Introducing Probability Theory through Heuristics: A Laboratory for High School Students

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Abstract: A more motivating way of introducing students to probability theory is through real-world problems. Since heuristics play an important role in our probabilistic judgement, knowing how to avoid incorrect probabilistic reasoning, which leads to several biases, could help students to develop more critical thinking skills. In this paper, we present a different way of introducing students from a linguistic lyceum (high school) in Italy to probability theory. In our laboratory, we introduced students to the three most studied heuristics: representation, availability, and anchoring. The aims of our laboratory were two-fold: (1) motivate the students to learn probability theory through the presentation of the most common mistakes (biases) that are made due to erroneous probabilistic judgements; (2) increase students' awareness of the "tricky" questions that could be present in the national assessment of knowledge of mathematics for grade 10 students. The results of our laboratory show an increase in students' participation in the class activities, as well as increased motivation.

INTRODUCTION

The main goal of probability is to try to quantify uncertainty in order to perform better decision-making, both in our personal life and in our careers (Metz, 2010). Probability is used in various everyday situations—predicting the weather, lotteries, dice games, etc. (Amir & Williams, 1999; Sharma, 2015; Candelario-Aplaon, 2017; Fast, 1997)—and it reflects the applicability and the cross-curricular nature of mathematics. Furthermore, it offers a foundational theory for the development of statistics. Thus, probability is recognized in almost every school curriculum around the world (Capadia & Borovcnik, 1991). The teaching of probability and statistics has become increasingly important, especially in the last few decades, wherein an enormous amount of data is available every day (Metz, 2010). This is also due to the development of so-called "probability literacy", which includes the ability to interpret and critically analyze information that is related to probability and uncertainty, as well as the ability to carry out correct probabilistic reasoning (Gal, 2005, 2009; Borovcnik, 2016).

In the literature, there are several teaching methods and educational resources aimed at introducing probability- and statistics-related topics to students of different ages—from elementary school pupils to college students. Some of the most common resources include physical devices such as dice, coins, spinners, marbles in a bag, and cards which help create game-like scenarios that involve chance (Nilsson, 2014). Teachers using these devices in different experiments have noted an improvement in students' motivation. Konold (1994) presented the approach of teaching probability to high school and college students through real-world problems, which seems to considerably increase their motivation. Introducing real-world applications to mathematics classes is thought to be one of the most effective ways to motivate students (Halpern, 1987; Cai et al., 2020; Konold, 1994, Sheikh, 2019; Gürbüz & Birgin, 2012; Zetterqvist, 2017). There is also growing interest in using computers in the teaching of probability and statistics (Biehler, 1991; Hawkins, 1996; Bell Jr. & Glen, 2008; Sánchez 2002; Hsiao, 2001).

Despite studies highlighting the importance of teaching probability and investigating the most efficient and innovative ways of teaching this particular field of mathematics, both students and adults still have notable difficulties in understanding basic probability terms, concepts, and ideas (Mezhennaya & Pugachev, 2018; Gürbüz & Birgin, 2012).

Firstly, people have inappropriate intuitive convictions or beliefs concerning probability and the likelihood of uncertain events. Tversky and Kahneman (1974) have shown that people of all ages assess the probability of uncertain events using three heuristic principles that reduce the complexity of the task to simpler judgmental operations; however, these heuristics can lead to systematic errors or biases. Their conclusions have been confirmed by some of their other investigations (Kahneman & Tversky, 1972; Kahneman, 2003; Tversky & Kahneman, 1983). The main results of their empirical works show that participants evaluate probabilities of many events not based on mathematical reasoning, but rather with subjective assessments, which are known as "heuristics". These are mental processes which, with little or no reasoning, lead to a relatively quick answer to a (mathematical) problem. The three main heuristics are representativeness, availability, and adjustment or anchoring. In the representativeness heuristic, an occurrence is probable to the extent that it is representative or typical. For instance, when A is highly typical of B, the probability that A belongs to B is judged to be higher than when A is not typical of B. The availability heuristic judges the probability of an event by how easily instances of it can be brought to mind. For example, we assess the probability of getting a specific disease by recalling such occurrences in our memory. The adjustment or anchoring heuristic estimates the probability of an event by starting from an initial value and adjusting it to the final answer. Different starting points lead to different estimates, which are biased towards the initial value. Heuristics can be acquired with experience; however, it is difficult to eradicate them. For instance, Khazanov and Prado

(2010) conducted an experiment where the experimental group was given extra lectures about probability, while the control group did not have such lessons. The authors found that in both groups, some mistakes and misconceptions were still present while dealing with the assessment of probabilities, such as with the toss of a coin. Similar results have been found by Ang and Shahrill (2014).

