symmetry

Article

Maximal Elements of Preferences on Compact Spaces from Optimization of One-Way Utilities

Gianni Bosi [1,*], Gabriele Sbaiz [1] and Magalì Zuanon [2]

[1] DEAMS, Università di Trieste, 34127 Trieste, Italy; gabriele.sbaiz@deams.units.it
[2] DEM, Università di Brescia, 25121 Brescia, Italy; magali.zuanon@unibs.it
* Correspondence: gianni.bosi@deams.units.it; Tel.: +39-040-558-7115

Abstract

The search for maximal elements of preference relations has been recently related to the optimization of one-way utilities on compact topological spaces. In this paper, we deepen this study by referring to upper semicontinuous finite Richter–Peleg multi-utility representations of preorders. We provide necessary and sufficient conditions for the existence of representations of this kind and then, under the assumption of near-completeness, we characterize the identification of all maximal elements by maximizing all functions in an appropriate representation under compactness.

Keywords: maximal elements; Richter-Peleg multi-utility representation

MSC: 91B16 (Primary); 06A06 (Secondary)

1. Introduction

The search for *maximal elements* of binary relations, which usually represent *individual preferences*, is an old problem which has been naturally related to suitable *(semi)continuity conditions*, especially on *compact spaces*.

Such an approach was studied, for example, in the seminal papers by Bergstrom [1] and Campbell and Walker [2]. However, a number of authors were concerned with the existence of maximal elements of (possibly not transitive) preferences on topological spaces (see, e.g., Quartieri [3], Alcantud [4], Bosi and Zuanon [5,6], Andrikopoulos and Sampanis [7], Kukushin [8], Linares [9], Luc and Soubeyran [10], Mehta [11], Nosratabadi [12], Rodríguez-Palmero and García-Lapresta [13], and Ward [14], among the others).

We just mention that, more recently, when the *ground space* is a compact subset of $\mathbb{R}^n$, the maximal elements are identified as the *solutions* to suitable *variational inequalities* (see, e.g., Bueno et al. [15]).

Besides the existence of this, the problem arises of possibly *representing* the maximal elements. This means that, for instance, the maximal elements are detected belonging to the *argmax* of suitable *(transfer) upper semicontinuous functions* (see Tian and Zhou [16,17]), exhibiting some relevant properties with respect to the binary relation, like that of being *order-preserving*, or at least *one-way utilities* for its *strict part*.

As a matter of fact, the maximization of a preference order, possibly represented by a utility function, constitutes an initial step toward formalizing the notion of *purposive behaviour*. Indeed, an extension of this framework may require the consideration of multiple preference orderings, where an alternative selected from a feasible set is obtained from the simultaneous maximization of all admissible underlying preferences. When these

Academic Editors: Yanlin Li and Tiehong Zhao

Received: 17 December 2025
Revised: 9 January 2026
Accepted: 12 January 2026
Published: 14 January 2026

orders represent agents' rankings, the collection of selected alternatives coincides with the Pareto-efficient outcomes. Alternatively, the individual preference orders can be interpreted as possible preferences of a single agent. In this latter case, the choice set consists of the possibilities which cannot be rebuffed regardless of which ranking ultimately materializes. The arguments above highlight the central role of *maximal elements* with respect to the preference relation (that is *not* complete, in general).

The main supporting result for the representability issue is the so called *White Theorem* (see White [18]), according to which, if a preorder admits an order-preserving representation, then for every maximal element x_0 there is an order-preserving representation, which attains its maximum precisely at x_0. Bosi and Zuanon [19] presented a generalization of this result, which pledeges the existence of an order-preserving function, which is *upper semicontinuous* and at the same time exhibits the above property for every maximal element.

In this paper, we shall primarily refer to preorders, which admit particular multi-utility representations. The *upper semicontinuous/continuous multi-utility representation* of a preorder was originally elaborated by Ok [20] and Evren and Ok [21]. The existence of a countable collection of functions with the above property with respect to a preorder on a topological space has been recently studied by Bosi et al. [22]. The case of a *finite* family of functions was discussed by Kaminski [23] and more recently by Candeal [24,25].

Indeed, the problem concerning the existence of such multi-utility representations is fundamental in the modelling of incomplete preferences. It is particularly relevant for the analysis of behavioural phenomena such as intransitivity, multi-objective decisions, and models of risk, uncertainty, and theory of games. Real-world applications arise in collective decision-making contexts within representative and supranational institutions. Examples include the accession process of new member states to the European Union and the requirement of unanimous affirmative votes.

