LIE BI-ALGEBRAS ON THE NON-COMMUTATIVE TORUS

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ABSTRACT. Infinitesimal symmetries of a classical mechanical system are usually described by a Lie algebra acting on the phase space, preserving the Poisson brackets. We propose that a quantum analogue is the action of a Lie bi-algebra on the associative $*$-algebra of observables. The latter can be thought of as functions on some underlying non-commutative manifold. We illustrate this for the non-commutative torus $\mathbb{T}^2_\vartheta$. The canonical trace defines a Manin triple from which a Lie bi-algebra can be constructed. In the special case of rational $\vartheta = \frac{M}{N}$ this Lie bi-algebra is $\underline{GL}(N) = \underline{U}(N) \oplus \underline{B}(N)$, corresponding to unitary and upper triangular matrices. The Lie bi-algebra has a remnant in the classical limit $N \to \infty$: the elements of $\underline{U}(N)$ tend to real functions while $\underline{B}(N)$ tends to a space of complex analytic functions.

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1. INTRODUCTION

In the quantum theory the phase space cannot be a manifold in the usual sense, since momentum and position cannot be simultaneously measured. The appropriate generalization needed to describe the quantum phase space is non-commutative geometry. The algebra of quantum observables is an associative algebra which tends in the classical limit to the Poisson algebra of functions on the classical phase space; that is, a commutative algebra to order zero in $\hbar$ along with a Poisson bracket that gives the correction of order $\hbar$.

Topological properties of the manifold can be encoded into properties of the algebra of functions on it, which then carry over to the non-commutative case, as properties of the quantum

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observable algebra. There is a well-developed K-theory of operator algebras [1]: familiar invariants such as Chern numbers have non-commutative counterparts.

A third theme is that of symmetry. The continuous symmetries of a classical system form a Lie group which acts on the classical phase space. In simple cases, this Lie group extends to the quantum theory, with a representation on the Hilbert space of states. Sometimes, quantization modifies the symmetry. For example, the symmetry of the quantum theory can be the central extension of the classical one (the corresponding representation is a projective representation of the classical symmetry). Such deformations are called "anomalies" in the physics literature. In the most well-known case, the conformal anomaly leads to the Virasoro algebra [7]. In other examples the quantum symmetry is a new Lie group, of which the classical limit is a Wigner contraction.

An even more general possibility is that the symmetries of the quantum theory do not form a group at all. Instead, it is a "quantum group". More precisely, it is a Hopf algebra with an action (or co-action) on the algebra of quantum observables. Again, there is a classical remnant of this phenomenon: there is a Poisson bracket on the group itself, such that the group multiplication is a Poisson map [3]. Such a Poisson-Lie group A acts on the classical phase space M, such that the induced co-action $C(M) \to C(M) \otimes C(A)$ is a Poisson map. If the Poisson bracket on A is zero, this reduces to the familiar condition that the group action leave the Poisson bracket on M invariant; the generating function $f : M \to \underline{A}'$ of the infinitesimal action is the "moment map", valued in the dual $\underline{A}'$ of the Lie algebra $\underline{A}$ of A. More generally, the moment map is a function $f : M \to A'$ valued in a Lie group dual of A.

Infinitesimally, a Poisson-Lie group is a Lie bi-algebra. The infinitesimal group multiplications give the Lie bracket as usual. In addition, the Poisson bracket on the group A becomes a co-product on the Lie algebra $\underline{A}$. The product and co-product must satisfy an infinitesimal version of the condition that the group multiplication be a Poisson map.

These more general ideas about symmetries have not yet been fully utilized in physics, except in the narrow context of integrable systems. They have emerged recently in the very chaotic case of incompressible fluid mechanics [10] (Euler equation of an ideal fluid). It would be interesting to have more examples of physical systems with quantum group or Lie bi-algebra symmetry.

We study the simplest case of a phase space: a two dimensional torus. The quantum geometry is that of the non-commutative two-torus $\mathbb{T}^2_\vartheta$. We uncover a Lie bi-algebra hidden in $\mathbb{T}^2_\vartheta$. More precisely, for the algebra of function on $\mathbb{T}^2_\vartheta$, that is the complex associative $*$-algebra generated by two unitary elements P, Q satisfying the relations

$$PQ = \omega\, QP, \qquad \omega = e^{2\pi i\vartheta}. \tag{1.1}$$

The commutator of two elements $[F, G] = FG - GF$ yields a Lie algebra $\underline{S}_\vartheta$. The canonical invariant trace τ on $\mathbb{T}^2_\vartheta$ yields an invariant inner product

$$\langle F, G \rangle = \tau(FG) \tag{1.2}$$

for this Lie algebra. This inner product is not positive. In fact there are subspaces $\underline{A}$ and $\underline{B}$ on which it vanishes such that $\underline{S}_\vartheta = \underline{A} \oplus \underline{B}$; then $\underline{A}$ and $\underline{B}$ are dual to each other as vector spaces. That is, $(\underline{S}_\vartheta, \underline{A}, \underline{B})$ is a Manin triple. Another point of view is that $\underline{A}$ has a co-product (the mirror image of the Lie bracket in $\underline{B}$) which turns it into a Lie bi-algebra.

As $\vartheta \to 0$ this complex associative $*$-algebra tends to the Poisson algebra of complex-valued functions on the torus $\mathbb{T}^2$ with the bracket

$$\{F, G\} = \frac{\partial F}{\partial x_1}\frac{\partial G}{\partial x_2} - \frac{\partial F}{\partial x_2}\frac{\partial F}{\partial x_1} \tag{1.3}$$

and identification $P = e^{2\pi i x_1}, Q = e^{2\pi i x_2}$. We will see that this also is an (infinite dimensional) Lie bi-algebra. The invariant trace tends to the integral, yielding the invariant inner product

$$\langle F, G \rangle = \operatorname{Im} \int FG \frac{dx^1 dx^2}{(2\pi)^2}. \tag{1.4}$$

The classical limit of $\underline{A}$ is the sub-space of real-valued functions; the elements of the classical limit of $\underline{B}$ are complex-valued functions on the torus allowing an analytic continuation to the interior of the unit disc in the variable P (and satisfy a reality condition when independent of P). This Lie bi-algebra (for the Poisson structure) is the classical remnant of the quantum group symmetry of the non-commutative torus.

The physical meaning of $\underline{A} \subseteq \underline{S}_\vartheta$ is clear: it is the quantum counterpart to the Lie algebra of canonical transformations. The corresponding group A is the familiar group of unitary transformations. The physical meaning of $\underline{B}$ (or equivalently, the co-product on $\underline{A}$) remains somewhat mysterious. The co-product is the structure that allows us to combine two representations of $\underline{A}$ into a new one. In the simplest case (when the co-product is trivial) this is just the direct sum of representations. The surprise is that there are more general ways of combining representations, given the co-product. Physically, this means that there are new ways of combining two independent quantum systems to get a new ones, while preserving the invariance under unitary transformations. This ought to be important in the context of quantum computation, where the plan is to combine may small quantum systems to get a large processor.

What is the quantum group (Hopf algebra) associated to the Lie bi-algebra $\underline{S}_\vartheta$? Finite dimensional Lie bi-algebras can be exponentiated into quantum groups, as conjectured by Drinfeld (and Jimbo) originally. There is no such general construction in the infinite dimensional cases. We must approach this with a "regularization": approximate the infinite dimensional algebraby a sequence of finite dimensional ones, and then look at the limit as the dimension tends to infinity. A prototype of this is the approximation of real (irrational) numbers by rational ones.

So we look at the the case of rational $\vartheta = \frac{M}{N}$ (i.e., ω is a primitive Nth root of unity). Then

we can impose the additional relations $P^N = 1 = Q^N$, giving a finite dimensional algebra. There is then a simple explicit realization using clock-shift operators (j is taken modulo N)

$$P = \sum_{j=1}^{N} |j+1\rangle\langle j|, \qquad Q = \sum_{j=1}^{N} \omega^j |j\rangle\langle j|. \tag{1.5}$$

If $N = 2$ these are Pauli matrices (acting on "qubit" states), while for $N = 3$ they can be realized in terms of Gell-Mann matrices (acting on "qutrit" states). More generally, the commutators of P^aQ^b close on the N^2-dimensional complex algebra gl_N; in this basis, it is the "sine-algebra". There is a natural Lie bi-algebra structure on this gl_N. That is, a splitting $gl_N = \underline{A} \oplus \underline{B}$ into sub-algebras $\underline{A}$ and $\underline{B}$ which are isotropic and dual to each other w.r.t. an invariant inner product (a "Manin triple"). A natural choice is $\underline{A} = u_N$ (anti-hermitian matrices) and $\underline{B} = sb_N$, consisting of upper triangular matrices with real entries along the diagonal. It is known that this Lie algebra is the infinitesimal version of the well-known [3] quantum group $SL_\omega(N)$. There is already a well developed representation theory of this Hopf algebra, which should have interesting physical consequences.

The operators of the rational non-commutative torus defined in (1.5) appear, surprisingly, in experimental realizations of "Qudit" processors [2]. It is possible that the non-commutative torus provides an alternative model of quantum computation, instead of arrays of "qubits". Maintaining quantum coherence for many qubits is experimentally challenging. It might be easier to use instead a smaller number of "Qudit" systems, each one a rational NC torus for some $N > 2$. Other intriguing connections between operators algebras and complexity theoryof computing are also emerging recently, cf. [15].

Perhaps this theme plays out more generally: the analogue of diffeomorphisms for non-commutative geometries could be quantum groups; that is, there is a Hopf algebra co-action on the associative algebra of the non-commutative manifold. As a prototype of this, we recall in Appendix B the co-action of the Taft-Hopf algebra on the non-commutative torus. In the classical limit we don't just get commutative algebras and Lie groups; there is in addition a Poisson bracket with a Lie bi-algebra of symmetries.

