SUPPLEMENTARY MATERIAL

A. Maps for the wave function and for the density matrix

Let us consider a (N dimensional, for simplicity) Hilbert space $\mathbb{H}$ associated to the physical system of interest, and a generic norm-preserving map $T : \mathbb{H} \to \mathbb{H}$, which might not be linear. The map might be stochastic, as typical of collapse models; we will denote the stochastic average with a bar over the quantities to be averaged over the noise (if the map is deterministic, the bar can be removed from the proof).

Let us also consider the associated space $\mathcal{B}(\mathbb{H})$ of $N \times N$ matrices, to which the density matrix belongs. A density matrix $\hat{\rho}$ in general admits multiple decompositions in terms of pure states, for example:

$$\hat{\rho} = \sum_n p_n |\psi_n\rangle\langle\psi_n| = \sum_k q_k |\phi_k\rangle\langle\phi_k|, \tag{1}$$

where $\{|\psi_n\rangle\}$ and $\{|\phi_k\rangle\}$ are two different sets of normalized vectors in $\mathbb{H}$, while $\{p_n\}$ and $\{q_k\}$ are two sets of positive real numbers summing to 1, which represent our ignorance about the precise state of the system. The two statistical mixtures $\{(|\psi_n\rangle, p_n)\}$ and $\{(|\phi_k\rangle, q_k)\}$ are said to be *equivalent* since they correspond to the same statistical operator as per Eq. (1).

A generic map T need not preserve the equivalence between statistical mixtures, i.e.

$$\hat{\rho}'_{(1)} \equiv \sum_n p_n \overline{|T\psi_n\rangle\langle T\psi_n|} \neq \hat{\rho}'_{(2)} \equiv \sum_k q_k \overline{|T\phi_k\rangle\langle T\phi_k|}, \tag{2}$$

where $|T\psi_n\rangle = T[|\psi_n\rangle]$ and similarly for $|\phi_k\rangle$. However, if this happens it can be shown that, under the further assumption that measurements occur (either effectively or fundamentally) as provided by the Born rule and the von Neumann projection postulate, one can establish a protocol for superluminal signaling [1, 2]. Rejecting this possibility amounts to asking that the map T preserves the equivalence among statistical mixtures. In such a case, the map induces a map among density matrices according to:

$$\begin{aligned} \Lambda : \mathcal{B}(\mathbb{H}) \;&\to\; \mathcal{B}(\mathbb{H}) \\ \hat{\rho} \;&\to\; \Lambda[\hat{\rho}] = \sum_n p_n \overline{|T\psi_n\rangle\langle T\psi_n|}, \end{aligned} \tag{3}$$

where $\{(|\psi_n\rangle, p_n)\}$ is now *any* statistical mixture associated to $\hat{\rho}$.

We now show that the map Λ thus defined is *linear*, *positive* and *trace preserving*. Linearity is proven as follows: suppose that $\hat{\rho}$ is the convex sum of $\hat{\rho}_1$ and $\hat{\rho}_2$ according to: $\hat{\rho} = p_1\hat{\rho}_1 + p_2\hat{\rho}_2$, with $p_1, p_2 \geq 0$ and $p_1 + p_2 = 1$; consider also a statistical mixture $\{(|\psi_{1n}\rangle, p_{1n})\}$ associated to $\hat{\rho}_1$

and $\{(|\psi_{2n}\rangle, p_{2n})\}$ associated to $\hat{\rho}_2$. Then:

$$\begin{aligned}
\Lambda[\hat{\rho}] &= \Lambda\left[\sum_{j=1}^{2} p_j \sum_n p_{jn} |\psi_{jn}\rangle\langle\psi_{jn}|\right] = \\
&= \sum_{j=1}^{2} p_j \sum_n p_{jn} \overline{|T\psi_{jn}\rangle\langle T\psi_{jn}|} = p_1 \sum_n p_{1n} \overline{|T\psi_{1n}\rangle\langle T\psi_{1n}|} + p_2 \sum_n p_{2n} \overline{|T\psi_{2n}\rangle\langle T\psi_{2n}|} \\
&= p_1 \Lambda\left[\sum_n p_{1n} |\psi_{1n}\rangle\langle\psi_{1n}|\right] + p_2 \Lambda\left[\sum_n p_{2n} |\psi_{2n}\rangle\langle\psi_{2n}|\right] \\
&= p_1 \Lambda[\hat{\rho}_1] + p_2 \Lambda[\hat{\rho}_2]; \qquad (4)
\end{aligned}$$

in the first line we simply rewrote $\hat{\rho}$ in terms of the statistical mixture $\{(|\psi_{1n}\rangle, p_{1n})\} \cup \{(|\psi_{2n}\rangle, p_{2n})\}$; in going from the first to the second line we used Eq. (3) applied to $\hat{\rho}$; in going from the second to the third line we used again Eq. (3), this time applied to $\hat{\rho}_1$ and $\hat{\rho}_2$; in the last line, we used the fact that $\{(|\psi_{1n}\rangle, p_{1n})\}$ and $\{(|\psi_{2n}\rangle, p_{2n})\}$ are two statistical mixtures associated to $\hat{\rho}_1$ and $\hat{\rho}_2$ respectively.

The map Λ is automatically positive since it maps wave functions into wave functions; it is also trace preserving, given that it maps statistical mixtures into statistical mixtures. Now we discuss complete positivity.