A different application concerns decision-making under myopia, where lasting consequences of an alternative plan of action must be evaluated despite short-term biases (see, for example, Bosi et al. [22]).

In this paper, we tackle the problem concerning the representation of the whole set of the maximal elements relative to a preorder defined on a topological space by performing the maximization of a finite number of upper semicontinuous real-valued functions. More precisely, we concentrate our attention on a finite family $\mathbf{U} = \{u_1, ..., u_n\}$ of functions, which are all upper-semicontinuous, order-preserving, and at the same time fully represent a preorder defined on some topological space.

This is the case when all functions are order-preserving with respect to the preorder, so that their maximization leads to maximal elements. We analyze the upper semicontinuity properties of preorders, which admit this kind of representation.

We show that not only $\precsim$ must be *upper semiclosed* (i.e., for every point $x \in X$, $U_{\precsim}(x) = \{u \in X : x \precsim u\}$ is a closed set), but it must also be *upper semicontinuous* (i.e., for every point $x \in X$, $L_{\prec}(x) = \{u \in X \mid u \prec x\}$ is an open set).

It is of particular interest to ensure the existence of a full description of this kind, compatible with the general case of a nontotal preorder, which in addition has the property that all the maximal elements can be captured by maximizing all the functions in the representation. The interest in a finite representation is, so to say, of operational nature.

We provide necessary and sufficient conditions for the existence of representations of this kind and then, under the assumption of near-completeness, we characterize the identification of all maximal elements by maximizing all functions in an appropriate representation under compactness.

Needless to say, the assumption of compactness is essential in order to guarantee the existence of a maximum for each upper semicontinuous function. The description considered in this paper appears, in some sense, as the closest to the classical (upper semicontinuous) utility representation of a total preorder.

2. Notation and Preliminary Results

In the sequel, $\precsim$ stands for a *reflexive* binary relation on a set X. If $\precsim$ is also *transitive*, then it is called a *preorder*.

The *asymmetric part* (or *strict part*) $\prec$ of $\precsim$, is described in this way:

$$x \prec y \text{ if and only if } [(x \precsim y) \textit{ and } \textit{ not}(y \precsim x)] \ (x, y \in X),$$

while the *symmetric part* $\sim$ of $\precsim$ is described in the following way:

$$x \sim y \text{ if and only if } [(x \precsim y) \textit{ and } (y \precsim x)] \ (x, y \in X).$$

Clearly, the symmetric part $\sim$ of a preorder $\precsim$ is an *equivalence relation*.

The points $x, y \in X$ are defined to be *incomparable* (according to the binary relation $\precsim$) if

$$\textit{not}(x \precsim y) \textit{ and } \textit{not}(y \precsim x).$$

In general, $\precsim$ is defined to be *total* (or *complete*) if $x \precsim y$ or else $y \precsim x$ holds true for every choice of points $x, y \in X$.

Define, for all points $x \in X$,

$$L_{\prec}(x) = \{u \in X \mid u \prec x\}, \ U_{\prec}(x) = \{v \in X \mid x \prec v\},$$

$$L_{\precsim}(x) = \{u \in X \mid u \precsim x\}, \ U_{\precsim}(x) = \{v \in X \mid x \precsim v\}.$$

So, we have that $L_{\prec}(x)$ and $U_{\prec}(x)$ are the *strict sections* (respectively, *lower* and *upper*) of $\precsim$ at the point $x \in X$.

A set $D \subset X$ is said to be *decreasing* if $L_{\precsim}(x) \subset D$ or *increasing* if $U_{\precsim}(x) \subset D$ for each $x \in D$.

Definition 1. *If $\precsim$ is a reflexive binary relation on a set X, a function $u : (X, \precsim) \longrightarrow (\mathbb{R}, \leqslant)$ is said to be*

1. *Isotonic or increasing if, for every $x, y \in X$,*

$$x \precsim y \textit{ implies that } u(x) \leqslant u(y);$$

2. *A one-way utility for $\prec$ if, for every $x, y \in X$,*

$$x \prec y \textit{ implies that } u(x) < u(y);$$

3. *Order-preserving for $\precsim$ if it is at the same time increasing and a one-way utility for $\prec$;*

4. *A utility for $\precsim$ if, for every $x, y \in X$,*

$$x \precsim y \textit{ if and only if } u(x) \leq u(y).$$

Notice that, in the case of a one-way utility u for $\prec$, if $u(x) < u(y)$, then it occurs that either $x \prec y$ or else x and y are incomparable. If u is a utility for $\precsim$, then $u(x) < u(y)$ clearly implies that $x \prec y$. Order-preserving functions on $(X, \precsim)$ are also named *Richter–Peleg utilities* for $\precsim$.

https://doi.org/10.3390/sym18010155

Finally, we shall indicate with τ_{nat} the *natural topology* on $\mathbb{R}^n$ and $X_{|\sim}$ the quotient set modulo of the equivalence $\sim$.