Fischbein and Gazit (1984) stated that intuitive probabilistic judgments are factors that should not be ignored in the teaching process: if students develop a correct intuitive attitude, it can help them to integrate the corresponding concept. On the other hand, if the intuitive judgements are not correct or acceptable, they should be eliminated and should be replaced by a correct intuitive representation. Hence, teachers have great responsibility to eradicate misleading and false intuitive biases. If the teaching programs do not fulfil this goal, the students will continue to fail due to intuitive biases, even if the conceptual structure (formal theory) has been presented to them (ibidem).

Secondly, a significant difficulty concerns students' low motivation towards learning probability. Motivation refers to a person's desire to pursue a goal or perform a task, which is manifested by their choice of goals and effort in pursuing the goal (Keller, 2007).

In the present paper, we illustrate an alternative approach to teaching probability to high school students that takes into consideration all of the three main difficulties in learning probability, i.e., introducing probability through heuristics. In particular, for a second-year class from a linguistic lyceum (high school), we aimed to analyze the role that heuristics play in probabilistic reasoning and judgement and identify situations where probabilistic misconceptions and biases are most likely to appear. Afterwards, the class faced probability in a rigorous formal way. From the results of our research, we claim that making students aware of the most common biases in intuitive probabilistic judgments may have some indirect effects on the learning of formal probability.

THE LABORATORY

Aims of the Laboratory

Introducing students to formal probability theory can be a very long process, and the time available for teaching this topic is limited (Bognár & Nemetz, 1977). Hence, the time devoted to theory and applications should be constantly adjusted according to the specifics of the schools and classes: in a scientific lyceum, a more theoretical approach should be dominant, while in a linguistic lyceum, heuristic methods could represent a better choice (Bognár & Nemetz, 1977). Following this idea, we decided that for a linguistic lyceum, where students only study mathematics for three hours each week, a heuristic approach could be preferable.

The aim of our laboratory was to introduce students to formal probability theory through a new way that included the study of heuristics. We did so mainly for two reasons:

- The students from the second-year class at the linguistic lyceum were highly demotivated in learning mathematics; we wanted to present a new topic that could be considered closer to the humanities (psychology, in this case) and that could still be connected to a mathematical topic. The first aim of the laboratory was to increase students' motivation towards mathematics, such as by introducing some real-world problems. Halpern (1987) suggests that introducing real-world applications of mathematics is one of the best methods to motivate students. Real-world problems can generate more enthusiasm in introducing probability theory to students than classical considerations about dice and coins (Konold, 1994). Moreover, Konold (1994) states that there are some features that help to maintain students' high interest and help them to promote conceptual development—for instance, the fact that the results are counterintuitive. Finding surprising results that are counterintuitive and "illogical" helps students to be more motivated to understand and learn.
- In the second year of high schools, all grade 10 students are required to take the national assessment of knowledge of mathematics INVALSI, which has the aim of evaluating students' competence in and knowledge of four mathematical topics (similarly to the TIMSS assessments of knowledge): numbers & quantities, geometry, relations & functions, and data & prevision (Quadro di Riferimento, 2017). In particular, among the questions related to the topic "data and prevision", there are some questions that can be solved both by using formal mathematical procedures and by merely using logic. In the class we examined for the study, students often gave the wrong answer due to a bias in their probabilistic judgment. The second aim of our laboratory was thus to give students some basic knowledge about which are the most common mistakes that people make while dealing with a probability-related problem. In particular, we wanted to present the class with the major biases that occur due to erroneous probabilistic judgement. Where needed, we also wanted to eliminate incorrect intuitive biases and replace them with correct reasoning processes (as suggested by Fischbein & Gazit, 1984).

Subjects

The class we worked with had 19 students—6 boys and 13 girls. It was a class from a linguistic lyceum. The boys and girls in the class had an average age of 15.11 years, with a minimum of 14 years old and a maximum of 17. In the classroom, there were two students with specific learning disabilities: one student with dyslexia and one with dyscalculia.