The non-commutative torus also gives us an example of how topological properties (via K-theory) of the classical phase space lift to the quantum theory. It is well-known that $K_0(\mathrm{T}^2) = \mathbb{Z}^2$; vector bundles on a torus are classified by a pair of integers which are the rank and Chern number. This continues to be true in the non-commutative case: operator theoretic K-group $K_0(\mathrm{T}^2_\vartheta)$ is also $\mathbb{Z}^2$. For rational ϑ this can be understood in terms of Morita equivalence of vector bundles over T^2_ϑ to those over T^2, while for irrational ϑ the classes can be realised in terms of Heisenberg modules [12]. But for irrational ϑ we have some additional structure. The trace on T^2_ϑ embeds $\mathbb{Z}^2 \to \mathbb{R}$ as a subgroup of the real numbers:

$$\tau : (r, m) \mapsto r + m\vartheta.$$

In particular, it turns $K_0(\mathrm{T}^2)_\vartheta$ into an ordered group [13]. In the limit $\vartheta \to 0$ the embedding above is lost. Remarkably, $K_0(\mathrm{T}^2)$ is still an ordered group with a non-archimedean order. We will use such a lexicographic order in constructing a Lie bi-algebra structure for the Poisson algebra on the classical commutative torus while for a corresponding Lie bi-algebra structure for the non-commutative torus we use the order of $K_0(\mathrm{T}^2_\vartheta)$, which is most natural for T^2_ϑ. In the limit the latter ordering tends to the lexicographic one, as seen in Fig.1. The classical remnant of non-commutative geometry contained in the Poisson algebras needs to be studied further.

2. Manin triples and Lie bi-algebras

We recall the main definitions [3].

A *Lie bi-algebra* is a product $\Gamma : \underline{A} \otimes \underline{A} \to \underline{A}$ along with a co-product $\Delta : \underline{A} \to \underline{A} \otimes \underline{A}$

$$\Gamma(X_a, X_b) \equiv [X_a, X_b] = \Gamma^d_{ab} X_d, \qquad \Delta(X_a) = \Delta^{cd}_a X_c \otimes X_d$$

satisfying antisymmetry, $\Gamma^d_{ab} = -\Gamma^d_{ba}$ and $\Delta^{ab}_d = -\Delta^{ba}_d$, the Jacobi and co-Jacobi identities

$$\Gamma^d_{ab}\Gamma^e_{dc} + \Gamma^d_{bc}\Gamma^e_{da} + \Gamma^d_{ca}\Gamma^e_{db} = 0 \tag{2.1}$$

$$\Delta^{ab}_d\Delta^{dc}_e + \Delta^{bc}_d\Delta^{da}_e + \Delta^{ca}_d\Delta^{db}_e = 0 \tag{2.2}$$

and the compatibility condition that Δ be an infinitesimal automorphism of Γ:

$$[\Delta(X_a), X_b] + [X_a, \Delta(X_b)] = \Gamma^c_{ab}\Delta(X_c).$$

The latter amounts to the condition that Δ is closed in the Lie algebra cohomology $H^1(\underline{A}, \underline{A}\otimes\underline{A})$:

$$(\partial\Delta)^{be}_{ac} \equiv \left[\Gamma^b_{ad}\Delta^{de}_c + \Gamma^e_{ad}\Delta^{bd}_c - a \leftrightarrow c\right] - \Gamma^d_{ac}\Delta^{be}_d = 0 \tag{2.3}$$

A more familiar way to state these conditions is that $\underline{S} = \underline{A} \oplus \underline{B}$ (where $\underline{B}$ is the dual vector space of $\underline{A}$) is itself a Lie algebra; in a basis $X_a \in \underline{A}$ and the dual basis $X^a \in \underline{B}$ we must have

$$[X_a, X_b] = \Gamma^c_{ab} X_c, \qquad [X^a, X^b] = \Delta^{ab}_c X_c, \tag{2.4}$$

$$[X^a, X_b] = \Gamma^a_{bd} X^d - \Delta^{ad}_b X_d \tag{2.5}$$

The Jacobi identities from the first line (2.4) are the conditions (2.1)-(2.2) above; the mixed Jacobi identities give (2.3). The duality of $\underline{A}$ and $\underline{B}$ yield an invariant inner product on $\underline{S}$:

$$\langle X_a, X_b\rangle = 0 = \langle X^a, X^b\rangle, \quad \langle X_a, X^b\rangle = \delta^b_a.$$

A *Manin triple* is a Lie algebra $\underline{S}$ along with Lie sub-algebras $\underline{A}$, $\underline{B}$ such that $\underline{S} = \underline{A} \oplus \underline{B}$ as vector spaces; moreover $\underline{S}$ has an invariant inner product which vanishes when restricted to $\underline{A}$ or $\underline{B}$ (i.e.,

$\underline{A}$ and $\underline{B}$ are isotropic sub-spaces). If $\underline{S}$ is finite dimensional, every Manin triple admits [3] a basis satisfying (2.4)-(2.5); the notions of Manin triple and Lie bi-algebras are equivalent. However, in the infinite dimensional case, we cannot rely on this theorem; we must explicitly verify the commutation relations to obtain a Lie bi-algebra from a Manin triple.

Example 2.1. An example is the Lorentz Lie algebra $\underline{SL}(2, C)$ of traceless 2×2 matrices with complex entries (but viewed as a *real* Lie algebra). There is an invariant inner product

$$\langle U, V \rangle = \mathrm{Im}\,\mathrm{Tr}\, UV. \tag{2.6}$$

The Lie subalgebra $\underline{SU}(2)$ of anti-hermitian traceless matrices is isotropic and can be chosen as $\underline{A}$. The complementary space $\underline{B}$ cannot be the space of hermitian matrices, as it is not a Lie sub-algebra. Instead we can chose $\underline{B}$ to be $\underline{SB}(2, C)$, the Lie sub-algebra of traceless upper triangular matrices with real entries along the diagonal. That is,

$$\underline{SL}(2, C) = \underline{A} \oplus \underline{B}$$

where

$$\underline{A} = \underline{SU}(2) = \left\{ \begin{pmatrix} ia & b \\ -\bar{b} & -ia \end{pmatrix} \mid a \in \mathbb{R}, b \in \mathbb{C} \right\}$$

$$\underline{B} = \underline{SB}(2, C) = \left\{ \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix} \mid a \in \mathbb{R}, b \in \mathbb{C} \right\}$$

is a Manin triple. A basis for $\underline{A}$ can be built out of Pauli matrices, $X_a = -\frac{i}{2}\sigma_a$:

$$X_1 = \begin{pmatrix} 0 & -\frac{i}{2} \\ -\frac{i}{2} & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & -\frac{1}{2} \\ +\frac{1}{2} & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} -\frac{i}{2} & 0 \\ 0 & \frac{i}{2} \end{pmatrix}$$

which determines the dual basis for $\underline{B}$:

$$X^1 = \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix}, \quad X^2 = \begin{pmatrix} 0 & 2i \\ 0 & 0 \end{pmatrix}, \quad X^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

They satisfy

$$\langle X_a, X_b \rangle = 0 = \langle X^a, X^b \rangle, \quad \langle X_a, X^b \rangle = \delta_a^b$$

The structure constants Γ are the familiar ones from angular momentum theory

$$\Gamma^1_{23} = 1 = \Gamma^2_{31} = \Gamma^3_{12}$$

along with

$$\Delta_2^{23} = 2 = \Delta_1^{13}.$$

The remaining components of Γ, Δ are zero, unless related to these by anti-symmetry. Also, they satisfy the compatible commutation relations (2.5) above.

Example 2.2. Next, consider the Lie algebra $\underline{SL}(3, \mathbb{C})$ of traceless 3×3 matrices with complex entries (again, viewed as a real Lie algebra). The subalgebra $\underline{SU}(3)$ of anti-hermitian[1] traceless matrices is again isotropic and is dual to $\underline{SB}(3, \mathbb{C})$, the sub-algebra of upper triangular traceless matrices with real entries along the diagonal, which is isotropic as well.

$$\underline{SL}(3, \mathbb{C}) = \underline{SU}(3) \oplus \underline{SB}(3, \mathbb{C})$$

We take a basis of for $\underline{SU}(3)$ in terms of anti-hermitian Gell-Mann matrices [5], $X_a = -i\lambda_a$:

$$X_1 = \begin{pmatrix} 0 & -i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} -i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$X_4 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad X_5 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad X_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{pmatrix},$$

$$X_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad X_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} -i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 2i \end{pmatrix},$$

[1]We use a compromise between the physics and mathematical conventions for Lie algebras. For matrix Lie algebras $\underline{SU}(N)$, we use the mathematical convention that they consist of anti-hermitian matrices, whose commutators are also anti-hermitian. This differs by a factor *i* from the physics convention [5]. In the classical limit (Section 4), we use the physics convention that $\underline{SU}(N)$ elements tend to real functions on the torus.

A dual basis for $\underline{SB}(3,\mathbb{C})$ is worked out to be :

$$X^1 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X^2 = \begin{pmatrix} 0 & i & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X^3 = \frac{1}{2}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$X^4 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X^5 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$X^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & 0 & 0 \end{pmatrix}, \quad X^8 = \frac{1}{2\sqrt{3}}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

The bases are isotropic and dual to each other:

$$\langle X_a, X_b \rangle = 0, \quad \langle X^a, X^b \rangle = 0, \quad \langle X_a, X^b \rangle = \delta_a^b \qquad a, b \in \{1, \dots, 8\}.$$

The structure constants of the Gell-Mann matrices $[X_a, X_b] = \Gamma_{ab}{}^c X_c$ are well known: they are completely antisymmetric in the three indices and explicitly given by

$$\Gamma_{123} = 2\,, \quad \Gamma_{147} = \Gamma_{165} = \Gamma_{246} = \Gamma_{257} = \Gamma_{345} = \Gamma_{376} = 1\,, \quad \Gamma_{458} = \Gamma_{678} = \sqrt{3}\,.$$

As for the structure constants of the dual basis, $[X^a, X^b] = \Delta^{ab}{}_c X^c$, the non zero ones are:

$$\Delta^{13}{}_1 = \Delta^{23}{}_2 = \Delta^{61}{}_4 = \Delta^{71}{}_5 = \Delta^{62}{}_5 = \Delta^{27}{}_4 = 1,$$
$$\Delta^{43}{}_4 = \Delta^{53}{}_5 = \Delta^{36}{}_6 = \Delta^{37}{}_7 = \frac{1}{2},$$
$$\Delta^{84}{}_4 = \Delta^{85}{}_5 = \Delta^{86}{}_6 = \Delta^{87}{}_7 = \frac{\sqrt{3}}{2},$$

and their antisymmetric ones in the two upper indices. They may be verified to satisfy the compatibility condition $[X^a, X_b] = \Gamma_{bd}{}^a X^d - \Delta^{ad}{}_b X_d$.

