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Properties of Topologies for the Continuous Representability of All Weakly Continuous Preorders

Gianni Bosi *, Laura Franzoi and Gabriele Sbaiz

Department of Economics, Business, Mathematics and Statistics, University of Trieste, Via A. Valerio 4/1, 34127 Trieste, Italy; laura.franzoi@deams.units.it (L.F.); gabriele.sbaiz@deams.units.it (G.S.)
* Correspondence: gianni.bosi@deams.units.it; Tel.: +39-040-558-7115

Abstract: We investigate properties of *strongly useful topologies*, i.e., topologies with respect to which every *weakly continuous* preorder admits a continuous order-preserving function. In particular, we prove that a topology is strongly useful provided that the topology generated by every family of separable systems is countable. Focusing on normal Hausdorff topologies, whose consideration is fully justified and not restrictive at all, we show that strongly useful topologies are hereditarily separable on closed sets, and we identify a simple condition under which the Lindelöf property holds.

Keywords: strongly useful topology; weakly continuous preorder; hereditarily separable topology; Lindelöf property

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1. Introduction

Gerhard Herden inaugurated a very general approach to Mathematical Utility Theory and in particular to Real Representation of Preferences (see, e.g., Herden [1–3] and Herden and Pallack [4]). In particular, Herden was able to provide characterizations of the existence of continuous order-preserving functions for a not necessarily total preorder on a generic topological space, by using the fundamental notion of a *linear decreasing separable system*, which generalizes the concept of a *decreasing scale*, introduced by Burgess and Fitzpatrick [5].

The popular theorems by Eilenberg [6] and Debreu [7,8], which state that there exists a continuous order-preserving function (utility function) for every continuous total preorder on a connected and separable, and respectively on a second countable, topological space, are particular cases of Herden's axiomatization. We recall that a total preorder on a topological space is said to be continuous if all the weak lower and upper sections are closed (or, equivalently, if all the strict lower and upper sections are open).

The very important concept of a *useful topology* (i.e., a topology on a given set such that every continuous total preorder is representable by a continuous utility function) was inaugurated by Herden [3], who based his analysis on *linear separable systems* on a topological space. Indeed, to each linear separable system it is possible to associate a continuous total preorder in a very natural way. Useful topologies were recently studied and fully characterized in two subsequent papers by Bosi and Zuanon [9,10], after the seminal paper by Bosi and Herden [11], which introduced and applied the concept of a *complete separable system* on a topological space. This is the case of a particular linear separable system. According to Bosi and Zuanon, Theorem 2 in [10], a completely regular topology is useful if and only if it is separable and every chain of clopen sets is countable. It is very important to notice that the assumption of complete regularity is not restrictive, since a (total) preorder is continuous if and only if it is continuous with respect to the weak topology of continuous functions, which is completely regular (see Bosi and Zuanon, Lemma 3.1 in [9]). Other deep contributions to the study of useful topologies were presented by Herden and Pallack [12], Campión et al. [13–15], and Candeal et al. [16].

A very natural and general extension of the concept of a *continuous total preorder* to the nontotal case is the notion of a *weakly continuous preorder*, introduced by Herden and Pallack [4]. This is the case of a preorder $\precsim$ on a topological space (X, t) such that, for every $(x, y) \in \prec$, there exists a continuous increasing real-valued function u_{xy} such that $u_{xy}(x) < u_{xy}(y)$. In other words, points $(x, y) \in \prec$ are separated by continuous increasing functions. In the case of a total preorder, the notion of weak continuity is equivalent to that of continuity.

In order to generalize the concept of a useful topology, Bosi [17] introduced the notion of a *strongly useful topology* on a set. This is the case of a topology such that every weakly continuous and not necessarily total preorder admits a continuous order-preserving function. It is easy to see that a strongly useful topology is useful. The concept of a strongly useful topology was further examined by Bosi [17], who studied the bijective correspondence between weakly continuous preorders on one hand and (equivalence classes of) families of complete separable systems on the other hand. Therefore, some characterizations of strongly useful topology are already available. Actually, the situation is, needless to say, much more complicated when the preorder may fail to be total. However, there is some analogy with useful topologies, in the sense that, while in the case of useful topologies, it is not restrictive to deal with completely regular topologies, in the case of strongly useful topologies, it is not restrictive to deal with normal and Hausdorff spaces.

