the theory by Kowalczyk, Martel, Munoz and Van Den Bosch to the case when there is a large number of discrete modes in the cubic NLS with a trapping potential which is associated to a repulsive potential by a series of Darboux transformations. Even though, by its non translation invariance, our model avoids some of the difficulties related to the effect that translation has on virial inequalities of the kink stability problem for wave equations, it still is a classical model and it retains some of the main difficulties.

1. Introduction

In this paper, we consider the cubic nonlinear Schrödinger equation (NLS) with potential,

$$\mathrm{i}\partial_t u = -\partial_x^2 u + Vu + |u|^2 u, \quad (t,x) \in \mathbb{R}^{1+1}, \tag{1.1}$$

where the potential V satisfies

$$V \in \mathcal{S}(\mathbb{R},\mathbb{R}) \text{ (Schwartz functions)}, \ |V(x)| + |V'(x)| \le Ce^{-a_0|x|} \text{ for some } C, a_0 > 0, \tag{1.2}$$

and we assume that for $\sigma_{\mathrm{d}}(H)$, which is the set of discrete spectrum of $H := -\partial_x^2 + V$, we have

$$\sigma_{\mathrm{d}}(H) = \{\omega_j \mid j = 1, \cdots, N\}, \ \omega_1 < \cdots < \omega_N < 0, \ N \ge 2. \tag{1.3}$$

Remark 1.1. It is well known that $\sigma_d(H)$ is finite. Our assumption is that H has more than two eigenvalues. The case $N = 0$ has been treated by Naumkin [48], by Germain *et al.* [25], by Delort [19] and by Chen–Pusateri [8]. See also Masaki *et al.* [41] along with [12] for the case of a repulsive δ potential. The case $N = 1$ in the case of an attractive δ potential with rather general nonlinear terms, which include as a special case also the cubic nonlinearity, is treated in [12]. The case of a generic potential with $N = 1$, hence a less stringent hypothesis than Assumption 1.13 below, is discussed in Chen [7]. In this paper we focus only on the case $N \geq 2$, which is more delicate than the cases $N = 0, 1$.

Remark 1.2. In the sequel we will often use the notation $\dot{f} = \partial_t f$ and $f' = \partial_x f$. We also use the notation $a \lesssim b$, which means that there exists $C > 0$ s.t. $a \leq Cb$ with C not depending on important quantities. We also write $a \sim b$ if $a \lesssim b$ and $b \lesssim a$, and $a \lesssim_\alpha b$ if $a \leq C_\alpha b$ with the implicit constant C_α depending on α. Finally, $a \sim_\alpha b$ will mean $a \lesssim_\alpha b$ and $b \lesssim_\alpha a$.

The aim of this paper is to study the long time behavior of small solutions of (1.1). Here, we recall that from the energy

$$E(u) := \frac{1}{2} \int_{\mathbb{R}} \left(|\partial_x u|^2 + V|u|^2 + \frac{1}{2}|u|^4 \right) dx \tag{1.4}$$

and mass

$$Q(u) := \frac{1}{2} \int_{\mathbb{R}} |u|^2 \, dx \tag{1.5}$$

conservation, if we have $u_0 \in H^1$, then

$$\|u(t)\|_{H^1} \lesssim \|u_0\|_{H^1}. \tag{1.6}$$

Thus, global well-posedness of small solutions is trivial. In this paper, we seek a more precise understanding of the asymptotic behavior of $u(t)$. By elementary bifurcation argument, under appropriate hypotheses there exist N families of standing wave solutions (i.e. solutions with the form $u(t, x) = e^{\mathrm{i}\omega t}\phi(x)$) bifurcating from the eigenvalues of H. That is, there exist $\phi_j[z](x) = z\phi_j(x) + O(|z|^3)$ and $\omega_j(|z|^2) = \omega_j + O(|z|^2)$ ($j = 1, \cdots, N$) for small $z \in \mathbb{C}$ s.t. $e^{-\mathrm{i}\omega_j(|z|^2)t}\phi_j[z](x)$ are standing wave solutions of (1.1). In the linear case ($\mathrm{i}\partial_t u = Hu$), by the superposition principle, there also exist quasi-periodic solutions which are given by the sum of standing waves. However, it was shown that in the 3D case with smooth nonlinearity (corresponding to the 1D case with the nonlinearity $|u|^2u$ replaced by $g(|u|^2)u$ with $g \in C^\infty$ and $g(0) = g'(0) = 0$, e.g. $|u|^4u$), the solutions locally converge to the orbit of one single standing wave (selection of standing waves) and therefore there exist no quasi-periodic solution, see [11] and the references therein. Thus, even though the short time dynamics of small solutions of linear Schrödinger equation and NLS are similar, in 3D the long time dynamics are completely different. The aim of this paper is to prove a similar result also for 1D cubic NLS (1.1), under an additional hypothesis on the potential, see §1.1.2. This is more difficult than the 3D case because of the fact that $|u|^2u$ is a long range nonlinearity in 1D and by the weakness of linear dispersion in 1D. The main idea in this paper consists in a combined use of the dispersion theory of Kowalczyk, Martel and Munoz [29], from which we draw extensively, with the notion of Refined Profiles we introduced in [10] and which we discuss now, before stating the main result.

1.1. Set up. For $a \in \mathbb{R}$ and $\langle x \rangle := \sqrt{1+x^2}$, we set

$$\|u\|_{X_a} := \|e^{a\langle x\rangle} u\|_X, \text{ for } X = H^s, \ L^p \text{ and } a := 2^{-1}\sqrt{|\omega_N|}, \tag{1.7}$$
$$\Sigma^s := H^s_a, \tag{1.8}$$

where we make the convention that, when we write L^p, H^s or other analogous spaces like $L^{2,s}$ below, they are $L^p(\mathbb{R})$, $H^s(\mathbb{R})$, etc.

For any $s \in \mathbb{R}$, we will use also other weighted spaces, defined by the norm

$$\|u\|_{L^{2,s}} := \| \langle x\rangle^s u\|_{L^2(\mathbb{R})}. \tag{1.9}$$

We will consider repeatedly several Bochner spaces of the form $L^p(I, X)$, with $p \in [1, \infty]$, $I \subseteq \mathbb{R}$ an interval and X a Banach space, see [5, Chapter 1], with norms

$$\|u\|_{L^p(I,X)} := \| \|u\|_X \|_{L^p(I)}. \tag{1.10}$$

In particular, we will use spaces like $X = \widetilde{\Sigma}$, introduced in (3.1), and $X = X_a$ like in (1.7).

We recall that all eigenvalues of H are simple. We pick ϕ_j such that $\|\phi_j\|_{L^2} = 1$, $\mathbb{R}$-valued eigenfunctions of H associated with ω_j. Since $\phi_j^{(n)} \sim_n e^{-\sqrt{|\omega_j|}|x|}$, we have $\phi_j \in \cap_{s>0}\Sigma^s$ for all $j = 1, \cdots, N$.

1.1.1. Refined profile and Fermi Golden Rule assumption One of the keys is the notion of refined profile introduced in [10]. We set

$$\boldsymbol{\omega} = (\omega_1, \cdots, \omega_N). \tag{1.11}$$

For the discrete spectrum $\sigma_{\mathrm{d}}(-\partial_x^2 + V)$, we assume the following.

Assumption 1.3. For $\mathbf{m} := (m_1, \cdots, m_N) \in \mathbb{Z}^N \setminus \{0\}$, $\mathbf{m} \cdot \boldsymbol{\omega} = \sum_{j=1}^N m_j\omega_j \neq 0$.

Remark 1.4. In reality we need Assumption 1.3 for a restricted and finite set of indexes.

In the following, for $\mathbf{x} = (x_1, \cdots, x_N) \in X^N$, for $X = \mathbb{Z}, \mathbb{R}, \mathbb{C}$ we write $\|\mathbf{x}\| := \|\mathbf{x}\|_1 = \sum_{j=1}^N |x_j|$. We consider the sets of multi–indexes

$$\mathbf{NR}:=\{\mathbf{m} \in \mathbb{Z}^N \mid \sum_{j=1}^N m_j = 1, \ \mathbf{m}\cdot\boldsymbol{\omega} < 0\}, \ \mathbf{R}:=\{\mathbf{m} \in \mathbb{Z}^N \mid \sum_{j=1}^N m_j = 1, \ \mathbf{m}\cdot\boldsymbol{\omega}>0\}. \tag{1.12}$$

By Assumption 1.3, we have $\{\mathbf{m} \in \mathbb{Z}^N \mid \sum_{j=1}^N m_j = 1\} = \mathbf{NR} \cup \mathbf{R}$. Furthermore, we set

$$\mathbf{R}_{\min} := \{\mathbf{m} \in \mathbf{R} \mid \ \nexists \mathbf{n} \in \mathbf{R} \text{ s.t. } \mathbf{n} \prec \mathbf{m}\}, \tag{1.13}$$

where

$$\mathbf{n} \prec \mathbf{m} \ \Leftrightarrow \ \mathbf{n} \preceq \mathbf{m} \text{ and } \|\mathbf{n}\| < \|\mathbf{m}\|,$$
$$\mathbf{n} \preceq \mathbf{m} \ \Leftrightarrow \ \forall j = 1, \cdots, N, \ |n_j| \leq |m_j|.$$

Related to $\mathbf{R}_{\min}$ are the sets $\mathbf{I}$ and $\mathbf{NR}_1$, defined by

$$\mathbf{I} := \{\mathbf{m} \in \mathbf{R} \cup \mathbf{NR} \mid \exists \mathbf{n} \in \mathbf{R}_{\min} \text{ s.t. } \mathbf{n} \prec \mathbf{m}\}, \ \mathbf{NR}_1 := \mathbf{NR} \setminus \mathbf{I}. \tag{1.14}$$

Remark 1.5. The set $\mathbf{I}$ will be the collection of negligible multi–indexes, in the sense that for $\mathbf{n} \in \mathbf{I}$,

$$|\mathbf{z}^{\mathbf{n}}| \leq \|\mathbf{z}\| \sum_{\mathbf{m} \in \mathbf{R}_{\min}} |\mathbf{z}^{\mathbf{m}}| \text{ for } \|\mathbf{z}\| \leq 1,$$

where $\mathbf{z}^{\mathbf{n}}$ is defined right below. Here $\mathbf{R}$ stands for resonant, while $\mathbf{NR}$ stands for non resonant. The most significant elements of $\mathbf{NR}$ are those of $\mathbf{NR}_1$. The corresponding monomials $\mathbf{z}^{\mathbf{m}}$ for $\mathbf{m} \in \mathbf{NR}_1$ are eliminated and do not appear in the key equation (1.20) by a, rather elementary, normal forms argument, while the $\mathbf{z}^{\mathbf{m}}$ for $\mathbf{m} \in \mathbf{NR} \cap \mathbf{I}$ are small remainder terms, absorbed in $\mathcal{R}_{\mathrm{rp}}[\mathbf{z}]$ and easy to bound in the course of the proof.

For $\mathbf{z} \in \mathbb{C}^N$, we set

$$\mathbf{z}^{\mathbf{m}} := \prod_{j=1}^{N} z_j^{(m_j)}, \; z_j^{(m_j)} := \begin{cases} z_j^{m_j} & \text{if } m_j \geq 0, \\ \overline{z_j}^{-m_j} & \text{if } m_j < 0. \end{cases} \tag{1.15}$$

We inductively define $G_{\mathbf{m}}$ for $\mathbf{m} \in \mathbf{R}_{\min} \cup \mathbf{NR}_1$ by $G_{\mathbf{m}} = 0$ if $\|\mathbf{m}\| \leq 1$ and

$$G_{\mathbf{m}} = \sum_{\substack{\mathbf{m}^1, \mathbf{m}^2, \mathbf{m}^3 \in \mathbf{NR}_1, \\ \mathbf{m}^1 - \mathbf{m}^2 + \mathbf{m}^3 = \mathbf{m}, \; \|\mathbf{m}^1\| + \|\mathbf{m}^2\| + \|\mathbf{m}^3\| = \|\mathbf{m}\|}} \widetilde{\phi_{\mathbf{m}^1}} \, \widetilde{\phi_{\mathbf{m}^2}} \, \widetilde{\phi_{\mathbf{m}^3}}, \tag{1.16}$$

where

$$\widetilde{\phi_{\mathbf{m}}} = \begin{cases} 0, & \mathbf{m} = 0 \\ \phi_j, & \mathbf{m} = \mathbf{e}^j := (\delta_{j1}, \cdots, \delta_{jN}), \\ -(H - \mathbf{m} \cdot \boldsymbol{\omega})^{-1} G_{\mathbf{m}}, & \|\mathbf{m}\| \geq 2 \end{cases} \tag{1.17}$$

Example 1.6. We give the first few $G_{\mathbf{m}}$ for the case $N = 2$. When $\mathbf{m} \in \mathbf{R}_{\min} \cup \mathbf{NR}_1$ and $\|\mathbf{m}\| = 3$, we have $\mathbf{m} = (2, -1)$ or $(-1, 2)$ and

$$G_{(2,-1)} = \phi_1^2 \phi_2, \quad G_{(-1,2)} = \phi_1 \phi_2^2.$$

When $\mathbf{m} \in \mathbf{R}_{\min} \cup \mathbf{NR}_1$ and $\|\mathbf{m}\| = 5$, we have $\mathbf{m} = (3, -2)$ or $(-2, 3)$ and

$$G_{(3,-2)} = -2\phi_1\phi_2(H - (2\omega_1 - \omega_2))^{-1} \left(\phi_1^2\phi_2\right) - \phi_1^2(H - (-\omega_1 + 2\omega_2))^{-1}(\phi_1\phi_2^2),$$

$$G_{(-2,3)} = -2\phi_1\phi_2(H - (-\omega_1 + 2\omega_2))^{-1} \left(\phi_1\phi_2^2\right) - \phi_2^2(H - (2\omega_1 - \omega_2))^{-1}(\phi_1^2\phi_2),$$

By the inductive definition, we have the following.

Lemma 1.7. *For* $\mathbf{m} \in \mathbf{R}_{\min} \cup \mathbf{NR}_1$ *the* $G_{\mathbf{m}}$ *are* $\mathbb{R}$*-valued.*

Proof. If $\|\mathbf{m}\| \leq 1$, then the statement is obvious because we have chose ϕ_j' to be $\mathbb{R}$-valued. Next, for $\mathbf{m} \in \mathbf{R}_{\min} \cup \mathbf{NR}_1$, we assume that for all $\mathbf{n} \in \mathbf{NR}_1$ with $\|\mathbf{n}\| < \|\mathbf{m}\|$, $G_{\mathbf{n}}$ is $\mathbb{R}$-valued. Then, from (1.17), $\widetilde{\phi_{\mathbf{n}}}$ is $\mathbb{R}$-valued and by (1.16), $G_{\mathbf{m}}$ is also $\mathbb{R}$-valued. □

An important assumption, related to the Fermi Golden Rule (FGR), is the following.

Assumption 1.8. We assume that for all $\mathbf{m} \in \mathbf{R}_{\min}$,

$$\sum_{\pm} |\widehat{G_{\mathbf{m}}}\left(\pm\sqrt{\boldsymbol{\omega}\cdot\mathbf{m}}\right)| > 0,$$

where $\widehat{G_{\mathbf{m}}}$ is the distorted Fourier transform of $G_{\mathbf{m}}$ associated to the operator H, see [60].

For a Banach space X and $x \in X$, $r > 0$, we set

$$B_X(x,r) := \{y \in X \mid \|y - x\|_X < r\}. \tag{1.18}$$

For $F \in C^1(B_{\mathbb{C}^N}(0,\delta), X)$, $\mathbf{z} \in B_{\mathbb{C}^N}(0,\delta)$ and $\mathbf{w} \in \mathbb{C}^N$, we set $D_{\mathbf{z}}F(\mathbf{z})\mathbf{w} := \frac{d}{ds}\Big|_{s=0} F(\mathbf{z} + s\mathbf{w})$.

The following is proved in [10].

Proposition 1.9 (Refined profile). *For any $s > 0$, there exists $\delta_s > 0$ s.t. there exists $\phi[\cdot] \in C^\infty(B_{\mathbb{C}^N}(0,\delta_s), \Sigma^s)$ of the form*

$$\phi[\mathbf{z}] = \sum_{\mathbf{m}\in\mathbf{NR}_1} \mathbf{z}^{\mathbf{m}}\widetilde{\phi_{\mathbf{m}}} \text{ where } \widetilde{\phi_{\mathbf{m}}} \in \Sigma^s \text{ for all } s, \tag{1.19}$$

$\boldsymbol{\varpi} \in C^\infty(B_{\mathbb{C}^N}(0,\delta_s), \mathbb{R}^N)$ and $\mathcal{R}_{\mathrm{rp}}[\cdot] \in C^\infty(B_{\mathbb{C}^N}(0,\delta_s), \Sigma^s)$ s.t.

$$-\mathrm{i}D_{\mathbf{z}}\phi[\mathbf{z}]\,(\mathrm{i}\boldsymbol{\varpi}(\mathbf{z})\mathbf{z}) = H\phi[\mathbf{z}] + |\phi[\mathbf{z}]|^2\phi[\mathbf{z}] - \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}} G_{\mathbf{m}} - \mathcal{R}_{\mathrm{rp}}[\mathbf{z}], \tag{1.20}$$

and $\phi[0] = 0$, $D_{\mathbf{z}}\phi[0]\mathbf{e}^j = \phi_j$, $D_{\mathbf{z}}^2\phi[0] = 0$, $\boldsymbol{\varpi}(0) = \boldsymbol{\omega}$, $\phi[e^{\mathrm{i}\theta}\mathbf{z}] = e^{\mathrm{i}\theta}\phi[\mathbf{z}]$, $\boldsymbol{\varpi}(\mathbf{z}) = \boldsymbol{\varpi}(|z_1|^2, \cdots, |z_N|^2)$,

$$\|\mathcal{R}_{\mathrm{rp}}[\mathbf{z}]\|_{\Sigma^s} \lesssim_s \|\mathbf{z}\|^2 \sum_{\mathbf{m}\in\mathbf{R}_{\min}} |\mathbf{z}^{\mathbf{m}}|, \tag{1.21}$$

and $\boldsymbol{\varpi}(\mathbf{z})\mathbf{z} := (\varpi_1(\mathbf{z})z_1, \cdots, \varpi_N(\mathbf{z})z_N)$.

□

Remark 1.10. The Refined Profile generalizes the notion of standing waves, which are generated from the Refined Profile setting

$\phi_j(z_j) := \phi(z_j\mathbf{e}_j)$ for $z_j \in \mathcal{B}_{\mathbb{C}}(0,\delta_s)$ and $\mathbf{e}_j = (\delta_{1j}, ..., \delta_{Nj})$, with δ_{jk} the Kronecker delta.

It represents an effective way to represent the discrete component because it provides a modulation $u = \phi[\mathbf{z}] + \eta$ of the solution u, where in the equation of the continuous mode η, see (2.5) below, there are no monomials $\mathbf{z}^{\mathbf{m}}$ with $\mathbf{m} \in \mathbf{NR}_1$. It plays an analogous role to that of Fraiman's like ansatz in papers like Merle and Raphael [45] where it allows to bypass normal forms arguments in the course of the analysis of the Fermi Golden Rules.

1.1.2. The repulsivity hypothesis In order to use the dispersion theory of Kowalczyk *et al.* [27–30] we need the following, inspired by a more general notion in [30].

Definition 1.11. Let $\mathcal{V}$ be a potential like in (1.2). We say that $\mathcal{V}$ is repulsive if $\mathcal{V}$ is not identically zero and $x\mathcal{V}'(x) \le 0$ for all $x \in \mathbb{R}$.

Obviously the above notion, framed in terms of the origin, could be reframed with respect to any other $x_0 \in \mathbb{R}$, but we can always normalize by translation so that $x_0 = 0$.

Crucial in the theory in Kowalczyk *et al.* [29,30] is a mechanism of addition or subtraction of eigenvalues which can be traced to Darboux. Kowalczyk *et al.* [29,30] treat some very special situations and refer to the theory in Sect. 3.2–3.3 in Chang–Gustafson–Nakanishi–Tsai [6]. In reality, a systematic and quite general treatment of this topic is in Sect. 3 Deift–Truwobitz [16], to which we refer for the following, where we impose much stricter hypotheses than in [16].

Proposition 1.12. *Let $W \in \mathcal{S}(\mathbb{R}, \mathbb{R})$ s.t $\sigma_{\mathrm{d}}(-\partial_x^2 + W) \neq \emptyset$. Let ψ be the ground state (positive eigenfunction) of $-\partial_x^2 + W$ and set $A_W = \frac{1}{\psi}\partial_x(\psi\cdot)$. Then, there exists $W_1 \in \mathcal{S}(\mathbb{R}, \mathbb{R})$ s.t.*

$$A_W A_W^* = -\partial_x^2 + W - \omega, \ \ A_W^* A_W = -\partial_x^2 + W_1 - \omega,$$

where $\omega = \min \sigma_{\mathrm{d}}(-\partial_x^2 + W)$. Further, we have $\sigma_{\mathrm{d}}(-\partial_x^2 + W_1) = \sigma_{\mathrm{d}}(-\partial_x^2 + W) \setminus \{\omega\}$.

Using Proposition 1.12, we inductively define $V_j \in \mathcal{S}(\mathbb{R}, \mathbb{R})$ ($j = 1, \cdots, N+1$) by

1. $V_1 := V$, $H_1 := -\partial_x^2 + V_1$, $\psi_1 = \phi_1$ and $A_1 = A_{V_1}$.
2. Given V_k, we define

$$A_k := A_{V_k} \text{ and } H_{k+1} := -\partial_x^2 + V_{k+1} := A_k^* A_k + \omega_k. \tag{1.22}$$

From Proposition 1.12, we have

$$\sigma_{\mathrm{d}}(H_k) = \{\omega_j \mid j = k, \cdots, N\}, \ k = 1, \cdots, N, \text{ and } \sigma_{\mathrm{d}}(H_{N+1}) = \emptyset.$$

If ψ_k is the ground state of H_k and $A_k = \frac{1}{\psi_k}\partial_x(\psi_k\cdot)$ then, from

$$A_j^* H_j = A_j^*(A_j A_j^* + \omega_j) = (A_j^* A_j + \omega_j)A_j^* = H_{j+1}A_j^*, \tag{1.23}$$

we have the conjugation relation

$$\mathcal{A}^* H_1 = H_{N+1}\mathcal{A}^*, \tag{1.24}$$

where

$$\mathcal{A} = A_1 \cdots A_N \text{ and } \mathcal{A}^* = A_N^* \cdots A_1^*. \tag{1.25}$$

The crucial assumption to implement the theory in Kowalczyk *et al.* [29,30] and to overcome the strength of the cubic nonlinearity in 1D is the following.

Assumption 1.13. V_{N+1} is a repulsive potential in the sense of Definition 1.11.

Remark 1.14. By reverting the transformation given in Proposition 1.12, starting from any repulsive potential, for any N, one can construct a potential V satisfying Assumptions 1.3 and 1.13.

1.2. Main theorem. We are now in the position to state the main theorem of this paper.

Theorem 1.15. *Assume (1.2)–(1.3) and Assumptions 1.3, 1.8 and 1.13. Then for any* $\epsilon > 0$ *and* $a > 0$ *there exists* $\delta_0 > 0$ *s.t. for all* $u_0 \in H^1$ *with* $\|u_0\|_{H^1} := \delta < \delta_0$ *there are* $\mathbf{z} \in C^1(\mathbb{R}, \mathbb{C}^N)$*, and* $\eta \in C(\mathbb{R}, H^1)$ *s.t.*

$$\|\mathbf{z}\|_{W^{1,\infty}(\mathbb{R})} + \|\eta\|_{L^\infty(\mathbb{R}, H^1)} \lesssim \delta, \tag{1.26}$$

with, for all $t \geq 0$*,*

$$u(t) = \phi[\mathbf{z}(t)] + \eta(t). \tag{1.27}$$

Moreover, we have for $I = [0, \infty)$

$$\|\dot{\mathbf{z}} + \mathrm{i}\boldsymbol{\varpi}(\mathbf{z})\mathbf{z}\|_{L^2(I)} + \sum_{\mathbf{m} \in \mathbf{R}_{\min}} \|\mathbf{z}^{\mathbf{m}}\|_{L^2(I)} + \|\eta\|_{L^2(I, H^1_{-a})} \leq \epsilon. \tag{1.28}$$

Finally, there exists $j(u_0) \in \{1, \cdots, N\}$ *and* $\rho_+(u_0) \geq 0$ *such that*

$$\lim_{t \to \infty} |z_k(t)| = \begin{cases} 0 & k \neq j(u_0), \\ \rho_+(u_0) & k = j(u_0). \end{cases} \tag{1.29}$$

Remark 1.16. The fact that in the paper the cubic term is defocusing plays no role in our proof. Theorem 1.2 holds also with a focusing cubic term. Obviously, inverting time we conclude that (1.28) holds for $I = \mathbb{R}$. Notice also that it is not possible to prove decay rates because all the estimates need necessarily to be invariant by translation in time. Hence, all the literature which proves a rate of decay in time needs to take initial data in spaces smaller than $H^1(\mathbb{R})$.

The novel difficulties in Theorem 1.15 come from the cubic nonlinearity, which in dimension 1 is *long range*. For quintic or higher power, but always smooth, nonlinearities, which are short range, then the theory in [9–11,46] can be applied.

Assumption 1.13 is very important in the theory developed by Kowalczyk *et al.* in [29,30]. Here we are able to generalize their ideas thanks to the theory by Deift and Trubowitz [16, Sect. 3], which treats in great generality the transformations considered in Chang *et al.* [6]. It is then possible to develop the two virial inequalities of Kowalczyk *et al.* in [29] in the presence of general discrete spectra and to combine the theory of dispersion in [29] with the theory of nonlinear dissipation of the discrete modes in [9,10]. The latter was initiated by Buslaev and Perelman [4], was then considered by Soffer and Weinstein [51] and generalized in papers like [11]. The approach in [9,10], to which we refer for a more detailed discussion on this point, is far simpler than the previous literature, thanks to the notion of Refined Profile.

Recently, there has been a considerable interest on the asymptotic stability of patterns for dispersive equations with inhomogeneities and with long range nonlinearities in 1D, especially in view of the analysis of kinks of appropriate wave equations.

Kowalczyk *et al.* showed that a framework based on virial inequalities is very suitable and provides a very penetrating grasp of these problems, starting with their partial proof in [27] of the asymptotic stability of the kinks of the ϕ^4 model, with other insightful contributions in papers like [29,30]. See also the work of Alammari and Snelson [1,2,50] and of Martinez for long range Schrödinger and Hartree equations [39] and for Zakharov systems [40] in one dimension.

Quite different set ups from that of Kowalczyk *et al.* are in [8,24,25,41,48], in the absence of discrete modes. See also the series [32–37] and [54]. Recently Chen [7] considered the case of our NLS (1.1) with one eigenvalue mode, that is $N = 1$, without the repulsivity Assumption 1.13 while the book by Delort and Masmoudi [20] looked at the ϕ^4 model for long times but not asymptotically. The methods employed in all these works need yet to be tested in the case where there are more than one discrete modes, which potentially will slow the decay, see Gang Zhou and Sigal [22], creating additional difficulties. For the literature which uses dispersive estimates in the context of short range nonlinearities, the case $N = 2$ is significantly more complicated than the case $N = 1$, see Soffer and Weinstein [52], the series by Tsai and Yau [55–58] and, for special situations with $N > 2$, Nakanishi et al. [47]. This seems to be related to the need of using different weighted norms as the solution evolves through different stages. More general spectra than the special ones in these references are likely to complicate this kind of analysis. Obviously long range nonlinearities will add further complications.

Our main contribution in this paper involves the use of the notion of Refined Profile, which is significant only in the case of two or more eigenvalues, and so is not relevant to the problem considered in Chen [7] (where however, if the potential is repulsive after a Darboux transformation, the virial inequality argument provides an alternative proof of dispersion). As we show, the Refined Profile notion allows to avoid normal forms arguments and the search of canonical coordinates. As we have shown also elsewhere in [10] and expecially in [9], we provide a very simple alternative to [55–58], [52] and to the more general [11]. The additional complication here, compared to [9,10] is the long range nonlinearity, which does not allow to treat dispersion with a simple perturbation argument.

To prove dispersion we follow the framework of the virial inequalities of Kowalczyk *et al.* [29] which, while subtle, is simple and robust and, as a consequence, is shown here to apply easily in contexts with complicated spectra. Unfortunately, an important limitation is the repulsivity Assumption 1.13. Kowalczyk *et al.* [30] discuss how to avoid it in some cases, but we do not consider here the analogous situations. Our NLS problem is in some respect simpler than kink problems because virial inequalities like in [30], which involve three distinct functionals, are more complicated than the single one that suffices here.