Materials

In order to present the concept of heuristics and their relation to probabilistic judgement, we used several tools, such as PowerPoint presentations about the theory (Figure 1) and worksheets for the applicative part of the laboratory. A more detailed description of the materials used is provided in the procedure subsection.

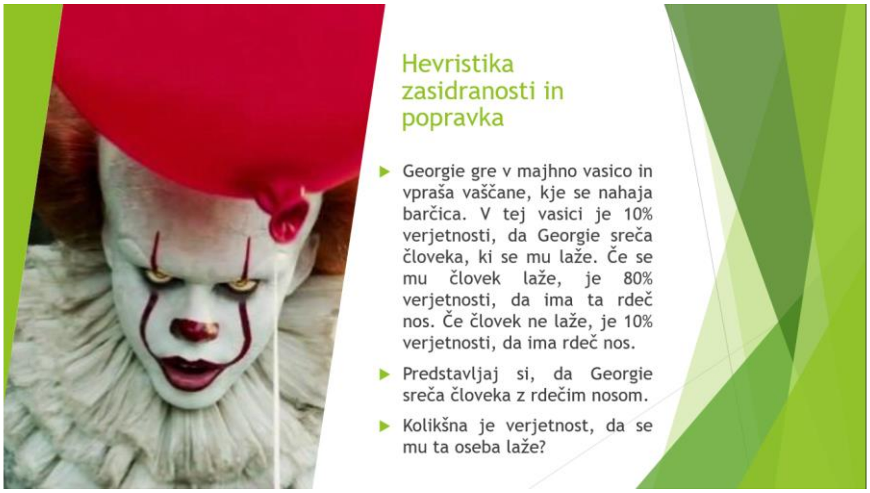

Figure 1: A slide from a PowerPoint presentation with Georgie's problem. A translation into English can be found in Table 11.

Procedure and findings

The total duration of the laboratory was about two class periods (of 50 minutes each), in two different weeks.

The laboratory was structured as a seminary where active participation of the students was required. Fischbein and Gazit (1984) stated that new intuitive attitudes (that are correct) can be developed only if the student is personally involved in a practical activity, since intuitions cannot be successfully modified through verbal explanation alone.

Students were asked some questions (e.g., to present some cases where probability is used to solve real-world problems) or were asked to solve some intuitive probability problems. The students were divided into two groups with equal abilities. The group members had to work together in order to solve a specific problem, using their communicative and argumentative skills. All the problems that we presented were solved by using some intuitive methods; some of them were also solved by using formal mathematical language and procedures.

After a short introduction to what probability is, we adopted the didactical method of brainstorming in order to gauge whether the students knew when probability was used in real-life situations. Students presented various answers (some correct) for where and how to use probability, mentioning, for instance, the lottery and predicting the results of a football game.

To show the two groups that some probability problems are not as easy as they may seem, we asked the students to solve the problem in Table 1.

In a school, students are tested for a disease. At the age of 15, the probability of having that disease is 0.1%. If a certain student was actually ill, the test would be positive 100% of the time; otherwise, if a student didn't have the disease, the test would still be positive 5% of the time. Suppose that a student did the test and that the result was positive. What is the probability that the student had the disease?

Table 1: Introductory problem.

All the answers put forward by the students were wrong. The correct answer to this problem could be determined through a frequentist approach, rather than using a probabilistic one (hence, we used the description "one person in 1000" rather than 0.1%). In the scientific literature, it is described that probabilistic judgement is facilitated when the information is presented in the form of natural frequencies (Slovic et al., 2000).

Before defining what heuristics are, and their role in everyday decision making, we presented a classical example of heuristics judgement (Table 2).

A notebook and a pen cost 1.10€ together. The notebook costs 1€ more than the pen. What's the price of the pen?

Table 2: A classic example of heuristics.

The majority of the students answered "10 cents", which is a very frequent wrong answer, as also found by Kahneman (2003). Kahneman and Frederick (2002) affirm that this problem does activate an intuitive, but wrong, answer, based on the natural division in 1 euro and 10 cents. People do not check whether the result is correct; in order to produce the correct answer, it is necessary to inhibit the intuitive answer and produce the correct answer based on deliberation and reasoning.

We explained to the students why there are so many wrong answers; this permitted us to introduce the concept of heuristics. Heuristics are simple procedures and intuitive strategies to make judgements under conditions of uncertainty. These procedures can generally lead to the formulation of correct and sensate judgements that align with the outcomes of that instance. In other circumstances, which are predictable, heuristics produce biases (i.e., systematic errors) in the judgement. Here, biases are understood as systematic deviations produced by regarding normative criteria for evaluation.

