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Hodge-Elliptic Genera, K3 Surfaces and Enumerative Geometry

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Abstract. K3 surfaces play a prominent role in string theory and algebraic geometry. The properties of their enumerative invariants have important consequences in black hole physics and in number theory. To a K3 surface, string theory associates an Elliptic genus, a certain partition function directly related to the theory of Jacobi modular forms. A multiplicative lift of the Elliptic genus produces another modular object, an Igusa cusp form, which is the generating function of BPS invariants of $\mathrm{K3}\times E$. In this note, we will discuss a refinement of this chain of ideas. The Elliptic genus can be generalized to the so-called Hodge-Elliptic genus which is then related to the counting of refined BPS states of $\mathrm{K3}\times E$. We show how such BPS invariants can be computed explicitly in terms of different versions of the Hodge-Elliptic genus, sometimes in closed form, and discuss some generalizations.

Contents

1. Introduction

String theory compactifications on K3 surfaces underlie a series of connections between gravitational physics, number theory and geometry. The Elliptic genus, a certain partition function in conformal field theory, transforms as a Jacobi modular form [39,47]; its coefficients are related to the irreducible representations of the Mathieu group [23]; its multiplicative lift, a certain Siegel modular form, counts BPS invariants which have the interpretation as black hole microstates in supergravity [18,20]. See Refs. [3,38,44] for topical reviews.

In this note, we will focus on the connection with enumerative geometry and investigate how it is modified when the Elliptic genus is replaced by the Hodge-Elliptic genus, introduced in Kachru and Tripathy [32]. In particular, we will focus on its relation with certain refined enumerative invariants of $\mathrm{K3} \times E$, with E an elliptic curve, see also Cheng et al. [13]. The Hodge-Elliptic genus for K3 is invariant under complex structure deformations, and therefore, its multiplicative lift should lead to objects invariant under such deformations.

The Hodge-Elliptic genus is, however, not invariant over the K3 moduli space, but jumps at specific jumping loci. The origin of these jumping phenomena, different from wall-crossing, is that the Hodge-Elliptic genus is not an index and is therefore sensitive to the appearance of boson-fermion pairs which would leave an index invariant. Remarkably such an object is often computable in closed form.

In this note, we will concentrate on three specific definitions of such refined partition functions: the generic, the orbifold and the complex Hodge-Elliptic genera. The generic Hodge-Elliptic genus was introduced in Wendland [46] and is conjectured to capture physically the refined partition function at a generic point in the moduli space. It can be computed under reasonable assumptions using conformal field theory methods and geometrically reflects the use of the chiral de Rham complex to model twisted sigma models [35]. The orbifold Hodge-Elliptic genus was introduced in Kachru and Tripathy [32] and captures the physics at an orbifold point. It can be computed in closed form and counts extra states with respect to the generic Hodge-Elliptic genus. The complex Hodge-Elliptic genus was also introduced in Kachru and Tripathy [32] as a refinement of the usual Elliptic genus, by replacing its geometric formulation as the Euler characteristic of a certain bundle, with the weighted sum of the dimension of certain cohomology groups. Also, this partition function contains extra states with respect to the generic Hodge-Elliptic genus. While we could not determine precisely the locus in the K3 moduli space where such states appear, it is reasonable to expect that an object invariant under complex structure deformations does indeed occur physically as a refinement of the Elliptic genus at some point of the moduli space. We therefore include it in our analysis, and indeed one of our results is an explicit closed form for the complex Hodge-Elliptic genus. In the process of doing so we also present an explicit computation of the ordinary Elliptic genus, by direct integration from its definition, which highlights the geometric origin of each of its terms. The closed-form computation of the complex Hodge-Elliptic genus can be extended to certain twining genera.

The Elliptic genus of a K3 surface determines the enumerative geometry of reduced Donaldson–Thomas invariants of K3 $\times$ E. We investigate how this relation is generalized in the case of the refined partition functions. Following Kachru and Tripathy [32], we conjecture that the resulting generating functions capture a refinement of ordinary Donaldson–Thomas theory. We exhibit closed-form formulas for the first terms of the generating functions in all cases, and a prescription to compute higher-order terms order by order. Interestingly, the specific form of the generic Hodge-Elliptic genus suggests that the moonshine phenomenon should be visible geometrically in the refined Donaldson–Thomas theory.

Ordinary Donaldson–Thomas theory is blind to the moonshine phenomenon due to a subtle interplay between the coefficients of the Elliptic genus, which we discuss in the text. It has been suggested in Kachru and Tripathy [36] that the situation could be different in the refined theory. However, a closer investigation reveals that this is not the case [27] for the generating function introduced in Katz et al. [36], but leaves the door open for higher-order generating functions. Our results seem to imply that this is indeed the case.

This paper is organized as follows. Section 2 contains a brief review of certain aspects of the geometry of K3 surfaces, of Elliptic genera and modular forms, and of enumerative geometry. We also connect these aspects explicitly, by showing how the terms in the elliptic genus count certain curves and how this counting is reflected in its (mock) modularity. In Sect. 3, we discuss Hodge-Elliptic genera and quickly review the orbifold and generic refined partition sums. We also present an explicit closed-form derivation of the complex Hodge-Elliptic genus; in order to do so we revisit the Elliptic genus and compute it explicitly from its integral definition. In doing so, we elucidate the geometric interpretation of each term as it arises from the characters of the symmetric product of the tangent bundle to the K3. Finally, we show explicitly that each Hodge-Elliptic genus admits a refined decomposition in terms of the holomorphic characters of the superconformal algebra. Section 4 contains an explicit derivation in closed form of certain terms in a refinement of Donaldson–Thomas theory, predicted using the Hodge-Elliptic genera. Higher-order terms can be derived with a similar prescription order by order. Finally, Sect. 5 extends the computation of the complex Hodge-Elliptic genus to certain twined case, obtained by working equivariantly with respect to certain finite-order symplectic automorphisms of K3 surfaces. The last Sect. 6 summarizes our findings and offers some discussions of open problems.

Two appendices contain some technical results. The supporting MATHEMATICA file contains most of the computations hereby discussed [16].

2. K3 Surfaces, the Elliptic Genus and Enumerative Geometry

In this section, we review some aspects of the relation between K3 surfaces and the elliptic genus. We will show how known facts about the Elliptic genus can be put in direct relation with certain Donaldson–Thomas invariants.

2.1. Some Geometric Aspects of K3 Surfaces

Here, we review some facts about K3 and geometry which will be used along the paper

Recall that for S a K3 surface $H^1(S,\mathcal{O}_S)=0$ and $\Omega_S^2 \simeq \mathcal{O}_S$. The cotangent sheaf Ω_S is locally free and of rank two, and its determinant is trivial $\mathcal{K}_S \simeq \mathcal{O}_S$. Furthermore, we have the isomorphism $\mathcal{T}_S \simeq \Omega_S$. The Hodge numbers are given by $h^{p,q}(S) = \dim H^q(S,\Omega_S^p)$. In particular $h^{p,q}=1$ for $(p,q)=(2,0),(0,2),(0,0),(2,2)$, $h^{1,1}=20$, and all the other Hodge numbers are vanishing. Note that $h^0(S,\Omega_S) = h^0(S,\mathcal{T}_S)=0$ and there are no global nontrivial vector fields on S.

For a rank r sheaf $\mathcal{E}$ on S, the splitting principle allows us to write the Chern polynomial as

$$c(\mathcal{E};\mathtt{t}) = \prod_{i=1}^{r} (1+a_i\,\mathtt{t}) \ , \tag{2.1}$$

with $\mathtt{t}$ a formal parameter. The Chern character is given by

$$\mathrm{ch}(\mathcal{E}) = r + c_1(\mathcal{E}) + \frac{1}{2}\left(c_1(\mathcal{E})^2 - 2c_2(\mathcal{E})\right) = \sum_{i=1}^{r} \mathrm{e}^{a_i} \ , \tag{2.2}$$

and the Todd class by

$$\mathrm{Todd}(\mathcal{E}) = 1 + \frac{1}{2}c_1(\mathcal{E}) + \frac{1}{12}\left(c_1(\mathcal{E})^2 + c_2(\mathcal{E})\right) = \prod_{i=1}^{r} \frac{a_i}{1-\mathrm{e}^{-a_i}} \, . \tag{2.3}$$

The Euler characteristic of a sheaf $\mathcal{E}$ can be computed via the Riemann–Roch theorem

$$\chi(\mathcal{E}) = \int \mathrm{ch}(\mathcal{E})\,\mathrm{Todd}(\mathcal{T}) \, , \tag{2.4}$$

where $\mathcal{T}$ is the holomorphic tangent space to S. This theorem can be used to compute geometrical quantities associated with the K3. For example, from the fact that $\chi(\mathcal{O})=2$ we deduce an expression for the integral of the second Chern class

$$\chi(\mathcal{O}) = 2 = \int \mathrm{ch}(\mathcal{O})\,\mathrm{Todd}(\mathcal{T}) = \frac{1}{12}\int c_2(\mathcal{T}) \, . \tag{2.5}$$

The tangent sheaf of a K3 surfaces satisfies

$$c(\mathcal{T};\mathsf{t}) = (1 + x\,\mathsf{t})(1 - x\,\mathsf{t}) = 1 - x^2\,\mathsf{t} \tag{2.6}$$

and therefore

$$\int x^2 = -\int c_2(\mathcal{T}) = -24\,, \tag{2.7}$$

a fact that we will use extensively when computing the Elliptic genus.

2.2. The Elliptic Genus

The definition of the Elliptic genus is rooted in conformal field theory. When the string propagates in a Calabi–Yau variety, the worldsheet theory is described by a $\mathcal{N} = 2$ superconformal algebra. This algebra consists of two copies, the left-moving and the right-moving sectors. If we let L_0 and J_0 denote, respectively, the zero modes of the Virasoro and R-symmetry current generators, the Elliptic genus is defined as a trace over the RR sector

$$\mathsf{Ell}(\tau, z) = \mathrm{Tr}_{RR}\left((-1)^{J_0+\overline{J}_0}\, y^{J_0}\, q^{L_0-\frac{c}{24}}\, \overline{q}^{\overline{L}_0-\frac{\overline{c}}{24}}\right)\,. \tag{2.8}$$

We have introduced the notation $q = \mathrm{e}^{2\pi\,\mathrm{i}\,\tau}$ and $y = \mathrm{e}^{2\pi\,\mathrm{i}\,z}$. Due to supersymmetry, this quantity is independent of $\overline{\tau}$ and depends holomorphically on τ and z. Physically the Elliptic genus counts RR states which are in the ground state in the left-moving sector and are unconstrained in the right-moving sector. It can be understood as computing the dimensions of certain cohomology groups graded by the L_0 and J_0 quantum numbers. The existence of an inner automorphism of the superconformal algebra, the spectral flow, implies that the Elliptic genus is a weak Jacobi form of weight zero and index $m = \frac{c}{6}$, where c is the central charge. The spectral flow symmetry also implies that Fourier expansion of the Elliptic genus has the structure

$$\mathsf{Ell}(\tau, z) = \sum_{n\in\mathbb{Z}_+,\,\ell\in\mathbb{Z}} c\,(n,\ell)\; q^n\, y^\ell = \sum_{n\in\mathbb{Z}_+,\,\ell\in\mathbb{Z}} c\left(4\,m\,n - \ell^2\right)\, q^n\, y^\ell \tag{2.9}$$

since the combination of operators which is invariant under spectral flow is precisely $4\,m\,L_0 - J_0^2$.

In the case of a K3 surface, supersymmetry is enhanced and the Elliptic genus is a weak Jacobi form of weight zero and index 1. The space of these forms is one-dimensional and generated by the form $\phi_{0,1}(\tau, z)$. Explicitly the Elliptic genus can be written as

$$\mathsf{Ell}(\tau, z) = 2\,\phi_{0,1}(\tau, z) = \frac{\theta_1(\tau, z)^2}{\eta(\tau)^3}\left[24\,\mu(\tau, z) + H(\tau)\right]\,. \tag{2.10}$$

Here,

$$\theta_1(\tau, z) = \mathrm{i}\, q^{\frac{1}{8}} y^{-\frac{1}{2}} \prod_{n=1}^{\infty} (1 - q^n)(1 - q^{n-1}y)(1 - q^n y^{-1})\,, \tag{2.11}$$

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)\,. \tag{2.12}$$

are a theta function and the Dedekind eta function. Furthermore, we have introduced the Appell–Lerch sum

$$\mu(\tau,z) = \frac{-\,\mathrm{i}\,y^{\frac{1}{2}}}{\theta_1(\tau,z)} \sum_{\ell=-\infty}^{+\infty} \frac{(-1)^\ell y^\ell\, q^{\ell(\ell+1)/2}}{1-y\,q^\ell}\,, \tag{2.13}$$

and the function

$$\begin{aligned} H(\tau) &= \frac{-2E_2(\tau)+48F_2^{(2)}}{\eta(\tau)^3} = q^{-\frac{1}{8}}\left(-2+\sum_{n=1}^{\infty} c_H(n) q^n\right) \\ &= 2q^{-\frac{1}{8}}\left(-1+45q+231q^2+770q^3+\cdots\right)\,, \end{aligned} \tag{2.14}$$

where $E_2(\tau)$ is the weight two (quasi-modular) Eisenstein series

$$E_2(\tau) = 1-24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}\,, \tag{2.15}$$

and

$$F_2^{(2)}(\tau) = \sum_{\substack{r-s=1 \bmod 2\\ r>s>0}} (-1)^r\, q^{\frac{rs}{2}}\,. \tag{2.16}$$

The function $H(\tau)$ is a mock modular form with shadow $24\,\eta(\tau)^3$. This means that

$$\widehat{H}(\tau) = H(\tau) + 24(4\,\mathrm{i}\,)^{-1/2}\int_{-\overline{\tau}}^{\infty}(z+\tau)^{-1/2}\,\overline{\eta(-\overline{z})}\mathrm{d}z \tag{2.17}$$

is a weight 1/2 modular form for $SL(2;\mathbb{Z})$. Similarly, $\mu(\tau,z)$ is a mock modular form with shadow $-\eta(\tau)^3$, so that

$$\widehat{\mu}(\tau,z) = \mu(\tau,z) - (4\,\mathrm{i}\,)^{-1/2}\int_{-\overline{\tau}}^{\infty}(z+\tau)^{-1/2}\,\overline{\eta(-\overline{z})}\mathrm{d}z \tag{2.18}$$

is a weight 1/2 modular form for $SL(2;\mathbb{Z})$. A striking fact about (2.10) is that the relative coefficients are such that the shadows of the two mock modular forms cancel exactly and the full Elliptic genus is an ordinary Jacobi form. See [18] for a more complete discussion.

Geometrically, the Elliptic genus is the holomorphic Euler characteristics of a certain sheaf, constructed out of a sheaf $\mathcal{E}$ on a variety Y. If we denote $r = \operatorname{rank}\mathcal{E}$ and $d = \dim_{\mathbb{C}} Y$, then

$$\mathsf{Ell}(Y;\tau,z) = \mathrm{i}^{\,r-d} q^{(r-d)/12}\, y^{-r/2}\chi\left(Y,\mathbb{E}\right)\,, \tag{2.19}$$

where explicitly

$$\mathbb{E} = \bigotimes_{n=1}^{\infty}\Lambda_{-yq^{n-1}}\mathcal{E}\otimes\bigotimes_{n=1}^{\infty}\Lambda_{-y^{-1}q^n}\mathcal{E}^\vee\otimes\bigotimes_{n=1}^{\infty}\mathcal{S}_{q^n}\mathcal{T}_Y\otimes\bigotimes_{n=1}^{\infty}\mathcal{S}_{q^n}\,\mathcal{T}_Y^*\,, \tag{2.20}$$

and we have introduced the notation

$$\Lambda_q\,\mathcal{E} = \bigoplus_{k=0}^{d} q^k\,\Lambda^k\,\mathcal{E}\,, \qquad \mathcal{S}_q\,\mathcal{E} = \bigoplus_{k=0}^{\infty} q^k\,\mathcal{S}^k\mathcal{E}\,. \tag{2.21}$$

We will often drop the explicit dependence on Y from the notation when it is clear from the context. In the following, we will take $\mathcal{E} = \mathcal{T}_Y$ the holomorphic tangent bundle. In the case of a K3 surface, we will simply call this $\mathcal{T}$. By using the Riemann–Roch theorem, the Elliptic genus can be written as

$$\mathsf{Ell}(Y;\tau,z) = \frac{1}{y}\int \mathrm{ch}(\mathbb{E})\,\mathrm{Todd}(\mathcal{T})\,. \tag{2.22}$$

The Elliptic genus captures information about certain BPS states in compactifications of the type II string on $\mathrm{K3}\times E$ Dijkgraaf et al. [20], where E is an elliptic curve. These are states corresponding to the partition function of the second quantized string theory on $\mathrm{K3}\times E$. The argument of [20] identifies such a partition function with the partition function of a single string which propagates on the symmetric product. The latter is expressed as

$$\sum_{n=0}^{\infty} p^n\,\mathsf{Ell}\left(\mathsf{Hilb}^n(\mathrm{K3});\tau,z\right) = p\,\frac{\varphi_{10,1}(\tau,z)}{\Phi_{10}(\sigma,z,\tau)} \tag{2.23}$$

in terms of the Siegel modular form of weight 10, computed as a multiplicative lift of the Elliptic genus

$$\Phi_{10}(\sigma,z,\tau) = p\,q\,y \prod_{(n,m,s)>0} (1-p^n\,y^s\,q^m)^{c(n\,m,s)}\,, \tag{2.24}$$

where the product is over all $s\in\mathbb{Z}$ and $n,m\geq 0$ such that one of the following two conditions holds

- $n>0$ or $m>0$,
- $n=m=0$ and $s<0$.