A map Λ on $\mathcal{B}(\mathbb{H})$ is completely positive if, for any $M \in \mathbb{N}$, the extended map $I_{\text{M}} \otimes \Lambda$ on $\mathcal{B}(\mathbb{H}_{\text{M}} \otimes \mathbb{H})$ is positive, where I_{M} is the identity map on $\mathcal{B}(\mathbb{H}_{\text{M}})$ and $\mathbb{H}_{\text{M}}$ is a M-dimensional Hilbert space. The Hilbert space $\mathbb{H}_{\text{M}}$ refers to any additional degree of freedom, which is not affected by the considered dynamics. Asking for an ancilla to exist, which is not affected by the considered dynamics, is in principle an additional assumption; anyhow, in all collapse models so far formulated, such an ancilla naturally exists in a strong sense (for example the spin of fermions is not affected by the collapse) and in a weak sense (systems like photons can have an arbitrarily weak coupling to the collapse noise, if their energy is arbitrarily low).

The map $\tilde{T} : \mathbb{H}_{\text{M}} \otimes \mathbb{H} \to \mathbb{H}_{\text{M}} \otimes \mathbb{H}$ must exist, otherwise by simply attaching the system of interest to an ancilla, there would be no dynamics for the wave function anymore; it must also preserve the equivalence among statistical mixtures, for the same reason spelled above. Then $\tilde{T}$ generates the map $\tilde{\Lambda} : \mathcal{B}(\mathbb{H}_{\text{M}} \otimes \mathbb{H}) \to \mathcal{B}(\mathbb{H}_{\text{M}} \otimes \mathbb{H})$ according to:

$$\begin{aligned}
\tilde{\Lambda} : \mathcal{B}(\mathbb{H}_{\text{M}} \otimes \mathbb{H}) &\to \mathcal{B}(\mathbb{H}_{\text{M}} \otimes \mathbb{H}) \\
\tilde{\hat{\rho}} &\to \tilde{\Lambda}\,[\tilde{\hat{\rho}}] = \sum_n \tilde{p}_n \overline{|\tilde{T}\tilde{\psi}_n\rangle\langle\tilde{T}\tilde{\psi}_n|}, \qquad (5)
\end{aligned}$$

with obvious meaning of symbols; again, $\tilde{\Lambda}$ is linear.

Since T on $\mathbb{H}$ is in general nonlinear, its extension $\tilde{T}$ on $\mathbb{H} \otimes \mathbb{H}_{\text{M}}$ (which is also nonlinear) is in general not uniquely determined by T, even if the ancilla does not evolve. However, locality, which is an instance of non-faster-than-light signaling, requires that, for *factorized* states, the map $\tilde{T}$ factorizes into:

$$\begin{aligned}
I_{\text{M}} \times T : \mathbb{H}_{\text{M}} \times \mathbb{H} &\to \mathbb{H}_{\text{M}} \times \mathbb{H} \\
|\phi\rangle \otimes |\psi\rangle &\to |\phi\rangle \otimes |T\psi\rangle \qquad (6)
\end{aligned}$$

(the ancilla could be arbitrarily far away). This map generates the following map among *factorized* density matrices:

$$\begin{aligned} I_{\text{M}} \times \Lambda : \mathcal{B}(\mathbb{H}_{\text{M}}) \times \mathcal{B}(\mathbb{H}) &\rightarrow \mathcal{B}(\mathbb{H}_{\text{M}}) \times \mathcal{B}(\mathbb{H}) \\ \rho_{\text{M}} \otimes \hat{\rho} &\rightarrow (I_{\text{M}} \times \Lambda)\,[\hat{\rho}_{\text{M}} \otimes \hat{\rho}] = \hat{\rho}_{\text{M}} \otimes \Lambda[\hat{\rho}], \end{aligned} \tag{7}$$

which can be extended by linearity to the whole tensor product space, thus defining:

$$I_{\text{M}} \otimes \Lambda : \mathcal{B}(\mathbb{H}_{\text{M}} \otimes \mathbb{H}) \rightarrow \mathcal{B}(\mathbb{H}_{\text{M}} \otimes \mathbb{H}). \tag{8}$$

The two maps $\tilde{\Lambda}$ and $I_{\text{M}} \otimes \Lambda$ are the same map, since they are both linear and coincide on a basis of $\mathcal{B}(\mathbb{H}_{\text{M}} \otimes \mathbb{H})$. Since $\tilde{\Lambda}$ is positive by construction, because it maps statistical mixtures in statistical mixtures, also $I_{\text{M}} \otimes \Lambda$ is. This proves that Λ is completely positive.

As a final note, we point out that in [3] it was shown that also positive but non completely-positive dynamics admit stochastic unravelings. This means that the stochastic unraveling T, which is not associated to a completely-positive dynamics, does not admit an extension $\tilde{T}$ to a larger Hilbert space, like the one discussed here above.

B. Derivation of Eq. (8) in the main text

In this section we derive the formulas for the expectation values of $\hat{p}_j$ and $\hat{p}_j^2$ with respect to $\Phi[\hat{\rho}]$. The reason why we include also the average momentum is that it is relevant for the analysis carried on in appendix C. We start from Eq. (5) in the main text, and we compute

$$\text{Tr}\,\{\hat{p}_j \Phi[\hat{\rho}]\} = \frac{1}{L^3}\int_{-\frac{L}{2}}^{+\frac{L}{2}} d\boldsymbol{x} \sum_k \sum_{\boldsymbol{n}} \langle \boldsymbol{n}|\hat{p}_j \hat{A}_k(\boldsymbol{x})\hat{\rho}\hat{A}_k^\dagger(\boldsymbol{x})|\boldsymbol{n}\rangle =$$