Alternative bases can be constructed out of clock and shift matrices in (1.5), which allows for a generalization to arbitrary dimension N in the following. Let $\omega = e^{2\pi i/3}$, a third root of unity, with $1 + \omega + \omega^2 = 0$, $\bar{\omega} = \omega^2$, $\omega - \omega^2 = i\sqrt{3}$. Then, a basis for $su(3)$ is given by:

$$X_0 = i(Q + Q^2) = i\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \tilde{X}_0 = Q - Q^2 = i\lambda\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$X_1 = i(P + P^2) = i\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad \tilde{X}_1 = P - P^2 = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix},$$

$$X_2 = i(PQ + \omega^2 P^2 Q^2) = i\begin{pmatrix} 0 & \omega & 1 \\ \omega^2 & 0 & \omega^2 \\ 1 & \omega & 0 \end{pmatrix}, \quad \tilde{X}_2 = PQ - \omega^2 P^2 Q^2 = \begin{pmatrix} 0 & \omega & -1 \\ -\omega^2 & 0 & \omega^2 \\ 1 & -\omega & 0 \end{pmatrix},$$

$$X_3 = i(PQ^2 + \omega P^2 Q) = i\begin{pmatrix} 0 & \omega^2 & 1 \\ \omega & 0 & \omega \\ 1 & \omega^2 & 0 \end{pmatrix}, \quad \tilde{X}_3 = PQ^2 - \omega^2 P^2 Q = \begin{pmatrix} 0 & \omega^2 & -1 \\ -\omega & 0 & \omega \\ 1 & -\omega^2 & 0 \end{pmatrix}.$$

One verifies directly that this basis is isotropic. These are not Gell-Mann matrices.

A dual basis for $\underline{SB}(3,\mathbb{C})$ can be found using Q and a matrix $R = P - |2\rangle\langle 0|$ satisfying

$$RQ = \omega QR, \quad R^3 = 0.$$

$$X = \frac{i}{2\lambda^2}(Q - Q) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -2\lambda \end{pmatrix},$$

Then,

$$X^0 = \frac{1}{2}(Q + \tilde{Q}_0^{2}) = \frac{1}{6}\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$$

$$X^1 = \frac{1}{3}(R + R^2) = \frac{1}{3}\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \tilde X^1 = \frac{\mathrm{i}}{3}(R - R^2) = \frac{\mathrm{i}}{3}\begin{pmatrix} 0 & -1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$X^2 = \frac{1}{3}(RQ + \omega^2 R^2 Q^2) = \frac{1}{3}\begin{pmatrix} 0 & \omega & 1 \\ 0 & 0 & \omega^2 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \tilde X^2 = \frac{\mathrm{i}}{3}(RQ - \omega^2 R^2 Q^2) = \frac{\mathrm{i}}{3}\begin{pmatrix} 0 & -\omega & 1 \\ 0 & 0 & -\omega^2 \\ 0 & 0 & 0 \end{pmatrix},$$

$$X^3 = \frac{1}{3}(RQ^2 + \omega R^2 Q) = \frac{1}{3}\begin{pmatrix} 0 & \omega^2 & 1 \\ 0 & 0 & \omega \\ 0 & 0 & 0 \end{pmatrix}, \qquad \tilde X^3 = \frac{\mathrm{i}}{3}(RQ^2 - \omega^2 R^2 Q) = \frac{\mathrm{i}}{3}\begin{pmatrix} 0 & -\omega^2 & 1 \\ 0 & 0 & -\omega \\ 0 & 0 & 0 \end{pmatrix}.$$

Again one verifies that the basis is isotropic. As for cross pairings one verifies that

$$\langle X_A, X^B \rangle = \delta_A^B \qquad A, B \in \{0, 1, 2, 3, \tilde 0, \tilde 1, \tilde 2, \tilde 3\}$$

thus the bases are dual to each other.

3. The rational non-commutative torus

We are ready to generalise the construction above out of of clock and shift matrices.

3.1. **Clock and shift matrices.** Let $\omega (= e^{2\pi \mathrm{i}/N})$ be a primitive N-the root of unity. This implies that $\sum_{j=0}^{N-1} \omega^{jm} = N\delta(m)$. Consider clock and shift matrices Q and P:

$$Q^j\,|m\rangle = \omega^{km}\,|m\rangle \qquad P^k\,|m\rangle = |m-k\rangle \tag{3.1}$$

and indices $m, k = 0, 1, \cdots, N-1$, defined modulo N. These matrices obey the relation

$$PQ = \omega\, QP,$$

are unitary, traceless and $Q^N = P^N = 1$. We shall also need the truncated matrix

$$R = P - |N-1\rangle\,\langle 0|$$

which is a strictly upper triangular matrix such that:

$$RQ = \omega\, QR, \qquad R^N = 0.$$

The matrices

$$e_{r,s} = \omega^{-\frac{1}{2}rs} P^r Q^s = \omega^{-\frac{1}{2}rs} P^r \sum_{m=0}^{N-1} \omega^{ms}\,|m\rangle\,\langle m| \tag{3.2}$$

generate the N^2-dimensional complex algebra $\underline{GL}_N$. They close on the sine-algebra. Indeed,

$$e_{j,k}\, e_{r,s} = \omega^{\frac{1}{2}(js-kr)}\, e_{j+r,k+s}$$

from which:

$$[e_{j,k}, e_{r,s}] = 2\mathrm{i}\sin\frac{\pi}{N}(js-kr)\, e_{j+r,k+s} \tag{3.3}$$

Also,

$$e_{r,s} = \omega^{\frac{1}{2}rs} Q^{-s} P^{-r} = \omega^{\frac{1}{2}rs} \sum_{m=0}^{N-1} \omega^{-ms}\,|m\rangle\,\langle m|\, P^{-r}$$
$$= e_{-r,-s}$$

and

$$e_{N\pm r,s} = (-1)^s e_{\pm r,s}, \qquad e_{r,N\pm s} = (-1)^r e_{r,\pm s} \quad \Rightarrow \tag{3.4}$$

$$e_{N+r,N+s} = (-1)^{N+r+s} e_{r,s},$$

$$e_{N-r,N-s} = (-1)^{N-r-s} e_{-r,-s} = (-1)^{N-r-s} e^*_{r,s}$$

with their conjugated.

We define 'truncated' matrices for $0 < a < N$ and $b \in \mathbb{Z}$, as follow

$$f_{a,b} = \omega^{-\frac{1}{2}ab} R^a Q^b = \omega^{-\frac{1}{2}ab} P^a \sum_{n=a}^{N-1} \omega^{bn} |n\rangle \langle n| \tag{3.5}$$

$$\tilde{f}_{a,b} = \omega^{\frac{1}{2}ab} Q^{-b} R^{N-a} = \omega^{\frac{1}{2}ab} \sum_{n=0}^{a-1} \omega^{-bn} |n\rangle \langle n| P^{(N-a)} \tag{3.6}$$

We may indeed define them also for $a = 0$ and $a = N$ observing from the above that

$$f_{0,b} = e_{0,b}, \quad f_{N,b} = 0, \qquad \tilde{f}_{0,b} = 0, \quad \tilde{f}_{N,b} = (-1)^b e^*_{0,b}. \tag{3.7}$$

When $a \neq 0, \neq N$, $f_{a,b}$ is the upper triangular part of $e_{a,b}$ while $\hat{f}_{a,b}$ is the upper triangular part of $e^*_{a,b}$. One finds, for $0 \le a \le N - 1$ with $0 \le N - a \le N - 1$ and $b \in \mathbb{Z}$,

$$f_{0,-b} = \tilde{f}_{0,b} \qquad \tilde{f}_{0,-b} = f_{0,b} \tag{3.8}$$

$$\begin{aligned}
f_{a,N\pm b} &= (-1)^{-a} f_{a,\pm b} \qquad \tilde{f}_{a,N\pm b} = (-1)^a \hat{f}_{a,\pm b} \\
f_{N-a,b} &= (-1)^{-b} \hat{f}_{a,-b} \qquad \tilde{f}_{N-a,b} = (-1)^b f_{a,-b} \\
\Rightarrow \quad \tilde{f}_{N-a,N-b} &= (-1)^{N+a-b} f_{a,b} \\
f_{N-a,N-b} &= (-1)^{-N-a+b} \tilde{f}_{a,b}.
\end{aligned} \tag{3.9}$$

Moreover,

$$\begin{aligned}
f_{a,b} f_{r,s} &= \omega^{\frac{1}{2}(as-br)} f_{a+r,b+s} \qquad = 0 \quad \text{if} \quad a + r \ge N \\
\tilde{f}_{a,b} \tilde{f}_{r,s} &= (-1)^{b+s} \omega^{\frac{1}{2}(as-br)} \tilde{f}_{a+r-N,b+s} \qquad = 0 \quad \text{if} \quad a + r \le N
\end{aligned} \tag{3.10}$$

The above imply:

$$\begin{aligned}
[f_{j,k}, f_{a,b}] &= 2\mathrm{i} \sin \frac{\pi}{N}(jb - ka) f_{j+a,k+b} \qquad = 0 \quad \text{if} \quad j + a \ge N \\
[f_{j,k}, \tilde{f}_{a,b}] &= -2\mathrm{i} \sin \frac{\pi}{N}(jb - ka) \tilde{f}_{a-j,b-k} \\
&= -(-1)^{k+b} 2\mathrm{i} \sin \frac{\pi}{N}(jb - ka) f_{N+j-a,k-b} \qquad = 0 \quad \text{if} \quad a - j \le 0 \\
[\tilde{f}_{j,k}, \tilde{f}_{a,b}] &= (-1)^{k+b} 2\mathrm{i} \sin \frac{\pi}{N}(jb - ka) \tilde{f}_{j+a-N,k+b} \qquad = 0 \quad \text{if} \quad j + a \le N.
\end{aligned} \tag{3.11}$$

3.2. **The Cartan sub-algebra.** The sub-algebra t of diagonal matrices with purely imaginary entries is given as:

$$U_{0,b} = \frac{\mathrm{i}}{\sqrt{N}} (e_{0,b} + e^*_{0,b}) = \frac{2\mathrm{i}}{\sqrt{N}} \sum_{m=0}^{N-1} \cos(2\pi \frac{mb}{N}) |m\rangle \langle m| \tag{3.12}$$

$$\tilde{U}_{0,b} = \frac{1}{\sqrt{N}} (e_{0,b} - e^*_{0,b}) = \frac{2\mathrm{i}}{\sqrt{N}} \sum_{m=0}^{N-1} \sin(2\pi \frac{mb}{N}) |m\rangle \langle m| \tag{3.13}$$

for $b = 0, 1, \cdots N - 1$. Being $\tilde{U}_{0,N-b} = -\tilde{U}_{0,b}$ and $U_{0,N-b} = U_{0,b}$, a basis of t is then given by

$$(\mathrm{i}H_b, \mathrm{i}H_{\tilde{b}}) = \begin{cases} \{U_{0,b},\ b = 0, 1, \cdots \boldsymbol{L}\} \bigcup \{U_{0,\tilde{b}},\ b = 1, \cdots, \boldsymbol{L}\} & N = 2\boldsymbol{L} + 1 \\ \{U_{0,b},\ b = 0, 1, \cdots \boldsymbol{L} - 1\} \bigcup \{U_{0,\tilde{b}},\ b = 1, \cdots, \boldsymbol{L}\} & N = 2\boldsymbol{L} \end{cases} \qquad (3.14)$$

with cardinality N in both cases. One has $H_L = 0$ when $N = 2L$.