We recall some concepts of upper/lower continuity already existing in the literature, some of them appearing meanwhile classical.

Definition 2. *Let $\precsim$ be a preorder on a topological space (X, τ). Then $\precsim$ is called the following:*

1. *Upper, respectively lower, semiclosed if, for all $x \in X$, $U_{\precsim}(x)$, respectively $L_{\precsim}(x)$, turns out to be a closed subset of X ;*
2. *Upper, respectively lower, semicontinuous if, for all $x \in X$, $L_{\prec}(x)$, respectively $U_{\prec}(x)$, turns out to be an open subset of X;*
3. *Continuous if it is both upper and lower semicontinuous;*
4. *Closed if $\precsim$ is closed as a subset of $X \times X$ endowed with the product topology $\tau \times \tau$;*
5. *Weakly upper semicontinuous if for every pair $(x, y) \in \prec$ there exists some open decreasing subset $O_{x,y}$ of X with the property that $x \in O_{x,y}$ and $y \in X \backslash O_{x,y}$;*
6. *Weakly upper continuous if, given any every pair $(x, y) \in \prec$, there is a set $V(x)$ containing x such that $V(x) \precsim y$;*
7. *Transfer weakly upper continuous if, given every pair $(x, y) \in \prec$, there is a point $y' \in X$ and an open set $V(x)$ which contains x and at the same time is such that $V(x) \precsim y'$;*
8. *Partially upper continuous if we have that, when we indicate by J the set of all jumps in $X_{|\sim}$, J is countable and, given any pair*
$$(x, y) \in \prec \backslash \{(v, w) : v \in [v'], w \in [w'], ([v'], [w']) \in J\},$$
there is an open set $V(x)$ which contains x and at the same time is such that $V(x) \precsim y$;
9. *Quasi upper semicontinuous if there is some upper semiclosed preorder $\lesssim$ on (X, τ) with $\prec \subset <$.*

Remark 1. *It is straightforward to see that a reflexive binary relation is weakly upper semicontinuous when it is either upper semicontinuous or upper semiclosed or else it has an upper semicontinuous order-preserving function. It is easily seen that all the notions of upper semicontinuity coincide when we deal with a total preorder $\precsim$.*

Remark 2. *Nosratabadi [12] showed that a preorder is transfer weakly upper continuous provided that it is partially upper continuous and it is defined on a compact space .*

Definition 3. *An element $x_0 \in X$ is defined to be maximal for a binary relation $\precsim$ if $x_0 \prec x$ is false for every $x \in X$ (i.e., $U_{\prec}(x_0) = \emptyset$).*

A preorder $\precsim$, which is defined on a compact topological space (X, τ), has a maximal element in case that it verifies any of the semicontinuity concepts of the above definition. To be more precise, the sufficiency of

- Upper semiclosedness was proven by Ward [14] (see also Evren and Ok [21]);
- Upper semicontinuity was proved by Bergstrom [1];
- Weak upper semicontinuity by Bosi and Zuanon [5];
- Weak upper continuity and transfer weak upper continuity by Tian and Zhou [17];
- Partial upper continuity by Nosratabadi [12];
- Quasi upper semicontinuity by Bosi and Zuanon [6].

Remark 3. *Consider that, if an upper semicontinuous order-preserving function for a preorder exists, then such a preorder is necessarily weakly upper semicontinuous. Upper semicontinuity*

https://doi.org/10.3390/sym18010155

or else upper semiclosedness is not implied by the existence of an upper semicontinuous order-preserving representation.

Let us now introduce the fundamental definition of a finite multi-utility description.

Definition 4. *A finite Richter–Peleg multi-utility representation of a preorder $\precsim$ on a set X is a finite collection $\mathbf{U} = \{u_1, ..., u_n\}$ of real-valued functions u_i on X, such that, for all $x, y \in X$,*

1. *$x \precsim y$ if and only if $u_i(x) \leqslant u_i(y)$, for every $i \in \{1, ..., n\}$;*
2. *$x \prec y$ if and only if $u_i(x) < u_i(y)$, for every $i \in \{1, ..., n\}$.*

If only property 1 above holds true, then $\mathbf{U} = \{u_1, ..., u_n\}$ is said to be a finite multi-utility representation of the preordered set $(X, \precsim)$.