For alternative proofs of dispersion, we notice that a rather simple framework is due to Ze Li [31], but the argument does not apply to the cubic nonlinearity and it too, needs to be tested in the presence of eigenvalues. Similar limitations are true for [15]. We do not discuss here the nonlinear Steepest Descent method of Deift and Zhou initiated in [17], which has been used extensively for integrable systems but for non integrable systems, to our knowledge, only in [18]. There is a large literature on PDE methods in the context of integrable systems. Here we mention only the recent paper on kinks of sine-Gordon by Lührmann and Schlag [38] and the paper on the black soliton for defocusing cubic NLS by Gravejat and Smets [23].

In the context of stability problems of ground states of the NLS, virial inequalities were introduced by Merle and Raphael [42–45]. The papers by Kowalczyk *et al.* [27, 29,30] developed further applications of virial inequalities in stability problems. In this paper we show that the theory can be applied in a relatively elementary fashion also in the presence of any number $N \geq 2$ of eigenvalues. The case $N = 1$ with Assumption 1.13 should be analogous to [29]. An analogue of Soffer and Weinstein [51], or of the more general [3], with Assumption 1.13, should hold in 1D for real valued solutions of the Nonlinear Klein Gordon with a quadratic nonlinearity, using arguments similar to the ones of this paper, which should simplify greatly [3,51] with the use of Refined Profiles. The same should hold for an analogue of [13] for the unitary invariant NLKG in 1D where there are small complex valued standing waves.

2. Modulation Coordinate and Transformed Problem

In this section we write the equation in modulation coordinates and consider the transformed problem induced by the conjugation relation (1.24). We start from the modulation coordinate. First of all, let

$$(u, v) := \int_{\mathbb{R}} u\overline{v}\, dx, \ \ \langle u, v\rangle := \Re\,(u, v)\,, \tag{2.1}$$

and set

$$P_{\mathrm{d}} := \sum_{j=1}^{N} \left(\cdot, \phi_j\right) \phi_j, \ \ P_{\mathrm{c}} := 1 - P_{\mathrm{d}}. \tag{2.2}$$

Then, the space $P_c L^2$ is the continuous space w.r.t. H. Recalling the refined profile $\phi[\mathbf{z}]$ from Proposition 1.9, we introduce the following analogue of nonlinear continuous space of Gustafson, Nakanishi and Tsai [26],

$$\mathcal{H}_c[\mathbf{z}] := \{v \in L^2 \mid \forall \widetilde{\mathbf{z}} \in \mathbb{C}^N, \ \langle \mathrm{i}v, D_{\mathbf{z}}\phi[\mathbf{z}]\widetilde{\mathbf{z}}\rangle = 0\}. \tag{2.3}$$

The modulation coordinates are given by decomposing u as follows.

Lemma 2.1. *There exist $\delta > 0$ and $\mathbf{z} \in C^\infty(\mathcal{B}_{\Sigma^{-1}}(0, \delta), \mathbb{C}^N)$ s.t. $\eta(u) := u-\phi[\mathbf{z}(u)] \in \mathcal{H}_c[\mathbf{z}(u)]$. Further, we have*

$$\|\mathbf{z}(u)\| + \|\eta(u)\|_{H^1} \sim \|u\|_{H^1}. \tag{2.4}$$

Proof. The proof is standard, so we omit it. □

In the following we write $\mathbf{z} = \mathbf{z}(u)$ and $\eta = \eta(u)$. Notice that the bound (1.26) is a straightforward consequence of Mass and Energy conservation, which imply $\|u\|_{H^1} \lesssim \delta$, and of (2.4).

Substituting $u = \phi[\mathbf{z}] + \eta$ in (1.1) and using (1.20), we obtain

$$\mathrm{i}\partial_t\eta + \mathrm{i}D_{\mathbf{z}}\phi[\mathbf{z}]\,(\dot{\mathbf{z}} + \mathrm{i}\boldsymbol{\varpi}(\mathbf{z})\mathbf{z}) = H[\mathbf{z}]\eta + \sum_{\mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}} G_{\mathbf{m}} + \mathcal{R}_{\mathrm{rp}}[\mathbf{z}] + F[\mathbf{z}, \eta] + |\eta|^2\eta, \tag{2.5}$$

where $H[\mathbf{z}] := H + L[\mathbf{z}]$,

$$L[\mathbf{z}] := 2|\phi[\mathbf{z}]|^2 + \phi[\mathbf{z}]^2\mathrm{C}, \ \text{with the complex conjugation operator } \mathrm{C}u = \overline{u} \text{ and} \tag{2.6}$$

$$F[\mathbf{z}, \eta] := 2\phi[\mathbf{z}]|\eta|^2 + \overline{\phi[\mathbf{z}]}\eta^2. \tag{2.7}$$

Following Gustafson, Nakanishi and Tsai [26], we can construct an inverse of P_c on $\mathcal{H}_c[\mathbf{z}]$.

Lemma 2.2. *There exists* $\delta > 0$ *s.t. there exists* $\{a_{jA}\}_{j=1,\cdots,N,A=\mathrm{R},\mathrm{I}} \in C^\infty(B_{\mathbb{C}^N(0,\delta)}, \Sigma^1)$ *s.t.*

$$\|a_{jA}(\mathbf{z})\|_{\Sigma^1} \lesssim \|\mathbf{z}\|^2, \ j = 1, \cdots, N, \ A = \mathrm{R}, \mathrm{I} \tag{2.8}$$

and

$$R[\mathbf{z}] := \mathrm{Id} - \sum_{j=1}^{N} \left(\langle \cdot, a_{j\mathrm{R}}(\mathbf{z}) \rangle \phi_j + \langle \cdot, a_{j\mathrm{I}}(\mathbf{z}) \rangle \mathrm{i}\phi_j \right), \tag{2.9}$$

satisfies $R[\mathbf{z}] P_c|_{\mathcal{H}_c[\mathbf{z}]} = \mathrm{Id}|_{\mathcal{H}_c[\mathbf{z}]}$, $P_c R[\mathbf{z}]|_{P_c L^2} = \mathrm{Id}|_{P_c L^2}$.

Proof. The proof, which we skip, is an analogue of that in [11], which in turn generalizes the one in [26]. □

We set $\widetilde{\eta} := P_c \eta$. Then, by Lemma 2.2 we have $\eta = R[\mathbf{z}]\widetilde{\eta}$ and furthermore, from the estimate (2.8), we have

$$\|\eta\|_{H^s_a} \sim \|\widetilde{\eta}\|_{H^s_a}, \tag{2.10}$$

for $s = 0, 1$ and $|a| \le a_1$. Applying P_c to (2.5), we obtain

$$\mathrm{i}\partial_t \widetilde{\eta} = H\widetilde{\eta} + \sum_{\mathbf{m} \in \mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}} P_c G_{\mathbf{m}} + \mathcal{R}_{\widetilde{\eta}}, \tag{2.11}$$

where

$$\mathcal{R}_{\widetilde{\eta}} = P_c \left(-\mathrm{i} D_{\mathbf{z}}\phi[\mathbf{z}] \left(\dot{\mathbf{z}} + \mathrm{i}\boldsymbol{\varpi}(\mathbf{z})\mathbf{z} \right) + \mathcal{R}_{\mathrm{rp}}[\mathbf{z}] + F[\mathbf{z}, \eta] + L[\mathbf{z}]\eta + |\eta|^2 \eta \right). \tag{2.12}$$

The rest of this section is framed like in [29].
We will consider constants $A, B, \varepsilon > 0$ satisfying

$$\log(\delta^{-1}) \gg \log(\epsilon^{-1}) \gg A \gg B^2 \gg B \gg \exp\left(\varepsilon^{-1}\right) \gg 1. \tag{2.13}$$

We will denote by $o_\varepsilon(1)$ constants depending on ε such that

$$o_\varepsilon(1) \xrightarrow{\varepsilon \to 0^+} 0. \tag{2.14}$$

For the two virial inequalities, we will use the approximations in [29] of $2^{-1} \langle \mathrm{i}\,(1/2 + x\partial_x)\, u(t), u(t) \rangle$, which is the quantized analogue of the form $2^{-1} x \cdot \xi$ for a finite dimensional hamiltonian system $\dot{x} = \nabla_\xi E$ and $\dot{\xi} = -\nabla_x E$ with energy $2^{-1}|\xi|^2 + V(x)$ (recall that $\frac{d}{dt}(2^{-1} x \cdot \xi) = |\xi|^2 - x \cdot \nabla V(x)$ where, if V is repulsive as in Definition 1.11, then $x \cdot \xi$ is strictly increasing for all t: this simple classical argument explains the heuristics around the notion of Virial Inequalities).
The first virial inequality, Sect. 4, involves a truncation of the function x outside an interval of size $\sim A^{-1}$ around 0. The fact that the initial potential $V = V_1$ is obviously not repulsive, makes necessary another virial inequality. This involves applying the operator $\mathcal{A}^*$ to (2.11), see (1.25), in order to exploit the conjugation (1.24), which transforms H_1 into the repulsive operator H_{N+1}. However, to offset the loss of regularity due to $\mathcal{A}^*$, which is a differential operator of order N, it is necessary to use a regularization and consider

$$\mathcal{T} := \langle \mathrm{i}\varepsilon\partial_x \rangle^{-N} \mathcal{A}^*, \tag{2.15}$$

for $\varepsilon > 0$. However, this does not work either, because symmetries of the nonlinear term $P_c(|\eta|^2\eta)$, used to get estimates like (4.18) below, do not hold any more. This is why the argument considers $\chi \in C_c^\infty(\mathbb{R}, \mathbb{R})$ such that

$$x\chi'(x) \le 0 \ \text{ and } 1_{|x|\le 1} \le \chi \le 1_{|x|\le 2}, \tag{2.16}$$

and $\chi_C := \chi(\cdot/C)$ for $C > 0$. Then multiplying equation (2.11) by χ_{B^2}, obtaining

$$\mathrm{i}\partial_t(\chi_{B^2}\widetilde{\eta}) = H(\chi_{B^2}\widetilde{\eta}) + \left(2\chi'_{B^2}\partial_x + \chi''_{B^2}\right)\tilde{\eta} + \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}}\chi_{B^2}P_cG_{\mathbf{m}} + \chi_{B^2}\mathcal{R}_{\widetilde{\eta}}, \tag{2.17}$$

setting

$$v := \mathcal{T}\chi_{B^2}\widetilde{\eta}, \tag{2.18}$$

and applying $\mathcal{T}$ to (2.17), we obtain

$$\mathrm{i}\partial_t v = H_{N+1}v + \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}}\widetilde{G}_{\mathbf{m}} + \mathcal{R}_v, \tag{2.19}$$

where

$$\widetilde{G}_{\mathbf{m}} := \mathcal{T}\chi_{B^2}P_cG_{\mathbf{m}}, \tag{2.20}$$

$$\mathcal{R}_v := \mathcal{T}\chi_{B^2}\mathcal{R}_{\widetilde{\eta}} + \langle \mathrm{i}\varepsilon\partial_x\rangle^{-N}[V_{N+1}, \langle \mathrm{i}\varepsilon\partial_x\rangle^N]v + \mathcal{T}\left(2\chi'_{B^2}\partial_x + \chi''_{B^2}\right)\tilde{\eta}. \tag{2.21}$$

Here it is possible to apply the second virial inequality, which involves a truncation of x in an interval centered in 0 of size $\sim B$. The technical fact that $A \gg B^2$ is required to work out the argument, see also [29].

3. The Continuation Argument

The proof of (1.28) in Theorem 1.15 is by means of a continuation argument. In particular, we will show the following.

Proposition 3.1. *There exists a $\delta_0 = \delta_0(\epsilon)$ s.t. if (1.28) holds for $I = [0, T]$ for some $T > 0$ and for $\delta \in (0, \delta_0)$ then in fact for $I = [0, T]$ inequality (1.28) holds for ϵ replaced by $\epsilon/2$.*

By completely routine arguments, which we skip, it is possible to show that Proposition 3.1 implies (1.28) with $I = [0, \infty)$. We reformulate the continuation argument. Let $a > 0$ be the one given in Theorem 1.15. Without loss of generality we can assume $a \le 2^{-1}\min(a_0, a_1)$ where a_0 is given in (1.2) and a_1 is given in (1.8). We introduce the following norm,

$$\|f\|^2_{\widetilde{\Sigma}} = \left\langle \left(-\partial_x^2 + \operatorname{sech}^2\left(\frac{ax}{10}\right)\right) f, f \right\rangle \sim \|f\|^2_{\dot{H}^1} + \|f\|^2_{L^2_{-\frac{a}{10}}}. \tag{3.1}$$

For $C = A, B$, we set

$$\zeta_C(x) := \exp\left(-\frac{|x|}{C}(1 - \chi(x))\right). \tag{3.2}$$

We consider the main variables in [29], given by

$$w := \zeta_A\widetilde{\eta} \text{ and } \xi := \chi_{B^2}\zeta_B v\ . \tag{3.3}$$

We will prove the following continuation argument.

Proposition 3.2. *For any small $\epsilon > 0$ there exists a $\delta_0 = \delta_0(\epsilon)$ s.t. if in $I = [0, T]$ we have*

$$\|\dot{\mathbf{z}} + \mathrm{i}\boldsymbol{\varpi}(\mathbf{z})\mathbf{z}\|_{L^2(I)} + \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \|\mathbf{z}^{\mathbf{m}}\|_{L^2(I)} + \|\xi\|_{L^2(I,\widetilde{\Sigma})} + \|w\|_{L^2(I,\widetilde{\Sigma})} \leq \epsilon \tag{3.4}$$

then for $\delta \in (0, \delta_0)$ inequality (3.4) holds for ϵ replaced by $o_\delta(1)\epsilon$ where $o_\delta(1) \xrightarrow{\delta\to 0^+} 0$.

Notice that Proposition 3.2 implies Proposition 3.1. In the following, we always assume the assumptions of the claim of Proposition 3.2.

In complete analogy to [29], we consider two virial estimates, one for w and the other for ξ. The first is based directly on the equation for $\widetilde{\eta}$, (2.11).

Proposition 3.3 (Virial estimate for $\widetilde{\eta}$). *We have*

$$\|w'\|_{L^2(I,L^2)} \lesssim A^{1/2}\delta + \|w\|_{L^2(I,L^2_{-\frac{a}{10}})} + \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \|\mathbf{z}^{\mathbf{m}}\|_{L^2(I)} + \epsilon^2. \tag{3.5}$$

The second virial estimate, involves the transformed problem (2.19).

Proposition 3.4 (Virial estimate for v). *Let $A \gg B^2$. We have*

$$\|\xi\|_{L^2(I,\widetilde{\Sigma})} \lesssim B\varepsilon^{-N}\delta + \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \|\mathbf{z}^{\mathbf{m}}\|_{L^2(I)} + o_\varepsilon(1)\left(\|w\|_{L^2(I,\widetilde{\Sigma})} + \|\dot{\mathbf{z}} + \mathrm{i}\boldsymbol{\varpi}(\mathbf{z})\mathbf{z}\|_{L^2(I)}\right). \tag{3.6}$$

The term $\dot{\mathbf{z}} + \mathrm{i}\varpi(\mathbf{z})\mathbf{z}$ can be controlled in term of the $\mathbf{z}^{\mathbf{m}}$, for $\mathbf{m} \in \mathbf{R}_{\min}$.

Proposition 3.5. *We have*

$$\|\dot{\mathbf{z}} + \mathrm{i}\varpi(\mathbf{z})\mathbf{z}\|_{L^2(I)} \lesssim \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \|\mathbf{z}^{\mathbf{m}}\|_{L^2(I)} + \delta^2\epsilon. \tag{3.7}$$

To bound the $\mathbf{z}^{\mathbf{m}}$, for $\mathbf{m} \in \mathbf{R}_{\min}$, we will use the Fermi Golden Rule.

Proposition 3.6 (FGR estimate). *We have*

$$\sum_{\mathbf{m}\in\mathbf{R}_{\min}} \|\mathbf{z}^{\mathbf{m}}\|_{L^2(I)} \lesssim \varepsilon^{-N} B^{2+2\tau}\delta + B^{-\frac{1}{2}}\epsilon + \epsilon^2. \tag{3.8}$$

In Sect. 4 we prove Proposition 3.3. In Sect. 6 we prove Proposition 3.4. Sections 4–6 are very close to [29]. Section 9 is the analogue of [29, Sect. 5.1]. The proofs of Propositions 3.5 and 3.6 are very similar to the discussion in [9]. As is in [4,51] and many other papers, most referenced in [11], at some point the continuous mode, in fact in this paper the variable v, needs to be decomposed in a part which resonates with the discrete mode $\mathbf{z}$ and a remainder which is supposed to be very small, and which we denote by g, see (8.6). To bound g we use Kato smoothing estimates, as in the previous literature. So, for example, Lemmas 8.3 and 8.4 are a typical tool, see [4,51] or the references in [11]. Some attention is needed in the use of the weights, to guarantee that some of the terms, i.e. the term in (8.24), are small. Finally, in Sect. 12, we prove the last sentence of Theorem 1.15.

4. Proof of Proposition 3.3

In this section, we prove the 1st virial estimate Proposition 3.3, which is a consequence of the estimate of the time derivative of the following functional,

$$\mathcal{I}(\widetilde{\eta}) := \frac{1}{2}\left\langle \widetilde{\eta}, \mathrm{i} S_A \widetilde{\eta}\right\rangle, \tag{4.1}$$

where the anti-symmetric operator S_A is defined by

$$S_A := \frac{\varphi_A'}{2} + \varphi_A \partial_x, \text{ where } \varphi_A(x) = \int_0^x \zeta_A^2(y)\, dy. \tag{4.2}$$

Proposition 3.3 is a direct consequence of the following estimate

Proposition 4.1 (1st virial estimate in differential form). *Under the assumptions of Proposition 3.1, for sufficiently small $\delta > 0$ we have*

$$\frac{d}{dt}\mathcal{I}(\widetilde{\eta}) + \frac{1}{2}\|w'\|_{L^2}^2 \lesssim \|w\|_{L^2_{-\frac{a}{10}}}^2 + \sum_{\mathbf{m}\in\mathbf{R}_{\min}} |\mathbf{z}^{\mathbf{m}}|^2 + \delta^2\|\dot{\mathbf{z}} + \mathrm{i}\boldsymbol{\varpi}(\mathbf{z})\mathbf{z}\|^2. \tag{4.3}$$

We first prove Proposition 3.3 from Proposition 4.1.

Proof of Proposition 3.3. We have $|\mathcal{I}(\widetilde{\eta})| \lesssim A\|\widetilde{\eta}\|_{H^1}^2 \lesssim A\delta^2$. Thus, integrating (4.3) over $[0, T]$, we obtain (3.5). ☐

The rest of this section is devoted for the proof of Proposition 4.1. First, from (2.11), we have

$$\begin{aligned}\frac{d}{dt}\mathcal{I}(\widetilde{\eta}) &= -\left\langle \mathrm{i}\partial_t\widetilde{\eta}, S_A\widetilde{\eta}\right\rangle \\ &= -\left\langle H\widetilde{\eta}, S_A\widetilde{\eta}\right\rangle - \sum_{\mathbf{m}\in\mathbf{R}_{\min}}\left\langle \mathbf{z}^{\mathbf{m}} P_c G_{\mathbf{m}}, S_A\widetilde{\eta}\right\rangle - \left\langle \mathcal{R}_{\widetilde{\eta}}, S_A\widetilde{\eta}\right\rangle.\end{aligned} \tag{4.4}$$

We will compute each terms in (4.4)

Lemma 4.2. *We have*

$$\left\langle H\widetilde{\eta}, S_A\widetilde{\eta}\right\rangle = \left\langle \left(-\partial_x^2 - \frac{\varphi_A}{2\zeta_A^2}V'\right) w, w\right\rangle + \frac{1}{2A}\left\langle V_0 w, w\right\rangle, \tag{4.5}$$

where

$$V_0(x) := \left(\chi''|x| + 2\chi'\frac{x}{|x|}\right). \tag{4.6}$$

Proof. By direct computation, we have

$$\left\langle H\widetilde{\eta}, S_A\widetilde{\eta}\right\rangle = \left\langle \varphi_A'\partial_x\widetilde{\eta}, \partial_x\widetilde{\eta}\right\rangle - \frac{1}{4}\left\langle \varphi_A'''\widetilde{\eta}, \widetilde{\eta}\right\rangle - \frac{1}{2}\left\langle \widetilde{\eta}, \varphi_A V'\widetilde{\eta}\right\rangle. \tag{4.7}$$

Following Lemma 1 of [29], we have

$$\left\langle \varphi_A'\partial_x\widetilde{\eta}, \partial_x\widetilde{\eta}\right\rangle = \left\langle w', w'\right\rangle + \left\langle \frac{\zeta_A''}{\zeta_A}w, w\right\rangle, \tag{4.8}$$

and

$$-\frac{1}{4}\langle\varphi_A'''\widetilde{\eta},\widetilde{\eta}\rangle=-\frac{1}{2}\left\langle\left(\frac{\zeta_A''}{\zeta_A}+\left(\frac{\zeta_A'}{\zeta_A}\right)^2\right)w,w\right\rangle, \tag{4.9}$$

so that

$$\langle\varphi_A'\partial_x\widetilde{\eta},\partial_x\widetilde{\eta}\rangle-\frac{1}{4}\langle\varphi_A'''\widetilde{\eta},\widetilde{\eta}\rangle=\left\langle-\partial_x^2 w,w\right\rangle+\frac{1}{2}\left\langle\left(\frac{\zeta_A''}{\zeta_A}-\left(\frac{\zeta_A'}{\zeta_A}\right)^2\right)w,w\right\rangle. \tag{4.10}$$

Substituting $w=\zeta_A\widetilde{\eta}$ also in $\langle\widetilde{\eta},\varphi_A V'\widetilde{\eta}\rangle$ and using the identity

$$A\left(\frac{\zeta_A''}{\zeta_A}-\left(\frac{\zeta_A'}{\zeta_A}\right)^2\right)=\chi''(x)|x|+2\chi'(x)\frac{x}{|x|}=V_0(x), \tag{4.11}$$

we obtain (4.5). □

Lemma 4.3. *We have*

$$\left|\langle\mathbf{z}^{\mathbf{m}}P_cG_{\mathbf{m}},S_A\widetilde{\eta}\rangle\right|\lesssim|\mathbf{z}^{\mathbf{m}}|\|w\|_{L^2_{-\frac{a}{10}}}. \tag{4.12}$$

Proof. Since $\|\zeta_A^{-1}S_AP_cG_{\mathbf{m}}\|_{L^2_{\frac{a}{10}}}\lesssim 1$, the conclusion is obvious. □

Lemma 4.4. *We have*

$$|\langle\mathcal{R}_{\widetilde{\eta}},S_A\widetilde{\eta}\rangle|\lesssim\delta^2\left(\left(\sum_{\mathbf{m}\in\mathbf{R}_{\min}}|\mathbf{z}^{\mathbf{m}}|+\|\dot{\mathbf{z}}+\mathrm{i}\boldsymbol{\varpi}(\mathbf{z})\mathbf{z}\|\right)\|w\|_{L^2_{-\frac{a}{10}}}+\|w\|^2_{L^2_{-\frac{a}{10}}}\right)+\delta^{2/3}\|w'\|^2_{L^2}. \tag{4.13}$$

Proof. We will estimate the contribution of each term in $\mathcal{R}_{\widetilde{\eta}}$, see (2.12). First, since, by $D_{\mathbf{z}}\phi[0]\widetilde{\mathbf{z}}=\boldsymbol{\phi}\cdot\widetilde{\mathbf{z}}$

$$\|P_cD_{\mathbf{z}}\phi[\mathbf{z}]\widetilde{\mathbf{z}}\|_{\Sigma^1}=\|P_c\left(D_{\mathbf{z}}\phi[\mathbf{z}]-D_{\mathbf{z}}\phi[0]\right)\widetilde{\mathbf{z}}\|_{\Sigma^1}\lesssim\delta^2\|\widetilde{\mathbf{z}}\|, \tag{4.14}$$

we have

$$|\langle P_c\left(-\mathrm{i}D_{\mathbf{z}}\phi[\mathbf{z}]\left(\dot{\mathbf{z}}+\mathrm{i}\boldsymbol{\varpi}(\mathbf{z})\mathbf{z}\right)\right),S_A\widetilde{\eta}\rangle|\lesssim\delta^2\|\dot{\mathbf{z}}+\mathrm{i}\boldsymbol{\varpi}(\mathbf{z})\mathbf{z}\|\|w\|_{L^2_{-\frac{a}{10}}}. \tag{4.15}$$

Next, from (1.21), we have

$$|\langle P_c\mathcal{R}_{\mathrm{rp}}[\mathbf{z}],S_A\widetilde{\eta}\rangle|\lesssim\delta^2\sum_{\mathbf{m}\in\mathbf{R}_{\min}}|\mathbf{z}^{\mathbf{m}}|\|w\|_{L^2_{-\frac{a}{10}}}. \tag{4.16}$$

We next estimate the contribution of $P_c\left(F[\mathbf{z},\eta]+L[\mathbf{z}]\eta+|\eta|^2\eta\right)$. First, since $P_c=1-P_d$ and $\|P_dS_A\widetilde{\eta}\|_{\Sigma^1}\lesssim\|w\|_{L^2_{-\frac{a}{10}}}$, we have

$$\begin{aligned}|\left\langle P_d\left(F[\mathbf{z},\eta]+L[\mathbf{z}]\eta+|\eta|^2\eta\right),S_A\widetilde{\eta}\right\rangle|&\lesssim\|F[\mathbf{z},\eta]+L[\mathbf{z}]\eta+|\eta|^2\eta\|_{\Sigma^{-1}}\|w\|_{L^2_{-\frac{a}{10}}}\\&\lesssim\delta^2\|w\|^2_{L^2_{-\frac{a}{10}}}.\end{aligned} \tag{4.17}$$

Next, by elementary integration by parts we have

$$\left\langle |\widetilde{\eta}|^2\widetilde{\eta}, S_A\widetilde{\eta}\right\rangle = 2^{-1}\left\langle |\widetilde{\eta}|^4, \varphi_A'\right\rangle + 2^{-2}\left\langle \left(|\widetilde{\eta}|^4\right)', \varphi_A\right\rangle = 4^{-1}\left\langle |\widetilde{\eta}|^4, \zeta_A^2\right\rangle$$

and by [29] and (1.26), see also Lemma 2.7 [12], we have

$$|\left\langle |\widetilde{\eta}|^2\widetilde{\eta}, S_A\widetilde{\eta}\right\rangle| \lesssim \delta^{2/3}\|w'\|_{L^2}^2. \tag{4.18}$$

For the remaining terms, by $\eta = \widetilde{\eta} + \widetilde{\eta}_1$ with $\widetilde{\eta}_1 = (R[z]-1)\widetilde{\eta}$, we can expand

$$\begin{aligned}
&F[\mathbf{z},\eta] + L[\mathbf{z}]\eta + |\eta|^2\eta - |\widetilde{\eta}|^2\widetilde{\eta} = v_0\widetilde{\eta}_1 + v_1\overline{\widetilde{\eta}_1} + v_2\widetilde{\eta} + v_3\overline{\widetilde{\eta}} + v_4\widetilde{\eta}^2 + v_5|\widetilde{\eta}|^2, \text{ where}\\
&v_0 = 2|\phi[\mathbf{z}]|^2 + 2\phi[\mathbf{z}]\overline{\widetilde{\eta}_1} + \overline{\phi[\mathbf{z}]}\widetilde{\eta}_1 + |\widetilde{\eta}_1|^2,\ v_1 = \phi[\mathbf{z}]^2,\ v_2 = 2|\phi[\mathbf{z}]|^2\\
&\quad + 2\phi[\mathbf{z}]\overline{\widetilde{\eta}_1} + \overline{\phi[\mathbf{z}]}\widetilde{\eta}_1 + 2|\widetilde{\eta}_1|,\\
&v_3 = \phi[\mathbf{z}]^2 + 2\phi[\mathbf{z}]\widetilde{\eta}_1 + \widetilde{\eta}_1{}^2,\ v_4 = \overline{\phi[\mathbf{z}]} + \overline{\widetilde{\eta}_1},\ v_5 = 2\phi[\mathbf{z}] + \widetilde{\eta}_1.
\end{aligned}\tag{4.19}$$

By Lemma 2.2, we have $\|\widetilde{\eta}_1\|_{\Sigma^1} \lesssim \|w\|_{L^2_{-\frac{a}{10}}}$ and

$$\|v_j\|_{\Sigma^1} \lesssim \delta^2 \text{ for } j = 0,1,2,3 \text{ and } \|v_j\|_{\Sigma^1} \lesssim \delta \text{ for } j = 4,5. \tag{4.20}$$

Thus, we have

$$|\langle v_0\widetilde{\eta}_1 + v_1\overline{\widetilde{\eta}_1}, S_A\widetilde{\eta}\rangle| \lesssim \|v_0\widetilde{\eta}_1 + v_1\overline{\widetilde{\eta}_1}\|_{L^2_{\frac{a}{5}}}\|w\|_{L^2_{-\frac{a}{10}}} \lesssim \delta^2\|w\|^2_{L^2_{-\frac{a}{10}}}, \tag{4.21}$$

$$|\left\langle v_2\widetilde{\eta} + v_3\overline{\widetilde{\eta}} + v_4\widetilde{\eta}^2 + v_5|\widetilde{\eta}|^2, \varphi_A'\widetilde{\eta}\right\rangle| \lesssim \delta^2\|w\|^2_{L^2_{-\frac{a}{10}}}, \tag{4.22}$$

and

$$|\langle\left(v_2 + v_4\widetilde{\eta} + v_5\overline{\widetilde{\eta}}\right)\widetilde{\eta}, \varphi_A\partial_x\widetilde{\eta}\rangle| = \frac{1}{2}\left|\int \zeta_A^{-2}\partial_x\left(\varphi_A\left(v_2 + v_4\widetilde{\eta} + v_5\overline{\widetilde{\eta}}\right)\right)|w|^2\right| \lesssim \delta^2\|w\|^2_{L^2_{-\frac{a}{10}}}, \tag{4.23}$$

$$|\langle v_3\overline{\widetilde{\eta}}, \varphi_A\partial_x\widetilde{\eta}\rangle| = \frac{1}{2}\left|\int \zeta_A^{-2}\partial_x\left(\varphi_A v_3\right)\overline{w}^2\right| \lesssim \delta^2\|w\|^2_{L^2_{-\frac{a}{10}}}. \tag{4.24}$$

Combining (4.17), (4.18), (4.21), (4.22), (4.23) and (4.24) we obtain

$$|\left\langle F[\mathbf{z},\eta] + L[\mathbf{z}]\eta + |\eta|^2\eta - |\widetilde{\eta}|^2\widetilde{\eta}, S_A\widetilde{\eta}\right\rangle| \lesssim \delta^2\|w\|^2_{L^2_{-\frac{a}{10}}} + \delta^{2/3}\|w'\|_{L^2}. \tag{4.25}$$

Therefore, from (4.15), (4.16) and (4.25) we have the conclusion. □

Proof of Proposition 4.1. By (4.4), Lemmas 4.2, 4.3 and 4.4 we obtain the estimate (4.3) for sufficiently small $\delta > 0$ and large A, satisfying (2.13). □

Before proving Proposition 3.4 we need some technical preliminaries, which we state in Sect. 5.