$$= \frac{1}{L^3}\int_{-\frac{L}{2}}^{+\frac{L}{2}} d\boldsymbol{x} \sum_k \sum_{\boldsymbol{n},\boldsymbol{\ell},\boldsymbol{m}} \langle \boldsymbol{n}|\hat{p}_j e^{\frac{i}{\hbar}\hat{\boldsymbol{p}}\cdot\boldsymbol{x}} A_k e^{-\frac{i}{\hbar}\hat{\boldsymbol{p}}\cdot\boldsymbol{x}}|\boldsymbol{\ell}\rangle\langle\boldsymbol{\ell}|\hat{\rho}|\boldsymbol{m}\rangle\langle\boldsymbol{m}|e^{\frac{i}{\hbar}\hat{\boldsymbol{p}}\cdot\boldsymbol{x}} A_k^\dagger e^{-\frac{i}{\hbar}\hat{\boldsymbol{p}}\cdot\boldsymbol{x}}|\boldsymbol{n}\rangle. \tag{9}$$

By exploiting $\hat{\boldsymbol{p}}|\boldsymbol{n}\rangle = (2\pi\hbar/L)\boldsymbol{n}|\boldsymbol{n}\rangle$ we find

$$\text{Tr}\,\{\hat{p}_j \Phi[\hat{\rho}]\} = \frac{2\pi\hbar}{L}\sum_k \sum_{\boldsymbol{n},\boldsymbol{\ell},\boldsymbol{m}} n_j \underbrace{\left(\frac{1}{L^3}\int_{-\frac{L}{2}}^{+\frac{L}{2}} d\boldsymbol{x} e^{\frac{2\pi i}{L}(\boldsymbol{m}-\boldsymbol{\ell})\cdot\boldsymbol{x}}\right)}_{\langle\boldsymbol{\ell}|\boldsymbol{m}\rangle} \langle\boldsymbol{n}|A_k|\boldsymbol{\ell}\rangle\langle\boldsymbol{\ell}|\hat{\rho}|\boldsymbol{m}\rangle\langle\boldsymbol{m}|A_k^\dagger|\boldsymbol{n}\rangle =$$

$$= \frac{2\pi\hbar}{L}\sum_{\boldsymbol{n},\boldsymbol{\ell}} n_j \sum_k |\langle\boldsymbol{n}|A_k|\boldsymbol{\ell}\rangle|^2 \langle\boldsymbol{\ell}|\hat{\rho}|\boldsymbol{\ell}\rangle = \frac{2\pi\hbar}{L}\sum_{\boldsymbol{n},\boldsymbol{\ell}} \widetilde{P}(\boldsymbol{n},\boldsymbol{\ell}) n_j \langle\boldsymbol{\ell}|\hat{\rho}|\boldsymbol{\ell}\rangle \tag{10}$$

where we defined

$$\widetilde{P}(\boldsymbol{n},\boldsymbol{\ell}) := \sum_k |\langle\boldsymbol{n}|A_k|\boldsymbol{\ell}\rangle|^2. \tag{11}$$

Performing the change of variables $\boldsymbol{m} = \boldsymbol{n} - \boldsymbol{\ell}$ and then relabelling $\boldsymbol{\ell} \rightarrow \boldsymbol{n}$ we finally get

$$\text{Tr}\,\{\hat{p}_j \Phi[\hat{\rho}]\} = \text{Tr}\,\{\hat{p}_j \hat{\rho}\} + \frac{2\pi\hbar}{L}\sum_{\boldsymbol{m},\boldsymbol{n}} P(\boldsymbol{m},\boldsymbol{n}) m_j \langle\boldsymbol{n}|\hat{\rho}|\boldsymbol{n}\rangle \tag{12}$$

where

$$P(\boldsymbol{m},\boldsymbol{n}) = \widetilde{P}(\boldsymbol{m}+\boldsymbol{n},\boldsymbol{n}) = \sum_k |\langle \boldsymbol{m}+\boldsymbol{n}|A_k|\boldsymbol{n}\rangle|^2. \tag{13}$$

is precisely Eq. (9) of the main text.

Recalling that $\tilde{m}_j = \frac{2\pi\hbar}{L} m_j$, it follows that:

$$d_{j,\hat{\rho}} := \mathrm{Tr}\left\{\hat{p}_j \Phi[\hat{\rho}]\right\} - \mathrm{Tr}\left\{\hat{p}_j \hat{\rho}\right\} = \sum_{\boldsymbol{m},\boldsymbol{n}} P(\boldsymbol{m},\boldsymbol{n}) \tilde{m}_j \langle \boldsymbol{n}|\hat{\rho}|\boldsymbol{n}\rangle\,, \tag{14}$$

A similar calculation leads to

$$\mathrm{Tr}\left\{\hat{p}_j^2\,\Phi[\hat{\rho}]\right\} = \frac{(2\pi\hbar)^2}{L^2} \sum_{\boldsymbol{n},\boldsymbol{\ell}} \widetilde{P}(\boldsymbol{n},\boldsymbol{\ell}) n_j^2 \langle \boldsymbol{\ell}|\hat{\rho}|\boldsymbol{\ell}\rangle \tag{15}$$