This paper is aimed to take advantage of this latter very important consideration in order to prove interesting properties of strongly useful topologies which are also normal and Hausdorff. Indeed, we show that strongly usefulness is a hereditary property on open sets when the topology is normal Hausdorff. This is proven by using the fact that strongly usefulness depends, loosely speaking, on the properties of families of complete separable systems and on the weakly continuous preorders which are naturally associated to them. We further prove that strongly useful normal Hausdorff topologies are hereditarily separable on closed sets, and we identify a simple condition concerning well-ordered families of open sets, which implies the Lindelöf property.

Incidentally, under fairly general conditions, we establish that a topology is strongly useful provided that the topology generated by every family of open sets is second countable, and we show that such condition is not necessary for strongly usefulness. We further furnish a very general characterization of the existence of continuous order-preserving functions, which is strictly related to the consideration of linear separable systems which are actually complete.

2. Notation and Preliminaries

The reader is assumed to be familiar with the definition of a *preorder* $\precsim$ on a set X and the related classical notation, according to which the associated *strict part* and *symmetric part* are denoted by $\prec$ and $\sim$, respectively. An *order* $\lesssim$ is an *antisymmetric* preorder, and a *chain* is a total order. However, all the detailed definitions can be found for example, in Bridges and Mehta [18].

Definition 1. *A real-valued function u on a preordered set $(X, \precsim)$ is said to be*

1. *increasing, if, for all $x, y \in X$,*

$$x \precsim y \Rightarrow u(x) \leq u(y);$$

2. *order-preserving, if u is increasing and, for all $x, y \in X$,*

$$x \prec y \Rightarrow u(x) < u(y);$$

3. *a utility function, if for all $x, y \in X$,*

$$x \precsim y \Leftrightarrow u(x) \leq u(y).$$

In the sequel, t_{nat} denotes the *natural topology* on $\mathbb{R}$. We shall consider, since it is not restrictive, continuous increasing functions or continuous order-preserving functions u taking values in $[0,1]$, i.e., we shall be authorized to use the notation $u : (X, \precsim, t) \rightarrow ([0,1], \leq, t_{nat})$ instead of $u : (X, \precsim, t) \rightarrow (\mathbb{R}, \leq, t_{nat})$.

Definition 2 (Herden and Pallack [4])**.** *A preorder $\precsim$ on a topological space (X, t) is defined to be weakly continuous if, given any pair $(x, y) \in \prec$, there is a continuous and increasing real-valued function $u_{xy} : (X, \precsim, t) \rightarrow ([0,1], \leq, t_{nat})$ with the property that $u_{xy}(x) < u_{xy}(y)$.*

It is urgent to check that weak continuity is a necessary condition for the existence of a continuous order-preserving function. Let us present the basic definition of a *strongly useful* topology.

Definition 3. *A topology t on a nonempty set X is defined to be strongly useful if, for every weakly continuous preorder on (X, t), there is a continuous order-preserving function $u : (X, \precsim, t) \rightarrow ([0,1], \leq, t_{nat})$ representing it.*

Characterizations of a strongly useful topology are presented, for example, in Bosi, Theorem 4.2 in [17].

If $(X \precsim)$ is any *preordered set*, then a subset D of X is said to be *decreasing* if, for all $x \in D$, $z \in D$ whenever $z \precsim x$ (for all $z \in X$).

In the sequel, if (X, t) is a topological space, then we shall denote by $\overline{E}$ the topological closure of any subset E of X. Further, we shall use the symbols $\subset$ and $\subsetneqq$ in order to denote inclusion and proper inclusion between sets, respectively.

Let us now present the definition of a *complete separable system* on a topological space (X, t).

Definition 4 (Bosi and Herden [11])**.** *A collection $\mathcal{E}$ of open subsets of a topological space (X, t), satisfying the property that $\bigcup_{E \in \mathcal{E}} E = X$, is defined to be a complete separable system on (X, t) if the following assertions are verified:*

- **S1***: There exist sets $E_1 \in \mathcal{E}$ and $E_2 \in \mathcal{E}$ with the property that $\overline{E_1} \subset E_2$;*
- **S2***: If E_1 and E_2 are any two sets of $\mathcal{E}$, and $\overline{E_1} \subset E_2$, then there is a set $E_3 \in \mathcal{E}$ with the property that $\overline{E_1} \subset E_3 \subset \overline{E_3} \subset E_2$;*
- **S3***: If E and E' are any two different sets of $\mathcal{E}$, then either $\overline{E} \subset E'$, or $\overline{E'} \subset E$.*

In addition, if $\precsim$ is a preorder on X, and every set $E \in \mathcal{E}$ is decreasing, then we refer to a complete decreasing separable system on $(X, \precsim, t)$.

