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On the Classical and Quantum Aspects of Memory Effects in Open Dynamical Systems

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ABSTRACT

Open systems that are strongly coupled to their environments generally manifest memory effects in their irreversible evolution, such as revivals of distances or divergences, which can be interpreted as backflows of information from the environment. This interpretation is universal for both classical and quantum systems. For instance, we shall study how, by reducing a Markovian quantum evolution to a fixed commutative subalgebra, one generally obtains a non-Markovian classical dynamics experiencing backflow of information. The latter is assisted by coherences built up by the quantum evolution which effectively act as an environment for the classical subalgebra. We shall demonstrate that this effect can be driven by a dissipative dynamics, yet capable of building enough coherence with respect to a suitable basis. Conversely, there exist memory effects with no classical counterpart. This is the case of backflow of information superactivating in bipartite systems whose subsystems do not exhibit revivals when observed individually. Nonetheless, such phenomenon can be assisted by a classical memory, as occurs in a sufficiently correlated Markov spin chain acting as a collisional environment. In this framework, the physical origin of the superactivation effect is investigated through the study of system–environment correlations. Generally, though, it is quite a hard task to infer the mechanisms behind information flows from the reduced dynamics, since the latter only represents the one-time marginal of an underlying multi-time quantum stochastic process. A possible way out of this problem is proposed by investigating system–environment information flows in terms of the Alicki–Lindblad–Fannes dynamical entropy, which extends the classical entropy of Kolmogorov and Sinai to quantum dynamical systems. After introducing the appropriate symbolic construction in presence of an external environment, exact results are provided in the framework of collisional models. The interpretation of the open-system dynamical entropy is then discussed in the state purification scheme known as GNS construction, whereby further interesting connections emerge with the superactivation of memory effects.

Contents

List of Acronyms

MASA	maximally Abelian subalgebra 8
GNS	Gelfand-Naimark-Segal construction 10
KS	Kolmogorov-Sinai dynamical entropy 17
OPU	operational partition of unity 28
POVM	positive operator-valued measurement 28
CPTP	positive trace-preserving map 30
ALF	Alicki-Lindblad-Fannes dynamical entropy 31, 32
GKLS	Gorini, Kossakowski, Lindblad and Sudarshan equation 41
BLP	Breuer-Laine-Piilo approach 46
BFI	backflow of information 46
SBFI	superactivation of backflow of information 80

List of Symbols

$\mathcal{A}$	C^*-algebra 7
ω	state 7
ρ	density matrix 8
$\mathcal{A}^{[a,b]}$	strictly local algebras 9
$M_D^{\mathbb{Z}}(\mathbb{C})$	quasi-local matrix algebra 9
σ	right-tensor shift 12
$\mathfrak{S}_\omega$	mean von-Neumann entropy rate 12

Introduction

The richness of quantum mechanics often emerges in attempts to extend well-established and useful notions from classical physics to the non-commutative context. In doing so, often, not a single possibility, but rather a multiplicity of non-equivalent ones arise. The theory of non-Markovian quantum processes – namely, processes influenced by memory effects – is one instance in which this feature is particularly evident.

A stochastic process can generally be reconstructed by multiple observations of an experimentally accessible system of interest. In the framework of *open systems*, the latter is part of a larger compound, possibly made of infinite degrees of freedom, that globally evolves reversibly. The so-called reduced dynamics, resulting from the elimination of the environment degrees of freedom, is irreversible and generally affected by noise and, in the quantum case, decoherence, with information being dissipated to the surrounding *environment*.

Classically, a memoryless stochastic process is clearly identified by the Markov property, expressing that the future evolution is independent of its past. The study of quantum Markovianity has a long history, stemming from earlier pioneering works on stochastic processes [1–3] and open systems [4–7]. The advances in quantum information theory and other rapidly evolving quantum technologies – among them quantum computation, cryptography, and communication – continue to fuel interest in open systems, together with experimental progress that made it possible to control single and strongly interacting degrees of freedom. In recent years, particular interest has been renewed to study open systems beyond the Markovian regime, with the aim of characterizing and quantifying memory effects from several points of view. Roughly speaking, indeed, Markovian behaviour amounts to information being unidirectionally dissipated into the environment. Conversely, memory effects represent reversals of such flow of information, so that, in principle, they could help in contrasting the detrimental effects of dissipation that are omnipresent in the majority of applications. Many attempts in exploiting non-Markovianity as a resource were already discussed in several works (see e.g. [8–10] and reviews [11, 12]).

Alongside the well-defined notion of Markovianity in classical stochastic processes, a whole hierarchy of non-equivalent concepts to characterize quantum Markovianity has thus emerged over recent years [13]. Comprehensive reviews of the progress in the field, with focus on different perspectives and proposals, can be found in [12, 14–19]. The emergent point of view in [14], that we shall also adopt in this work, is that quantum Markovianity should be understood, for all practical purposes, as a highly context-dependent concept. For example, if the coupling with the environment is weak, a standard procedure is to perform systematic Markovian approximations in order to obtain the reduced dynamics of the system, resulting in a semigroup of completely positive and trace-preserving maps [6, 20]. Mathematically, such an evolution has been fully characterized in the celebrated result of Gorini, Kossakowski, Sudarshan [4] and Lindblad [5]. Notably, the weak-coupling regime constrains not only the reduced dynamics of the system, but much more, in that one can reconstruct the full multi-time statistics of the quantum stochastic process out of the dynamical map [21–23]. This is the so-called Quantum Regression regime, which offers a very strong characterization of Markovianity and that reduces to the familiar definition for classical systems. On the other hand, beyond the weak-coupling limit, different notions along the aforementioned hierarchy come useful to characterize memory effects, and the most appropriate one could well be context-dependent. It is though essential to clarify and deepen the understanding of the different aspects of non-Markovian behaviour, as well as the sources of discrepancy with respect to the classical regime.

This thesis revolves around three main lines, having as common object of investigation the information flow between system and environment, with particular emphasis on the interplay of its classical and quantum aspects. As we shall see, in treating memory effects, most conceptual ideas and mathematical structures can be framed in a universal way in both quantum and classical regimes.

Memory effects often emerge as properties of the reduced dynamics of the open system, such as revivals of distances or divergences, which are typically interpreted as a backflow of information (shortly, BFI) previously stored in the environment during earlier stages of the evolution [24], and that are due to the loss of the so-called *divisibility* of the dynamics [25, 26]. This kind of approaches offers a legitimate *intrinsic* characterization of Markovianity in terms of the dynamical map that might be sufficient in several problems [12, 14]. For instance, in Chapter 3, we shall see that a quantum dynamics that does not exhibit BFI on its own can display it when restricted to a suitable commutative subalgebra. This effect is due to the ability of the quantum dynamics to build up coherence, but, as will be demonstrated, it can also be driven by a genuinely irreversible evolution. In Chapter 4, we will focus on another type of memory effect emerging in the reduced dynamics; this time, however, with no classical analogue. Indeed, it may occur that two dynamically independent parties, which do not exhibit BFI when observed independently, do show it when jointly considered. We call this phenomenon *superactivation* of backflow of information (SBFI in the following). We shall first extensively study such effect from the point view of the dynamical map and highlight its quantum character, which is somewhat hidden since no quantum entanglement is needed.

The reduced dynamics of open systems represents only the one-time marginal of a stochastic process. In particular, the physical mechanisms underlying memory effects, as those governing the SBFI effect, cannot be fully appreciated from it. At least some information about the full system-environment evolution is required for this purpose. The extraction of the full multi-time statistics of the process involves a sequence of repeated measurements that are intertwined with the system-environment dynamics. In a non-commutative context, though, observations do not commute with the dynamics nor leave the quantum state invariant. Thus, they would themselves generally affect the dynamics and, accordingly, would be somewhat indistinguishable from it. This is a ubiquitous feature of quantum mechanics. An analogue situation is encountered when trying to generalize a fundamental concept of classical ergodic theory like that of dynamical coarse-graining to the quantum scenario. Classically, due to the finite precision of measurements, one effectively performs a partition of the phase-space, resulting in a symbolic description of the dynamics. In this context, the Kolmogorov-Sinai *dynamical entropy* (KS entropy for short) provides an estimate of the predictability of the dynamics or, equivalently, of the average amount of information extractable per time step through repeated observations that do not interfere with the system dynamics.

Many inequivalent proposals to extend the KS entropy to the non-commutative setting have been put forward. Roughly speaking, they differ in answering the basic question whether measurements or not should be explicitly considered in extracting information. We shall be particularly interested in the proposal by Alicki, Lindblad and Fannes (ALF entropy) which provides a quantum counterpart to the KS entropy in that the partitioning of the phase space is replaced by positive operator-valued measurements. Thus, in such approach, measurements enter actively into the game. The germ of the Alicki-Fannes construction can be found in the pioneering work of Lindblad [7], which was in fact concerned with non-Markovian processes and their entropy. Most later applications of ALF entropy, however, focused on reversible dynamical systems in the context of quantum chaos [27–30]. One of the main goals of this thesis is to adapt-back the ALF to the open system scenarios and exploit it in the investigation of system-environment information flows.

The work is divided in two Parts: the first one consists in two Chapters devoted to review the necessary mathematical tools for the subsequent developments. The second Part consists of three Chapters that include the main findings of the research work underlying the thesis.

In Chapter 1, the so-called algebraic approach [27, 31] to quantum dynamical systems is reviewed. This powerful approach allows to treat, within the same formalism, quantum and classical dynamical systems, made of either finite or infinite degrees of freedom. The proper tool to deal with irreversible behaviour in quantum systems are positive and completely positive maps, whose essential properties are also reviewed in this Chapter. Finally, symbolic models for both classical and quantum systems will be discussed together with the definition of dynamical entropy of Kolmogorov and Sinai and that of Alicki, Lindblad and Fannes, which will be illustrated by paradigmatic examples.

In Chapter 2, the dynamics of open systems is properly framed within the algebraic approach. First, the focus will be put on quantum dynamical maps beyond the Markovian regime. Here, we shall stress the role that positive maps, whose usefulness usually emerges as entanglement witnesses, play in characterizing the non-contractive behaviour of non-Markovian evolutions. This usually amounts in revivals of distances and divergences that is identified as backflow of information. Furthermore, symbolic models for open quantum systems will be introduced. The multi-time statistics of the open system is encoded through a quantum many-body state. In this context the proper characterization of Markovianity is provided by the Quantum Regression regime, which we examine in detail.

In Chapter 3, the aforementioned classical reduction of quantum dynamical maps will be discussed through several examples in the single qubit case. A highly-constrained family of maps will be constructed so to find a purely dissipative evolution showing BFI in its classical reductions.

Chapter 4 will be devoted to the study of SBFI both from the abstract perspective of the quantum dynamical map and from the physical perspective of a concrete collisional model. Interestingly, this purely quantum effect can be assisted by a classical memory, such as a classical Markov chain collisional environment, whereby it is triggered by sufficiently strong correlations between first-neighboring spins. The collisional approach to open quantum systems is reformulated through the algebraic approach appropriate for many-body physics. This treatment, on the one hand, facilitates the investigation of the SBFI phenomenon through the evaluation of system–environment correlations. Moreover, this framework is well-suited to broaden the perspective of the reduced dynamics to the full multi-time statistics of the process.

Chapter 5 is devoted to the study of non-Markovianity and BFI by incorporating the information contained in multi-time correlation functions. This information is eventually encoded in the *open-system ALF entropy*. Some exact results are obtained within the collisional model framework. Their interpretation is then discussed through the state-purification scheme provided by the so-called GNS representation; in the latter, moreover, interesting connections to the SBFI effect appear.

Part I

Preliminaries

CHAPTER 1

Quantum Dynamical Systems

A unifying way to deal with both classical or quantum dynamical systems, made of either finite or infinite degrees of freedom, is offered by the so-called algebraic approach typical of many-body physics.

1.1 Algebraic tools

The aim of this Section is to provide a concise review of the necessary definitions and algebraic tools required for the subsequent Chapters, without attempting an exhaustive treatment. For comprehensive sources on operator algebras and the algebraic approach to quantum mechanics of infinite degrees of freedom, we refer to [31–33].

Definition 1.1. *A Banach *-algebra $\mathcal{A}$, namely a Banach algebra endowed with an anti-linear* involution $\mathcal{A} \ni X \mapsto X^\dagger$,

$$(X^\dagger)^\dagger = X, \quad (\lambda X + \mu Y)^\dagger = \overline{\lambda} X^\dagger + \overline{\mu}\, Y^\dagger\,, \qquad (XY)^\dagger = Y^\dagger X^\dagger\,, \tag{1.1}$$

is called a C^*-algebra if

$$\left\|X^\dagger X\right\| = \|X\|^2\,, \qquad (C^*\text{-}property)\,. \tag{1.2}$$

$\mathcal{A}$ is called unital if it possesses an identity $\mathbb{1}_\mathcal{A}$ such that $\mathbb{1}_\mathcal{A} X = X \mathbb{1}_\mathcal{A} = X$, for all $X \in \mathcal{A}$.

C^*-algebra postulate. Quantum systems are described by a *quantum probability space* $(\mathcal{A}, \omega)$, formed by *(i)* a unital C^*-algebra $\mathcal{A}$, whose self-adjoint elements are the observables of the quantum system and *(ii)* a *state* $\omega : \mathcal{A} \to \mathbb{C}$, namely a positive semi-definite and normalized functional on $\mathcal{A}$,

$$\omega(A) \geq 0\,, \qquad \forall\, A = X^\dagger X\,,\; X \in \mathcal{A}\,, \tag{1.3}$$

$$\omega(\mathbb{1}_\mathcal{A}) = 1\,. \tag{1.4}$$

For $X = X^\dagger \in \mathcal{A}$, $\omega(X)$ is interpreted as the expectation of the observable X when the system is in the state ω.

1.1.1 Subalgebras

Definition 1.2. *A subalgebra $\mathcal{B} \subseteq \mathcal{A}$ is called a *-subalgebra of $\mathcal{A}$ if it is closed with respect to the involution and contains $\mathbb{1}_{\mathcal{A}}$.*

Let now $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$, a subalgebra of the bounded operators on a Hilbert space $\mathcal{H}$.

Definition 1.3. *Given a C^*-algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$, the *-subalgebra*

$$\mathcal{A}' = \{X \in \mathcal{B}(\mathcal{H}) : [A, X] = 0, \ \forall\, A \in \mathcal{A}\} \tag{1.5}$$

is called the commutant *of $\mathcal{A}$.*
The bicommutant *$\mathcal{A}''$ is the commutant of $\mathcal{A}'$. If $\mathcal{A} = \mathcal{A}''$, $\mathcal{A}$ is a called a* von Neumann algebra.
If $\mathcal{A} \subseteq \mathcal{A}'$, $\mathcal{A}$ is called Abelian.
If $\mathcal{A} = \mathcal{A}'$, $\mathcal{A}$ is called maximally Abelian. *If $\mathcal{A}' = \{\alpha \mathbb{1}_{\mathcal{A}}, \alpha \in \mathbb{C}\}$, $\mathcal{A}$ is called* irreducible.