Now t is the maximal abelian sub-algebra of $\underline{U}_N$, the skew symmetric matrices, and $h = t \oplus it$ is the Cartan sub-algebra of $\underline{GL}_N$.

3.3. **Upper-triangular matrices and Borel sub-algebra.** Due to (3.7), we could also write

$$H_b = \frac{1}{\sqrt{N}}(f_{0,b} + \tilde{f}_{N,b}) \qquad H_{\tilde{b}} = -\frac{\mathrm{i}}{\sqrt{N}}(f_{0,b} - \hat{f}_{N,b}). \tag{3.15}$$

These diagonal matrices have real entries and are 'diagonal upper triangular' matrices. The sub-algebra n_+ of strictly upper-triangular matrices is made of the matrices,

$$T^{a,b} = \frac{1}{\sqrt{N}}(f_{a,b} + \tilde{f}_{a,b}) \qquad \tilde{T}^{a,b} = -\frac{\mathrm{i}}{\sqrt{N}}(f_{a,b} - \tilde{f}_{a,b}) \qquad 0 < a < N. \tag{3.16}$$

Owing to relations (3.8) not all matrices in (3.16) are independent (some of them may indeed vanish). A basis for the strict upper triangular ones is given by

$$\left\{T^{r,s},\ \tilde{T}^{r,s},\ (r,s) \in \{1,\cdots,L\} \times \{0,1,\cdots,N-1\}\right\} \qquad N = 2L+1 \tag{3.17}$$

$$\left\{T^{r,s},\ \tilde{T}^{r,s},\ (r,s) \in \{1,\cdots,L-1\} \times \{0,1,\cdots,N-1\}\right\} \bigcup \{T^{L,s},\ s = 1,\cdots,L-1\} \bigcup \{\tilde{T}^{L,s},\ s = 0,\cdots,L\} \qquad N = 2L. \tag{3.18}$$

Note that both $T^{L,0}$ and $T^{L,L}$ vanish when $N = 2L$. The basis is of cardinality $N(N-1)$ in both cases.

The (upper) Borel sub-algebra is then $\underline{B}_N = \mathrm{i}t \oplus n_+$, with basis the union of (3.17), (3.18), and real matrices $(H_b, H_{\tilde{b}})$ as in (3.15) adding up to N^2 elements.

3.4. **The anti-hermitian matrices.** The sub-algebra u_N of anti-hermitian matrices:

$$U_{r,s} = \frac{\mathrm{i}}{\sqrt{N}}(e_{r,s} + e^*_{r,s}) \qquad \tilde{U}_{r,s} = \frac{1}{\sqrt{N}}(e_{r,s} - e^*_{r,s}) \tag{3.19}$$

In particular we have

$$U_{0,s} = \frac{\mathrm{i}}{\sqrt{N}}(e_{0,s} + e^*_{0,s}) = \mathrm{i}H_s \qquad \tilde{U}_{0,s} = \frac{1}{\sqrt{N}}(e_{0,s} - e^*_{0,s}) = \mathrm{i}H_{\tilde{s}}$$

making up the maximal abelian N-dimensional sub-algebra t of u_N.

Again, due to relations (3.4) of the matrices in (3.19) the independent ones are $N(N-1)$ in number. A basis for the non-diagonal ones is given by

$$\left\{U_{r,s},\ \tilde{U}_{r,s},\ (r,s) \in \{1,\cdots,L\} \times \{0,1,\cdots,N-1\}\right\} \qquad N = 2L+1 \tag{3.20}$$

$$\left\{U_{r,s},\ \tilde{U}_{r,s},\ (r,s) \in \{1,\cdots,L-1\} \times \{0,1,\cdots,N-1\}\right\} \bigcup \{U_{L,s},\ s = 1,\cdots,L-1\} \bigcup \{\tilde{U}_{L,s},\ s = 0,\cdots,L\} \qquad N = 2L. \tag{3.21}$$

Now both $U_{L,0}$ and $U_{L,L}$ vanish when $N = 2L$. The basis is of cardinality $N(N-1)$ in both cases, adding to N^2, the dimension of $\underline{U}_N$, with the N diagonal ones in (3.14).

3.5. **The Lie bi-algebra structure.** From the above it follows that as real vector spaces

$$\underline{GL}_N = \underline{U}_N \oplus \underline{B}_N \tag{3.22}$$

Lemma 3.1. *The subspaces $\underline{U}_N$ and $\underline{B}_N$ are isotropic and dual via the invariant pairing:*

$$\langle X, Y \rangle = \operatorname{Im}\operatorname{Tr}(XY) \tag{3.23}$$

the imaginary part of the trace of the product.

The invariance is just: $\langle [Z,X], Y \rangle + \langle X, [Z,Y] \rangle = 0$. The explicit proof is in Appendix A whereit is shown that both u_N and b_N are isotropic and that the only non vanishing pairings are

$$\left\langle T^{a,b}, U_{a,b} \right\rangle = 1 = \left\langle \tilde{T}^{a,b}, \tilde{U}_{a,b} \right\rangle \qquad a,b = 0,1,\cdots,N-1. \tag{3.24}$$

Thus, the basis (3.17), (3.18) for the strictly upper triangular matrices of b_N is dual to the basis (3.20), (3.21) of anti-hermitian non diagonal matrices, while the basis $(H_b, H_{\tilde{b}})$ in (3.14) for the real diagonal matrices is dual to the basis $(\mathrm{i}H_b, \mathrm{i}H_{\tilde{b}})$ for the diagonal purely imaginary matrices.

The data $(\underline{GL}_N, \underline{U}_N, \underline{B}_N)$ makes up a Manin triple and then [3, Prop. 1.3.4] a Lie bi-algebra.

4. The classical limit $N \to \infty$

Consider the large N limit of the construction of the previous section. As for the commutation relations (3.3), for large N with j, k, r, s fixed, one gets

$$[e_{j,k}, e_{r,s}] \sim \mathrm{i}\frac{2\pi}{N}(js - kr)\, e_{j+r,k+s}$$

This goes to the usual Poisson structure on the torus

$$\{e_{j,k}, e_{r,s}\} \sim \frac{\mathrm{i}}{\mathrm{k}}[e_{j,k}, e_{r,s}]\Big|_{\mathrm{k}=0} = -(js - kr)\, e_{j+r,k+s} \tag{4.1}$$

and $\frac{2\pi}{N}$ is the analogue of k. Also, the limit of anti-hermitian matrices $U_{N=\infty}$ is the Lie algebra of area preserving vector fields on the commutative torus. We shall next describe the Lie bi-algebra structure on the (commutative) algebra $C^\infty(\mathbb{T}^2)$ obtained in the limit.

4.1. **The algebra.** The (commutative) algebra $C^\infty(\mathbb{T}^2)$ of complex valued smooth functions on the torus $\mathbb{T}^2$ is made of all elements of the form

$$\varphi = \sum_{(m_1,m_2)\in\mathbb{Z}^2} \varphi_{m_1 m_2}\, e^{\mathrm{i}m_1x_1 + m_2x_2} = \sum_{m\in\mathbb{Z}^2} \varphi_m\, e^{\mathrm{i}m.x} \tag{4.2}$$

with coefficients $\{\varphi_{mn}\} \in S(\mathbb{Z}^2)$ a complex-valued Schwartz function on $\mathbb{Z}^2$. This means that the sequence of complex numbers $\{\varphi_{m,n} \in \mathbb{C} \mid (m,n) \in \mathbb{Z}^2\}$ decreases rapidly at 'infinity', that is for any $k = 0, 1, 2, \cdots$, one has bounded semi-norms

$$||\varphi||_k = \sup_{(m,n)\in\mathbb{Z}^2} |\varphi_{m,r}| \left(1 + |m| + |r|\right)^k < \infty\,. \tag{4.3}$$

The algebra $C^\infty(\mathbb{T}^2)$ is a Poisson algebra with brackets:

$$\{\varphi, \psi\} = \partial_1\varphi\partial_2\psi - \partial_2\varphi\partial_1\psi$$

For the basis elements $e_m = e^{\mathrm{i}m.x}$ one gets:

$$\{e_k, e_m\} = -k \wedge m\, e_{k+m} \qquad k \wedge m = k_1m_2 - k_2m_1 \tag{4.4}$$

An invariant real valued inner product, the counterpart of (3.23), is given by

$$\langle \varphi, \psi \rangle = \mathrm{Im} \int \varphi\psi \frac{d^2x}{(2\pi)^2} = \frac{1}{2\mathrm{i}} \sum_{m\in\mathbb{Z}^2} (\varphi_m\psi_{-m} - \varphi^*_{-m}\psi^*_m). \tag{4.5}$$

The sum is finite due to the condition (4.3) on the coefficients. The invariance means that

$$\langle \{\eta, \varphi\}, \psi \rangle + \langle \varphi, \{\eta, \psi\} \rangle = 0.$$

The inner product is non degenerate but is not positive definite. We use the inner product to break the Poisson Lie algebra $C^\infty(\mathbb{T}^2)$, regarded as a *real* Lie algebra, as the sum of two real sub-algebras which are isotropic and paired via the inner product. We seek a splitting of the kind in (3.22) of a 'real' and 'upper triangular' parts (rather then a 'purely imaginary' part).