Note that the function (2.24) is symmetric in the exchange $p\leftrightarrow q$. Finally, the Jacobi form $\varphi_{10,1}(\tau,z)$ which appears in (2.23) can also be thought of as the generating function of χ_y genera

$$\begin{aligned}\sum_{n=0}^{\infty} q^n\,\chi_{-y}\left(\mathsf{Hilb}^n(\mathrm{K3})\right) &= \prod_{k=1}^{\infty}\frac{1}{(1-y\,q^k)^2\,(1-q^k)^{20}\,(1-q^k y^{-1})^2}\\ &= q\,\frac{y-2+y^{-1}}{\varphi_{10,1}(\tau,z)}\end{aligned} \tag{2.25}$$

with

$$\chi_y(Y) = \sum_{p,q}(-1)^q h^{p,q}(Y) y^p\,. \tag{2.26}$$

2.3. Donaldson–Thomas Theory of K3 $\times$ E

Donaldson–Thomas theory is the mathematical formalism underlying BPS states formed by bound states of D-branes with electric and magnetic charges. In its simplest version, one considers bound states of a single D6 brane with a gas of D2-D0 branes on a Calabi–Yau X. These configurations are parametrized by the Hilbert scheme

$$\mathrm{Hilb}^{C,n}(X) = \{Z\subset X\,|\,[Z]=C\in H_2(X),\ n=\chi(\mathcal{O}_Z)\}\,. \tag{2.27}$$

The Donaldson–Thomas invariant is defined via virtual integration over this moduli space

$$\mathtt{DT}_{C,n}(X) = \int_{[\mathrm{Hilb}^{C,n}(X)]^{\mathrm{vir}}} 1 \ . \tag{2.28}$$

In the case at hand $X = \mathrm{K3} \times E$, with E an elliptic curve. This is the main example that we will consider and therefore we shall often omit the specification of the variety X from the notation. In this case, one can parametrize the curve C as $C = \beta + dE$, where β is (the push-forward of) a primitive curve in the K3 with $\beta^2 = \langle \beta, \beta \rangle = 2h - 2$, in terms of the intersection pairing. In this case, all the relevant invariants vanish; this follows from the localization formulas, since the torus E acts on itself, and therefore on the moduli space, freely. Physically this vanishing is related to the presence of an extra fermionic zero mode $\mathrm{K3} \times E$ compactifications.

To avoid this problem, the relevant moduli space is the quotient of the Hilbert scheme $\mathsf{Hilb}^{C,n}(\mathrm{K3} \times E)/E$. Then, one can introduce (reduced) BPS invariants as $\mathtt{DT}_{h,d,n}(X) := \mathtt{DT}_{\beta+dE,n}(X)$ corresponding to the Hilbert scheme $\mathrm{Hilb}^{h,d,n}(X) := \mathrm{Hilb}^{\beta+dE,n}(X)/E$. It is useful to encode this information in the generating function

$$\mathtt{DT}(X) = \sum_{h=0}^{\infty} \mathtt{DT}_h(X) p^{h-1} = \sum_{h,d \geq 0, n \in \mathbb{Z}} \mathtt{DT}_{h,d,n}(X) p^{h-1} q^{d-1} (-y)^n \tag{2.29}$$

where we have introduced the counting parameter $p = \mathrm{e}^{2\pi\,\mathrm{i}\,\sigma}$.

The counting parameters have the following physical interpretation. BPS states on X are parametrized by a certain vector of electric and magnetic charges. However, due to T-duality this information is redundant, and only three T-duality invariant scalar combinations of the charges are physical, as reviewed, for example, in Sen [44]. By an appropriate T-duality rotation, we can parametrize the T-duality invariant charges as

$$Q^2 = q_a C^{ab} q_b = 2h - 2 \qquad P^2 = -2q_1 p^0 = -2d \qquad Q \cdot P = p^0 q_0 = n \tag{2.30}$$

which are then the parameters which enter in the generating function (2.29). Here, $p^0 = 1$ is the D6 brane charge, $q_0 = n$ the number of D0 branes and the remaining q_1 and q_a with $a = 2, \ldots, 23$ are the charges of D2 branes wrapping the elliptic curve or 2-cycles in the K3, respectively. Here, C^{ab} is the signature $(3, 19)$ intersection product on $H_2(\mathrm{K3})$. Note that in the generating function (2.29) the functions $\mathtt{DT}_h$ have fixed electric charge Q, which is different from what one normally does in the physics literature. The geometrical reason for this is that it is easier to count invariants by fixing the curve class in the K3 and summing over the covering of the elliptic curve than vice versa. Ultimately this should be just a matter of conventions since the generating function is symmetric in the exchange $p \leftrightarrow q$.

The generating function of reduced Donaldson–Thomas invariants is conjecturally given by Bryan [8]

$$\mathtt{DT}(\mathrm{K3}\times E)=-\frac{1}{\Phi_{10}(\sigma,z,\tau)} \tag{2.31}$$

in terms of the Siegel modular form (2.24). See also [9,10]. The Gromov–Witten formulation of this conjecture [37,42] was proven in [43].

In particular, Bryan proves Bryan [8]

$$\begin{aligned}\mathtt{DT}_0(\mathrm{K3}\times E)&=-\frac{1}{q}\,\frac{y}{(1-y)^2}\prod_{m=1}^{\infty}(1-q^m)^{-20}\\ &\quad(1-y\,q^m)^{-2}(1-y^{-1}\,q^m)^{-2}=-\frac{1}{\varphi_{10,1}(\tau,z)}\,, \end{aligned}\tag{2.32}$$

$$\begin{aligned}\mathtt{DT}_1(\mathrm{K3}\times E)&=-\frac{24}{q}\prod_{m=1}^{\infty}(1-q^m)^{-24}\\ &\quad\left(\frac{1}{12}+\frac{y}{(1-y)^2}+\sum_{d=1}^{\infty}\sum_{k|d}k(y^k-2+y^{-k})\,q^d\right)\,,\end{aligned}\tag{2.33}$$

by a direct toric localization computation. This is quite remarkable since K3 is not a toric manifold. However, the relevant moduli space admits a stratification where each stratum is separately toric, even if the torus action does not glue to a globally defined action. These results, as we will see momentarily, give a very good control on the precise geometric interpretation of every BPS state.

In particular, since the sum in (2.23) starts from $n=0$ and since $\mathsf{Hilb}^1(K3)=K3$, we find

$$\mathsf{Ell}(\mathrm{K3};\tau,z)=\frac{\mathtt{DT}_1}{\mathtt{DT}_0}\,. \tag{2.34}$$

We will now use this fact to derive a few non-trivial identities.

Noting that

$$\frac{\theta_1^2(\tau,z)}{\eta^3(\tau)}=-q^{\frac{1}{8}}\frac{(1-y)^2}{y}\prod_{m=1}^{\infty}\frac{(1-y\,q^m)^2(1-q^m\,\frac{1}{y})^2}{(1-q^n)}\,, \tag{2.35}$$

we can then rewrite the Elliptic genus as

$$\begin{aligned}\mathsf{Ell}(\tau,z)&=\left(\frac{\theta_1^2}{\eta^3}\right)\left(-\frac{24}{\eta^3}\right)\left(\frac{1}{12}+\frac{y}{(1-y)^2}+\sum_{d=1}^{\infty}\sum_{k|d}k(y^k-2+y^{-k})\,q^d\right)\\ &=\left(\frac{\theta_1^2}{\eta^3}\right)\left(-\frac{24}{\eta^3}\right)\wp(\tau,y)\end{aligned}\tag{2.36}$$

where dove $\wp(\tau,y)$ is the Weierstrass $\wp$-function. Let us consider this expression term by term and compare it with (2.10). It is easy to see that

$$\sum_{d=1}^{\infty}\sum_{k|d}k(-2)\,q^d=\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}k(-2)\,q^{m\,k}=-2\sum_{n=1}^{\infty}\frac{n\,q^n}{(1-q^n)} \tag{2.37}$$

and therefore

$$-\frac{24}{\eta^3(\tau)}\left(\frac{1}{12}+\sum_{d=1}^{\infty}\sum_{k|d}k(-2)\,q^d\right)$$
$$=-\frac{24}{\eta^3(\tau)}\left(\frac{1}{12}-2\sum_{n=1}^{\infty}\frac{nq^n}{(1-q^n)}\right)=-2\frac{E_2(\tau)}{\eta^3(\tau)}\,. \tag{2.38}$$

By comparing with the known expression of the Elliptic genus, we deduce the fairly non-trivial identity

$$-\frac{24}{\eta^3(\tau)}\left(\frac{y}{(1-y)^2}+\sum_{d=1}^{\infty}\sum_{k|d}k(y^k+y^{-k})\,q^d\right)$$
$$=24\,\mu(q,y)+24\,\frac{2F_2^{(2)}(\tau)}{\eta^3(\tau)} \tag{2.39}$$

which we will use in the following. Note that for this identity to hold it is crucial that the $2F_2^{(2)}(\tau)/\eta^3(\tau)$ term cancels the y-independent terms in the Appell–Lerch sum (2.13), since in the left-hand side every term in the q-expansion has a y-dependent coefficient. The above relation can also be proven indirectly using modularity, by using the fact that the ratio of Jacobi forms $\varphi_{0,1}/\varphi_{-2,1}$ is proportional to the Weierstrass $\wp$-function, see Dabholkar et al. [18], for example.

The function $H(\tau)$ (2.14), whose coefficient encodes the dimensions of certain irreducible representations of the Mathieu group $\mathbb{M}_{24}$, does not appear to have an independent geometrical interpretation. This fact underlies the difficulty in finding a geometric interpretation of the Mathieu group representations. We will discuss more aspects of the geometry of curves in the target space $\mathrm{K3}\times E$ momentarily.

2.4. Enumerative Aspects of Jacobi Forms

We will now give an example of what kind of information we can gain from the explicit expression of the enumerative invariants (2.32). It was shown in Dabholkar et al. [18] that the coefficients of the expansion of $1/\Phi_{10}$ at fixed magnetic charges are meromorphic Jacobi forms. These can be decomposed into a finite part, a certain finite linear combination of classical theta series where the coefficients are mock modular forms, and a polar part, that is completely determined by the poles of the original meromorphic Jacobi form. Physically the finite part counts microstates associated with immortal black holes, defined over all the moduli space, while the polar part captures the degeneracies associated with two centered black holes. While this decomposition is perfectly clear from the point of view of modular forms, it begs for a geometric explanation. For example, one can ask what is the enumerative content of single-centered black holes, or if we can give a precise mapping between enumerative invariants and black hole microstates. In other words, given a certain Donaldson–Thomas invariant, possibly in terms of a moduli space of schemes, how can we know in purely geometrical terms if it corresponds to a single center black hole or not?

In our case the situation is slightly different, since we are considering the coefficients of the expansion of $1/\Phi_{10}$ at fixed electric charges; nevertheless, Φ_{10} is symmetric under this exchange and the results of Dabholkar et al. [18] formally still hold. Such an expansion is somewhat unnatural from the physical point of view and possibly needs a clearer interpretation. (Possibly it holds in a different region of the moduli space). Nevertheless, we will show explicitly such decomposition for $\mathtt{DT}_1$.

Before addressing the issue, let us clarify the geometrical content of the generating functions (2.32), following [8]. Consider $X = K3 \times E$ and let $p_{1,2}$ be the projections onto the first and second factors. Then, a curve C is called *vertical* if its projection $p_2 : C \longrightarrow E$ has degree zero, so that the curve lies in $K3 \times pt$. Similarly a curve C is called *horizontal* if the projection $p_1 : C \longrightarrow K3$ has degree zero. A curve which is neither vertical nor horizontal is called *diagonal*. This classification determines a decomposition

$$\mathrm{Hilb}^{h,d,n}(X) = \mathrm{Hilb}^{h,d,n}_{\mathrm{vert}}(X) \times \mathrm{Hilb}^{h,d,n}_{\mathrm{diag}}(X)\,. \tag{2.40}$$

This decomposition follows from the fact that since β is irreducible, any subscheme Z will have a unique component which is either diagonal or vertical, while all the other components are horizontal. In particular, if $h = 0$, only vertical components are possible.

Let us now focus on the $\mathtt{DT}_1$ contribution. Consider the vertical part in the decomposition (2.40). In the case $h = 1$, we have $\beta^2 = 0$. Equivalently the K3 are elliptically fibered and β is the class of the fiber. The fibration $S \longrightarrow \mathbb{P}^1$ has 24 singular fibers. As proven in Bryan [8], the computation of the BPS invariants receives contributions from subschemes whose unique vertical component is a smooth fiber:

$$-\,22 \prod_{m=1}^{\infty} (1-q^m)^{-24}\ , \tag{2.41}$$

where 22 is the Euler characteristics of the base $\mathbb{P}^1$ minus 24 points, and from subschemes whose unique vertical component is a nodal fiber:

$$24 \prod_{m=1}^{\infty} (1-q^m)^{-24} \left[1 + \frac{y}{(1-y)^2} + \sum_{d=1}^{\infty} \sum_{k|d} k(y^k + y^{-k})\, q^d \right] . \tag{2.42}$$

To these, one must add the contribution from the diagonal curves

$$24 \prod_{m=1}^{\infty} (1-q^m)^{-24} \sum_{d=1}^{\infty} \sum_{k|d} (-2k)\, q^d \tag{2.43}$$

These three terms give a concrete geometrical interpretation of (2.33). We will now show how precisely these terms contribute to the black hole partition function. It follows from Dabholkar et al. [18] that we expect that each $\mathtt{DT}_h$

splits into a polar part

$$\mathtt{DT}_h^P = \frac{p_{24}(h+1)}{\eta^{24}(\tau)} \sum_{s\in\mathbb{Z}} \frac{q^{hs^2+s}\, y^{2hs+1}}{(1-q^h\, y)^2}\,, \tag{2.44}$$

and a finite part $\mathtt{DT}_h^F = \mathtt{DT}_h - \mathtt{DT}_h^{(P)}$. Indeed in our case

$$\mathtt{DT}_1 = \mathtt{DT}_0\, \mathsf{Ell} = -2\frac{\varphi_{0,1}}{\varphi_{10,1}} = -2\frac{\varphi_{0,1}}{\varphi_{-2,1}}\frac{1}{\eta(\tau)^{24}}\,. \tag{2.45}$$

Then, we have (from Dabholkar et al. [18] eq 8.54; both $\varphi_{0,1}$ and $\varphi_{-2,1}$ here are defined with the opposite sign as there)

$$\frac{\varphi_{0,1}}{\varphi_{-2,1}} = 12\mathrm{Av}^{(0)}\left[\frac{y}{(1-y)^2}\right] + E_2(\tau)\,, \tag{2.46}$$

where

$$\mathrm{Av}^{(m)}[f(y)] = \sum_k q^{mk+2}\, y^{2km} f(q^k y) \tag{2.47}$$

is the averaging operator. In our case, it is easy to see that

$$\mathrm{Av}^{(0)}\left[\frac{y}{(1-y)^2}\right] = \sum_{s\in\mathbb{Z}} \frac{q^s y}{(1-q^s y)^2} = \frac{y}{(y-1)^2} + \sum_{d=1}^{\infty}\sum_{k|d} k\left(y^k + y^{-k}\right)\, q^d \tag{2.48}$$

We conclude that

$$\mathtt{DT}_1^{(P)} = -\frac{24}{\eta(\tau)^{24}}\,\mathrm{Av}^{(0)}\left[\frac{y}{(1-y)^2}\right] \tag{2.49}$$

$$\mathtt{DT}_1^{(F)} = -\frac{2}{\eta(\tau)^{24}}\, E_2(\tau) \tag{2.50}$$

Now, we have an explicit enumerative interpretation of the Jacobi form $\mathtt{DT}_1$: The BPS states counted by the finite part correspond to the diagonal curves (2.43) as well as those curves whose vertical component is a smooth fiber (2.41) or a nodal curve without points, that is, the y^0 term in (2.42). The rest of the vertical curves in (2.42) are associated with the polar part.

This example shows that by carefully analyzing the interplay between modularity and geometry we can give a very explicit description of the BPS states.

3. Hodge-Elliptic Genera on K3 Surfaces

In this section, we introduce various Hodge-Elliptic genera for K3 surfaces, following [32, 46], and discuss some of their properties. The Hodge-Elliptic genus is invariant under complex structure deformations but depends sensitively on the Kähler structure. The orbifold Hodge-Elliptic genus [32] corresponds to K3 surfaces which are resolutions of a certain quotient, while the generic conformal field theoretic Hodge-Elliptic genus [46] is conjectured to capture the large radius conformal field theory. Both are explicitly computable. We also discuss the complex Hodge-Elliptic genus [32] and express it in closed form by using representation theory arguments. In passing we also clarify the structure of each contribution to the Elliptic genus from bundles on K3.

3.1. Refined Partition Functions

Counting functions of BPS states can usually be refined by keeping track of additional quantum numbers. For example, the function $1/\varphi_{10,1}$ can be interpreted as counting 1/4-BPS states in type II string compactifications on $\mathrm{K3}\times E$ with fixed electric charge and transforming in a certain spin representation of $\mathrm{SU}(2)_L$. The function $\varphi_{10,1}$ admits a refinement which was derived by Katz, Klemm and Pandharipande [36]

$$\begin{aligned}\phi_{KKP}(\tau,z,\nu) = q\,&(y^{\frac{1}{2}}u^{\frac{1}{2}}-y^{-\frac{1}{2}}u^{-\frac{1}{2}})(y^{-\frac{1}{2}}u^{\frac{1}{2}}-y^{\frac{1}{2}}u^{-\frac{1}{2}})\\ &\times\prod_{m=1}(1-q^m)^{20}(1-q^m u\,y)(1-q^m u\,y^{-1})\\ &\times(1-q^m y\,u^{-1})(1-q^m u^{-1}y^{-1}),\end{aligned} \tag{3.1}$$

where we have introduced the new counting parameter $u=\mathrm{e}^{2\pi\,\mathrm{i}\,\nu}$. Physically $1/\phi_{KKP}(\tau,z,\nu)$ refines the counting of $1/\varphi_{10,1}$ in the sense that the fugacity u keeps track of the spin representation now in $\mathrm{SU}(2)_R$. This function has also a geometrical interpretation as

$$\begin{aligned}\sum_{n\geq 0}\chi_{\mathrm{Hodge}}\,(\mathsf{Hilb}^n\mathrm{K3})\,q^n &= \frac{q\,(u-y-y^{-1}+u^{-1})}{\phi_{KKP}(\tau,z,\nu)}\\ &=\prod_{k=1}^{\infty}(1-q^k)^{-20}(1-u\,y\,q^k)^{-1}(1-u^{-1}y\,q^k)^{-1}\\ &\quad\times(1-u\,y^{-1}q^k)^{-1}(1-u^{-1}y^{-1}q^k)^{-1}\end{aligned} \tag{3.2}$$

where the Hodge polynomial for a variety Y is given by

$$\chi_{hodge}(Y)=u^{-d/2}\,y^{-d/2}\sum_{p,q}(-u)^q\,(-y)^p\,h^{p,q}(Y) \tag{3.3}$$

For a K3 surface, $\chi_{\mathrm{Hodge}}(\mathrm{K3})=1/(uy)+y/u+20+u/y+y\,u$, which reduces to $\chi_y(\mathrm{K3})=2\,y+20+2/y$ for $u=1$.