Then again by performing the change of variables $\boldsymbol{m} = \boldsymbol{n} - \boldsymbol{\ell}$ and then relabelling $\boldsymbol{\ell} \to \boldsymbol{n}$ we get:

$$\begin{aligned} \mathrm{Tr}\left\{\hat{p}_j^2\,\Phi[\hat{\rho}]\right\} &= \frac{(2\pi\hbar)^2}{L^2} \sum_{\boldsymbol{m},\boldsymbol{n}} P(\boldsymbol{m},\boldsymbol{n}) \left(m_j + n_j\right)^2 \langle \boldsymbol{n}|\hat{\rho}|\boldsymbol{n}\rangle \\ &= \mathrm{Tr}\left\{\hat{p}_j^2\,\hat{\rho}\right\} + \sum_{\boldsymbol{m},\boldsymbol{n}} P(\boldsymbol{m},\boldsymbol{n}) \tilde{m}_j^2 \langle \boldsymbol{n}|\hat{\rho}|\boldsymbol{n}\rangle + 2\sum_{\boldsymbol{m},\boldsymbol{n}} P(\boldsymbol{m},\boldsymbol{n}) \tilde{m}_j \langle \boldsymbol{n}|\hat{p}_j\hat{\rho}|\boldsymbol{n}\rangle, \end{aligned} \tag{16}$$

which implies

$$D_{j,\hat{\rho}} = \mathrm{Tr}\left\{\hat{p}_j^2\,\Phi[\hat{\rho}]\right\} - \mathrm{Tr}\left\{\hat{p}_j^2\,\hat{\rho}\right\} = \sum_{\boldsymbol{m},\boldsymbol{n}} P(\boldsymbol{m},\boldsymbol{n}) \tilde{m}_j^2 \langle \boldsymbol{n}|\hat{\rho}|\boldsymbol{n}\rangle + 2\sum_{\boldsymbol{m},\boldsymbol{n}} P(\boldsymbol{m},\boldsymbol{n}) \tilde{m}_j \langle \boldsymbol{n}|\hat{p}_j\hat{\rho}|\boldsymbol{n}\rangle. \tag{17}$$

When $d_{j,\hat{\rho}} = 0$ for all $\hat{\rho}$, by taking $\hat{\rho} = |\boldsymbol{n}_0\rangle\langle\boldsymbol{n}_0|$ one finds $\sum_{\boldsymbol{m}} P(\boldsymbol{m},\boldsymbol{n}_0)\tilde{m}_j = 0$ for any $\boldsymbol{n}_0$; In such a case the last term of Eq. (17) is always equal to zero, from which Eq. (8) in the main text follows.

C. General proof of the theorem, with $d_{j,\hat{\rho}} \neq 0$

In the main text, we proved the theorem with the simplifying assumption $d_{j,\hat{\rho}} = 0$. Here we want to generalise the first part of the theorem also to maps that change the average momentum. Typical examples of this kind of maps are dissipative dynamics; in the context of collapse models, dissipative extensions of the Ghirardi-Rimini-Weber (GRW) model, CSL model and QMUPL model were introduced in [4–8]. Non-interferometric tests of the dissipative CSL model were studied in [9].

We will prove the following theorem: consider a map of the form in Eq. (5) in the main text (i.e. fulfilling conditions (*i*) and (*ii*) of the main text); consider the difference in the momentum spread after and before the application of the map i.e.

$$\Delta_{j,\hat{\rho}} := \Delta p_{i,\Phi[\hat{\rho}]} - \Delta p_{i,\hat{\rho}} = D_{j,\hat{\rho}} - d_{j,\hat{\rho}}^2 - 2\langle \hat{p}_j\rangle d_{j,\hat{\rho}} \tag{18}$$

where $\Delta p_{j,\hat{\rho}} = \mathrm{Tr}(\hat{p}_j^2\hat{\rho}) - [\mathrm{Tr}(\hat{p}_j\hat{\rho})]^2$ and $d_{j,\hat{\rho}}$ and $D_{j,\hat{\rho}}$ are defined, respectively, in Eqs. (14) and (17); If any state of the form:

$$\hat{\rho} = |\psi\rangle\langle\psi| \quad \text{with} \quad |\psi\rangle = a|\boldsymbol{n}_0\rangle + b|\boldsymbol{m}_0\rangle \tag{19}$$

(where a, b are generic complex coefficients which satisfy the normalization condition $|a|^2+|b|^2 = 1$) satisfy $\Delta_{j,\hat{\rho}} = 0$ (no spread in the momentum) then the map Φ is such that:

$$\Phi[|\boldsymbol{n}_0\rangle\langle\boldsymbol{n}_0|] = |\boldsymbol{\gamma}(\boldsymbol{n}_0) + \boldsymbol{n}_0\rangle\langle\boldsymbol{\gamma}(\boldsymbol{n}_0) + \boldsymbol{n}_0| \tag{20}$$

with $\boldsymbol{\gamma}(\boldsymbol{n}_0) \in \mathbb{Z}^3$. This implies that this map does not collapse the momentum eigenstates (it just acts on them as a boost) hence it cannot be a satisfactory dynamics for the wave function collapse in space.

Proof. It is convenient to introduce the following notation:

$$\overline{m_j(\boldsymbol{n})} := \sum_{\boldsymbol{m}} P(\boldsymbol{m},\boldsymbol{n})\tilde{m}_j \qquad \overline{m_j^2(\boldsymbol{n})} := \sum_{\boldsymbol{m}} P(\boldsymbol{m},\boldsymbol{n})\tilde{m}_j^2\,, \tag{21}$$

which exploits the fact that, for each $\boldsymbol{n}$, $P(\boldsymbol{m},\boldsymbol{n})$ is a probability distribution of the variable $\boldsymbol{m}$, in such a way that $\overline{m_j(\boldsymbol{n})}$ and $\overline{m_j^2(\boldsymbol{n})}$ represent averages with respect to it.