We presented the three principal typologies of heuristics to the two groups of students, as proposed by the research program "Heuristics and Biases" (Kahneman et al., 1992; Gilovich et al., 2002): availability, representation, and anchoring heuristics.

We started by presenting the availability heuristics. In order to introduce this formal concept, we presented an example of this kind of heuristic. Each student received a worksheet with a list of 14 names of well-known personalities of both genders (7 females and 7 males). However, the two groups of students received different lists: the first group (Group A) had a list in which the male personalities were relatively more famous than the female ones, whilst the second group (Group B) had a list in which the female personalities were relatively more famous (see Table 3). “Being famous”, in this context, means that the person is well-known to teenagers because they hear their names frequently in everyday life (e.g., William Shakespeare is more famous to teenagers than Alfred Sisley).

Group A (men more famous)	Group B (women more famous)
Donald Trump	Taylor Swift
Margaret Hamilton	Queen Elizabeth II
Kate Bosworth	Riccardo Patrese
William Shakespeare	Daniel Kahneman
Gigi Buffon	J. K. Rowling
Sofia Vassilieva	Howard Carter
Usain Bolt	Michelle Obama
Emily Brontë	Arthur Ashkin
Elizabeth Cook	Cara Delevingne
Tiziano Ferro	Alfred Sisley
Adam Sandler	Hartley Coleridge
Paola Turci	Ilary Blasi
Naomi Watts	Donatella Versace
Sigmund Freud	Buzz Aldrin

Table 3: The men–women problem.

Each participant had 30 seconds to read the list of the names written on the worksheets. When they were asked whether the list contained names of men or women, the students did not recognize that there was an equal number of people of both genders. Specifically, in each of the lists, the subjects erroneously judged that the gender that had the more famous personalities was the most numerous. These results are consistent with those found by Tversky and Kahneman (1974).

In order to better understand the concept of the availability heuristic, we presented another experiment. Both groups were asked to determine which event from each couple in Table 4 causes more deaths.

a) homicide / diabetes
b) tornado / lightning
c) car accident / stomach cancer
d) terroristic attack / cocaine overdose

Table 4: An example of the availability heuristic.

The answers given by the students of both groups showed the presence of a bias, as also underlined by one of the students: the answers given favored those events that have a stronger presence on different media (Instagram, YouTube, newspapers, etc.). For instance, students judged that terrorist attacks cause more deaths than cocaine overdoses, since terrorist attacks are more likely to appear in newspapers and TV news. These results are also in agreement with those found by Tversky and Kahneman (1974).

The next step was to present and understand the concept of the representativeness heuristic. In order to introduce the students to this heuristic, we started with an adaptation of the problem presented by Tversky and Kahneman (1974) (Table 5).

Steve is very shy and withdrawn, and invariably helpful, but with little interest in people or in the real world. A meek and tidy soul, he has a need for order and structure, and a passion for detail. Which is, among the following, most likely Steve's job? • ice skater • surgeon • librarian • workman • airline pilot

Table 5: Steve's problem (Tversky & Kahneman, 1974)

The students were given 30 seconds to discuss the possible options and answer the question. Both groups answered "librarian"; some students from both groups also added the option "surgeon". These results agree with the observations of Tversky and Kahneman (1974), who affirm that people judge and estimate probability by also considering various stereotypes that do not influence the objective probability of an event occurring. For instance, the probability that Steve is a librarian is assessed by the degree to which he is representative of (or similar to) the stereotype of a librarian.

The next problem that students had to solve was Linda's problem, as shown in Table 6. The problem was presented by Tversky and Kahneman (1983) and it has been revisited and discussed by many other researchers (Charness et al., 2010; Sides et al., 2002; Tentori et al., 2004). The students were asked to order the possible answers in Table 6 in descending order, from the most likely to the least likely job.

Linda is 31 years old, single, outspoken, and very bright. She majored in philosophy. As a student, she was deeply concerned with issues of discrimination and social justice, and also participated in anti-nuclear demonstrations. a. Linda is a teacher in elementary school. b. Linda works in a bookstore and takes Yoga classes. c. Linda is active in the feminist movement. d. Linda is a psychiatric social worker. e. Linda is a member of the League of Women Voters.

f.	Linda is a bank teller.
g.	Linda is an insurance salesperson.
h.	Linda is a bank teller and is active in the feminist movement.