Precisely as the generating function $\varphi_{10,1}$ admits the refinement ϕ_{KKP}, also the elliptic genus and the associated generating functions can be refined. Kachru and Tripathy [32] introduce the conformal field theory Hodge-Elliptic genus as a refinement of the Elliptic genus

$$\begin{aligned}\mathsf{H\text{-}Ell}(\tau,z,\nu) &= \mathrm{Tr}\left((-1)^{J_0+\overline{J}_0}\,y^{J_0}u^{\overline{J}_0}q^{L_0-\frac{c}{24}}\right)\\ &=\sum_{n,s,m}c\,(n,\ell,m)\,q^n y^\ell u^m\,.\end{aligned} \tag{3.4}$$

The trace is taken only over states which are arbitrary in the left-moving sector but in a Ramond ground state in the right-moving sector. Such a quantity takes its name from the fact that its first term in the q expansion is the Hodge polynomial of the K3. It is, however, not a genus in the strict mathematical sense. (For example, it is non-vanishing on the torus.) This quantity is moduli dependent and jumps over the Calabi–Yau moduli space, for example, at those points where the left chiral CFT algebra is enhanced.

At first sight, (3.4) is similar to (2.8) and the variable u appears to play a role similar to the variable y. Therefore, it is tempting to promote it to an elliptic variable and try to interpret (3.4) as a Jacobi form with two elliptic variables. However, this symmetry is not really there, due to the fact that the trace is taken only over those states which are in a Ramond ground state in the right-moving sector. As a consequence, the range of values of the variable m in (3.4) is restricted to a finite number (we will see that $m = -1, 0, 1$ with the appropriate normalization) and (3.4) is only a Laurent polynomial in u. This argument, however, does not rule out that u could transform in a more complicated way under the modular group. In this paper, we take the simplest route and assume that u is invariant under the modular group.

The Hodge-Elliptic genus can also be given a geometric definition on a variety Y using the geometric realization of the Elliptic genus as the Euler characteristic of the sheaf (2.20) [32]:

$$
\begin{aligned}
\mathsf{H\text{-}Ell}^c(Y;\tau,z,\nu) = {} & \mathrm{i}^{\,r-d} q^{(r-d)/12}\, y^{-r/2}\, u^{-d/2} \\
& \sum_{j=0}^{r} (-u)^j \dim H^j \left(Y, \bigotimes_{n=1}^{\infty} \Lambda_{-yq^{n-1}} \mathcal{E} \otimes \bigotimes_{n=1}^{\infty} \Lambda_{-y^{-1}q^n} \mathcal{E}^\vee \otimes \bigotimes_{n=1}^{\infty} \right. \\
& \left. \mathcal{S}_{q^n} \mathcal{T}_X \otimes \bigotimes_{n=1}^{\infty} \mathcal{S}_{q^n}\, \mathcal{T}_X^* \right) .
\end{aligned}
\tag{3.5}
$$

As for the Elliptic genus, we will only consider the case where Y is a K3 surface and $\mathcal{E} = \mathcal{T}$ its holomorphic tangent bundle. We will refer to this as the complex Hodge-Elliptic genus. In general, a direct computation of the Hodge-Elliptic genus will give different results when carried out at different points of the moduli space, as we will discuss momentarily. Remarkably in certain cases the Hodge-Elliptic genus is known in closed form. A direct computation of (3.4) was done in Kachru and Tripathy [32] at a certain orbifold point and in Wendland [46] in the strict large radius limit. We will refer to these as $\mathsf{H\text{-}Ell}^{\mathrm{orb}}$ and $\mathsf{H\text{-}Ell}^{g}$; when writing $\mathsf{H\text{-}Ell}$ without any further specification we refer to properties which hold for each of the Hodge-Elliptic genera described so far.

The main interest of this paper is how Hodge-Elliptic genera are related to enumerative geometry. As shown in Kachru and Tripathy [32], essentially following the argument of Dijkgraaf et al. [20] one can define a generating function

$$
\begin{aligned}
\sum_{k=0}^{\infty} p^k\, \mathsf{H\text{-}Ell}\left(\mathrm{Hilb}^{[k]}(\mathrm{K3})\right) &= \prod_{r>0,s\geq 0,t,v} \frac{1}{(1-q^s y^t p^r u^v)^{c(r\,s,t,v)}} \\
&= \frac{p\,\phi_{KKP}(\tau,z,\nu)}{\Phi^{\mathrm{ref}}(\tau,z,\nu,\sigma)} \, .
\end{aligned}
\tag{3.6}
$$

The authors of Kachru and Tripathy [32] conjecture that this $\Phi^{\mathrm{ref}}(\tau,z,\nu,\sigma)$ is the generating function of motivic/refined Donaldson–Thomas invariants.

Since it is defined in terms of the Hodge-Elliptic genus, its precise form will depend on the Calabi–Yau moduli. In this note, we will study various $\Phi^{\mathrm{ref}}(\tau, z, \nu, \sigma)$ corresponding to different version of the Hodge-Elliptic genus and use it to make prediction for enumerative invariants. These matters will be discussed in Sect. 4.

3.2. BPS Jumping Loci

The flavored partition function (3.4) is not an index and can exhibit jumping behavior at certain points in the moduli space [33,34]. This behavior is generically different from wall-crossing and is due to the unprotected nature of the new partition function. For example, the indexed count can differ from the flavored partition function at generic points in the moduli space due to the presence of boson-fermion pairs (where the statistics refers to the fermion number appearing in the index, so typically we really are talking about different multiplets) whose contribution to the Elliptic genus cancels exactly. Additionally, an extra chiral current can appear at special points in the moduli space, leading to more state appearing (always in Bose–Fermi pairs to leave the indexed count invariant). A physical interpretation could be that for special loci in the moduli space some particles in the full spectrum are now annihilated by a certain supercharge (or a combination thereof). Another difference with wall-crossing is that jumping loci are typically of higher codimension.

For example, the moduli space of type IIA compactification on K3 surfaces has the form of a locally symmetric space

$$\mathcal{M}(p,q) = O(p,q;\mathbb{Z})\backslash O(p,q;\mathbb{R})/\left(O(p)\times O(q)\right) \tag{3.7}$$

which can be understood as the moduli space of lattices $\Gamma^{p,q}$ of signature (p,q). By adopting this perspective, the appearance of extra chiral currents corresponds to loci in the moduli space where the lattice generated by a collection of k vectors becomes purely left moving, corresponding to subvarieties of the form $\mathcal{M}(p,q-k)\subset\mathcal{M}(p,q)$ and called *special cycles*. In algebraic geometry, such loci where the rank of the lattice changes abruptly are known as Noether–Lefschetz loci, or generalization thereof, and are often Shimura varieties. Remarkably the formal generating functions of these loci, that is sums whose coefficients are special cycles, are in certain cases (mock) modular forms valued in $H^\bullet(\mathcal{M}(p,q))$ [34].

3.3. The Orbifold Hodge-Elliptic Genus

In Kachru and Tripathy [32], the Hodge-Elliptic genus is computed at a certain point in the moduli space where the K3 is a resolution of the quotient of $\mathbb{T}^4$ (a product of two square tori with unit volume) by the $\mathbb{Z}_2$ inversion. Using CFT techniques, one finds explicitly[1]

$$\mathsf{H}\text{-}\mathsf{Ell}^{\mathrm{orb}}(K3) = 8\left[-\left(\frac{\theta_1(\tau,z)}{\theta_1^*(\tau,0)}\frac{u^{-1/2}-u^{1/2}}{2}\right)^2+\left(\frac{\theta_2(\tau,z)}{\theta_2(\tau,0)}\frac{u^{-1/2}+u^{1/2}}{2}\right)^2\right.$$

[1]The sign difference with Kachru and Tripathy [32] is due to a different convention in the definition of $\theta_1(\tau, z)$.

$$+\left(\frac{\theta_3(\tau,z)}{\theta_3(\tau,0)}\right)^2+\left(\frac{\theta_4(\tau,z)}{\theta_4(\tau,0)}\right)^2\Bigg] \tag{3.8}$$

with $\theta_1^*(\tau,0)=-2q^{1/8}\prod_{n=1}^{\infty}(1-q^n)^3$. We can rewrite this as

$$\begin{aligned}\mathsf{H\text{-}Ell}^{\mathrm{orb}}(K3) &= 2\left(\frac{1}{u}+u-2\right)\left(-\left(\frac{\theta_1(\tau,z)}{\theta_1^*(\tau,0)}\right)^2+\left(\frac{\theta_2(\tau,z)}{\theta_2(\tau,0)}\right)^2\right)+2\,\varphi_{0,1}(\tau,z)\\ &= 2\left(\frac{1}{u}+u-2\right)\left(\frac{1}{4}\varphi_{-2,1}(\tau,z)+\left(\frac{\theta_2(\tau,z)}{\theta_2(\tau,0)}\right)^2\right)+2\,\varphi_{0,1}(\tau,z)\end{aligned} \tag{3.9}$$

Define the function[2] $\Lambda_N(\tau)\in M_2(\Gamma_0(N))$ as

$$\Lambda_N(q)=N\frac{\mathrm{d}}{\mathrm{d}q}\log\left(\frac{\eta(q^N)}{\eta(q)}\right)=\frac{N}{24}\left(NE_2(N\tau)-E_2(\tau)\right) \tag{3.10}$$

Then, using the identity [21]

$$\left(\frac{\theta_2(\tau,z)}{\theta_2(\tau,0)}\right)^2=\frac{1}{12}\,\varphi_{0,1}(\tau,z)+2\,\Lambda_2(\tau)\,\varphi_{-2,1}(\tau,z) \tag{3.11}$$

one can write

$$\begin{aligned}\mathsf{H\text{-}Ell}^{\mathrm{orb}}(K3) &= \left(\frac{5}{3}+\frac{1}{6u}+\frac{u}{6}\right)\varphi_{0,1}(\tau,z)\\ &\quad-\left(1-\frac{1}{2u}-\frac{u}{2}\right)(1+8\Lambda_2(\tau))\,\varphi_{-2,1}(\tau,z) \qquad (3.12)\\ &= \frac{1}{24}\left(\frac{2}{u}+20+2u\right)\mathsf{Ell}(\tau,z)\\ &\quad+\left(1-\frac{1}{2u}-\frac{u}{2}\right)\frac{\theta_1(\tau,z)^2}{\eta(\tau)^6}\,(1+8\Lambda_2(\tau)) \qquad (3.13)\end{aligned}$$

which has the structure of the sum of two Jacobi forms with coefficients which are u-dependent and a weight 2 modular form on $\Gamma_0(2)$.

3.4. The Generic Hodge-Elliptic Genus

The generic conformal field theory Hodge-Elliptic genus corresponds to the partition function (3.4) computed in the infinite volume limit. It was computed in Wendland [46] by a careful analysis of the space of ground states and of the representation theory of the superconformal algebra. The result is

$$\mathsf{H\text{-}Ell}^{g}(\tau,z,\nu)=\mathsf{Ell}(\tau,z)+\left(2-\frac{1}{u}-u\right)\frac{\theta_1(\tau,z)^2}{\eta(\tau)^3}\left[q^{-1/8}-2\mu(\tau,z)\right] \tag{3.14}$$

where the second term on the right-hand side is proportional to the character $\mathrm{ch}_{\frac{1}{4},\frac{1}{2}}(\tau,z)$ of the superconformal algebra, which will, however, appear later

[2]Here $M_2(\Gamma_0(N))$ denotes the space of weight 2 modular forms on $\Gamma_0(N)$, the congruence subgroup of the modular group of level N. Also, recall that $\Gamma(N)\subset\Gamma_1(N)\subset\Gamma_0(N)\subset\mathrm{SL}(2;\mathbb{Z})$ and $\Gamma(N')\subset\Gamma(N)$ whenever $N|N'$. If $\Gamma\subset\Gamma'$, then $M_k(\Gamma')\subset M_k(\Gamma)$.

on. The latter can also be written as the quantity $\eta(\tau)^3\left[q^{-1/8}-2\mu(\tau,z)\right]$ multiplying the Jacobi form $\varphi_{-2,1}(\tau,z)=\theta_1(\tau,z)^2/\eta(\tau)^6$. Therefore, also the generic Hodge-Elliptic genus has the structure of the sum of two Jacobi forms with u-dependent coefficients. However, now the u-dependent part spoils the modular properties, due to the presence of the term $q^{-1/8}$ and of the mock modular form $\mu(\tau,z)$.

Remarkably (3.14) has a geometric interpretation in terms of the chiral de Rham complex Ω^{ch} of K3 introduced in Malikov [40]. Recall that the chiral de Rham complex is a sheaf of vertex operator algebras obtained by gluing together local $(bc-\beta\gamma)$-systems. Taking the sheaf homology $H^\bullet(\mathrm{K3},\Omega^{ch})$ provides a model for the sigma model Hilbert space of states. Then, it is shown in Wendland [46] that

$$\mathsf{H}\text{-}\mathsf{Ell}^g(\tau,z,\nu)=(y\,u)^{-1}\sum_{j=0}^{2}(-u)^j\,\mathrm{Tr}_{H^j(\mathrm{K3},\Omega^{ch})}\left((-1)^{J_0}y^{J_0}q^{L_0-\frac{1}{2}J_0}\right) \tag{3.15}$$

where the combination $L_0-\frac{1}{2}J_0$ signals that the chiral de Rham complex carries the action of a topologically twisted superconformal algebra.

By using (2.10), we can write (3.14) in a form similar to (3.13)

$$\begin{aligned}\mathsf{H}\text{-}\mathsf{Ell}^g(\tau,z,\nu)&=\frac{1}{24}\left(20+\frac{2}{u}+2u\right)\mathsf{Ell}(\tau,z)\\&\quad+\left(2-\frac{1}{u}-u\right)\frac{\theta_1(\tau,z)^2}{\eta(\tau)^3}\left[\frac{1}{12}H(\tau)+q^{-\frac{1}{8}}\right].\end{aligned} \tag{3.16}$$

3.5. The Complex Hodge-Elliptic Genus and Representation Theory

We will now outline a procedure to compute (3.5) directly from the definition, as an expansion in q. The idea is to reduce the computation to a sum of factors which can be read of from the ordinary elliptic genus expansion, with different weights.

A K3 surface has strict SU(2) holonomy and therefore SU(2) acts on the tangent space $\mathcal{T}$. By using the splitting principle, we can write the character $\mathrm{ch}(\mathcal{T})=t+1/t$ which is then identified with the character of the fundamental representation of SU(2). In the case of SU(2) characters, one can compute explicitly the generating function

$$\sum_{n=0}s^n\chi_n(t)=\frac{1}{(1-st)(1-\frac{s}{t})}. \tag{3.17}$$

We can think of t as a one-dimensional module under the action of the diagonal $\mathrm{U}(1)\subset\mathrm{SU}(2)$. Here $\chi_n(t)=\frac{t^{n+1}-t^{-n-1}}{t-t^{-1}}=\sum_{i=0}^n t^{2i-n}$.

Now, we look explicitly at the bundle $\mathbb{E}$ in (2.20). It has a form of a direct sum of bundles whose coefficients are weighted by q^n. The antisymmetric factors only contain a finite number of terms. We can then write

$$\mathbb{E} = \bigotimes_{n=1}^{\infty} \left(\mathcal{O} - y q^{n-1} \mathcal{T} + y^2 q^{2(n-1)} \mathcal{O} \right) \left(\mathcal{O} - \frac{q^n}{y} \mathcal{T} + \frac{q^{2n}}{y^2} \mathcal{O} \right)$$
$$\left(\bigoplus_{k=0} q^{nk} \mathcal{S}^k(\mathcal{T}) \right) \left(\bigoplus_{l=0} q^{nl} \mathcal{S}^l(\mathcal{T}) \right) . \tag{3.18}$$

Under the SU(2) action $\mathcal{S}^k(\mathcal{T})$ corresponds to the character χ_k. The reason why it is useful to think in this terms is that $H^0(X, \mathcal{S}^m \mathcal{T}) = 0 \ \forall m > 0$, by a classic result of Kobayashi. Therefore to compute the Hodge-Elliptic genus, it is sufficient to single out the terms which admit global sections. The terms which do not admit global sections have trivial H^0 and therefore by Serre's duality trivial H^2. The strategy of the computation is to use the representation theory of SU(2) to decompose a generic term $\mathcal{T}^k$ into terms which admit global sections (that are given by $\mathcal{O}$ and correspond to the trivial representation) and terms which do not (which have the form $\mathcal{S}^m \mathcal{T}$ and correspond to non-trivial characters). Note that products of terms of the form $\mathcal{S}^m \mathcal{T}$ may contain a copy of the trivial bundle in their decomposition.

In order to extract the full contribution of the trivial bundle, we will formally write

$$\bigoplus_{k=0}^{\infty} q^{nk} \mathcal{S}^k \mathcal{T} = \frac{1}{1 - \mathcal{T} q^n + q^{2n}} = \exp\left(- \log(1 - \mathcal{T} q^n + q^{2n}) \right) . \tag{3.19}$$

More precisely, we formally identify $\mathcal{T}$ with the character of the fundamental representation of SU(2) and interpret the above formula as a formal power series in its generator. This rewriting is convenient since now by expanding we have succeeded in writing $\mathbb{E}$ as a direct sum whose summands are all of the form $\mathcal{T}^k$ for some $k \in \mathbb{N}$.