5. Technical Estimates

In this section, we collect estimates used in the sequel of the paper.

Lemma 5.1. *Let $U \geq 0$ be a non–zero potential $U \in L^1(\mathbb{R}, \mathbb{R})$. Then there exists a constant $C_U > 0$ such that for any function $0 \leq W$ such that $\langle x \rangle W \in L^1(\mathbb{R})$ then*

$$\langle Wf, f \rangle \leq C_U \| \langle x \rangle W \|_{L^1(\mathbb{R})} \left\langle (-\partial_x^2 + U)f, f \right\rangle. \tag{5.1}$$

In particular, for $a > 0$ the constant in the norm $\| \cdot \|_{\widetilde{\Sigma}}$ in (3.1), there exists a constant $C(a) > 0$ such that

$$\langle Wf, f \rangle \leq C(a) \| \langle x \rangle W \|_{L^1(\mathbb{R})} \| f \|_{\widetilde{\Sigma}}^2. \tag{5.2}$$

Proof. Let J be a compact interval where $I_U := \int_J U(x)dx > 0$. Let then $x_0 \in J$ s.t.

$$|f(x_0)|^2 \leq I_U^{-1} \int_J |f(x)|^2 U(x) dx.$$

Then,

$$|f(x)| \leq |x - x_0|^{\frac{1}{2}} \|f'\|_{L^2(\mathbb{R})} + |f(x_0)| \leq |x - x_0|^{\frac{1}{2}} \|f'\|_{L^2(\mathbb{R})} + I_U^{-1/2} \langle Uf, f \rangle^{\frac{1}{2}}.$$

Taking second power and multiplying by W it is easy to conclude the following, which after integration yields (5.1),

$$W(x)|f(x)|^2 \leq C_U \langle x \rangle W(x) \left\langle (-\partial_x^2 + U)f, f \right\rangle \text{ where } C_U = 2\left(1 + |x_0| + I_U^{-1}\right).$$

□

A direct consequence of Lemma 5.1 is the following. Recall $A \gg B^2 \gg B \gg a^{-1}$, with a like in the previous lemma.

Corollary 5.2. *We have*

$$\|\widetilde{\eta}\|_{L^2_{-\frac{a}{10}}} + \|w\|_{L^2_{-\frac{a}{20}}} \lesssim \|w\|_{\widetilde{\Sigma}} \text{ and } \|\zeta_B^{-1}\xi\|_{L^2_{-\frac{a}{10}}} \lesssim \|\xi\|_{\widetilde{\Sigma}}.$$

We will use the following standard fact.

Lemma 5.3. *Consider a 0 order Pseudodifferential Operator (ΨDO)*

$$p(x, \mathrm{i}\partial_x)f(x) = \int_{\mathbb{R}^2} e^{-\mathrm{i}k(x-y)} p(x,k) f(y) dk dy \tag{5.3}$$

with symbol $p(x,k)$ such that

$$|\partial_x^\alpha \partial_k^\beta p(x,k)| \leq C_{\alpha\beta} \langle k \rangle^{-\beta} \text{ for all } (x,k) \in \mathbb{R}^2 \text{ and for all } (\alpha, \beta). \tag{5.4}$$

Then, for any $m \in \mathbb{R}$ and for any $\varepsilon \in (0, 1]$

$$\| \langle \varepsilon x \rangle^{-m} p(\varepsilon x, \mathrm{i}\partial_x) f \|_{L^2(\mathbb{R})} \leq C_m \| \langle \varepsilon x \rangle^{-m} f \|_{L^2(\mathbb{R})} \text{ for all } f \in L^2(\mathbb{R}), \tag{5.5}$$

where each constant C_m depends on finitely many of the constants $C_{\alpha\beta}$ and is independent from $\varepsilon \in (0, 1]$.

Proof (sketch). We write

$$p(\varepsilon x, \mathrm{i}\partial_x) f(x) = P_1 f + P_2 f$$
$$P_j f(x) = \int_{\mathbb{R}^2} e^{\mathrm{i}k(x-x')} p(\varepsilon x, k) \chi_j \left(x - x'\right) f(x') dk dx', \tag{5.6}$$

with $\chi_1 \in C_c^\infty(\mathbb{R}, [0, 1])$ a cutoff with $\chi_1 = 1$ near 0 and with $\chi_2 := 1 - \chi_1$. Then

$$\| \left(\langle \varepsilon x\rangle^{-m} P_1 \langle \varepsilon x\rangle^{m}\right) \langle \varepsilon x\rangle^{-m} f \|_{L^2(\mathbb{R})} \lesssim \| \langle \varepsilon x\rangle^{-m} f \|_{L^2(\mathbb{R})},$$

because $\langle \varepsilon x\rangle^{-m} P_1 \left\langle \varepsilon x'\right\rangle^{m}$ is a ΨDO with symbol

$$p(\varepsilon x, k) \chi_1 \left(x - x'\right) \langle \varepsilon x\rangle^{-m} \left\langle \varepsilon x'\right\rangle^{m} \in S^0_{1,0,0}(\mathbb{R} \times \mathbb{R} \times \mathbb{R}),$$

see Definition 3.5 p. 43 [59], with, for fixed constants $C_{\alpha\alpha'\beta}$,

$$\left| \partial_x^\alpha \partial_{x'}^{\alpha'} \partial_k^\beta \left(p(\varepsilon x, k) \chi_1 \left(x - x'\right) \langle \varepsilon x\rangle^{-m} \left\langle \varepsilon x'\right\rangle^{m}\right)\right| \le C_{\alpha\alpha'\beta} \langle k\rangle^{-\beta},$$

for all $(\alpha, \alpha', \beta, x, x', k)$ and all $\varepsilon \in (0, 1]$. Then, by the theory in Ch. II [59], this ΨDO defines an operator from $L^2(\mathbb{R})$ into itself whose norm can be bounded in terms of finitely many of the $C_{\alpha\alpha'\beta}$, and so has a finite upper bound independent from $\varepsilon \in (0, 1]$. We also have, integrating by parts with respect to k in (5.6) for $j = 2$,

$$\begin{aligned} \| \langle \varepsilon x\rangle^{-m} P_2 f \|_{L^2(\mathbb{R})} &\lesssim \left\| \langle \varepsilon x\rangle^{-m} \int_{\mathbb{R}} \left\langle x - x'\right\rangle^{-10-m} |f(x')| dx' \right\|_{L^2(\mathbb{R})} \\ &\lesssim \left\| \int_{\mathbb{R}} \left\langle x - x'\right\rangle^{-10} \left\langle \varepsilon x'\right\rangle^{-m} |f(x')| dx' \right\|_{L^2(\mathbb{R})} \le \| \langle x\rangle^{-10} \|_{L^1(\mathbb{R})} \| \langle \varepsilon x\rangle^{-m} f \|_{L^2(\mathbb{R})}, \end{aligned}$$

where the last inequality follows from Young's inequality for convolutions, and all the constants are independent from $\varepsilon \in (0, 1]$. □

Following [29] we will will use a regularizing operator $\langle \mathrm{i}\varepsilon\partial_x\rangle^{-N}$, with $\varepsilon > 0$ a small constant. We will use the following lemma.

Lemma 5.4. *Consider a Schwartz function $\mathcal{V} \in \mathcal{S}(\mathbb{R}, \mathbb{C})$. Then, for any $L \in \mathbb{N} \cup \{0\}$ there exists a constant C_L s.t. we have for all $\varepsilon \in (0, 1]$*

$$\| \langle \mathrm{i}\varepsilon\partial_x\rangle^{-N} [\mathcal{V}, \langle \mathrm{i}\varepsilon\partial_x\rangle^{N}] \|_{L^{2,-L}(\mathbb{R}) \to L^{2,L}(\mathbb{R})} \le C_L \varepsilon, \tag{5.7}$$

where $L^{2,s}(\mathbb{R})$ is defined in (1.9).

Proof. Let us consider case $L = 0$,

$$\left\| \langle \mathrm{i}\varepsilon\partial_x\rangle^{-N} \left[\mathcal{V}, \langle \mathrm{i}\varepsilon\partial_x\rangle^{N}\right] \right\|_{L^2(\mathbb{R}) \to L^2(\mathbb{R})} \lesssim \varepsilon. \tag{5.8}$$

Taking Fourier transform, it is equivalent to prove the above $L^2 \to L^2$ bound for the operator

$$\int_{\mathbb{R}} H(k, \ell) f(\ell) d\ell \text{ with } H(k, \ell) = \langle \varepsilon k\rangle^{-N} \widehat{\mathcal{V}}(k - \ell) \left(\langle \varepsilon k\rangle^{N} - \langle \varepsilon \ell\rangle^{N}\right).$$

Multiplying numerator and denominator by $\langle \varepsilon k\rangle^N + \langle \varepsilon \ell\rangle^N$ we obtain $H(k,\ell) = \varepsilon \widetilde{H}(k,\ell)$

$$\widetilde{H}(k,\ell) = \langle \varepsilon k\rangle^{-N} \widehat{\mathcal{V}}(k-\ell)(k-\ell)\frac{P(\varepsilon k, \varepsilon \ell)}{\langle \varepsilon k\rangle^N + \langle \varepsilon \ell\rangle^N}$$

where P is a $2N-1$ degree polynomial. It is elementary that the integral operator with integral kernel $\widetilde{H}(k,\ell)$ is uniformly bounded in ε from $L^p(\mathbb{R})$ to itself, for any $p \in [1,\infty]$, by Young's inequality, see Theorem 0.3.1 [53]. Indeed, it is enough to prove that there exists a constant $C > 0$ independent from small $\varepsilon > 0$ s.t.

$$\sup_{k\in\mathbb{R}} \int_{\mathbb{R}} |\widetilde{H}(k,\ell)| d\ell < C, \tag{5.9}$$

since by symmetry a similar bound can be proved interchanging the role of k and ℓ. Now, for $M \geq N+1$, we have for fixed k

$$\begin{aligned}\int_{\mathbb{R}} |\widetilde{H}(k,\ell)| d\ell \lesssim & \int_{|\ell|\in\left[\frac{|k|}{2}, 2|k|\right]} \langle \varepsilon k\rangle^{-N} \langle k-\ell\rangle^{-M} \left(\langle \varepsilon k\rangle^{N-1} + \langle \varepsilon \ell\rangle^{N-1}\right) d\ell \\ & + \int_{|\ell|\notin\left[\frac{|k|}{2}, 2|k|\right]} \langle \varepsilon k\rangle^{-N} \langle k-\ell\rangle^{-M} \left(\langle \varepsilon k\rangle^{N-1} + \langle \varepsilon \ell\rangle^{N-1}\right) d\ell.\end{aligned}$$

The first integral can be bounded above, for appropriate C_N, by the elementary inequality

$$C_N \int_{|\ell|\in\left[\frac{|k|}{2}, 2|k|\right]} \langle \varepsilon k\rangle^{-1} \langle k-\ell\rangle^{-M} d\ell \leq C_N \int_{\mathbb{R}} \langle \ell\rangle^{-M} d\ell = C_N \| \langle x\rangle^{-M} \|_{L^1(\mathbb{R})},$$

while the second can be bounded above by

$$\begin{aligned}& 2\int_{|\ell|\leq\frac{|k|}{2}} \langle \varepsilon k\rangle^{-N} \frac{\langle \varepsilon k\rangle^{N-1}}{\langle \ell\rangle^M} d\ell + 2\int_{|\ell|\geq 2|k|} \frac{\langle \varepsilon \ell\rangle^{N-1}}{\langle \ell\rangle^M} \\ & \leq 4\int_{\mathbb{R}} \langle \ell\rangle^{N-1-M} d\ell = 4\| \langle x\rangle^{-M-1+N} \|_{L^1(\mathbb{R})},\end{aligned}$$

with all the constants independent from $\varepsilon > 0$. This completes the case $L = 0$.

Let us consider now the case with $L \geq 1$. From the proof in the case $L = 0$, we have

$$\langle \mathrm{i}\varepsilon\partial_x\rangle^{-N} [\mathcal{V}, \langle \mathrm{i}\varepsilon\partial_x\rangle^N] v = \varepsilon \int_{\mathbb{R}} K^0(x,y) v(y) dy$$

where for $\sigma \geq 0$ we set

$$K^\sigma(x,y) = \int_{\mathbb{R}^2} e^{\mathrm{i}xk - \mathrm{i}y\ell} \langle \varepsilon k\rangle^{-N-\sigma} \widehat{\mathcal{V}}(k-\ell)(k-\ell)\frac{P(\varepsilon k, \varepsilon \ell)}{\langle \varepsilon k\rangle^N + \langle \varepsilon \ell\rangle^N} dk d\ell. \tag{5.10}$$

Notice that the integral is absolutely convergent for $\sigma > 0$. Let us consider case $\sigma > 0$. When $|x| \geq 1$, integrating by parts we have, up to constants which we ignore,

$$\begin{aligned}& K^\sigma(x,y) = \frac{1}{x^L} K^\sigma_{1,0}(x,y) \text{ with } K^\sigma_{a,b}(x,y) = \int_{\mathbb{R}} e^{\mathrm{i}xk - \mathrm{i}y\ell} \widetilde{H}_{a,b}(k,\ell) dk d\ell \text{ with} \\ & \widetilde{H}^\sigma_{a,b}(k,\ell) = (\partial_k^L)^a (\partial_\ell^L)^b \left(\langle \varepsilon k\rangle^{-N-\sigma} \widehat{\mathcal{V}}(k-\ell)(k-\ell)\frac{P(\varepsilon k, \varepsilon \ell)}{\langle \varepsilon k\rangle^N + \langle \varepsilon \ell\rangle^N}\right),\end{aligned}$$

where $\widetilde{H}^{\sigma}_{a,b}(k,\ell)$ for $a, b \in \{0, 1\}$ has the properties of $\widetilde{H}(k,\ell)$. Then, for $\chi_0 \in C^{\infty}_c(\mathbb{R}, [0, 1])$ with $\chi_0 = 1$ near 0 and $\chi_1 = 1 - \chi_0$ and ignoring irrelevant constants, we have

$$K^{\sigma}(x, y) = \sum_{a,b=0,1} \chi_a(x)\chi_b(y)x^{-aL}y^{-bL}K^{\sigma}_{a,b}(x, y).$$

Then, for $f_b(y) = \chi_b(y)y^{-bL} f(y)$,

$$\begin{aligned}\left\| \langle x\rangle^L \int K^{\sigma}(x, y) f(y)dy \right\|_{L^2(\mathbb{R})} &\le \sum_{a,b=0,1} \Big\| \langle x\rangle^L x^{-aL}\chi_a(x) \\ &\quad \int K^{\sigma}_{a,b}(x, y)\chi_b(y)y^{-bL} f(y)dy \Big\|_{L^2(\mathbb{R})} \\ &\le \sum_{a,b=0,1} \left\| \int K^{\sigma}_{a,b}(x, y)\chi_b(y)y^{-bL} f(y)dy \right\|_{L^2(\mathbb{R})} \\ &\le \sum_{a,b=0,1} \left\| \int \widetilde{H}^{\sigma}_{a,b}(k,\ell)\widehat{f_b}(\ell)d\ell \right\|_{L^2(\mathbb{R})} \\ &\le C_L \sum_{b=0,1} \| f_b\|_{L^2(\mathbb{R})} \lesssim C_L \| \langle x\rangle^{-L} f\|_{L^2(\mathbb{R})},\end{aligned}$$

where the constants in the last line are uniform on σ by the argument used to prove (5.9). Since, furthermore, for a sequence $\sigma_n \to 0$ then $\int_{\mathbb{R}} K^{\sigma_n}(x, y) f(y)dy \xrightarrow{n\to+\infty} \int_{\mathbb{R}} K^0(x, y) f(y)dy$ point–wise for $f \in C^0_c(\mathbb{R})$, we can assume, by the Fatou lemma and by the density of $C^0_c(\mathbb{R})$ in $L^{2,-L}(\mathbb{R})$,

$$\left\| \langle x\rangle^L \int K^0(x, y) f(y)dy \right\|_{L^2(\mathbb{R})} \le C_L \| \langle x\rangle^{-L} f\|_{L^2(\mathbb{R})} \text{ for all } f \in L^{2,-L}(\mathbb{R}).$$

This yields (5.12) and ends the proof of Lemma 5.4. □

We will need in Appendix A a variation of Lemma 5.4.

Lemma 5.5. *Suppose that the function $\mathcal{V}$ in Lemma 5.4 has the additional property that for $M \ge N + 1$ its Fourier transform satisfies*

$$\begin{aligned}&|\widehat{\mathcal{V}}(k_1 + ik_2)| \le C_M \langle k_1\rangle^{-M-1} \text{ for all } (k_1, k_2) \in \mathbb{R} \times [\mathbf{b}, \mathbf{b}] \text{ and} \\ &\widehat{\mathcal{V}} \in C^0(\mathbb{R} \times [-\mathbf{b}, \mathbf{b}]) \cap H(\mathbb{R} \times (-\mathbf{b}, \mathbf{b})),\end{aligned} \tag{5.11}$$

with $H(\Omega)$ the set of holomorphic functions in an open subset $\Omega \subseteq \mathbb{C}$ and with a number $\mathbf{b} > 0$. Then

$$\| \langle \mathrm{i}\varepsilon\partial_x\rangle^{-N} [\mathcal{V}, \langle \mathrm{i}\varepsilon\partial_x\rangle^N]e^{\mathbf{b}\langle y\rangle}\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} \le C_{\mathbf{b}}\varepsilon. \tag{5.12}$$

Proof. Formula (5.10) continues to hold for all $\sigma \in (0, 1]$, as a path integral

$$K^{\sigma}(x, y) = e^{y\ell_2} \int_{\mathbb{R}^2} e^{\mathrm{i}xk_1 - \mathrm{i}y\ell_1} \langle \varepsilon k\rangle^{-N-\sigma} \widehat{\mathcal{V}}(k - \ell)(k - \ell)\frac{P(\varepsilon k, \varepsilon\ell)}{\langle \varepsilon k\rangle^N + \langle \varepsilon\ell\rangle^N}dk_1d\ell_1 \tag{5.13}$$

with $k_2 = 0$ and $|\ell_2| \le \mathbf{b}$. Then, adjusting to the sign of y, we conclude that

$$|K^\sigma(x,y)| \le e^{-|y|\mathbf{b}} \left| \int_{\mathbb{R}^2} e^{\mathrm{i}xk_1 - \mathrm{i}y\ell_1} \langle \varepsilon k \rangle^{-N-\sigma} \widehat{\mathcal{V}}(k-\ell)(k-\ell) \frac{P(\varepsilon k, \varepsilon \ell)}{\langle \varepsilon k \rangle^N + \langle \varepsilon \ell \rangle^N} dk_1 d\ell_1 \right|.$$

This implies

$$\left\| \int K^\sigma(x,y) e^{|y|\mathbf{b}} e^{-|y|\mathbf{b}} f(y) dy \right\|_{L^2(\mathbb{R})} \le C_{\mathbf{b}} \| e^{-|x|\mathbf{b}} f \|_{L^2(\mathbb{R})}.$$

Like in the proof of Lemma 5.4 we can take the limit $\sigma \to 0^+$ obtaining (5.12). □

We next give estimates on the operator $\mathcal{T}$.

Lemma 5.6. *There exist constants C_0 and C_N such that for $\varepsilon >$ small enough we have*

$$\|\mathcal{T}\|_{L^2 \to L^2} \le C_0 \varepsilon^{-N} \text{ and } \|\mathcal{T}\|_{\Sigma^N \to \Sigma^0} \le C_N. \tag{5.14}$$

Furthermore, let $K_\varepsilon(x,y) \in \mathcal{D}'(\mathbb{R} \times \mathbb{R})$ be the Schwartz kernel of $\mathcal{T}$. Then, we have

$$|K_\varepsilon(x,y)| \le C_0 e^{-\frac{|x-y|}{3\varepsilon}} \text{ for all } x, y \text{ with } |x-y| \ge 1. \tag{5.15}$$

Proof. First, for $\mathcal{T}_1 = \langle \mathrm{i}\partial_x \rangle^{-N} \mathcal{A}^*$, we have

$$\|\mathcal{T}\|_{L^2 \to L^2} \le \| \langle \mathrm{i}\varepsilon\partial_x \rangle^{-N} \langle \mathrm{i}\partial_x \rangle^N \|_{L^2 \to L^2} \|\mathcal{T}_1\|_{L^2 \to L^2}.$$

Since $\| \langle \mathrm{i}\varepsilon\partial_x \rangle^{-N} \langle \partial_x \rangle^N \|_{L^2 \to L^2} = \| \langle \varepsilon k \rangle^{-N} \langle k \rangle^N \|_{L^\infty(\mathbb{R})} \lesssim \varepsilon^{-N}$ and $\|\mathcal{T}_1\|_{L^2 \to L^2} \lesssim 1$ because $\mathcal{T}_1$ is a degree 0 ΨDO, we have the first inequality in (5.14).
It is enough to prove (5.15) for operators $\langle \mathrm{i}\varepsilon\partial_x \rangle^{-N} (\mathrm{i}\partial_x)^m$ for $0 \le m \le N$, which, up to irrelevant constant factors, are convolutions, for $x \neq 0$ by the generalized integrals

$$K_\varepsilon(x) = \int_{\mathbb{R}} e^{\mathrm{i}xk_1} \frac{k_1^n}{(1+\varepsilon^2 k_1^2)^{N/2}} dk_1 = e^{-xk_2} \int_{\mathbb{R}} e^{\mathrm{i}xk_1} \frac{(k_1 + \mathrm{i}k_2)^n}{(1 - \varepsilon^2 k_2^2 + \varepsilon^2 k_1^2 + 2\varepsilon^2 \mathrm{i} k_1 k_2)^{N/2}} dk_1,$$

which are well defined for $|k_2| \le 2^{-1}\varepsilon^{-1}$. For $|x| \ge 1$ and if $k_2 = 2^{-1}\varepsilon^{-1}\mathrm{sign}(x)$, it is elementary to show, by standard arguments with cutoffs and integration by parts, that the above is $\lesssim e^{-\frac{|x|}{3\varepsilon}}$, yielding (5.15).
We turn now to the second inequality in (5.14). We have, for functions $b_n(x)$ bounded with all their derivatives,

$$\|\mathcal{T} f\|_{\Sigma^0} = \|\widetilde{\mathcal{T}} \left(e^{a\langle x \rangle} \sum_{n=0}^{N} b_n \partial_x^n f \right) \|_{L^2} \text{ where } \widetilde{\mathcal{T}} := e^{a\langle x \rangle} \langle \mathrm{i}\varepsilon\partial_x \rangle^{-N} e^{-a\langle x \rangle}.$$

For $\widetilde{\mathcal{T}}(x,y)$ the integral kernel of $\widetilde{\mathcal{T}}$, we have, for $m(k) = \langle k \rangle^{-N}$ and for $m^\vee$ its inverse Fourier transform, using (5.15) we obtain

$$\begin{aligned} |\widetilde{\mathcal{T}}(x,y)| &= |\widetilde{\mathcal{T}}(x,y)| \chi_{[0,1]}(|x-y|) + |\widetilde{\mathcal{T}}(x,y)| \chi_{[1,\infty)}(|x-y|) \\ &\lesssim \varepsilon^{-1} m^\vee \left(\frac{x-y}{\varepsilon} \right) \chi_{[0,1]}(|x-y|) + e^{-\frac{|x-y|}{3\varepsilon}} e^{a\langle x \rangle} e^{-a\langle y \rangle} \\ &\lesssim \varepsilon^{-1} m^\vee \left(\frac{x-y}{\varepsilon} \right) \chi_{[0,1]}(|x-y|) + e^{-\frac{|x-y|}{4\varepsilon}} \end{aligned}$$

for $\varepsilon > 0$ small enough. This implies, from Young's inequality, [53, Theorem 0.3.1], that

$$\|\mathcal{T} f\|_{\Sigma^0} \lesssim \|e^{a\langle x\rangle} \sum_{n=0}^{N} b_n \partial_x^n f\|_{L^2} \lesssim \|f\|_{\Sigma^N},$$

yielding the 2nd inequality in (5.14). □

The following technical estimates are related to analogous ones in Sect. 4.4 [29].