Start with the real Lie sub-algebra of real functions:

$$\underline{A} = \Big\{ \varphi = \sum_{m\in\mathbb{Z}^2} \varphi_m e_m \mid \varphi^*_m = \varphi_{-m} \Big\}$$

This is isotropic:

$$\langle \varphi, \psi \rangle = \frac{1}{2i} \sum_{m \in \mathbb{Z}^2} (\varphi_m \psi_{-m} - \varphi_m \psi_{-m}) = 0.$$

It is convenient to use the trigonometric basis for $\underline{A}$. To avoid over-counting indices are restricted, following a lexicographic ordering:

$$U_m = \begin{cases} \frac{1}{2}(e_m + e^*_m) = \cos(m\cdot x) & m_1 > 0, m_2 \in \mathbb{Z} \\ \frac{1}{2}(e_{0,m_2} + e^*_{0,m_2}) = \cos(m_2 x_2) & m_2 > 0 \\ 1 & m = 0 \end{cases} \tag{4.6}$$

$$\tilde{U}_m = \begin{cases} -\frac{i}{2}(e_m - e^*_m) = \sin(m\cdot x) & m_1 > 0, m_2 \in \mathbb{Z} \\ -\frac{i}{2}(e_{0,m_2} - e^*_{0,m_2}) = \sin(m_2 x_2) & m_2 > 0 \end{cases} \tag{4.7}$$

Next, consider the real Lie sub-algebra of functions:

$$\underline{B} = \left\{ \psi = \sum_{m_1 \geq 0, m_2 \in \mathbb{Z}} \psi_m e_m \mid \psi^*_{0,m_2} = -\psi_{0,-m_2} \right\}$$

So, $\underline{B}$ is made of functions on the torus that can be continued holomorphically to the disk in the first variable, and are purely imaginary when averaged over x_1:

$$\underline{\psi} = \sum_{m_1 \geq 0, m_2 \in \mathbb{Z}} \psi_m z_1^{m_1} e^{\mathrm{i} m_2 x_2}, \quad z_1 = e^{\mathrm{i}x_1}, \quad |z_1| < 1.$$

The sub-algebra B is isotropic as well:

$$\langle \varphi, \psi \rangle = \frac{1}{2\mathrm{i}} \sum_{n\in\mathbb{Z}} (\varphi_{0,n}\psi_{0,-n} - \varphi_{0,n}\psi_{0,-n}) = 0.$$

A basis for $\underline{B}$ is given by:

$$T^n = \begin{cases} 2\mathrm{i} e_n & n_1 > 0, n_2 \in \mathbb{Z} \\ \mathrm{i}(e_{0,n_2} + e^*_{0,n_2}) = 2\mathrm{i}\cos(n_2 x_2) & n_2 > 0 \\ \mathrm{i} & n = 0 \end{cases} \tag{4.8}$$

$$\tilde{T}^n = \begin{cases} 2e_n & n_1 > 0, n_2 \in \mathbb{Z} \\ e_{0,n_2} - e^*_{0,n_2} = 2\mathrm{i}\sin(n_2 x_2) & n_2 \geq 0 \end{cases} \tag{4.9}$$

The basis (4.6) and (4.7) is dual to the basis (4.8) and (4.9) for the inner product (4.5) as it can be checked directly. The only non vanishing pairings are:

$$\langle T^m, U_m \rangle = 1 = \left\langle \tilde{T}^m, \tilde{U}_m \right\rangle. \tag{4.10}$$

4.2. **The commutation relations.** Out of (4.4), we workout explicitly the commutation relations of the generators U's of the algebra A in (4.6) and (4.7), and of the generators T's of the algebra B in (4.8) and (4.9). Let $m = (m_1, m_2)$, $n = (n_1, n_2)$ be two elements in $\mathbb{Z}^2$.

4.2.1. *The algebra $\underline{A}$.* Firstly: $\{U_m, U_n\} = \{U_m, \tilde{U}_n\} = \{\tilde{U}_m, \tilde{U}_n\} = 0$, when $m_1 = n_1 = 0$. Then,

$$\begin{aligned} m_1 > 0,\ n_1 > 0 \qquad \{U_m, U_n\} &= \tfrac{1}{2} m\wedge n\,(-U_{m+n} + U_{m-n}) && \text{if} \quad m_1 > n_1 \\ &= \tfrac{1}{2} m\wedge n\,(-U_{m+n} + U_{n-m}) && \text{if} \quad m_1 < n_1 \end{aligned} \tag{4.11}$$

$$\begin{aligned} m_1 = n_1 > 0 \qquad \{U_m, U_n\} &= \tfrac{1}{2} m_1(n_2 - m_2)\,(-U_{m+n} + U_{0,m_2-n_2}) && \text{if} \quad m_2 > n_2 \\ &= \tfrac{1}{2} m_1(n_2 - m_2)\,(-U_{m+n} + U_{0,n_2-m_2}) && \text{if} \quad m_2 < n_2. \end{aligned} \tag{4.12}$$

Relations (4.11) is valid also when either $m_1 = 0$ or $n_1 = 0$, but not both.

Next,

$$m_1 > 0,\ n_1 > 0 \qquad \{U_m, \tilde{U}_n\} = -\tfrac{1}{2} m \wedge n\, (\tilde{U}_{m+n} + \tilde{U}_{m-n}) \qquad \text{if} \quad m_1 > n_1$$

$$= -\tfrac{1}{2} m \wedge n\, (\tilde{U}_{m+n} - \tilde{U}_{n-m}) \qquad \text{if} \quad m_1 < n_1 \qquad (4.13)$$

$$m_1 = n_1 > 0 \qquad \{U_m, \tilde{U}_n\} = -\tfrac{1}{2} m_1(n_2 - m_2)\,(\tilde{U}_{m+n} + \tilde{U}_{0,(m-n)_2}) \qquad \text{if} \quad m_2 > n_2$$
$$= -\tfrac{1}{2} m_1(n_2 - m_2)\,(\tilde{U}_{m+n} - \tilde{U}_{0,(n-m)_2}) \qquad \text{if} \quad m_2 < n_2. \qquad (4.14)$$

Relations (4.13) is valid also when either $m_1 = 0$ or $n_1 = 0$, but not both.

Finally,

$$m_1 > 0,\ n_1 > 0 \qquad \{\tilde{U}_m, \tilde{U}_n\} = \tfrac{1}{2} m \wedge n\,(U_{m+n} + U_{m-n}) \qquad \text{if} \quad m_1 > n_1$$
$$= \tfrac{1}{2} m \wedge n\,(U_{m+n} + U_{n-m}) \qquad \text{if} \quad m_1 < n_1 \qquad (4.15)$$

$$m_1 = n_1 > 0 \qquad \{\tilde{U}_m, \tilde{U}_n\} = \tfrac{1}{2} m_1(n_2 - m_2)\,(U_{m+n} + U_{0,(m-n)_2}) \qquad \text{if} \quad m_2 > n_2$$
$$= \tfrac{1}{2} m_1(n_2 - m_2)\,(U_{m+n} + U_{0,(n-m)_2}) \qquad \text{if} \quad m_2 < n_2. \qquad (4.16)$$

Relations (4.15) is valid also when either $m_1 = 0$ or $n_1 = 0$, but not both.

4.2.2. *The algebra* $\underline{B}$. Firstly: $\{T^m, T^n\} = \{T^m, \tilde{T}^n\} = \{\tilde{T}^m, \tilde{T}^n\} = 0$ when $m_1 = n_1 = 0$.

Then,

$$m_1 > 0,\ n_1 > 0 \qquad \{T^m, T^n\} = 2m \wedge n\, \tilde{T}^{m+n} \qquad (4.17)$$
$$m_2 > 0,\ n_1 > 0 \qquad \{T^{(0,m_2)}, T^n\} = -m_2 n_1\,(\tilde{T}^{(n_1,n_2+m_2)} - \tilde{T}^{(n_1,n_2-m_2)}). \qquad (4.18)$$

Next,

$$m_1 > 0,\ n_1 > 0 \qquad \{T^m, \tilde{T}^n\} = -2m \wedge n\, T^{m+n} \qquad (4.19)$$
$$m_2 > 0,\ n_1 > 0 \qquad \{T^{(0,m_2)}, \tilde{T}^n\} = m_2 n_1\,(T^{(n_1,n_2+m_2)} - T^{(n_1,n_2-m_2)}). \qquad (4.20)$$
$$m_2 > 0,\ n_1 > 0 \qquad \{\tilde{T}^{(0,m_2)}, T^n\} = m_2 n_1\,(T^{(n_1,n_2+m_2)} + T^{(n_1,n_2-m_2)}). \qquad (4.21)$$

Finally,

$$m_1 > 0,\ n_1 > 0 \qquad \{\tilde{T}^m, \tilde{T}^n\} = -2m \wedge n\, \tilde{T}^{m+n} \qquad (4.22)$$
$$m_2 > 0,\ n_1 > 0 \qquad \{\tilde{T}^{(0,m_2)}, \tilde{T}^n\} = m_2 n_1\,(\tilde{T}^{(n_1,n_2+m_2)} + \tilde{T}^{(n_1,n_2-m_2)}). \qquad (4.23)$$

4.3. **The Lie bi-algebra structure.** We indicate by Γ the structure constants of the sub- algebra $\underline{A}$ and by Δ those of the subalgebra $\underline{B}$. The mixed commutators ought to be of the form

$$\{T^a, U_b\} = \Gamma^a_{bd} T^d - \Delta^{ad}_b U_d.$$

Let $m = (m_1, m_2)$, $n = (n_1, n_2)$ be two elements in $\mathbb{Z}^2$.

- $m_1 > 0, n_1 > 0, m_1 \neq n_1$:

$$\{T^m, U_n\} = -\mathrm{i}\, m \wedge n(e_{m+n} - e_{m-n})$$

$m_1 < n_1$: the admissible structure constants are

$$\Delta^{m,\widetilde{n-m}}_n, \quad \Gamma^m_{n,n+m}, \quad \Gamma^m_{n,n-m}$$

then

$$\Gamma^m_{n,n+m} T^{n+m} + \Gamma^m_{n,n-m} T^{n-m} - \Delta^{m,\widetilde{n-m}}_n \tilde{U}_{n-m} = -\mathrm{i}\, m \wedge n(e_{m+n} - e_{m-n})$$

$m_1 > n_1$: the admissible structure constants are

$$\Gamma^m_{n,m+n}, \quad \Gamma^m_{n,m-n}$$

then

$\Gamma^m_{n,m+n}$

- $m_1 = n_1 > 0$:

$T^{m+n} + \Gamma^{m}_{n,m-n}$ T^{m-n} =
$-\mathrm{i}\, m \wedge$
$n(e_{m+n}$
$- e_{m-n})$

$$\{T^{m}, U_n\} = \mathrm{i}\, m_1(m_2 - n_2)(e_{m+n} - e_{0,m_2-n_2})$$

$m_2 < n_2$: the admissible structure constants are

$$\Delta_n^{m,(0,\widetilde{n_2-m_2})},\quad \Gamma^m_{n,n+m},\quad \Gamma^m_{n,(0,n_2-m_2)}$$

then

$$\Gamma^m_{n,n+m} T^{n+m} + \Gamma^m_{(0,n_2-m_2)} T^{(0,n_2-m_2)} - \Delta_n^{m,(0,\widetilde{n_2-m_2})} \tilde{U}_{(0,n_2-m_2)} = \mathrm{i}\, m_1(m_2-n_2)(e_{m+n} - e_{0,m_2-n_2})$$