Now to extract the contribution of the trivial bundle $\mathcal{O}$, we have to use repeatedly the tensor product decomposition rules, recalling that $\mathcal{T}$ transforms as the character of the fundamental representation under the SU(2) action. One starts by $\mathcal{T} \otimes \mathcal{T} = \mathcal{O} \oplus \mathcal{S}^2 \mathcal{T}$, where the second factor does not admit global sections and can therefore be discarded. Tensoring again by $\mathcal{T}$ one gets $\mathcal{T}^3 = 2\mathcal{T} \oplus \mathcal{S}^3 \mathcal{T}$. Similarly, $\mathcal{T}^4 = 2\mathcal{O} \oplus 3\mathcal{S}^2 \mathcal{T} \oplus \mathcal{S}^4 \mathcal{T}$. It is easy to see that any time we tensor $\mathcal{T}^k$ with k odd with $\mathcal{T}$, and the resulting decomposition of $\mathcal{T}^{k+1}$ has one $\mathcal{O}$ factor whose coefficient is the same as the coefficient of the $\mathcal{T}$ factor in the decomposition of $\mathcal{T}^k$. Similarly, any time we tensor $\mathcal{T}^n$ with n even with $\mathcal{T}$, the decomposition $\mathcal{T}^{n+1}$ has one $\mathcal{T}$ factor whose coefficient is the sum of the coefficients of $\mathcal{O}$ and $S^2 \mathcal{T}$ in $\mathcal{T}^n$. The coefficient of $\mathcal{S}^2 \mathcal{T}$ in $\mathcal{T}^n$ is, however, the sum of the coefficients of $\mathcal{T}$ and $\mathcal{S}^3 \mathcal{T}$ in $\mathcal{T}^{n-1}$. All these facts follow immediately from the tensor product decomposition of products of the fundamental representation. In summary we see that

$$\begin{aligned} \mathcal{T}^{2i} &= C_i \ \mathcal{O} \oplus \cdots \\ \mathcal{T}^{2i+1} &= C_{i+1} \ \mathcal{T} \oplus \cdots \end{aligned} \tag{3.20}$$

where $i \in \mathbb{N}$ and $C_i = \frac{(2i)!}{(i+1)!i!}$ is the i^{th} Catalan number. The dots denote terms which are sums of factors of the form $\mathcal{S}^k\mathcal{T}$ with $k > 1$.

We are finally ready to put all of our results together. The Hodge-elliptic genus has two contributions: the contribution from the $\mathcal{S}^{k>0}\mathcal{T}_X$ bundles, which is equal to their contribution to the elliptic genus (the u factor cancels with the overall $1/u$ normalization of (3.5)) since for these bundles only H^1 is non-trivial; and the contribution from the $\mathcal{O}_X$ factors which by Serre duality is equal to their contribution to the elliptic genus weighted by $\frac{1}{2}\left(u + \frac{1}{u}\right)$, since for these bundles only H^0 and H^2 are non-trivial and they are weighted, respectively, by $1/u$ and u.

Therefore, an equivalent and perhaps simpler way of computing the Hodge-elliptic genus is to compute the ordinary elliptic genus and add the contribution of the trivial bundles weighted by $\frac{1}{2}\left(u + \frac{1}{u}\right) - 1$. We formally express this by writing

$$\mathsf{H}\text{-}\mathsf{Ell}^c(\tau, z, \nu) = \mathsf{Ell}(\tau, z) - \left[1 - \frac{1}{2}\left(u + \frac{1}{u}\right)\right] \mathsf{Ell}(\tau, z)\Big|_{\mathcal{O}_X} \tag{3.21}$$

where $\mathsf{Ell}(\tau, z)\big|_{\mathcal{O}_X}$ is the contribution to the Elliptic genus from the flat bundle $\mathcal{O}_X$, and this equation is intended as an equivalence between formal power series.

This strategy can be easily implemented to compute $\mathsf{H}\text{-}\mathsf{Ell}^c(\tau, z, \nu)$ as a power series in q

$$\begin{aligned}
\mathsf{H}\text{-}\mathsf{Ell}^c(\tau, z, \nu) = &\left(uy + \frac{u}{y} + \frac{y}{u} + \frac{1}{uy} + 20\right) \\
&+ q\left(uy + \frac{y}{u} + \frac{u}{y} + \frac{1}{uy} - 2u - \frac{2}{u}\right. \\
&\left.+20y^2 + \frac{20}{y^2} - 130y - \frac{130}{y} + 220\right) \\
&+ q^2\left(uy^3 + \frac{y^3}{u} + \frac{u}{y^3} + \frac{1}{uy^3} - 2uy^2 - \frac{2y^2}{u}\right. \\
&- \frac{2u}{y^2} - \frac{2}{uy^2} + 4uy + \frac{4y}{u} + \frac{4u}{y} + \frac{4}{uy} \\
&\left.-6u - \frac{6}{u} + 220y^2 + \frac{220}{y^2} - 1034y - \frac{1034}{y} + 1628\right) \\
&+ q^3\left(uy^3 + \frac{y^3}{u} + \frac{u}{y^3} + \frac{1}{uy^3} - 6uy^2 - \frac{6y^2}{u} - \frac{6u}{y^2} - \frac{6}{uy^2}\right. \\
&+ 13uy + \frac{13y}{u} + \frac{13u}{y} + \frac{13}{uy} - 16u - \frac{16}{u} - 130y^3 - \frac{130}{y^3} \\
&\left.+1628y^2 + \frac{1628}{y^2} - 5530y - \frac{5530}{y} + 8064\right) + \cdots
\end{aligned} \tag{3.22}$$

We will show momentarily how to obtain $\mathsf{H}\text{-}\mathsf{Ell}^c(\tau, z, \nu)$ in closed form, by finding a way to implement the above arguments systematically. Before that we have to revisit the computation of the Elliptic genus.

3.6. The Elliptic Genus Revisited

The Elliptic genus integrand is determined, up to a normalization, by the Chern character of the bundle (2.20). By using (3.19), we can write

$$\frac{1}{y}\mathrm{ch}(\mathbb{E}) = \frac{1}{y}\prod_{n=1}^{\infty}\frac{\left(1-\frac{y}{\zeta}q^{n-1}\right)\left(1-\frac{\zeta}{y}q^{n}\right)\left(1-\frac{1}{y\zeta}q^{n}\right)\left(1-y\,\zeta q^{n-1}\right)}{\left(1-\zeta q^{n}\right)^{2}\left(1-\frac{1}{\zeta}q^{n}\right)^{2}} \tag{3.23}$$

where we have used the splitting principle to write $\mathrm{ch}(\mathcal{T}) = \zeta + 1/\zeta$, where $\zeta = \mathrm{e}^{x}$ and x is the integration variable. Now, following [17] we use the following denominator formulas from Kac and Wakimoto [31] (see Example 4.1)

$$\begin{aligned}
\sum_{j\in\mathbb{Z}}\frac{\zeta^{j}}{1-\frac{q^{j}}{y}} &= \frac{\zeta}{1-\zeta}\prod_{n=1}^{\infty}\frac{(1-q^{n})^{2}\left(1-\frac{\zeta}{y}q^{n}\right)\left(1-\frac{y}{\zeta}q^{n-1}\right)}{(1-\zeta\,q^{n})\left(1-\frac{1}{\zeta}\,q^{n}\right)\left(1-\frac{1}{y}\,q^{n}\right)(1-y\,q^{n-1})}\\
\sum_{j\in\mathbb{Z}}\frac{\zeta^{j}}{1-y\,q^{j}} &= \frac{1}{1-\zeta}\prod_{n=1}^{\infty}\frac{(1-q^{n})^{2}\left(1-\zeta\,y\,q^{n-1}\right)\left(1-\frac{1}{y\,\zeta}q^{n}\right)}{(1-\zeta\,q^{n})\left(1-\frac{1}{\zeta}\,q^{n}\right)\left(1-\frac{1}{y}\,q^{n}\right)(1-y\,q^{n-1})}
\end{aligned} \tag{3.24}$$

Noting that

$$\begin{aligned}
\theta_1^{2}(\tau,z) &= -q^{\frac{1}{4}}\frac{1}{y}\prod_{n=1}^{\infty}(1-q^{n})^{2}\left(1-y\,q^{n-1}\right)^{2}\left(1-\frac{1}{y}q^{n}\right)^{2}\\
\eta^{6}(\tau) &= q^{\frac{1}{4}}\prod_{n=1}^{\infty}(1-q^{n})^{6},
\end{aligned} \tag{3.25}$$

we can now write (3.23) as

$$\begin{aligned}
\frac{1}{y}\mathrm{ch}(\mathbb{E}) &= -\frac{(1-\zeta)^{2}}{\zeta}\,\frac{\theta_1^{2}(\tau,z)}{\eta(\tau)^{6}}\sum_{i,j\in\mathbb{Z}}\frac{\zeta^{i+j}}{(1-y\,q^{j})(1-\frac{1}{y}q^{i})} = -\frac{\theta_1(\tau,z)^{2}}{\eta(\tau)^{6}}\\
&\quad\sum_{N\in\mathbb{Z}}\sum_{i\in\mathbb{Z}}\frac{\zeta^{N-1}-2\zeta^{N}+\zeta^{N+1}}{(1-y\,q^{i})(1-\frac{1}{y}q^{N-i})}\\
&= -\frac{\theta_1(\tau,z)}{\eta(\tau)^{3}}\sum_{N\in\mathbb{Z}}\zeta^{N}\left(s_{N+2}(\tau,z)-2s_{N+1}(\tau,z)+s_{N}(\tau,z)\right)
\end{aligned} \tag{3.26}$$

by changing summation variable $i+j=N$. In the last step, we have introduced the functions

$$s_N(\tau,z) = \frac{\theta_1(\tau,z)}{\eta(\tau)^{3}}\sum_{i\in\mathbb{Z}}\frac{1}{(1-yq^{i})(1-\frac{1}{y}q^{N-1-i})}. \tag{3.27}$$

As we have discussed, we would like to rewrite the Chern character of $\mathbb{E}$ in a form where the contribution from each $S^{N}\mathcal{T}$ bundle is highlighted:

$$\frac{1}{y}\mathrm{ch}(\mathbb{E}) = \sum_{N=0}^{\infty}\mathrm{ch}(S^{N}\mathcal{T})\mathscr{F}_N(\tau,z) \tag{3.28}$$

in terms of certain functions $\mathscr{F}_N$ to be determined by comparing with (3.28). In order to do so note that on the right-hand side of (3.28) the coefficient of ζ^k is of the form $\sum_{i=k}^{\infty} \mathscr{F}_{2i}$. Therefore, it follows that we can obtain $\mathscr{F}_N$ by taking the difference between the coefficient of ζ^N and the coefficient of ζ^{N+2} in (3.26), since all the other terms cancel. Therefore,

$$\begin{aligned}\mathscr{F}_N(\tau, z) = &-\frac{\theta_1(\tau, z)}{\eta(\tau)^3} \\ &(s_N(\tau, z) - 2s_{N+1}(\tau, z) + 2s_{N+3}(\tau, z) - s_{N+4}(\tau, z))\end{aligned} \tag{3.29}$$

In particular, now we know how to isolate the trivial character which corresponds to the contribution of the trivial bundle.

We prove in Appendix A that

$$s_N(\tau, z) = \frac{\theta_1(\tau, z)}{\eta(\tau)^3}\widetilde{s}_N(\tau) + \delta_{N,1}\, \theta_1(\tau, z)\, \mu(\tau, z) \tag{3.30}$$

with

$$\widetilde{s}_N(\tau) = \begin{cases} \frac{q}{1-q} & \text{if} \quad N = 0\,, \\ 2F_2^{(2)}(\tau) & \text{if} \quad N = 1\,, \\ \frac{N-1}{1-q^{N-1}} & \text{otherwise.} \end{cases} \tag{3.31}$$

Therefore, we can write the full Elliptic genus as

$$\mathsf{Ell}(\tau, z) = \frac{1}{y}\int \mathrm{ch}(\mathbb{E})\mathrm{Todd}(\mathcal{T}) = \sum_{N=0}^{\infty} \mathscr{F}_N(\tau, z) \int \mathrm{ch}(S^N\mathcal{T})\mathrm{Todd}(\mathcal{T})\,. \tag{3.32}$$

The integration is now elementary:

$$\begin{aligned}\int \mathrm{ch}(S^N\mathcal{T})\mathrm{Todd}(\mathcal{T}) &= \int \frac{\mathrm{e}^{(N+1)x} - \mathrm{e}^{-(N+1)x}}{\mathrm{e}^x - \mathrm{e}^{-x}} \frac{-x^2}{(1-\mathrm{e}^x)(1-\mathrm{e}^{-x})} \\ &= \frac{1}{12}(2N^3 + 6N^2 + 3N - 1)\int x^2 = -2(2N^3 + 6N^2 + 3N - 1)\end{aligned} \tag{3.33}$$

where we have used (2.7).

We conclude that the Elliptic genus can be written as

$$\mathsf{Ell}(\tau, z) = -\sum_{N=0}^{\infty} 2(2N^3 + 6N^2 + 3N - 1)\ \mathscr{F}_N(\tau, z)\,. \tag{3.34}$$

It is convenient to rearrange terms as

$$\begin{aligned}&\sum_{N=0}^{\infty} 2(2N^3 + 6N^2 + 3N - 1)\ (s_N(\tau, z) - 2s_{N+1}(\tau, z) + 2s_{N+3}(\tau, z) - s_{N+4}(\tau, z)) \\ &\quad = -2s_0(\tau, z) + 24s_1(\tau, z) + 50s_2(\tau, z) + 48\sum_{N=3}^{\infty} s_N(\tau, z)\,.\end{aligned} \tag{3.35}$$

The first three contributions can be checked directly. All the other follow from some boring but straightforward algebra

$$\begin{aligned}&\big(2(2N^3+6N^2+3N-1)\big)-2\left(2(2(N-1)^3+6(N-1)^2+3(N-1)-1)\right)\\&\quad+2\left(2(2(N-3)^3+6(N-3)^2+3(N-3)-1)\right)\\&\quad-\big(2(2(N-4)^3+6(N-4)^2+3(N-4)-1)\big)=48\,.\end{aligned} \tag{3.36}$$

To summarize, we have shown

$$\begin{aligned}\mathsf{Ell}(\tau,z)=&\frac{\theta_1(\tau,z)^2}{\eta(\tau)^3}\\&\left[\frac{1}{\eta(\tau)^3}\left(-2\widetilde{s}_0+24\widetilde{s}_1+50\widetilde{s}_2+48\sum_{N=3}^{\infty}\widetilde{s}_N\right)+24\mu(\tau,z)\right].\end{aligned} \tag{3.37}$$

Furthermore, the combination

$$-2\widetilde{s}_0+50\widetilde{s}_2+48\sum_{N=3}^{\infty}\widetilde{s}_N=2+48\frac{1}{1-q}+48\sum_{n=2}^{\infty}\frac{n}{1-q^n}=-2E_2(\tau)\,, \tag{3.38}$$

gives the Eisenstein series $E_2(\tau)$. Indeed the latter can be written as[3]

$$\begin{aligned}E_2&=1-24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}=1-24\sum_{n=1}^{\infty}\left(\frac{n}{1-q^n}-n\frac{1-q^n}{1-q^n}\right)\\&=1-24\left(\sum_{n=1}^{\infty}\frac{n}{1-q^n}+\frac{1}{12}\right)\\&=-1-24\sum_{n=1}^{\infty}\frac{n}{1-q^n}\end{aligned} \tag{3.39}$$

where we have used $\sum_{n=1}^{\infty}n=\zeta(-1)=-\frac{1}{12}$ in terms of the analytically continued Riemann zeta function $\zeta(s)$. By putting everything together, we finally have

$$\mathsf{Ell}(\tau,z)=\frac{\theta_1(\tau,z)^2}{\eta(\tau)^3}\left(\frac{-2E_2(\tau)+48F_2^{(2)}(\tau)}{\eta(\tau)^3}+24\mu(\tau,z)\right) \tag{3.40}$$

as expected.

The above result was obtained by using the zeta function regularization. One may wonder if this procedure introduces ambiguities in the final expression; for example, a different prescription could shift the Eisenstein function by a constant. However, such shifts are not compatible with modular invariance, which apparently forces this specific form of regularization. Nevertheless, it would be desirable to find a better argument which does not assume modular invariance, for example, by avoiding divergent expressions altogether.

---

[3]Note that this identity does *not* hold as a truncated expansion, because of the analytical continuation.