Lemma 5.7. *We have*

$$\|w\|_{L^2(|x|\le 2B^2)} \lesssim B^2 \|w\|_{\widetilde{\Sigma}} \text{ for any } w, \tag{5.16}$$

$$\|\xi\|_{\widetilde{\Sigma}}^2 \lesssim \left\langle (-\partial_x^2 - 2^{-2}\chi_{B^2}^2 x V_{N+1}')\xi, \xi \right\rangle \lesssim \|\xi\|_{\widetilde{\Sigma}}^2 \text{ for any } \xi, \tag{5.17}$$

$$\|v\|_{L^2(\mathbb{R})} \lesssim \varepsilon^{-N} B^2 \|w\|_{\widetilde{\Sigma}}\ , \tag{5.18}$$

$$\|v'\|_{L^2(\mathbb{R})} \lesssim \varepsilon^{-N} \|w\|_{\widetilde{\Sigma}}\ , \tag{5.19}$$

$$\| \langle x\rangle^{-M} v\|_{H^1(\mathbb{R})} \lesssim \|\xi\|_{\widetilde{\Sigma}} + \varepsilon^{-N} \langle B\rangle^{-M+3} \|w\|_{\widetilde{\Sigma}} \text{ for } M \in \mathbb{N},\ M \ge 4. \tag{5.20}$$

Proof. The proof of (5.16)–(5.17) is exactly the same in Lemma 4 [29] and is a consequence of Lemma 5.1. Now we consider, still following [29], the proof of (5.18), which is rather immediate. Indeed, by (5.14), (5.16) and $A \gg B^2$, we have

$$\|v\|_{L^2(\mathbb{R})} \lesssim \varepsilon^{-N} \|\chi_{B^2}\widetilde{\eta}\|_{L^2(\mathbb{R})} \lesssim \varepsilon^{-N} \left\| \frac{\chi_{B^2}}{\zeta_A} w \right\|_{L^2(\mathbb{R})} \lesssim \varepsilon^{-N} B^2 \|w\|_{\widetilde{\Sigma}}. \tag{5.21}$$

More complicated is the proof of (5.19). We have

$$v' = \mathcal{T} \left(\chi_{B^2}\widetilde{\eta}\right)' + \langle \mathrm{i}\varepsilon\partial_x\rangle^{-N} [\partial_x, \mathcal{A}^*]\chi_{B^2}\widetilde{\eta}. \tag{5.22}$$

To bound the first term in the right hand side, we use the inequality

$$|(\chi_{B^2}\widetilde{\eta})'| = \left| \left(\frac{\chi_{B^2}}{\zeta_A} w\right)' \right| \le \left| \frac{\chi_{B^2}}{\zeta_A} w' \right| + \left| \left(\frac{\chi_{B^2}}{\zeta_A}\right)' w \right| \lesssim |w'| + B^{-2}|w|\chi_{|x|\le 2B^2},$$

where we used $A \gg B^2$. Then

$$\begin{aligned} \|\mathcal{T} \left(\chi_{B^2}\widetilde{\eta}\right)' \|_{L^2(\mathbb{R})} &\lesssim \varepsilon^{-N} \| \left(\chi_{B^2}\widetilde{\eta}\right)' \|_{L^2(\mathbb{R})} \lesssim \varepsilon^{-N} \left(\|w'\|_{L^2(\mathbb{R})} + B^{-2}\|w\|_{L^2(|x|\le 2B^2)} \right) \\ &\lesssim \varepsilon^{-N} \|w\|_{\widetilde{\Sigma}} \end{aligned} \tag{5.23}$$

by (5.16). To bound the second term in the right hand side in (5.22), we use formula

$$[\partial_x, \mathcal{A}^*] = \sum_{j=1}^{N} \prod_{i=0}^{N-1-j} A_{N-i}^* \left(\log \psi_j\right)'' \prod_{i=1}^{j-1} A_{j-i}^*.$$

with the convention $\prod_{i=0}^{l} B_i = B_0 \circ \dots \circ B_l$. Then we have

$$\left\| \langle \mathrm{i}\varepsilon\partial_x\rangle^{-N} \prod_{i=0}^{N-1-j} A_{N-i}^* \left(\log \psi_j\right)'' \prod_{i=1}^{j-1} A_{j-i}^* \chi_{B^2}\eta \right\|_{L^2(\mathbb{R})}$$

$$= \varepsilon^{\frac{3}{2}-N} \Bigg\| \langle \mathrm{i}\partial_y \rangle^{-N} \prod_{i=0}^{N-1-j} \psi_{N-i}(\varepsilon y) \circ$$

$$\partial_y \circ \frac{1}{\psi_{N-i}(\varepsilon y)} \left(\log \psi_j\right)''(\varepsilon y) \prod_{i=1}^{j-1} \psi_{j-i}(\varepsilon y) \circ \partial_y \circ \frac{1}{\psi_{j-i}(\varepsilon y)} \chi_{B^2}(\varepsilon y)\eta(\varepsilon y) \Bigg\|_{L^2(\mathbb{R})} . \tag{5.24}$$

Now write the operator inside the last term as

$$\mathcal{P} \langle \mathrm{i}\partial_y \rangle^{-j+1} \left(\log \psi_j\right)''(\varepsilon y) \prod_{i=1}^{j-1} \psi_{j-i}(\varepsilon y) \circ \partial_y \circ \frac{1}{\psi_{j-i}(\varepsilon y)} \chi_{B^2}(\varepsilon y), \text{ where}$$

$$\mathcal{P} := \langle \mathrm{i}\partial_y \rangle^{-N} \prod_{i=0}^{N-1-j} \psi_{N-i}(\varepsilon y) \partial_y \circ \frac{1}{\psi_{N-i}(\varepsilon y)} \langle \mathrm{i}\partial_y \rangle^{j-1} .$$

We have $\|\mathcal{P}\|_{L^2 \to L^2} \lesssim 1$. Then the term in (5.24) is

$$\begin{aligned} &\lesssim \varepsilon^{\frac{3}{2}-N} \Bigg\| \left(\log \psi_j\right)''(\varepsilon y) \langle \mathrm{i}\partial_y \rangle^{-j+1} \prod_{i=1}^{j-1} \psi_{j-i}(\varepsilon y) \circ \partial_y \circ \frac{1}{\psi_{j-i}(\varepsilon y)} \chi_{B^2}(\varepsilon y)\eta(\varepsilon y) \Bigg\|_{L^2(\mathbb{R})} \\ &+ \varepsilon^{\frac{3}{2}-N} \Bigg\| \langle \mathrm{i}\partial_y \rangle^{-j+1} \left[\left(\log \psi_j\right)''(\varepsilon y), \langle \partial_y \rangle^{j-1} \right] \\ &\langle \mathrm{i}\partial_y \rangle^{-j+1} \prod_{i=1}^{j-1} \psi_{j-i}(\varepsilon y) \circ \partial_y \circ \frac{1}{\psi_{j-i}(\varepsilon y)} \chi_{B^2}(\varepsilon y)\eta(\varepsilon y) \Bigg\|_{L^2(\mathbb{R})} . \end{aligned} \tag{5.25}$$

By Lemma 5.1 and $A \gg B^2$, the term in the first line is

$$\lesssim \varepsilon^{\frac{3}{2}-N} \left\| \langle \varepsilon y \rangle^{-3} \chi_B(\varepsilon y)\eta(\varepsilon y) \right\|_{L^2(\mathbb{R})} = \varepsilon^{1-N} \left\| \langle x \rangle^{-3} \frac{\chi_{B^2}}{\zeta_A} w \right\|_{L^2(\mathbb{R})} \lesssim \varepsilon^{1-N} \|w\|_{\widetilde{\Sigma}} .$$

By Lemmas 5.1, 5.3 and 5.4, the term in the last two lines of (5.25) is

$$\begin{aligned} &\le \varepsilon^{j-N} \Bigg\| \langle \mathrm{i}\varepsilon\partial_x \rangle^{-j+1} \left[\left(\log \psi_j\right)'', \langle \mathrm{i}\varepsilon\partial_x \rangle^{j-1} \right] \langle \mathrm{i}\varepsilon\partial_x \rangle^{-j+1} \prod_{i=1}^{j-1} \psi_{j-i} \circ \partial_x \circ \frac{1}{\psi_{j-i}} \chi_{B^2}\eta \Bigg\|_{L^{2,3}(\mathbb{R})} \\ &\lesssim \varepsilon^{j+1-N} \Bigg\| \langle x \rangle^{-3} \langle \mathrm{i}\varepsilon\partial_x \rangle^{-j+1} \prod_{i=1}^{j-1} \psi_{j-i} \circ \partial_x \circ \frac{1}{\psi_{j-i}} \chi_{B^2}\eta \Bigg\|_{L^2(\mathbb{R})} \\ &= \varepsilon^{5/2-N} \Bigg\| \langle \varepsilon y \rangle^{-3} \langle \mathrm{i}\partial_y \rangle^{-j+1} \prod_{i=1}^{j-1} \psi_{j-i}(\varepsilon y) \circ \partial_y \circ \frac{1}{\psi_{j-i}(\varepsilon y)} \chi_{B^2}(\varepsilon y)\eta(\varepsilon y) \Bigg\|_{L^2(\mathbb{R})} \\ &\lesssim \varepsilon^{5/2-N} \left\| \langle \varepsilon y \rangle^{-3} \chi_{B^2}(\varepsilon y)\eta(\varepsilon y) \right\|_{L^2(\mathbb{R})} = \varepsilon^{2-N} \left\| \langle x \rangle^{-3} \chi_{B^2}\eta \right\|_{L^2(\mathbb{R})} \lesssim \varepsilon^{2-N} \|w\|_{\widetilde{\Sigma}}. \end{aligned}$$

This completes the proof of (5.19).
We finally consider the proof of (5.20). We have

$$\| \langle x \rangle^{-M} v \|_{H^1(\mathbb{R})} \le \| \langle x \rangle^{-M} \chi_B \xi \|_{H^1(\mathbb{R})} + \| \langle x \rangle^{-M} (1 - \chi_B) v \|_{H^1(\mathbb{R})}$$

$$\lesssim \|\xi\|_{\widetilde{\Sigma}} + \langle B\rangle^{-M+3} \| \langle x\rangle^{-3} v\|_{H^1(\mathbb{R})}, \tag{5.26}$$

where we used Lemma 5.1. To evaluate the last term, we observe, introducing $x = \varepsilon y$, that

$$\begin{aligned}\langle x\rangle^{-3} (\mathcal{T} f)(x) &= (-1)^N \langle \varepsilon y\rangle^{-3} \varepsilon^{-N} \left\langle \mathrm{i}\partial_y\right\rangle^{-N} \prod_{i=0}^{N-1} \psi_{N-i}(\varepsilon y) \circ \partial_y \circ \frac{1}{\psi_{N-i}(\varepsilon y)} (f(\varepsilon\cdot))(y) \\ &= \langle \varepsilon y\rangle^{-3} \varepsilon^{-N} p(\varepsilon y, \mathrm{i}\partial_y)(f(\varepsilon\cdot))(y)).\end{aligned}$$

Then, we can apply Lemma 5.3, concluding that

$$\begin{aligned}\| \langle x\rangle^{-3} \mathcal{T} f\|_{L^2_x} &= \varepsilon^{\frac{1}{2}-N} \| \langle \varepsilon y\rangle^{-3} p(\varepsilon y, \mathrm{i}\partial_y)(f(\varepsilon\cdot))(y))\|_{L^2_y} \\ &\lesssim \varepsilon^{\frac{1}{2}-N} \| \langle \varepsilon y\rangle^{-3} f(\varepsilon y)\|_{L^2_y} \lesssim \varepsilon^{-N} \| \langle x\rangle^{-3} f\|_{L^2_x}.\end{aligned} \tag{5.27}$$

By $v = \mathcal{T}\chi_{B^2}\widetilde{\eta}$, Lemma 5.1 and $A \gg B^2$, this by implies

$$\begin{aligned}B^{-M+3}\| \langle x\rangle^{-3} \mathcal{T}\chi_{B^2}\widetilde{\eta}\|_{L^2(\mathbb{R})} &\lesssim B^{-M+3}\varepsilon^{-N} \left\| \langle x\rangle^{-3} \frac{\chi_{B^2}}{\zeta_A}\zeta_A\widetilde{\eta}\right\|_{L^2(\mathbb{R})} \\ &\lesssim B^{-M+3}\varepsilon^{-N}\| \langle x\rangle^{-3} w\|_{L^2(\mathbb{R})} \lesssim B^{-M+3}\varepsilon^{-N}\|w\|_{\widetilde{\Sigma}}.\end{aligned} \tag{5.28}$$

Next,

$$\left(\langle x\rangle^{-3} \mathcal{T}\left(\chi_{B^2}\widetilde{\eta}\right)\right)' = \left(\langle x\rangle^{-3}\right)' \mathcal{T}\left(\chi_{B^2}\widetilde{\eta}\right) + [\partial_x, \mathcal{T}]\left(\chi_{B^2}\widetilde{\eta}\right) + \langle x\rangle^{-3} \mathcal{T}\left(\chi_{B^2}\widetilde{\eta}\right)'. \tag{5.29}$$

Since $\left(\langle x\rangle^{-3}\right)' \sim \langle x\rangle^{-4}$, the first term in the right can be treated like (5.27). The second term in the right hand side of (5.29) coincides with the second term in the right in (5.22). So we conclude, using Lemma 5.1 and (5.16),

$$\begin{aligned}\langle B\rangle^{-M+3} \| \left(\langle x\rangle^{-3} \mathcal{T}\chi_{B^2}\widetilde{\eta}\right)' \|_{L^2(\mathbb{R})} &\lesssim \langle B\rangle^{-M+3} \varepsilon^{-N} \left(\|w\|_{\widetilde{\Sigma}} + \left\|\langle x\rangle^{-3}\left(\frac{\chi_{B^2}}{\zeta_A} w\right)'\right\|_{L^2(\mathbb{R})}\right) \\ &\lesssim \langle B\rangle^{-M+3} \varepsilon^{-N} \left(\|w\|_{\widetilde{\Sigma}} + \| \langle x\rangle^{-3} w'\|_{L^2(\mathbb{R})} + \left\|\langle x\rangle^{-3}\left(\frac{\chi_{B^2}}{\zeta_A}\right)' w\right\|_{L^2(|x|\le 2B^2)}\right) \\ &\lesssim \langle B\rangle^{-M+3} \varepsilon^{-N} \|w\|_{\widetilde{\Sigma}}.\end{aligned} \tag{5.30}$$

Entering (5.28)–(5.30) in (5.26), we obtain (5.20). □

6. Proof of Proposition 3.4

For the proof of the 2nd virial estimate Proposition 3.4, we use the following functional,

$$\mathcal{J}(v) := \frac{1}{2}\left\langle v, \mathrm{i}\widetilde{S}_B v\right\rangle, \tag{6.1}$$

where the anti-symmetric operator $\widetilde{S}_B$ is defined by

$$\widetilde{S}_B := \frac{\psi_B'}{2} + \psi_B\partial_x, \ \ \psi_B(x) := \chi_{B^2}^2(x)\varphi_B(x). \tag{6.2}$$

Then, by the equation of v in (2.19), we have

$$\frac{d}{dt}\mathcal{J}(v) = -\left\langle H_{N+1}v, \widetilde{S}_B v\right\rangle - \left\langle \mathcal{R}_v, \widetilde{S}_B v\right\rangle - \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \left\langle \mathbf{z}^{\mathbf{m}}\widetilde{G}_{\mathbf{m}}, \widetilde{S}_B v\right\rangle \tag{6.3}$$

where by remainder formulas (2.12) and (2.21), we have

$$\left\langle \mathcal{R}_v, \widetilde{S}_B v\right\rangle = \sum_{j=1}^{7}\left\langle \mathcal{R}_{vj}, \widetilde{S}_B v\right\rangle, \tag{6.4}$$

with

$$\mathcal{R}_{v1} = -\mathrm{i}\mathcal{T}\chi_{B^2}P_c D_{\mathbf{z}}\phi[\mathbf{z}]\left(\dot{\mathbf{z}} + \mathrm{i}\boldsymbol{\varpi}(\mathbf{z})\mathbf{z}\right), \tag{6.5}$$
$$\mathcal{R}_{v2} = \mathcal{T}\chi_{B^2}P_c\mathcal{R}_{\mathrm{rp}}[\mathbf{z}], \tag{6.6}$$
$$\mathcal{R}_{v3} = \mathcal{T}\chi_{B^2}P_c\left(2\phi[\mathbf{z}]|\eta|^2 + \overline{\phi[\mathbf{z}]}\eta^2\right), \tag{6.7}$$
$$\mathcal{R}_{v4} = \mathcal{T}\chi_{B^2}P_c|\eta|^2\eta, \tag{6.8}$$
$$\mathcal{R}_{v5} = \mathcal{T}\chi_{B^2}P_c L[\mathbf{z}]\eta, \tag{6.9}$$
$$\mathcal{R}_{v6} = \langle\varepsilon\mathrm{i}\partial_x\rangle^{-N}\left[V_{N+1}, \langle\varepsilon\mathrm{i}\partial_x\rangle^{N}\right]v, \tag{6.10}$$
$$\mathcal{R}_{v7} = \mathcal{T}\left(2\chi'_{B^2}\partial_x + \chi''_{B^2}\right)\tilde{\eta}. \tag{6.11}$$

Proposition 3.4 follows from the following three lemmas.

Lemma 6.1. *We have*

$$\left\langle H_{N+1}v, \widetilde{S}_B v\right\rangle \geq 2^{-1}\left\langle -\xi'' - \frac{1}{2}\chi_{B^2}^2 x V'_{N+1}\xi, \xi\right\rangle + B^{-1/2}O\left(\|\xi\|_{\tilde{\Sigma}}^2 + \|w\|_{\tilde{\Sigma}}^2\right). \tag{6.12}$$

Proof. Like in Lemma 4.2 the l.h.s. of (6.12) equals

$$\left\langle H_{N+1}v, \widetilde{S}_B v\right\rangle = \left\langle \psi'_B v', v'\right\rangle - \frac{1}{4}\left\langle \psi'''_B v, v\right\rangle - \frac{1}{2}\left\langle v, \psi_B V'_{N+1}v\right\rangle. \tag{6.13}$$

For the 1st and 2nd term in the right hand side of (6.13), we have

$$\begin{aligned}\left\langle \psi'_B v', v'\right\rangle - \frac{1}{4}\left\langle \psi'''_B v, v\right\rangle &= \left\langle \chi_{B^2}^2\zeta_B^2 v', v'\right\rangle - \frac{1}{4}\left\langle \chi_{B^2}^2\left(\zeta_B^2\right)'' v, v\right\rangle \\ &+ \left\langle \left(\chi_{B^2}^2\right)'\varphi_B v', v'\right\rangle - \frac{1}{4}\left\langle \left((\chi_{B^2}^2)'''\varphi_B + 3\left(\chi_{B^2}\right)''\zeta_B^2 + 3(\chi_{B^2})'(\zeta_B^2)'\right)v, v\right\rangle.\end{aligned} \tag{6.14}$$

By Lemma 5.7, the last term of (6.14) can by bounded as

$$\begin{aligned}&\left|\left\langle \left((\chi_{B^2}^2)'''\varphi_B + 3\left(\chi_{B^2}\right)''\zeta_B^2 + 3(\chi_{B^2})'(\zeta_B^2)'\right)v, v\right\rangle\right| \\ &\lesssim \left(B^{-5} + B^{-4}e^{-2B} + B^{-4}e^{-B}\right)B^4\varepsilon^{-2N}\|w\|_{\tilde{\Sigma}}^2 \lesssim B^{-1/2}\|w\|_{\tilde{\Sigma}}^2.\end{aligned}$$

For the 1st term of the 2nd line of (6.14), we have

$$\left\langle \left|(\chi_{B^2}^2)'\varphi_B\right|, |v'|^2\right\rangle \lesssim B^{-1}\|v'\|_{L^2(|x|\leq 2B^2)}^2 \lesssim \varepsilon^{-2N}B^{-1}\|w\|_{\tilde{\Sigma}}^2.$$

We consider now the 1st and 2nd term of (6.14). Using $\chi_{B^2}\zeta_B v' = \partial_x \xi - \chi_{B^2}\zeta_B' v - \chi_{B^2}'\zeta_B v$, we have

$$\left\langle \chi_{B^2}^2 \zeta_B^2 v', v' \right\rangle - \frac{1}{4}\left\langle \chi_{B^2}^2 \left(\zeta_B^2\right)'' v, v \right\rangle = \left\langle -\xi'', \xi \right\rangle + \frac{1}{2B}\left\langle V_0 \xi, \xi \right\rangle$$
$$+ 2\left\langle \xi', \chi_{B^2}'\zeta_B v \right\rangle - 2\left\langle \zeta_B' \xi, \chi_{B^2}' v \right\rangle + \left\langle \chi_{B^2}\zeta_B v, \chi_{B^2}'\zeta_B v \right\rangle. \tag{6.15}$$

The 2nd line of (6.15) can be bounded as

$$\begin{aligned} &|\left\langle \xi', \chi_{B^2}'\zeta_B v \right\rangle| \lesssim B^{-2}B^4 e^{-B} \|\xi\|_{\widetilde{\Sigma}} \|w\|_{\widetilde{\Sigma}} \lesssim B^{-1}\|\xi\|_{\widetilde{\Sigma}} \|w\|_{\widetilde{\Sigma}}, \\ &|\left\langle \zeta_B' \xi, \chi_{B^2}' v \right\rangle| \lesssim B^{-2}B^8 e^{-B} \| \langle x\rangle^{-3} \xi \|_{L^2} \varepsilon^{-N} \|w\|_{\widetilde{\Sigma}} \lesssim B^{-1}\|\xi\|_{\widetilde{\Sigma}} \|w\|_{\widetilde{\Sigma}}, \\ &|\left\langle \chi_{B^2}'\zeta_B v, \chi_{B^2}'\zeta_B v \right\rangle| \lesssim B^{-2} e^{-B} B^4 \|w\|_{\widetilde{\Sigma}}^2 \lesssim B^{-1}\|w\|_{\widetilde{\Sigma}}^2, \end{aligned} \tag{6.16}$$

where we have used $\|\zeta_B' \xi\|_{L^2} \lesssim B\|\xi\|_{\widetilde{\Sigma}}$ by (5.1).
Summing up, we obtain

$$\begin{aligned} \left\langle H_{N+1} v, \widetilde{S}_B v \right\rangle &= \left\langle -\xi'' + \frac{1}{2B} V_0 \xi - 2^{-1}\psi_B \zeta_B^{-2} V_{N+1}' \xi, \xi \right\rangle + B^{-1/2} O\left(\|\xi\|_{\widetilde{\Sigma}}^2 + \|w\|_{\widetilde{\Sigma}}^2 \right) \\ &\geq \left\langle (-\partial_x^2 - 2^{-1}\chi_{B^2}^2 x V_{N+1}')\xi, \xi \right\rangle + \frac{1}{2B}\left\langle V_0 \xi, \xi \right\rangle + B^{-1/2} O\left(\|\xi\|_{\widetilde{\Sigma}}^2 + \|w\|_{\widetilde{\Sigma}}^2 \right), \end{aligned}$$

where, like in Lemma 3 in [29], since $\frac{\varphi_B(x)}{\zeta_B^2(x)} \geq x$ for $x \geq 0$, for B large enough, by Lemma 5.1,

$$\left\langle (-\partial_x^2 - 2^{-1}\chi_{B^2}^2 x V_{N+1}')\xi, \xi \right\rangle \geq B^{-1}\left\langle V_0 \xi, \xi \right\rangle.$$

So

$$\left\langle -\xi'' + \frac{1}{2B} V_0 \xi - 2^{-1}\psi_B \zeta_B^{-2} V_{N+1}' \xi, \xi \right\rangle \geq 2^{-1}\left\langle -\xi'' - 2^{-1}\chi_{B^2}^2 x V_{N+1}' \xi, \xi \right\rangle.$$

□

Lemma 6.2. *We have*

$$\sum_{j=1,2} |\left\langle \mathcal{R}_{vj}, \widetilde{S}_B v \right\rangle| \lesssim \varepsilon^{-N} B\delta^2 \left(\|w\|_{\widetilde{\Sigma}} + \|\xi\|_{\widetilde{\Sigma}} \right) \left(\sum_{\mathbf{m}\in\mathbf{R}_{\min}} |\mathbf{z}^{\mathbf{m}}| + \|\dot{\mathbf{z}} + \mathrm{i}\boldsymbol{\varpi}(\mathbf{z})\mathbf{z}\| \right), \tag{6.17}$$

$$\sum_{j=3,4,5} |\left\langle \mathcal{R}_{vj}, \widetilde{S}_B v \right\rangle| \lesssim \varepsilon^{-N} B^3 \delta^2 \|w\|_{\widetilde{\Sigma}} \left(\|w\|_{\widetilde{\Sigma}} + \|\xi\|_{\widetilde{\Sigma}} \right); \tag{6.18}$$

$$|\left\langle \mathcal{R}_{v6}, \widetilde{S}_B v \right\rangle| \lesssim \varepsilon \left(\|\xi\|_{\widetilde{\Sigma}}^2 + \|w\|_{\widetilde{\Sigma}}^2 \right); \tag{6.19}$$

$$|\left\langle \mathcal{R}_{v7}, \widetilde{S}_B v \right\rangle| \lesssim \varepsilon^{-N} B^{-1} \|w\|_{\widetilde{\Sigma}} \left(\|w\|_{\widetilde{\Sigma}} + \|\xi\|_{\widetilde{\Sigma}} \right). \tag{6.20}$$

Proof. First we claim

$$\|\widetilde{S}_B v\|_{L^2} \lesssim \varepsilon^{-N} B \|w\|_{\widetilde{\Sigma}} + B\|\xi\|_{\widetilde{\Sigma}}. \tag{6.21}$$

The proof of (6.21) is like in [29]. By (5.19) and $\|\psi_B\|_{L^\infty} \lesssim B$ we have

$$\|\widetilde{S}_B v\|_{L^2} \lesssim \|\psi_B' v\|_{L^2} + \|\psi_B v'\|_{L^2} \lesssim \|\psi_B' v\|_{L^2} + \varepsilon^{-N} B\|w\|_{\widetilde{\Sigma}}.$$

Next, we have

$$|\psi_B'| = |2\chi_{B^2}'\chi_{B^2}\varphi_B + \chi_{B^2}^2\zeta_B^2| \lesssim B^{-1}\chi_{B^2} + \chi_{B^2}^2\zeta_B^2. \tag{6.22}$$

Then

$$B^{-1}\|v\|_{L^2} \lesssim B\varepsilon^{-N} B\|w\|_{\widetilde{\Sigma}}$$

by (5.18), by Lemma 5.1 we have

$$\|\chi_{B^2}^2\zeta_B^2 v\|_{L^2} = \|\chi_{B^2}\zeta_B \xi\|_{L^2} \lesssim \sqrt{\|\langle x\rangle\, \chi_{B^2}\zeta_B\|_{L^1}}\|\xi\|_{\widetilde{\Sigma}} \sim B\|\xi\|_{\widetilde{\Sigma}}$$

and, finally,

$$\|\chi_{B^2}^2\varphi_B v'\|_{L^2} \lesssim B\|v'\|_{L^2} \lesssim B\varepsilon^{-N}\|w\|_{\widetilde{\Sigma}}$$

by (5.19), so that so that we get (6.21).
We have $\|P_c D_{\mathbf{z}}\phi[\mathbf{z}]\|_{L^2} = O(\|\mathbf{z}\|^2)$ by Proposition 1.9. Then, using (5.18)–(5.19) and $\|\psi_B\|_{L^\infty} \lesssim B$, we have

$$\begin{aligned}\sum_{j=1,2} |\langle \mathcal{R}_{vj}, \widetilde{S}_B v\rangle| &\lesssim \sum_{j=1,2} \|\mathcal{R}_{vj}\|_{L^2}\|\widetilde{S}_B v\|_{L^2} \lesssim \sum_{j=1,2} \|\mathcal{R}_{vj}\|_{L^2}\left(\varepsilon^{-N}B\|w\|_{\widetilde{\Sigma}} + B\|\xi\|_{\widetilde{\Sigma}}\right)\\ &\lesssim \varepsilon^{-N}B\delta^2\left(\|w\|_{\widetilde{\Sigma}} + \|\xi\|_{\widetilde{\Sigma}}\right)\left(\sum_{\mathbf{m}\in\mathbf{R}_{\min}} |\mathbf{z}^{\mathbf{m}}| + \|\dot{\mathbf{z}} + \mathrm{i}\boldsymbol{\varpi}(\mathbf{z})\mathbf{z}\|\right).\end{aligned}$$

We claim

$$\begin{aligned}\sum_{j=3,4,5} |\langle \mathcal{R}_{vj}, \widetilde{S}_B v\rangle| &\lesssim \sum_{j=3,4,5} \|\mathcal{R}_{vj}\|_{L^2}\left(\varepsilon^{-N}B\|w\|_{\widetilde{\Sigma}} + B\|\xi\|_{\widetilde{\Sigma}}\right)\\ &\lesssim \varepsilon^{-N}B\left(\|\mathbf{z}\|\|\eta\|_{H^1} + B^2\|\eta\|_{H^1}^2 + \|\mathbf{z}\|^2\right)\|w\|_{\widetilde{\Sigma}}\left(\|w\|_{\widetilde{\Sigma}} + \|\xi\|_{\widetilde{\Sigma}}\right).\end{aligned} \tag{6.23}$$