Table 6: Linda's problem (Tversky & Kahneman, 1983)

In this case, there was not a unique correct answer; the aim of the problem was to identify that the statement "Linda is a bank teller and is active in the feminist movement" is less probable than both the statements "Linda is active in the feminist movement" and "Linda is a bank teller". One group did indeed make this judgement.

Furthermore, we presented another example of the representativeness heuristic, depicted in Table 7, which is an adaptation of the Tversky and Kahneman (1974) heads-or-tails problem. The students had to identify which of the dice outcomes was the most likely.

A die has 4 green (G) faces and 2 red (R) faces. We roll the die 20 times and each time we write down the outcome. Which of the following sequences of outcomes is the most likely to happen?
a. RGRRR
b. GRRRRR
c. GRGRRR

Table 7: The dice problem.

We wanted the students to notice that the third option was already included in the first outcome, since it adds only a green outcome at the very beginning. However, students believed that the GRGRRR option was the most likely outcome to occur, since it is a more representative sequence for a random dice toss.

In relation to this activity, we presented another game in which there were a total of 40 cards, with 24 (60%) red cards and 16 (40%) black cards. We took a card from the set 10 times, each time returning it back to the set. We asked the students to write down a sequence of possible outcomes; for each correct guess, the group received a prize.

The aim of this activity was to present the "law of small numbers" (Tversky & Kahneman, 1974): the belief of an individual that a sample (a small group of objects/outcomes) should still include the characteristics of the entire group (Kahneman & Tversky, 1972). The best strategy would be to write always "red", since the card taken is always returned to the set and there are, overall, more red cards. One group of students did write always "red", while the other put 6 "reds" and 4 "blacks", which corresponds to the same proportion of red and black cards in the total set of 40 cards. In order to evaluate whether students understood the connection between the representativeness heuristic and casual events, we presented another problem from Tversky and Kahneman (1974), which is shown in Table 8.

A certain town is served by two hospitals. In the larger hospital, about 45 babies are born each day, and in the smaller hospital, about 15 babies are born each day. As you know, about 50 percent of all babies are boys. However, the exact percentage varies from day to day. Sometimes it may be higher than 50 percent, sometimes lower. For a period of 1 year, each hospital recorded the days on which more than 60 percent of the babies born were boys. Which hospital do you think recorded more such days?

Table 8: The hospital problem (Tversky & Kahneman, 1974).

The next heuristic presented was the anchoring type. In order to present the major problems that could occur while dealing with this kind of reasoning, we started by presenting the following problem from Tversky and Kahneman (1974), which is shown in Table 9. Each group was given a randomly selected whole number between 1 and 100 (in our case, 8 and 60), and then they had to estimate the percentage of African countries in the United Nations. Firstly, the groups had to evaluate whether the percentage was greater or lower than the given number; secondly, they had to make a rough estimation of the real value.

Is the percentage of African countries in the United Nations (UN) greater or smaller than the number you received on the piece of paper? What is the correct percentage?

Table 9: African countries in the UN.

After a minute, the two groups had to present their results, which demonstrated very low agreement with the observations of Tversky and Kahneman (1974). In contrast, the "anchoring" effect was strong in the second activity, which was also presented by Tversky and Kahneman (1974). In this activity, the groups each received one piece of paper with the following numerical expressions:

$$1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8$$

$$8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

The students had 10 seconds to roughly evaluate the value of the given expression. The results agreed with those presented by Tversky and Kahneman (1974): the students that had the expression starting from 8 were more likely to give an answer which was greater than the answer given by the students who had the expression starting from 1.

The next activity was centered on an observation of Goodie and Fantino (1995, 1996), based on ignoring base rates. The problem that was shown to the class is presented in Table 10.

A taxi was involved in a hit and run accident at night. Two taxi companies, green and the blue, operate in the city. - Eighty-five percent of the taxis are green and 15% are blue. - A witness identified the taxi as blue. - The witness identifies the correct color 80% of the time and fails 20% of the time.

What is the probability that the taxi was blue?

Table 10: The taxi problem (Nicholls, 1999).

The problem can be solved by applying Bayes' theorem; however, we instead used a logical and intuitive way to solve the problem, using frequentist language. Since the problem was complex for the participants, we decided to formulate a similar problem in order to evaluate whether the groups understood the mechanism for solving it. This problem is presented in Table 11.