3.7. The Complex Hodge-Elliptic Genus

We have just shown that

$$\begin{aligned}\mathsf{Ell}(\tau,z) &= \frac{1}{y}\chi(\mathbb{E}) = \frac{1}{y}\sum_{j=0}^{2}(-1)^j \dim H^j(\mathbb{E}) \\ &= \sum_{N=0}^{\infty}\mathscr{F}_N(\tau,z)\sum_{j=0}^{2}(-1)^j \dim H^j(S^N\mathcal{T})\,. \end{aligned} \tag{3.41}$$

Similarly, we can write for the complex Hodge-Elliptic genus

$$\begin{aligned}\mathsf{H\text{-}Ell}^c(\tau,z,\nu) &= \frac{1}{yu}\sum_{j=0}^{2}(-u)^j \dim H^j(\mathbb{E}) \\ &= \sum_{N=0}^{\infty}\mathscr{F}_N(\tau,z)\left(\frac{1}{u}\dim H^0\left(S^N\mathcal{T}\right) - \dim H^1\left(S^N\mathcal{T}\right) + u\dim H^2\left(S^N\mathcal{T}\right)\right) \\ &= \frac{1}{2}\left(u+\frac{1}{u}\right)\mathscr{F}_0(\tau,z)\,\chi(\mathcal{O}) + \sum_{N=1}^{\infty}\mathscr{F}_N(\tau,z)\,\chi(S^N\mathcal{T}) \\ &= \mathsf{Ell}(\tau,z) - \left[1-\frac{1}{2}\left(u+\frac{1}{u}\right)\right]\mathscr{F}_0(\tau,z)\,\chi(\mathcal{O}) \end{aligned} \tag{3.42}$$

where we have used

$$\begin{aligned}&\frac{1}{u}\dim H^0\left(S^N\mathcal{T}\right) - \dim H^1\left(S^N\mathcal{T}\right) + u\dim H^2\left(S^N\mathcal{T}\right) \\ &\quad = \begin{cases}\frac{1}{2}\left(\frac{1}{u}+u\right)\chi(\mathcal{O}) & \text{if } N=0 \\ \chi(S^N\mathcal{T}) & \text{if } N\neq 0\end{cases}\end{aligned} \tag{3.43}$$

Now, we can compute the trivial bundle contribution

$$\begin{aligned}&\left[1-\frac{1}{2}\left(u+\frac{1}{u}\right)\right]\mathscr{F}_0(\tau,z)\,\chi(\mathcal{O}) \\ &\quad = \left[1-\frac{1}{2}\left(u+\frac{1}{u}\right)\right]\left[\left(-2\frac{\theta_1(\tau,z)^2}{\eta(\tau)^6}\right)\left(\frac{q}{1-q}+\frac{4}{1-q^2}-\frac{3}{1-q^3}-4F_2^{(2)}(\tau)\right)\right. \\ &\qquad \left. +4\frac{\theta_1(\tau,z)^2}{\eta(\tau)^3}\,\mu(\tau,z)\right] \\ &\quad = \left[-2+u\quad+\frac{1}{u}\right]\frac{\theta_1(\tau,z)^2}{\eta(\tau)^6}\left(\frac{1+2q+6q^2+2q^3+q^4}{(1-q)(1+q)(1+q+q^2)}-4F_2^{(2)}(\tau)\right) \\ &\qquad + \left[4-2\left(u+\frac{1}{u}\right)\right]\frac{\theta_1(\tau,z)^2}{\eta(\tau)^3}\,\mu(\tau,z)\end{aligned} \tag{3.44}$$

Finally, putting everything together we conclude that

$$\begin{aligned}\mathsf{H\text{-}Ell}^c(\tau,z,\nu;K3) &= \frac{\theta_1(q,y)^2}{\eta(q)^3}\left[\left(20+2\left(\frac{1}{u}+u\right)\right)\mu(q,y)+H(q)\right] \\ &\quad + \left(2-\left(\frac{1}{u}+u\right)\right)\frac{\theta_1(q,y)^2}{\eta(q)^6}\left(\frac{1+2q+6q^2+2q^3+q^4}{(1-q)(1+q)(1+q+q^2)}-4F_2^{(2)}(\tau)\right)\end{aligned} \tag{3.45}$$

Equivalently, we can use the definition (2.14) to write

$$\begin{aligned}\mathsf{H}\text{-}\mathsf{Ell}^c(\tau,z,\nu;K3) &= \frac{1}{24}\left(20+2\left(u+\frac{1}{u}\right)\right)\mathsf{Ell}(\tau,z)\\ &+\left(2-\left(\frac{1}{u}+u\right)\right)\frac{\theta_1(\tau,z)^2}{\eta(\tau)^6}\left(\frac{1+2q+6q^2+2q^3+q^4}{(1-q)(1+q)(1+q+q^2)}-\frac{1}{6}E_2(\tau)\right)\end{aligned} \tag{3.46}$$

Recall that $\mu(\tau,z)$ and $H(\tau)$ are mock modular forms with shadows $-\eta(\tau)^3$ and $24\eta(\tau)^3$. Remarkably the u-dependence factors out in the first line and again the two shadows cancel exactly to give a Jacobi form with a u-dependent coefficient which reduces to one in the limit $u \to 1$. In the second line, $\varphi_{-2,1} = -\theta_1^2/\eta^6$ is also a Jacobi form, and $E_2(\tau)$ is a quasi-modular form, and it transforms as a modular form of weight 2 when we add $-3/\pi\mathrm{Im}(\tau)$. Modular properties are, however, spoiled by the presence of the rational function.

Note that the form (3.46) of the u-dependent factor $\frac{1}{24}\left(20+2u+\frac{2}{u}\right)$ multiplying the Elliptic genus plus a correction is common to all Hodge-Elliptic genera.

3.8. Character Decomposition and Mathieu Moonshine

Among the connections between string theory and number theory, perhaps one of the most striking is the observation by Eguchi, Ooguri and Tachikawa [23] that certain coefficients in the character expansion of the Elliptic genus are twice the dimensions of certain irreducible representations of the largest Mathieu group $\mathbb{M}_{24}$. We would like to investigate how this statement is modified when the Elliptic genus is refined into its Hodge-Elliptic counterpart. The main motivation in doing so is that often in string theory refined enumerative invariant helps to understand the Hilbert space of BPS states[4] and could help in identifying the physical and geometrical reason the Mathieu group appears. We will now show directly that all the Hodge-Elliptic genera we have seen can be decomposed as a sum over the same superconformal characters as the Elliptic genus with u dependent coefficients.

The Elliptic genus admits the following decomposition [23]

$$\mathsf{Ell}(\tau;z) = 20\,\mathrm{ch}_{\frac14,0}(\tau;z) - 2\,\mathrm{ch}_{\frac14,\frac12}(\tau;z) + \sum_{n=1}^{\infty} c_H(n)\,\mathrm{ch}_{\frac14+n,\frac12}(\tau,z) \tag{3.47}$$

in terms of the characters of the superconformal algebra. These have the form

$$\mathrm{ch}_{h,\ell}(\tau,z) = \mathrm{Tr}_{V_{h,\ell}}\left((-1)^{J_0}\,y^{J_0}q^{L_0-\frac{c}{24}}\right), \tag{3.48}$$

where an irreducible representation $V_{h,\ell}$ of the $\mathcal{N}=4$ algebra is labeled by the quantum numbers h and ℓ, the eigenvalues of L_0 and J_0^3, respectively. For central charge $c=6$, the massless representations have quantum numbers

[4]On general ground one expects that the categorification of Donaldson–Thomas invariants will provide a model for the Hilbert space of BPS states. Making sense of this statement is one of the goals of Donaldson–Thomas theory.

$(h,\ell) = (\frac{1}{4},0)$ and $(h,\ell) = (\frac{1}{4},\frac{1}{2})$, while the massive representations have $(h,\ell) = (\frac{1}{4}+n,0)$ with $n = 1,2,\ldots$. Their characters are [24–26]

$$\begin{aligned}
\mathrm{ch}_{\frac{1}{4},0}(\tau;z) &= \frac{\theta_1(\tau;z)^2}{\eta(\tau)^3}\ \mu(\tau,z)\,,\\
\mathrm{ch}_{\frac{1}{4},\frac{1}{2}}(\tau;z) &= q^{-\frac{1}{8}}\frac{\theta_1(\tau;z)^2}{\eta(\tau)^3} - 2\frac{\theta_1(\tau;z)^2}{\eta(\tau)^3}\ \mu(\tau,z)\,,\\
\mathrm{ch}_{\frac{1}{4}+n,\frac{1}{2}}(\tau;z) &= q^{-\frac{1}{8}+n}\frac{\theta_1(\tau;z)^2}{\eta(\tau)^3}\,.
\end{aligned} \tag{3.49}$$

In (3.47), the coefficients $c_H(n)$ are defined via (2.14), or equivalently

$$\sum_{n=1}^{\infty} c_H(n)\,\mathrm{ch}_{\frac{1}{4}+n,\frac{1}{2}}(\tau,z) = \frac{\theta_1(\tau;z)^2}{\eta(\tau)^3}H(\tau) + 2q^{-\frac{1}{8}}\frac{\theta_1(\tau;z)^2}{\eta(\tau)^3}\,. \tag{3.50}$$

Now, let us consider the Hodge-Elliptic genera (3.46), (3.13) and (3.14). Differently from (3.47), the Hodge-Elliptic genus will involve also right-moving characters of the form

$$\mathrm{ch}_{h,\ell}(\overline{\tau},\nu) = \mathrm{Tr}_{V_{h,\ell}}\left((-1)^{\overline{J}_0}\,u^{\overline{J}_0}\overline{q}^{\overline{L}_0-\frac{\overline{c}}{24}}\right)\,. \tag{3.51}$$

However, since the sum is constrained to states with $\overline{L}_0 = \frac{\overline{c}}{24}$, the net effect is that we expect a decomposition similar to that of (3.47) whose coefficients depend now on the fugacity u. In the following, we will exhibit explicitly this structure for the three Hodge-Elliptic partition functions we know in closed form. The results have the form of a refined Mathieu moonshine. It is natural to hope that the u-dependent coefficients can now be interpreted as a refined trace over the $\mathbb{M}_{24}$ module which appear in the decomposition (3.47). Indeed from the geometrical point of view in the complex and generic definitions (3.5) and (3.16), the refinement consists in an extra parameter which weight differently the cohomology groups. We will see that this structure is preserved in the holomorphic character decomposition.

Consider first the complex Hodge-Elliptic genus (3.46). Using the definitions of the characters and (3.47), we see that

$$\begin{aligned}
\mathsf{H\text{-}Ell}^c(\tau,z,\nu) &= 20\,\mathrm{ch}_{\frac{1}{4},0}(\tau;z) + 2\left(\frac{1}{u}+u\right)\frac{\theta_1(\tau,z)^2}{\eta(\tau)^3}\mu(\tau,z) +\\
&+ \frac{1}{24}\left(20+2\left(u+\frac{1}{u}\right)\right)\left[\sum_{n=1}^{\infty} c_H(n)\,\mathrm{ch}_{\frac{1}{4}+n,\frac{1}{2}}(\tau,z) - 2q^{-\frac{1}{8}}\frac{\theta_1(\tau;z)^2}{\eta(\tau)^3}\right]\\
&+ \left(2-\left(\frac{1}{u}+u\right)\right)\frac{\theta_1(\tau,z)^2}{\eta(q)^6}\left(\frac{1+2q+6q^2+2q^3+q^4}{(1-q)(1+q)(1+q+q^2)} - \frac{1}{6}E_2(\tau)\right).
\end{aligned} \tag{3.52}$$

Adding and subtracting $(\frac{1}{u}+u)q^{-\frac{1}{8}}\frac{\theta_1(\tau,z)^2}{\eta(\tau)^3}$, we can write

$$\begin{aligned}\mathsf{H\text{-}Ell}^c(\tau,z,\nu) &= 20\,\mathrm{ch}_{\frac14,0}(\tau;z) - \left(\frac{1}{u}+u\right)\mathrm{ch}_{\frac14,\frac12}(\tau;z)\\ &\quad + \frac{1}{24}\left(20+2u+\frac{2}{u}\right)\sum_{n=1}^{\infty} c_H(n)\,\mathrm{ch}_{\frac14+n,\frac12}(\tau,z)\\ &\quad + \left(2-\frac{1}{u}-u\right) q^{-\frac18}\frac{\theta_1(\tau,z)^2}{\eta(\tau)^3}\\ &\quad \left[\frac{q^{1/8}}{\eta(\tau)^3}\left(\frac{1+2q+6q^2+2q^3+q^4}{(1-q)(1+q)(1+q+q^2)} - \frac16\,E_2(\tau)\right) - \frac56\right]\end{aligned} \tag{3.53}$$

We can interpret the second line as a correction to the coefficients $c_H(n)$. Indeed, it is clear that the terms in square brackets have a q expansion without constant term, since in the q expansion the zeroth-order terms coming from the rational function and from the Eisenstein series precisely cancel the 5/6 factor:

$$\begin{aligned}&\left[\frac{q^{1/8}}{\eta(\tau)^3}\left(\frac{1+2q+6q^2+2q^3+q^4}{(1-q)(1+q)(1+q+q^2)} - \frac16\,E_2(\tau)\right) - \frac56\right]\\ &\quad = \sum_{n=1}^{\infty} a_H(n) q^n\\ &\quad = \frac{15}{2}q + \frac{79}{2}q^2 + \frac{385}{3}q^3 + \frac{761}{2}q^4 + 969q^5\\ &\qquad + \frac{13927}{6}q^6 + \frac{10293}{2}q^7 + 10936q^8 + \cdots .\end{aligned} \tag{3.54}$$

Therefore, the whole expression can be understood as a correction to the coefficients of the characters $\mathrm{ch}_{\frac14+n,\frac12}$. We can write the complex Hodge-Elliptic genus as

$$\begin{aligned}\mathsf{H\text{-}Ell}^c(\tau,z,\nu) &= 20\,\mathrm{ch}_{\frac14,0}(\tau:z) - \left(\frac{1}{u}+u\right)\mathrm{ch}_{\frac14,\frac12}(\tau;z)\\ &\quad + \sum_{n=1}^{\infty} \tilde{c}_H(n)\,\mathrm{ch}_{\frac14+n,\frac12}(\tau,z)\end{aligned} \tag{3.55}$$

where now the coefficients

$$\tilde{c}_H(u;n) = \left(\frac{1}{24}\left(20+2u+\frac{2}{u}\right)c_H(n) + \left(2-\frac1u-u\right)a_H(n)\right) \tag{3.56}$$

are functions of u.

Equivalently, we could have started from the expression (3.45) and obtained

$$\begin{aligned}\mathsf{H\text{-}Ell}^c(\tau,z,\nu) &= 20\,\mathrm{ch}_{\frac14,0}(\tau;z) - \left(\frac{1}{u}+u\right)\mathrm{ch}_{\frac14,\frac12}(\tau;z)\\ &\quad + \sum_{n=1}^{\infty} c_H(n)\,\mathrm{ch}_{\frac14+n,\frac12}(\tau,z)\end{aligned}$$

$$+\left(2-\frac{1}{u}-u\right)\,q^{-\frac{1}{8}}\frac{\theta_1(\tau,z)^2}{\eta(\tau)^3}$$
$$\times\left[\frac{q^{1/8}}{\eta(\tau)^3}\left(\frac{1+2q+6q^2+2q^3+q^4}{(1-q)(1+q)(1+q+q^2)}-4\,F_2^{(2)}(\tau)\right)-1\right] \tag{3.57}$$

Now, the term in square brackets has an *integral* expansion in q without constant term

$$\left[\frac{q^{1/8}}{\eta(\tau)^3}\left(\frac{1+2q+6q^2+2q^3+q^4}{(1-q)(1+q)(1+q+q^2)}-4\,F_2^{(2)}(\tau)\right)-1\right]=\sum_{n=1}^{\infty}b_H(n)q^n$$
$$=q^2+q^4+3q^5+2q^6+6q^7+11q^8+13q^9+24q^{10}+\cdots \tag{3.58}$$

so that the whole expression can be understood as a shift of the coefficients of the characters $\mathrm{ch}_{\frac{1}{4}+n,\frac{1}{2}}$. We can therefore write the Hodge-Elliptic genus as

$$\mathsf{H\text{-}Ell}=20\,\mathrm{ch}_{\frac{1}{4},0}(\tau:z)-\left(\frac{1}{u}+u\right)\mathrm{ch}_{\frac{1}{4},\frac{1}{2}}(\tau;z)+\sum_{n=1}^{\infty}\tilde{c}_H(n)\,\mathrm{ch}_{\frac{1}{4}+n,\frac{1}{2}}(\tau,z) \tag{3.59}$$

where now

$$\tilde{c}_H(u;n)=\left(c_H(n)+\left(2-\frac{1}{u}-u\right)\,b_H(n)\right) \tag{3.60}$$

and clearly coincide with (3.56). Regardless of which expression for these coefficients we find more convenient, we can write down the first few terms

$$\sum_{n=1}^{\infty}\tilde{c}_H(u;n)\,\mathrm{ch}_{\frac{1}{4}+n,\frac{1}{2}}$$
$$=2\left[45q+q^2\left(232-\frac{u}{2}-\frac{1}{2u}\right)+770q^3+q^4\left(2278-\frac{u}{2}-\frac{1}{2u}\right)\right.$$
$$+q^5\left(5799-\frac{3u}{2}-\frac{3}{2u}\right)$$
$$+q^6\left(13917-u-\frac{1}{u}\right)+q^7\left(30849-3u-\frac{3}{u}\right)$$
$$+q^8\left(65561-\frac{11u}{2}-\frac{11}{2u}\right)+q^9\left(132838-\frac{13u}{2}-\frac{13}{2u}\right)$$
$$\left.+q^{10}\left(260592-12u-\frac{12}{u}\right)+\cdots\right]. \tag{3.61}$$

It remains to be understood if these coefficients have any interpretation in the context of Mathieu Moonshine.