We have for example, using Lemma 2.2 and inequality (5.16),

$$\begin{aligned}\|\mathcal{T}\chi_{B^2}P_c|\eta|^2\eta\|_{L^2} &\lesssim \varepsilon^{-N}\left(\|P_c|\eta|^2\,(R[\mathbf{z}]-1)\,\widetilde{\eta}\|_{L^2} + \|\chi_{B^2}P_c|\eta|^2\widetilde{\eta}\|_{L^2}\right)\\ &\lesssim \varepsilon^{-N}\left(\|\mathbf{z}\|^2\|\eta\|_{L^\infty}^2\|\widetilde{\eta}\|_{L^2_{-a}} + \|P_d|\eta|^2\widetilde{\eta}\|_{L^2} + \|\chi_{B^2}|\eta|^2\widetilde{\eta}\|_{L^2}\right)\\ &\lesssim \varepsilon^{-N}\left(\delta^2\|\mathbf{z}\|^2\|w\|_{\widetilde{\Sigma}} + \|\eta\|_{L^\infty}^2\|\widetilde{\eta}\|_{L^2_{-a}} + \|\chi_{B^2}|\eta|^2 w\|_{L^2}\right)\\ &\lesssim \varepsilon^{-N}\left(\left(\|\mathbf{z}\|^2 + \|\eta\|_{H^1}^2\right)\|w\|_{\widetilde{\Sigma}} + \|\eta\|_{H^1}^2\|w\|_{L^2(|x|\le 2B^2)}\right)\\ &\lesssim \varepsilon^{-N}\left(\|\mathbf{z}\|^2 + B^2\|\eta\|_{H^1}^2\right)\|w\|_{\widetilde{\Sigma}},\end{aligned}$$

with better the bounds for the other terms in the r.h.s. of (6.23).
Using Lemma 5.4, (5.20) and (6.22) we obtain (6.19):

$$|\langle \mathcal{R}_{v\eta 6}, \widetilde{S}_B v\rangle|$$

$$\lesssim \left(\|\psi_B'\|_{L^\infty}\| \langle x\rangle^{-10} v\|_{L^2} + \| \langle x\rangle^{-10} \psi_B\|_{L^\infty}\| \langle x\rangle^{-10} v'\|_{L^2}\right) \| \langle x\rangle^{20} \langle \mathrm{i}\varepsilon\partial_x\rangle^{-N}$$
$$[V_{N+1}, \langle \mathrm{i}\varepsilon\partial_x\rangle^N]v\|_{L^2}$$
$$\lesssim \varepsilon\| \langle x\rangle^{-10} v\|_{H^1}^2 \lesssim \varepsilon \left(\|\xi\|_{\widetilde{\Sigma}}^2 + \|w\|_{\widetilde{\Sigma}}^2\right).$$

Finally, the proof of (6.20) is the same as in [29]. We write

$$|\langle \mathcal{R}_{v7}, \widetilde{S}_B v\rangle| \lesssim \varepsilon^{-N} \left(\|\chi_{B^2}'\tilde{\eta}'\|_{L^2} + \|\chi_{B^2}''\tilde{\eta}\|_{L^2}\right) \left(\|\psi_B' v\|_{L^2} + \|\psi_B v'\|_{L^2}\right). \tag{6.24}$$

We claim

$$\|\chi_{B^2}'\widetilde{\eta}'\|_{L^2} + \|\chi_{B^2}''\widetilde{\eta}\|_{L^2} \lesssim B^{-2}\|w\|_{\widetilde{\Sigma}}. \tag{6.25}$$

Indeed from $w = \zeta_A\widetilde{\eta}$ we have

$$w' = \zeta_A'\widetilde{\eta} + \zeta_A\widetilde{\eta}',$$

so, for $|x| \le A$,

$$|\eta'| \lesssim A^{-1}|\eta| + |w'| = A^{-1}\zeta_A^{-1}|w| + |w'|.$$

By $A \gg B^2$ and (5.16), we have

$$\|\chi_B'\widetilde{\eta}'\|_{L^2} \lesssim B^{-2}\|\widetilde{\eta}'\|_{L^2(B^2\le|x|\le 2B^2)}$$
$$\lesssim B^{-2}\left(\|w'\|_{L^2(\mathbb{R})} + B^{-2}\|w\|_{L^2(B^2\le|x|\le 2B^2)}\right) \lesssim B^{-2}\|w\|_{\widetilde{\Sigma}}$$

and the following

$$\|\chi_{B^2}''\widetilde{\eta}\|_{L^2} \lesssim B^{-4}\|\widetilde{\eta}\|_{L^2(B^2\le|x|\le 2B^2)} \lesssim B^{-4}\|w\|_{L^2(|x|\le 2B^2)} \lesssim B^{-2}\|w\|_{\widetilde{\Sigma}}.$$

The next step is to prove the following, which with (6.25) yields (6.20),

$$\|\psi_B' v\|_{L^2} + \|\psi_B v'\|_{L^2} \lesssim B\varepsilon^{-N}\|w\|_{\widetilde{\Sigma}} + B\|\xi\|_{\widetilde{\Sigma}}. \tag{6.26}$$

From $\xi = \chi_{B^2}^2\zeta_B v$, we have by (5.18), (5.1) and (6.22),

$$\|\psi_B' v\|_{L^2} \lesssim B^{-1}\|v\|_{L^2} + \|\zeta_B\xi\|_{L^2} \lesssim B\varepsilon^{-N}\|w\|_{\widetilde{\Sigma}} + B\|\xi\|_{\widetilde{\Sigma}}.$$

Using (5.19) and $|\psi_B| \lesssim B$, we get the following, which completes the proof of (6.26),

$$\|\psi_B v'\|_{L^2} \lesssim B\|v'\|_{L^2} \lesssim \varepsilon^{-N}B\|w\|_{\widetilde{\Sigma}}.$$

□

Lemma 6.3. *We have*

$$\left|\langle \mathbf{z}^{\mathbf{m}}\widetilde{G}_{\mathbf{m}}, \widetilde{S}_B v\rangle\right| \lesssim |\mathbf{z}^{\mathbf{m}}| \left(\|\xi\|_{\widetilde{\Sigma}} + e^{-B/2}\|w\|_{\widetilde{\Sigma}}\right). \tag{6.27}$$

Proof. We have

$$\begin{aligned}&\left|\langle \mathbf{z}^{\mathbf{m}}\widetilde{G}_{\mathbf{m}},\psi_B' v+2\psi_B v'\rangle\right|\\&\lesssim \left|\left\langle \mathbf{z}^{\mathbf{m}}\widetilde{G}_{\mathbf{m}},\left(\chi_{B^2}^2\right)'\varphi_B v\right\rangle\right|+\left|\left\langle \mathbf{z}^{\mathbf{m}}\widetilde{G}_{\mathbf{m}},\chi_{B^2}^2\zeta_B^2 v\right\rangle\right|+\left|\langle \mathbf{z}^{\mathbf{m}}\widetilde{G}_{\mathbf{m}},\psi_B v'\rangle\right|. \end{aligned}\tag{6.28}$$

We now we examine the three terms in line (6.28). Using (5.18), $|\varphi_B|\le B$, $1_{|x|\le 1}\le \chi\le 1_{|x|\le 2}$ and $\chi_{B^2}:=\chi(B^{-2}\cdot)$, we get

$$\begin{aligned}\left|\left\langle \mathbf{z}^{\mathbf{m}}\widetilde{G}_{\mathbf{m}},\left(\chi_{B^2}^2\right)'\varphi_B v\right\rangle\right| &\lesssim B^{-1}|\mathbf{z}^{\mathbf{m}}\|\widetilde{G}_{\mathbf{m}}\|_{L^2(B^2\le|x|\le 2B^2)}\|v\|_{L^2}\\ &\lesssim \varepsilon^{-N}B|\mathbf{z}^{\mathbf{m}}|\|\widetilde{G}_{\mathbf{m}}\|_{L^2(B^2\le|x|\le 2B^2)}\|w\|_{\widetilde{\Sigma}}\end{aligned}$$

Now we claim $\|\widetilde{G}_{\mathbf{m}}\|_{L^2(B^2\le|x|\le 2B^2)}\le e^{-B}$, so that $\varepsilon^{-N}Be^{-B}\le e^{-B/2}$. To prove our claim we split

$$\begin{aligned}&\|\widetilde{G}_{\mathbf{m}}\|_{L^2(B^2\le|x|\le 2B^2)}\le \|\mathcal{T}1_{|x|\le B^2/2}\chi_{B^2}P_cG_{\mathbf{m}}\|_{L^2(B^2\le|x|\le 2B^2)}\\&+\|\mathcal{T}1_{|x|\ge B^2/2}\chi_{B^2}P_cG_{\mathbf{m}}\|_{L^2}.\end{aligned}$$

Using (5.15) we have

$$\|\mathcal{T}1_{|x|\le B^2/2}\chi_{B^2}P_cG_{\mathbf{m}}\|_{L^2(B^2\le|x|\le 2B^2)}\le e^{-3B}\|G_{\mathbf{m}}\|_{L^2}\le e^{-2B}$$

while

$$\|\mathcal{T}1_{|x|\ge B^2/2}\chi_{B^2}P_cG_{\mathbf{m}}\|_{L^2}\lesssim \varepsilon^{-N}\|1_{|x|\ge B^2/2}P_cG_{\mathbf{m}}\|_{L^2}\le \varepsilon^{-N}e^{-3B}\le e^{-2B}.$$

Next, we consider the 2nd term in (6.28). Using (5.14) and (5.20)

$$\begin{aligned}&\left|\left\langle \langle x\rangle^{20}\mathbf{z}^{\mathbf{m}}\widetilde{G}_{\mathbf{m}},\chi_{B^2}^2\zeta_B^2\langle x\rangle^{-20}v\right\rangle\right|\le |\mathbf{z}^{\mathbf{m}}|\ \|\mathcal{T}\chi_{B^2}P_cG_{\mathbf{m}}\|_{\Sigma^0}\|\ \langle x\rangle^{-20}v\|_{L^2}\\&\lesssim |\mathbf{z}^{\mathbf{m}}|\ \|\chi_{B^2}P_cG_{\mathbf{m}}\|_{\Sigma^N}\left(\|\xi\|_{\widetilde{\Sigma}}+\langle B\rangle^{-10}\|w\|_{\widetilde{\Sigma}}\right)\lesssim |\mathbf{z}^{\mathbf{m}}|\left(\|\xi\|_{\widetilde{\Sigma}}+\langle B\rangle^{-10}\|w\|_{\widetilde{\Sigma}}\right).\end{aligned}$$

Finally, we consider the last term in line (6.28). Like in the estimate of J_2 in Sect. 4.4 [29], from

$$\xi'=\chi_{B^2}\zeta_B v'+\left(\chi_{B^2}\zeta_B\right)'v$$

we obtain

$$|\chi_{B^2}\zeta_B v'|\lesssim|\xi'|+|\left(\chi_{B^2}\zeta_B\right)'v|\lesssim|\xi'|+B^{-1}|\chi_{B^2}\zeta_B v|+B^{-2}|\chi_{[B^2\le|x|\le 2B^2]}\zeta_B v|,$$

so that

$$|\chi_{B^2}^2\zeta_B v'|\lesssim|\xi'|+B^{-1}|\xi|.$$

Then, using (5.19) and the above estimates, we have

$$\begin{aligned}&\left|\langle \mathbf{z}^{\mathbf{m}}\widetilde{G}_{\mathbf{m}},\psi_B v'\rangle\right|\le\left|\left\langle \mathbf{z}^{\mathbf{m}}\psi_B\zeta_B^{-1}\widetilde{G}_{\mathbf{m}},\chi_{B^2}^2\zeta_B v'\right\rangle\right|+\left|\left\langle \mathbf{z}^{\mathbf{m}}\psi_B\zeta_B^{-1}\widetilde{G}_{\mathbf{m}},\left(1-\chi_{B^2}^2\right)\zeta_B v'\right\rangle\right|\\&\lesssim\left|\left\langle \mathbf{z}^{\mathbf{m}}\psi_B|\widetilde{G}_{\mathbf{m}}|,|\xi'|+B^{-1}|\xi|\right\rangle\right|+|\mathbf{z}^{\mathbf{m}}|\|\left(1-\chi_{B^2}^2\right)\psi_B\widetilde{G}_{\mathbf{m}}\|_{L^2}\|v'\|_{L^2}\\&\lesssim|\mathbf{z}^{\mathbf{m}}|\left(\|\xi'\|_{L^2}+B^{-1}\|\xi\|_{\widetilde{\Sigma}}+e^{-B}\varepsilon^{-N}\|w\|_{\widetilde{\Sigma}}\right).\end{aligned}$$

□

Proof of Proposition 3.4. Using (6.3), Lemmas 6.1–6.3 and (2.13)

$$\begin{aligned}\frac{d}{dt}\mathcal{J}(v) &= -\left\langle H_{N+1}v, \widetilde{S}_B v\right\rangle - \left\langle \mathcal{R}_v, \widetilde{S}_B v\right\rangle - \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \left\langle \mathbf{z}^{\mathbf{m}}\widetilde{G}_{\mathbf{m}}, \widetilde{S}_B v\right\rangle \\ &\lesssim -\left\langle -\xi'' - \frac{1}{4}\chi_{B^2}^2 x V_{N+1}'\xi, \xi\right\rangle \\ &\quad + \sum_{\mathbf{m}\in\mathbf{R}_{\min}} |\mathbf{z}^{\mathbf{m}}| \left(\|\xi\|_{\widetilde{\Sigma}} + e^{-B/2}\|w\|_{\widetilde{\Sigma}}\right) + o_\varepsilon(1)\left(\|\xi\|_{\widetilde{\Sigma}}^2 + \|w\|_{\widetilde{\Sigma}}^2 + \|\dot{\mathbf{z}} + \mathrm{i}\boldsymbol{\varpi}(\mathbf{z})\mathbf{z}\|^2\right) \\ &\lesssim -\|\xi\|_{\widetilde{\Sigma}}^2 + \sum_{\mathbf{m}\in\mathbf{R}_{\min}} |\mathbf{z}^{\mathbf{m}}|^2 + o_\varepsilon(1)\left(\|w\|_{\widetilde{\Sigma}}^2 + \|\dot{\mathbf{z}} + \mathrm{i}\boldsymbol{\varpi}(\mathbf{z})\mathbf{z}\|^2\right),\end{aligned}$$

so that integrating in time we obtain inequality (3.6) concluding the proof of Proposition 3.4. ◻

Our next task is to estimate the discrete modes, that is the contributions from $\mathbf{z}$. While so far in the paper we have drawn from Kowalczyk, Martel and Munoz [29], we now start drawing from [9].

7. Proof of Proposition 3.5

Proposition 3.5 is an immediate consequence of the following lemma which is taken from [9].

Lemma 7.1. *Under the assumption of Proposition 3.1, we have*

$$\dot{z}_j + \mathrm{i}\varpi_j(|\mathbf{z}|^2)z_j = -\mathrm{i}\sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}}\left\langle G_{\mathbf{m}}, \phi_j\right\rangle + r_j(\mathbf{z}, \eta), \tag{7.1}$$

where $r_j(\mathbf{z}, \eta)$ satisfies

$$\|r_j(\mathbf{z}, \eta)\|_{L^2(I)} \lesssim \delta^2\epsilon.$$

Proof. The proof is in [9], but for completeness we reproduce it here. Recall that $\phi[\mathbf{z}]$ satisfies identically equation (1.20). Furthermore, differentiating (1.20) w.r.t. $\mathbf{z}$ in any given direction $\widetilde{\mathbf{z}} \in \mathbb{C}^N$, we obtain

$$\begin{aligned}H[\mathbf{z}]D_{\mathbf{z}}\phi[\mathbf{z}]\widetilde{\mathbf{z}} &= \mathrm{i}D_{\mathbf{z}}^2\phi[\mathbf{z}](-\mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}, \widetilde{\mathbf{z}}) + \mathrm{i}D_{\mathbf{z}}\phi[\mathbf{z}]\left(D_{\mathbf{z}}(-\mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z})\widetilde{\mathbf{z}}\right) \\ &\quad + \sum_{\mathbf{m}\in\mathbf{R}_{\min}} D_{\mathbf{z}}(\mathbf{z}^{\mathbf{m}})\widetilde{\mathbf{z}}G_{\mathbf{m}} + D_{\mathbf{z}}\mathcal{R}_{\mathrm{rp}}(\mathbf{z})\widetilde{\mathbf{z}},\end{aligned} \tag{7.2}$$

with $H[\mathbf{z}]$ defined under (2.5). By $\eta \in \mathcal{H}_c[\mathbf{z}]$ we obtain the orthogonality relation

$$\langle \mathrm{i}\dot{\eta}, D_{\mathbf{z}}\phi[\mathbf{z}]\widetilde{\mathbf{z}}\rangle = -\left\langle \mathrm{i}\eta, D_{\mathbf{z}}^2\phi[\mathbf{z}](\dot{\mathbf{z}}, \widetilde{\mathbf{z}})\right\rangle.$$

By applying the inner product $\langle \eta, \cdot\rangle$ to equation (7.2), we have

$$\begin{aligned}\langle H[\mathbf{z}]\eta, D_{\mathbf{z}}\phi(\mathbf{z})\widetilde{\mathbf{z}}\rangle &= \left\langle \mathrm{i}\eta, D_{\mathbf{z}}^2\phi(\mathbf{z})(\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}, \widetilde{\mathbf{z}})\right\rangle + \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \left\langle \eta, \left(D_{\mathbf{z}}\left(\mathbf{z}^{\mathbf{m}}\right)\widetilde{\mathbf{z}}\right)G_{\mathbf{m}}\right\rangle \\ &\quad + \left\langle \eta, D_{\mathbf{z}}\mathcal{R}_{\mathrm{rp}}(\mathbf{z})\widetilde{\mathbf{z}}\right\rangle,\end{aligned}$$

where we exploited the selfadjointness of $H[\mathbf{z}]$ and the orthogonality in Lemma 2.1. Thus, applying $\langle \cdot, D_{\mathbf{z}}\phi(\mathbf{z})\widetilde{\mathbf{z}}\rangle$ to equation (2.5) for η and using the last two equalities, we obtain

$$\begin{aligned}
&\left\langle \mathrm{i}D_{\mathbf{z}}\phi(\mathbf{z})(\dot{\mathbf{z}}+\mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}), D_{\mathbf{z}}\phi(\mathbf{z})\widetilde{\mathbf{z}}\right\rangle = \left\langle \mathrm{i}\eta, D_{\mathbf{z}}^2\phi(\mathbf{z})\left(\dot{\mathbf{z}}+\mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}, \widetilde{\mathbf{z}}\right)\right\rangle + \left\langle \eta, D_{\mathbf{z}}\mathcal{R}_{\mathrm{rp}}[\mathbf{z}]\widetilde{\mathbf{z}}\right\rangle \\
&\quad + \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \left\langle \eta, \left(D_{\mathbf{z}}\left(\mathbf{z}^{\mathbf{m}}\right)\widetilde{\mathbf{z}}\right)G_{\mathbf{m}}\right\rangle + \left\langle \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}}G_{\mathbf{m}} + \mathcal{R}_{\mathrm{rp}}[\mathbf{z}], D_{\mathbf{z}}\phi(\mathbf{z})\widetilde{\mathbf{z}}\right\rangle \\
&\quad + \left\langle F(\mathbf{z},\eta) + |\eta|^2\eta, D_{\mathbf{z}}\phi[\mathbf{z}]\widetilde{\mathbf{z}}\right\rangle.
\end{aligned} \tag{7.3}$$

First since $D_{\mathbf{z}}\phi[0]\widetilde{\mathbf{z}} = \widetilde{\mathbf{z}}\cdot\boldsymbol{\phi}$, we have

$$\left\langle \mathrm{i}D_{\mathbf{z}}\phi[\mathbf{z}](\dot{\mathbf{z}}+\mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}), D_{\mathbf{z}}\phi(\mathbf{z})\widetilde{\mathbf{z}}\right\rangle = \sum_{j=1}^{N}\Re(\mathrm{i}(\dot{z}_j + \mathrm{i}\varpi_j(|\mathbf{z}|^2)z_j)\overline{\widetilde{z}_j}) + r(\mathbf{z},\widetilde{\mathbf{z}}), \tag{7.4}$$

where

$$\begin{aligned}
r(\mathbf{z},\widetilde{\mathbf{z}}) = &\left\langle \mathrm{i}\left(D_{\mathbf{z}}\phi(\mathbf{z}) - D_{\mathbf{z}}\phi(0)\right)(\dot{\mathbf{z}}+\mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}), D_{\mathbf{z}}\phi(\mathbf{z})\widetilde{\mathbf{z}}\right\rangle \\
&+ \left\langle \mathrm{i}D_{\mathbf{z}}\phi(0)(\dot{\mathbf{z}}+\mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}), \left(D_{\mathbf{z}}\phi(\mathbf{z}) - D_{\mathbf{z}}\phi(0)\right)\widetilde{\mathbf{z}}\right\rangle.
\end{aligned} \tag{7.5}$$

Since $\|D_{\mathbf{z}}\phi(\mathbf{z}) - D_{\mathbf{z}}\phi(0)\|_{L^2} \lesssim |\mathbf{z}|^2 \lesssim \delta^2$ by Proposition 1.9 and inequality (1.26), by assumption (3.4) we have

$$\|r(\mathbf{z},\widetilde{\mathbf{z}})\|_{L^2(I)} \lesssim \delta^2\epsilon \text{ for all } \widetilde{\mathbf{z}} = \mathbf{e}_1, \mathrm{i}\mathbf{e}_1, \cdots, \mathbf{e}_N, \mathrm{i}\mathbf{e}_N. \tag{7.6}$$

Setting

$$\begin{aligned}
&\widetilde{r}(\mathbf{z},\widetilde{\mathbf{z}},\eta) := \left\langle \mathrm{i}\eta, D_{\mathbf{z}}^2\phi(\mathbf{z})\left(\dot{\mathbf{z}}+\mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}, \widetilde{\mathbf{z}}\right)\right\rangle + \left\langle \eta, D_{\mathbf{z}}\mathcal{R}_{\mathrm{rp}}(\mathbf{z})\widetilde{\mathbf{z}}\right\rangle \\
&\quad + \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \left\langle \eta, \left(D_{\mathbf{z}}\left(\mathbf{z}^{\mathbf{m}}\right)\widetilde{\mathbf{z}}\right)G_{\mathbf{m}}\right\rangle \\
&\quad + \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \left\langle \mathbf{z}^{\mathbf{m}}G_{\mathbf{m}}, \left(D_{\mathbf{z}}\phi(\mathbf{z}) - D_{\mathbf{z}}\phi(0)\right)\widetilde{\mathbf{z}}\right\rangle + \left\langle \mathcal{R}_{\mathrm{rp}}(\mathbf{z}), D_{\mathbf{z}}\phi(\mathbf{z})\widetilde{\mathbf{z}}\right\rangle + \left\langle F(\mathbf{z},\eta), D_{\mathbf{z}}\phi(\mathbf{z})\widetilde{\mathbf{z}}\right\rangle,
\end{aligned} \tag{7.7}$$

by by assumption (3.4) we have we have

$$\|\widetilde{r}(\mathbf{z},\widetilde{\mathbf{z}},\eta)\|_{L^2(I)} \lesssim \delta^2\epsilon \text{ for all } \widetilde{\mathbf{z}} = \mathbf{e}_1, \mathrm{i}\mathbf{e}_1, \cdots, \mathbf{e}_N, \mathrm{i}\mathbf{e}_N. \tag{7.8}$$

Therefore, since $D\phi(0)\mathrm{i}^k\mathbf{e}_j = \mathrm{i}^k\phi_j$ $(k = 0, 1)$, we have

$$\begin{aligned}
-\mathrm{Im}\left(\partial_t z_j + \mathrm{i}\varpi_j(|\mathbf{z}|^2)z_j\right) &= \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \left\langle \mathbf{z}^{\mathbf{m}}G_{\mathbf{m}}, \phi_j\right\rangle - r(\mathbf{z},\mathbf{e}_j) + \widetilde{r}(\mathbf{z},\mathbf{e}_j,\eta), \\
\mathrm{Re}\left(\partial_t z_j + \mathrm{i}\varpi_j(|\mathbf{z}|^2)z_j\right) &= \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \left\langle \mathbf{z}^{\mathbf{m}}G_{\mathbf{m}}, \mathrm{i}\phi_j\right\rangle - r(\mathbf{z},\mathrm{i}\mathbf{e}_j) + \widetilde{r}(\mathbf{z},\mathrm{i}\mathbf{e}_j,\eta).
\end{aligned}$$

Since $G_{\mathbf{m}}$ and ϕ_j are $\mathbb{R}$-valued (see Lemma 1.7), we have

$$\dot{z}_j + \mathrm{i}\varpi_j(|\mathbf{z}|^2)z_j = -\mathrm{i}\sum_{\mathbf{m}}\left\langle G_{\mathbf{m}}, \phi_j\right\rangle\mathbf{z}^{\mathbf{m}} - r(\mathbf{z},\mathrm{i}\mathbf{e}_j) + \mathrm{i}r(\mathbf{z},\mathbf{e}_j) + \widetilde{r}(\mathbf{z},\mathrm{i}\mathbf{e}_j,\eta) - \mathrm{i}\widetilde{r}(\mathbf{z},\mathbf{e}_j,\eta).$$

Therefore, from (7.6) and (7.8), we have the conclusion with $r_j(\mathbf{z},\eta) = -r(\mathbf{z},\mathrm{i}\mathbf{e}_j) + \mathrm{i}r(\mathbf{z},\mathbf{e}_j) + \widetilde{r}(\mathbf{z},\mathrm{i}\mathbf{e}_j,\eta) - \mathrm{i}\widetilde{r}(\mathbf{z},\mathbf{e}_j,\eta)$. □

Our next task, is to examine the terms $\mathbf{z}^{\mathbf{m}}$. We need to show that these terms satisfy $\mathbf{z}^{\mathbf{m}} \xrightarrow{t\to+\infty} 0$, that is they are damped by nonlinear interaction with the radiation terms. In order to do so, we expand the variable v, defined in (2.19), in a part resonating with the discrete modes $\mathbf{z}$, which will yield the damping, and a remainder which we denote by g.

8. Smoothing Estimate for g

In analogy to [4,11,51] and a large literature, we will introduce and bound an auxiliary variable, g here. It appears to be impossible to bound g or analogues of g by means of virial type inequalities. We will use instead Kato–smoothing, as in [4,11,51]. Fortunately, the fact that the cubic nonlinearity is long range is immaterial, thanks to the cutoff χ_{B^2} in front of $|\eta|^2\eta$ in the equation of v.

The following is elementary and the proof is skipped.

Lemma 8.1. *0 is neither an eigenvalue nor a resonance for the operator H_{N+1}.*

□

We recall that we have the kernel for $x < y$, with an analogous formula for $x > y$,

$$R_{H_{N+1}}(z)(x,y) = \frac{T(\sqrt{z})}{2\mathrm{i}\sqrt{z}} f_-(x,\sqrt{z}) f_+(y,\sqrt{z}) = \frac{T(\sqrt{z})}{2\mathrm{i}\sqrt{z}} e^{\mathrm{i}\sqrt{z}(x-y)} m_-(x,\sqrt{z}) m_+(y,\sqrt{z}), \tag{8.1}$$

where the Jost functions $f_\pm(x,\sqrt{z}) = e^{\pm\mathrm{i}\sqrt{z}x} m_\pm(x,\sqrt{z})$ solve $(-\Delta + V_{N+1})\, u = zu$ with

$$\lim_{x\to+\infty} m_+(x,\sqrt{z}) = 1 = \lim_{x\to-\infty} m_-(x,\sqrt{z}).$$

These functions satisfy, see Lemma 1 p. 130 [16],

$$|m_\pm(x,\sqrt{z}) - 1| \le C_1 \langle \max\{0,\mp x\}\rangle \langle\sqrt{z}\rangle^{-1} \left| \int_x^{\pm\infty} \langle y\rangle |V_{N+1}(y)| dy \right| \tag{8.2}$$

$$|m_\pm(x,k) - 1| \le \left\langle\sqrt{z}\right\rangle^{-1} \left| \int_x^{\pm\infty} |V_{N+1}(y)| dy \right| \exp\left(\left\langle\sqrt{z}\right\rangle^{-1} \left| \int_x^{\pm\infty} |V_{N+1}(y)| dy \right| \right), \tag{8.3}$$

while, by Lemma 8.1, $T(k) = \alpha k(1+o(1))$ near $k = 0$ for some $\alpha \in \mathbb{R}$, see [60, formula (2.45)], and $T(k) = 1 + O(1/k)$ for $k \to \infty$ and $T \in C^0(\mathbb{R})$, see Theorem 1 [16].