Example 1.1.

1. *Let $\mathcal{A} = M_d(\mathbb{C})$, the algebra of $d \times d$ matrix. It is a C^*-algebra with respect to the matrix norm. Moreover, $\mathcal{A}' = \{\alpha \mathbb{1}_d\}$, so that $\mathcal{A}$ is non-Abelian and irreducible. Since $\mathcal{A}'' = \mathcal{A}$, $\mathcal{A}$ is a von Neumann algebra. All states on $M_d(\mathbb{C})$ can be written as* density matrices
$$\omega(X) = \mathrm{Tr}\,(\rho X)\,, \qquad \mathrm{Tr}\,(\rho) = 1, \quad \rho \geq 0\,. \tag{1.6}$$
2. *Let $\mathcal{A} = \left\{\sum_i \alpha_i P_i, \alpha_i \in \mathbb{C}\right\} \subseteq M_d(\mathbb{C})$ be a *-subalgebra of $M_d(\mathbb{C})$, where P_i are orthogonal rank-1 projectors such that $P_i P_j = \delta_{ij} P_i$, $i = 1, \ldots, d$. Then, $\mathcal{A}' = \mathcal{A} = \mathcal{A}''$, so that $\mathcal{A}$ is reducible and a maximally Abelian subalgebra (MASA) of $M_d(\mathbb{C})$.*

Remark 1.1. *The algebra of $d \times d$ matrices can also be given the structure of a Hilbert space when endowed with the Hilbert-Schmidt scalar product* $\mathrm{Tr}(X^\dagger Y)$ *of two matrices X, Y. One can always choose $d^2 - 1$ traceless operators F_i, $i = 1, \ldots d^2 - 1$ that, along with $F_0 = \mathbb{1}_d / \sqrt{d}$, form a Hilbert-Schmidt orthonormal basis,* $\mathrm{Tr}(F_i^\dagger F_j) = \delta_{ij}$, *$i, j = 0, \ldots, d^2 - 1$. Then, every matrix can be decomposed along such basis, $M_d(\mathbb{C}) \ni X = x_0 \mathbb{1} + \sum_i x_i F_i$, where $x_i = \mathrm{Tr}(F_i^\dagger X)$. The $d^2 - 1$ matrices spanning the traceless subspace can be always chosen to be Hermitian, $F_i = F_i^\dagger$. In fact, given a set of matrix units $\{E_{jk}\}_{j,k=1}^d \in M_d(\mathbb{C})$, $E_{jk} = |j\rangle\langle k|$, that constitute themselves a Hilbert-Schmidt basis, take*

$$G_{jk} = \begin{cases} \frac{1}{\sqrt{2}}(E_{jk} + E_{kj})\,, & 1 \leq j < k \leq d\,, \\ \frac{1}{\sqrt{2}i}(E_{jk} - E_{kj})\,, & 1 \leq k < j \leq d\,, \\ \frac{1}{\sqrt{j(j+1)}}\left(\sum_{l=1}^{j} E_{ll} - jE_{j+1,j+1}\right)\,, & 1 \leq j = k \leq d-1\,. \end{cases} \tag{1.7}$$

The $d^2 - 1$ Hermitian matrices in (1.7) *are traceless, normalized, and mutually orthogonal; together with $G_{dd} = \mathbb{1}_d / \sqrt{d}$ they form n orthonormal basis. For $d = 2$ they yield the (normalized) Pauli matrices $\{\sigma_\alpha / \sqrt{2}\}_{\alpha=0}^3$ with*

$$\sigma_0 = \mathbb{1}_2\,, \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1.8}$$

while for $d = 3$ they correspond to the (normalized) Gell-Mann matrices. In the qubit case, $d = 2$, the state-space can be given a particularly simple form, since every density matrix can be written in the Bloch form

$$\rho = \frac{\mathbb{1}_2 + \boldsymbol{r} \cdot \boldsymbol{\sigma}}{2} \tag{1.9}$$

where $\boldsymbol{r}$ is the so-called Bloch vector of ρ, $\|\boldsymbol{r}\| \leq 1$. *Every pure state, namely a rank-1 projector* $P_{\boldsymbol{r}}$ *with Bloch vector* $\boldsymbol{r}$, $\|\boldsymbol{r}\| = 1$, *is then sufficient to select a MASA of* $M_2(\mathbb{C})$, *since the only other orthogonal projector is* $P_{-\boldsymbol{r}} = \mathbb{1}_2 - P_{\boldsymbol{r}}$.

1.1.2 Quantum spin chains

The following example helps to appreciate the necessity of refining the notion of states as positive normalized expectation functionals over a C^*-algebra [34].

Example 1.2. *Consider a finite chain of* N *qubits, each in a Gibbs state*

$$\rho_\beta^{(k)} = \frac{e^{-\beta\sigma_3}}{2\cosh(\beta)}, \quad k \in [1, N]. \tag{1.10}$$

The state of N *such qubits, localized from site* a *to site* $a + N$, *is naturally described by the density matrix*

$$\rho_\beta^{[a,a+N]} = \bigotimes_{k=0}^{N-1} \frac{e^{-\beta\sigma_3^{(a+k)}}}{2\cosh(\beta)} = \frac{e^{-\beta\sum_{k=0}^{N-1}\sigma_3^{(a+k)}}}{\mathrm{Tr}\left(e^{-\beta\sum_{k=0}^{N}\sigma_3}\right)}. \tag{1.11}$$

Note that, as $N \to \infty$, $\left\|\rho_\beta^{[1,N]}\right\| \xrightarrow{N\to\infty} 0$*: this signals the fact that the state of infinitely many independent spins, each in the state* (1.10), *cannot be described by a single density matrix.*

We now give a sense to the expression (1.11) in the framework of algebraic quantum spin chains, which are prototypical infinite systems. They are described by the algebra of *quasi-local observables*, constructed as follows [31].

1. Consider first *strictly local algebras*

$$\mathcal{A}^{[a,b]} := \bigotimes_{k=a}^{b} \mathcal{A}^{(k)} \tag{1.12}$$

built, for the sake of simplicity, out of a same matrix algebra $\mathcal{A}^{(k)} = M_D(\mathbb{C})$. Local algebras which are supported by intervals $[a, b]$ of integers $a \leq k \leq b$, are generated by tensor products of the form $X_{\boldsymbol{i}_{[a,b]}}^{[a,b]} = \bigotimes_{k=a}^{b} X_{i_k}^{(k)}$, where the upper index refers to the site at which the operator X_{i_k} is located. These local operators can be embedded within the infinite chain as $\mathbb{1}_E^{-a-1]} \otimes X_{\boldsymbol{i}_{[-a,b]}}^{[-a,b]} \otimes \mathbb{1}^{[b+1}$, where $\mathbb{1}^{-a-1]} = \otimes_{k=-\infty}^{-a-1}\mathbb{1}^{(k)}$ and $\mathbb{1}^{[b+1} = \otimes_{k=b+1}^{+\infty}\mathbb{1}^{(k)}$. By omitting the identities of the embeddings, one then introduces the algebra of local observables,

$$\mathcal{A}^{\mathrm{loc}} = \bigcup_{a\leq b} \mathcal{A}^{[a,b]}. \tag{1.13}$$

Physically speaking, experimentally accessible operators belong to local algebras. Nevertheless, to accommodate observables emerging in the thermodynamic limit, one needs to consider the so-called quasi-local algebra $\mathcal{A}$ obtained by taking the norm closure,

$$\mathcal{A} = \overline{\mathcal{A}^{\mathrm{loc}}}^{\|\cdot\|}. \tag{1.14}$$

In the following, we shall also denote such algebra as $\mathcal{A} = M_D^{\mathbb{Z}}(\mathbb{C})$.

2. States on the quantum spin chain are fully specified by their restriction to local algebras $\mathcal{A}^{[a,b]}$, where they can be represented by a consistent set of density matrices $\rho^{[a,b]} \in \mathcal{A}^{[a,b]}$,

$$\omega\left(A^{[a,b]}\right) = \mathrm{Tr}\left(\rho^{[a,b]} A^{[a,b]}\right) , \tag{1.15}$$

where $A^{[a,b]} \in \mathcal{A}^{[a,b]}$. Consistency means that the following condition holds:

$$\mathrm{Tr}_b\left(\rho^{[a,b]}\right) = \rho^{[a,b-1]} \quad \forall \quad \text{sites} \quad a, b , \tag{1.16}$$

where Tr_k defines the partial trace over the k-th site. Vice versa, a family of density matrices $\rho^{[a,b]} \in \mathcal{A}^{[a,b]}$ gives rise to a state ω over the chain if (1.16) holds, for all $a \leq b$. Such states are called *locally normal* [31]. In Example 1.2, one thus defines an expectation $\omega_\beta : \mathcal{A} \to \mathbb{C}$ on the infinite spin chain $M_2^{\mathbb{Z}}(\mathbb{C}) := \bigotimes_{k=-\infty}^{+\infty} M_2(\mathbb{C})$. Expectations of local observables are then obtained from density matrices (1.11) through

$$\omega_\beta\left(A^{[a,a+N]}\right) = \mathrm{Tr}\left(\rho_\beta^{[a,a+N]} A^{[a,a+N]}\right) , \tag{1.17}$$

whence defining a locally normal state ω_β.

1.1.3 The GNS construction

The C^*-algebraic structure of the observables results to be somewhat more fundamental than the familiar Hilbert space formulation of quantum mechanics. This fact is particularly evident when dealing with infinite systems, whereby inequivalent *representations* of the algebra might arise (and so, different Hilbert spaces), that are identified with distinct phases. Nevertheless, one retrieves the Hilbert space formulation from the abstract algebraic one through the construction by Gelfand, Naimark and Segal (GNS construction). For an exhaustive treatment of the following results, together with their proofs, we refer to [31] and [32].

Definition 1.4. *A *-homomorphism is a map $\alpha : \mathcal{A} \to \mathcal{B}$ such that*

$$\begin{aligned} \alpha(\lambda X + \mu Y) &= \lambda \alpha(X) + \mu\, \alpha(Y), \quad \lambda, \mu \in \mathbb{C} , \\ \alpha(XY) &= \alpha(X)\alpha(Y) , \quad \alpha(X^\dagger) = \alpha(X)^\dagger , \end{aligned}$$

Definition 1.5. *A* representation $(\mathcal{H}, \pi)$ *of a C^*-algebra $\mathcal{A}$ is given by a Hilbert space $\mathcal{H}$ and a *-homomorphism $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$, $\mathcal{B}(\mathcal{H})$ being the C^*-algebra of bounded operators on $\mathcal{H}$.*

Definition 1.6. *Let $(\mathcal{H}, \pi)$ be a representation of $\mathcal{A}$. A vector $|\Omega\rangle$ is called* cyclic *if the linear subspace*

$$\pi(\mathcal{A})\,|\Omega\rangle := \{\pi(X)\,|\Omega\rangle : X \in \mathcal{A}\} \tag{1.18}$$

is norm dense in $\mathcal{H}$. $(\mathcal{H}, \pi)$ is then called a cyclic representation for $\mathcal{A}$.

Remark 1.2. *A vector $|\Omega\rangle$ is cyclic for a the *-subalgebra $\pi(\mathcal{A})$ if and only if it is* separating *for $\pi'(\mathcal{A})$, namely $\pi'(X)\,|\Omega\rangle = 0 \implies X = 0$.*

Theorem 1.1 (Gelfand, Naimark, Segal). *Let $(\mathcal{A}, \omega)$ be a quantum probability space. Then, there exist a Hilbert space $\mathcal{H}_\omega$, a representation $\pi_\omega : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\omega)$ and a cyclic vector $|\Omega_\omega\rangle$ such that*

$$\omega(X) = \langle \Omega_\omega | \pi_\omega(X) | \Omega_\omega \rangle , \qquad \forall\, X \in \mathcal{A}. \tag{1.19}$$

The cyclic representation $(\mathcal{H}_\omega, \pi_\omega, |\Omega_\omega\rangle)$ is called GNS representation and is unique up to an isomorphism.

Proposition 1.2. *Let $(\mathcal{H}, \pi, |\Omega\rangle)$ be a triple such that $\omega(X) = \langle\Omega|\pi(X)|\Omega\rangle \ \forall X \in \mathcal{A}$. Then, there exists a unitary operator $U : \mathcal{H} \to \mathcal{H}_\omega$, $UU^\dagger = \mathbb{1}_{\mathcal{H}_\omega}, U^\dagger U = \mathbb{1}_{\mathcal{H}}$, such that*

$$U^\dagger \pi_\omega(X) U = \pi(X), \ X \in \mathcal{A}, \qquad U|\Omega\rangle = |\Omega_\omega\rangle, \qquad U^\dagger |\Omega_\omega\rangle = |\Omega\rangle . \tag{1.20}$$

Example 1.3. *Consider a finite level system whose state is described by an invertible density matrix $M_d(\mathbb{C}) \ni \rho = \sum_\alpha r_\alpha |\alpha\rangle\langle\alpha|$, $0 < r_\alpha \leq 1$, so that the state is* faithful, *namely* $\mathrm{Tr}\left(\rho X^\dagger X\right) = 0 \implies X = 0$. *The GNS construction is then achieved through a state purification. Indeed by means of the Hilbert-Schmidt scalar product* $\mathrm{Tr}\left(X^\dagger Y\right) = \langle X|Y\rangle$ *operators $X \in M_d(\mathbb{C})$ are readily identified as vectors in $\mathbb{C}^d \otimes \mathbb{C}^d$. One introduces representations*

$$\pi(X)|Y\rangle = |XY\rangle , \qquad \pi'(X)|Y\rangle = |YX\rangle , \tag{1.21}$$