Also, the orbifold Hodge-Elliptic genus (3.13) can be written in terms of the superconformal characters

$$
\begin{aligned}
\mathsf{H\text{-}Ell}^{\mathrm{orb}}(\tau,z,\nu) = & \frac{1}{24}\left(\frac{2}{u}+20+2u\right) \\
& \left[20\,\mathrm{ch}_{\frac{1}{4},0}(\tau,z)-2\mathrm{ch}_{\frac{1}{4},\frac{1}{2}}(\tau,z)+\sum_{n=1}^{\infty}c_H(n)\mathrm{ch}_{\frac{1}{4}+n,\frac{1}{2}}(\tau,z)\right] \\
& +\left(1-\frac{1}{2u}-\frac{u}{2}\right)\frac{\theta_1(\tau,z)^2}{\eta(\tau)^3}q^{-\frac{1}{8}}\left[\frac{q^{\frac{1}{8}}}{\eta(\tau)^3}\left(1+8\Lambda_2(\tau)\right)\right].
\end{aligned} \tag{3.62}
$$

The term on the second line has a q expansion which starts with a constant term. We can, however, write it in terms of the superconformal characters as

$$
\begin{aligned}
\frac{\theta_1(\tau,z)^2}{\eta(\tau)^3}q^{-\frac{1}{8}}\left[\frac{q^{\frac{1}{8}}}{\eta(\tau)^3}\left(1+8\Lambda_2(\tau)\right)\right] &= \frac{\theta_1(\tau,z)^2}{\eta(\tau)^3}q^{-\frac{1}{8}}\left[\frac{5}{3}+\sum_{n=1}^{\infty}d_H(n)q^n\right] \\
&= \frac{5}{3}\left(\mathrm{ch}_{\frac{1}{4},\frac{1}{2}}(\tau,z)+2\mathrm{ch}_{\frac{1}{4},0}(\tau,z)\right)+\sum_{n=1}^{\infty}d_H(n)\mathrm{ch}_{\frac{1}{4}+n,\frac{1}{2}}(\tau,z)
\end{aligned} \tag{3.63}
$$

where

$$
\sum_{n=1}^{\infty}d_H(n)q^n = 21q+79q^2+\frac{878}{3}q^3+789q^4+2068q^5+\frac{14449}{3}q^6+\dots \tag{3.64}
$$

which has no constant term. Putting everything together, the orbifold Hodge-Elliptic genus has the simple expression

$$
\begin{aligned}
\mathsf{H\text{-}Ell}^{\mathrm{orb}}(\tau,z,\nu) = & \, 20\,\mathrm{ch}_{\frac{1}{4},0}(\tau,z)-\left(\frac{1}{u}+u\right)\mathrm{ch}_{\frac{1}{4},\frac{1}{2}}(\tau,z) \\
& +\sum_{n=1}^{\infty}\tilde{c}_H^{\mathrm{orb}}(n)\,\mathrm{ch}_{\frac{1}{4}+n,\frac{1}{2}}(\tau,z)
\end{aligned} \tag{3.65}
$$

where

$$
\begin{aligned}
\tilde{c}_H(u,n) = & \frac{1}{24}\left(\frac{2}{u}+20+2u\right)c_H(n)+\left(1-\frac{1}{2u}-\frac{u}{2}\right)d_H(n) \\
= & \, 2\left[q\left(48-\frac{3u}{2}-\frac{3}{2u}\right)+q^2\left(232-\frac{u}{2}-\frac{1}{2u}\right)+q^3\left(788-9u-\frac{9}{u}\right)\right. \\
& +q^4\left(2292-\frac{15u}{2}-\frac{15}{2u}\right)+q^5\left(5864-34u-\frac{34}{u}\right) \\
& +q^6\left(14004-\frac{89u}{2}-\frac{89}{2u}\right)+q^7\left(31092-\frac{249u}{2}-\frac{249}{2u}\right) \\
& +q^8\left(65908-179u-\frac{179}{u}\right) \\
& \left.+q^9\left(133624-\frac{799u}{2}-\frac{799}{2u}\right)+q^{10}\left(261804-618u-\frac{618}{u}\right)+\cdots\right]
\end{aligned} \tag{3.66}
$$

Finally, let us consider the generic Hodge-Elliptic genus (3.14)

$$\begin{aligned}\mathsf{H}\text{-}\mathsf{Ell}^g(\tau,z,\nu) &= \mathsf{Ell}(\tau,z) + \left(2-\frac{1}{u}-u\right)\mathrm{ch}_{\frac{1}{4},\frac{1}{2}}(\tau,z)\\ &= 20\,\mathrm{ch}_{\frac{1}{4},0}(\tau;z) - \left(u+\frac{1}{u}\right)\mathrm{ch}_{\frac{1}{4},\frac{1}{2}}(\tau;z) + \sum_{n=1}^{\infty} c_H(n)\,\mathrm{ch}_{\frac{1}{4}+n,\frac{1}{2}}(\tau,z)\,.\end{aligned} \tag{3.67}$$

In the case only, the coefficient of $\mathrm{ch}_{\frac{1}{4},\frac{1}{2}}(\tau;z)$ is modified.

Overall, we find that the Hodge-Elliptic genera we have consider always admit a character expansion of the form (3.47) but where now the coefficients are u dependent.

4. Refined Dyons and Enumerative Geometry

In this section, we will discuss a refined version of the counting function Φ_{10} and use it to make some predictions for enumerative geometry. We will discuss this procedure in general, and then, we will specialize to the several Hodge-Elliptic genera we have encountered in the previous Section. To each one, we can associate a different set of enumerative invariants, conjecturally related by wall-crossing.

Now that we have an explicit form for the Hodge-Elliptic partition function, we can use it to define a refined version of the Igusa cusp form Φ_{10}. We expand the Hodge-Elliptic genus as

$$\mathsf{H}\text{-}\mathsf{Ell}(\tau,z,\nu) = \sum_{s,t,v} c(s,t,v)\, q^s\, y^t\, u^v\,, \tag{4.1}$$

and use the coefficients of the expansion to define the function

$$\begin{aligned}\Phi^{\mathrm{ref}}(\tau,z,\nu,\sigma) &= p\,q\,y \prod_{(s,t,r,v)>0} \left(1-q^s y^t p^r u^v\right)^{c(r\,s,t,v)}\\ &= p\,q\,y\left(1-\frac{u}{y}\right)\left(1-\frac{1}{uy}\right)\prod_{\substack{s>0,r>0\\(t,v)\in\mathbb{Z}}}\left(1-q^s y^t p^r u^v\right)^{c(r\,s,t,v)}\,,\end{aligned} \tag{4.2}$$

where the notation $(s,t,r,v)>0$ means that the product is over all the $t,v\in\mathbb{Z}$ and $s,r\geq 0$ such that one of the following two conditions hold

- $s>0$ or $r>0$,
- $s=r=0$ and $t<0$.

Note that by definition the range of v is just $v=-1,0,1$ since otherwise the coefficients $c(r\,s,t,v)$ vanish, by direct inspection of the Hodge-Elliptic genera. Of course the function Φ^{ref} will contain arbitrary powers of u. We also stress that the function Φ^{ref} so defined, including the range of indices, is again formally symmetric for the exchange of q and p.

Then, the main conjecture of Kachru and Tripathy [32] is that

$$\Phi^{\mathrm{ref}}(\tau, z, \nu, \sigma) = -\frac{1}{\mathtt{DT}^{\mathrm{ref}}} \tag{4.3}$$

where

$$\mathtt{DT}^{\mathrm{ref}} = \sum_{h=0}^{\infty} \mathtt{DT}_h^{\mathrm{ref}}(\mathrm{K3} \times E)\, p^{h-1} = \sum \mathtt{DT}_{h,d,n}^{\mathrm{ref}}\; p^{h-1} q^{d-1} (-y)^n \tag{4.4}$$

is the generating function of (reduced) refined Donaldson–Thomas invariants for $K3 \times E$.

This conjecture requires, however, some explanation. Since the Hodge-Elliptic genus depends on the Kähler moduli, the same is true for the generating function of refined BPS invariants. While the unrefined partition function already has some mild wall-crossing behavior when crossing codimension 1 walls of marginal stability, the refined partition function is expected to be sensitive also to BPS jumping loci, higher codimension loci in the moduli space where certain extra BPS states appear, leaving the rest of the spectrum unchanged [32–34]. Across such loci the full generating function is expected to change also for $\mathcal{N} = 4$ compactifications.

The origin of this phenomenon is the fact that the Hodge-Elliptic genus is not an index and in particular jumps at points in the moduli space where extra chiral currents appear. For example, the generic Hodge-Elliptic genus should capture the physics of the infinite volume limit, while the orbifold Hodge-Elliptic genus is defined at a particular point where the orbifold $\mathbb{T}^4/\mathbb{Z}_2$ is resolved. Therefore, one expects, and indeed finds, two distinct partition functions $\mathtt{DT}^{\mathrm{ref}}$. Less clear is the situation for the complex Hodge-Elliptic genus for which a certain SU(2) vertex operator algebra extends the $\mathcal{N} = 4$ superconformal algebra [17]. Since this quantity is independent of complex structure deformations [32, 46] and explicitly computable, it is natural to expect that it captures the physics somewhere in the K3 moduli space. Unfortunately, we could not find explicitly this locus or show its existence; at this stage this is only a conjecture as no deformation of the large radius sigma model is known which would lead to such a point.

Geometrically, we expect the BPS invariants to be defined in terms of the intersection theory of the reduced Hilbert scheme $\mathrm{Hilb}^{h,d,n}(X)/E$. Let us denote by $\mathscr{M}_\gamma$ the relevant moduli spaces for a charge vector γ. The general problem in extracting enumerative invariants from the spaces $\mathscr{M}_\gamma$ is that these are poorly understood with several singularities and branches of different dimensions. If $\mathscr{M}_\gamma$ were smooth, one could define the refined Donaldson–Thomas invariants as χ_u genera of such moduli spaces, appropriately normalized. Following [32], it is natural to identify the invariants $\mathtt{DT}_{h,d,n}^{\mathrm{ref}}$ with motivic invariants. Roughly speaking, one considers $[\mathscr{M}_\gamma]$ as a class in the abelian K-theory group of varieties, generated by isomorphism classes of complex varieties modulo the scissor relations, extended by $\mathbb{L}^{-1}$, the formal inverse of the Lefschetz motive (the class of the affine line). Taking the symmetrized Poincaré polynomials

$$\mathtt{DT}^{\mathrm{ref}}_{h,d,n} = \mathrm{P}\left([\mathscr{M}_\gamma]\right) \in \mathbb{Q}[u, u^{-1}] \tag{4.5}$$

provides a working definition for the refined invariants.

It is tempting to speculate that such objects could more naturally be obtained in terms of the M2-brane index, defined in [41] for smooth toric manifolds and in [15] in noncommutative chambers. Indeed, it was shown in Nekrasov and Okounkov [8] that the unrefined invariants with $h = 0, 1$ can be computed directly via toric localization. While the compact geometry is not toric, there exist in this case a stratification of the relevant moduli spaces where each strata is separately toric. Roughly speaking, the strategy of Refs. [15,41] is to perform the localization computation by keeping all the toric weights and then identifying the product of all the toric weights with the refined parameter. Such product is kept finite in a scaling limit which sends all the weights to zero or to infinity. While in Nekrasov and Okounkov [8] the toric action is different in each strata, one can speculate that a refined parameter introduced in this way should be identified across different strata and would correspond to a square root of the canonical bundle of $\mathrm{K3} \times E$. We leave such speculations for future work.

Regrettably, speculations notwithstanding, we do not have a precise definition of the refined invariants nor we understand how to precisely associate a counting function with a point in the moduli space. Presumably, this would entail introducing a theory of stability conditions on the reduced Hilbert scheme of $\mathrm{K3} \times E$. In this note, we take a working approach and compute such invariants in the hope that this could help clarifying their geometrical meaning.

It is worth mentioning that certain refined BPS invariants can also be derived by duality constraints [1,2]. As already discussed in Sect. 2.4 in order to compare with the geometrical results of Bryan [8], we use an expansion in a different variable from what is commonly used in the literature. This give results which are not immediately comparable, although we hope to return to this problem in the future. Finally, in Sen [45] it was proposed to interpret the counting of refined invariants from a representation theory perspective as a refinement of a certain helicity supertrace.

Let us go back to the definition (4.3) and try to elucidate its structure. As in Katz et al. [32], we have

$$\begin{aligned}\sum_{k=0}^{\infty} p^k \, \mathsf{H}\text{-}\mathsf{Ell}\left(\mathrm{Hilb}^{[k]}(\mathrm{K3})\right) &= \prod_{r>0, s\geq 0, t, v} \frac{1}{(1 - q^s y^t p^r u^v)^{c(r\,s,t,v)}} \\ &= \frac{p\,\phi_{KKP}(\tau, z, \nu)}{\Phi^{\mathrm{ref}}(\tau, z, \nu, \sigma)}\,.\end{aligned} \tag{4.6}$$

Note that

$$\prod_{r>0,s\geq 0,t,v}(1-q^s y^t p^r u^v)^{c(r\,s,t,v)}$$
$$= \exp\sum_{r\geq 1}\sum_{s\geq 0}\sum_{t,v\in\mathbb{Z}} c(r\,s,t,v)\log\left(1-q^s y^t p^r u^v\right)$$
$$= \exp\sum_{r\geq 1}\sum_{s\geq 0}\sum_{t,v\in\mathbb{Z}} c(r\,s,t,v)\left(-\sum_{d=1}^{\infty}\frac{q^{ds}y^{dt}p^{dr}u^{dv}}{d}\right)$$
$$= \exp\left(-\sum_{r\geq 1}\sum_{s\geq 0}\sum_{t,v\in\mathbb{Z}}\sum_{d|(r,s,t,v)}^{\infty}\frac{1}{d}c\left(\frac{rs}{d^2},\frac{t}{d},\frac{v}{d}\right)q^s y^t p^r u^v\right). \tag{4.7}$$

Equivalently

$$\Phi^{\mathrm{ref}}(\tau,z,\nu,\sigma) = p\,\phi_{KKP}(\tau,z,\nu)\,\exp\left(-\sum_{r=1}^{\infty}p^r\left(\mathsf{H\text{-}Ell}|V_r\right)(\tau,z,\nu,\sigma)\right) \tag{4.8}$$

where we have introduced the Hecke-like operators

$$V_r\ :\ \sum_{s,t,v}c(s,t,v)q^s y^t u^v \longrightarrow \sum_{s\geq 0}\sum_{t,v\in\mathbb{Z}}\sum_{d|(r,s,t,v)}^{\infty}\frac{1}{d}c\left(\frac{rs}{d^2},\frac{t}{d},\frac{v}{d}\right)q^s y^t u^v \tag{4.9}$$

which implement the plethystic symmetrization. If we introduce the notation

$$\zeta_r = \left(\mathsf{H\text{-}Ell}|V_r\right)(\tau,z,\nu,\sigma) \tag{4.10}$$

, we can then express

$$\mathsf{H\text{-}Ell}(\mathrm{Hilb}^{[m]}) = s_m(\zeta_1,\ldots,\zeta_m) \tag{4.11}$$

in terms of the Schur polynomials (for the symmetric representations). The first few are

$$\begin{aligned} s_1(\zeta_1) &= \zeta_1\,, \\ s_2(\zeta_1,\zeta_2) &= \frac{1}{2}\zeta_1^2+\zeta_2\,, \\ s_3(\zeta_1,\zeta_2,\zeta_3) &= \frac{1}{6}\zeta_1^3+\zeta_1\zeta_2+\zeta_3\,. \end{aligned} \tag{4.12}$$

Explicitly now

$$\begin{aligned} \mathsf{H\text{-}Ell}\left(\mathrm{Hilb}^{[1]}(\mathrm{K3})\right) &= \mathsf{H\text{-}Ell}(\tau,z,\nu)\,, \\ \mathsf{H\text{-}Ell}\left(\mathrm{Hilb}^{[2]}(\mathrm{K3})\right) &= \frac{1}{2}\mathsf{H\text{-}Ell}(\tau,z,\nu)^2+\frac{1}{2} \\ &\quad\times\left[\mathsf{H\text{-}Ell}\left(\frac{\tau}{2},z,\nu\right)+\mathsf{H\text{-}Ell}(2\tau,2z,2\nu)+\mathsf{H\text{-}Ell}\left(\frac{\tau+1}{2},z,\nu\right)\right]\,, \\ \mathsf{H\text{-}Ell}\left(\mathrm{Hilb}^{[3]}(\mathrm{K3})\right) &= \frac{1}{6}\mathsf{H\text{-}Ell}(\tau,z,\nu)^3 \\ &\quad+\frac{1}{2}\mathsf{H\text{-}Ell}(\tau,z,\nu) \end{aligned}$$

$$\times \left[\mathsf{H}\text{ -}\mathsf{Ell}\left(\frac{\tau}{2}, z, \nu\right) + \mathsf{H}\text{ -}\mathsf{Ell}(2\tau, 2z, 2\nu) + \mathsf{H}\text{ -}\mathsf{Ell}\left(\frac{\tau+1}{2}, z, \nu\right)\right]$$
$$+ \frac{1}{3}\left[\mathsf{H}\text{ -}\mathsf{Ell}\left(\frac{\tau}{3}, z, \nu\right) + \mathsf{H}\text{ -}\mathsf{Ell}(3\tau, 3z, 3\nu) + \mathsf{H}\text{ -}\mathsf{Ell}\left(\frac{\tau+1}{3}, z, \nu\right)\right.$$
$$\left. + \mathsf{H}\text{ -}\mathsf{Ell}\left(\frac{\tau+2}{3}, z, \nu\right)\right]. \tag{4.13}$$

In particular, comparing term to term

$$\begin{aligned}\mathtt{DT}^{\mathrm{ref}} &= \frac{1}{p}\mathtt{DT}_0^{\mathrm{ref}} + \mathtt{DT}_1^{\mathrm{ref}} + p\,\mathtt{DT}_2^{\mathrm{ref}} + \cdots \\ &= -\frac{1}{p}\frac{1}{\phi_{KKP}(\tau, z, \nu)}\left(\mathsf{H}\text{ -}\mathsf{Ell}\left(\mathrm{Hilb}^{[0]}(\mathrm{K3})\right)\right. \\ &\quad \left. + p\,\mathsf{H}\text{ -}\mathsf{Ell}\left(\mathrm{Hilb}^{[1]}(\mathrm{K3})\right) + p^2\,\mathsf{H}\text{ -}\mathsf{Ell}\left(\mathrm{Hilb}^{[2]}(\mathrm{K3})\right) + \cdots\right)\end{aligned} \tag{4.14}$$

we see that we can write explicitly the refined invariants in terms of the Hodge-Elliptic genus and its powers (computed at different values of the arguments)

$$\begin{aligned}\mathtt{DT}_0^{\mathrm{ref}} &= -\frac{1}{\phi_{KKP}(\tau, z, \nu)}, \\ \mathtt{DT}_1^{\mathrm{ref}} &= \mathtt{DT}_0^{\mathrm{ref}}\,\mathsf{H}\text{ -}\mathsf{Ell}(\tau, z, \nu), \\ \mathtt{DT}_2^{\mathrm{ref}} &= \mathtt{DT}_0^{\mathrm{ref}}\frac{1}{2} \\ &\quad \left[\mathsf{H}\text{ -}\mathsf{Ell}(\tau, z, \nu)^2 + \mathsf{H}\text{ -}\mathsf{Ell}\left(\frac{\tau}{2}, z, \nu\right) + \mathsf{H}\text{ -}\mathsf{Ell}(2\tau, 2z, 2\nu) + \mathsf{H}\text{ -}\mathsf{Ell}\left(\frac{\tau+1}{2}, z, \nu\right)\right].\end{aligned} \tag{4.15}$$

Using the explicit form of the Hodge-Elliptic genus, we have a prediction for the full series of refined Donaldson–Thomas invariants, order by order. Now, we will specialize to different version of the Hodge-Elliptic genus and try to extract geometrical information from the counting functions.

The first prediction was actually proven in Katz et al. [36]

$$\mathtt{DT}_0^{\mathrm{ref}} = -\frac{1}{\phi_{KKP}(\tau, z, \nu)} = \frac{1}{\eta(\tau)^{18}\ \theta_1(\tau, z+\nu)\ \theta_1(\tau, z-\nu)} \tag{4.16}$$

and is universal, independent of the particular Hodge-Elliptic genus one is considering.