In this new notation Eqs. (14) and (17) become:

$$d_{j,\hat{\rho}} = \sum_{\boldsymbol{n}} \overline{m_j(\boldsymbol{n})}\langle\boldsymbol{n}|\hat{\rho}|\boldsymbol{n}\rangle, \tag{22}$$

$$D_{j,\hat{\rho}} = \sum_{\boldsymbol{n}} \left(\overline{m_j^2(\boldsymbol{n})} + 2\overline{m_j(\boldsymbol{n})}\tilde{n}_j\right)\langle\boldsymbol{n}|\hat{\rho}|\boldsymbol{n}\rangle. \tag{23}$$

Given a state of the form in Eq. (19) one has:

$$\langle\boldsymbol{n}|\hat{\rho}|\boldsymbol{n}\rangle = |a|^2\delta_{\boldsymbol{n},\boldsymbol{n}_0} + |b|^2\delta_{\boldsymbol{n},\boldsymbol{m}_0} \tag{24}$$

By choosing $a = 1$ and $b = 0$ one gets

$$\Delta_{j,\hat{\rho}} = \overline{m_j^2(\boldsymbol{n}_0)} - \left(\overline{m_j(\boldsymbol{n}_0)}\right)^2 \tag{25}$$

and requiring $\Delta_{j,\hat{\rho}} = 0$ implies:

$$\overline{m_j^2(\boldsymbol{n}_0)} = \left(\overline{m_j(\boldsymbol{n}_0)}\right)^2. \tag{26}$$

We can now compute $\Delta_{j,\hat{\rho}}$ for generic coefficients a and b using condition (26). This leads to:

$$\Delta_{j,\hat{\rho}} = |a|^2|b|^2\left(\overline{m_j(\boldsymbol{n}_0)} - \overline{m_j(\boldsymbol{m}_0)}\right)\left[\left(\overline{m_j(\boldsymbol{n}_0)} - \overline{m_j(\boldsymbol{m}_0)}\right) - 2\left(\tilde{m}_{0j} - \tilde{n}_{0j}\right)\right] \tag{27}$$

and the condition $\Delta_{j,\hat{\rho}} = 0$ implies:

$$\overline{m_j(\boldsymbol{n}_0)} - \overline{m_j(\boldsymbol{m}_0)} = \begin{cases} 0 \\ 2\left(\tilde{m}_{0j} - \tilde{n}_{0j}\right) \end{cases}. \tag{28}$$

In is now convenient to rewrite Eqs. (21) as follows:

$$\overline{m_j(\boldsymbol{n}_0)} = \sum_{m_j} P_j(m_j,\boldsymbol{n}_0)\tilde{m}_j \qquad \overline{m_j^2(\boldsymbol{n}_0)} = \sum_{m_j} P_j(m_j,\boldsymbol{n}_0)\tilde{m}_j^2 \tag{29}$$

where P_j are the marginals defined in Eq. (11) of the main text. As a consequence, condition (26) reads

$$\sum_{m_j} P_j(m_j, \boldsymbol{n}_0) m_j^2 - \left(\sum_{m_j} P_j(m_j, \boldsymbol{n}_0) m_j \right)^2 = 0\,, \tag{30}$$

which implies that each $P_j(m_j, \boldsymbol{n}_0)$ is a distribution with zero variance, i.e.

$$P_j(m_j, \boldsymbol{n}_0) = \delta_{m_j, \gamma_j(\boldsymbol{n}_0)} \tag{31}$$

By replacing this identity in Eq. (28) one finds

$$\overline{m_j(\boldsymbol{n}_0)} - \overline{m_j(\boldsymbol{m}_0)} = \frac{2\pi\hbar}{L} \left(\gamma_j(\boldsymbol{n}_0) - \gamma_j(\boldsymbol{m}_0) \right) = \begin{cases} 0 \\ -2 \frac{2\pi\hbar}{L} \left(n_{0j} - m_{0j} \right) \end{cases} \tag{32}$$

for all $\boldsymbol{n}_0, \boldsymbol{m}_0$. This implies that either:

$$\gamma_j(\boldsymbol{n}_0) = \gamma_j(\boldsymbol{m}_0) \implies \gamma_j(\boldsymbol{n}_0) = \gamma_j \tag{33}$$

or

$$\gamma_j(\boldsymbol{n}_0) - \gamma_j(\boldsymbol{m}_0) = -2 \left(n_{0j} - m_{0j} \right). \tag{34}$$

Since this equation must hold for all $\boldsymbol{n}_0$ and $\boldsymbol{m}_0$, it follows that $\gamma_j(\boldsymbol{n}_0)$ must depend only on the component n_{0j}, which in turn implies

$$\gamma_j(\boldsymbol{n}_0) = \gamma_j - 2n_{0j}\,, \tag{35}$$

with γ_j arbitrary real constants. To summarize, we found that the marginals are equal to:

$$P_j(m_j, \boldsymbol{n}_0) = \begin{cases} \delta_{m_j, \gamma_j} \\ \delta_{m_j, \gamma_j - 2n_{0j}}\,, \end{cases} \tag{36}$$

and since they are Kronecker deltas, the joint distribution is simply

$$P(\boldsymbol{m}, \boldsymbol{n}_0) = \prod_{j=1}^{3} P_j(m_j, \boldsymbol{n}_0)\,. \tag{37}$$