Denote by $\mathbb{S}_C(X, t)$ and $\mathbb{F}_C(X, t)$ the collection of all complete separable systems and the collection of all families of complete separable systems on (X, t), respectively.

Let us now present the definition of the weakly continuous preorder associated to a family of complete separable systems.

Definition 5. *If we consider any family $\mathbb{E} := \{\mathcal{E}_i\}_{i \in I} \in \mathbb{F}_C(X, t)$, then the weakly continuous preorder $\precsim_{\mathbb{E}}$ naturally associated to $\mathbb{E}$ is defined as follows, for all points $x, y \in X$:*

$$x \precsim_{\mathbb{E}} y \Leftrightarrow \forall i \in I,\ \forall E \in \mathcal{E}_i :\ (y \in E \Rightarrow x \in E).$$

We now present a characterization of weak continuity of a preorder on a topological space.

Proposition 1. *Let $\precsim$ be a preorder on a topological space (X, t). Then the following conditions are equivalent:*

1. *$\precsim$ is weakly continuous;*

2. *For every $(x,y) \in \prec$, there exists a complete decreasing separable system $\mathcal{E}_{xy}$ on (X,t) such that there exists $E \in \mathcal{E}_{xy}$ with $x \in E$, $y \notin E$.*

Proof. $1 \Rightarrow 2$. Let $\precsim$ be a weakly continuous preorder on a topological space (X,t). Consider any pair $(x,y) \in \prec$, and a continuous and increasing real-valued function $u_{xy} : (X,\precsim,t) \to ([0,1],\leq,t_{nat})$ such that $u_{xy}(x) < u_{xy}(y)$. Then just define

$$\mathcal{E}_{xy} := \{\{u_{xy}^{-1}([0,q[)\}_{q\in\mathbb{Q}\cap[0,1[} \cup X\}$$

in order to immediately verify that condition 2 holds.

$2 \Rightarrow 1$. Assume that, given any pair $(x,y) \in \prec$, there exists a complete decreasing separable system $\mathcal{E}_{xy}$ on (X,t) with the indicated property. Then, there exists a complete decreasing separable subsystem $\mathcal{E}'_{xy} \subset \mathcal{E}_{xy}$ such that $x \in E' \subset E$ for every $E' \in \mathcal{E}'_{xy}$. Therefore, we have that $\mathcal{E}'_{xy}$ is a complete decreasing separable system on (X,t) such that $x \in E'$, $y \notin E'$ for every $E' \in \mathcal{E}'_{xy}$. Then the thesis follows from Bosi and Zuanon Remark 2.21, 2 in [19], since to every complete decreasing separable system we can associate a continuous increasing function in a very natural way. □

Remark 1. *The weak continuity of the preorder $\precsim_{\mathbb{E}}$ presented in Definition 5 is an immediate consequence of the previous proposition.*

Based on the above Proposition 1, it is easy to prove the following theorem, which presents a general characterization of the existence of a continuous order-preserving function. This theorem provides a slight generalization of the analogous characterizations presented by Herden,Theorem 4.1 in [1], and Herden, Theorem 3.1 in [2]. We omit the proof that is based on the proof of the aforementioned theorems and on the above Proposition 1.