The following characterization of the existence of a finite Richter–Peleg multi-utility representation holds true.

Proposition 1. *A family $\mathbf{U} = \{u_1, ..., u_n\}$ of functions $u_i : (X, \precsim) \longrightarrow (\mathbb{R}, \leqslant)$ is a finite Richter–Peleg multi-utility representation of the preordered set $(X, \precsim)$ if and only if the following properties hold:*

1. *u_i is an order-preserving function for $\precsim$ for every index $i \in \{1, ..., n\}$;*
2. *For all $x, y \in X$, if $not(y \precsim x)$, then there is an index $i \in \{1, ..., n\}$ with $u_i(x) < u_i(y)$.*

Proof. Assume that $\mathbf{U} = \{u_1, ..., u_n\}$ is a finite Richter–Peleg multi-utility representation of $\precsim$. Then it is clear that every function u_i is order-preserving, since it is increasing and, whenever $x \prec y$, $u_i(x) < u_i(y)$ for every index $i \in \{1, ..., n\}$. Further, if $not(y \precsim x)$, then it is not true that $u_i(y) \leq u_i(x)$ for all $i \in \{1, ..., n\}$. Conversely, let $\mathbf{U} = \{u_1, ..., u_n\}$ be a family of real-valued functions on $(X, \precsim)$, which satisfies conditions 1 and 2. Since condition 1 holds, it is clear that, for all $x, y \in X$,

$$x \precsim y \Rightarrow u_i(x) \leqslant u_i(y) \quad \text{for every } i \in \{1, ..., n\}.$$

In order to show that, for all $x, y \in X$,

$$u_i(x) \leqslant u_i(y) \quad \text{for every } i \in \{1, ..., n\} \Rightarrow x \precsim y,$$

assume that $not(x \precsim y)$. Then by condition 2, $u_i(y) < u_i(x)$ for some index i. □

Remark 4. *Observe that, if $\mathbf{U} = \{u_1, ..., u_n\}$ is a finite Richter–Peleg multi-utility representation, then each function u_i is order-preserving for $\precsim$ on X.*

Remark 5. *The interest in the existence of a (finite) Richter–Peleg multi-utility representation is related to the obvious fact that the maximization (when possible) of any order-preserving function leads to a maximal element for the preorder, and this is the case of every function in a representation of this kind.*

The concept of a *multi-utility representation with the Richter–Peleg property* is essentially a folk idea among scholars. The terminology is due to the requirement according to which all the functions are order-preserving, while order-preserving functions were first studied by Richter [26] and Peleg [27]. This kind of representation was also studied by Minguzzi [28].

Definition 5. *If (X, τ) is a topological space, and $\precsim$ is a preorder on (X, τ), then a finite Richter–Peleg multi-utility representation of $\precsim$ is called upper semicontinuous if each function u_i is upper semicontinuous.*

3. Finite Multi-Utilities and Representation of Maximal Elements

It is well known that the prominent interest of multi-utilities is related to the fact that they fully represent not necessarily total preorders. Needless to say, the finite Richter–Peleg case is even more attractive for obvious reasons. Indeed, the maximization of any order-preserving function for a preorder leads to a maximal element for the preorder itself.

We are now going to incorporate topological considerations in our treatment. The following theorem presents a characterization of an upper semicontinuous finite multi-utility representation exhibiting the Richter–Peleg property.

Theorem 1. *If $\precsim$ is a preorder on a topological space (X, τ), then we have that the following conditions are equivalent:*

1. *There exists an upper semicontinuous finite Richter–Peleg multi-utility representation $\mathbf{U} = \{u_1, ..., u_n\}$ of $\precsim$;*
2. *There exist exactly n total preorders $\precsim_i$ $(i = 1, ..., n)$ with respective upper semicontinuous utility functions u_i, and the following properties are verified:*
   - *(a)* $\precsim = \bigcap_{i=1}^{n} \precsim_i$;
   - *(b)* $\prec = \bigcap_{i=1}^{n} \prec_i$;
3. *There exist exactly n total preorders $\precsim_i$ $(i = 1, ..., n)$ satisfying*
   - *(c)* $L_{\precsim}(x) = \bigcap_{i=1}^{n} L_{\precsim_i}(x)$ *for every* $x \in X$;
   - *(d)* $L_{\prec}(x) = \bigcap_{i=1}^{n} L_{\prec_i}(x)$ *for every* $x \in X$;
   - *(e)* $L_{\prec_i}(x)$ *is an open set for each* $x \in X$ *and, for every* $i \in \{1, ..., n\}$, *there exists a countable subset* $\mathcal{D}_i$ *of* X *such that, for all* $x, y \in X$ *with* $x \prec_i y$, *there is* $d_i \in \mathcal{D}_i$ *with* $x \prec_i d_i \precsim_i y$.