In order to establish how this requirement constrains the Kraus map in Eq. (1) in the main text, we recall that:

$$P(\boldsymbol{m}, \boldsymbol{n}) = \sum_k |\langle \boldsymbol{m} + \boldsymbol{n} | \hat{A}_k | \boldsymbol{n} \rangle|^2 \tag{38}$$

and from all analysis above

$$P(\boldsymbol{m}, \boldsymbol{n}) = \delta_{\boldsymbol{m}, \boldsymbol{\gamma}(\boldsymbol{n})} \quad \text{where} \quad \gamma_j(n_j) = \begin{cases} \gamma_j \\ \gamma_j - 2n_j \end{cases}\,, \tag{39}$$

which imply that for each k

$$|\langle \boldsymbol{m}+\boldsymbol{n}|\hat{A}_k|\boldsymbol{n}\rangle|^2 = c_k(\boldsymbol{n})\delta_{\boldsymbol{m},\boldsymbol{\gamma}(\boldsymbol{n})}\,, \tag{40}$$

with $\sum_k c_k(\boldsymbol{n}) = 1$, $c_k(\boldsymbol{n}) \geq 0$. By decomposing the matrix element as

$$\langle \boldsymbol{m}+\boldsymbol{n}|\hat{A}_k|\boldsymbol{n}\rangle = R_k(\boldsymbol{m},\boldsymbol{n})e^{i\varphi_k(\boldsymbol{m},\boldsymbol{n})}\,, \tag{41}$$

one finds that the condition (40) does not restrict $\varphi_k(\boldsymbol{m},\boldsymbol{n})$ but implies

$$R_k(\boldsymbol{m},\boldsymbol{n}) = \sqrt{c_k(\boldsymbol{n})}\delta_{\boldsymbol{m},\boldsymbol{\gamma}(\boldsymbol{n})}. \tag{42}$$

Accordingly, the Kraus operator can be expressed as:

$$\hat{A}_k = \sum_{\boldsymbol{m},\boldsymbol{n}} |\boldsymbol{m}+\boldsymbol{n}\rangle\langle \boldsymbol{m}+\boldsymbol{n}|A_k|\boldsymbol{n}\rangle\langle \boldsymbol{n}| = \sum_{\boldsymbol{n}} \sqrt{c_k(\boldsymbol{n})}|\boldsymbol{\gamma}(\boldsymbol{n})+\boldsymbol{n}\rangle\langle \boldsymbol{n}|e^{i\varphi_k(\boldsymbol{\gamma}(\boldsymbol{n}),\boldsymbol{n})}\,, \tag{43}$$

which once replaced in Eq. (4) in the main text gives

$$\hat{A}_k(\boldsymbol{x}) = \sum_{\boldsymbol{n}} \sqrt{c_k(\boldsymbol{n})}|\boldsymbol{\gamma}(\boldsymbol{n})+\boldsymbol{n}\rangle\langle \boldsymbol{n}|e^{i\varphi_k(\boldsymbol{\gamma}(\boldsymbol{n}),\boldsymbol{n})}e^{\frac{2\pi i}{L}\boldsymbol{\gamma}(\boldsymbol{n})\cdot\boldsymbol{x}} \tag{44}$$

and the translation covariant CP map reads

$$\Phi[\hat{\rho}] = \frac{1}{L^3}\int_{-\frac{L}{2}}^{+\frac{L}{2}} d\boldsymbol{x} \sum_k \hat{A}_k(\boldsymbol{x})\hat{\rho}\hat{A}_k^\dagger(\boldsymbol{x}) =$$

$$= \sum_{\boldsymbol{n},\boldsymbol{\ell}}\sum_k \sqrt{c_k(\boldsymbol{n})c_k(\boldsymbol{\ell})}e^{i[\varphi_k(\boldsymbol{\gamma}(\boldsymbol{n}),\boldsymbol{n})-\varphi_k(\boldsymbol{\gamma}(\boldsymbol{\ell}),\boldsymbol{\ell})]}\left(\frac{1}{L^3}\int_{-\frac{L}{2}}^{+\frac{L}{2}} d\boldsymbol{x} e^{\frac{2\pi i}{L}[\boldsymbol{\gamma}(\boldsymbol{n})-\boldsymbol{\gamma}(\boldsymbol{\ell})]\cdot\boldsymbol{x}}\right)\langle \boldsymbol{n}|\hat{\rho}|\boldsymbol{\ell}\rangle|\boldsymbol{\gamma}(\boldsymbol{n})+\boldsymbol{n}\rangle\langle\boldsymbol{\gamma}(\boldsymbol{\ell})+\boldsymbol{\ell}|.$$

$$= \sum_{\boldsymbol{n},\boldsymbol{\ell}}\sum_k \sqrt{c_k(\boldsymbol{n})c_k(\boldsymbol{\ell})}e^{i[\varphi_k(\boldsymbol{\gamma}(\boldsymbol{n}),\boldsymbol{n})-\varphi_k(\boldsymbol{\gamma}(\boldsymbol{\ell}),\boldsymbol{\ell})]}\left(\delta_{\boldsymbol{\gamma}(\boldsymbol{n}),\boldsymbol{\gamma}(\boldsymbol{\ell})}\right)\langle \boldsymbol{n}|\hat{\rho}|\boldsymbol{\ell}\rangle|\boldsymbol{\gamma}(\boldsymbol{n})+\boldsymbol{n}\rangle\langle\boldsymbol{\gamma}(\boldsymbol{\ell})+\boldsymbol{\ell}|. \tag{45}$$

Since according to Eq. (39) each $\gamma_j(n_j)$ can take two possible values we have several possibilities. However, this is not really important for us since, if we consider as initial state a plane wave $\hat{\rho} = |\boldsymbol{n}_0\rangle\langle\boldsymbol{n}_0|$, which is the most delocalized state in space, we have:

$$\Phi[\hat{\rho}] = |\boldsymbol{\gamma}(\boldsymbol{n}_0)+\boldsymbol{n}_0\rangle\langle\boldsymbol{\gamma}(\boldsymbol{n}_0)+\boldsymbol{n}_0| \tag{46}$$

which means the dynamics does not collapse plane waves, it just give a boost $\boldsymbol{\gamma}(\boldsymbol{n}_0)$. This conclude our proof.