$m_2 > n_2$: the admissible structure constants are

$$\Delta_n^{m,(0,\widetilde{m_2-n_2})},\quad \Gamma^m_{n,m+n},\quad \Gamma^m_{n,(0,m_2-n_2)}$$

then

$$\Gamma^m_{n,m+n} T^{m+n} + \Gamma^m_{(0,m_2-n_2)} T^{(0,m_2-n_2)} - \Delta^{m,(0,\widetilde{m_2-n_2})} \tilde{U}_{(0,m_2-n_2)} = \mathrm{i}\, m_1(m_2-n_2)(e_{m+n} - e_{0,m_2-n_2})$$

- $m_1 = 0,\ n_1 > 0$:

  $$\{T^{(0,m_2)}, U_n\} = \frac{\mathrm{i}}{2} m_2 n_1 (e_{(n_1,m_2+n_2)} + e_{(-n_1,-m_2-n_2)} - e_{(n_1,-m_2+n_2)} - e_{(-n_1,m_2-n_2)})$$

the admissible structure constants are

$$\Delta_n^{(0,m_2),(\widetilde{n_1,n_2+m_2})},\quad \Delta_n^{(0,m_2),(\widetilde{n_1,n_2-m_2})},\quad \Gamma^{(0,m_2)}_{n,(n_1,n_2+m_2)},\quad \Gamma^{(0,m_2)}_{n,(n_1,n_2-m_2)}$$

then

$$\Gamma^{(0,m_2)}_{n,(n_1,n_2+m_2)} T^{n_1,n_2+m_2} + \Gamma^{(0,m_2)}_{n,(n_1,n_2-m_2)} T^{n_1,n_2-m_2} - \Delta_n^{(0,m_2),(\widetilde{n_1,n_2+m_2})} \tilde{U}_{n_1,n_2+m_2} - \Delta_n^{(0,m_2),(\widetilde{n_1,n_2-m_2})} \tilde{U}_{n_1,n_2-m_2}$$
$$= \frac{\mathrm{i}}{2} m_2 n_1 (e_{(n_1,m_2+n_2)} + e_{(-n_1,-m_2-n_2)} - e_{(n_1,-m_2+n_2)} - e_{(-n_1,m_2-n_2)})$$

- $m_1 > 0,\ n_1 = 0$:

  $$\{T^m, U_n\} = -\mathrm{i}\, m_1 n_2 (e_{(m_1,m_2+n_2)} - e_{m_1,m_2-n_2})$$

the admissible structure constants are

$$\Gamma^m_{(0,n_2),(m_1,m_2+n_2)},\quad \Gamma^m_{(0,n_2),(m_1,m_2-n_2)}$$

then

$$\Gamma^m_{(0,n_2),(m_1,m_2+n_2)} T^{m_1,m_2+n_2} + \Gamma^m_{(0,n_2),(m_1,m_2-n_2)} T^{m_1,m_2-n_2} = -\mathrm{i}\, m_1 n_2 (e_{(m_1,m_2+n_2)} - e_{m_1,m_2-n_2})$$

All other commutators, $\{T^a, \tilde{U}_b\}$, $\{\tilde{T}^a, U_b\}$, $\{\tilde{T}^a, \tilde{U}_b\}$, go along the same lines.

5. The non-commutative torus

5.1. **The algebra.** Let ϑ be a real number. The algebra $A_\vartheta = C^\infty(\mathbb{T}^2)$ of smooth functions on the non-commutative torus $\mathbb{T}^2$ is the associative algebra made up of all elements of the form,

$$a = \sum_{(m,n)\in\mathbb{Z}^2} a_{mn}\, Q^m P^n, \tag{5.1}$$

with two generators Q and P that satisfy

$$P\,Q = e^{2\pi i\vartheta}\, QP. \tag{5.2}$$

As in the commutative case, the coefficients $\{a_{mn}\} \in S(\mathbb{Z}^2)$ form a complex-valued Schwartz function on $\mathbb{Z}^2$. The algebra A_ϑ can be made into a $*$-algebra by defining an involution by

$$Q^\dagger := Q^{-1}, \quad P^\dagger := P^{-1}, \tag{5.3}$$

so that Q and P are unitary. Heuristically, the non-commutative relation (5.2) of the torus is the exponential of the Heisenberg commutation relation $[x_2, x_1] = i\vartheta/2\pi$. The algebra A_ϑ can be represented as bounded operators on the Hilbert space $H = L^2(\mathbb{R})$ by

$$(Qf)(t) = e^{2\pi i t} f(t) \qquad (Pf)(t) = f(t+\vartheta). \tag{5.4}$$

getting the commutation relations (5.2).

From (5.2) one sees that A_ϑ is commutative if and only if ϑ is an integer, and one identifies A_0 with the algebra $C^\infty(\mathbb{T}^2)$ of complex-valued smooth functions on an ordinary square two-torus $\mathbb{T}^2$ with coordinate functions given by $Q = e^{ix_1}$ and $P = e^{ix_2}$, recovering then the Fourier expansion (4.2) of any such a function.

When the deformation parameter is a rational number, $\vartheta = M/N$, with M and N positive integers (taken to be relatively prime, say) also the algebra $A_{M/N}$ is related to the algebra $C^\infty(\mathbb{T}^2)$. More precisely, $A_{M/N}$ is Morita equivalent to $C^\infty(\mathbb{T}^2)$, that is $A_{M/N}$ is a twisted matrix bundle over $C^\infty(\mathbb{T}^2)$ of Chern number M whose fibers are $N \times N$ complex matrix algebras. The algebra $A_{M/N}$ has a 'huge' center $C(A_{M/N})$ which is generated by the elements Q^N and P^N . One identifies $C(A_{M/N})$ with the algebra $C^\infty(\mathbb{T}^2)$ = of the torus winding N times over itself, while there are finite dimensional representations given as copies of those in (3.1).

Denote $\omega = e^{2\pi i\vartheta}$. It is convenient to change basis to

$$\hat{e}_m = \omega^{-\frac{1}{2}m_1 m_2} P^{m_1} Q^{m_2} \qquad m = (m_1, m_2) \in \mathbb{Z}.$$

Then $\hat{e}^\dagger_m = \hat{e}_{-m}$. From (5.4) they act on $H = L^2(\mathbb{R})$ as

$$(\hat{e}_m f)(t) = \omega^{\frac{1}{2}m_1 m_2} e^{2\pi i m_2 t} f(t + m_1\vartheta). \tag{5.5}$$

These elements yield an infinite-dimensional Lie algebra (the sine-algebra). Indeed, one checks

$$\hat{e}_k \hat{e}_m = \omega^{\frac{1}{2}k\wedge m} \hat{e}_{k+m} \qquad k \wedge m = k_1 m_2 - k_2 m_1$$

This implies

$$\hat{e}_k\hat{e}_m - \hat{e}_m\hat{e}_k = 2i \sin(\pi\vartheta\, k \wedge m)\, \hat{e}_{k+m}. \tag{5.6}$$

The sine-algebra has the role of the hamiltonian vector fields on $\mathbb{T}^2$ for the canonical Poisson structure. The above is indeed seen as the quantisation of the canonical Poisson structure on $\mathbb{T}^2$. As a vector space A_ϑ and $C^\infty(\mathbb{T}^2)$ are the same. With an abuse of notation on generators, $\hat{e}_m \to e_m$, the Poisson structure is recovered as

$$\begin{aligned} \{e_k, e_m\} &= \frac{i}{\mathrm{k}}(e_k e_m - e_m e_k)\Big|_{\mathrm{k}\sim\theta=0} \\ &= -\frac{2\pi}{\mathrm{k}}\vartheta\, k \wedge m\, e_{k+m} \\ &= -k \wedge m\, e_{k+m} \end{aligned} \tag{5.7}$$

with the identification $\mathrm{k} = 2\pi\vartheta$, in parallel with (4.1). This is just the Poisson structure (4.4).

5.2. **The trace and the K-theory.** On the algebra A_ϑ there is a (unique if ϑ is irrational) normalized, positive definite trace, $\tau : A_\vartheta \to \mathbb{C}$, given by

$$\tau\Big(\sum_{m\in\mathbb{Z}^2} a_m \hat{e}_m\Big) := a_0. \tag{5.8}$$

Then, for any $a, b \in A_\vartheta$, one checks that

$$\tau(a\,b) = \sum_{m\in\mathbb{Z}^2} a_m b_{-m} = \tau(b\,a). \tag{5.9}$$

Also, $\tau(1) = 1$, $\tau(a^\dagger a) > 0$, for $a \neq 0$ and $\tau(a^\dagger a) = 0$ if and only if $a = 0$ (the trace is faithful).

This trace is invariant under the natural action of the commutative torus $\mathbb{T}^2$ on $\mathcal{A}_\vartheta$ whose infinitesimal form is generated by two commuting derivations ∂_1, ∂_2 acting as

$$\partial_1(P) = 2\pi i P, \quad \partial_1(Q) = 0, \qquad \partial_1(P) = 0, \quad \partial_1(Q) = 2\pi i Q \tag{5.10}$$

Invariance is just the statement that $\tau(\partial_1(a)) = 0 = \tau(\partial_1(a))$ for $a \in A_\vartheta$.

A remarkable fact about the non-commutative two-torus algebra A_ϑ is that it contains not trivial projections. In fact it contains a representative projection for each equivalence classe in the K-theory of the torus. The archetype of all such projections is the Powers-Rieffel projection [11]. To construct it, observe first that there is an injective algebra homomorphism

$$\begin{aligned} \rho : C^\infty(\mathbb{S}^1) &\longrightarrow A_\vartheta , \\ f(x_1) = \sum_{m\in\mathbb{Z}} f_m e^{2\pi i m x_1} &\longrightarrow \rho(f) = \sum_{m\in\mathbb{Z}} f_m Q^m . \end{aligned} \tag{5.11}$$

From the commutation relations (5.2) it follows that if $f(x_1)$ is mapped to $\rho(f)$, then $P\rho(f)P^{-1}$ is the image of the shifted function $f(x_1 + \vartheta)$. One now looks for projections of the form

$$p_\vartheta = P^{-1}\rho(g) + \rho(f) + \rho(g)P . \tag{5.12}$$

In order that (5.12) defines a projection operator $p^2 = p$, the functions $f, g \in C^\infty(\mathbb{S}^1)$ must satisfy some conditions. These conditions are satisfied by the choice

$$f(x_1) = \begin{cases} \text{smoothly increasing from 0 to 1} & 0 \le x_1 \le 1-\vartheta \\ 1 & 1-\vartheta \le x_1 \le \vartheta \\ 1 - f(x_1-\vartheta) & \vartheta \le x_1 \le 1 \end{cases} ,$$

$$g(x_1) = \begin{cases} 0 & 0 \le x_1 \le \vartheta \\ \sqrt{f(x_1) - f(x_1)^2} & \vartheta \le x_1 \le 1 \end{cases} . \tag{5.13}$$

It is straighforward to check that the rank (i.e. the trace) of p_ϑ is just ϑ. From (5.12) and the expressions in (5.13) one finds

$$\tau(p_\vartheta) = f_0 = \int_0^1 dx\, f(x) = \vartheta . \tag{5.14}$$

Furthermore, the monopole charge (i.e. first Chern number) of p_ϑ is 1. This is computed as the index of a Fredholm operator [4] given by

$$\begin{aligned} c_1(p_\vartheta) &:= -\frac{1}{2\pi i}\, \tau\, p_\vartheta\,(\partial_1 p_\vartheta\, \partial_2 p_\vartheta - \partial_2 p_\vartheta\, \partial_1 p_\vartheta) \\ &= -6 \int_0^1 dx\, g(x)^2 f'(x) = 1 , \end{aligned} \tag{5.15}$$

with the last equality following from expression (5.13) of the function f.