By vectorizing operators as $|X\rangle = \sum_{\alpha\beta} X_{\alpha\beta} |\alpha \otimes \beta\rangle$ the above representations are factor representations,

$$\pi(X) = X \otimes \mathbb{1}_d , \qquad \pi'(X) = \mathbb{1}_d \otimes \overline{X} , \tag{1.22}$$

where $\left\langle\alpha\middle|\overline{X}\middle|\beta\right\rangle \equiv \overline{\langle\alpha|X|\beta\rangle}$. Then, for all $X \in M_d(\mathbb{C})$,

$$\mathrm{Tr}(\rho X) = \mathrm{Tr}\left(\sqrt{\rho}\, X \sqrt{\rho}\right) = \left\langle\sqrt{\rho}\middle|\pi(X)\middle|\sqrt{\rho}\right\rangle , \tag{1.23}$$

and the GNS-vector is identified with

$$\left|\sqrt{\rho}\right\rangle = \sum_\alpha \sqrt{r_\alpha}\, |\alpha \otimes \alpha\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d , \tag{1.24}$$

that provides a so-called purification of ρ. Moreover, for all $X \in M_d(\mathbb{C})$,

$$\left\langle\sqrt{\rho}\middle|X^\dagger X \otimes \mathbb{1}_d\middle|\sqrt{\rho}\right\rangle = \left\langle\sqrt{\rho}\middle|\mathbb{1}_d \otimes X^\dagger X\middle|\sqrt{\rho}\right\rangle = \mathrm{Tr}\left(\rho\, X^\dagger X\right) = 0 \iff X = 0 .$$

so that (1.24) *is both cyclic and separating for $\pi(\mathcal{A})$ and the commutant representation $\pi'(\mathcal{A})$.*

Remark 1.3. *Given two C^*-algebras $\mathcal{A}_1$ and $\mathcal{A}_2$, there generally exist multiple norms to complete their algebraic tensor product $\mathcal{A}_1 \otimes \mathcal{A}_2$ to a C^*-algebra, thus making the tensor product of C^*-algebras a delicate matter. Nevertheless, in all the concrete instances that we shall encounter in the following Chapters, one of the parties, say, $\mathcal{A}_1$, is a so-called* nuclear C^**-algebra, for which the C^*-norm on the tensor product $\mathcal{A}_1 \otimes \mathcal{A}_2$ is unique. Nuclear algebras include: the algebra of $d \times d$ matrices $M_d(\mathbb{C})$, the one of bounded operators on a Hilbert space $\mathcal{B}(\mathcal{H})$, and that of quasi-local observables obtained as inductive limit of finite dimensional algebras as in Section 1.1.2. The next result helps in determining the GNS representation of C^*-tensor products endowed with product states [32].*

Proposition 1.3. *Consider quantum probability spaces $(\mathcal{A}, \omega_A)$, $(\mathcal{B}, \omega_B)$ and associated GNS representations $(\mathcal{H}_A, \pi_A, |\Omega_A\rangle)$, $(\mathcal{H}_B, \pi_B, |\Omega_B\rangle)$. Let then $\mathcal{A} \otimes \mathcal{B}$ be a C^*-tensor product (with respect to a suitable norm). Then, the GNS triple for $(\mathcal{A} \otimes \mathcal{B}, \omega_A \otimes \omega_B)$ is unitarily equivalent to $(\mathcal{H}_A \otimes \mathcal{H}_B, \pi_A \otimes \pi_B, |\Omega_A \otimes \Omega_B\rangle)$.*

Proof. Let $\omega = \omega_A \otimes \omega_B$ and let $(\mathcal{H}_\omega, \pi_\omega, |\Omega_\omega\rangle)$ be the GNS triple associated to $(\mathcal{A} \otimes \mathcal{B}, \omega)$. For $a \in \mathcal{A}$ and $b \in \mathcal{B}$, one has

$$\begin{aligned}\langle\Omega_\omega|\pi_\omega(a \otimes b)|\Omega_\omega\rangle &= \omega(a \otimes b) = \omega_A(a)\omega_B(b) = \langle\Omega_A|\pi_A(a)\Omega_A\rangle\langle\Omega_B|\pi_B(b)\Omega_B\rangle \\ &= \langle\Omega_A \otimes \Omega_B|\pi_A \otimes \pi_B(a \otimes b)|\Omega_A \otimes \Omega_B\rangle\ . \end{aligned} \tag{1.25}$$

One then defines an operator $V : \pi_A \otimes \pi_B(\mathcal{A} \otimes \mathcal{B})\,|\Omega_A \otimes \Omega_B\rangle \to \pi_\omega(\mathcal{A} \otimes \mathcal{B})\,|\Omega_\omega\rangle$ such that

$$V\pi_A \otimes \pi_B(C)\,|\Omega_A \otimes \Omega_B\rangle = \pi_\omega(C)\,|\Omega_\omega\rangle\,, \qquad C \in \mathcal{A} \otimes \mathcal{B}\,.$$

Accordingly, (1.25) ensures that V is a well defined isometry. By density of $\pi_A \otimes \pi_B(\mathcal{A} \otimes \mathcal{B})\,|\Omega_A \otimes \Omega_B\rangle$ in $\mathcal{H}_A \otimes \mathcal{H}_B$ and of $\pi_\omega(\mathcal{A} \otimes \mathcal{B})\,|\Omega_\omega\rangle$ in $\mathcal{H}_\omega$, V can be extended to a unitary operator $\mathcal{H}_A \otimes \mathcal{H}_B \to \mathcal{H}_\omega$ such that $V^\dagger\pi_A \otimes \pi_B(C)V = \pi_\omega(C)$ for all $C \in \mathcal{A} \otimes \mathcal{B}$. □

1.2 Closed Dynamical Systems

A quantum probability space $(\mathcal{A}, \omega)$ only specifies the kinematics of a quantum system. We now come to the description of the dynamics. Reversible dynamics is given by an one-parameter group of automorphisms of the algebra $\Theta_t : \mathcal{A} \to \mathcal{A}$,

$$\Theta_t \circ \Theta_s = \Theta_{t+s}\,. \tag{1.26}$$

The latter specifies the evolution of observables in the Heisenberg picture. Equivalently, one can let evolve the states in the Schrödinger picture according to

$$\omega \longmapsto \omega_t := \omega \circ \Theta_t\,, \tag{1.27}$$

ensuring that expectations of observables agree in both pictures. In the following, we shall consider both continuous and discrete-time families, namely t either in $\mathbb{R}$ or $\mathbb{Z}$. To fix the ideas, let us focus for the moment on a discrete-time group of automorphisms, namely $\Theta_t = \Theta^t$, $t \in \mathbb{Z}$. The triple $(\mathcal{A}, \omega, \Theta)$ defines a discrete-time *quantum dynamical system*.

Often, one refers to a time-invariant state $\omega \circ \Theta = \omega$. Other states can be obtained by suitable perturbations of the invariant one.

Example 1.4. *Consider the quantum spin chain introduced in Section 1.1.2, described by the quasi-local algebra $\mathcal{A} = M_d^{\mathbb{Z}}(\mathbb{C})$ and a locally normal state ω. It is turned into a shift dynamical system $(\mathcal{A}, \omega, \sigma)$ by endowing it with the right shift on the quasi-local algebra $\mathcal{A}$, namely the automorphism $\sigma : \mathcal{A} \to \mathcal{A}$ acting on local observables as*

$$\sigma\Big(X_{i_a}^{(a)} \otimes \cdots \otimes X_{i_b}^{(b)}\Big) = X_{i_a}^{(a+1)} \otimes \cdots \otimes X_{i_b}^{(b+1)}\,. \tag{1.28}$$

ω *is* shift-invariant, *namely $\omega \circ \sigma = \omega$, if and only if $\mathrm{Tr}_a\,\rho^{[a,b]} = \rho^{[a+1,b]} = \rho^{[a,b-1]}$. Given such a stationary state, consider the von Neumann entropy of the n-site local density matrix,*

$$S_n := S\left(\rho^{[1,n]}\right), \tag{1.29}$$

where $S(\rho) = -\mathrm{Tr}\,\rho\log\rho$. Then, from stationarity and strong subadditivity of the von Neumann entropy, one proves the existence of the mean entropy rate *[35],*

$$\mathfrak{S}_\omega := \lim_n \frac{S_n}{n} = \lim_n\,(S_{n+1} - S_n)\,, \tag{1.30}$$

where both limits are achieved by the infimum of the decreasing and bounded sequences S_n/n and $S_{n+1} - S_n$. Notice that for a classically stationary source the quantity $S_{n+1} - S_n$ corresponds to the conditional Shannon entropy of the source.

Fixing an invariant state proves convenient, as in the following important

Proposition 1.4. *Let $(\mathcal{A}, \omega, \Theta)$ be a discrete dynamical system with time-invariant state $\omega = \omega \circ \Theta$ and $(\mathcal{H}_\omega, \pi_\omega, |\Omega_\omega\rangle)$ the associated GNS representation. Then, there exists a unique unitary operator $U_\omega : \mathcal{H}_\omega \to \mathcal{H}_\omega$ implementing the automorphism Θ*

$$U_\omega^\dagger \pi_\omega(X) U_\omega = \pi_\omega(\Theta(X)), \quad X \in \mathcal{A}, \tag{1.31}$$

$$U_\omega^\dagger |\Omega_\omega\rangle = U_\omega |\Omega_\omega\rangle = |\Omega_\omega\rangle . \tag{1.32}$$

Proof. From Θ-invariance of the state, one has

$$\langle\Omega_\omega|\pi_\omega(X)|\Omega_\omega\rangle = \omega(X) = \omega(\Theta(X)) = \langle\Omega_\omega|\pi_\omega \circ \Theta(X)|\Omega_\omega\rangle , \quad X \in \mathcal{A} .$$

Thus, triples $(\mathcal{H}_\omega, \pi_\omega, |\Omega_\omega\rangle)$ and $(\mathcal{H}_\omega, \pi_\omega \circ \Theta, |\Omega_\omega\rangle)$ must be unitary equivalent by Proposition 1.2, namely, there exists a unitary operator $U_\omega : \mathcal{H}_\omega \to \mathcal{H}_\omega$ such that

$$U_\omega^\dagger \pi_\omega(X) U_\omega = \pi_\omega \circ \Theta(X), \quad X \in \mathcal{A} . \tag{1.33}$$

and $U_\omega^\dagger |\Omega_\omega\rangle = U_\omega |\Omega_\omega\rangle = |\Omega_\omega\rangle$. □

When the automorphism Θ is implemented unitarily one says that there exists a *covariant representation* of the quantum dynamical system $(\mathcal{A}, \omega, \Theta)$. The following example illustrates how to concretely achieve such representation for finite-level systems.

Example 1.5. *Consider the finite-level system $(M_d(\mathbb{C}), \rho, \Theta)$ of Example 1.3, $\rho > 0$ with Θ an automorphism of $M_d(\mathbb{C})$ that leaves ρ invariant $\Theta(A) = V^\dagger A V$, $V \rho V^\dagger = \rho$. The GNS implementation of Θ is then achieved by taking*

$$U = V \otimes \overline{V}, \qquad \langle\alpha|\overline{V}|\beta\rangle := \overline{\langle\alpha|V|\beta\rangle} . \tag{1.34}$$

Indeed,

$$\begin{aligned} U^\dagger \pi(A) \left|\sqrt{\rho}\right\rangle &= \sum_\alpha \sqrt{r_\alpha} V^\dagger A \, |\alpha\rangle \otimes \overline{V}^\dagger |\alpha\rangle = \sum_{\alpha,\beta} \sqrt{r_\alpha} V^\dagger A |\alpha\rangle \langle\alpha| V |\beta\rangle \otimes |\beta\rangle \\ &= \sum_\beta V^\dagger A \sqrt{\rho} V |\beta\rangle \otimes |\beta\rangle = V^\dagger A V \otimes \mathbb{1}_d \left|\sqrt{\rho}\right\rangle = \pi(\Theta(A)) \left|\sqrt{\rho}\right\rangle . \end{aligned} \tag{1.35}$$

and, setting $A = \mathbb{1}_d$, $U^\dagger \left|\sqrt{\rho}\right\rangle = U_\rho \left|\sqrt{\rho}\right\rangle = \left|\sqrt{\rho}\right\rangle$. Let us evaluate U in two concrete instances.

1. *Let $V = e^{iH}$ with $H = \sum_\alpha h_\alpha |\alpha\rangle$, that is H and V are diagonal in the basis $\{|\alpha\rangle\}_\alpha$ that appear in (1.34). Then, $\overline{V} = V^\dagger$ and $U = V \otimes V^\dagger$.*

2. *Consider instead the unitary "shift" operator on a basis $\{|\alpha\rangle\}_{\alpha=0}^{d-1}$*

$$V = \sum_{\alpha=0}^{d-1} |\alpha\rangle\langle\alpha+1| , \qquad V^\dagger |\alpha\rangle = |\alpha+1\rangle , \tag{1.36}$$

(where the sum mod d is intended) and $\rho = \frac{\mathbb{1}}{d}$*, which is the only density matrix left invariant by* V*. Then,* $\langle\alpha| V |\beta\rangle = \delta_{a,b+1}$ *so that* $V = \overline{V}$ *and* $U = V \otimes V$*. Accordingly, the maximally entangled state* $\left|\psi_+^{(d)}\right\rangle = \frac{1}{\sqrt{d}} \sum_\alpha |\alpha \otimes \alpha\rangle$ *is left invariant by the action of* $U^\dagger$*, since*

$$U^\dagger \left|\psi_+^{(d)}\right\rangle = \frac{1}{\sqrt{d}} \sum_{\alpha=0}^{d-1} V^\dagger |\alpha\rangle \otimes V^\dagger |\alpha\rangle = \frac{1}{\sqrt{d}} \sum_{\alpha=0}^{d-1} |\alpha+1\rangle \otimes |\alpha+1\rangle = \left|\psi_+^{(d)}\right\rangle . \tag{1.37}$$

Remark 1.4. *If* $\{\Theta_t\}_t$ *is a (strongly) continuous one-parameter family of automorphisms, its GNS implementation with respect to an invariant state is achieved through a unique* Hamiltonian*, namely a self-adjoint operator* H *on* $\mathcal{H}_\omega$ *such that*

$$U_t^\dagger \pi_\omega(X) |\Omega_\omega\rangle := \pi_\omega(\Theta_t(X)) |\Omega_\omega\rangle , \qquad U_t = e^{-itH} .$$

Then, the family $\{U_t\}_t$ *forms a (strongly) continuous group in* $\pi(\mathcal{A})''$ *[27].*

1.3 Classical dynamical systems: dynamical shifts

This Section is devoted to an essential review of basic ideas from classical ergodic theory, such as the coarse-graining of the phase-space through a suitable discretization and the concept of dynamical entropy [36–38]. Here, we aim to catch *two birds with one stone*: on the one hand, through the study of shift dynamical systems, we shall review the classical definition of Markovianity and its algebraic formulation in terms of classical spin chains. On the other hand, we prepare the ground for extending the procedure of dynamical coarse-graining to quantum systems, which will be treated in Section 1.5.