Consider first the complex Hodge-Elliptic partition function (3.45). Multiplying it by (4.16) and using the identities (2.38) and (2.39), we find the prediction

$$\mathtt{DT}_1^{ref,c} = -\left(\frac{\theta_1(q,y)^2}{\theta_1(q,uy)\,\theta_1(q,\frac{y}{u})}\right)\frac{1}{\eta(q)^{24}}\left[\left(20 + 2u + \frac{2}{u}\right)\right.$$
$$\left(1 + \frac{y}{(1-y)^2} + \sum_{d=1}^{\infty}\sum_{k|d} k(y^k + y^{-k})\,q^d\right)$$

$$
- \left(18 + 2u + \frac{2}{u}\right) + 24 \sum_{d=1}^{\infty} \sum_{k|d} k(-2)\, q^d + \left(2 - \frac{1}{u} - u\right)
\left(\frac{q^4 + 2q^3 + 6q^2 + 2q + 1}{(q-1)(q+1)(q^2+q+1)}\right)\Bigg]. \tag{4.17}
$$

One of the effects of the refinement is to produce an overall ratio of theta functions. We see that the vertical curves with nodal fibers are refined also by a multiplicative factor $\left(20 + 2u + \frac{2}{u}\right)$, which replaces the 24 which counts the number of singular fibers. This factor is replaced by $\left(18 + 2u + \frac{2}{u}\right)$ for smooth vertical curves. The refinement of the diagonal curves is more complicated. The contribution from the diagonal curves captured by the Eisenstein series should really be paired with the rational function, since they are the only q-dependent terms without any y-dependence (up to the constant term in the expansion of the rational function). Indeed, we show in Appendix B that the rational function can be interpreted as shifting the first terms in the Eisenstein sum, when this is rephrased in terms of cyclotomic polynomials. However, we do not have a simple geometrical interpretation of this.

Consider now the Kummer surface, the $\mathbb{Z}_2$ orbifold of complex two-tori. In this case, we consider the orbifold Hodge-Elliptic partition function (3.13) multiplied by (4.16). By using again the identities (2.38) and (2.39), we find the prediction

$$
\begin{aligned}
\mathtt{DT}_1^{\mathrm{ref,orb}} = &- \left(\frac{\theta_1(q,y)^2}{\theta_1(q,uy)\,\theta_1(q,\frac{y}{u})}\right) \frac{1}{\eta(q)^{24}} \Bigg[\left(20 + 2u + \frac{2}{u}\right) \\
&\left(1 + \frac{y}{(1-y)^2} + \sum_{d=1}^{\infty} \sum_{k|d} k(y^k + y^{-k})\, q^d\right) \\
&- \left(20 + u + \frac{1}{u}\right) + \left(28 - \frac{2}{u} - 2u\right)\left(\sum_{d=1}^{\infty} \sum_{k|d} k(-2)\, q^d\right) \\
&- \left(16 - \frac{8}{u} - 8u\right)\left(\sum_{d=1}^{\infty} \sum_{k|d} k(-2)\, q^{2d}\right)\Bigg]
\end{aligned} \tag{4.18}
$$

We see that the contribution of the vertical curves with nodal fibers has the same structure as in the previous case (4.17), while now smooth vertical curves contribute with a weight $\left(20 + u + \frac{1}{u}\right)$. More complicated is the situation for diagonal curves. Similarly to what happens in (4.17), the effect of the refinement on diagonal curves involves correcting the coefficients of the Eisenstein series by u-dependent shifts. As above it is not clear what is the geometrical interpretation.

Finally, let us consider the generic Hodge-Elliptic genus (3.16). As before multiplying by (4.16), we obtain

$$\begin{aligned}\mathtt{DT}_1^{\mathrm{ref,g}} = -\left(\frac{\theta_1(\tau,z)^2}{\theta_1(\tau,z+\nu)\,\theta_1(\tau,z-\nu)}\right)\frac{1}{\eta(\tau)^{24}}\Bigg[-\frac{22}{24}\left(20+\frac{2}{u} \quad +2u\right) \\ +\left(20+\frac{2}{u}+2u\right)\sum_{d=1}^{\infty}\sum_{k|d}(-2k)\,q^d + \left(20+2u+\frac{2}{u}\right) \\ \left(1+\frac{y}{(1-y)^2}+\sum_{d=1}^{\infty}\sum_{k|d}k(y^k+y^{-k})\,q^d\right) \\ -\left(2-\frac{1}{u}-u\right)\eta^3(\tau)\left(\frac{1}{12}H(\tau)+q^{-\frac{1}{8}}\right)\Bigg] \end{aligned} \tag{4.19}$$

The first two lines are a simple multiplicative refinement of the Elliptic genus, and we recognize immediately the contributions of the various types of curves. The last line is more challenging to interpret. Also in the two previous cases, one can start with a multiplicative refinement of the Elliptic genus, as in (3.46) or (3.13), but then the extra terms conspire precisely to alter the weights of the contributions of various curves. Here, the situation is different, due to the appearance of the function $H(\tau)$. As we have remarked in Sect. 2.3, this function has a clear interpretation in terms of the geometry of $\mathrm{K3}\times E$ only when in combination with the Appell–Lerch sum $\mu(\tau,z)$. Its appearance in the refined Donaldson–Thomas generating function leads to the surprising prediction that in the refined setting its coefficients, which are related to the dimensions of the irreducible representations of the Mathieu group $\mathbb{M}_{24}$, should have a direct geometric representation in terms of the reduced Hilbert scheme on $\mathrm{K3}\times E$. Note that this does not happen in $\phi_{KKP}(\tau,z,\nu)$ but only in the higher-order terms in the Donaldson–Thomas expansion. Furthermore, the term $(2-\frac{1}{u}-u)$ guarantees that the unrefined theory is blind to the action of $\mathbb{M}_{24}$ on the Hilbert space of BPS states.

Consider now the Hilbert space decomposition $\mathcal{H}_{h=1}=\bigoplus_{d,n}\mathcal{H}_{d,n}q^{d-1}(-y)^n$ where the Donaldson–Thomas invariant is interpreted as a refined Witten index associated to each Hilbert space subfactor. Each $\mathcal{H}_{d,n}$ can be in turn decomposed as a sum of factors according to the structure of (4.19) and (4.16), for example, by keeping track of the diagonal and vertical curve contributions. Due to the presence of the function $H(\tau)$ in (4.19), we see that one of these factors carries the action of the Mathieu group. It would be interesting to develop this further.

Higher-order enumerative invariants can be computed similarly, using the expansion (4.14). The first few orders for $\mathtt{DT}_2^{\mathrm{ref}}$ and $\mathtt{DT}_3^{\mathrm{ref}}$ for all three Hodge-Elliptic genera can be found in the supporting MATHEMATICA file [16].

TABLE 1. Data characterizing the non-trivial minimal subgroups and their $\tilde{T}_g(\tau)$ functions

Conjugacy class	Number of fixed points	Eigenvalues at the fixed points	$\tilde{T}_g(\tau)$
2A	8	$8 \times (-1,-1)$	$16\Lambda_2$
3A	6	$6 \times (\zeta_3, \zeta_3^{-1})$	$6\Lambda_3$
4B	4	$4 \times (\mathrm{i}, -\mathrm{i})$	$4\,(-\Lambda_2 + \Lambda_4)$
5A	4	$2 \times (\zeta_5, \zeta_5^{-1})$ and $2 \times (\zeta_5^2, \zeta_5^{-2})$	$2\Lambda_5$
6A	2	$2 \times (\zeta_6, \zeta_6^{-1})$	$2\,(-\Lambda_2 - \Lambda_3 + \Lambda_6)$
7AB	3	$(\zeta_7, \zeta_7^{-1}), (\zeta_7^2, \zeta_7^{-2})$ and $(\zeta_7^3, \zeta_7^{-3})$	Λ_7
8A	2	(ζ_8, ζ_8^{-1}) and $(\zeta_8^3, \zeta_8^{-3})$	$-\Lambda_4 + \Lambda_8$

5. The Mathieu Group and Twined Complex Hodge-Elliptic Genera

In this section, we will consider the case where there is a finite symmetry group G which commutes with the superconformal symmetries of the model. In this case, one can define the elliptic genus twined by $g \in G$, see [14,22,28,29] for a sample of the literature. We can generalize (2.8) and define the elliptic genus twined by any $g \in G$ as

$$\mathsf{Ell}_g = \mathrm{Tr}_{\mathcal{H}_{RR}} \left(g\,(-1)^{J_0 + \overline{J}_0} y^{J_0} q^{L_0 - \frac{c}{24}} \overline{q}^{\overline{L}_0 - \frac{c}{24}} \right) . \tag{5.1}$$

In this section, we will introduce twined complex Hodge-Elliptic genera and compute explicitly several examples.

In particular, there is a class of twining genera which are induced by certain geometric automorphisms of K3 surfaces. A geometric automorphism which acts trivially on the holomorphic 2-forms of the K3 is called symplectic. The automorphisms which have finite order, so that they only have isolated fixed points, are classified and correspond to certain subgroups of the Mathieu group $\mathbb{M}_{23}$, with certain conditions. The non-trivial minimal subgroups are given in the following table, together with their ATLAS names as elements of $\mathbb{M}_{24}$, the number of fixed points and their eigenvalues at the fixed point:

The corresponding Elliptic genus can be computed as before, with the result

$$\mathsf{Ell}_g(\tau, z) = \frac{1}{12} \chi_g(\mathrm{K3})\, \varphi_{0,1}(\tau, z) + \tilde{T}_g(\tau)\, \varphi_{-2,1}(\tau, z) \tag{5.2}$$

where $\chi_g(\mathrm{K3})$, counting the number of fixed points, and the function $\tilde{T}_g(\tau)$ can be read off table 1. The functions $\Lambda_N(\tau)$ were introduced in (3.10).

This result follows from the fixed point formula applied to the integral (3.32) by writing

$$\int \mathrm{ch}(S^N \mathcal{T}) \mathrm{Todd}(\mathcal{T}) = \sum_{\lambda_i} \frac{\chi_N(\lambda_i)}{(1-\lambda_i)\,(1-\lambda_i^{-1})} , \tag{5.3}$$

where the eigenvalues λ_i can be read off table (1), and χ_N denotes the SU(2) characters. The result can be compactly written as an equivariant Euler characteristic

$$\mathsf{Ell}_g(\tau,z)=\sum_{N=0}^{\infty}\mathscr{F}_N(\tau,z)\sum_{\lambda_i}\frac{\chi_N(\lambda_i)}{(1-\lambda_i)\left(1-\lambda_i^{-1}\right)}\,, \tag{5.4}$$

where the function $\mathscr{F}_N(\tau,z)$ was defined in (3.29) in terms of the functions (3.30) and (3.31). The computation proceeds as in Section 3.6. For example, in the case $[g]=2A$

$$\begin{aligned}\mathsf{Ell}_{[g]=2A}(\tau,z)&=\frac{\theta_1(\tau,z)^2}{\eta(\tau)^3}\\ &\left[\frac{1}{\eta(\tau)^3}\left(-2\widetilde{s}_0(\tau)+8\widetilde{s}_1(\tau)-14\widetilde{s}_2(\tau)+16\sum_{N=3}^{\infty}(-1)^{N+1}\widetilde{s}_N(\tau)\right)+8\mu(\tau,z)\right].\end{aligned} \tag{5.5}$$

The quantity in round brackets can be written as

$$\begin{aligned}&8\widetilde{s}_1(\tau)+\frac{1}{3}\left(-2\widetilde{s}_0(\tau)+50\widetilde{s}_2(\tau)+48\sum_{N=3}^{\infty}\widetilde{s}_N(\tau)\right)\\ &\qquad-\frac{1}{3}\left(4\widetilde{s}_0(\tau)+92\widetilde{s}_2(\tau)+96\sum_{N=2}^{\infty}\widetilde{s}_{2N}(\tau)\right)\\ &\quad=\frac{1}{3}(-2E_2(\tau)+48F_2^{(2)}(\tau))-16\Lambda_2(\tau)\end{aligned} \tag{5.6}$$

where we have repeatedly used the analytical continuation[5] $\sum_{n=1}^{\infty}n=-1/12$ and (3.39) to write

$$\begin{aligned}16\Lambda_2(\tau)&=\frac{4}{3}\left(-1-48\sum_{n=1}^{\infty}\frac{n}{1-q^{2n}}+24\sum_{n=1}^{\infty}\frac{n}{1-q^n}\right)\\ &=\frac{4}{3}\left(-1+\frac{24}{1-q}+24\sum_{N=1}^{\infty}\frac{2N-1}{1-q^{2N-1}}\right)\\ &=\frac{1}{3}\left(4\widetilde{s}_0(\tau)+92\widetilde{s}_2(\tau)+96\sum_{N=2}^{\infty}\widetilde{s}_{2N}(\tau)\right).\end{aligned} \tag{5.7}$$

By putting everything together, we can write

$$\begin{aligned}\mathsf{Ell}_{[g]=2A}(\tau,z)&=\frac{1}{3}\mathsf{Ell}(\tau,z)-16\Lambda_2(\tau)\frac{\theta_1(\tau,z)^2}{\eta(\tau)^6}\\ &=\frac{\chi_{[g]=2A}(\mathrm{K3})}{12}\varphi_{0,1}(\tau,z)+16\Lambda_2(\tau)\varphi_{-2,1}(\tau,z).\end{aligned} \tag{5.8}$$

The remaining cases can all be treated similarly and are left to the reader.

[5] We stress again that the above identities do not hold in a truncated form.

We can now define a twining Hodge-Elliptic genus as

$$\mathsf{H\text{-}Ell}_g(\tau, z, \nu) = \mathrm{Tr}\left(g\,(-1)^{J_0+\overline{J}_0} y^{J_0} u^{\overline{J}_0} q^{L_0-\frac{c}{24}}\right) \tag{5.9}$$

where as before the right movers are in the ground states. Geometrically, we can start from the relation (3.42) to define

$$\mathsf{H\text{-}Ell}_g(\tau, z, \nu) = \mathsf{Ell}_g(\tau, z) - \left[1 - \frac{1}{2}\left(u + \frac{1}{u}\right)\right] \mathscr{F}_0(\tau, z)\,\chi_g(\mathcal{O})\,. \tag{5.10}$$

The computation of the complex Hodge-Elliptic genus proceeds as in Sect. 3.7, by weighting the contribution of the flat bundles with an appropriate factor. The flat bundle contribution is the $N = 0$ term in the sum (5.4). One can see by direct computation that this term is the same for all conjugacy classes and is determined by the combination

$$-2\widetilde{s}_0(\tau) + 4\widetilde{s}_1(\tau) - 4\widetilde{s}_3(\tau) + 2\widetilde{s}_4(\tau)\,. \tag{5.11}$$

Then, repeating the computations of Sect. 3.7 step by step now gives the flat bundle contribution

$$\begin{aligned}\mathsf{Flat}(\tau, z) = {} & \frac{1}{2}\frac{\theta_1(q, y)^2}{\eta(q)^3} 8\mu(q, y) + \frac{\theta_1(q, y)^2}{\eta(q)^6} \\ & \times \left[\frac{2(1 + 2\,q + 6\,q^2 + 2\,q^3 + q^4)}{(q - 1)(q + 1)(1 + q + q^2)} + 8F_2^{(2)}(q)\right]\end{aligned} \tag{5.12}$$

so that the Hodge-Elliptic twining genera are given by

$$\mathsf{HEll}_g(\mathrm{K3}) = \frac{1}{12}\chi_g(\mathrm{K3})\,\varphi_{0,1} + \tilde{T}_g\,\varphi_{-2,1} - \left(1 - \frac{1}{2}\left(u + \frac{1}{u}\right)\right) \mathsf{Fl}(q, y) \tag{5.13}$$

Equivalently, we can rewrite this as

$$\begin{aligned}\mathsf{HEll}_g(\mathrm{K3}) = {} & \frac{1}{24}\chi_{g,u}(\mathrm{K3})\mathsf{Ell}_g(\tau, z) + \tilde{T}_g\,\varphi_{-2,1} \\ & - 2\left(1 - \frac{1}{2}\left(u + \frac{1}{u}\right)\right) \frac{\theta_1(\tau, z)^2}{\eta(\tau)^6} \left[\frac{(1 + 2\,q + 6\,q^2 + 2\,q^3 + q^4)}{(q - 1)(q + 1)(1 + q + q^2)} + \frac{1}{6}E_2(\tau)\right]\end{aligned} \tag{5.14}$$

in terms of the equivariant χ_u genus

$$\chi_{g,u}(\mathrm{K3}) = \frac{1}{u}\sum_{\lambda_i} \frac{(1 - u\lambda_i)\left(1 - u\lambda_i^{-1}\right)}{(1 - \lambda_i)\left(1 - \lambda_i^{-1}\right)} = 2u + \chi_g(\mathrm{K3}) - 4 + \frac{2}{u} \tag{5.15}$$

evaluated by localization.