Proof. $1 \Rightarrow 2$. Let $\mathbf{U} = \{u_1, \ldots, u_n\}$ be an upper semicontinuous finite Richter–Peleg multi-utility representation of $\precsim$. Then, for every index $i \in \{1, \ldots, n\}$, consider the following total preorder $\precsim_i$:

$$x \precsim_i y \text{ if and only if } u_i(x) \leq u_i(y) \quad (x, y \in X).$$

We have that $\precsim_i$ is upper semicontinuous for all $i \in \{1, \ldots, n\}$ since u_i is upper semicontinuous for all $i \in \{1, \ldots, n\}$. Therefore, this part of the proof directly follows from Definition 4.

$2 \Rightarrow 1$. This part of the proof is perfectly analogous to the previous one.

$2 \Leftrightarrow 3$. It is immediate to check that condition *(a)* holds true if and only if it is the case of condition *(c)*, and condition *(b)* holds true if and only if it is the case of *(d)*. Finally, condition *(e)* is valid if and only if every total preorder $\precsim_i$ has an upper semicontinuous utility function u_i (see, for example, Bosi and Sbaiz [29]). □

A corollary to Theorem 1 is in order now, which concerns the case of a *second countable topological space* (X, τ) (i.e., (X, τ) has a countable basis of open sets). We utilize the famous *Rader Theorem* [30], which guarantees that every upper semicontinuous total preorder $\precsim$ admits an upper semicontinuous utility function, provided that it is defined on a second countable topological space (X, τ). For example, every *separable metric space* has this property.

Corollary 1. *If $\precsim$ is a preorder on a second countable topological space (X, τ), then we have that the following conditions are equivalent:*

1. *There is a finite upper semicontinuous Richter–Peleg multi-utility representation $\mathbf{U} = \{u_1, ..., u_n\}$ of $\precsim$;*
2. *There exist exactly n upper semicontinuous total preorders $\precsim_i$ ($i = 1, \ldots, n$), and the following properties are verified:*
   (a) $\precsim = \bigcap_{i=1}^{n} \precsim_i$;
   (b) $\prec = \bigcap_{i=1}^{n} \prec_i$.

Proof. Just consider that, by the Rader theorem, every upper semicontinuous total preorder $\precsim_i$ admits an upper semicontinuous utility function u_i, and then apply Theorem 1. □

We are now going to use the previous considerations in order to represent all the maximal elements of a preorder. Generally speaking, the collection of all maximal elements for the binary relation $\precsim$ on the set X will be indicated by $X_M^{\precsim}$.

The following proposition presents the semicontinuity properties of a preorder which has an upper semicontinuous finite Richter–Peleg multi-utility representation on an arbitrary topological space.

Proposition 2. *Consider a preorder $\precsim$, which admits an upper semicontinuous finite Richter–Peleg multi-utility representation $\mathbf{U} = \{u_1, \ldots, u_n\}$. Then we have that*

1. *$\precsim$ is upper semiclosed;*
2. *$\precsim$ is upper semicontinuous.*

Proof. Let $\precsim$ be a preorder with the indicated property.

In order to show that $\precsim$ is upper semiclosed, let any two points $x, z \in X$ be given, such that $z \notin U_{\precsim}(x)$. Then, since we have that $not(x \precsim z)$, there exists an index $i \in \{1, \ldots, n\}$ such that $u_i(z) < u_i(x)$, so that, since u_i is an upper semicontinuous function, $u_i^{-1}(]-\infty, u_i(x)[)$ is an open subset of X which contains z, and in addition $w \notin U_{\precsim}(x)$ for every $w \in u_i^{-1}(]-\infty, u_i(x)[)$.

To prove that $\precsim$ is also upper semicontinuous, consider that, from Definition 4, we have that

$$\begin{aligned} L_{\prec}(x) = \{z \in X : z \prec x\} &= \{z \in X : u_i(z) < u_i(x) \text{ for all } i \in \{1, \ldots, n\} \\ &= \bigcap_{i=\{1,\ldots,n\}} u_i^{-1}(]-\infty, u_i(x)[) \end{aligned}$$

is an open subset of X, since it is the intersection of finitely many open sets. So, the proof is complete. □

Remark 6. *Notice that, in the general case of a nontotal preorder, given any element $x \in X$, the complement of $L_{\prec}(x)$ not only consists of all the elements in $U_{\precsim}(x)$, but also of all the elements which are incomparable with x.*

With a view to the identification of all maximal elements of a preorder by using the maximization of all the (upper semicontinuous) functions in a finite Richter–Peleg multi-utility representation, we recall a concept which is known in the technical literature.