Classical dynamical systems are typically identified by (1) a probability space $(\mathfrak{X}, \Sigma, \mu)$ consisting of a measurable space $\mathfrak{X}$, a so-called Σ-algebra Σ and a probability measure μ and (2) an invertible measure-preserving map $\mathcal{T} : \mathfrak{X} \to \mathfrak{X}$

$$\mu(A) = \mu(\mathcal{T}(A)) = \mu(\mathcal{T}^{-1}(A)), \qquad A \in \mathfrak{X} . \tag{1.38}$$

Remark 1.5. *Measure-theoretical dynamical systems can be given an algebraic description by the triple* $(L^\infty_\mu(\mathfrak{X}), \omega_\mu, \Theta_\mathcal{T})$ *where:*
1. the (maximally Abelian) C^**-algebra of observables is given by the so-called essentially bounded functions* $L^\infty_\mu(\mathfrak{X})$ *on* $(\mathfrak{X}, \mu)$*;*
2. $L^\infty_\mu(\mathfrak{X}) \ni f \mapsto \omega_\mu(f) = \int_\mathfrak{X} \mathrm{d}\mu(x) f(x)$ *defines a state on* $L^\infty_\mu(\mathfrak{X})$*;*
3. $\Theta_\mathcal{T}(f) := f(\mathcal{T}^{-1}(x))$ *defines the automorphic dynamics on the algebra.*

In the following Section, we shall mostly deal with a phase-space with finitely many states. The corresponding dynamical systems admit an algebraic description via classical spin chains. From an operational standpoint, this corresponds to the typical situation arising through a coarse-grained description of the dynamics, owing to the finite precision with which measurements can be performed.

1.3.1 Symbolic models

Consider a finite *partition* of the phase-space. The latter consists in a finite collection $\mathcal{P} = \{P_i\}_{i=1}^{|\mathcal{P}|}$ of *atoms*, namely subsets $P_i \subseteq \mathfrak{X}$ such that $\bigcup_{i=1}^{|\mathcal{P}|} P_i = \mathfrak{X}$, $\bigcap_{i=1}^{|\mathcal{P}|} P_i = \emptyset$. Given two

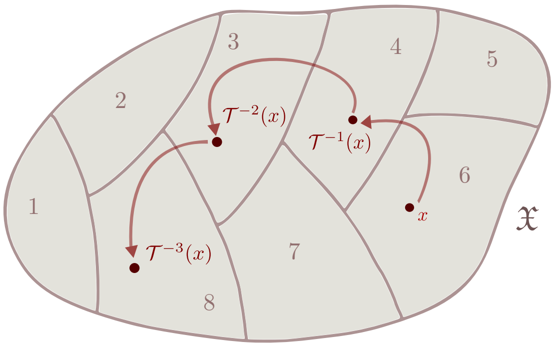

FIGURE 1.1: The phase-space $\mathfrak{X}$ is partitioned in 8 atoms. The trajectory of a point $x \in \mathfrak{X}$, namely $x \mapsto \mathcal{T}^{-1}(x) \mapsto \mathcal{T}^{-2}(x) \mapsto \mathcal{T}^{-3}(x)$, is encoded by the symbolic sequence 6, 4, 3, 8.

partitions $\mathcal{P}$ and $\mathcal{Q}$, one defines their coarsest *refinement* as a new partition $\mathcal{P} \vee \mathcal{Q} = \{P_i \cap Q_j\}_{i,j=1}^{|\mathcal{P}|,|\mathcal{Q}|}$. When the time evolution is taken into account, one considers *time-refinements*

$$\mathcal{P}^{(n)} = \bigvee_{k=0}^{n-1} \mathcal{T}^{-j}(\mathcal{P}) \,, \tag{1.39}$$

whose atoms read

$$\mathcal{P}^{(n)} \ni P^{(n)}_{\boldsymbol{a}_{[0,n-1]}} = P_{a_0} \cap \mathcal{T}^{-1}(P_{a_1}) \cap \ldots \cap \mathcal{T}^{-n+1}(P_{a_{n-1}}) \,. \tag{1.40}$$

The time-refined partition allows one to extract out of $(\mathfrak{X}, \mu, \mathcal{T})$ a *symbolic* model by the following ingredients:

1. $\boldsymbol{a}_{[0,n-1]} := a_0 \ldots a_{n-1}$ in (1.40) are sequences of symbols belonging to a p-letter alphabet, $a_k \in \{1, \ldots, p\}$, where we set $p := |\mathcal{P}|$, that correspond to trajectories in the phase-space (see Figure 1.1).

2. $\Omega_p^{\mathbb{Z}} := \{a_k\}_{k\in\mathbb{Z}}$, is the collection of all possible strings over the p-letter alphabet. The latter is endowed with the Σ-algebra generated by the so-called cylinder sets

$$C^{[i,j]}_{\boldsymbol{a}_{[i,j]}} = \left\{\boldsymbol{k}_{[i,j]} : k_i = a_i \ldots , k_j = a_j\right\} , \quad i \leq j \,. \tag{1.41}$$

3. $\sigma_{\mathbb{Z}}$ is the shift on sequences

$$\sigma_{\mathbb{Z}}(\boldsymbol{a}_{[0,n-1]}) = \sigma_{\mathbb{Z}}(\boldsymbol{a}_{[1,n]}) \,.$$

4. $\pi_{\mathcal{P}}$ a probability measure fully specified by the local marginals

$$\pi_{\mathcal{P}}^{[0,n-1]} = \left\{p_{\boldsymbol{a}_{[0,n-1]}}\right\}_{\boldsymbol{a}_{[0,n-1]}} \quad \text{where } p_{\boldsymbol{a}_{[0,n-1]}} := \mu\left(P^{(n)}_{\boldsymbol{a}_{[0,n-1]}}\right) . \tag{1.42}$$

The so obtained dynamical system $(\Omega_p^{\mathbb{Z}}, \pi_{\mathcal{P}}, \sigma_{\mathbb{Z}})$ provides a coarse-grained description of the original dynamical system $(\mathfrak{X}, \mu, \mathcal{T})$.

Definition 1.7. *A stochastic process* $\pi_{\mathcal{P}}^{[0,n-1]} = \left\{p_{\boldsymbol{a}_{[0,n-1]}}\right\}_{\boldsymbol{a}_{[0,n-1]}}$, $n \geq 1$, *is a Markov process if*

$$p^{[0,n-1]}_{\boldsymbol{a}_{[0,n-1]}} = \left(\prod_{k=1}^{n-1} T_{a_k\, a_{k-1}}(k, k-1)\right) p_{a_0}\,, \tag{1.43}$$

where $T(k, k-1) := [T_{ab}(k, k-1)]_{ab}$ *are* stochastic matrices,

$$T_{ab}(k, k-1) \geq 0, \qquad \sum_a T_{ab}(k, k-1) = 1\,. \tag{1.44}$$

The Markov process is called homogeneous if $T(k, k-1) = T(1, 0) \equiv T$.

Remark 1.6.

1. *The Markov property* (1.43) *can be equivalently defined through conditional probabilities*

$$p^{(n|n-1\dots 0)}_{a_n|a_{n-1}\dots a_0} := \frac{p^{[0,n]}_{\boldsymbol{a}_{[0,n]}}}{p^{[0,n-1]}_{\boldsymbol{a}_{[0,n-1]}}}$$

   by the fact that the probability of the n*-th symbol conditioned on all the previous ones depends only on the* $n-1$*-th symbol,*

$$p^{(n|n-1\dots 0)}_{a_n|a_{n-1}\dots a_0} = p^{(n|n-1)}_{a_n|a_{n-1}} =: T_{a_n a_{n-1}}(n, n-1)\,.$$

   Non-homogeneous Markov processes will be reprised in Section 1.3.3

2. *A stationary Markov process is necessarily homogeneous. Moreover, a necessary and sufficient condition for the homogenous probability distribution to be stationary is* $\sum_b T_{ab} p_b = p_a$, *namely the probability vector with components* p_a *is an eigenvector of the stochastic matrix* T. *In such case, the probability distribution reads*

$$p_{\boldsymbol{a}_{[0,n-1]}} := p^{[0,n-1]}_{\boldsymbol{a}_{[0,n-1]}} = \left(\prod_{k=0}^{n-1} T_{a_k\, a_{k-1}}\right) p_{a_0}\,, \tag{1.45}$$

   and is thus independent of the interval considered.

3. *The two conditions in* (1.44) *ensure that probability vectors are mapped into probability vectors. Moreover,* T *is a stochastic matrix if and only if it contracts the* ℓ_1*-norm* $\|T\boldsymbol{x}\|_1 \leq \|\boldsymbol{x}\|_1$, *where* $\|\boldsymbol{x}\|_1 = \sum_i |x_i|$.

1.3.2 Kolmogorov-Sinai dynamical entropy

A partition $\mathcal{P}$ represents the collection of outcomes of an experiment. Then, the Shannon entropy of the partition

$$H(\mathcal{P}) = -\sum_{i=1}^{|\mathcal{P}|} \mu(P_i) \log \mu(P_i)\,, \tag{1.46}$$

is a measure of the expected uncertainty about the outcomes or, equivalently, the information gained by performing the experiment. It is maximal for equiprobable partitions and, in such case, equals $\log(|\mathcal{P}|)$. Refinement $\mathcal{P} \vee \mathcal{Q}$ then represents the compound experiment obtained by performing $\mathcal{P}$ and $\mathcal{Q}$ simultaneously. Accordingly, the time-refinement (1.39) represents repetitions of the same experiment at subsequent instants of time. Thus, the Shannon entropy rate

$$\mathfrak{h}_\mu(\mathcal{T}, \mathcal{P}^{(n)}) := \lim_n \frac{1}{n} H_\mu\left(\mathcal{P}^{(n)}\right), \tag{1.47}$$

which is well defined as a limit [36], can be interpreted as the time-average of the information content of experiment $\mathcal{P}$. To get a quantity which is representative of the information content of the evolution, one optimizes (1.47) over all possible partitions of the phase-space, so to get a partition-independent quantity:

$$\mathfrak{h}_\mu^{(KS)}(\mathcal{T}) := \sup_{\mathcal{P} \subseteq \mathcal{X}} \mathfrak{h}_\mu\left(\mathcal{T}, \mathcal{P}^{(n)}\right). \tag{1.48}$$

The latter is the so-called Kolmogorov–Sinai dynamical entropy (shortly, KS entropy) and depends only on the dynamics $\mathcal{T}$ and on the initial probability measure μ. It is also referred to as the Kolmogorov–Sinai invariant, since isomorphic dynamical systems share the same KS entropy [37].

Remark 1.7. *The KS entropy can be viewed as a measure of the degree of instability of a given dynamics on phase-space. Indeed, under certain assumptions, the existence of positive Lyapunov exponents can be ascertained directly by means of a positive KS entropy through the so-called Pesin formula [39].*

Remark 1.8. *Practically speaking, one should be able to devise an experiment $\mathcal{P}$ that maximises* (1.48). *The Kolmogorov-Sinai theorem [36] establishes that the maximum is achieved for* generating *partitions, namely those generating the full Σ-algebra.*

It is instructive to review the KS entropy of some paradigmatic, and oppositely behaving, classical dynamical systems. For an exhaustive treatment of these examples, as well as many others, see [37, 40, 41].

Example 1.6.

1. Rotation on the unit circle. *Let $\mathfrak{X}$ be the interval $[0,1)$ and μ be the Lebesgue measure. Let $\mathcal{T}(x) = x + \alpha \mod 1$. If $\alpha = p/q, p, q \in \mathbb{N}$, $\mathcal{T}$ is periodic with period q. Then the time refinement has no more than $|\mathcal{P}|^q$ elements. Then, since the entropy of a partition cannot exceed the* log *of its cardinality,*

$$\frac{1}{n} H\left(\mathcal{P}^{(n)}\right) \le \frac{q}{n} \log(|\mathcal{P}|) \xrightarrow{n\to\infty} 0\,.$$

If α is irrational, take the partition $\widetilde{\mathcal{P}}$ of $[0,1)$ in two intervals $\widetilde{P}_1 = [0, 1/2), \widetilde{P}_2 = [1/2, 1)$, which is generating for the Borel Σ-algebra obtained from the open subsets of $[0,1)$. Inductively, one sees that time-refinements $\widetilde{\mathcal{P}}^{(n)}$ consist in a partition of $[0,1)$ into $2n$ segments. It is trivially true for $n = 1$. Assume that it is true for $n = m$: then one has to determine the intersection of $2m$ segments in $\mathcal{P}^{(m)}$ with $\mathcal{T}^{-(m+1)}(\mathcal{P})$: the atoms of the latter have (distinct) end points $\mathcal{T}^{-(m+1)}(0)$ and $\mathcal{T}^{-(m+1)}(\frac{1}{2})$, generating 2 more segments. The total intersection consists then of $2(m+1)$ segments. Hence,

$$\frac{1}{n} H\left(\mathcal{P}^{(n)}\right) \le \frac{1}{n} \log(2n) \xrightarrow{n\to\infty} 0. \tag{1.49}$$

For all $\alpha \in \mathbb{R}$, then, one has $\mathfrak{h}_\mu^{KS}(\mathcal{T}) = 0$.