A final comment about the relation between this theorem and Heisenberg's uncertainty principle might be useful. The spread can be computed either at the wave function level, or at the density matrix level; for a stochastic dynamics—as typical for collapse models—the first case refers to a single realization of the noise, while the second case refers to the average over all realizations, and is the one which is associated to experiments. Here we are interested in this second case.

At the density matrix level, the spread in position in general does not decrease after the collapse (whose main effect is to cancel the off-diagonal elements of the density matrix in the position basis, not the diagonal ones), actually it increases more than what expected by the Schrödinger's dynamics alone [10]. As such, Heisenberg's principle does not require the spread in momentum to increase after the collapse; yet our theorem shows that it must, under the specified assumptions.

Even at the wave function level, it is not true that a collapse in position increases the spread in momentum, unless the wave function starts in a state of minimum uncertainty. What happens in general [4, 11] is that a collapse in position also localizes the wave function in momentum, so that any initial state converges asymptotically to a state with (almost) the minimum uncertainty allowed by Heisenberg's uncertainty principle.

D. Proof of the theorem for a Lindblad dynamics

Here we prove the theorem for the case of CP and space-translation covariant Quantum Dynamical Semigroup $\{\Phi_t, t \geq 0\}$, whose generator is of the Lindblad type [12, 13]. According to Holevo's theorem [14–18], it takes the form $\dot{\hat{\rho}}(t) = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}(t)] + \mathcal{L}[\hat{\rho}(t)]$, with

$$\mathcal{L}[\hat{\rho}(t)] = \int d\boldsymbol{Q} \sum_{j=1}^{\infty} \left(e^{\frac{i}{\hbar}\boldsymbol{Q}\cdot\hat{\boldsymbol{x}}} \hat{L}_j \hat{\rho}(t) \hat{L}_j^\dagger e^{-\frac{i}{\hbar}\boldsymbol{Q}\cdot\hat{\boldsymbol{x}}} - \frac{1}{2}\left\{ \hat{L}_j^\dagger \hat{L}_j, \hat{\rho}(t) \right\} \right), \tag{47}$$

where we used the shorthand notation $\hat{L}_j = \hat{L}_j(\boldsymbol{Q}, \hat{\boldsymbol{p}})$, and such operators satisfy

$$\int d\boldsymbol{Q} \sum_{j=1}^{\infty} |\hat{L}_j(\boldsymbol{Q}, \cdot)|^2 < \infty. \tag{48}$$

We recall that we are considering a single particle or, alternatively, the center of mass of a composite object. The Hamiltonian evolution might change the spread of the particle in momentum, but here we are interested only in the diffusive contribution given by $\mathcal{L}[\hat{\rho}(t)]$.

Similarly to what we did above, we carry out the calculation by confining the system in a box of size L with periodic boundary conditions. This implies that in Eqs. (47) and (48) the variable $\boldsymbol{Q}$ is discrete: $\boldsymbol{Q} = (2\pi\hbar/L)\boldsymbol{\ell} := \tilde{\boldsymbol{\ell}}$; integration over $\boldsymbol{Q}$ is replaced by a sum over $\boldsymbol{\ell} \in \mathbb{Z}^3$ and, as before, the eigenvalues of the momentum operator take only discrete values $\boldsymbol{p} = (2\pi\hbar/L)\boldsymbol{n} := \tilde{\boldsymbol{n}}$ with $\boldsymbol{n} \in \mathbb{Z}^3$.

Then Eq. (47) becomes:

$$\mathcal{L}[\hat{\rho}(t)] = \left(\frac{2\pi\hbar}{L}\right)^3 \sum_{\boldsymbol{\ell}} \sum_{j=1}^{\infty} \left(e^{\frac{i}{\hbar}\tilde{\boldsymbol{\ell}}\cdot\hat{\boldsymbol{x}}} \hat{L}_j \hat{\rho}(t) \hat{L}_j^\dagger e^{-\frac{i}{\hbar}\tilde{\boldsymbol{\ell}}\cdot\hat{\boldsymbol{x}}} - \frac{1}{2}\left\{ \hat{L}_j^\dagger \hat{L}_j, \hat{\rho}(t) \right\} \right) \tag{49}$$

with $\hat{L}_j = \hat{L}_j(\tilde{\boldsymbol{\ell}}, \hat{\boldsymbol{p}})$.