The following definition was presented by Ok [20].

Definition 6. *A preorder $\precsim$ defined on a set X is called near-complete if every collection A of elements of X, such that any two elements of A are incomparable, is finite.*

The notion of near-completeness is a little bit restrictive, while it is still compatible with the case of a nontotal preorder, obviously.

Ok ([20], Theorem 3) proved that every near-complete upper semicontinuous preorder $\precsim$ has an upper semicontinuous finite multi-utility representation, provided that $\precsim$ is defined on a second countable topological space (X, τ). Observe that, in this case, there exists an upper semicontinuous order-preserving function u for the preorder $\precsim$. Indeed, if $\mathbf{U} = \{u_1, \ldots, u_m\}$ is an upper semicontinuous multi-utility representation for $\precsim$, then we have that the function

$$u = \sum_{i=1}^{n} u_i$$

is an upper semicontinuous order-preserving function for $\precsim$.

To illustrate this fact, consider that u is increasing, since every function u_i is increasing. In addition, for every $x, y \in X$, if $x \prec y$, then there exists some index $\bar{i}$ such that $u_{\bar{i}}(x) < u_{\bar{i}}(y)$, so that

$$u(x) = \sum_{i=1}^{n} u_i(x) < \sum_{i=1}^{n} u_i(y) = u(y).$$

Finally, the fact that the sum of upper semicontinuous functions is an upper semicontinuous function is well known.

Therefore, since every upper semicontinuous function attains its maximum on a compact space, and a point of maximum of any order-preserving function (when it exists) is a maximal element, we have that, if (X, τ) is compact, then $X_M^{\precsim}$ is nonempty. Clearly, when we deal with a *compact metric space* (in particular, any compact subset of $\mathbb{R}^n$), we have that the space is also second countable.

Thus, the following result is in order now, which shows the representability of all the maximal elements of a preorder by the maximization of the functions in a finite Richter–Peleg upper semicontinuous multi-utility representation.

We need a generalization of the *White theorem* [18] to the upper semicontinuous case. We recall that, as it was previously illustrated in the introduction, the White theorem guarantees that, if there exists a maximal element x_0 for a preorder $\precsim$, which at the same time admits an order-preserving function u', then there exists an order-preserving function u which attains its maximum precisely at the point x_0.

As regards the notation, if u is a real-valued function on a set X, then, as usual, we denote by

$$\textit{arg max}\, u = \{x \in X \mid u(z) \leq u(x) \text{ for all } z \in X\}$$

the (possibly empty) set of all the elements of X at which u attains its maximum.

Theorem 2. *Let $\precsim$ be a near-complete preorder on a compact topological space (X, τ), and assume that $\precsim$ has an upper semicontinuous finite Richter–Peleg multi-utility representation $\mathbf{U} = \{u'_1, \ldots, u'_m\}$. Further, assume that $[x_0]_\sim = \{u \in X \mid u \sim x_0\}$ is a closed subset of X for each point $x_0 \in X_M^{\precsim}$. Then there exists an upper semicontinuous finite Richter–Peleg multi-utility representation $\mathbf{U} = \{u_1, \ldots, u_n\}$, with the property that*

$$X_M^{\precsim} = \bigcup_{i=1}^{n} \textit{arg max}\, u_i.$$

Proof. Consider that, by near-completeness of $\precsim$, and using the obvious observation according to which a point x belongs to $X_M^{\precsim}$ as soon as $x \sim x_0$ for some $x_0 \in X_M^{\precsim}$, we have that there are $k \in \mathbb{N}^+$ and $x_1, \ldots, x_k \in X_M^{\precsim}$ with

$$X_M^{\precsim} = \bigcup_{i=1}^{k} [x_i]_\sim.$$

By Bosi and Zuanon ([19], Theorem 3.2), for every $x_i \in X_M^{\precsim}$ there is an upper semicontinuous order-preserving function u_i with the property that

$$\arg\max u_i = [x_i]_\sim,$$

where, obviously, $[x_i]_\sim = \{z \in X : z \sim x_i\}$. If there is an upper semicontinuous finite Richter–Peleg multi-utility representation $\mathbf{U} = \{u'_1, \ldots, u'_m\}$, then it is immediate to observe that