2. Markov shifts. *Take the shift dynamical system described by sequences of symbols in* $\Omega_p^{\mathbb{Z}}$*, equipped with the Markov distribution* (1.45)*. Choose the partition made of simple cylinder sets:*

$$C_{i_0}^{(0)} = \left\{\{k_j\}_j : k_0 = i_0\right\} .$$

Via the action of the shift, they generate the full Σ*-algebra of cylinders. A typical element of the time-refinement is then the cylinder set*

$$C_{\boldsymbol{i}_{[0,n-1]}}^{[0,n-1]} = \bigcap_{j=0}^{n-1} \sigma_{\mathbb{Z}}^{-j} C_{i_j}^{(0)} = \left\{\{k_j\}_j : k_0 = i_0, \dots, k_{n-1} = i_{n-1}\right\},$$

which has measure $p_{\boldsymbol{i}_{[0,n-1]}}$*. Then, exploiting compatibility and stationarity of the Markov distribution, the KS entropy reads*

$$h_\pi^{KS}(\sigma_{\mathbb{Z}}) = -\sum_{\boldsymbol{i}_{[0,n-1]}} p_{\boldsymbol{i}_{[0,n-1]}} \log\left(p_{\boldsymbol{i}_{[0,n-1]}}\right) = -\sum_{ij} p_j T_{ij} \log T_{ij} .$$

Note that, for $T_{ij} = p_i$*, the Markov shift reduces to the so-called Bernoulli shift, describing an independent and identically distributed source, with entropy* $h_\pi^{KS}(\sigma_{\mathbb{Z}}) = -\sum_j p_j \log p_j$*.*

3. Baker's map. *Let* $\mathfrak{X} = [0,1) \times [0,1)$ *endowed with the Lebesgue measure and the invertible transformation*

$$T(x,y) = \begin{cases} (2x, \frac{y}{2}) & x < \frac{1}{2} \\ (2x-1, \frac{y}{2} + \frac{1}{2}) & x \geq \frac{1}{2} \end{cases} . \tag{1.50}$$

Then, by writing x *and* y *in binary digits,* $x = .a_0 a_{-1} a_{-2} \dots$ *and* $y = .a_1 a_2 a_3 \dots$*,* $a_k, b_k \in \{0,1\}$*, the action of* T *translates into the shift on doubly infinite sequences*

$$\dots a_{-2} a_{-1} a_0 a_1 a_2 \dots$$

in $\{0,1\}^{\mathbb{Z}}$*. One indeed proves that the dynamical system* $(\mathfrak{X}, T, \mu)$ *is isomorphic to the Bernoulli shift on* $\{0,1\}^{\mathbb{Z}}$ *with* $p_0 = p_1 = 1/2$ *[40]. Due to invariance of the KS entropy, Example 1.6.2 thus yields* $h_\mu(T) = \log(2)$ *for the entropy of the Baker's map.*

Remark 1.9. *Example 1.6.1 is a paradigmatic instance of regular motion, for either periodic (*α *rational) or quasi-periodic rotations (*α *irrational), having zero KS entropy. In fact, all systems with finite spectrum have zero entropy [36]. On the other hand, the Baker's map of Example 1.6.3 is a prototype of chaotic dynamical system, as witnessed by its positive entropy. In fact, two nearby initial conditions lying in the same atom of the partition – though indistinguishable from the coarse-grained picture – evolve into exponentially separated trajectories. In this sense, information about the initial condition is lost irretrievably. We stress that this "irreversible" behaviour is due to the coarse-graining procedure: indeed, if one had full control over the fine-grain structure of the evolution, there would be no uncertainty about the future [42, 43].*

1.3.2.1 Classical spin chains

An algebraic description of the symbolic dynamics $(\Omega_p^{\mathbb{Z}}, \sigma_{\mathbb{Z}}, \pi)$ can be given in terms of classical spin chains. This is done by associating to each symbol $a \in \{1, \dots, p\}$ a rank-one projector $\Pi_a = |a\rangle\langle a|$. This selects a MASA $D_p(\mathbb{C})$ of $p \times p$ diagonal matrices spanned by $\{\Pi_a\}_{a=1}^p$,

$\Pi_a\Pi_b = \delta_{ab}\Pi_a$ whose elements are of the form $A = \sum_{a=1}^{p} a_i\Pi_i$. Then, as in Section 1.1.2, one constructs Abelian strictly local algebras $\mathcal{D}^{[a,b]} = \bigotimes_{k=a}^{b}(D_{|\mathcal{P}|}(\mathbb{C}))_k$. Their union $\mathcal{D}^{\rm loc} = \bigcup_{a\leq b}\mathcal{D}^{[a,b]}$ can be closed to give the quasi-local Abelian C^*-algebra $\mathcal{D} \equiv D_p^{\mathbb{Z}}$. We can endow the latter with a locally normal state

$$\omega_{\mathcal{P}}(A^{[0,n-1]}) = {\rm Tr}\left(\rho_{\mathcal{P}}^{[0,n-1]}A^{[0,n-1]}\right), \qquad \rho_{\mathcal{P}}^{[0,n-1]} = \sum_{\boldsymbol{a}_{[0,n-1]}} p_{\boldsymbol{a}_{[0,n-1]}}\,\Pi^{[0,n-1]}_{\boldsymbol{a}_{[0,n-1]}}, \tag{1.51}$$

where $\Pi^{[i,j]}_{\boldsymbol{a}_{[i,j]}} := \bigotimes_{k=i}^{j}\Pi_{a_k}$. The spectrum of local density matrices $\rho_{\mathcal{P}}^{[0,n-1]}$ amounts to stationary probability distribution $\pi_{\mathcal{P}}^{[0,n-1]}$ defined in (1.42). Shift invariance of π translates into translational invariance of $\omega_{\mathcal{P}}$ under the tensor shift σ introduced in (1.28).

Remark 1.10. *Consider a shift dynamical systems $(\Omega_p^{\mathbb{Z}},\pi,\sigma_{\mathbb{Z}})$ and its algebraic rendering $(\mathcal{D},\omega_{\mathcal{P}},\sigma)$ constructed as above. Then, one proves that KS entropy of the shift on symbols corresponds to the mean Shannon entropy of $\omega_{\mathcal{P}}$ [44], namely*

$$\mathfrak{h}_\mu^{KS}(\sigma) = \mathfrak{S}_\omega\,, \qquad \mathfrak{S}_\omega = \lim_n \frac{1}{n}H\left(\pi_{\mathcal{P}}^{[0,n-1]}\right) = \lim_n\frac{1}{n}S\left(\rho_{\mathcal{P}}^{[0,n-1]}\right). \tag{1.52}$$

where

$$H\left(\pi_{\mathcal{P}}^{[0,n-1]}\right) = -\sum_{\boldsymbol{a}_{[0,n-1]}} p_{\boldsymbol{a}_{[0,n-1]}}\log p_{\boldsymbol{a}_{[0,n-1]}}\,.$$

An indirect proof will be provided in Section 1.5.

1.3.3 Time-inhomogeneous Markov chains

Consider now a non-shift invariant state ω on the half-chain $D_p^{\mathbb{N}}$, defined through localized density matrices

$$\rho^{[0,n-1]} = \sum_{\boldsymbol{a}_{[0,n-1]}} p^{[0,n-1]}_{\boldsymbol{a}_{[0,n-1]}}\,\Pi^{[0,n-1]}_{\boldsymbol{a}_{[0,n-1]}}, \tag{1.53}$$

with $p^{[0,n-1]}_{\boldsymbol{a}_{[0,n-1]}}$ satisfying the Markov property (1.43). Interpreting again n as the discrete time variable and letting $X = \sum_k x_k\Pi_k$, one evaluates,

$$\omega(\mathbb{1}^{(0)}\otimes\ldots\otimes\mathbb{1}^{(n-2)}\otimes X^{(n-1)}) = \sum_{a_{n-1}}\left(\sum_{a_0\ldots a_{n-2}} p^{[0,n-1]}_{\boldsymbol{a}^{[0,n-1]}}\right)x_{a_{n-1}} = \left\langle \boldsymbol{p}^{(n-1)}\middle|X\right\rangle$$

where $p^{(n-1)}_{a_{n-1}} = \sum_{a_0\ldots a_{n-2}} p^{[0,n-1]}_{\boldsymbol{a}^{[0,n-1]}}$, and $\boldsymbol{p}^{(n-1)}$ is the one-time marginal evolving according to

$$\boldsymbol{p}^{(n-1)} = T(n-1,n-2)\ldots T(2,1)T(1,0)\,\boldsymbol{p}^{(0)}\,. \tag{1.54}$$

which yields,

$$\boldsymbol{p}^{(n)} = T(n,m)\,\boldsymbol{p}^{(m)} \geq 0, \qquad T(n,m) := \prod_{k=m+1}^{n} T(k,k-1)\,, \quad n\geq m\,. \tag{1.55}$$

Equivalently, let

$$p^{(n-1|0)}_{i_{n-1}i_0} = \frac{1}{p^{(0)}_{i_0}} \sum_{i_1 \dots i_{n-2}} p^{[0,n-1]}_{\boldsymbol{i}_{[0,n-1]}} =: T_{i_{n-1}i_0}(n-1)\,, \tag{1.56}$$

be the conditional probability of the $n-1$-th symbol conditioned on the 0-th one, defining a stochastic matrix $T(n-1)$. Interpreting n as the time variable, the latter maps one-time marginals from time 0 to time $n-1$,

$$p^{(n-1)}_{i_{n-1}} = \sum_{i_0} T_{i_{n-1}i_0}(n-1)\, p^{(0)}_{i_0}\,,$$

Accordingly, by (1.55), one recovers the so-called Chapman-Kolmogorov equation,

$$T(n) = T(n,m)\,T(m)\,, \qquad n \geq m\,. \tag{1.57}$$

1.3.4 Classical divisibility

The Chapman-Kolmogorov equation represents only a necessary condition for the process $p^{[0,n-1]}_{\boldsymbol{a}_{[0,n-1]}}$ to be Markovian [45, 46].

Example 1.7. *Let $n = 3$ and consider the process described by the probability distribution*

$$p^{[0,2]}_{a_0a_1a_2} = p^{(2|1,0)}_{a_2|a_1a_0}\, p^{(1|0)}_{a_1|a_0}\, p^{(0)}_{a_0}\,,$$

defined on binary alphabet, $a_k \in \{0,1\}$. In particular, we choose $p^{(0)}_0 = p^{(0)}_1 = 1/2$ and conditional probabilities

$$p^{(1|0)}_{a_1|a_0} = T_{a_1a_0}(1) = \frac{1}{2} \quad\Longrightarrow\quad T(1) = |+\rangle\langle+| = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\,, \tag{1.58}$$

and

$$p^{(2|10)}_{a_2|a_1a_0} = \begin{cases} T_{a_2a_1}(1)\,, & a_0 = 0, \\ 2\epsilon\delta_{a_2a_1} + (1-2\epsilon)T_{a_2a_1}(1) & a_0 = 1\,. \end{cases} \tag{1.59}$$

Hence, since $p^{(2|10)}_{a_2|a_1a_0}$ depends on a_0, the process is non-Markovian for $0 < \epsilon \leq 1/2$. Nevertheless, one computes

$$T_{a_2a_0}(2) = \frac{1}{2} \quad\Longrightarrow\quad T(2) = |+\rangle\langle+|\,.$$

Moreover, define the transition matrix

$$T_{a_2a_1}(2,1) := \frac{\sum_{a_0} p^{(2|10)}_{a_2|a_1a_0} p^{(1|0)}_{a_1|a_0} p^{(0)}_{a_0}}{\sum_{b_0} p^{(1|0)}_{a_1|b_0} p^{(0)}_{b_0}} = \epsilon\delta_{a_2a_1} + (1-\epsilon)\frac{1}{2} \implies T(2,1) = \frac{1+\epsilon}{2}\mathbb{1}_2 + \frac{1-\epsilon}{2}\sigma_1\,, \tag{1.60}$$

so that

$$T(2,1)T(1) = |+\rangle\langle+| = T(2)\,,$$

namely, the Chapman Kolmogorov equation is satisfied.

Definition 1.8. *Let* $T_{ij}(n) := p^{(n|0)}_{i|j}$ *be the stochastic matrix defined from conditional probabilities* $p^{(n|0)}_{i|j}$. *A process is P-divisible if, for all* $n \geq m \geq 0$, *there exists a stochastic matrix* $T(n,m)$ *such that*

$$T(n) = T(n,m)\,T(m)\,, \qquad n \geq m\,. \tag{1.61}$$

Remark 1.11. *If* $T(n)$ *is an invertible matrix for all* n, *the intertwining matrix is uniquely determined by matrix inversion,* $T(n,m) = T(n)\,T(m)^{-1}$. *Of course, the resulting matrix* $T(n,m)$ *need not be stochastic, since the inverse of a positive-entrywise matrix is not generally positive-entrywise. It follows from* (1.57) *that a Markov process is P-divisible. Therefore, if* $T(n,m)$ *fails to be stochastic, one can conclude that the process is non-Markovian. On the other hand, as in Example 1.7, there exist processes that are P-divisible but non-Markovian.*

1.3.5 Divisibility in continuous time

The definition of P-divisibility easily extends to continuous-time processes. In such case, 1-time probability vectors evolve according to a 1-parameter family of stochastic matrices $\{T(t)\}_{t\geq 0}$,

$$\boldsymbol{p} \longmapsto \boldsymbol{p}^{(t)} := T(t)\boldsymbol{p}\,. \tag{1.62}$$

If $T(t)$ is invertible for all $t \geq 0$, one can define propagators by $T(t,s) = T(t)T(s)^{-1}$, so that

$$T(t) = T(t,s)T(s)\,, \quad t \geq s \geq 0\,. \tag{1.63}$$

Furthermore, one can find a time-local *master equation*,

$$\dot{T}(t) = L(t)\,T(t)\,, \qquad T(0) = \mathbb{1}\,, \tag{1.64}$$

where $L(t) = \dot{T}(t)T(t)^{-1}$ is the *generator* of the family $\{T(t)\}_{t\geq 0}$. The formal solution for the two parameter family of propagators $T(t,s)$, $t \geq s$ is found through the Dyson series,

$$T(t,s) = \mathcal{T}_{\leftarrow}\, e^{\int_s^t \mathrm{d}u L(u)} \equiv \mathbb{1} + \sum_{k=1}^{\infty} \int_s^t \mathrm{d}u_1 \int_s^{u_1} \mathrm{d}u_2 \ldots \int_s^{u_{k-1}} \mathrm{d}u_k\, L(u_1)L(u_2)\ldots L(u_k)\,, \tag{1.65}$$

and $T(t) = T(t,0)$. As in Definition 1.8, but with continuous time, the process is called P-divisible if $T(t,s)$ is a stochastic matrix for all $t \geq s \geq 0$. Generators of P-divisible families can be characterized as follows [47].