Looking at the equation for v, (2.19), we introduce the functions

$$\rho_{\mathbf{m}} := -R^+_{H_{N+1}}(\boldsymbol{\omega}\cdot\mathbf{m})\widetilde{G}_{\mathbf{m}}, \tag{8.4}$$

which solve

$$(H_{N+1} - \boldsymbol{\omega}\cdot\mathbf{m})\rho_{\mathbf{m}} = -\widetilde{G}_{\mathbf{m}} \tag{8.5}$$

and we set

$$g = v + Z(\mathbf{z}) \text{ where } Z(\mathbf{z}) := -\sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}} \rho_{\mathbf{m}}, \tag{8.6}$$

which is analogous the expansion of $\overrightarrow{h}$ in p. 86 in Buslaev and Perelman [4] or also to the formula under (4.5) in Merle and Raphael [45]. An elementary computation yields

$$\mathrm{i}\partial_t g = H_{N+1}g - \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \left(\mathrm{i}\partial_t \left(\mathbf{z}^{\mathbf{m}}\right) - \boldsymbol{\omega}\cdot\mathbf{m}\,\mathbf{z}^{\mathbf{m}}\right)\rho_{\mathbf{m}} + \mathcal{R}_v$$

or, equivalently,

$$g(t) = e^{-\mathrm{i}tH_{N+1}}v(0) + \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}}(0)e^{-\mathrm{i}tH_{N+1}}R^{+}_{H_{N+1}}(\boldsymbol{\omega}\cdot\mathbf{m})\widetilde{G}_{\mathbf{m}} \tag{8.7}$$

$$- \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathrm{i}\int_0^t e^{-\mathrm{i}(t-t')H_{N+1}}\left(\partial_t\left(\mathbf{z}^{\mathbf{m}}\right) + \mathrm{i}\boldsymbol{\omega}\cdot\mathbf{m}\,\mathbf{z}^{\mathbf{m}}\right)\rho_{\mathbf{m}}dt' \tag{8.8}$$

$$- \mathrm{i}\int_0^t e^{-\mathrm{i}(t-t')H_{N+1}}\mathcal{T}\left(2\chi'_{B^2}\partial_x + \chi''_{B^2}\right)\tilde{\eta}dt' \tag{8.9}$$

$$- \mathrm{i}\int_0^t e^{-\mathrm{i}(t-t')H_{N+1}}\left(\langle \mathrm{i}\varepsilon\partial_x\rangle^{-N}[V_{N+1}, \langle \mathrm{i}\varepsilon\partial_x\rangle^{N}]v + \mathcal{T}\chi_{B^2}\mathcal{R}_{\widetilde{\eta}}\right). \tag{8.10}$$

We will prove the following, where we use the weighted spaces defined in (1.9).

Proposition 8.2. *For $S > 4$ There exist constants $c_0 > 0$ and $C(C_0)$ such that*

$$\|g\|_{L^2(I,L^{2,-S}(\mathbb{R}))} \lesssim \varepsilon^{-N}B^{2+2\tau}\delta + \varepsilon\epsilon + \epsilon^2. \tag{8.11}$$

To prove Proposition 8.2 we will need to bound one by one the terms in (8.7)–(8.10) in various lemmas.

Lemma 8.1 implies that H_{N+1} is a *generic* operator, and that in particular the following Kato smoothing holds, which is sufficient for our purposes. The proof is standard, is similar for example to Lemma 3.3 [14] and we skip it.

Lemma 8.3. *For any $S > 3/2$ there exists a fixed $c(S)$ s.t.*

$$\|\langle x\rangle^{-S}e^{-\mathrm{i}H_{N+1}t}f\|_{L^2(\mathbb{R}^2)} \le c(S)\|f\|_{L^2(\mathbb{R})} \text{ for all } f\in L^2(\mathbb{R}). \tag{8.12}$$

□

Lemma 8.3, inequality (5.18), the definition of w in (3.3), Lemma 2.2, the Modulation Lemma 2.1 and the conservation of mass and of energy yield

$$\begin{aligned}\|e^{-\mathrm{i}tH_{N+1}}v(0)\|_{L^2(\mathbb{R},L^{2,-S}(\mathbb{R}))} &\lesssim \|v(0)\|_{L^2} \lesssim \varepsilon^{-N}B^2\|w(0)\|_{\widetilde{\Sigma}} \lesssim \varepsilon^{-N}B^2\|\widetilde{\eta}(0)\|_{H^1}\\ &\lesssim \varepsilon^{-N}B^2\|\eta(0)\|_{H1} \lesssim \varepsilon^{-N}B^2\|u_0\|_{H^1} \le \varepsilon^{-N}B^2\delta.\end{aligned} \tag{8.13}$$

Next, we have the following lemma, which is standard in this theory, see [4,11,51].

Lemma 8.4. *Let Λ be a finite subset of $(0,\infty)$ and let $S > 4$. Then there exists a fixed $c(S,\Lambda)$ s.t. for every $t \ge 0$ and $\lambda \in \Lambda$*

$$\|e^{-\mathrm{i}H_{N+1}t}R^{+}_{H_{N+1}}(\lambda)f\|_{L^{2,-S}(\mathbb{R})} \le c(S,\Lambda)\langle t\rangle^{-\frac{3}{2}}\|f\|_{L^{2,S}(\mathbb{R})} \text{ for all } f\in L^{2,S}(\mathbb{R}). \tag{8.14}$$

Proof (sketch). This lemma is similar to Proposition 2.2 [51]. We will consider case $t \geq 1$, while we skip the simpler case $t \in [0, 1]$.
Consider $\Lambda \subset (a, b)$ with $[a, b] \subset \mathbb{R}_+$. Let $\mathbf{g} \in C^\infty((a/2, +\infty), [0, 1])$ such that $\mathbf{g} \equiv 1$ in $[a, +\infty)$. Let $\mathbf{g}_1 \in C_c^\infty(\mathbb{R}, [0, 1])$ with $\mathbf{g}_1 = 1 - \mathbf{g}$ in $\mathbb{R}_+$. Next we consider

$$\begin{aligned}
&\| \langle x\rangle^{-S} e^{-\mathrm{i}H_{N+1}t} R^+_{H_{N+1}}(\lambda)\mathbf{g}_1(H_{N+1}) f\|_{L^2(\mathbb{R})} \\
&\quad\lesssim \| \langle x\rangle^{-S+2} e^{-\mathrm{i}H_{N+1}t} R^+_{H_{N+1}}(\lambda)\mathbf{g}_1(H_{N+1}) f\|_{L^\infty(\mathbb{R})} \\
&\quad\lesssim t^{-\frac{3}{2}} \| \langle x\rangle R^+_{H_{N+1}}(\lambda)\mathbf{g}_1(H_{N+1}) f\|_{L^1(\mathbb{R})} \\
&\quad\lesssim t^{-\frac{3}{2}} \| \langle x\rangle^2 R^+_{H_{N+1}}(\lambda)\mathbf{g}_1(H_{N+1}) f\|_{L^2(\mathbb{R})} \lesssim t^{-\frac{3}{2}} \| \langle x\rangle^2 f\|_{L^2(\mathbb{R})}
\end{aligned}$$

where we used Theorem 3.1 [49] and Lemma 5.3, since $R^+_{H_{N+1}}(\lambda)\mathbf{g}_1(H_{N+1}) = \mathbf{g}_2(H_{N+1})$, with $\mathbf{g}_2 \in C_c^\infty(\mathbb{R}, \mathbb{R})$, is a 0 order ΨDO with symbols satisfying the inequalities (5.4) uniformly as λ takes finitely many values. Here we used Theorem 8.7 in Dimassi and Sjöstrand [21].
Next we consider

$$\begin{aligned}
&\langle x\rangle^{-S}\mathbf{g}(H_{N+1}) e^{-\mathrm{i}H_{N+1}t} R^+_{H_{N+1}}(\lambda)\langle y\rangle^{-S} \\
&\quad= \lim_{\sigma\to 0^+} e^{-\mathrm{i}\lambda t}\langle x\rangle^{-S}\int_t^{+\infty} e^{-\mathrm{i}(H_{N+1}-\lambda-\mathrm{i}\sigma)s}\mathbf{g}(H_{N+1})\, ds\langle y\rangle^{-S}.
\end{aligned} \tag{8.15}$$

Using the distorted plane waves $\psi(x, k)$ associated to H_{N+1}, see (1.9) [60], we can write the following integral kernel, ignoring irrelevant constants,

$$\begin{aligned}
&\langle x\rangle^{-S}\left(e^{-\mathrm{i}(H_{N+1}-\lambda-\mathrm{i}\sigma)s}\mathbf{g}(H_{N+1})\right)(x, y)\langle y\rangle^{-S} \\
&\quad= \langle x\rangle^{-S}\langle y\rangle^{-S}\int_{\mathbb{R}_+} e^{-\mathrm{i}(k^2-\lambda-\mathrm{i}\sigma)s-\mathrm{i}k(x-y)}\mathbf{g}\left(k^2\right)\overline{m_+(x, k)}m_+(y, k)dk \\
&\quad+ \langle x\rangle^{-S}\langle y\rangle^{-S}\int_{\mathbb{R}_-} e^{-\mathrm{i}(k^2-\lambda-\mathrm{i}\sigma)s-\mathrm{i}k(x-y)}\mathbf{g}\left(k^2\right)\overline{m_-(x, -k)}m_-(y, -k)dk.
\end{aligned} \tag{8.16}$$

Take for example the first term in the right hand side of (8.16). Then, from $\frac{\mathrm{i}}{2ks}\frac{d}{dk}e^{-\mathrm{i}k^2s} = e^{-\mathrm{i}k^2s}$ and taking the limit $\sigma \to 0^+$, we can write it as

$$\langle x\rangle^{-S}\langle y\rangle^{-S}\int_{\mathbb{R}_+} e^{-\mathrm{i}(k^2-\lambda)s}\left(-\frac{d}{dk}\frac{\mathrm{i}}{2ks}\right)^3\left(e^{-\mathrm{i}k(x-y)}\mathbf{g}\left(k^2\right)\overline{m_+(x, k)}m_+(y, k)\right)dk,$$

which, using for instance the bounds on the k derivatives of $m_\pm$ in Lemma 2.1 [25], is absolutely integrable in k ($\mathbf{g}$ is constant outside a bounded interval) and is bounded in absolute value by

$$\lesssim \langle x\rangle^{-S+3}\langle y\rangle^{-S+3}s^{-3}.$$

Integrating in $[t, \infty)$ we obtain an upper bound $\sim \langle x\rangle^{-S+3}\langle y\rangle^{-S+3}t^{-2}$ for integral kernel of the operator of the corresponding part in (8.16), which gives an upper bound of t^{-2} in the corresponding contribution in (8.14). So we get the desired result for $t \geq 1$. $\square$

Lemma 8.4, (2.20), (2.15) imply that

$$\sum_{\mathbf{m}\in\mathbf{R}_{\min}} |\mathbf{z}^{\mathbf{m}}(0)| \| e^{-\mathrm{i}tH_{N+1}} \rho_{\mathbf{m}} \|_{L^2(\mathbb{R},L^{2,-S}(\mathbb{R}))} \lesssim \sum_{\mathbf{m}\in\mathbf{R}_{\min}} |\mathbf{z}^{\mathbf{m}}(0)| \| \widetilde{G}_{\mathbf{m}} \|_{L^{2,S}(\mathbb{R})}$$
$$\lesssim \delta^2 \| \langle x\rangle^S \langle \mathrm{i}\varepsilon\partial_x\rangle^{-N} \langle x\rangle^{-S} \|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} \sup_{\mathbf{m}\in\mathbf{R}_{\min}} \| \langle x\rangle^S \mathcal{A}^* \chi_{B^2} G_{\mathbf{m}} \|_{L^2(\mathbb{R})} \lesssim \delta^2, \tag{8.17}$$

where we used the fact that $T := \langle x\rangle^S \langle \mathrm{i}\varepsilon\partial_x\rangle^{-N} \langle x\rangle^{-S}$ has integral kernel

$$T(x,y) = \varepsilon^{-1} \langle x\rangle^S \langle y\rangle^{-S} f\left(\varepsilon^{-1}(x-y)\right), \text{ where } \widehat{f}(k) = \langle k\rangle^{-N} \tag{8.18}$$

where f is a continuous rapidly decreasing function, so that it is easy to see that Young's inequality, see [53, Theorem 0.3.1], gives $\|T\|_{L^2\to L^2} \lesssim 1$ uniformly in $\varepsilon \in (0,1]$.

Notice that in this way we gave a bound on the contribution of the terms in the right hand side in (8.7) to (8.11) .

It is easy to bound the contribution to (8.11) of the term (8.8). Indeed, using the identity

$$\left(D_{\mathbf{z}}\mathbf{z}^{\mathbf{m}}\right)(\mathrm{i}\boldsymbol{\omega}\mathbf{z}) = \mathrm{i}\mathbf{m}\cdot\boldsymbol{\omega}\,\mathbf{z}^{\mathbf{m}}\ , \text{ where } \boldsymbol{\omega}\mathbf{z} := (\omega_1 z_1, \cdots, \omega_N z_N), \tag{8.19}$$

we have

$$\begin{aligned}\partial_t\left(\mathbf{z}^{\mathbf{m}}\right) + \mathrm{i}\boldsymbol{\omega}\cdot\mathbf{m}\mathbf{z}^{\mathbf{m}} &= D_{\mathbf{z}}\mathbf{z}^{\mathbf{m}}\left(\partial_t\mathbf{z} + \boldsymbol{\omega}\mathbf{z}\right) = D_{\mathbf{z}}\mathbf{z}^{\mathbf{m}}\left(\dot{\mathbf{z}} + \mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}\right) \\ &\quad + D_{\mathbf{z}}\mathbf{z}^{\mathbf{m}}\mathrm{i}\left(\boldsymbol{\omega} - \boldsymbol{\varpi}(|\mathbf{z}|^2)\right)\mathbf{z} \\ &= D_{\mathbf{z}}\mathbf{z}^{\mathbf{m}}\left(\dot{\mathbf{z}} + \mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}\right) + \mathrm{i}\mathbf{m}\cdot\left(\boldsymbol{\omega} - \boldsymbol{\varpi}(|\mathbf{z}|^2)\right)\mathbf{z}^{\mathbf{m}}.\end{aligned}$$

From this and Lemma 8.4 and the bound $\|\widetilde{G}_{\mathbf{m}}\|_{L^{2,S}(\mathbb{R})} \lesssim 1$ in (8.17) we obtain

$$\sum_{\mathbf{m}\in\mathbf{R}_{\min}} \left\| \int_0^t e^{-\mathrm{i}(t-t')H_{N+1}} \left(\partial_t\left(\mathbf{z}^{\mathbf{m}}\right) + \mathrm{i}\boldsymbol{\omega}\cdot\mathbf{m}\mathbf{z}^{\mathbf{m}}\right)\rho_{\mathbf{m}} \right\|_{L^2(I,L^{2,-S}(\mathbb{R}))}$$
$$\lesssim \delta^2 \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \left(\|\dot{\mathbf{z}} + \mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}\|_{L^2(I,)} + \|\mathbf{z}^{\mathbf{m}}\|_{L^2(I,)} \right) \|\widetilde{G}_{\mathbf{m}}\|_{L^{2,S}(\mathbb{R})} \lesssim \delta^2\epsilon. \tag{8.20}$$

Now we look at the contribution to (8.11) of the term (8.9). We will need the following result about the Limiting Absorption Principle. The following is related to Lemma 5.7 [14].

Lemma 8.5. *For $S > 5/2$ and $\tau > 1/2$ we have*

$$\sup_{z\in\mathbb{R}} \|R^{\pm}_{H_{N+1}}(z)\|_{L^{2,\tau}(\mathbb{R})\to L^{2,-S}(\mathbb{R})} < \infty. \tag{8.21}$$

Proof. It is equivalent to show $\sup_{z\in\mathbb{R}} \| \langle x\rangle^{-S} R^{\pm}_{H_{N+1}}(z) \langle y\rangle^{-\tau} \|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} < \infty$. We will consider only the + case. We consider the square of the Hilbert–Schmidt norm

$$\int_{\mathbb{R}} dx\, \langle x\rangle^{-2S} \int_{\mathbb{R}} |R^+_{H_{N+1}}(x,y,z)|^2 \langle y\rangle^{-2\tau}\, dy = \int_{\mathbb{R}} dx\, \langle x\rangle^{-2S}$$

$$\int_{-\infty}^{x} |R^+_{H_{N+1}}(x,y,z)|^2 \langle y\rangle^{-2\tau} dy$$
$$+ \int_{\mathbb{R}} dx \langle x\rangle^{-2S} \int_x^{+\infty} |R^+_{H_{N+1}}(x,y,z)|^2 \langle y\rangle^{-2\tau} dy.$$

We will bound only the second term in the right hand side: for the first term the argument is similar. Recalling formula (8.1), we have to bound

$$\left|\frac{T(\sqrt{z})}{2\mathrm{i}\sqrt{z}}\right| \int_{x<y} \langle x\rangle^{-2S} |m_-(x,\sqrt{z})m_+(y,\sqrt{z})|^2 \langle y\rangle^{-2\tau} dxdy$$
$$\lesssim \int_{x<y} \langle x\rangle^{-2S} \langle y\rangle^{-2\tau} (1+\max(x,0)+\max(-y,0))^2 dxdy,$$

where we used the bound (8.2). Now, in the last integral we can distinguish the region $|y| \lesssim |x|$, where the corresponding contribution can be bounded by

$$\int_{\mathbb{R}^2} \langle x\rangle^{-2(S-2)} \langle y\rangle^{-2\tau} dxdy < \infty \text{ for } S > 5/2 \text{ and } \tau > 1/2,$$

and the region $|y| \gg |x|$, where we have the same bound, because $x < y$ and $|y| \gg |x|$ imply that $y > 0$, and hence $\max(-y,0) = 0$. □

We will also need the following formulas that we take from Mizumachi [46, Lemma 4.5] and to which we refer for the proof.

Lemma 8.6. *Let for* $g \in \mathcal{S}(\mathbb{R}\times\mathbb{R},\mathbb{C})$

$$U(t,x) = \frac{1}{\sqrt{2\pi \mathrm{i}}} \int_{\mathbb{R}} e^{-\mathrm{i}\lambda t} \left(R^-_{H_{N+1}}(\lambda) + R^+_{H_{N+1}}(\lambda)\right) \mathcal{F}_t^{-1} g(\lambda,\cdot) d\lambda,$$

where $\mathcal{F}_t^{-1}$ *is the inverse Fourier transform in* t. *Then*

$$2\int_0^t e^{-\mathrm{i}(t-t')H_{N+1}} g(t')dt' = U(t,x) - \int_{\mathbb{R}_-} e^{-\mathrm{i}(t-t')H_{N+1}} g(t')dt' + \int_{\mathbb{R}_+} e^{-\mathrm{i}(t-t')H_{N+1}} g(t')dt'. \tag{8.22}$$

□

The last two lemmas give us the following smoothing estimate.

Lemma 8.7. *For* $S > 5/2$ *and* $\tau > 1/2$ *there exists a constant* $C(S,\tau)$ *such that we have*

$$\left\|\int_0^t e^{-\mathrm{i}(t-t')H_{N+1}} g(t')dt'\right\|_{L^2(\mathbb{R},L^{2,-S}(\mathbb{R}))} \le C(S,\tau)\|g\|_{L^2(\mathbb{R},L^{2,\tau}(\mathbb{R}))}. \tag{8.23}$$

Proof. We can use formula (8.22) and bound U, with the bound on the last two terms in the right hand side of (8.22) similar. So we have, taking Fourier transform in t,

$$\|U\|_{L^2_t L^{2,-S}} \le 2\sup_{\pm} \|R^{\pm}_{H_{N+1}}(\lambda)\widehat{g}(\lambda,\cdot)\|_{L^2_\lambda L^{2,-S}}$$
$$\le 2\sup_{\pm}\sup_{\lambda\in\mathbb{R}} \|R^{\pm}_{H_{N+1}}(\lambda)\|_{L^{2,\tau}\to L^{2,-S}} \|\widehat{g}(\lambda,x)\|_{L^{2,\tau}L^2_\lambda} \lesssim \|g\|_{L^2_t L^{2,\tau}}.$$

Notice that, while Lemma 8.6 is stated for $g \in \mathcal{S}(\mathbb{R}\times\mathbb{R},\mathbb{C})$, the estimate (8.23) extends to all $g \in L^2(\mathbb{R}, L^{2,\tau}(\mathbb{R}))$ by density. □

Remark 8.8. The above is basically Lemma 3.4 [14], which in turn is based on an argument in [46]. Unfortunately Lemma 3.4 [14] has a mistake, which however can be corrected using Lemma 8.6, as we did here.

We now examine the term in (8.9). By Lemma 8.7 we have

$$\begin{aligned}\| \int_0^t e^{-\mathrm{i}(t-t')H_{N+1}} \mathcal{T}\left(2\chi'_{B^2}\partial_x + \chi''_{B^2}\right)\tilde{\eta}dt'\|_{L^2(I,L^{2,-S}(\mathbb{R}))}\\ \lesssim \|\mathcal{T}\left(2\chi'_{B^2}\partial_x + \chi''_{B^2}\right)\tilde{\eta}\|_{L^2(I,L^{2,\tau}(\mathbb{R}))}.\end{aligned}$$

In order to bound the right hand side we expand

$$\begin{aligned}\mathcal{T}\left(2\chi'_{B^2}\partial_x + \chi''_{B^2}\right)\tilde{\eta} = \mathcal{T}\left(2\chi'_{B^2}\partial_x + \chi''_{B^2}\right) w \left(\zeta_A^{-1} - 1\right) - 2\mathcal{T}\chi'_{B^2}\zeta_A^{-2}\zeta'_A w\\ + \mathcal{T}\left(2\chi'_{B^2}\partial_x + \chi''_{B^2}\right) w.\end{aligned}$$

By $|\chi'_{B^2}\zeta_A^{-2}\zeta'_A| \lesssim A^{-1}|\chi'_{B^2}|$ and $1_{|x|\le 2B^2}|\zeta_A^{-1} - 1| \lesssim 1_{|x|\le 2B^2}\frac{B^2}{A}$, both of which are small, the main term is the one in the last line, which is the only one we discuss explicitly, because the others are similar, simpler and smaller. We decompose

$$\begin{aligned}\mathcal{T}\left(2\chi'_{B^2}\partial_x + \chi''_{B^2}\right) w = \mathcal{I} + \mathcal{II} \text{ where } \mathcal{I} := 1_{2^{-1}B^2\le|x|\le 3B^2}\mathcal{T}\left(2\chi'_{B^2}\partial_x + \chi''_{B^2}\right) w,\\ \mathcal{II} := 1_{\{|x|\le 2^{-1}B^2\}\cup\{|x|\ge 3B^2\}}\mathcal{T}.\end{aligned} \tag{8.24}$$

By Lemmas 5.1 and 5.6, we have

$$\begin{aligned}&\| \langle x\rangle^\tau \mathcal{I}\|_{L^2(I,L^2(\mathbb{R}))} = \| \langle x\rangle^\tau 1_{2^{-1}B^2\le|x|\le 3B^2}\mathcal{T}\left(2\chi'_{B^2}\partial_x + \chi''_{B^2}\right) w\|_{L^2(I,L^2(\mathbb{R}))}\\ &\lesssim B^{2\tau}\|\mathcal{T}\left(2\chi'_{B^2}\partial_x + \chi''_{B^2}\right) w\|_{L^2(I,L^2(\mathbb{R}))} \lesssim \varepsilon^{-N}B^{2\tau}\|2\chi'_{B^2}w' + \chi''_{B^2}w\|_{L^2(I,L^2(\mathbb{R}))}\\ &\lesssim \varepsilon^{-N}B^{2\tau-2}\|w'\|_{L^2(I,L^2(\mathbb{R}))} + \varepsilon^{-N}B^{2\tau-4}\|1_{B^2\le|x|\le 2B^2}w\|_{L^2(I,L^2(\mathbb{R}))}\\ &\lesssim \varepsilon^{-N}B^{2\tau-2}\|w\|_{L^2(I,\widetilde{\Sigma})} + \varepsilon^{-N}B^{2\tau-4}\| \langle x\rangle 1_{B^2\le|x|\le 2B^2}\|^{\frac12}_{L^1(\mathbb{R})}\|w\|_{L^2(I,\widetilde{\Sigma})}\\ &\lesssim \varepsilon^{-N}B^{2\tau-2}\|w\|_{L^2(I,\widetilde{\Sigma})} \le B^{-\frac12}\epsilon.\end{aligned} \tag{8.25}$$

By Lemma 5.6 we have

$$\begin{aligned}&\| \langle x\rangle^\tau \mathcal{II}\|_{L^2(\mathbb{R})} \lesssim \| \langle x\rangle^\tau 1_{\{|x|\le 2^{-1}B^2\}\cup\{|x|\ge 3B^2\}}\int e^{-\frac{|x-y|}{2\varepsilon}}\left(2\chi'_{B^2}w' + \chi''_{B^2}w\right)\|_{L^2(\mathbb{R})}\\ &\lesssim e^{-B^2}\|\mathcal{K}\left(\langle y\rangle^\tau\left(2\chi'_{B^2}w' + \chi''_{B^2}w\right)\right)\|_{L^2(\mathbb{R})},\end{aligned}$$

where the operator $\mathcal{K}f = \int \mathcal{K}(x,y)f(y)dy$ has integral kernel

$$\mathcal{K}(x,y) = \langle x\rangle^\tau e^{-|x-y|}\langle y\rangle^{-\tau}.$$

Since we have

$$\|\mathcal{K}\|^2_{L^2(\mathbb{R})\to L^2(\mathbb{R})} \le \|\mathcal{K}(\cdot,\cdot)\|^2_{L^2(\mathbb{R}\times\mathbb{R})} < +\infty,$$

by the bounds implicit in (8.25), we have

$$\| \langle x\rangle^\tau \mathcal{II}\|_{L^2(I,L^2(\mathbb{R}))} \lesssim e^{-B^2}\| \langle x\rangle^\tau\left(2\chi'_{B^2}w' + \chi''_{B^2}w\right)\|_{L^2(I,L^2(\mathbb{R}))}$$

$$\lesssim e^{-B^2} B^{2\tau} \| \left(2\chi'_{B^2} w' + \chi''_{B^2} w\right) \|_{L^2(I, L^2(\mathbb{R}))} \lesssim e^{-B^2/2} \epsilon. \tag{8.26}$$

We next consider the terms in (8.10), starting with

$$\begin{aligned}
&\| \langle x \rangle^{-S} \int_0^t e^{-\mathrm{i}(t-t')H_{N+1}} \langle \mathrm{i}\varepsilon \partial_x \rangle^{-N} [V_{N+1}, \langle \mathrm{i}\varepsilon \partial_x \rangle^N] v \|_{L^2(I, L^2(\mathbb{R}))} \\
&\quad \lesssim \| \langle x \rangle^{\tau} \langle \mathrm{i}\varepsilon \partial_x \rangle^{-N} [V_{N+1}, \langle \mathrm{i}\varepsilon \partial_x \rangle^N] v \|_{L^2(I, L^2(\mathbb{R}))} \\
&\lesssim \varepsilon \| \langle x \rangle^{-100} v \|_{L^2(I, L^2(\mathbb{R}))} \lesssim \varepsilon \left(\|\xi\|_{L^2(I, \widetilde{\Sigma})} + B^{-1} \|\xi\|_{L^2(I, \widetilde{\Sigma})} \right) \lesssim \varepsilon \epsilon,
\end{aligned} \tag{8.27}$$

where we used Lemma 5.12 in the first inequality in the last line, and (5.20) for the second inequality.