$$\mathbf{U} = \{u_1, \ldots, u_n\} = \{u'_1, \ldots, u'_m\} \cup \{u_1, \ldots, u_k\}$$

is an upper semicontinuous finite Richter–Peleg multi-utility representation $\mathbf{U} = \{u_1, \ldots, u_n\}$ such that

$$X_M^{\precsim} = \bigcup_{i=1}^{n} \arg\max u_i.$$

So the proof is complete. □

Remark 7. *Consider that near-completeness is invoked in order to guarantee that all the maximal elements are "reached" in a suitable upper semicontinuous finite Richter–Peleg multi-utility representation.*

Remark 8. *The assumption of compactness of the topological space is adopted in order to ensure that every upper semicontinuous function attains its maximum, so that we can represent all the maximal elements of a preorder under certain conditions.*

We finish this paper by recalling the possibility of invoking a particular upper semicontinuous representation of a total preorder on a compact subset of a metric space. Indeed, the utility representation method *à la Arrow–Hahn*, as described for example by Gori and Pianigiani [31], gives rise to a particular upper semicontinuous utility representation of an upper semicontinuous total preorder on a metric space (X, d) (so, in particular, as far as we are concerned, on a compact subset of $\mathbb{R}^n$).

The following theorem holds and uses the fundamental consideration, according to which any upper semicontinuous (not necessarily total) preorder can be extended by means of an appropriate upper semicontinuous total preorder.

A preorder $\lesssim$ is said to *extend* a preorder $\precsim$ on a set X if, for all points $x, y \in X$,

$$x \precsim y \text{ implies that } x \lesssim y,$$

$$x \prec y \text{ implies that } x < y.$$

Theorem 3. *If $\precsim$ is an upper semicontinuous preorder on a compact metric space (X, d), then there is an upper semicontinuous total preorder $\lesssim$ extending $\precsim$, and, for every maximal element x_0 for $\lesssim$, the following function u_{x_0} is an upper semicontinuous order-preserving function for $\precsim$:*

$$u_{x_0}(x) = -d(x_0, L_{\lesssim}(x)) \quad (x \in X).$$

Proof. From Bosi and Sbaiz ([29], Theorem 5.4), $\precsim$ admits an extension by a total upper semicontinuous preorder $\lesssim$, which therefore has a utility representation of the form above described by Gori and Pianigiani ([31], Theorem 2). Since u_{x_0} is a utility function for $\lesssim$, and $\lesssim$ extends $\precsim$, then it is immediate to check that u_{x_0} is an order-preserving function for $\precsim$. □

4. Conclusions

In this paper, we have considered the problem concerning the representation of all the maximal elements of a preorder on a topological space by means of the maximization of all functions in a finite collection of order-preserving functions, which fully represents the given preorder (i.e., it is at the same time a multi-utility representation).

The upper semicontinuous case naturally comes into consideration, since it guarantees the existence of maxima on compact spaces. Incidentally, we have investigated the upper semicontinuity properties of a preorder admitting a representation of this kind.

However, the consideration of transfer upper semicontinuous functions, and therefore representations, would guarantee a much more general approach. We are confident to study these topics in a future paper.

Author Contributions: Conceptualization, G.B. and G.S.; Methodology, G.S. Writing—review and editing, M.Z.; Supervision, G.B. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: The original contributions presented in this study are included in the article. Further inquiries can be directed to the corresponding author.