Proposition 1.5 (Kolmogorov). $L(t) = [L_{ij}(t)]_{ij}$ *is the generator of a P-divisible family if and only if*

$$L_{ij}(t) \geq 0\,, \quad i \neq j\,, \qquad \sum_i L_{ij}(t) = 0\,, \qquad t \geq 0\,. \tag{1.66}$$

Proof. The "only if" part follows from the first order expansion of $T(t+\epsilon,t)$, as in (1.65), for $0 < \epsilon \ll 1$,

$$T(t+\epsilon,t) = \mathbb{1} + \epsilon\, L(t) + O(\epsilon^2)\,.$$

If $T(t+\epsilon,t)$ is stochastic, its entries read

$$T_{ij}(t+\epsilon,t) = \delta_{ij} + \epsilon\, L_{ij}(t) + O(\epsilon^2) \implies L_{ij}(t) \geq 0 \quad \text{if} \quad i \neq j\,.$$

Conversely, conditions (1.66) imply that the matrix

$$\mathbb{1} + \alpha L(s), \qquad \alpha > 0, s \geq 0, \tag{1.67}$$

is stochastic for α sufficiently small. Hence, for fixed $s \geq 0$,

$$t \longmapsto T_s(t) := e^{tL(s)} = \lim_n \left(\mathbb{1} + \frac{t}{n} L(s)\right)^n, \tag{1.68}$$

is a semigroup of stochastic matrices, since the product of stochastic matrices is stochastic. Finally, consider a sequence of partitions of the interval $[s,t]$,

$$t \equiv t_N \geq t_{N-1} \geq \ldots \geq t_0 \equiv s, \quad \text{with} \quad \limsup_N (t_k - t_{k-1}) = 0,$$

that allows one to express the Dyson series as a time-ordered matrix product [48, Theorem 1.4.3],

$$T(t,s) = \mathcal{T}_{\leftarrow} e^{\int_s^t \mathrm{d}u L(u)} = \lim_N \prod_{k=N-1}^{0} T_{\delta t_k}(t_k), \qquad \delta t_k \equiv t_{k+1} - t_k.$$

Since $T_{t-t_k}(t_k)$ is stochastic for all $t \geq t_k$ and all $k = 0, \ldots, N-1$, and the set of stochastic matrices in $M_d(\mathbb{C})$ is compact, it follows that $T(t,s)$ is stochastic. □

If $T(t)$ is an invertible matrix for all $t \geq 0$, one can equivalently characterize P-divisibility in terms of the distinguishability between two time-evolving probability vectors. Consider the weighted difference between a pair of probability vectors $\boldsymbol{p}, \boldsymbol{q}$

$$\boldsymbol{\delta}_\mu(\boldsymbol{p},\boldsymbol{q}) = \mu\, \boldsymbol{p} - (1-\mu)\, \boldsymbol{q}, \qquad 0 \leq \mu \leq 1, \tag{1.69}$$

Then, the Kolmogorov distance between $\boldsymbol{p}$ and $\boldsymbol{q}$, when picked with a-priori biases μ and $1-\mu$, respectively, amounts to the ℓ_1-norm of the vector (1.69):

$$\left\|\boldsymbol{\delta}_\mu(\boldsymbol{p},\boldsymbol{q})\right\|_1 = \sum_i \left|\langle i|\boldsymbol{\delta}_\mu(\boldsymbol{p},\boldsymbol{q})\rangle\right|. \tag{1.70}$$

Proposition 1.6. *Let $T(t)$ be invertible. Then, the process is P-divisible if and only if*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|T(t)\boldsymbol{\delta}_\mu(\boldsymbol{p},\boldsymbol{q})\right\|_1 \leq 0, \tag{1.71}$$

for all $t \geq 0$, all probability vectors $\boldsymbol{p}, \boldsymbol{q}$ and weights $0 \leq \mu \leq 1$.

Proof. First, note that any vector $\boldsymbol{x} \in \mathbb{R}^d$ is proportional to the weighted difference of two probability vectors $\boldsymbol{p}$, $\boldsymbol{q}$ (1.69),

$$\boldsymbol{x} = \alpha \boldsymbol{\delta}_\mu(\boldsymbol{p},\boldsymbol{q}), \qquad \alpha > 0. \tag{1.72}$$

Then, let $\boldsymbol{x}$ be a vector in $\mathbb{R}^d$ and consider its time-evolution under a P-divisible stochastic evolution $\boldsymbol{x}_t := T(t)\boldsymbol{x}$. Then, P-divisibility implies that one can decompose the dynamics by entrywise-positive propagators $T(t,s)$, so that

$$\|x_t\|_1 = \|T(t)\boldsymbol{x}\|_1 = \|T(t,s)\boldsymbol{x}_s\|_1 = \left\|\sum_{ij} T_{ij}(t,s)x_j(s)\right\|_1$$

$$\leq \sum_j \left(\sum_i T_{ij}(t,s)\right)|x_j(s)| = \|\boldsymbol{x}_s\|_1 ,$$

where the last equality follows from $\sum_i T_{ij}(t,s) = 1$. Hence, (1.71) follows. When $T(t)$ is an invertible matrix, the converse implication is easily proved. □

1.4 Quantum irreversibility

In order to include in the description of quantum dynamical systems those affected by an irreversible dynamics, it is necessary to introduce the concepts of positive and completely positive maps between operator algebras.

Definition 1.9. *Let $\mathcal{A}, \mathcal{B}$ be C^* algebras and $\Lambda : \mathcal{A} \to \mathcal{B}$ be a linear map.*
Λ is called positive *if $\Lambda[X^\dagger X] \geq 0$ for all $X \in \mathcal{A}$.*
Λ is called unital *if $\Lambda[\mathbb{1}_\mathcal{A}] = \mathbb{1}_\mathcal{B}$.*

Remark 1.12. *A linear map on the algebra of observables $\Lambda : \mathcal{A} \to \mathcal{B}$ describes a transformation in the Heisenberg picture. One can then canonically associate a dual action in the Schrödinger picture; namely, given a state $\omega : \mathcal{B} \to \mathbb{C}$*

$$\omega(\Lambda[X]) = \Lambda^\ddagger[\omega](X) , \qquad X \in \mathcal{B}. \tag{1.73}$$

If $\mathcal{A} = \mathcal{B}(\mathcal{H})$, the C^*-algebra of bounded operators on $\mathcal{H}$, states are described by density matrices within the trace class operators of $\mathcal{B}(\mathcal{H})$. Then, the dual $\Lambda^\ddagger$ of a linear map $\Lambda : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ on trace-class operators is defined through the relation

$$\mathrm{Tr}\,(\rho\,\Lambda[X]) = \mathrm{Tr}\left(\Lambda^\ddagger[\rho]\,X\right) . \tag{1.74}$$

From the latter, one deduces that Λ is unital if and only if its dual $\Lambda^\ddagger$ is trace preserving, namely $\mathrm{Tr}\left(\Lambda^\ddagger[\rho]\right) = \mathrm{Tr}\,(\rho)\,, \forall\rho$. Moreover, Λ is positive if and only if $\Lambda^\ddagger$ is.

1.4.1 Positive maps as contractions

A very important feature of positive maps is contractivity: the physical consequences of this fact will emerge later on (see Section 2.1.3). For the following developments, it will suffice to consider contractivity over matrix algebras $\mathcal{A} = \mathcal{B} = M_d(\mathbb{C})$. For a more general formulation for positive maps on operator algebras, see [49].

Proposition 1.7 (Choi). *Let $\Lambda : M_d(\mathbb{C}) \to M_d(\mathbb{C})$ being a positive, unital map. Then, for all normal matrices $X \in M_d(\mathbb{C})$, $X^\dagger X = XX^\dagger$,*

$$\Lambda[X^\dagger X] \geq \Lambda[X^\dagger]\Lambda[X] , \tag{1.75}$$

Proof. Since X is normal, we can consider its spectral decomposition, $X = \sum_i x_i \, |x\rangle\langle x|_i$, $x_i \in \mathbb{C}$, and consider the block-matrix

$$\begin{pmatrix} X^\dagger X & X \\ X^\dagger & \mathbb{1}_d \end{pmatrix} = \sum_i \begin{pmatrix} |x_i|^2 & x_i \\ \overline{x_i} & 1 \end{pmatrix} \otimes |x_i\rangle\langle x_i| \ ,$$

which is positive, since the submatrix $\begin{pmatrix} |x_i|^2 & x_i \\ x_i & 1 \end{pmatrix}$ has eigenvalues $1 + |x_i|^2$ and 0. Then,

$$\begin{pmatrix} \Lambda[X^\dagger X] & \Lambda[X] \\ \Lambda[X^\dagger] & \mathbb{1}_d \end{pmatrix} = \sum_i \begin{pmatrix} |x_i|^2 & x_i \\ \overline{x_i} & 1 \end{pmatrix} \otimes \Lambda[|x_i\rangle\langle x_i|] \, ,$$

is again positive and (1.75) follows. □

Proposition 1.8. *Let $\mathcal{A} = M_d(\mathbb{C})$. A unital map Λ is a contraction in the matrix (uniform) norm,*

$$\|\Lambda[X]\| \leq \|X\| \, , \qquad \forall X \in M_d(\mathbb{C}) \, . \tag{1.76}$$

if and only if it is positive.

Remark 1.13. *Recall that the matrix norm, $\|X\| = \sup_{\|\psi\|=1} \big\| X \, |\psi\rangle \big\|$ and the trace norm*

$$\|X\|_1 = \mathrm{Tr}\,(|X|) \ , \qquad |X| := \sqrt{X^\dagger X} \, ,$$

are dual norms [50], namely

$$\|Y\|_1 = \sup_{\|X\|=1} |\mathrm{Tr}\,(YX)| \quad \textit{and} \quad \|X\| = \sup_{\|Y\|_1=1} |\mathrm{Tr}\,(YX)| \, .$$

Therefore,

$$|\mathrm{Tr}\,(Y\Lambda[X])| \leq \max\{\|Y\|_1 \|\Lambda[X]\|, \big\|\Lambda^\ddagger[Y]\big\|_1 \|X\|\}, \tag{1.77}$$

from which one deduces that Λ contracts the uniform norm if and only if $\Lambda^\ddagger$ contracts the trace norm.

Proof of Proposition 1.8. Suppose that Λ is positive and unital. Consider first a unitary matrix U which is, in particular, normal

$$U^\dagger U = UU^\dagger = \mathbb{1}_d \implies \mathbb{1}_d = \Lambda[U^\dagger U] \geq \Lambda[U^\dagger]\Lambda[U] \implies \|\Lambda[U]\| \leq 1 \, , \tag{1.78}$$

thanks to inequality (1.75). Then, to verify contractivity, it suffices to consider X, $\|X\| = 1$. Since its singular values are between 0 and 1, every such matrix can be written as the average of two unitaries U and V: $X = \dfrac{U+V}{2}$. Accordingly, one has:

$$\|\Lambda[X]\| = \left\| \frac{\Lambda[U+V]}{2} \right\| \leq \frac{\|\Lambda[U]\| + \|\Lambda[V]\|}{2} \leq \frac{\|U\| + \|V\|}{2} = 1 = \|X\|. \tag{1.79}$$

Conversely, let

$$\omega_{\Lambda,\phi}[X] := \mathrm{Tr}\left(\Lambda^\ddagger[|\phi\rangle\langle\phi|]X\right) \, , \quad \langle\phi|\phi\rangle = 1.$$

Note that, from duality and the assumed contractivity,

$$\left|\omega_{\Lambda,\phi}[X]\right| = \left|\langle\phi|\Lambda[X]|\phi\rangle\right| \leq \|\Lambda[X]\| \leq \|X\|.$$

But Λ is unital, so that

$$\left\|\Lambda^{\ddagger}[|\phi\rangle\langle\phi|]\right\|_1 = \max_{\|X\|=1} \left|\omega_{\Lambda,\phi}[X]\right| = 1 = \mathrm{Tr}\left(\Lambda^{\ddagger}[|\phi\rangle\langle\phi|]\right). \tag{1.80}$$

Thus, $\omega_{\Lambda,\phi}$ is a state, that is $\Lambda^{\ddagger}[|\phi\rangle\langle\phi|] \geq 0$, yielding positivity of $\Lambda^{\ddagger}$ (and of its dual) due to the arbitrariness of ϕ. □

In order to check for contractivity of the trace norm, one might as well restrict to Hermitian matrices [51].

Corollary 1.1. *Let $\Lambda^{\ddagger}$ be trace and Hermiticity preserving. Then, it is positive if and only if it contracts the trace-norm of Hermitian matrices.*

Proof. The only if part follows from Proposition 1.8 and Remark 1.13. Suppose that $\Lambda^{\ddagger}$ contracts the trace norm of Hermitian matrices. Then, for $X \geq 0$,

$$\left\|\Lambda^{\ddagger}[X]\right\|_1 \leq \|X\|_1 = \mathrm{Tr}\,(X) = \mathrm{Tr}\left(\Lambda^{\ddagger}[X]\right) \implies \Lambda^{\ddagger} \geq 0. \qquad \square$$

1.4.2 Completely positive maps

Positivity could be seen as a minimal mathematical requirement in order to preserve the consistency in the probabilistic description of quantum mechanics when describing general operations on quantum states. On the other hand, when dealing with compound systems, one realizes that positivity is not enough, due to the existence of quantum entanglement.

Definition 1.10. *A linear map $\Lambda : \mathcal{A} \to \mathcal{B}$ is k-positive if the lifted map $\Lambda \otimes \mathrm{id}_k : \mathcal{A} \otimes M_k(\mathbb{C}) \to \mathcal{B} \otimes M_k(\mathbb{C})$ is positive. Λ is completely positive (CP) if it is k-positive for all $k \in \mathbb{N}$.*

Remark 1.14. *Complete positivity essentially amounts to the robustness of quantum operations against tensor products. This is crucial in order to guarantee the physical consistency of the description due to the presence of quantum entanglement. For the sake of simplicity, consider $M_d(\mathbb{C}) \otimes M_d(\mathbb{C})$ and take a bipartite density matrix of the form*

$$\rho_{sep} = \sum_i \lambda_i \,\rho_i \otimes \sigma_i\,, \qquad 0 \leq \lambda_i \leq 1\,,\ \sum_i \lambda_i = 1\,. \tag{1.81}$$

Then, take two positive maps Λ and $\widetilde{\Lambda}$ and evaluate the action of their tensor product on (1.81),

$$\Lambda \otimes \widetilde{\Lambda}[\rho_{sep}] = \sum_i \lambda_i \,\Lambda[\rho_i] \otimes \widetilde{\Lambda}[\sigma_i]\,, \tag{1.82}$$

which is positive, being the convex mixture of positive matrices $\Lambda[\rho_i] \otimes \widetilde{\Lambda}[\sigma_i]$. States (1.81) *are called* separable. *All states in $M_d(\mathbb{C}) \otimes M_d(\mathbb{C})$ which are not in the form* (1.81) *are called* entangled. *Let now T be the transposition map on $M_2(\mathbb{C})$,*

$$\mathsf{T}\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] = \begin{pmatrix} a & c \\ b & d \end{pmatrix}. \tag{1.83}$$

which is a positive map. Then, consider the lifting $\mathsf{T}\otimes\mathrm{id}_2 : M_2(\mathbb{C})\otimes M_2(\mathbb{C}) \to M_2(\mathbb{C})\otimes M_2(\mathbb{C})$ *and its action on the projector* $P_+^{(2)} = \left|\psi_+^{(2)}\middle\rangle\middle\langle\psi_+^{(2)}\right|$ *onto fully symmetric maximally entangled state* $\left|\psi_+^{(2)}\right\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$,

$$P_+^{(2)} = \frac{1}{2}\begin{pmatrix}1&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&1\end{pmatrix} \longmapsto \mathsf{T}\otimes\mathrm{id}_d[P_+^{(2)}] = \frac{1}{2}\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}. \tag{1.84}$$

The latter matrix has a negative eigenvalue, so that $\mathsf{T}\otimes\mathrm{id}_2$ *cannot be positive; therefore,* T *is not completely positive. Notably, the second tensor power* $\mathsf{T}\otimes\mathsf{T}$*, which is the transposition map on* $M_2(\mathbb{C})\otimes M_2(\mathbb{C}) \simeq M_4(\mathbb{C})$*, is again a positive map. The previous arguments show that the necessity of complete positivity is due to the existence of quantum entanglement and, in this sense, represents its dynamical counterpart.*

Complete positivity becomes relevant only in a fully quantum mechanical setting. That is, if either the domain or the range of Λ are commutative algebras, complete positivity coincides with simple positivity.