We now consider remaining contributions of (8.10) to (8.11). To start with, by Lemma 8.3 we have

$$\| \langle x \rangle^{-S} \int_0^t e^{-\mathrm{i}(t-t')H_{N+1}} \mathcal{T} \chi_{B^2} \mathcal{R}_{\widetilde{\eta}} dt' \|_{L^2(I, L^2(\mathbb{R}))} \lesssim \| \langle x \rangle^{\tau} \mathcal{T} \chi_{B^2} \mathcal{R}_{\widetilde{\eta}} \|_{L^2(I, L^2(\mathbb{R}))}.$$

The right hand side il less than $I + II$ where

$$\begin{aligned}
I &= \| 1_{|x| \le 3B^2} \langle x \rangle^{\tau} \mathcal{T} \chi_{B^2} \mathcal{R}_{\widetilde{\eta}} \|_{L^2(I, L^2(\mathbb{R}))} \\
II &= \| 1_{|x| \ge 3B^2} \langle x \rangle^{\tau} \mathcal{T} \chi_{B^2} \mathcal{R}_{\widetilde{\eta}} \|_{L^2(I, L^2(\mathbb{R}))}
\end{aligned}$$

We have

$$\begin{aligned}
&I \lesssim B^{2\tau} (I_1 + I_2) \\
&I_1 = \| P_c \left(-\mathrm{i} D_{\mathbf{z}} \phi[\mathbf{z}] \left(\dot{\mathbf{z}} + \mathrm{i} \boldsymbol{\varpi}(\mathbf{z}) \mathbf{z} \right) + \mathcal{R}_{\mathrm{rp}}[\mathbf{z}] + F[\mathbf{z}, \eta] + L[\mathbf{z}] \eta \right) \|_{L^2(I, L^2(\mathbb{R}))} \\
&I_2 = \| \chi_{B^2} P_c |\eta|^2 \eta \|_{L^2(I, L^2(\mathbb{R}))}.
\end{aligned}$$

By $\| P_c D_{\mathbf{z}} \phi[\mathbf{z}] \|_{\widetilde{\Sigma}} = O\left(\|\mathbf{z}\|^2 \right)$ because of $D_{\mathbf{z}} \phi[0] \widetilde{\mathbf{z}} = \boldsymbol{\phi} \cdot \widetilde{\mathbf{z}}$ for any $\widetilde{\mathbf{z}} \in \mathbb{C}^N$, it is easy to conclude

$$I_1 \lesssim \delta^2 \epsilon.$$

We have

$$\begin{aligned}
I_2 &\lesssim \| \chi_{B^2} |\eta|^2 \eta \|_{L^2(I, L^2(\mathbb{R}))} + \| \chi_{B^2} P_d |\eta|^2 \eta \|_{L^2(I, L^2(\mathbb{R}))} \\
&\lesssim \sum_{j=1}^N \| \langle x \rangle^{\tau} \chi_{B^2} \phi_j (|\eta|^2 \eta, \phi_j) \|_{L^2(I, L^2(\mathbb{R}))} + \|\eta\|^2_{L^\infty(\mathbb{R}, H^1(\mathbb{R}))} \|w\|_{L^2(I, L^2(|x| \le 2B^2))} \\
&\lesssim \sum_{j=1}^N \|\eta\|^2_{L^\infty(\mathbb{R}, H^1(\mathbb{R}))} \|w\|_{L^2(I, \widetilde{\Sigma})} + B^2 \|\eta\|^2_{L^\infty(\mathbb{R}, H^1(\mathbb{R}))} \|w\|_{L^2(I, \widetilde{\Sigma})} \\
&\lesssim B^2 \|\eta\|^2_{L^\infty(\mathbb{R}, H^1(\mathbb{R}))} \|w\|_{L^2(I, \widetilde{\Sigma})} \lesssim B^2 \delta^2 \epsilon.
\end{aligned}$$

So we conclude

$$I \lesssim B^{2\tau+2} \delta^2 \epsilon. \tag{8.28}$$

Turning to the analysis of II, we have

$$\begin{aligned} II &\lesssim \| 1_{|x|\geq 3B^2} \langle x \rangle^{\tau} \mathcal{T} \langle x \rangle^{-\tau} 1_{|x|\leq 2B^2} \|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} \| \langle x \rangle^{\tau} \chi_{B^2} \mathcal{R}_{\widetilde{\eta}} \|_{L^2(I,L^2(\mathbb{R}))} \\ &\lesssim \| \langle x \rangle^{\tau} \chi_{B^2} \mathcal{R}_{\widetilde{\eta}} \|_{L^2(I,L^2(\mathbb{R}))} \lesssim B^{2\tau+2} \delta^2 \epsilon \end{aligned} \tag{8.29}$$

by an analysis similar to the operator $\mathcal{K}$ above and to the analysis of I.

Taken together, (8.13), (8.17), (8.20), (8.26)–(8.29) yield Proposition 8.2, and so its proof is completed. □

Before the proof of Propositions 3.2 and 3.6 we need an analogue of the coercivity results in Sect. 5 [29].

9. Coercivity Results

Our main aim is to prove the following.

Proposition 9.1. *We have*

$$\|w\|_{L^2_{-\frac{a}{10}}} \lesssim \|\xi\|_{\widetilde{\Sigma}} + e^{-\frac{B}{20}} \|w'\|_{L^2}. \tag{9.1}$$

Before proving Proposition 9.1 we consider the following partial inversion of (2.18), which is our analogue of Formula (62) in [29].

Lemma 9.2. *We have*

$$P_c\left(\chi_{B^2}\widetilde{\eta}\right) = \prod_{j=1}^{N} R_H(\omega_j) P_c \mathcal{A} \langle \mathrm{i}\varepsilon\partial_x \rangle^N v. \tag{9.2}$$

Proof. We first claim

$$\mathcal{A}\mathcal{A}^* = A_1 \circ \cdots \circ A_N \circ A_N^* \circ \cdots \circ A_1^* = \prod_{j=1}^{N} (H - \omega_j). \tag{9.3}$$

Then, using (9.3), from (2.15) and (2.18) we have

$$\begin{aligned} &\prod_{j=1}^{N} R_H(\omega_j) P_c A_1 \circ \cdots \circ A_N \langle \mathrm{i}\varepsilon\partial_x \rangle^N v \\ &\quad = \prod_{j=1}^{N} R_H(\omega_j) P_c A_1 \circ \cdots \circ A_N \circ A_N^* \circ \cdots \circ A_1^* \chi_{B^2} \widetilde{\eta} \\ &\quad = \prod_{j=1}^{N} R_H(\omega_j) P_c \prod_{j=1}^{N} (H - \omega_j) \chi_{B^2} \widetilde{\eta} = P_c\left(\chi_{B^2}\widetilde{\eta}\right). \end{aligned}$$

Thus, it remains to prove (9.3). First, from (1.22), we have

$$A_N \circ A_N^* = H_N - \omega_N.$$

For $2 \le j \le N$, we assume (notice that the Schrödinger operator H_j is fixed)

$$A_j \circ \cdots \circ A_N \circ A_N^* \circ \cdots A_j^* = \prod_{k=j}^{N} (H_j - \omega_k).$$

Then, by

$$\begin{aligned} A_{j-1}(H_j - \omega_k) = A_{j-1}(A_{j-1}^* A_{j-1} + \omega_{j-1} - \omega_k) &= (A_{j-1}A_{j-1}^* + \omega_{j-1} - \omega_k)A_{j-1} \\ &= (H_{j-1} - \omega_k)A_{j-1}, \end{aligned}$$

we have

$$\begin{aligned} A_{j-1} \circ \cdots \circ A_N \circ A_N^* \circ \cdots A_{j-1}^* &= A_{j-1} \prod_{k=j}^{N} (H_j - \omega_k) A_{j-1}^* \\ &= \prod_{k=j}^{N} (H_{j-1} - \omega_k) A_{j-1} \circ A_{j-1}^* \\ &= \prod_{k=j}^{N} (H_{j-1} - \omega_k)\, (H_{j-1} - \omega_{j-1}) = \prod_{k=j-1}^{N} (H_{j-1} - \omega_k). \end{aligned}$$

Therefore, we have (9.3) by induction. □

The proof of Lemma 9.3 is postponed to Appendix A.

Lemma 9.3. *We have* $\| \prod_{j=1}^{N} R_H(\omega_j) P_c \mathcal{A} \langle \mathrm{i}\varepsilon \partial_x \rangle^N \|_{L^2_{-\frac{a}{20}} \to L^2_{-\frac{a}{10}}} \lesssim 1$ *uniformly for* $0 < \varepsilon \le 1$.

□

We continue this section, assuming Lemma 9.3.

Lemma 9.4. *We have*

$$\|\chi_{B^2} \widetilde{\eta}\|_{L^2_{-\frac{a}{10}}} \lesssim \|v\|_{L^2_{-\frac{a}{20}}} + e^{-B} \|\widetilde{\eta}\|_{L^2_{-\frac{a}{10}}}.$$

Proof. First,

$$\|\chi_{B^2} \widetilde{\eta}\|_{L^2_{-\frac{a}{10}}} \le \|e^{-\frac{a}{10}\langle x \rangle} P_c(\chi_{B^2} \widetilde{\eta})\|_{L^2} + \|e^{-\frac{a}{10}\langle x \rangle} P_d \left(\chi_{B^2} \widetilde{\eta}\right) \|_{L^2}. \tag{9.4}$$

Then, by Lemmas 9.2 and 9.3, we have

$$\|e^{-\frac{a}{10}\langle x \rangle} P_c(\chi_{B^2} \widetilde{\eta})\|_{L^2} \lesssim \|v\|_{L^2_{-\frac{a}{20}}}. \tag{9.5}$$

On the other hand, from $P_d \widetilde{\eta} = 0$ and (2.2), we have

$$P_d(\chi_{B^2} \widetilde{\eta}) = \sum_{j=1}^{N} (\chi_{B^2} \widetilde{\eta}, \phi_j)\phi_j = \sum_{j=1}^{N} (\widetilde{\eta}, \left(\chi_{B^2} - 1\right) \phi_j)\phi_j.$$

Then, since $\|e^{\frac{a}{10}\langle x \rangle}(\chi_{B^2} - 1)\phi_j\|_{L^2} \lesssim e^{-(a_1 - \frac{a}{10})B^2} \lesssim e^{-B}$, we have

$$\|e^{-\frac{a}{10}\langle x \rangle} P_d \left(\chi_{B^2} \widetilde{\eta}\right) \|_{L^2} \lesssim e^{-B} \|\widetilde{\eta}\|_{L^2_{-\frac{a}{10}}}. \tag{9.6}$$

By (9.4), (9.5) and (9.6) we have the conclusion. □

Proof of Proposition 9.1. First we split

$$\|w\|_{L^2_{-\frac{a}{10}}} \le \|\chi_{B^2} w\|_{L^2_{-\frac{a}{10}}} + \|(1-\chi_{B^2})e^{-\frac{a}{10}\langle x\rangle} w\|_{L^2}. \tag{9.7}$$

For the 2nd term of r.h.s. of (9.7), using Corollary 5.2, we have

$$\|(1-\chi_{B^2})e^{-\frac{a}{10}\langle x\rangle} w\|_{L^2} \le \|(1-\chi_{B^2})e^{-\frac{a}{20}}\|_{L^\infty}\|e^{-\frac{a}{20}\langle x\rangle} w\|_{L^2} \lesssim e^{-\frac{aB^2}{20}}\|w\|_{\widetilde{\Sigma}}. \tag{9.8}$$

For the 1st term of the r.h.s. of (9.7), by $\|\zeta_A\|_{L^\infty} \le 1$ and Lemma 9.4,

$$\begin{aligned}\|\chi_{B^2} w\|_{L^2_{-\frac{a}{10}}} &\le \|\chi_{B^2} e^{-\frac{a}{10}\langle x\rangle}\widetilde{\eta}\|_{L^2} \lesssim \|v\|_{L^2_{-\frac{a}{20}}} + e^{-B}\|\widetilde{\eta}\|_{L^2_{-\frac{a}{10}}}\\ &\lesssim \|\chi_B v\|_{L^2_{-\frac{a}{20}}} + \|\,(1-\chi_B)\,v\|_{L^2_{-\frac{a}{20}}} + e^{-B}\|\zeta_A^{-1} e^{-\frac{a}{20}\langle x\rangle}\|_{L^\infty}\|e^{-\frac{a}{20}\langle x\rangle} w\|_{L^2}\end{aligned} \tag{9.9}$$

From $A \gg a^{-1}$ and Corollary 5.2, the 3rd term of line (9.9) can be bounded as

$$e^{-B}\|\zeta_A^{-1} e^{-\frac{a}{20}\langle x\rangle}\|_{L^\infty}\|e^{-\frac{a}{20}\langle x\rangle} w\|_{L^2} \lesssim e^{-B}\|w\|_{\widetilde{\Sigma}}. \tag{9.10}$$

For the 2nd term of line (9.9), by Lemma 5.7,

$$\|\,(1-\chi_B)\,v\|_{L^2_{-\frac{a}{20}}} \le \|e^{-\frac{a}{20}\langle x\rangle}(1-\chi_B)\|_{L^\infty}\|v\|_{L^2} \lesssim e^{-\frac{B}{20}}\varepsilon^{-N}B^2\|w\|_{\widetilde{\Sigma}}. \tag{9.11}$$

Finally, for the 1st term of line (9.9), by the definition of ζ_B in (3.2), see also the definition of χ in (2.16), and of ξ in (3.3), we have

$$\begin{aligned}\|\chi_B v\|_{L^2_{-\frac{a}{20}}} &\le \|\zeta_B^{-1}\|_{L^\infty(|x|\le 2B}\|\chi_B\zeta_B v\|_{L^2_{-\frac{a}{20}}} = \|\zeta_B^{-1}\|_{L^\infty(|x|\le 2B}\|\xi\|_{L^2_{-\frac{a}{20}}}\\ &\lesssim \|\xi\|_{L^2_{-\frac{a}{20}}} \lesssim \|\xi\|_{\widetilde{\Sigma}},\end{aligned} \tag{9.12}$$

where in the last inequality we applied Lemma 5.1. Collecting the estimates (9.8), (9.10), (9.11) and (9.12) we have the conclusion. □

10. Proof of Proposition 3.6: Fermi Golden Rule

We substitute $\widetilde{\mathbf{z}} = \mathrm{i}\varpi(|\mathbf{z}|^2)\mathbf{z}$ in (7.3) and we make various simplifications. The first, by $\langle f, \mathrm{i}f\rangle = 0$ the left hand side of (7.3) can be rewritten as

$$\left\langle \mathrm{i}D_{\mathbf{z}}\phi[\mathbf{z}](\dot{\mathbf{z}} + \mathrm{i}\varpi(|\mathbf{z}|^2)\mathbf{z}),\, D_{\mathbf{z}}\phi[\mathbf{z}]\mathrm{i}\varpi(|\mathbf{z}|^2)\mathbf{z}\right\rangle = \left\langle \mathrm{i}D_{\mathbf{z}}\phi[\mathbf{z}]\dot{\mathbf{z}},\, D_{\mathbf{z}}\phi[\mathbf{z}]\mathrm{i}\varpi(|\mathbf{z}|^2)\mathbf{z}\right\rangle. \tag{10.1}$$

Next, we consider the 2nd term in the 2nd line of (7.3), which we rewrite as

$$\begin{aligned}&\left\langle \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}} G_{\mathbf{m}} + \mathcal{R}_{\mathrm{rp}}[\mathbf{z}],\, D_{\mathbf{z}}\phi[\mathbf{z}]\mathrm{i}\varpi(|\mathbf{z}|^2)\mathbf{z}\right\rangle\\ &\quad= \left\langle \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}} G_{\mathbf{m}} + \mathcal{R}_{\mathrm{rp}}[\mathbf{z}],\, D_{\mathbf{z}}\phi[\mathbf{z}]\left(\dot{\mathbf{z}} + \mathrm{i}\varpi(|\mathbf{z}|^2)\mathbf{z}\right)\right\rangle\\ &\qquad - \left\langle \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}} G_{\mathbf{m}} + \mathcal{R}_{\mathrm{rp}}[\mathbf{z}],\, D_{\mathbf{z}}\phi[\mathbf{z}]\dot{\mathbf{z}}\right\rangle.\end{aligned} \tag{10.2}$$

The term in the 1st line of the r.h.s. of (10.2) can be written as

$$\left\langle \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}} G_{\mathbf{m}}, D_{\mathbf{z}}\phi[0]\left(\dot{\mathbf{z}}+\mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}\right)\right\rangle + R_1(\mathbf{z}), \tag{10.3}$$

where

$$\begin{aligned} R_1(\mathbf{z}) =& \left\langle \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}} G_{\mathbf{m}}, (D_{\mathbf{z}}\phi[\mathbf{z}] - D_{\mathbf{z}}\phi[0])\left(\dot{\mathbf{z}}+\mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}\right)\right\rangle \\ &+\left\langle \mathcal{R}_{\mathrm{rp}}[\mathbf{z}], D_{\mathbf{z}}\phi[\mathbf{z}]\left(\dot{\mathbf{z}}+\mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}\right)\right\rangle, \end{aligned}$$

by (1.21), inequalities (1.26) and (3.4), Proposition 3.5 and $\|D_{\mathbf{z}}\phi[\mathbf{z}] - D_{\mathbf{z}}\phi[0]\|_{H^1} = O(\|\mathbf{z}\|^2)$ by (1.19), satisfies

$$\int_0^T |R_1(\mathbf{z}(t))|\, dt \lesssim \delta^2\epsilon^2. \tag{10.4}$$

Using the stationary Refined Profile equation (1.20), the last line of (10.2) can be written as

$$-\left\langle H\phi[\mathbf{z}] + |\phi[\mathbf{z}]|^2\phi[\mathbf{z}], D_{\mathbf{z}}\phi[\mathbf{z}]\dot{\mathbf{z}}\right\rangle + \left\langle D_{\mathbf{z}}\phi(\mathbf{z})(\mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2))\mathbf{z}, \mathrm{i}D_{\mathbf{z}}\phi[\mathbf{z}]\dot{\mathbf{z}}\right\rangle. \tag{10.5}$$

Notice that the 2nd term of (10.5) coincides with the right hand side of (10.1), which lies in the left hand side of (7.3), so that the two cancel each other. On the other hand, we have

$$\left\langle H\phi[\mathbf{z}] + |\phi[\mathbf{z}]|^2\phi[\mathbf{z}], D_{\mathbf{z}}\phi[\mathbf{z}]\dot{\mathbf{z}}\right\rangle = \frac{d}{dt}E(\phi[\mathbf{z}]). \tag{10.6}$$

Therefore, from (7.3) with $\widetilde{\mathbf{z}} = \mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}$, (10.1), (10.2), (10.3), (10.5) and (10.6), we have

$$\begin{aligned} \frac{d}{dt}E(\phi[\mathbf{z}]) - \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{m}\cdot\boldsymbol{\omega}\langle\eta, \mathrm{i}\mathbf{z}^{\mathbf{m}}G_{\mathbf{m}}\rangle =& \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \left\langle \mathbf{z}^{\mathbf{m}} G_{\mathbf{m}}, D_{\mathbf{z}}\phi[0]\left(\dot{\mathbf{z}}+\mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}\right)\right\rangle \\ &+ R_2(\mathbf{z},\eta), \end{aligned} \tag{10.7}$$

where

$$\begin{aligned} R_2(\mathbf{z},\eta) =& R_1(\mathbf{z}) + \left\langle \mathrm{i}\eta, D^2_{\mathbf{z}}\phi[\mathbf{z}]\left(\dot{\mathbf{z}}+\mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}, \mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}\right)\right\rangle + \left\langle \eta, D_{\mathbf{z}}\mathcal{R}_{\mathrm{rp}}[\mathbf{z}]\mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}\right\rangle \\ &+ \sum_{\mathbf{m}\in\mathbf{R}_{\min}} (\boldsymbol{\varpi}(|\mathbf{z}|^2) - \boldsymbol{\omega})\langle\eta, \mathbf{z}^{\mathbf{m}}G_{\mathbf{m}}\rangle + \left\langle L[\mathbf{z}]\eta + F(\mathbf{z},\eta) + |\eta|^2\eta, D_{\mathbf{z}}\phi[\mathbf{z}]\mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}\right\rangle \end{aligned} \tag{10.8}$$

satisfies

$$\int_I |R_2(\mathbf{z}(t),\eta(t))|\, dt \lesssim \delta^2\epsilon^2. \tag{10.9}$$

We consider the first term in the right hand side of (10.7). By (1.19) we have $D_{\mathbf{z}}\phi[0]\widetilde{\mathbf{z}} = \boldsymbol{\phi}\cdot\widetilde{\mathbf{z}}$, this term is the left hand side of (10.10) below.

Lemma 10.1. *We have*

$$\sum_{\mathbf{m}\in\mathbf{R}_{\min}} \left\langle \mathbf{z}^{\mathbf{m}} G_{\mathbf{m}}, \boldsymbol{\phi}\cdot\left(\dot{\mathbf{z}} + \mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}\right)\right\rangle = \partial_t A_1(\mathbf{z}) + R_4(\mathbf{z},\eta) \tag{10.10}$$

where:

$$A_1(\mathbf{z}) = \sum_{\substack{\mathbf{m},\mathbf{n}\in\mathbf{R}_{\min}\\ \mathbf{m}\neq\mathbf{n}}} \sum_{j=1}^{N} \frac{1}{(\mathbf{n}-\mathbf{m})\cdot\boldsymbol{\omega}} \mathrm{Re}(\mathbf{z}^{\mathbf{m}}\overline{\mathbf{z}^{\mathbf{n}}}) g_{\mathbf{m},j} g_{\mathbf{n},j}, \tag{10.11}$$

for $g_{\mathbf{m},j} := \langle G_{\mathbf{m}}, \phi_j\rangle$*;*

$$R_4(\mathbf{z},\eta) = R_3(\mathbf{z}) + \sum_{\mathbf{m},\mathbf{n}\in\mathbf{R}_{\min}} \sum_{j=1}^{N} \langle \mathbf{z}^{\mathbf{m}} G_{\mathbf{m}}, r_j(\mathbf{z},\eta)\phi_j\rangle \text{ where} \tag{10.12}$$

$$R_3(\mathbf{z}) = \sum_{\substack{\mathbf{m},\mathbf{n}\in\mathbf{R}_{\min}\\ \mathbf{m}\neq\mathbf{n}}} \sum_{j=1}^{N} \mathrm{Re}\left(\mathrm{i} r_{\mathbf{n},\mathbf{m}}(\mathbf{z})\right) g_{\mathbf{m},j} g_{\mathbf{n},j} \text{ for} \tag{10.13}$$

$$\begin{aligned} r_{\mathbf{n},\mathbf{m}}(\mathbf{z}) &= -\frac{(\mathbf{m}-\mathbf{n})\cdot\left(\boldsymbol{\varpi}(|\mathbf{z}|^2)-\boldsymbol{\omega}\right)}{(\mathbf{m}-\mathbf{n})\cdot\boldsymbol{\omega}} \mathbf{z}^{\mathbf{n}}\overline{\mathbf{z}^{\mathbf{m}}} \\ &+ \frac{\mathrm{i}}{(\mathbf{m}-\mathbf{n})\cdot\boldsymbol{\omega}} \left(D_{\mathbf{z}}(\mathbf{z}^{\mathbf{n}})(\dot{\mathbf{z}} + \mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2\mathbf{z}))\overline{\mathbf{z}^{\mathbf{m}}} + \mathbf{z}^{\mathbf{n}} \overline{D_{\mathbf{z}}(\mathbf{z}^{\mathbf{m}})((\dot{\mathbf{z}} + \mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2\mathbf{z})))}\right); \end{aligned} \tag{10.14}$$

we have

$$\int_I |R_4(\mathbf{z}(t),\eta(t))|\, dt \lesssim \delta^2\epsilon^2. \tag{10.15}$$

Proof. The left hand side of (10.10) equals

$$\begin{aligned} &\sum_{\mathbf{m},\mathbf{n}\in\mathbf{R}_{\min}} \sum_{j=1}^{N} \langle \mathbf{z}^{\mathbf{m}} G_{\mathbf{m}}, \phi_j \left(-\mathrm{i}\mathbf{z}^{\mathbf{n}} g_{\mathbf{n},j} + r_j(\mathbf{z},\eta)\right)\rangle \\ &= \sum_{\substack{\mathbf{m},\mathbf{n}\in\mathbf{R}_{\min}\\ \mathbf{m}\neq\mathbf{n}}} \sum_{j=1}^{N} \mathrm{Re}\left(\mathrm{i}\mathbf{z}^{\mathbf{m}}\overline{\mathbf{z}^{\mathbf{n}}}\right) g_{\mathbf{m},j} g_{\mathbf{n},j} + \sum_{\mathbf{m},\mathbf{n}\in\mathbf{R}_{\min}} \sum_{j=1}^{N} \langle \mathbf{z}^{\mathbf{m}} G_{\mathbf{m}}, r_j(\mathbf{z},\eta)\phi_j\rangle, \end{aligned}$$

used the fact that $\langle \mathbf{z}^{\mathbf{m}} G_{\mathbf{m}}, -\mathrm{i}\mathbf{z}^{\mathbf{m}}\phi_j\rangle = 0$ due to ϕ_j and $G_{\mathbf{m}}$ being $\mathbb{R}$ valued, see [10]. Since $(\mathbf{m}-\mathbf{n})\cdot\boldsymbol{\omega} \neq 0$ for $\mathbf{m}\neq\mathbf{n}$ by Assumption 1.3, we have

$$\mathbf{z}^{\mathbf{n}}\overline{\mathbf{z}^{\mathbf{m}}} = \frac{1}{\mathrm{i}((\mathbf{m}-\mathbf{n})\cdot\boldsymbol{\omega})} \partial_t(\mathbf{z}^{\mathbf{n}}\overline{\mathbf{z}^{\mathbf{m}}}) + r_{\mathbf{n},\mathbf{m}}(\mathbf{z}), \tag{10.16}$$

with $r_{\mathbf{n},\mathbf{m}}(\mathbf{z})$ given by (10.14) and satisfying

$$\int_I |r_{\mathbf{m},\mathbf{n}}(\mathbf{z})|\, dt \lesssim \delta^2\epsilon^2. \tag{10.17}$$

We have

$$\sum_{\substack{\mathbf{m},\mathbf{n}\in\mathbf{R}_{\min}\\ \mathbf{m}\neq\mathbf{n}}}\sum_{j=1}^{N}\mathrm{Re}\left(\mathrm{i}\mathbf{z}^{\mathbf{m}}\overline{\mathbf{z}^{\mathbf{n}}}\right)g_{\mathbf{m},j}g_{\mathbf{n},j}=\partial_t A_1(\mathbf{z})+R_3(\mathbf{z}).$$

for $A_1(\mathbf{z})$ and $R_3(\mathbf{z})$ defined above. Finally (10.15) is straightforward. □

We focus now on (10.7).