6. Conclusions

In this note, we have studied the implications of three Hodge-Elliptic genera for the refined enumerative geometry of $\mathrm{K3} \times E$. We have found explicit formulas showing how the ordinary Donaldson–Thomas sums are modified, in the process elucidating also certain aspects of the ordinary Elliptic genus. The results should function as a starting point to study refinements of Donaldson–Thomas theory on K3 $\times$ E. We conclude with a few comments and open problems:

- It would be very interesting to reproduce the results obtained here via a direct localization computation. As already mentioned in the text, a possible strategy is to use the full K-theory vertex and afterward take a certain scaling limit of the toric weights. A reason this could work is that the localization computation of the ordinary enumerative invariants is carried on by using a stratification of the moduli space, where strata are separately toric. While the toric action will not glue globally, the scaling limit should individuate a parameter associated with the square root of the canonical bundle and therefore should make sense globally. Note that there are several possible scaling limits of the toric weights, a fact perhaps explaining why different versions of the Hodge-Elliptic genus can appear.
- We do not really understand the modular properties of Hodge-Elliptic genera. They fail to be modular but in a very specific way. It would be interesting to explore this issue further.
- The relation between jumping phenomena, wall-crossing and the analytic properties of the refined BPS counting function should be established precisely.
- An intriguing aspect of the construction is the appearance of the function $H(\tau)$ whose coefficients are related to the representations of the Mathieu group $\mathbb{M}_{24}$. This function appears in the generic Hodge-Elliptic genus and therefore in the enumerative geometry of K3 $\times$ E. In contrast with what happens in the case of the Elliptic genus, in the refinement this function appears unpaired with the Appell–Lerch sum $\mu(\tau, z)$. In the unrefined case, the combination of these two functions decomposes in a non-trivial way to capture the geometry of curves in the target space. Unfortunately in this decomposition the individual coefficients are mixed up and all information of the representations of $\mathbb{M}_{24}$ is lost. Apparently, this is not the case for the generic Hodge-Elliptic genus. Our results imply that it should be possible to see the Mathieu Moonshine phenomenon geometrically when one considers refined invariants which have nonzero indices in both the K3 and the elliptic curve directions. This phenomenon appears only starting from $\mathtt{DT}_1$ but persists at higher orders.
- As we have explained, it is natural to conjecture that the refined partition functions that we have constructed in the course of the paper are related to a refinement of Donaldson–Thomas theory. This leads naturally to the further conjecture that the refined counting of BPS states is related to the refined topological string. As was conjectured in Oberdieck and Pandharipande [42] and then proven in Oberdieck and Pixton [43], the BPS counting function Φ_{10} can be expressed in terms of the (disconnected, reduced) Gromov–Witten invariants.

  One can define the generating function

$$\mathsf{GW} = \sum_{g\in\mathbb{Z}} \sum_{h\geq 0} \sum_{d\geq 0} \mathsf{GW}_{g,h,d}\ \lambda^{2g-2} p^{h-1} q^{d-1} \tag{6.1}$$

where the labels have the same meaning as in (2.29). The generating function of Gromov–Witten invariants is related to the Igusa cusp form as

$$\mathsf{GW} = -\frac{1}{\Phi_{10}} \tag{6.2}$$

where the genus counting parameter λ appears in the right-hand side via the substitution $q = -\mathrm{e}^{\mathrm{i}\,\lambda}$.

It is natural to propose that the GW-DT correspondence also holds in the refined setting for $\mathrm{K3} \times E$. This corresponds to considering the $\mathcal{N} = 4$ string on the Ω-background and taking an appropriate limit [41]. That such a construction should be possible is suggested by the fact that in the Donaldson–Thomas side the relevant moduli space admits a stratification where each strata admits a toric action, even if this does not glue globally. Since the GW-DT correspondence should hold in the large volume limit, it should be compared with the chiral Hodge-Elliptic genus. As an aside comment, it is an interesting open problem to define the worldsheet theory of the refined $\mathcal{N} = 4$ topological string, generalizing the construction of [4].

- We have computed certain twining complex Hodge-Elliptic genera associated with geometrical automorphisms of K3 surfaces. It would be interesting to extend this computation to all the available twisted-twined genera, as well as the generic and orbifold Hodge-Elliptic genera, to predict enumerative invariants which would refine the partition functions computed in Refs. [11,12,19,30].

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Data Availability All data generated or analyzed during this study are included in this article (and its supplementary information files). Alternatively, they are freely available for download at https://cirafici.dmg.units.it/HEsupporting.nb, as well as with the arXiv submission.

Declarations

Conflict of interest The corresponding author states that there is no conflict of interest.

A Some Technical Identities

In this appendix, we prove some of the technical identities used in the main text

Lemma A.1. *The following identities hold*

$$\sum_{i=-k}^{k} \frac{1}{(1-yq^i)(1-\frac{q^{-i-1}}{y})} = \frac{q}{(1-q^{k+1}y)(1-\frac{q^k}{y})} \frac{q^{2k+1}-1}{q-1}\,, \tag{A.2}$$

$$\sum_{i=-k}^{k} \frac{1}{(1-yq^i)(1-\frac{q^{N-1-i}}{y})} = \sum_{i=0}^{N-2} \frac{1}{(1-q^{k-i}y)(1-\frac{q^{k+(N-1)-i}}{y})} \frac{q^{2k+(N-1)-2i}-1}{q^{N-1}-1}\,, \tag{A.3}$$

for $k \geq 0$ *and* $N \geq 2$.

Proof. Consider first (A.2). We prove it by induction. Setting $k=0$, we see

$$\frac{1}{1-y}\frac{1}{1-\frac{1}{qy}} = \frac{1}{1-qy}\frac{1}{1-\frac{1}{y}} q\,, \tag{A.4}$$

which indeed holds. Now, we assume that (A.2) holds for a certain k and we prove the identity for $k+1$. For $k+1$, we use the induction hypothesis to write the left-hand side of (A.2) as

$$\begin{aligned}
&\sum_{i=-k-1}^{k+1} \frac{1}{(1-yq^i)(1-\frac{q^{-i-1}}{y})} \\
&\quad = \frac{1}{1-q^{-k-1}y}\frac{1}{1-\frac{q^k}{y}} + \frac{q}{(1-q^{k+1}y)(1-\frac{q^k}{y})}\frac{q^{2k+1}-1}{q-1} \\
&\qquad + \frac{1}{1-q^{k+1}y}\frac{1}{1-\frac{q^{-k-2}}{y}} \\
&\quad = \frac{q}{(1-q^{k+2}y)(1-\frac{q^{k+1}}{y})}\frac{q^{2k+3}-1}{q-1}\,,
\end{aligned} \tag{A.5}$$

which holds by direct computation.

Consider now (A.3). Again we proceed by induction. For $k = 0$, we have to prove the identity

$$\frac{1}{(1-y)}\frac{1}{(1-\frac{q^{N-1}}{y})} = \sum_{i=0}^{N-2}\frac{1}{(1-q^{-i}y)}\frac{1}{(1-\frac{q^{N-1-i}}{y})}\frac{q^{N-1-2i}-1}{q^{N-1}-1}\,, \qquad \text{(A.6)}$$

or equivalently

$$\frac{q^{N-1}-1}{1-y}\frac{1}{1-\frac{q^{N-1}}{y}} = \sum_{i=0}^{N-2}\frac{q^{N-1-2i}-1}{1-q^{-i}y}\frac{1}{1-\frac{q^{N-1-i}}{y}}\,. \qquad \text{(A.7)}$$

This identity holds if and only if the following identity

$$\sum_{i=1}^{N-2}\frac{q^{N-1-2i}-1}{1-q^{-i}y}\frac{1}{1-\frac{q^{N-1-i}}{y}} = 0 \qquad \text{(A.8)}$$

is true and only the $i = 0$ term in the sum is non-trivial. To prove this, let us consider separately the cases where N is odd or even.

If N is odd, we can split the sum as

$$\sum_{i=1}^{N-2} = \sum_{i=1}^{\frac{N-3}{2}} + \sum_{i=\frac{N-1}{2}} + \sum_{i=\frac{N+1}{2}}^{N-2} \qquad \text{(A.9)}$$

The middle term is identically zero, since $q^{(N-1)-2i}$ equals 1 when $i = \frac{N-1}{2}$. To evaluate the last term, let us relabel $i = (N-1) - j$. Then, since

$$\begin{aligned}\frac{q^{N-1-2i}-1}{1-q^{-i}y}\frac{1}{1-\frac{q^{N-1-i}}{y}} &= \frac{q^{N-1-2((N-1)-j)}-1}{1-q^{-((N-1)-j)}y}\frac{1}{1-\frac{q^{N-1-((N-1)-j)}}{y}} \\ &= -\frac{q^{N-1-2j}-1}{1-q^{-j}y}\frac{1}{1-\frac{q^{N-1-j}}{y}},\end{aligned} \qquad \text{(A.10)}$$

the first and third sums cancel exactly and (A.8) is proven.

Consider now N even. Now, we simply split

$$\sum_{i=1}^{N-2} = \sum_{i=1}^{\frac{N-2}{2}} + \sum_{i=\frac{N-2}{2}+1}^{N-2}\,. \qquad \text{(A.11)}$$

By using the same relabeling $i = (N-1) - j$, we see that again the sums cancel each other. Therefore, the first step of the induction is proven.

Now, let us assume that the identity (A.3) holds for a generic k. To simplify the exposition introduces the notation

$$g_{k,N}(i) = \frac{1}{(1-q^{k-i}y)(1-\frac{q^{k+(N-1)-i}}{y})}\frac{q^{2k+(N-1)-2i}-1}{q^{N-1}-1}\,. \qquad \text{(A.12)}$$

Using the induction hypothesis, we can write

$$\sum_{i=-k-1}^{k+1} \frac{1}{(1-yq^i)(1-\frac{q^{N-i-1}}{y})} = \frac{1}{(1-yq^{-k-1})(1-\frac{q^{N+k}}{y})} + \sum_{i=0}^{N-2} g_{k,N}(i) + \frac{1}{(1-yq^{k+1})(1-\frac{q^{N-2-k}}{y})}. \tag{A.13}$$

We have to prove that this is equivalent to $\sum_{i=0}^{N-2} g_{k+1,N}(i)$. However, since in $g_{k,N}(i)$, k and i only appear in the combination $k-i$, it is easy to see that $\sum_{i=0}^{N-2} g_{k+1,N}(i) = \sum_{i=0}^{N-2} g_{k,N}(i-1)$. Therefore, we have to prove that

$$\frac{1}{(1-yq^{-k-1})(1-\frac{q^{N+k}}{y})} + \sum_{i=0}^{N-2} g_{k,N}(i) + \frac{1}{(1-yq^{k+1})(1-\frac{q^{N-2-k}}{y})} - \sum_{i=0}^{N-2} g_{k,N}(i-1) = 0. \tag{A.14}$$

It is easy to see that

$$\sum_{i=0}^{N-2} (g_{k,N}(i) - g_{k,N}(i-1)) = g_{k,N}(N-2) - g_{k,N}(-1) \tag{A.15}$$

since the terms in the sum cancel pairwise. Then, it is only a matter of brute force to show that the identity

$$\frac{1}{(1-yq^{-k-1})(1-\frac{q^{N+k}}{y})} + \frac{1}{(1-yq^{k+1})(1-\frac{q^{N-2-k}}{y})} + g_{k,N}(N-2) - g_{k,N}(-1) = 0 \tag{A.16}$$

holds. □

We now can prove our

Proposition A.17. *We have that*

$$\tilde{s}_0(\tau) = \frac{q}{1-q}, \tag{A.18}$$

$$\tilde{s}_N(\tau) = \frac{N-1}{1-q^{N-1}}, \tag{A.19}$$

where $N \geq 2$

Proof. By definition, we have

$$\tilde{s}_M(\tau) = \lim_{k\to\infty} \sum_{i=-k}^{k} \frac{1}{(1-yq^i)(1-\frac{q^{N-1-i}}{y})}, \tag{A.20}$$

with $M = 0$ or $M \geq 2$. By the results of the previous lemma

$$\tilde{s}_0(\tau) = \lim_{k\to\infty} \frac{1}{(1-q^{k+1}y)(1-\frac{q^k}{y})} q \frac{q^{2k+1}-1}{q-1}, \tag{A.21}$$

$$\tilde{s}_N(\tau) = \lim_{k\to\infty} \sum_{i=0}^{N-2} \frac{1}{(1-q^{k-i}y)(1-\frac{q^{k+(N-1)-i}}{y})} \frac{q^{2k+(N-1)-2i}-1}{q^{N-1}-1} \,. \tag{A.22}$$

To take the limit, recall that $q = \mathrm{e}^{2\pi\,\mathrm{i}\,\tau}$ is an analytic function of τ defined in the upper half plane. Therefore, $\lim_{k\to\infty} q^k = 0$ since the oscillatory part is bounded.

Then, the result (A.18) follows immediately. To derive (A.19), we commute the limit with the finite sum and evaluate the limit at N and i fixed. Each limit gives the result $\frac{1}{1-q^{N-1}}$; evaluating the finite sum gives the result. □

Finally, we have

Proposition A.23. *We have the following identity*

$$s_1(\tau,z) = \frac{\theta_1(\tau,z)}{\eta(\tau)^3}\, 2F_2^{(2)}(\tau) + \theta_1(\tau,z)\,\mu(\tau,z)\,. \tag{A.24}$$

Proof. From the definition of $s_1(\tau,z)$, we have

$$s_1(\tau,z) = \frac{\theta_1(\tau,z)}{\eta^3(\tau)} \sum_{i\in\mathbb{Z}} \frac{1}{(1-yq^i)(1-\frac{1}{y}q^{-i})} = -\frac{\theta_1(\tau,z)}{\eta^3(\tau)} \sum_{i\in\mathbb{Z}} \frac{yq^i}{(1-yq^i)^2} \,. \tag{A.25}$$

We can write

$$\sum_{i\in\mathbb{Z}} \frac{yq^i}{(1-yq^i)^2} = \frac{y}{(1-y)^2} + \sum_{i=1}^{\infty} \frac{yq^i}{(1-yq^i)^2} + \sum_{i=1}^{\infty} \frac{yq^{-i}}{(1-yq^{-i})^2} \,. \tag{A.26}$$

By using the expansions

$$\frac{xy}{(1-xy)^2} = \sum_{n=1}^{\infty} n x^n y^n \,, \qquad \frac{x^{-1}y}{(1-x^{-1}y)^2} = \sum_{n=1}^{\infty} n \frac{x^n}{y^n} \,, \tag{A.27}$$

we find

$$\begin{aligned} &\sum_{i=1}^{\infty} \frac{yq^i}{(1-yq^i)^2} + \sum_{i=1}^{\infty} \frac{yq^{-i}}{(1-yq^{-i})^2} \\ &\quad = \sum_{i=1}^{\infty}\sum_{n=1}^{\infty} n \left(y^n + \frac{1}{y^n}\right) q^{ni} \\ &\quad = \sum_{d=1}^{\infty}\sum_{d|k} k \left(y^k + \frac{1}{y^k}\right) q^d \,. \end{aligned} \tag{A.28}$$

Putting all together, we see

$$s_1(\tau,z) = -\frac{\theta_1(\tau,z)}{\eta^3(\tau)} \left(\frac{y}{(1-y)^2} + \sum_{d=1}^{\infty}\sum_{d|k} k \left(y^k + \frac{1}{y^k}\right) q^d \right) \tag{A.29}$$

and the proposition follows from the identity (2.39) □

B The Eisenstein Series $E_2(\tau)$ and the Cyclotomic Polynomials

To have a different perspective on the rational function in (3.46), let us consider a truncated expansion of the Eisenstein series $E_2(\tau)$ in terms of cyclotomic polynomials. Recall that the cyclotomic polynomials are defined as

$$\Phi_n(x) = \prod_{\substack{1\leq k\leq n\\ \gcd(k,n)=1}} \left(x - \mathrm{e}^{2\pi\,\mathrm{i}\,\frac{k}{n}}\right) . \tag{B.1}$$

In particular, one has

$$x^n - 1 = \prod_{d|n} \Phi_d(x) . \tag{B.2}$$

Consider now the following truncation of the Eisenstein series

$$E_2^{(N)}(\tau) = 1 - 24\sum_{n=1}^{N} \frac{nq^n}{1-q^n} = 1 - 24\sum_{n=1}^{N} \frac{n}{1-q^n} + 24\sum_{n=1}^{N} n . \tag{B.3}$$

Using (B.2), we can write

$$-\frac{n}{1-q^n} = n\prod_{d|n} \frac{1}{\Phi_d(q)} . \tag{B.4}$$

It is easy to see that

$$n\prod_{d|n} \frac{1}{\Phi_d(q)} = \sum_{d|n} \frac{\mathrm{Reminder}\,(q\,\Phi_d'(q);\Phi_d(q))}{\Phi_d(q)} , \tag{B.5}$$

expressed in terms of the reminder of the division of the polynomial $q\,\Phi_d'(q)$ by $\Phi_d(q)$. Indeed by taking the logarithmic derivative of (B.2), one finds

$$\frac{n\,q^n}{\prod_{d|n}\Phi_d(q)} = \sum_{d|n} \frac{q\,\Phi_d'(q)}{\Phi_d(q)} , \tag{B.6}$$

which implies by adding and subtracting 1 to the numerator on the left-hand side

$$\frac{n}{\prod_{d|n}\Phi_d(q)} = -n + \sum_{d|n} \frac{q\,\Phi_d'(q)}{\Phi_d(q)} . \tag{B.7}$$

The result follows from the fact that

$$\frac{q\,\Phi_d'(q)}{\Phi_d(q)} = \phi(d) + \frac{\mathrm{Reminder}\,(q\,\Phi_d'(q);\Phi_d(q))}{\Phi_d(q)} \tag{B.8}$$

which, for example, can be seen directly using the long polynomial division algorithm. Here, $\phi(d)$ denotes the degree of $\Phi_d(q)$. Finally, (B.5) follows by substituting (B.8) into (B.7) and using $\sum_{d|n}\phi(d) = n$.

Therefore, we can write

$$E_2^{(N)}(\tau) = 1 + 24\sum_{n=1}^{N}\sum_{d|n} \frac{\mathrm{Reminder}\,(q\,\Phi_d'(q);\Phi_d(q))}{\Phi_d(q)} + 24\sum_{n=1}^{N} n$$

$$= 1 + 24 \sum_{n=1}^{N} r_N(n) \frac{\text{Reminder}\,(q\,\Phi_n'(q); \Phi_n(q))}{\Phi_n(q)} + 24 \sum_{n=1}^{N} n \,, \qquad \text{(B.9)}$$

where the coefficients $r_N(n)$ count the number of times an integer n appears as a divisor of the set of integers $\{1, \ldots, N\}$

$$r_N(n) = \text{Coeff} \left(\sum_{k=1}^{N} \sum_{i|k} x^i \ ; \ x^n \right) . \qquad \text{(B.10)}$$

Since we can write

$$\frac{q^4 + 2q^3 + 6q^2 + 2q + 1}{(q-1)(q+1)(q^2+q+1)} = -1 - 2\frac{1}{\Phi_1(q)} - 2\frac{1}{\Phi_2(q)} + \frac{\text{Reminder}\,(q\,\Phi_3'(q); \Phi_3(q))}{\Phi_3(q)} \,, \qquad \text{(B.11)}$$

the rational function can be thought of as a shift of the first coefficients of the Eisenstein series.

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