Proposition 1.9 (Stinespring). *Let* $\Lambda : \mathcal{A} \to \mathcal{B}$ *be a positive map. If either* $\mathcal{A}$ *or* $\mathcal{B}$ *is Abelian,* Λ *is completely positive.*

Remark 1.15. *It turns out that to extend* (1.75) *to all* $X \in M_d(\mathbb{C})$*, positivity is not enough: indeed, one needs the stronger requirement of 2-positivity so to fulfil the so-called Schwarz inequality,*

$$\Lambda[X^\dagger X] \geq \Lambda[X^\dagger]\Lambda[X], \qquad X \in M_d(\mathbb{C}). \tag{1.85}$$

Remark 1.16. *Consider the* C^* *algebra* $\mathcal{A}\otimes M_n(\mathbb{C})$*, whose elements can be written as*

$$X = \sum_{ij} X^{(ij)} \otimes E_{ij}, \qquad X^{(ij)} \in \mathcal{A}, \tag{1.86}$$

where $\{E_{ij}\}_{ij=0}^{d-1}$ *is a system of matrix units in* $M_n(\mathbb{C})$*. Suppose that* $X \geq 0$*. Then, it can be always written as*

$$X = Y^\dagger Y = \sum_i \sum_{jk} Y^{(ij)\dagger}\, Y^{(ik)} \otimes E_{jk}. \tag{1.87}$$

Hence, $X \geq 0$ *iff it is the sum of elementary elements of the form* $Y_{el} = \sum_{jk} Y_j^\dagger Y_k \otimes E_{jk}$*. Moreover, let* π *be a faithful representation of* $\mathcal{A}$ *on a Hilbert space* $\mathcal{H}$*. Then, take* $|\phi\rangle \in \mathcal{H}$ *and let* $|\psi\rangle = \sum_k \pi(Z_k)\,|\phi\rangle \otimes |k\rangle \in \mathcal{H}\otimes\mathbb{C}^d$*, for some fixed* Z_k *in* $\mathcal{A}$*. Hence,*

$$0 \leq \langle\psi|\pi\otimes\mathrm{id}_d(X)|\psi\rangle = \sum_{ij} \langle\phi|\,\pi(Z_i)\pi(X^{(ij)})\pi(Z_j)\,|\phi\rangle = \langle\phi|\,\pi\left(\sum_{ij} Z_i X^{(ij)} Z_j\right)|\phi\rangle\,.$$

Thus, positivity of X *is equivalent to that of* $\sum_{ij} Z_i X^{(ij)} Z_j$ *for all* $Z_i \in \mathcal{A}$*. Consider now a map* $\Lambda : \mathcal{A} \to \mathcal{A}$*. The action of* Λ *on a elementary element of* $\mathcal{A}$ *reads*

$$\Lambda\otimes\mathrm{id}_d[Y_{el}] = \sum_{ij} \Lambda[Y_i^\dagger Y_j] \otimes E_{ij}, \tag{1.88}$$

so that, complete positivity of Λ *is equivalent to the condition*

$$\sum_{ij=0}^{d-1} Z_i^\dagger \, \Lambda[Y_i^\dagger Y_j] \, Z_j \geq 0\,, \quad \forall\, Z_i, Y_i \in \mathcal{A},\ i = 0, \dots, d-1\,. \tag{1.89}$$

Example 1.8. *Endomorphisms* $\Theta : \mathcal{A} \to \mathcal{A}$*, namely linear maps such that* $\Theta(XY) = \Theta(X)\Theta(Y)$*, are completely positive maps, since*

$$\sum_{ij=0}^{d-1} Z_i^\dagger \, \Theta(Y_i^\dagger Y_j) Z_j = \left(\sum_{i=0}^{d-1} Z_i^\dagger \Theta(Y_i^\dagger)\right)\left(\sum_{j=0}^{d-1} \Theta(Y_j) Z_j\right) \geq 0\,.$$

Example 1.9. *A positive, unital map* $\Gamma : \mathcal{A} \to \mathcal{B}$*, where* $\mathcal{A}, \mathcal{B}$ *are unital* C^**-algebras and* $\mathcal{B} \subseteq \mathcal{A}$*, is called a* conditional expectation *if [52]*

$$\Gamma[CAB] = C\Gamma[A]B, \qquad A \in \mathcal{A},\ B, C \in \mathcal{B}\,. \tag{1.90}$$

Conditional expectations are instances of completely positive maps. Indeed, let Γ *be a conditional expectation for arbitrary* $A_i \in \mathcal{A}$ *and* $B_i \in \mathcal{B}$*,*

$$\sum_{ij}^{d-1} B_i^\dagger \Gamma[A_i^\dagger A_j] B_j = \sum_{ij}^{d-1} \Gamma[B_i^\dagger A_i^\dagger A_j B_j] = \Gamma\left[\left(\sum_i A_i B_i\right)^\dagger \sum_j A_j B_j\right] \geq 0\,. \tag{1.91}$$

since Γ *is positive. An immediate example of conditional expectation is obtained by lifting a state to* $M_d(\mathbb{C}) \otimes \mathcal{A} \to M_d(\mathbb{C})$*,*

$$\mathrm{id}_d \otimes \omega(X \otimes A) = \omega(A) X\,, \qquad X \in M_d(\mathbb{C}),\ A \in \mathcal{A}\,, \tag{1.92}$$

for which (1.90) *can be easily checked.*

In principle, complete positivity amounts to check positivity of all liftings $\Lambda \otimes \mathrm{id}_k$, $k \in \mathbb{N}$. Luckily, for maps from $M_d(\mathbb{C})$ to a C^*-algebra $\mathcal{A}$, complete positivity is equivalent to d-positivity [53]. Furthermore, it is sufficient to check positivity of just one operator in $M_d(\mathcal{A})$.

Theorem 1.10 (Choi). *A map* $\Lambda : M_d(\mathbb{C}) \to \mathcal{A}$ *is completely positive if and only if*

$$\mathcal{A} \otimes M_d(\mathbb{C}) \ni \Lambda \otimes \mathrm{id}_d \left[\sum_{ij=1}^{d} E_{ij} \otimes E_{ij}\right] = \sum_{ij=1}^{d} \Lambda[E_{ij}] \otimes E_{ij} \geq 0\,, \tag{1.93}$$

namely, if and only if $\left[\Lambda[E_{ij}]\right]$ *is positive in* $M_d(\mathcal{A})$*.*

For $\mathcal{A} = M_d(\mathbb{C})$, the previous result amounts to check for positivity of the $d^2 \times d^2$ Choi matrix $\Lambda \otimes \mathrm{id}_d[P_+^{(d)}]$, with $P_+^{(d)}$ being the projector onto the fully symmetric maximally entangled state $\left|\psi_+^{(d)}\right\rangle = d^{-1/2} \sum_{i=0}^{d-1} |i \otimes i\rangle$.

The most general structure of completely positive maps between bounded operators is given by the so-called Kraus-Stinespring form.

Theorem 1.11. *A linear map* $\Lambda : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ *is completely positive if and only if it can be written as*

$$\Lambda[X] = \sum_a F_a^\dagger X F_a\,, \tag{1.94}$$

where the sum, if infinite, converges strongly. If $\mathcal{H} = \mathbb{C}^d$*, the sum can be limited to* d^2 *terms.*

1.4.3 Quantum measurements

Measurements in quantum mechanics are a paradigmatic instance of irreversible evolution, mapping pure states into statistical mixtures.

Definition 1.11 (Operational partition of unity). *A finite subset $\mathcal{X} = \{X_a\}_{a=1}^{|\mathcal{X}|} \subseteq \mathcal{A}$ is called a operational partition of unity (OPU) if*

$$\sum_{a=1}^{|\mathcal{X}|} X_a^\dagger X_a = \mathbb{1}_{\mathcal{A}} . \tag{1.95}$$

where $|\mathcal{X}|$ is the cardinality of $\mathcal{X}$.

We then naturally associate a CP unital map $\mathbb{X}^\ddagger$ to the OPU $\mathcal{X}$ by identifying X_a as its Kraus operators,

$$\mathbb{X}^\ddagger[\rho] = \sum_a X_a^\dagger \, \rho \, X_a .$$

The dual $\mathbb{X}$ in the Schrödinger picture corresponds to a so-called positive operator-valued measurement (POVM) of quantum information theory. There, one is usually interested in finite-level systems; let us then focus for the moment on $\mathcal{A} = M_d(\mathbb{C})$. Consider the OPU $\mathcal{P} = \{\Pi_a\}_{a=1}^{|\mathcal{P}|}$ made of orthogonal projections, $\Pi_a \Pi_b = \delta_{a,b} \Pi_a$. The latter corresponds to so-called projective measurement,

$$\mathbb{P}\rho = \sum_{a=1}^{|\mathcal{P}|} P_a \, \rho \, P_a = \sum_a p_a \, \rho_a' , \qquad \rho_a' = \frac{P_a \rho P_a}{\mathrm{Tr}\,(P_a \rho)} , \qquad p_a' = \mathrm{Tr}\,(P_a \rho) . \tag{1.96}$$

which is itself a projector on $M_d(\mathbb{C})$, $\mathbb{P} \circ \mathbb{P} = \mathbb{P}$.

Remark 1.17. *The OPU $\mathcal{P}$ made of orthogonal projectors P_a, $a = 1 \ldots |\mathcal{P}|$ selects an Abelian *-subalgebra of $M_d(\mathbb{C})$. If $P_a = |a\rangle\langle a|$ are one-dimensional projectors, so that $|\mathcal{P}| = d$, $\mathcal{P}$ identifies a MASA of $M_d(\mathbb{C})$. In particular, the map $\mathbb{P}$ provides a conditional expectation from $M_d(\mathbb{C})$ onto such MASA. Indeed, by letting $A = \sum_i a_i \Pi_i$ and $B = \sum_i b_i \Pi_i$ and $X \in M_d(\mathbb{C})$,*

$$\mathbb{P}[AXB] = \sum_i \langle i|AXB|i\rangle \, |i\rangle\langle i| = \sum_i a_i b_i \, \langle i|X|i\rangle \, |i\rangle\langle i| = A\mathbb{P}[X]B . \tag{1.97}$$

Being orthogonal, measuring the projectors P_a has a straightforward interpretation as the perfectly distinguishable positions of a pointer on a grade; it is less so in the case of measurements described by a generic OPU $\mathcal{X} = \{X_a\}_a$; in such a case sorting out the labels a associated with generically non-orthogonal components X_a requires the dilation to a larger Hilbert space. This can be seen by means of the following simple model of measurement. Let $|0\rangle$ be a fixed vector in $\mathbb{C}^{|\mathcal{X}|}$ and define the isometry $V : \mathbb{C}^d \otimes |0\rangle \to \mathbb{C}^d \otimes \mathbb{C}^{|\mathcal{X}|}$,

$$V \, |\psi\rangle \otimes |0\rangle = \sum_{a=1}^{|\mathcal{X}|} X_a \, |\psi\rangle \otimes |a\rangle . \tag{1.98}$$

$$\rho' := V \rho \otimes |0\rangle\langle 0| \, V^\dagger = \sum_{a,b} X_a \rho X_b^\dagger \otimes |a\rangle\langle b| , \qquad \mathrm{Tr}_2 \, \rho' = \mathbb{X}[\rho] . \tag{1.99}$$

The isometry V provides a Stinespring-like dilation of the map $\mathbb{X}$ and can be extended to a unitary on the whole $\mathbb{C}^d \otimes \mathbb{C}^{|\mathcal{X}|}$, so that (1.99) can be seen as a joint evolution of system and measurement apparatus [54]. Performing then a projective measurement $\mathbb{P}$, with $P_a = |a\rangle\langle a|$, on the apparatus (which does not influence the state of the system) yields,

$$\mathrm{id}_d \otimes \mathbb{P}[\rho'] = \sum_a X_a \rho X_a^\dagger \otimes |a\rangle\langle a| \,. \tag{1.100}$$

Thus, measuring the projectors $\mathbb{1} \otimes |a\rangle\langle a|$, the post-measurement state of the system reads

$$\rho'_a = \frac{X_a \rho X_a^\dagger}{\mathrm{Tr}\left(\rho X_a^\dagger X_a\right)} \,,$$

with probability $p_a := \mathrm{Tr}\left(\rho X_a^\dagger X_a\right)$. The non-selective version of the measurement corresponds indeed to the action of the CPTP map $\rho \mapsto \mathbb{X}[\rho] = \mathrm{Tr}_2(\rho')$.

1.4.3.1 Coarse-grained density matrix: entropy exchange

Suppose to trace away from (1.99) the degrees of freedom of the apparatus, instead:

$$\rho\,[\mathcal{X}] := \rho'_2 = \mathrm{Tr}_2(\rho') = \sum_{ab=1}^{|\mathcal{X}|} \mathrm{Tr}\left(\rho X_b^\dagger X_a\right) |a\rangle\langle b| \,. \tag{1.101}$$

Since the partial trace is a completely positive map, $\rho\,[\mathcal{X}]$ is a bona-fide density matrix and we shall refer to it as *coarse-grained* density matrix associated to the OPU $\mathcal{X}$. The latter is a correlation matrix, its diagonal entries being the outcome probabilities p_a, while the off-diagonal terms are cross-correlations containing further information provided by the measurement. In fact, the von Neumann entropy of a density matrix is always less or equal to the Shannon entropy of its diagonal,

$$S\left(\rho[\mathcal{X}]\right) \;\leq\; H\left(\{p_a\}_{a=1}^{|\mathcal{X}|}\right) , \tag{1.102}$$

the inequality being saturated when the OPU $\mathcal{X}$ is made of orthogonal projections. The quantity $S(\rho[\mathcal{X}])$ is sometimes called *entropy exchange* [54, 55]. It represents the entropy increase in the apparatus, initially in the pure state $|0\rangle\langle 0|$ and thus at zero entropy, due to the measurement process that sends it into the mixed state $\rho[\mathcal{X}]$. When the state ρ of the system is pure, so is the compound state ρ' in (1.102) emerging from the measuring unitary interaction; then, $S(\rho[\mathcal{X}])$ equals the entropy growth in the system, $S(\mathbb{X}[\rho])$, due to the non-selective measurement process $\rho \mapsto \mathbb{X}[\rho]$ that sent a pure state into a statistical mixture. We now show, following [56], that the entropy gained by the measurement apparatus is at least as large as the information gained during the measurement. Recalling the definition of quantum relative entropy $S(\rho\|\sigma) = \mathrm{Tr}\left(\rho\left(\log\rho - \log\sigma\right)\right)$, consider

$$S(\rho'\|\rho'_1 \otimes \rho'_2) = S(\mathbb{X}[\rho]) + S(\rho[\mathcal{X}]) - S(\rho) \,, \tag{1.103}$$

where we used the fact that $S(\rho') = S(\rho)$. Further, consider the action of the full decoherence map $\mathbb{P}$ on the second party,

$$\mathrm{id} \otimes \mathbb{P}[\rho'] = \sum_{a=1}^{|\mathcal{X}|} p_a\, \rho'_a \otimes |a\rangle\langle a| \,, \qquad \rho'_a = \frac{X_a \rho X_a^\dagger}{p_a} \,, \tag{1.104}$$

so that

$$S(\mathrm{id}\otimes\mathbb{P}[\rho']\|\mathrm{id}\otimes\mathbb{P}[\rho_1'\otimes\rho_2']) = S(\mathbb{X}[\rho]) - \sum_a p_a S(\rho_a') \,. \tag{1.105}$$

Hence, due to the monotonicity of the relative entropy under CPTP maps, the difference

$$S\left(\mathrm{id}\otimes\mathbb{P}[\rho']\middle\|\mathrm{id}\otimes\mathbb{P}[\rho_1'\otimes\rho_2']\right) - S\left(\rho'\middle\|\rho_1'\otimes\rho_2'\right) = S(\rho) - \sum_a p_a S(\rho_a') - S(\rho[\mathcal{X}]) \leq 0 \,,$$

or, equivalently,

$$S(\rho) - \sum_a p_a S(\rho_a') \;\leq\; S(\rho[\mathcal{X}]) \,. \tag{1.106}$$

On the left-hand side, one has the average information gain about the state ρ when the measurement associated with $\mathcal{X}$ is performed. Hence, the entropy exchange $S(\rho[\mathcal{X}])$ provides an universal upper bound for the average information gain.