When ϑ is irrational the projection p_ϑ, together with the trivial projection 1, generates the K_0 group. The trace on A_ϑ gives a map

$$\begin{aligned} \mathbb{Z} : K_0(A_\vartheta) &\longrightarrow \mathbb{Z} + \mathbb{Z}\vartheta , \\ r[1] + m[p_\vartheta] &\longrightarrow \tau(1) + m\,\tau(p_\vartheta) = r + m\vartheta \end{aligned} \tag{5.16}$$

which is an isomorphism of ordered groups [9]. The class $m[p_\vartheta]$ can be represented by a Powers-

Rieffel projection in the algebra itself A_ϑ with suitable functions in (5.12) of the kind (5.13). The positive cone is the collection of (equivalence classes of) projections with non-negative trace,

$$K_0^+(A_\vartheta) = \{ (r, m) \in \mathbb{Z}^2 \cdot r + m\,\vartheta \geq 0 \} . \quad (5.17)$$

5.3. **The splitting of the algebra as a Lie bi-algebra.** We use the trace τ to define a real valued inner product on the algebra A_ϑ:

$$\langle a, b\rangle = \operatorname{Im}\tau(ab) = \frac{1}{2i}\sum_{m\in Z^2}(a_m b_{-m} - a^*_{-m} b^*_m). \tag{5.18}$$

This inner product is non degenerate but, as in the commutative case, it is not positive definite.

We shall denote $\underline{S}_\vartheta$ the non-commutative torus algebra A_ϑ when thought of as a Lie algebra with commutator (5.6). As before, we aim at using the inner product to break for a splitting $\underline{S}_\vartheta = \underline{A} \oplus \underline{B}$ into real sub-algebras which are isotropic and paired via the inner product. We seek a splitting of a 'purely imaginary part' (anti-hermitian operators are closed for the commutator while hermitian ones) and a 'upper triangular' part.

It is known [8, Thm. 3.11] that the ordered group $(K_0(A_\vartheta), K_0^+(A_\vartheta))$ characterizes non-commutative tori up to Rieffel-Morita equivalence: two non-commutative tori are Rieffel-Morita equivalent if and only if their ordered K_0-groups are isomorphic. Algebras which are equivalent in this sense are usually thought of as having the same geometry. It is then only proper to use the order of the K_0-group to label natural bases of the sub-algebras $\underline{A}$ and $\underline{B}$ in the splitting $\underline{S}_\vartheta = \underline{A} \oplus \underline{B}$. As mentioned, in the limit $\vartheta \to 0$ the K-theoretical ordering tends to the lexicographic ordering we used earlier for the Lie bi-algebra of the commutative torus.

Start with the real Lie sub-algebra of anti-hermitian:

$$\underline{A} = \left\{ a = \sum_{m\in Z^2} a_m\,\hat{e}_m \,|\, a^*_m = -a_{-m} \right\},$$

which is clearly isotropic for the inner product (5.18). To avoid over counting, we use a real basis of $\underline{A}$ labelled by the positive cone (5.17) of $K_0(A_\vartheta)$. That is, if $m = (m_1, m_2) \in \mathbb{Z}$ we take

$$\begin{aligned} U_{(0,0)} &= i \\ U_m &= \tfrac{i}{2}(\hat{e}_m + \hat{e}^\dagger_m) \qquad m_1 + m_2\vartheta > 0 \\ \tilde{U}_m &= -\tfrac{1}{2}(\hat{e}_m - \hat{e}^\dagger_m) \qquad m_1 + m_2\vartheta > 0. \end{aligned} \tag{5.19}$$

In the limit $\vartheta \to 0$ this tends to the lexicographic ordering we used earlier, as illustrated in fig.1.

The dual real Lie sub-algebra $\underline{B}$ is the real span of the basis elements

$$\begin{aligned} T^{(0,0)} &= 1 \\ T^n &= 2\,\hat{e}_n \qquad n_1 + n_2\vartheta > 0 \\ \tilde{T}^n &= 2i\,\hat{e}_n \qquad n_1 + n_2\vartheta > 0. \end{aligned} \tag{5.20}$$

For the inner product (5.18) the basis (5.19) is isotropic and is dual to the basis (5.20) which is isotropic as well, as it can be checked directly. The only non vanishing pairings are:

$$\langle T^m, U_m\rangle = 1 = \left\langle \tilde{T}^m, \tilde{U}_m \right\rangle. \tag{5.21}$$

Next, we compute the structure constants of both the Lie algebras A and B starting from the commutators (5.6). For $m, n \in \mathbb{Z}$, let us use the short notation

$$s(m, n) = \sin(\pi\vartheta\, m \wedge n) \tag{5.22}$$

(an odd function on each argument) and the convention $m > n$ if and only if $m_1+m_2\vartheta > n_1+n_2\vartheta$.

For the algebra $\underline{A}$: $[X_a, X_b] = \Gamma^c_{ab} X_c$, the only non vanishing Γ's are computed to be

$$\begin{aligned} &\Gamma^{m+n}_{m,n} = -s(m,n), && \Gamma^{\widetilde{m+n}}_{\tilde{m},n} = -s(m,n), && \Gamma^{m+n}_{\tilde{m},\tilde{n}} = s(m,n), && \\ &\Gamma^{m-n}_{m,n} = s(m,n), && \Gamma^{\widetilde{m-n}}_{m,n} = -s(m,n), && \Gamma^{m-n}_{m,n} = s(m,n), && \text{for} \quad m > n, \\ &\Gamma^{n-m}_{m,n} = s(m,n), && \Gamma^{\widetilde{n-m}}_{m,\tilde{n}} = -s(m,n), && \Gamma^{n-m}_{\tilde{m},\tilde{n}} = s(m,n), && \text{for} \quad m < n, \end{aligned} \tag{5.23}$$

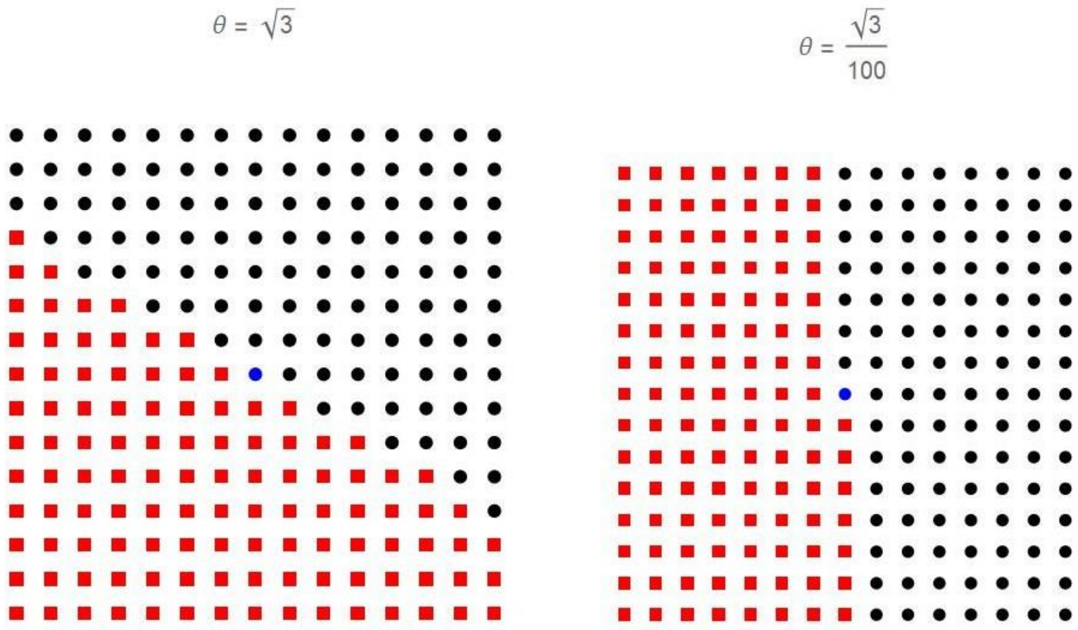

FIGURE 1. The squares represent elements of $(m_1, m_2) \in \mathbb{Z}^2$ with $m_1 + \vartheta m_2 < 0$, while the disks represent the positive elements. In the limit $\vartheta \to 0$ this tends to the lexicographic ordering on $\mathbb{Z}^2$.

and their antisymmetric ones in the two lower indices.

For the algebra $\underline{B}$: $[X^a, X^b] = \Delta^{ab}_{c} X_c$, the only non vanishing Δ's are computed to be

$$\Delta^{m,n}_{\widetilde{m+n}} = 4s(m,n), \qquad \Delta^{m,n}_{m+n} = \tilde{4}s(m,n), \qquad \Delta^{m,n}_{\widetilde{m+n}} = -4s(m,n) \tag{5.24}$$

and their antisymmetric ones in the two upper indices.