Theorem 1. *Consider a preorder $\precsim$ on a topological space (X,t). Then the following assertions are equivalent:*

1. *There is a continuous order-preserving function $u : (X,\precsim,t) \to ([0,1],\leq,t_{nat})$;*
2. *There is a countable complete decreasing separable system $\mathcal{E} = \{E_n\}_{n\in\mathbb{N}}$ with the property that, for every $(x,y) \in \prec$, there is $n \in \mathbb{N}$ with $x \in E_n$, $y \notin E_n$;*
3. *There is a countable family $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$ of complete decreasing separable systems on (X,t) with the property that, for every $(x,y) \in \prec$, there exist $n \in \mathbb{N}$ and $E \in \mathcal{E}_n$ with $x \in E$, $y \notin E$;*
4. *There is a countable family $\{u_n : (X,\precsim,t) \to ([0,1],\leq,t_{nat})\}_{n\in\mathbb{N}\setminus\{0\}}$ of continuous increasing functions with the property that, for every $(x,y) \in \prec$, there exists $n \in \mathbb{N}$ with $u_n(x) < u_n(y)$.*

Denote by $\mathbf{F}_{\precsim}(X,t)$ the family of all continuous increasing functions for a preorder $\precsim$ on a topological space (X,t). Let us now define the family of complete separable systems associated to a (weakly continuous) preorder.

Definition 6. *Let $\precsim$ be a weakly continuous preorder on a topological space (X,t). Then the family $\mathbb{E}_{\precsim}(X,t) \in \mathbb{F}_C(X)$ which is naturally associated to $\precsim$ is defined as follows:*

$$\mathbb{E}_{\precsim}(X,t) := \{\{u^{-1}([0,q[)\}_{q\in\mathbb{Q}\cap[0,1[} \cup X\}_{u\in\mathbf{F}_{\precsim}(X,t)}.$$

Definition 7. *Let $\precsim$ and $\precsim'$ be two preorders on a set X. Then $\precsim'$* is said to extend *$\precsim$ if $\precsim \subset \precsim'$ and $\prec \subset \prec'$.*

Remark 2. *Let $\precsim$ be a weakly continuous preorder on a topological space (X,t). Then the preorder $\precsim_{\mathbb{E}_{\precsim}(X,t)}$ is a weakly continuous preorder which extends $\precsim$.*

Definition 8. *Given a topological space (X, t), the weak topology on X, $\sigma(X, C(X, t, \mathbb{R}))$ is defined to be the coarsest topology on X such that every continuous real-valued function on (X, t) remains continuous. Two points $x, y \in X$ are considered equivalent if $f(x) = f(y)$ for all continuous real-valued functions f. In this case, we write $x \sim_C y$.*

The reader is assumed to be familiar with the fundamental concepts of general topology, like those of *basis*, *subbasis*, *complete regularity*, and *Hausdorff*. However, for the reader's convenience, we recall the definition of normal topological space since it is central in the present study.

Definition 9. *A topology t on a nonempty set X is defined to be normal if, given any pair (A, B) of closed subsets of X such that $A \cap B = \emptyset$, there exist two open sets U, V such that $A \subseteq U$, $B \subseteq V$ and $U \cap V = \emptyset$.*

The famous *Urysohn's Lemma* states that a topology t on a set X is normal if and only if for every pair of disjoint nonempty closed sets A, $B \subset X$ there exists a continuous function $f : (X, t) \Rightarrow ([0, 1], t_{nat})$ such that $f(x) = 0$ for all $x \in A$ and $f(x) = 1$ for all $x \in B$.

Bosi, Proposition 4.1 and Proposition 4.2 in [17], presented the following result.

Proposition 2. *If (X, t) is any topological space, then the following assertions are verified:*

1. *$\sigma(X, C(X, t, \mathbb{R}))$ is the coarsest topology on X such that all weakly continuous preorders on (X, t) are still continuous;*
2. *If t is strongly useful, then $(X_{|\sim_C}, \sigma(X, C(X, t, \mathbb{R}))_{|\sim_C})$ is a normal Hausdorff-space.*

Notice that, according to the previous proposition, it is not restrictive to limit ourselves to take into account normal Hausdorff spaces when dealing with strongly useful topology. This consideration is of vital importance and motivates the analysis performed in the next section.

3. Properties of Strongly Useful Topologies

We now present a sufficient condition for the strong usefulness of a topology. This condition holds in the general case, without the aforementioned normality assumption. We recall that the topology $t_{\mathbb{E}}$ generated by any family $\mathbb{E} \in \mathbb{F}_C(X)$ of complete separable systems on a topological space (X, t) is the topology on X which has $\mathbb{E}$ as a subbasis of open sets.