Georgie lives in a small city and he asks its inhabitants where his little paper boat is. In the city, there is a 10% chance of meeting a person that lies. If a person lies, there is an 80% chance that the inhabitant has a red nose (It!). If a person doesn't lie, there is still a 10% chance that he/she has a red nose. - Georgie meets a person with a red nose. What is the probability that Georgie met a liar?

Table 11: Georgie's problem.

RESULTS AND DISCUSSION

While conducting the laboratory, we tried to follow the method proposed by Spitznagel Jr. (1968). We also tried to choose examples and experiments that would be amusing, so that the students would not be bored. The students were active during the whole laboratory and answered all the questions posed to them. Some students also exhibited a very high level of maturity, since they tried to anticipate the result we wanted to obtain. For example, some students recognized the problem of the availability heuristic and affirmed that since Instagram and YouTube present more news about one of the categories, it would be more expected to answer in such a way, even if the correct answer was the other one.

During the laboratory, we also had the opportunity to verify Konold's (1994) proposal that students who encounter some "strange", "illogical", and counterintuitive problems are more motivated to learn and more interested in the subject. Our observations confirm this thesis, since we noticed that while working on Linda's and Steve's problems, where students would choose the most "natural" answer, they were surprised to learn that their answer was not correct. After a short discussion, students learnt why their reasoning was wrong and how their judgement can lead to biases. Indeed, great interest was observed over the whole experiment: students were curious to understand why their reasoning was wrong and were generally very active during discussions.

At the end of the laboratory, the students were asked to undertake both a questionnaire and a test with some exercises that were similar to the ones discussed during the laboratory. The questionnaire consisted of 10 questions with a 5-point Likert scale (1 = don't agree at all; 5 = completely agree), composed in order to measure the effectiveness of the laboratory activity.

Descriptive statistics are presented in Table 12.

Item	Mean	Median	SD
I enjoyed the laboratory activities.	4.28	4.50	.895
I was active during the whole duration of the laboratory activities.	4.33	4	.686
I understood the topics.	4.56	5	.616
I liked the way the topics were presented.	4.61	5	.502
The studied topics were presented in a clear and systematic way.	4.33	4	.686
Having participated in the laboratory, I now enjoy probability more than I did before.	3.61	4	1.33
Having participated in the laboratory, I am now more motivated to learn probability.	3.83	4	1.04
I listened carefully to the explanations.	4.61	5	.502
I wish more math-related activities were like this (i.e., in laboratory form).	4.72	5	.461
Through these activities, I have learnt more and learnt it faster than I would have done in a traditional classroom.	4.33	5	.840
Studying probability is useful for solving real-world problems.	4.17	5	1.10

Table 12: Descriptive statistics for the questionnaire.

As can be deduced from Table 12, the students enjoyed the laboratory, had actively participated in the laboratory activities, and understood the topics. Moreover, they wish that more mathematics topics would be presented in laboratory form, i.e., through practical activities. The students also felt that such laboratory activities made them learn more and learn it faster than traditional classes could. They also recognized that understanding probability is useful for solving several real-world problems. Moreover, students felt more motivated to learn probability and enjoyed it more than they did before. Nevertheless, such results should be interpreted carefully, since they rely solely on students' reported levels of agreement; furthermore, no pre-test was employed in order to compare students' answers before and after the laboratory.

We might conclude that our first goal was reached: after (and during) the laboratory, we noticed that students' motivation in learning mathematics had increased. Students seemed to be more enthusiastic to learn formal probability theory, and we noticed that they tried to connect the learnt concepts with some examples we saw during the laboratory. In particular, when we studied the conjunction (intersection) of events and their probability, the majority of the students recalled Linda's problem and presented it as a concrete example of how the probability of the intersection of events is less than or equal to the probabilities of the single events.

After the questionnaire, the students took a test of probability knowledge, which consisted of nine questions aimed at measuring their understanding of heuristics and probability. The results of the test showed that the majority of the students learnt which heuristic is involved in each problem; moreover, they were able to solve simple tasks without biases. However, they demonstrated greater difficulty in solving the two problems that required a frequentist approach and a "numerical answer" (problems similar to the "Taxi" and "Georgie" problems).

In Table 13, we present the frequencies of correct and incorrect answers to these two questions.