Lemma 10.2. *There exists a constant $\Gamma_0>0$ such that*

$$\sum_{\mathbf{m}\in\mathbf{R}_{\min}}\mathbf{m}\cdot\boldsymbol{\omega}\left\langle\eta,\mathrm{i}\mathbf{z}^{\mathbf{m}}G_{\mathbf{m}}\right\rangle\leq-\Gamma_0\sum_{\mathbf{m}\in\mathbf{R}_{\min}}|\mathbf{z}^{\mathbf{m}}|^2+\mathcal{E}_1+\mathcal{E}_2+\mathcal{E}_3,\tag{10.18}$$

where for some constants $c_{\mathbf{m},\mathbf{n}}$ the term $\mathcal{E}_1$ is of the form

$$\mathcal{E}_1=\sum_{\substack{\mathbf{m},\mathbf{n}\in\mathbf{R}_{\min}\\ \mathbf{m}\neq\mathbf{n}}}c_{\mathbf{m},\mathbf{n}}\mathbf{z}^{\mathbf{n}}\overline{\mathbf{z}^{\mathbf{m}}},$$

$$\mathcal{E}_2=\sum_{\mathbf{m}\in\mathbf{R}_{\min}}\boldsymbol{\omega}\cdot\mathbf{m}\left\langle\mathrm{i}\mathbf{z}^{\mathbf{m}}\left\langle\mathrm{i}\varepsilon\partial_x\right\rangle^N\mathcal{A}^*\prod_{j=1}^{N}R_H(\omega_j)P_cG_{\mathbf{m}},g\right\rangle,$$

$$|\mathcal{E}_3|\leq o_\varepsilon(1)\left(\sum_{\mathbf{m}\in\mathbf{R}_{\min}}|\mathbf{z}^{\mathbf{m}}|^2+\|w\|^2_{\tilde{\Sigma}}\right).$$

Proof. First of all, notice that

$$\eta=P_c\eta+P_d(R[\mathbf{z}]-1)P_c\eta,$$

where the following term can be absorbed in $\mathcal{E}_3$,

$$\left|\left\langle\mathrm{i}\boldsymbol{\omega}\cdot\mathbf{m}\mathbf{z}^{\mathbf{m}}G_{\mathbf{m}},P_d(R[\mathbf{z}]-1)P_c\eta\right\rangle\right|\lesssim\delta|\mathbf{z}^{\mathbf{m}}|\|w\|_{\tilde{\Sigma}}$$

Now we consider

$$\left\langle\mathrm{i}\boldsymbol{\omega}\cdot\mathbf{m}\mathbf{z}^{\mathbf{m}}G_{\mathbf{m}},P_c\eta\right\rangle=\left\langle\mathrm{i}\boldsymbol{\omega}\cdot\mathbf{m}\mathbf{z}^{\mathbf{m}}G_{\mathbf{m}},P_c\chi_{B^2}\eta\right\rangle+\left\langle\mathrm{i}\boldsymbol{\omega}\cdot\mathbf{m}\mathbf{z}^{\mathbf{m}}G_{\mathbf{m}},P_c\left(1-\chi_{B^2}\right)\eta\right\rangle.$$

Then, by (9.2) we have

$$\begin{aligned}\left\langle\mathrm{i}\boldsymbol{\omega}\cdot\mathbf{m}\mathbf{z}^{\mathbf{m}}G_{\mathbf{m}},P_c\chi_{B^2}\eta\right\rangle&=\left\langle\mathrm{i}\boldsymbol{\omega}\cdot\mathbf{m}\mathbf{z}^{\mathbf{m}}G_{\mathbf{m}},\prod_{j=1}^{N}R_H(\omega_j)P_c\mathcal{A}\left\langle\mathrm{i}\varepsilon\partial_x\right\rangle^N v\right\rangle\\&=\sum_{\mathbf{m}\in\mathbf{R}_{\min}}\boldsymbol{\omega}\cdot\mathbf{m}|\mathbf{z}^{\mathbf{m}}|^2\left\langle\mathrm{i}G_{\mathbf{m}},\prod_{j=1}^{N}R_H(\omega_j)P_c\mathcal{A}\left\langle\mathrm{i}\varepsilon\partial_x\right\rangle^N\rho_{\mathbf{m}}\right\rangle\qquad(10.19)\\&+\sum_{\mathbf{m}\in\mathbf{R}_{\min}}\boldsymbol{\omega}\cdot\mathbf{m}\left\langle\mathrm{i}\mathbf{z}^{\mathbf{m}}G_{\mathbf{m}},\prod_{j=1}^{N}R_H(\omega_j)P_c\mathcal{A}\left\langle\mathrm{i}\varepsilon\partial_x\right\rangle^N g\right\rangle\qquad(10.20)\end{aligned}$$

$$+\sum_{\substack{\mathbf{m},\mathbf{n}\in\mathbf{R}_{\min}\\ \mathbf{m}\neq\mathbf{n}}} \boldsymbol{\omega}\cdot\mathbf{m}\left\langle \mathrm{i}\mathbf{z}^{\mathbf{m}}G_{\mathbf{m}}, \mathbf{z}^{\mathbf{n}}\prod_{j=1}^{N} R_H(\omega_j)P_c\mathcal{A}\left\langle \mathrm{i}\varepsilon\partial_x\right\rangle^N \rho_{\mathbf{n}}\right\rangle. \tag{10.21}$$

Obviously the remainder term in line (10.20) can be absorbed in $\mathcal{E}_2$ and the remainder term in line (10.21) can be absorbed in $\mathcal{E}_1$. We now examine the main term, in the line (10.19). We have

$$\begin{aligned}
&\left\langle \mathrm{i}G_{\mathbf{m}}, \prod_{j=1}^{N} R_H(\omega_j)P_c\mathcal{A}\left\langle \mathrm{i}\varepsilon\partial_x\right\rangle^N \rho_{\mathbf{m}}\right\rangle\\
&\quad= -\left\langle \mathrm{i}\left\langle \mathrm{i}\varepsilon\partial_x\right\rangle^N\mathcal{A}^*\prod_{j=1}^{N} R_H(\omega_j)P_cG_{\mathbf{m}}, R^+_{H_{N+1}}(\boldsymbol{\omega}\cdot\mathbf{m})\left\langle \mathrm{i}\varepsilon\partial_x\right\rangle^{-N}\mathcal{A}^*\chi_{B^2}G_{\mathbf{m}}\right\rangle\\
&\quad= -\left\langle \mathrm{i}\mathcal{A}^*\prod_{j=1}^{N} R_H(\omega_j)P_cG_{\mathbf{m}}, R^+_{H_{N+1}}(\boldsymbol{\omega}\cdot\mathbf{m})\mathcal{A}^*\chi_{B^2}G_{\mathbf{m}}\right\rangle \qquad (10.22)\\
&\quad\quad -\left\langle \mathrm{i}\left\langle \mathrm{i}\varepsilon\partial_x\right\rangle^N\mathcal{A}^*\prod_{j=1}^{N} R_H(\omega_j)P_cG_{\mathbf{m}}, \left[R^+_{H_{N+1}}(\boldsymbol{\omega}\cdot\mathbf{m}), \left\langle \mathrm{i}\varepsilon\partial_x\right\rangle^{-N}\right]\mathcal{A}^*\chi_{B^2}G_{\mathbf{m}}\right\rangle, \qquad (10.23)
\end{aligned}$$

where we see now that the quantity in (10.23) is a $o_\varepsilon(1)$ and its contribution to (10.18) can be absorbed in $\mathcal{E}_3$. Indeed the quantity in (10.23) can be bounded by the product $A\ B$, where

$$\begin{aligned}
A &= \|\left\langle \mathrm{i}\varepsilon\partial_x\right\rangle^N\mathcal{A}^*\prod_{j=1}^{N} R_H(\omega_j)P_cG_{\mathbf{m}}\|_{L^{2,\ell}} \text{ and}\\
B &= \|R^+_{H_{N+1}}(\boldsymbol{\omega}\cdot\mathbf{m})\left[V_{N+1}, \left\langle \mathrm{i}\varepsilon\partial_x\right\rangle^{-N}\right]R^+_{H_{N+1}}(\boldsymbol{\omega}\cdot\mathbf{m})\mathcal{A}^*\chi_{B^2}G_{\mathbf{m}}\|_{L^{2,-\ell}},
\end{aligned}$$

for $\ell\geq 2$. We have

$$\begin{aligned}
B \leq &\|R^+_{H_{N+1}}(\boldsymbol{\omega}\cdot\mathbf{m})\|^2_{L^{2,\ell}\to L^{2,-\ell}}\|\left\langle \mathrm{i}\varepsilon\partial_x\right\rangle^{-N}\left[V_{N+1}, \left\langle \mathrm{i}\varepsilon\partial_x\right\rangle^{N}\right]\|_{L^{2,-\ell}\to L^{2,\ell}}\\
&\times\|\left\langle \mathrm{i}\varepsilon\partial_x\right\rangle^{-N}\|_{L^{2,-\ell}\to L^{2,-\ell}}\|\mathcal{A}^*\chi_{B^2}G_{\mathbf{m}}\|_{L^{2,\ell}}\lesssim\varepsilon,
\end{aligned}$$

where the ε comes from the commutator term in the first line, by Lemma 5.4, while the other terms are uniformly bounded, with $\|\left\langle \mathrm{i}\varepsilon\partial_x\right\rangle^{-N}\|_{L^{2,-\ell}\to L^{2,-\ell}}\lesssim 1$ uniformly in $\varepsilon\in(0,1]$, because we have an operator like in (8.18). On the other hand, uniformly in $\varepsilon\in(0,1]$, we have

$$A\leq\|\left\langle \mathrm{i}\varepsilon\partial_x\right\rangle^N\left\langle \mathrm{i}\partial_x\right\rangle^{-2N}\|_{L^{2,\ell}\to L^{2,\ell}}\|\left\langle \mathrm{i}\partial_x\right\rangle^{2N}\mathcal{A}^*\prod_{j=1}^{N} R_H(\omega_j)P_cG_{\mathbf{m}}\|_{L^{2,\ell}}\lesssim 1.$$

We consider the main term (10.22). Essentially by (1.24), it equals

$$-\left\langle \mathrm{i}\mathcal{A}^*\prod_{j=1}^{N} R_H(\omega_j)P_cG_{\mathbf{m}}, \mathcal{A}^*R^+_H(\boldsymbol{\omega}\cdot\mathbf{m})\chi_{B^2}G_{\mathbf{m}}\right\rangle$$

$$= -\left\langle \mathrm{i}\mathcal{A}\mathcal{A}^* \prod_{j=1}^{N} R_H(\omega_j) P_c G_{\mathbf{m}}, R_H^+(\boldsymbol{\omega}\cdot\mathbf{m})\chi_{B^2} G_{\mathbf{m}} \right\rangle = -\left\langle \mathrm{i} P_c G_{\mathbf{m}}, R_H^+(\boldsymbol{\omega}\cdot\mathbf{m})\chi_{B^2} G_{\mathbf{m}} \right\rangle,$$

where we used (9.3). By the Limit Absorption Principle and the Sokhotski–Plemelj Formula, the last term equals

$$\begin{aligned} &- \left\langle \mathrm{i} P_c G_{\mathbf{m}}, R_H^+(\boldsymbol{\omega}\cdot\mathbf{m}) G_{\mathbf{m}} \right\rangle + \left\langle \mathrm{i} P_c G_{\mathbf{m}}, R_H^+(\boldsymbol{\omega}\cdot\mathbf{m}) \left(1-\chi_{B^2}\right) G_{\mathbf{m}} \right\rangle \\ &\quad = -\pi \left\langle P_c G_{\mathbf{m}}, \delta(H-\boldsymbol{\omega}\cdot\mathbf{m}) G_{\mathbf{m}} \right\rangle \qquad (10.24) \\ &\qquad + \left\langle \mathrm{i} P_c G_{\mathbf{m}}, R_H^+(\boldsymbol{\omega}\cdot\mathbf{m}) \left(1-\chi_{B^2}\right) G_{\mathbf{m}} \right\rangle, \qquad (10.25) \end{aligned}$$

where the quantity (10.25) is of the form $O(B^{-1})$ and so also of the form $o_\varepsilon(1)$ and the corresponding contribution to (10.18) can be absorbed in $\mathcal{E}_3$. Finally, by elementary computation we have

$$-\pi \left\langle P_c G_{\mathbf{m}}, \delta(H-\boldsymbol{\omega}\cdot\mathbf{m}) G_{\mathbf{m}} \right\rangle = -\frac{\pi}{2\sqrt{\boldsymbol{\omega}\cdot\mathbf{m}}} \left(\left|\widehat{G}_{\mathbf{m}}(\sqrt{\boldsymbol{\omega}\cdot\mathbf{m}})\right|^2 + \left|\widehat{G}_{\mathbf{m}}(-\sqrt{\boldsymbol{\omega}\cdot\mathbf{m}})\right|^2 \right) < 0, \qquad (10.26)$$

where $\widehat{G}_{\mathbf{m}}$ is the distorted Fourier transform associated to the operator H and where the inequality follows from Assumption 1.8. The corresponding contribution to (10.18) can be absorbed in the first term in the right hand side. □

Lemma 10.3. *For a constant $C_{V,\Gamma_0} > 0$ we have*

$$\int_I \mathcal{E}_1 dt \lesssim \delta^2 \epsilon^2, \qquad (10.27)$$

$$\int_I \mathcal{E}_2 dt \le 2^{-1}\Gamma_0 \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \|\mathbf{z}^{\mathbf{m}}\|_{L^2(I)}^2 + C_{V,\Gamma_0} \left(\varepsilon^{-N} B^{4+4\tau} \delta^2 + B^{-1}\epsilon^2 + \varepsilon^4 \right), \qquad (10.28)$$

$$\int_I \mathcal{E}_3 dt \lesssim o_\varepsilon(1)\epsilon^2. \qquad (10.29)$$

Proof. Inequality (10.29) is straightforward and so is (10.27), thanks to (10.16) and (10.17).
Turning to (10.28), we have, for constants $C'_{V,\Gamma_0} > 0$ and $C_{V,\Gamma_0} > 0$,

$$\begin{aligned} \int_I \mathcal{E}_3\, dt &\lesssim 2^{-1}\Gamma_0 \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \|\mathbf{z}^{\mathbf{m}}\|_{L^2(I)}^2 \\ &\quad + C'_{V,\Gamma_0} \|g\|_{L^2(I, L^{2,-S}(\mathbb{R}))}^2 \sup_{\mathbf{m}\in\mathbf{R}_{\min}} \| \langle \mathrm{i}\varepsilon\partial_x \rangle^N \mathcal{A}^* \prod_{j=1}^{N} R_H(\omega_j) P_c G_{\mathbf{m}} \|_{L^{2,-S}(\mathbb{R})} \\ &\le 2^{-1}\Gamma_0 \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \|\mathbf{z}^{\mathbf{m}}\|_{L^2(I)}^2 + C_{V,\Gamma_0} \left(\varepsilon^{-N} B^{4+4\tau} \delta^2 + B^{-1}\epsilon^2 + \varepsilon^4 \right). \end{aligned}$$

□

Conclusion of the proof of Proposition 3.6. From (10.7), (10.10) and (10.18), we have

$$\frac{d}{dt} E(\phi[\mathbf{z}]) = \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{m}\cdot\boldsymbol{\omega} \left\langle \eta, \mathrm{i}\mathbf{z}^{\mathbf{m}} G_{\mathbf{m}} \right\rangle + \partial_t A_1(\mathbf{z}) + R_4(\mathbf{z}, \eta) + R_2(\mathbf{z}, \eta)$$

$$\leq -\Gamma_0 \sum_{\mathbf{m}\in\mathbf{R}_{\min}} |\mathbf{z}^{\mathbf{m}}|^2 + \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 + \partial_t A_1(\mathbf{z}) + R_4(\mathbf{z},\eta) + R_2(\mathbf{z},\eta). \tag{10.30}$$

So, integrating and using (10.9), (10.15) and Lemma 10.3, we have

$$2^{-1}\Gamma_0 \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \|\mathbf{z}^{\mathbf{m}}\|^2_{L^2(I)} \leq (A_1(\mathbf{z}) - E(\phi[\mathbf{z}]))_0^T + \int_I (|R_2(\mathbf{z},\eta)| + |R_4(\mathbf{z},\eta)|)\, dt$$
$$+ C_{V,\Gamma_0}\left(\varepsilon^{-N} B^{2+2\tau}\delta^2 + B^{-1}\epsilon^2 + \epsilon^4\right).$$

From $(A_1(\mathbf{z}) - E(\phi[\mathbf{z}]))_0^T = O(\delta^2)$, we conclude

$$\sum_{\mathbf{m}\in\mathbf{R}_{\min}} \|\mathbf{z}^{\mathbf{m}}\|^2_{L^2(I)} \lesssim \varepsilon^{-N} B^{4+4\tau}\delta^2 + B^{-1}\epsilon^2 + \varepsilon^4.$$

This completes the proof of Proposition 3.6.

11. Proof of Proposition 3.2

By (3.8) and by the relation between A, B, ε, ϵ and δ in (2.13), we have

$$\sum_{\mathbf{m}\in\mathbf{R}_{\min}} \|\mathbf{z}^{\mathbf{m}}\|_{L^2(I)} \lesssim \varepsilon^{-N} B^{2+2\tau}\delta + B^{-\frac{1}{2}}\epsilon + \epsilon^2 \lesssim o_\varepsilon(1)\epsilon. \tag{11.1}$$

Inserting this in (3.7) we obtain

$$\|\dot{\mathbf{z}} + \mathrm{i}\varpi(\mathbf{z})\mathbf{z}\|_{L^2} \lesssim \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \|\mathbf{z}^{\mathbf{m}}\|_{L^2} + \delta^2\epsilon \lesssim o_\varepsilon(1)\epsilon. \tag{11.2}$$

By (3.5), (9.1) and (11.2)

$$\|w'\|_{L^2L^2} \lesssim A^{1/2}\delta + \|w\|_{L^2L^2_{-\frac{a}{10}}} + \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \|\mathbf{z}^{\mathbf{m}}\|_{L^2} + \epsilon^2 \lesssim o_\varepsilon(1)\epsilon + \|\xi\|_{\widetilde{\Sigma}} + o_\varepsilon(1)\|w'\|_{L^2L^2},$$

so that

$$\|w'\|_{L^2L^2} \lesssim o_\varepsilon(1)\epsilon + \|\xi\|_{\widetilde{\Sigma}}. \tag{11.3}$$

By (3.6), (9.1), (11.1)–(11.3)

$$\|\xi\|_{L^2\widetilde{\Sigma}} \lesssim B\varepsilon^{-N}\delta + \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \|\mathbf{z}^{\mathbf{m}}\|_{L^2} + o_\varepsilon(1)\left(\|w\|_{L^2\widetilde{\Sigma}} + \|\dot{\mathbf{z}} + \mathrm{i}\boldsymbol{\varpi}(\mathbf{z})\mathbf{z}\|_{L^2}\right)$$
$$\lesssim o_\varepsilon(1)\epsilon + o_\varepsilon(1)\|\xi\|_{\widetilde{\Sigma}}$$

which implies

$$\|\xi\|_{L^2\widetilde{\Sigma}} \lesssim o_\varepsilon(1)\epsilon, \tag{11.4}$$

which fed in (11.3) yields

$$\|w'\|_{L^2L^2} \lesssim o_\varepsilon(1)\epsilon. \tag{11.5}$$

Obviously, (11.1), (11.2), (11.4), (11.5) and (9.1) imply

$$\|\dot{\mathbf{z}} + \mathrm{i}\boldsymbol{\varpi}(\mathbf{z})\mathbf{z}\|_{L^2(I)} + \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \|\mathbf{z}^{\mathbf{m}}\|_{L^2(I)} + \|\xi\|_{L^2(I,\widetilde{\Sigma})} + \|w\|_{L^2(I,\widetilde{\Sigma})} \le o_\varepsilon(1)\epsilon. \quad (11.6)$$

In the above discussion we can take $\varepsilon = \varepsilon(\delta)$ with $\varepsilon(\delta) \xrightarrow{\delta\to 0^+} 0$, so that for the upper bound in (11.6) we have $o_\varepsilon(1)\epsilon = o_\delta(1)\epsilon$ and the conclusion of of Proposition 3.2 is true. □

12. Proof of (1.29)

Up to here, we have proved (1.26) and (1.28). It remains to prove (1.29).

Proof of (1.29). By the equality in the 1st line of (10.30) and (1.28), we have $\frac{d}{dt}(E(\phi[\mathbf{z}]) - A_1(\mathbf{z})) \in L^1(\mathbb{R}_+)$. Furthermore, $E(\phi[\mathbf{z}]) - A_1(\mathbf{z}) \in L^\infty(\mathbb{R}_+)$, by (1.26). Thus, $\lim_{t\to+\infty}(E(\phi[\mathbf{z}]) - A_1(\mathbf{z}))$ exists and is finite. We have $A_1(\mathbf{z}) \xrightarrow{t\to+\infty} 0$ by (1.26), (1.28) and (10.11). This implies that $\lim_{t\to+\infty} E(\phi[\mathbf{z}])$ exists and is finite. Now, from (1.4) and Proposition 1.9, we have

$$E(\phi[\mathbf{z}]) = \sum_{l=1}^{N} \omega_l |z_l|^2 + O(\|\mathbf{z}\|^4).$$

Thus, taking $\delta > 0$ small enough, we have

$$\frac{1}{2}|E(\phi[\mathbf{z}])| \le \sum_{j=1}^{N} |\omega_j|\, |z_j|^2 \le 2|E(\phi[\mathbf{z}])|. \quad (12.1)$$

Now, if $\lim_{t\to+\infty} E(\phi[\mathbf{z}(t)]) = 0$, we have $|z_j(t)| \to 0$ for all $j = 1, \cdots, N$ and we are done. Thus, we can assume

$$\lim_{t\to+\infty} E(\phi[\mathbf{z}(t)]) = -c^2, \text{ with } c > 0. \quad (12.2)$$

Notice that we have $c \lesssim \delta$. From (12.1) and (12.2), there exists $T_1 > 0$ s.t. for all $t \ge T_1$, there exists at least one $j(t) \in \{1, \cdots, N\}$ s.t.

$$\frac{c}{\sqrt{4N|\omega_1|}} \le |z_{j(t)}(t)|. \quad (12.3)$$

Next, from (1.13) and (1.28), there exists $M \in \mathbb{N}$ s.t. for any j, k with $j \ne k$ we have $|z_j z_k|^M \in L^1(\mathbb{R})$. Further, by (1.26), we have $(z_j z_k)^M \in W^{1,\infty}(\mathbb{R})$. Thus, we conclude

$$z_j(t) z_k(t) \xrightarrow{t\to+\infty} 0. \quad (12.4)$$

In particular, there exists $T_2 \ge T_1$ s.t. for all $t \ge T_2$ and all $j, k = 1, \cdots, N$ with $j \ne k$, we have

$$|z_j(t) z_k(t)| \le \frac{c^2}{8N|\omega_1|}. \quad (12.5)$$

Combining (12.3) and (12.5), for $t > T_2$ and $k \neq j(t)$, we have

$$|z_k(t)| \leq \frac{c}{2\sqrt{4N|\omega_1|}}.$$

Thus, we see that j satisfying (12.3) is unique. Moreover by continuity, we have $j(t) = j(T_2)$ for all $t \geq T_2$. Going back to (12.4), we have

$$\lim_{t\to+\infty} z_k(t) = 0, \tag{12.6}$$

for all $k \neq j(T_2)$. Finally, by (12.6) we have $\left(E(\phi[\mathbf{z}(t)]) - E(\phi_{j(T_2)}[z_{j(T_2)}(t)])\right) \xrightarrow{t\to+\infty} 0$, which implies the convergence of $E(\phi_{j(T_2)}[z_{j(T_2)}(t)])$. For small $|z_{j(T_2)}|$, the map $|z_{j(T_2)}| \mapsto E(\phi_j[z_{j(T_2)}])$ is one to one with continuous inverse. Thus, $\lim_{t\to+\infty} |z_{j(T_2)}(t)|$ exists. $\square$

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Declarations

Conflict of Interest The authors declare that there was no conflict of interest.

A Appendix: Proof of Lemma 9.3

It is equivalent to show that there is a constant $C > 0$ such that for all v

$$\left\| \operatorname{sech}\left(\frac{ax}{10}\right) \prod_{j=1}^{N} R_H(\omega_j) P_c \mathcal{A} \langle \mathrm{i}\varepsilon\partial_x \rangle^N v \right\|_{L^2(\mathbb{R})} \leq C \left\| \operatorname{sech}\left(\frac{ax}{20}\right) v \right\|_{L^2(\mathbb{R})}. \tag{A.1}$$

By (8.1), for $x < y$ we have the formula

$$\begin{aligned} R_H(z^2)(x,y) &= \frac{T(z)}{2\mathrm{i}z} f_-(x,z) f_+(y,z) \\ &= \frac{1}{z^2+\omega_j} \frac{f_-(x,\mathrm{i}\sqrt{|\omega_j|}) f_+(y,\mathrm{i}\sqrt{|\omega_j|})}{\int_{\mathbb{R}} f_-(x',\mathrm{i}\sqrt{|\omega_j|}) f_+(x',\mathrm{i}\sqrt{|\omega_j|}) dx'} + \widetilde{R}_H(z^2)(x,y), \end{aligned} \tag{A.2}$$

where $\frac{T(z)}{2\mathrm{i}z} = \frac{1}{[f_+(x,z), f_-(x,z)]}$, where in the denominator in the r.h.s. we have the Wronskian, where $\widetilde{R}_H(z^2)(x, y)$ is not singular in $z = \mathrm{i}\sqrt{|\omega_j|}$. On the other hand,

$$T(z) = \frac{\mathrm{Res}(T, \mathrm{i}\sqrt{|\omega_j|})}{z - \mathrm{i}\sqrt{|\omega_j|}} + \widetilde{T}(z),$$

with $\widetilde{T}(z)$ non singular and with residue, see p. 146 [16],

$$\mathrm{Res}(T, \mathrm{i}\sqrt{|\omega_j|}) = \mathrm{i} \left(\int_{\mathbb{R}} f_-(x', \mathrm{i}\sqrt{|\omega_j|}) f_+(x', \mathrm{i}\sqrt{|\omega_j|}) dx' \right)^{-1}.$$

It is elementary to conclude, comparing the terms in (A.2), that

$$\begin{aligned} &\widetilde{R}_H(\omega_j)(x, y) = K_j(x, y) + C(\omega_j)\phi_j(x)\phi_j(y) \text{ with} \\ &K_j(x, y) = \frac{1}{2\mathrm{i}\sqrt{|\omega_j|}} \frac{\partial_z \left(f_-(x, z) f_+(y, z)\right)|_{z=\mathrm{i}\sqrt{|\omega_j|}}}{\int_{\mathbb{R}} f_-(x', \mathrm{i}\sqrt{|\omega_j|}) f_+(x', \mathrm{i}\sqrt{|\omega_j|}) dx'}. \end{aligned} \tag{A.3}$$

for some constant $C(\omega_j)$. For $x > y$ we obtain the same formula, interchanging x and y. Denoting by K_j the operator with the kernel (A.3) for $x < y$ and the formula obtained from (A.3) interchanging x and y if $x > y$, we notice that

$$\prod_{j=1}^{N} R_H(\omega_j) P_c = K_1 ... K_N.$$

It is also easy to check, following the discussion in p. 134 [16], that there is a fixed $C > 0$ s.t. $|K_j(x, y)| \le C \langle x - y \rangle e^{-\sqrt{|\omega_j|}|x-y|}$. Then, for any value $a \in [0, \sqrt{|\omega_N|}]$ we have

$$\begin{aligned} &\|\mathrm{sech}\left(\frac{ax}{10}\right) \prod_{j=1}^{N} R_H(\omega_j) P_c \mathcal{A} \langle \mathrm{i}\varepsilon\partial_x \rangle^N v\|_{L^2} \\ &\lesssim \| \prod_{j=1}^{N} R_H(\omega_j) P_c \mathrm{sech}\left(\frac{ax}{10}\right) \mathcal{A} \langle \mathrm{i}\varepsilon\partial_x \rangle^N v\|_{L^2}. \end{aligned}$$

We have

$$\mathrm{sech}\left(\frac{ax}{10}\right) \mathcal{A} = P_N(x, \mathrm{i}\partial_x) \mathrm{sech}\left(\frac{ax}{10}\right),$$

for an N–th order differential operator with smooth and bounded coefficients.
Next, we write

$$\mathrm{sech}\left(\frac{ax}{10}\right) \langle \mathrm{i}\varepsilon\partial_x \rangle^N = \langle \mathrm{i}\varepsilon\partial_x \rangle^N \mathrm{sech}\left(\frac{ax}{10}\right) + \langle \mathrm{i}\varepsilon\partial_x \rangle^N \langle \mathrm{i}\varepsilon\partial_x \rangle^{-N} \left[\mathrm{sech}\left(\frac{ax}{10}\right), \langle \mathrm{i}\varepsilon\partial_x \rangle^N\right],$$

so that

$$\left\| \mathrm{sech}\left(\frac{ax}{10}\right) \prod_{j=1}^{N} R_H(\omega_j) P_c \mathcal{A} \langle \mathrm{i}\varepsilon\partial_x \rangle^N v \right\|_{L^2(\mathbb{R})}$$

$$
\begin{aligned}
&\lesssim \left\| \prod_{j=1}^{N} R_H(\omega_j) P_c P_N(x, \mathrm{i}\partial_x) \langle \mathrm{i}\varepsilon \partial_x \rangle^N \operatorname{sech}\left(\frac{ax}{10}\right) v \right\|_{L^2(\mathbb{R})} \\
&\quad + \left\| \prod_{j=1}^{N} R_H(\omega_j) P_c P_N(x, \mathrm{i}\partial_x) \langle \mathrm{i}\varepsilon \partial_x \rangle^N \langle \mathrm{i}\varepsilon \partial_x \rangle^{-N} \left[\operatorname{sech}\left(\frac{ax}{10}\right), \langle \mathrm{i}\varepsilon \partial_x \rangle^N \right] v \right\|_{L^2(\mathbb{R})} \\
&=: I + II.
\end{aligned}
$$

We have

$$
\begin{aligned}
I &\leq \left\| \prod_{j=1}^{N} R_H(\omega_j) P_c P_N(x, \mathrm{i}\partial_x) \langle \mathrm{i}\varepsilon \partial_x \rangle^N \right\|_{L^2 \to L^2} \left\| \operatorname{sech}\left(\frac{ax}{10}\right) v \right\|_{L^2(\mathbb{R})} \\
&\leq C \left\| \operatorname{sech}\left(\frac{ax}{10}\right) v \right\|_{L^2(\mathbb{R})}
\end{aligned}
$$

with a fixed constant C independent from $\varepsilon \in (0, 1)$. Next, we have

$$
II \leq \left\| \langle \mathrm{i}\varepsilon \partial_x \rangle^{-N} \left[\operatorname{sech}\left(\frac{ax}{10}\right), \langle \mathrm{i}\varepsilon \partial_x \rangle^N \right] v \right\|_{L^2(\mathbb{R})} \leq C \left\| \operatorname{sech}\left(\frac{ax}{20}\right) v \right\|_{L^2(\mathbb{R})}
$$

by Lemma 5.5, because $\int e^{-\mathrm{i}kx} \operatorname{sech}(x) dx = \pi \operatorname{sech}\left(\frac{\pi}{2} k\right)$ (which can be proved by an elementary application of the Residue Theorem) so that in the strip $k = k_1 + \mathrm{i}k_2$ with $|k_2| \leq \mathbf{b} := a/20$, then $\operatorname{sech}\left(\frac{\pi}{2} \frac{10}{a} k\right)$ satisfies the estimates required on $\widehat{\mathcal{V}}$ in (5.11). This completes the proof of (A.1).

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