Proposition 3. *A topology t on a set X is strongly useful provided that the topology $t_{\mathbb{E}}$ generated by every family $\mathbb{E} \in \mathbb{F}_C(X)$ is second countable.*

Proof. Let $\precsim$ be a weakly continuous preorder on a topological space (X, t), and consider the family $\mathbb{E}_{\precsim}(X, t)$ of complete separable systems associated to $\precsim$ as in Definition 6. Since the topology $t_{\mathbb{E}_{\precsim}(X,t)}$ is second countable, we have that there exists a countable family $\{u_n\}_{n \in \mathbb{N} \setminus \{0\}}$ of continuous increasing functions such that the topology generated by $\{\{u_n^{-1}([0, q[)\}_{q \in \mathbb{Q} \cap]0,1[} \cup X\}_{n \in \mathbb{N} \setminus \{0\}}$ coincides with $t_{\mathbb{E}_{\precsim}(X,t)}$. Therefore, based on this family $\{u_n\}_{n \in \mathbb{N} \setminus \{0\}}$, we have that, for every $(x, y) \in \prec$, there exists $n \in \mathbb{N}$ with $u_n(x) < u_n(y)$. Hence, the function

$$u := \sum_{n=1}^{\infty} 2^{-n} u_n$$

is a continuous order-preserving function for $\precsim$. This consideration completes the proof. □

The previous proposition is analogous to the sufficient part of the characterization of useful topologies presented by Bosi and Herden, Theorem 3.1 in [11], according to which a topology is useful if and only if the topology generated by every complete separable system is second countable. However, the converse of Proposition 3 does not hold in the case of a normal Hausdorff space, as the following example shows.

Example 1. *Consider a normal Hausdorff topology on a countable set X, which is not second countable. This is the case, for example, of Appert's topology t_A on the positive integers $X := \mathbb{N} \setminus \{0\}$ (see Steen and Seebach [20]). We have that t_A is strongly useful, since X is countable. But since t_A is not second countable, the countability of X implies the existence of at least one point $x \in X$ having a basis $\mathbf{B}(x) := \{O_i(x)\}_{i \in I}$ of (open) neighborhoods of x, the cardinality of which is not countable. Let now, for every $i \in I$, some (countable) complete separable system (chain) $\mathcal{E}_i(O_i(x))$ be chosen in such a way that $O_i(x)$ is the minimum element of $(\mathcal{E}(O_i(x)), \subsetneqq)$. Since (X, t_A) is a normal topological space, the arbitrary construction of these chains does not make any difficulties. But now we are already done. Indeed, the topology $t_{\mathbb{E}}$ that is generated by the family $\mathbb{E} := \{\mathcal{E}_i\}_{i \in I}$ cannot be second countable.*

Another general property of strongly useful topologies is in order now. We shall denote by $C((X,t),[0,1])$ the set of all continuous function on the topological space (X,t) taking values in the real interval $[0,1]$.

Proposition 4. *If (X,t) is a strongly useful normal Hausdorff topological space, then for every open subset O of X, the subspace $\left(O, t_{|O}\right)$ of (X,t) is strongly useful.*

Proof. Let a (nonempty) open subset O of X be arbitrarily chosen. First of all, we notice that our assumption (X,t), to be a normal Hausdorff-space, allows us to choose a complete separable system $\mathcal{C}$ on $\left(O, t_{|O}\right)$ in such a way that there exists some countable strictly increasing chain $\emptyset \subsetneqq C_0 \subsetneqq C_1 \subsetneqq \cdots \subsetneqq C_n \subsetneqq \cdots \subsetneqq O$ of closed subsets of O having the properties that $\bigcup_{n \in \mathbb{N}} C_n$ coincides with O and $X \setminus C_n$ belongs to $\mathcal{C}$ for every $n \in \mathbb{N}$. Let now $\mathcal{E}_O$ be a complete separable system on $\left(O, t_{|O}\right)$, i.e., the open and closed sets with respect to $\mathcal{E}_O$ are open and closed with respect to the relativized topology $t_{|O}$ on O. At this point, we consider for every $n \in \mathbb{N}$ the complete separable system $\mathcal{E}_O^n := \{E \cap C_n | E \in \mathcal{E}_O\}$ that is induced by $\mathcal{E}_O$. Since O is open and every set C_n, where n runs through the set $\mathbb{N}$ of natural numbers, is closed, we may apply our assumption (X,t) to being strongly useful in order to conclude that for every $n \in \mathbb{N}$ there exists a function $u_n \in C((X,t),[0,1])$ that represents the total preorder $\precsim_{\mathcal{E}_O^n}$ (see Bosi and Zuanon, Remark 2.21 in [19]). Since $O = \bigcup_{n \in \mathbb{N}} C_n$, the function $u_O := \sum_{n \in \mathbb{N}} \frac{1}{2^n} u_n \in C((X,t),[0,1])$ is a function such that its restriction to O represents the total preorder $\precsim_{\mathcal{E}_O}$. Summarizing our considerations, it follows that $\mathcal{E}_O$ is the restriction of some complete separable system that is defined on (X,t), and, needless to say, such consideration naturally extends to the case of any family of complete separable systems on (X,t). This means, however, that the case of an arbitrary open subset O of X to be given reduces to the situation that O coincides with X. Hence, we have completed the proof. □