Question	Correct answers	Incorrect answers
Similar to the Taxi problem*	6 (33.3%)	12 (66.7%)
Similar to Georgie's problem**	4 (22.2%)	14 (77.8%)

*In a city, there are two bus companies: St. Andrew's and St. Barbara's.

- Seventy percent of the buses are St. Andrew's and the rest are St. Barbara's.
- Maria thought she saw a St. Andrews' bus.
- However, Maria identifies the correct company 90% of the time and fails 10% of the time.

What is the probability that the bus was St. Andrew's?

**A new infectious disease is spreading across a small city. There is a 15% chance of becoming infected. If a person becomes infected, there is a 75% chance that they will get headaches. If a person is not infected, there is still a 5% chance of having headaches. In a hospital, there is a person with a headache. What is the probability that the patient is infected with the disease?

Table 13: The problems from the test similar to the "Taxi" and "Georgie" problems, along with statistics for the responses.

Another question that demonstrated greater difficulties among the students is presented in Table 14. With this question, we wanted to assess the students' understanding of the "law of small numbers".

Lindersday is a multinational company which works in different fields. In the field "Analysis", 70% of the employees have a degree in philosophy, while 30% of the employees have a degree in law. We randomly chose 10 employees in the field "Analysis" and asked them to introduce themselves. For each of the following descriptions, decide whether it is more likely that the employee has a degree in philosophy or law. a. I'm Bryan and I'm 42. I live with my cat, Universe. I'm very motivated, I like my job a lot, and my colleagues appreciate my work. b. Hello, I'm Matthew. I'm 32 and I live in the city center, because I like to live near my company and it is easier for me to get around during the day. I'm very active and I like to try new and exotic things. c. My name is Lara. My relatives still live in Greece and I love poetry. I've been working for this company for 3 years and I have a good relationship with my boss. d. I'm Robert. I'm 42. I'm married, but I live by myself because my wife is way too disorganized and messy; I can't stand mess. I like to be with my friends because I like to talk with them about everything, especially about yoga. e. Hi, my name in Karla. I'm 39 years old and I'm more of a morning person. I'm good at my job because I'm very competitive; I'm also quite aggressive. f. Hello, I'm Stephany. I'm a feminist, vegan, and pacifist. My colleagues describe me as a very strong, independent woman, but I would describe myself as rather a sentimental person. g. Good morning, I'm Barbara. I'm 24. I'm the youngest employee in the company. I think that age is just a number written on a document, since I know many 80-year-old people that feel like teenagers; they are full of energy and are not afraid of death. On the other hand, I also know… h. My name is Hasan. I'm 55 years old and I have a PhD. I'm very meticulous and I fight for the things I believe in. I like people, which is the reason I chose this job. I can't stand injustice! i. I'm Paul and I'm 37. I'm very relaxed and before sleeping, I always think about things and meditate because it helps me purify my soul. j. Hello, I'm Carol. I like reading newspapers and criminal novels. I'm very good at my job and I'm the best employee in the company for sure. I have a lot of friends.

Table 14: The "Lindersday" problem from the test.

We expected that the students would recognize this problem as another version of Steve's problem, so that they would not judge a person's job from their description but would instead consider just the probabilities. Contrary to what we expected, the majority of students (61.1%) labeled each different person with the job that would "most suit their character", instead of considering that the probability of having an employee with a degree in philosophy is higher. Thus, with the exception of a small number of students, the class overall confirmed the results of Tversky and Kahneman (1974) about the law of small numbers. When the students were asked to show their work and to explain their reasoning, they stated that since

70% of employees had a degree in philosophy, there should be more or less 7 philosophers among the 10 random chosen employees and, consequently, more or less 3 law graduates. In Table 15, we present the frequencies of correct and incorrect answers to these two questions.

Question	Correct answers	Incorrect answers
The Lindersday problem	7 (38.9%)	11 (61.1%)

Table 15: Statistics for responses to the "Lindersday" problem.

In contrast, the students achieved excellent results for the other exercises, proving that they understood where biases are present and how to avoid some typical heuristic mistakes. For instance, all the students correctly solved an exercise similar to the "notebook and pen" problem (Kahneman, 2003) and a problem similar to the "deadliest" problem (i.e., the availability heuristic problem, Tversky & Kahneman, 1974); see Table 16 for details.