Acknowledgments: G. Sbaiz is a member of the INdAM (Italian Institute for Advanced Mathematics) group.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Bergstrom, T.C. Maximal elements of acyclic relations on compact sets. *J. Econom. Theory* **1975**, *10*, 403–404. [CrossRef]
2. Campbell, D.E.; Walker, M. Maximal elements of weakly continuous relations. *J. Econom. Theory* **1990**, *50*, 459–464. [CrossRef]
3. Quartieri, F. Secure and perfect maximality. *J. Math. Psych.* **2025**, *125*, 102922. [CrossRef]
4. Alcantud, J.C.R. Characterizations of the existence of maximal elements of acyclic relations. *Econom. Theory* **2002**, *19*, 407–416. [CrossRef]
5. Bosi, G.; Zuanon, M. Existence of maximal elements of semicontinuous preorders. *Int. J. Math. Anal.* **2013**, *7*, 1005–1007. [CrossRef]
6. Bosi, G.; Zuanon, M. Maximal elements of quasi upper semicontinuous preorders on compact spaces. *Econom. Theory Bull.* **2017**, *5*, 109–117. [CrossRef]
7. Andrikopoulos, A.; Sampanis, N. A characterization framework for stable sets and their variants. *arXiv* **2025**, arXiv:2508.09798v1. [CrossRef]
8. Kukushkin, N.S. Maximizing an interval order on compact subsets of its domain. *Math. Soc. Sci.* **2008**, *56*, 195–206. [CrossRef]
9. Llinares, J.-V. Unified treatment of the problem of existence of maximal elements in binary relations: A characterization. *J. Math. Econom.* **1998**, *29*, 285–302. [CrossRef]
10. Luc, D.T.; Soubeyran, A. Variable preference relations: Existence of maximal elements. *J. Math. Econom.* **2013**, *49*, 251–262. [CrossRef]
11. Mehta, G. Maximal elements in Banach spaces. *Indian J. Pure Appl. Math.* **1989**, *20*, 690–697.
12. Nosratabadi, H. Partially upper continuous preferences: Representation and maximal elements. *Econom. Lett.* **2014**, *125*, 408–411. [CrossRef]
13. Rodríguez-Palmero, C.; García-Lapresta, J.-L. Maximal elements for irreflexive binary relations on compact sets. *Math. Soc. Sci.* **2002**, *43*, 45–60. [CrossRef]
14. Ward, L.E., Jr. Partially ordered topologicl spaces. *Proc. Amer. Math. Soc.* **1954**, *5*, 144–161. [CrossRef]

15. Bueno, O.; Cotrina, J.; García, Y. Existence and Uniqueness of Maximal Elements for Preference Relations: Variational Approach. *J. Optim. Theory Appl.* **2023**, *198*, 1246–1263. [CrossRef]
16. Tian, G.; Zhou, J. Transfer Method for Characterizing the Existence of Maximal Elements of Binary Relations on Compact or Noncompact Sets. *SIAM J. Optim.* **1992**, *3*, 360–375. [CrossRef]
17. Tian, G.; Zhou, J. Transfer continuities, generalizations of the Weierstrass and maximum theorems: A full characterization. *J. Math. Econom.* **1995**, *24*, 281–303. [CrossRef]
18. White, D.J. Notes in decision theory: Optimality and efficiency II. *Eur. J. Oper. Res.* **1980**, *4*, 426–427. [CrossRef]
19. Bosi, G.; Zuanon, M. Upper semicontinuous representability of maximal elements for nontransitive preferences. *J. Optim. Theory Appl.* **2019**, *181*, 758–765. [CrossRef]
20. Ok, E.A. Utility representation of an incomplete preference relation. *J. Econom. Theory* **2002**, *104*, 429–449. [CrossRef]
21. Evren, O.; Ok, E.A. On the multi-utility representation of preference relations. *J. Math. Econom.* **2011**, *47*, 554–563. [CrossRef]
22. Bosi, G.; Induráin, E.; Munárriz, A.; Rincón, Y.R. Characterization of countable and continuous Richter-Peleg multi-utility representations. *J. Math. Psych.* **2025**, *126*, 102940. [CrossRef]
23. Kaminski, B. On quasi-orderings and multi-objective functions. *Eur. J. Oper. Res.* **2007**, *177*, 1591–1598. [CrossRef]
24. Candel, J.C. Bi-utility representable orderings on a countable set. *Econom. Lett.* **2022**, *217*, 110658. [CrossRef]
25. Candel, J.C. Finite multi-utility representable preferences and Pareto orderings. *Econom. Lett.* **2024**, *242*, 111855. [CrossRef]
26. Richter, M.K. Revealed preference theory. *Econometrica* **1966**, *34*, 635–645. [CrossRef]
27. Peleg, B. Utility functions for partially ordered topological spaces. *Econometrica* **1970**, *38*, 93–96. [CrossRef]
28. Minguzzi, E. Normally Preordered Spaces and Utilities. *Order* **2013**, *30*, 137–150. [CrossRef]
29. Bosi, G.; Sbaiz, G. Upper semicontinuous utilities for all upper semicontinuous total preorders. *Math. Soc. Sci.* **2025**, *134*, 31–41. [CrossRef]
30. Rader, T. The existence of a utility function to present preferences. *Rev. Econom. Stud.* **1963**, *30*, 229–232. [CrossRef]
31. Gori, M.; Pianigiani, G. On the Arrow-Hahn utility representation method. *Math. Soc. Sci.* **2010**, *59*, 282–287. [CrossRef]