Let us now present other properties relative to the assumption of normality of the topological space. The appropriateness of such hypothesis has been previously discussed.

The reader may recall that a topology t on a set X is said to be *hereditarily separable on closed sets* if every subspace $(A, t_{|A})$ of (X,t) is separable whenever A is a closed subset of X.

Proposition 5. *Let (X,t) be a strongly useful normal Hausdorff topological space. Then t is hereditarily separable on closed sets.*

Proof. Consider a strongly useful normal Hausdorff topological space (X, t). Assume, by contraposition, that t is not hereditarily separable on closed sets. Then, there exists some (nonempty) closed subset C of X such that the induced topology $t_{|C}$ is not separable. Consider, however, that $t_{|C}$ is a normal Hausdorff topology, since normality is a hereditary property on closed sets. Then, from Bosi and Zuanon, Theorem 3.1, statement 3 in [9], since $t_{|C}$ is in particular a completely regular topology, there exists a continuous total preorder $\precsim_{|C}$ which does not admit any continuous order-preserving function (i.e., utility function) $u^C : (X, \precsim_{|C}, t_{|C}) \rightarrow ([0,1], \leq, t_{nat})$. Define the preorder $\precsim$ on X as follows, for all $x, y \in X$:

$$x \precsim y \Leftrightarrow (x, y \in C) \text{ and } (x \precsim_{|C} y).$$

Notice that, since (X, t) is a normal Hausdorff topological space, every increasing function $u^C : (X, \precsim_{|C}, t_{|C}) \rightarrow ([0,1], \leq, t_{nat})$ can be extended, by the Tietze–Urysohn Extension Theorem, to a continuous increasing function $u : (X, \precsim, t) \rightarrow [0,1], \leq, t_{nat})$ (see e.g., Engelking, Theorem 2.1.8 in [21]). Therefore $\precsim$ is a weakly continuous preorder on (X, t), and clearly it cannot admit any continuous order-preserving function $u : (X, \precsim, t) \rightarrow [0,1], \leq, t_{nat})$ (otherwise, its restriction to C would be a continuous order-preserving function for $\precsim_{|C}$). Hence, the topology t on X is not strongly useful, and the proof is complete. □

Since from Bosi, Theorem 4.2 in [17], a topology t is strongly useful if and only if for every family $\mathbb{E}$ of complete separable systems there exists a complete separable system $\mathcal{L}$ for which $t_{\mathcal{L}}$ is second countable and $t_{\mathbb{E}} \subset t_{\mathcal{L}}$, then we have that the following proposition holds.

Proposition 6. *Let (X, t) be a strongly useful topological space. Then for every family $\mathcal{O} = \{O_\alpha\}_{\alpha \in I}$ of clopen subsets of X, there exists a complete separable system $\mathcal{E}$ on (X, t) such that $t_{\mathcal{O}} \subset t_{\mathcal{E}}$ and $t_{\mathcal{E}}$ is second countable.*

In order to proceed, we consider for every subset S of X the set $\mathbf{\Sigma}(S)$ of all well-ordered chains $\boldsymbol{O}(S) := S \subset O_0 \subsetneqq O_1 \subsetneqq ... \subsetneqq O_\alpha \subsetneqq ... \subsetneqq X$ of open subsets O_α of X such that $\bigcup\limits_{\alpha} O_\alpha = X$. Then the *length* $\boldsymbol{\lambda}(\boldsymbol{O}(S))$ of $\boldsymbol{O}(S)$ is the cardinality of $\boldsymbol{O}(S)$, or equivalently the least ordinal which can be order-embeddable in $(\boldsymbol{O}(S), \subsetneqq)$.