A straightforward calculation, making use of the cyclicity of the trace and of the identity $e^{-\frac{i}{\hbar}\boldsymbol{Q}\cdot\hat{\boldsymbol{x}}} \hat{\boldsymbol{p}} e^{\frac{i}{\hbar}\boldsymbol{Q}\cdot\hat{\boldsymbol{x}}} = \hat{\boldsymbol{p}} + \boldsymbol{Q}$, leads to

$$\mathrm{Tr}\left[\hat{\boldsymbol{p}}\mathcal{L}[\hat{\rho}(t)]\right] = \left(\frac{2\pi\hbar}{L}\right)^3 \sum_{\boldsymbol{\ell}} \sum_{\boldsymbol{n}} f(\tilde{\boldsymbol{\ell}}, \tilde{\boldsymbol{n}}) \tilde{\boldsymbol{\ell}} \langle \boldsymbol{n} | \hat{\rho}(t) | \boldsymbol{n} \rangle \tag{50}$$

and

$$\mathrm{Tr}\left[\hat{\boldsymbol{p}}^2\mathcal{L}[\hat{\rho}(t)]\right] = \left(\frac{2\pi\hbar}{L}\right)^3 \sum_{\boldsymbol{\ell}} \sum_{\boldsymbol{n}} f(\tilde{\boldsymbol{\ell}}, \tilde{\boldsymbol{n}}) (\tilde{\boldsymbol{\ell}}^2 + 2\tilde{\boldsymbol{n}}\cdot\tilde{\boldsymbol{\ell}}) \langle \boldsymbol{n} | \hat{\rho}(t) | \boldsymbol{n} \rangle, \tag{51}$$

where we defined

$$f(\tilde{\boldsymbol{\ell}},\tilde{\boldsymbol{n}}) := \sum_{j=1}^{\infty} |L_j(\tilde{\boldsymbol{\ell}},\tilde{\boldsymbol{n}})|^2, \tag{52}$$

with $L_j(\tilde{\boldsymbol{\ell}},\tilde{\boldsymbol{n}})$ eigenvalues of the $\hat{L}_j$ operators i.e. $\hat{L}_j(\tilde{\boldsymbol{\ell}},\hat{\boldsymbol{p}})|\boldsymbol{n}\rangle = L_j(\tilde{\boldsymbol{\ell}},\tilde{\boldsymbol{n}})|\boldsymbol{n}\rangle$.

Similarly to the proof in the main text, we assume that $\text{Tr}\,[\hat{\boldsymbol{p}}\mathcal{L}[\hat{\rho}(t)]] = 0$ for any $\hat{\rho}(t)$. Accordingly, Eq. (50) implies $\sum_{\boldsymbol{\ell}} f(\tilde{\boldsymbol{\ell}},\tilde{\boldsymbol{n}})\tilde{\boldsymbol{\ell}} = 0$ and Eq. (51) reduces to

$$\text{Tr}\left[\hat{\boldsymbol{p}}^2\mathcal{L}[\hat{\rho}(t)]\right] = \left(\frac{2\pi\hbar}{L}\right)^3 \sum_{\boldsymbol{\ell}}\sum_{\boldsymbol{n}} f(\tilde{\boldsymbol{\ell}},\tilde{\boldsymbol{n}})\tilde{\boldsymbol{\ell}}^2\langle\boldsymbol{n}|\hat{\rho}(t)|\boldsymbol{n}\rangle. \tag{53}$$

The condition of having no diffusion, i.e. $\text{Tr}\left[\hat{\boldsymbol{p}}^2\mathcal{L}[\hat{\rho}(t)]\right] = 0$ for any $\hat{\rho}(t)$, implies

$$\sum_{\boldsymbol{\ell}} f(\tilde{\boldsymbol{\ell}},\tilde{\boldsymbol{n}})\tilde{\boldsymbol{\ell}}^2 = 0. \tag{54}$$

Since $f(\tilde{\boldsymbol{\ell}},\tilde{\boldsymbol{n}})$ is, by definition, a positive function, Eq. (54) implies $f(\tilde{\boldsymbol{\ell}},\tilde{\boldsymbol{n}}) = \lambda(\tilde{\boldsymbol{n}})(2\pi\hbar/L)^{-3}\delta_{\boldsymbol{\ell},0}$ for all $\tilde{\boldsymbol{n}}$ (the factor $(2\pi\hbar/L)^{-3}$ is necessary to obtain a well defined limit when $L\to\infty$ in the final result in Eq. (56)). Given Eq. (52), this implies

$$L_j(\tilde{\boldsymbol{\ell}},\tilde{\boldsymbol{n}}) = |L_j(\tilde{\boldsymbol{\ell}},\tilde{\boldsymbol{n}})|e^{i\varphi_j(\tilde{\boldsymbol{\ell}},\tilde{\boldsymbol{n}})} = \sqrt{\lambda_j(\tilde{\boldsymbol{n}})}\left(\frac{2\pi\hbar}{L}\right)^{-\frac{3}{2}}\delta_{\boldsymbol{\ell},0}e^{i\varphi_j(\tilde{\boldsymbol{n}})} \tag{55}$$

with $\sum_j \lambda_j(\tilde{\boldsymbol{n}}) = \lambda(\tilde{\boldsymbol{n}})$. From this it follows that the Lindbladian in Eq. (49) is of the form:

$$\mathcal{L}[\hat{\rho}(t)] = \sum_{j=1}^{\infty}\left(\hat{L}_j(\hat{\boldsymbol{p}})\hat{\rho}(t)\hat{L}_j^\dagger(\hat{\boldsymbol{p}}) - \frac{1}{2}\left\{\hat{L}_j^\dagger(\hat{\boldsymbol{p}})\hat{L}_j(\hat{\boldsymbol{p}}),\hat{\rho}(t)\right\}\right), \tag{56}$$

with $\hat{L}_j(\hat{\boldsymbol{p}}) = \sum_{\boldsymbol{n}}\sqrt{\lambda_j(\tilde{\boldsymbol{n}})}e^{i\varphi_j(\tilde{\boldsymbol{n}})}|\boldsymbol{n}\rangle\langle\boldsymbol{n}|$. This result is consistent with what found in Eq. (17) in the main text.

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