Finally, for the mixed compatible ones $[X^a, X_b] = \Gamma^a_{bd} X^d - \Delta^{ad}_{b} X_d$, one computes

$$[T^m, U_n] = \begin{cases} \Gamma^m_{n,m+n} T^{m+n} + \Gamma^m_{n,m-n} T^{m-n} & m > n \\ \Gamma^m_{n,m+n} T^{m+n} + \Gamma^m_{n,n-m} T^{n-m} - \Delta^{n}_{m,n-m} \tilde{U}^{n-m} & m < n \end{cases} \tag{5.25}$$

$$[T^m, \tilde{U}_n] = \begin{cases} \Gamma^m_{n,m+n} \tilde{T}^{m+n} + \Gamma^m_{n,m-n} \tilde{T}^{m-n} & m > n \\ \Gamma^m_{\tilde{n},m+n} \tilde{T}^{m+n} + \Gamma^m_{\tilde{n},n-m} \tilde{T}^{n-m} - \Delta^{m,n-m}_{\tilde{n}} U_{n-m} & m < n \end{cases} \tag{5.26}$$

$$[\tilde{T}^m, U_n] = \begin{cases} \Gamma^{\tilde{m}}_{n,m+n} \tilde{T}^{m+n} + \Gamma^{\tilde{m}}_{n,m-n} \tilde{T}^{m-n} & m > n \\ \quad \tilde{m},n-m \end{cases} \tag{5.27}$$

$[\tilde{T}^m, U_n] = \Gamma^m_{n,m+n}$ $\tilde{n},m+n$ $\tilde{n},m+n$

$$_{n,n-m} \quad \Gamma^{\tilde{m}} \tilde{T}^{m+n} + \Gamma^{\tilde{m}} \tilde{T}^{n-m} - \Delta_n \quad U_{n-m} \quad m < n$$

$$T^{m+n} + \Gamma^{m} \quad T^{m-n} \quad \tilde{n}, m-n$$

(5.28)

$$T^{m+n} + \Gamma^{m}_{\tilde{n},n-m}$$

$m > n$

T^{n-m} -

$\Delta^{\tilde{m}}$ $\tilde{n}$

$_{,\widetilde{n-m}}\tilde{U}_{n-m}$

$m < n$

The above shows the Lie-bi-algebra structure of the non-commutative torus $S_{\vartheta} = A \oplus \bar{B}$.

Appendix A. The pairings: rational case

We need the scalar product between E's in (3.2) and F's in (3.5), (3.6). One finds explicitly

$$\mathrm{Tr}(f_{a,b}\, e^*_{r,s}) = \delta(a-r)\, \omega^{-\frac{1}{2}a(b-s)} \sum_{n=a}^{N-1} \omega^{n(b-s)}$$

$$\mathrm{Tr}(\tilde{f}_{a,b}\, e_{r,s}) = \delta(a-r)\, \omega^{\frac{1}{2}a(b-s)} \sum_{n=0}^{a-1} \omega^{-n(b-s)} \tag{A.1}$$

$$\mathrm{Tr}(f_{a,b}\, e_{r,s}) = \delta(a+r)\, \omega^{-\frac{1}{2}a(b+s)} \sum_{n=a}^{N-1} \omega^{n(b+s)}$$

$$\mathrm{Tr}(\tilde{f}_{a,b}\, e^*_{r,s}) = \delta(a+r)\, \omega^{\frac{1}{2}a(b+s)} \sum_{n=0}^{a-1} \omega^{-n(b+s)} \tag{A.2}$$

Putting these together, and using $\sum_{j=0}^{N-1} \omega^{jm} = N\delta(m)$, we get:

For $a = r$:

$$\mathrm{Tr}(f_{a,b}\, e^*_{r,s} + \tilde{f}_{a,b}\, e_{r,s}) = \omega^{-\frac{1}{2}a(b-s)} \sum_{n=a}^{N-1} \omega^{n(b-s)} + \omega^{\frac{1}{2}a(b-s)} \sum_{n=0}^{a-1} \omega^{-n(b-s)} \tag{A.3}$$

$$= \begin{cases} N & b = s \\ -\omega^{-\frac{1}{2}a(b-s)} \sum_{n=0}^{a-1} \omega^{n(b-s)} + \omega^{\frac{1}{2}a(b-s)} \sum_{n=0}^{a-1} \omega^{-n(b-s)} \in i\mathbb{R} & b \neq s \end{cases}$$

$$\mathrm{Tr}(f_{a,b}\, e^*_{r,s} - \tilde{f}_{a,b}\, e_{r,s}) = \omega^{-\frac{1}{2}a(b-s)} \sum_{n=a}^{N-1} \omega^{n(b-s)} - \omega^{\frac{1}{2}a(b-s)} \sum_{n=0}^{a-1} \omega^{-n(b-s)} \tag{A.4}$$

$$= \begin{cases} N - 2a & b = s \\ -\omega^{-\frac{1}{2}a(b-s)} \sum_{n=0}^{a-1} \omega^{n(b-s)} - \omega^{\frac{1}{2}a(b-s)} \sum_{n=0}^{a-1} \omega^{-n(b-s)} \in \mathbb{R} & b \neq s \end{cases}$$

For $a - r$:

$$\mathrm{Tr}(f_{a,b}\, e_{r,s} + \tilde{f}_{a,b}\, e^*_{r,s}) = \omega^{-\frac{1}{2}a(b+s)} \sum_{n=a}^{N-1} \omega^{n(b+s)} + \omega^{\frac{1}{2}a(b+s)} \sum_{n=0}^{a-1} \omega^{-n(b+s)} \tag{A.5}$$

$$= \begin{cases} N & b = -s \\ -\omega^{-\frac{1}{2}a(b+s)} \sum_{n=0}^{a-1} \omega^{n(b+s)} + \omega^{\frac{1}{2}a(b+s)} \sum_{n=0}^{a-1} \omega^{-n(b+s)} \in i\mathbb{R} & b \neq -s \end{cases}$$

$$\mathrm{Tr}(f_{a,b}\, e_{r,s} - \tilde{f}_{a,b}\, e^*_{r,s}) = \omega^{-\frac{1}{2}a(b+s)} \sum_{n=a}^{N-1} \omega^{n(b+s)} - \omega^{\frac{1}{2}a(b+s)} \sum_{n=0}^{a-1} \omega^{-n(b+s)} \tag{A.6}$$

$$= \begin{cases} N - 2a & b = -s \\ -\omega^{-\frac{1}{2}a(b+s)} \sum_{n=0}^{a-1} \omega^{n(b+s)} - \omega^{\frac{1}{2}a(b+s)} \sum \end{cases}$$

$$\sum_{n=0}^{a-1} \omega^{-n(b+s)} \in \mathbb{R} \qquad b \neq -s$$

From (A.1) and (A.2) the pairing of a $T^{a,b}$ (either without or with a tilde) and a $T^{a,b}$ (again either without or with a tilde) is zero unless $r = a$ or $r = a$. Then, on the one end, from (A.4)and (A.6), for any s,

$$\mathrm{Tr}(\tilde{T}^{a,b}\, U_{\pm a,s}) \in \mathbb{R} \qquad \mathrm{Tr}(T^{a,b}\, \tilde{U}_{\pm a,s}) \in \mathbb{R}$$

and these lead for any s to

$$\left\langle \tilde{T}^{a,b}, U_{\pm a,s} \right\rangle = \left\langle T^{a,b}, \tilde{U}_{\pm a,s} \right\rangle = 0$$

On the other end, from (A.3) we get

$$\mathrm{Tr}(T^{a,b}\, U_{a,b}) = \mathrm{i} \qquad \mathrm{Tr}(T^{a,b}\, U_{a,s}) \in \mathbb{R} \quad s \neq b$$

$$\mathrm{Tr}(\tilde{T}^{a,b}\, \tilde{U}_{a,b}) = \mathrm{i} \qquad \mathrm{Tr}(\tilde{T}^{a,b}\, \tilde{U}_{a,s}) \in \mathbb{R} \quad s \neq b$$

that is the only non vanishing pairings are

$$\left\langle T^{a,b}, U_{a,b} \right\rangle = 1 = \left\langle \tilde{T}^{a,b}, \tilde{U}_{a,b} \right\rangle \tag{A.7}$$

In particular for the diagonal matrices, t and it are isotropic and dually paired with pairings:

$$\langle\, H_A, \mathrm{i}H_B \rangle \; = \delta_{A,B} \qquad A = a, \tilde{a},\;\; B = b, \tilde{b} \tag{A.8}$$

with the ranges of the indices as in (3.14).

Appendix B. Taft algebras and their action

Let $\omega(= e^{2\pi \mathrm{i}/N})$ be a primitive N-the root of unity. The *Taft algebra* T_N, introduced in [14], is a Hopf algebra which is neither commutative nor co-commutative. Firstly, T_N is the N^2-dimensional unital algebra generated by generators R, G subject to the relations:

$$R^N = 0, \quad G^N = 1, \quad RG - \omega\, GR = 0.$$

It is a Hopf algebra with coproduct:

$$\Delta(R) := 1 \otimes R + R \otimes G, \qquad \Delta(G) := G \otimes G\,;$$

counit: $\varepsilon(R) := 0$, $\varepsilon(G) := 1$, and antipode: $S(R) := -RG^{-1}$, $S(Q) := G^{-1}$. The four dimensional algebra T_2 is also known as the *Sweedler algebra*.

For any $s \in \mathbb{C}$, let A_s be the unital algebra generated by elements r, g with relations:

$$r^N = s, \quad g^N = 1, \quad rg - \omega\, gr = 0.$$

When $s = 1$ this is just the algebra of rational non-commutative torus of Section 3.
The algebra A_s is a right T_N-comodule algebra, with coaction $\delta^A : A_s \to A_s \otimes T_N$ defined by

$$\delta^A(r) := 1 \otimes R + r \otimes G, \qquad \delta^A(g) := g \otimes G. \tag{B.1}$$

The algebra of corresponding coinvariant (invariant for the coaction) elements, that is elements $x \in A_s$ such that $\delta^A(x) = x \otimes 1$ is just the algebra $\mathbb{C}$. Moreover, the canonical map,

$$\chi : A_s \otimes A_s \to A_s \otimes T_N, \quad \chi(x \otimes y) = (x \otimes 1)\, \delta^A(y)$$

is an isomorphism. This states that the "coaction is free and transitive" and the extension $\mathbb{C} = (A_s)^{T_N} A_s$ is a non-commutative principal bundle over a point whose algebra of function is the coinvariant algebra $\mathbb{C}$. One also says that A_s is a T_N-Galois object.

Contrary to the commutative case, these are not trivial. It is known (see [6], Prop. 2.17 and Prop. 2.22) that any T_N-Galois object is isomorphic to A_s for some $s \in \mathbb{C}$ and that any two such Galois objects A_s and A_t are isomorphic if and only if $s = t$. Thus the equivalence classes of T_N-Galois objects are in bijective correspondence with the abelian group $\mathbb{C}$. The translation map of the coaction, $\tau := (\chi^{-1})_{|1\otimes T_N} : T_N \to A_s \otimes A_s$, is given on generators by

$$\tau(G) = g^{-1} \otimes g, \quad \tau(R) = 1 \otimes r - rg^{-1} \otimes g. \tag{B.2}$$

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