The reader may recall that a topology t on X is said to be a *Lindelöf-topology* if for every open covering $\mathbf{C}$ of X there exists a countable covering $\mathbf{C}' \subset \mathbf{C}$ of X.

In order to prove the following theorem, which in some sense illustrates how close strongly useful normal Hausdorff topological spaces are to Lindelöf topological spaces, the following lemma is needed, which provides a sufficient condition for the non-strong usefulness of a topology.

Lemma 1. *Let t be a topology on X. If there exists a complete separable system $\mathcal{E}$ on (X, t) which contains an uncountable well-ordered sub-chain $\mathcal{E}'$, then t is not strongly useful.*

Proof. Consider the weakly continuous total preorder $\precsim_{\mathcal{E}}$ associated to the complete separable system $\mathcal{E}$ (see Definition 5). Since there exists an uncountable well-ordered sub-chain $\mathcal{E}'$ of $\mathcal{E}$, then $\precsim_{\mathcal{E}}$ does not admit a continuous order-preserving function (utility function) by the above Theorem 1. This consideration completes the proof. □

As usual, $\aleph_1$ stands for the first uncountable ordinal number.

Theorem 2. *In order for a strongly useful normal Hausdorff topological space (X, t) to be Lindelöf, it is sufficient that every chain $\boldsymbol{O}(\emptyset) := \emptyset \subset O_0 \subsetneqq O_1 \subsetneqq ... \subsetneqq O_\alpha \subsetneqq ... \subsetneqq X$, such that the length $\boldsymbol{\lambda}(\boldsymbol{O}(\emptyset))$ exceeds $\aleph_0$, contains some open set O_α having the property that $\overline{O_\alpha} \subset O_\lambda$ for some $\lambda > \alpha$.*

Proof. First consider that, if (X, t) is not Lindelöf, then there exists an open cover of X which does not admit a countable subcover of X, and therefore a standard transfinite induction argument guarantees the existence of a well-ordered chain

$$\boldsymbol{O}(\emptyset) := \emptyset \subset O_0 \subsetneqq O_1 \subsetneqq ... \subsetneqq O_\alpha \subsetneqq ... \subsetneqq X$$

such that $\bigcup_{\alpha < \aleph_1} O_\alpha = X$. Therefore, it suffices to exclude this possibility given our hypotheses.

Let us consider any well-ordered chain $\boldsymbol{O}(\emptyset)$ which covers X. We have to show that the length $\boldsymbol{\lambda}(\boldsymbol{O}(\emptyset))$ is not greater than $\aleph_0$. Therefore, we distinguish between the case $\boldsymbol{\lambda}(\boldsymbol{O}(\emptyset)) \leq \aleph_0$ and the case $\boldsymbol{\lambda}(\boldsymbol{O}(\emptyset)) > \aleph_0$. In the first case, we are done. In the second case, there exists some set $O_\alpha \in \boldsymbol{O}(\emptyset)$ having the property that $\overline{O_\alpha} \subset O_\lambda$ for some $\lambda > \alpha$. Hence, we may consider the disjoint closed sets $\overline{O_\alpha}$ and $X \setminus O_\lambda$. Since (X, t) is a normal Hausdorff-space, there exists some function $f_0 \in C((X, t), [0, 1])$ such that $f_{0_{|\overline{O_\alpha}}} = 0$ and $f_{0_{|X \setminus O_\lambda}} = 1$. We, thus, may proceed by transfinite induction.