Question	Correct answers	Incorrect answers
Notebook and pen*	17 (94.4%)	1 (5.6%)—answer not given
Deadliest**	18 (100.0%)	0 (0.0%)
*A laptop and a smartphone together cost 1200€; the laptop costs 200€ more than the smartphone. What is the cost of the smartphone?		
**From the following pairs of events, identify which is more likely to result in a high number of victims. a) Volcano eruption / car accident b) Plane crash / train crash c) Drowning / ship sinking d) Falling from stairs / skydiving		

Table 16: The "Notebook and pen" and "Deadliest" problems from the test.

Concerning our second goal, i.e., that the students become more aware of "tricky" questions that might appear in the national assessment of mathematical knowledge INVALSI, we can only arrive at a partial assessment. Since the results of the INVALSI assessment of knowledge of mathematics were not available at the time of the research, we cannot state that the laboratory had a positive effect on students' performance for it. However, we were able to evaluate whether the laboratory influenced students' intuitive probabilistic judgements. Overall, the results of the test show that the students still had some problems with specific tasks that required multi-step and out-of-the-box thinking, especially while dealing with more complex probabilistic problems. Nevertheless, the students were able to solve the simpler problems that required probabilistic judgements, and did so without biased reasoning based on heuristics.

Hence, we might conclude that the second goal of the laboratory was partially fulfilled. Since no pre-test was implemented in order to evaluate the students' initial knowledge of probability and heuristics, we cannot state whether they have eliminated their erroneous biases; this assessment is made merely by observing students' answers during and after the laboratory.

CONCLUSIONS AND FUTURE WORK

An introduction to probability theory through a heuristic (intuitive) approach is favorable for classes in the linguistic lyceums of Italy. A formal and axiomatic introduction to probability theory could be useful while dealing with students from a scientific lyceum; in other high schools, where students are less motivated toward studying mathematics, an intuitive introduction could represent a better choice (see Bognár & Nemetz, 1977).

In this work, we wanted to present the idea of a laboratory for probability theory focused on heuristics. In particular, we wanted to introduce students to what heuristics are, where they are present in our everyday life, and why they can represent an obstacle while dealing with probabilistic judgements. Moreover, we wanted the students to understand which types of heuristics are "more dangerous", and how to avoid them in order to produce correct probabilistic reasoning.

There were two main aims of our laboratory. Firstly, we wanted to motivate our class and make them appreciate probability. This first goal was fully achieved: we noticed an increase in the students' participation in class and their motivation was also positively impacted. The students were very active in all parts of the laboratory, interacting with the speakers and asking different questions related to the presented topics. We also noticed that many students tried to explain the observed phenomena and why they occur. Our second goal was to prepare students for the national assessment of knowledge of mathematics INVALSI, which is a mandatory assessment for all grade 10 students in Italy. In particular, some questions in these examinations are related to data analysis, probability, and statistics ("Data and prevision", Quadro di Riferimento, 2017). Some of the questions can evoke biases; when students are not able to produce correct probabilistic reasoning, they try to give the most intuitive answer, which can oftentimes be incorrect. Hence, we wanted to increase students' awareness of different types of mistakes that are commonly associated with heuristic judgements. Regarding this second goal, we cannot state whether it has been achieved since the results of the INVALSI examinations are presented in an aggregated form, which means that the progress of a single student cannot be monitored. On the other hand, the results of the test did demonstrate positive feedback. However, since we did not make the students take a pre-test, we cannot assume that their probabilistic judgement has been modified by our laboratory.

Considering the final remarks from the class and their results for the final questionnaire, we might state that introducing formal probability theory through a laboratory focused on heuristics seemed to be a good choice for the class analyzed in this work. Through observation, we determined an increase in students' motivation and participation: students were considerably interested in the new topic and, at the same time, surprised about the outcomes of the experiments and scenarios posed to them.

The laboratory represents, from our point of view, a good way to introduce students to probability theory. Indeed, our results confirm Konold's (1994) ideas: starting from real-world problems and trying to analyze them seems to be more motivating for students (see also Halpern, 1987); moreover, unexpected and counterintuitive results can help the teacher to maintain a high level of interest from the students. Even if the results of our laboratory have shown an increase in motivation and interest toward probability theory, further research should be conducted in order to understand whether the proposed laboratory could represent a better way of introducing students to an intuitive (and heuristic) approach to probability. In particular, future research should aim to determine whether the proposed laboratory could help to eradicate erroneous probabilistic judgements based on a heuristic reasoning, replacing these ways of thinking with correct interpretations of posed problems.

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