Let $0 < \alpha < \boldsymbol{\lambda}(\boldsymbol{O}(\emptyset))$ be not a limit ordinal and let some function $f_{\alpha-1} \in C((X, t), [0, 1])$ with corresponding sets $O_\gamma \in \boldsymbol{O}(\emptyset)$ and $O_\chi \in \boldsymbol{O}(\emptyset)$ such that $\overline{O_\gamma} \subset O_\chi$ have already been defined. Then we consider the chain $\boldsymbol{O}(O_\chi) := O_\chi \subsetneqq O_{\chi+1} \subsetneqq O_{\chi+2} \subsetneqq ... \subsetneqq O_\xi \subsetneqq ... \subsetneqq X$ and distinguish between the case $\boldsymbol{\lambda}(\boldsymbol{O}(O_\chi)) \leq \aleph_0$ and the case $\boldsymbol{\lambda}(\boldsymbol{O}(O_\chi)) \geq \aleph_1$. In the first case, nothing remains to be proved. In the second case, there exists some set $O_\eta \in \boldsymbol{O}(O_\chi)$ having the property that $\overline{O_\eta} \subset O_\rho$ for some $\rho > \eta$. Using the above normality argument, it follows that in this case there exists some function $f_\alpha \in C((X, t), [0, 1])$ satisfying the equations $f_{\alpha_{|\overline{O_\eta}}} = 0$ and $f_{\alpha_{|X \setminus O_\rho}} = 1$.

Let $0 < \alpha < \boldsymbol{\lambda}(\boldsymbol{O}(\emptyset))$ be a limit ordinal. Now we may assume that for every ordinal number $\beta < \alpha$, functions $f_\beta \in C((X, t), [0, 1])$ have already been defined with respect to corresponding sets $\overline{O_\mu} \subset O_\nu$. Let τ be the minimum of all ordinal numbers that are not smaller than any of the ordinal numbers ν that already have been considered. Then we consider the chain $\boldsymbol{O}(O_\tau) := O_\tau \subsetneqq O_{\tau+1} \subsetneqq O_{\tau+2} \subsetneqq ... \subsetneqq O_\theta \subsetneqq ... \subsetneqq X$ and distinguish, in analogy to the above considerations, between the case $\boldsymbol{\lambda}(\boldsymbol{O}(O_\tau)) \leq \aleph_0$ and the case $\boldsymbol{\lambda}(\boldsymbol{O}(O_\tau)) \geq \aleph_1$. In the first case, the limit step is done. In the second case, there again exist sets $O_\tau \subset \overline{O_\sigma} \subset O_\delta$ and a corresponding function $f_\alpha \in C((X, t), [0, 1])$ such that $f_{\alpha_{|\overline{O_\sigma}}} = 0$ and $f_{\alpha_{|X \setminus O_\delta}} = 1$.

Having finished the transfinite induction procedure, we must distinguish between the case that there exists an ordinal number θ such that the transfinite induction process stops at θ and the case that there exists no such ordinal number. In the first case, it follows that $\boldsymbol{\lambda}(\boldsymbol{O}(\emptyset)) < \aleph_1$ and everything is shown. In the second case, however, the complete separable system

$$\mathcal{E} := \{l(x)\}_{x \in X} := \{\{z \in X | \forall \alpha\, (f_\alpha(z) \leq f_\alpha(x)) \wedge \exists \beta\, (f_\beta(z) < f_\beta(x))\}\}_{x \in X}$$

on (X, t) that is induced by the functions $f_\alpha \in C((X, t), [0, 1])$ contains some well-ordered sub-chain, the length of which is at least $\aleph_1$. This conclusion contradicts our assumption (X, t) to be strongly useful since the above Lemma 1 applies. Therefore, the proof is complete. □

4. Conclusions

We have presented some new properties of topologies such that every weakly continuous preorder admits a continuous order-preserving function, namely, strongly useful topologies. Majority of the results concern the case of normal Hausdorff topologies, whose interest arises in connection with the consideration of weak topologies of continuous functions. Indeed, the concept of weak continuity of a preorder is based on the consideration of continuous (increasing) functions.

Hopefully, we shall be able to arrive at some characterization of strong usefulness simpler than existing ones in a future paper. Indeed, we formulate the conjecture according to which a normal Hausdorff topology is strongly useful if and only if it is hereditarily separable on closed sets (a condition whose necessity is proven in this paper) and, in addition, every family of clopen sets is (at most) countable. A result of this type would be the counterpart to the best existing characterization of completely regular useful topologies. We recall that a topology is useful if every continuous total preorder admits a continuous order-preserving function (utility function). According to this latter characterization, a completely regular topology is useful if and only if it is separable and every chain of clopen sets is (at most) countable (see the already mentioned Bosi and Zuanon, Theorem 2 in [10]).

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