Remark 1.18. *An alternative expression for* $S(\rho[\mathcal{X}])$ *is given by*

$$S(\rho[\mathcal{X}]) = S\left(\mathbb{X}\otimes\mathrm{id}_d\left[|\sqrt{\rho}\rangle\langle\sqrt{\rho}|\right]\right) \,, \tag{1.107}$$

where $|\sqrt{\rho}\rangle$ *is a purification of* ρ *(see Example 1.3); the proof is reported in Appendix C.1. Let now* $\rho > 0$ *be an invertible density matrix and consider the CPTP map*

$$\mathbb{F}[\sigma] = \mathrm{Tr}\,(\sigma)\,\rho = \sum_{ij} \sqrt{r_i}R_{ij}\,\sigma\,\sqrt{r_i}R_{ij}^\dagger = \sum_{ij} F_{ij}\sigma F_{ij}^\dagger \,, \qquad F_{ij} = \sqrt{r_i}R_{ij} \,, \tag{1.108}$$

corresponding to the OPU $\mathcal{F} = \{F_{ij}\}_{ij=1}^d$, $|\mathcal{F}| = d^2$. *Clearly,* $\mathbb{F}[\rho] = \rho$. *We now show that [57]*

$$\max_{\mathcal{X}\,:\,\mathbb{X}[\rho]=\rho} S(\rho[\mathcal{X}]) = S(\rho[\mathcal{F}]) \,, \tag{1.109}$$

where the the maximization is constrained to the OPUs that leave ρ *invariant. It suffices to consider the positive trace-preserving map (CPTP) map* $\mathbb{X}$ *associated to the OPU* $\mathcal{X}$ *and evaluate*

$$\begin{aligned} 0 &\leq S\left(\mathbb{X}\otimes\mathrm{id}_d\left[|\sqrt{\rho}\rangle\langle\sqrt{\rho}|\right]\|\rho\otimes\rho\right) \\ &= -S\left(\mathbb{X}\otimes\mathrm{id}_d\left[|\sqrt{\rho}\rangle\langle\sqrt{\rho}|\right]\right) - \mathrm{Tr}\left(\mathbb{X}\otimes\mathrm{id}_d\left[|\sqrt{\rho}\rangle\langle\sqrt{\rho}|\right]\log(\rho\otimes\rho)\right) \\ &= -S(\rho[\mathcal{X}]) + 2S(\rho) = -S(\rho[\mathcal{X}]) + S\left(\mathbb{F}\otimes\mathrm{id}_d\left[|\sqrt{\rho}\rangle\langle\sqrt{\rho}|\right]\right) \,, \end{aligned}$$

yielding $S(\rho[\mathcal{X}]) \leq S(\rho[\mathcal{F}])$, *for all OPUs* $\mathcal{X}$. *The second to last equality was obtained by exploiting the invariance* $\rho = \mathbb{X}[\rho]$, *while the last equality follows from* $\mathbb{F}\otimes\mathrm{id}_d\left[|\sqrt{\rho}\rangle\langle\sqrt{\rho}|\right] = \rho\otimes\rho$. *Hence, among those partitions that leave* ρ *invariant, the maximal entropy exchange is obtained by* (1.108).

1.5 Quantum symbolic models

As seen in 1.3.2, the Kolmogorov-Sinai entropy measures the optimal rate of information provided by the symbolic dynamics arising from a coarse-graining of the classical phase-space. There have been many attempts to a non-commutative extension of the KS entropy. For a comparative discussion of the different quantum dynamical entropies see [38, 58]. Among the different proposals, two most prominent ones are that by Connes, Narnhofer and Thirring [59] and Alicki, Lindblad and Fannes. They provide inequivalent extensions to the KS entropy to the quantum setting, since they yield different results for different dynamical systems. In the following, we shall exclusively treat the proposal by Alicki and Fannes [27, 60, 61], based on earlier pioneering work by Lindblad [7, 57]. The Alicki-Lindblad-Fannes dynamical entropy (ALF for short) provides a quantum counterpart to the KS entropy in that the coarse-graining of the phase-space is replaced by POVM-measurements at successive time-steps of the Schrödinger dynamics. Let $(\mathcal{A}, \Theta, \omega)$ be a discrete-time quantum dynamical system, with $\Theta : \mathcal{A} \to \mathcal{A}$ an automorphism of $\mathcal{A}$ and ω a Θ-invariant state. Consider then an OPU $\mathcal{X} = \{X_a\}_{a=1}^{|\mathcal{X}|}$, $\sum_{a=1}^{|\mathcal{X}|} X_a^\dagger X_a = \mathbb{1}_{\mathcal{A}}$. As discussed in Section 1.4.3, the latter corresponds to nothing but a POVM measurement in the language of quantum information. As for classical partitions (see Section 1.3.1), given two OPUs $\mathcal{X}$ and $\mathcal{Y}$, one defines their *refinement* as the OPU

$$\mathcal{X} \circ \mathcal{Y} := \{X_a Y_b\}_{a=1,\, b=1}^{|\mathcal{X}|,\, |\mathcal{Y}|} \,. \tag{1.110}$$

In fact, property (1.95) is easily checked. Since we shall be now interested in the dynamics, we define the time evolution of an OPU $\mathcal{X}$ as

$$\Theta(\mathcal{X}) = \{\Theta(X_a)\}_{a=1}^{|\mathcal{X}|} \,. \tag{1.111}$$

Finally, the *time-refinement* of $\mathcal{X}$ after n steps of the dynamics is the refinement of time evolutions of $\mathcal{X}$ under iterates of Θ,

$$\mathcal{X}^{(n)} = \Theta^{n-1}(\mathcal{X}) \circ \ldots \Theta(\mathcal{X}) \circ \mathcal{X} = \left\{X_{\boldsymbol{a}}^{(n)}\right\}_{\boldsymbol{a}} \,, \tag{1.112}$$

where $\boldsymbol{a} = a_0 a_1 \cdots a_{n-1}$ defines the symbolic sequence and the time-refined OPU elements read

$$X_{\boldsymbol{a}}^{(n)} := \Theta^{n-1}(X_{a_{n-1}}) \,\ldots\, \Theta(X_{a_1})\, X_{a_0} \,. \tag{1.113}$$

As shown in Section 1.4.3 for finite-level systems, given a state ω and the OPU $\mathcal{X} = \{X_a\}_{a=1}^{|\mathcal{X}|}$, one associates the *coarse-grained density matrix*

$$\rho[\mathcal{X}] = \sum_{a,b=1}^{|\mathcal{X}|} \omega\left(X_b^\dagger X_a\right) |a\rangle\langle b| \;\in M_{|\mathcal{X}|}(\mathbb{C}) \,, \tag{1.114}$$

$\{|a\rangle\}_{a=1}^{|\mathcal{X}|}$ being the standard basis of $\mathbb{C}^{|\mathcal{X}|}$. Accordingly, vectors $\left|\boldsymbol{a}_{[0,n-1]}\right\rangle = |a_0\rangle \otimes \cdots \otimes |a_{n-1}\rangle$ form a basis in $\left(\mathbb{C}^{|\mathcal{X}|}\right)^{\otimes n-1}$ so that the coarse-grained density matrices associated with the n-step refinements read

$$\rho\left[\mathcal{X}^{(n)}\right] = \sum_{\boldsymbol{a},\boldsymbol{b}}^{|\mathcal{X}|} \omega\left(X_{\boldsymbol{b}}^{(n)\dagger} X_{\boldsymbol{a}}^{(n)}\right) \left|\boldsymbol{a}_{[0,n-1]}\right\rangle\!\left\langle \boldsymbol{b}_{[0,n-1]}\right| \;\in\; M_{|\mathcal{X}|}^{[0,n-1]}(\mathbb{C}) \,. \tag{1.115}$$

Their matrix elements are the multi-time correlation functions

$$\begin{aligned} \rho\left[\mathcal{X}^{(n)}\right]_{\boldsymbol{a},\boldsymbol{b}} &= \omega\left(X_{\boldsymbol{b}}^{(n)\dagger} X_{\boldsymbol{a}}^{(n)}\right) \\ &= \omega\left(X_{b_0}^\dagger \Theta(X_{b_1}^\dagger) \Theta^2(X_{b_2}^\dagger) \cdots \Theta^{n-1}(X_{b_{n-1}}^\dagger X_{a_{n-1}}) \cdots \Theta^2(X_{a_2}) \Theta(X_{a_1}) X_{a_0}\right) . \end{aligned} \tag{1.116}$$

Remark 1.19. *Using that* $\Theta(XY) = \Theta(X)\Theta(Y)$, *one can recast* (1.116) *in a "nested" fashion*

$$\rho\big[\mathcal{X}^{(n)}\big]_{\boldsymbol{a},\boldsymbol{b}} = \omega\left(X^\dagger_{b_0}\Theta\left(X^\dagger_{b_1}\Theta\left(X^\dagger_{b_2}\cdots\Theta\left(X^\dagger_{b_{n-1}}X_{a_{n-1}}\right)\cdots X_{a_2}\right)X_{a_1}\right)X_{a_0}\right). \tag{1.117}$$

Furthermore, one passes from the Heisenberg to the Schrödinger picture by introducing the following dual maps on the space of states of the system: $\omega \mapsto \mathbb{E}^\ddagger_{ab}[\omega]$ *and* $\omega \mapsto \Theta^\ddagger[\omega]$ *such that*

$$\mathbb{E}^\ddagger_{ab}[\omega](X) = \omega(X^\dagger_b\, X\, X_a)\,, \quad \Theta^\ddagger[\omega](X) = \omega(\Theta[X])\,.$$

Then, one can rewrite

$$\rho\big[\mathcal{X}^{(n)}\big]_{\boldsymbol{a},\boldsymbol{b}} = \mathbb{E}_{a_{n-1}b_{n-1}}\circ\Theta^\ddagger\circ\cdots\mathbb{E}^\ddagger_{a_1b_1}\circ\Theta^\ddagger\circ\mathbb{E}^\ddagger_{a_0b_0}[\omega](\mathbb{1})\,. \tag{1.118}$$

The diagonal entries of $\rho\big[\mathcal{X}^{(n)}\big]$ *are then the probabilities of sequences of selective measurements of the various OPU's terms followed by the one-step reversible dynamics.*

The density matrices $\rho[\mathcal{X}^{(n)}] \in M^{\otimes[0,n-1]}_{|\mathcal{X}|}(\mathbb{C})$, $n \geq 1$ form a consistent family in the sense that $\mathrm{Tr}_{n-1}\left(\rho[\mathcal{X}^{(n)}]\right) = \rho[\mathcal{X}^{(n-1)}] \in M^{\otimes[0,n-2]}_{|\mathcal{X}|}(\mathbb{C})$. They thus define a state $\omega_\mathcal{X}$ on the quantum spin half-chain $\mathcal{M}$ generated by the nested strictly local algebras $M^{\otimes[0,n-1]}_{|\mathcal{X}|}(\mathbb{C})$ through the expectations

$$\omega_\mathcal{X}\left(Y^{[0,n-1]}\right) = \mathrm{Tr}\left(\rho[\mathcal{X}^{(n)}]\,Y^{[0,n-1]}\right)\,, \quad Y \in M^{\otimes[0,n-1]}_{|\mathcal{X}|}(\mathbb{C})\,.$$

Remark 1.20. *The previous construction provides a coarse-grained description of quantum dynamical system* $(\mathcal{A},\Theta,\omega)$ *by means of a symbolic model* $(M^{\mathbb{N}}_{|\mathcal{X}|}(\mathbb{C}),\sigma,\omega_\mathcal{X})$ *on the half-chain* $M^{\mathbb{N}}_{|\mathcal{X}|}(\mathbb{C})$ *with the shift-dynamics* σ. *If*

$$\omega\circ\Theta = \omega \qquad \omega\circ\mathbb{X}^\ddagger = \omega\,,$$

namely if both the dynamics and the measurement leave ω *invariant, the state* $\omega_\mathcal{X}$ *is shift invariant. On the other hand, in the quantum mechanical framework, one can hardly only restrict to OPUs that leave* ω *invariant [60]. Then, if the reference state* ω *is only* Θ*-invariant and not invariant under the OPU,* $\omega(\mathbb{X}^\ddagger[Y]) \neq \omega(Y)$, $\omega_\mathcal{X}$ *will not generally be shift-invariant,* $\omega_\mathcal{X} \neq \omega_\mathcal{X}\circ\sigma$.

1.5.1 ALF entropy

Since $\rho\big[\mathcal{X}^{(n)}\big]$ is a well-defined density matrix living in $M^{[0,n-1]}_{|\mathcal{X}|}(\mathbb{C})$, one can consider its von Neumann entropy rate

$$\mathfrak{h}_\omega(\Theta,\mathcal{X}) := \limsup_n \frac{1}{n}\,S\left(\rho\big[\mathcal{X}^{(n)}\big]\right)\,, \qquad S(\rho) = -\,\mathrm{Tr}\left(\rho\log\rho\right)\,. \tag{1.119}$$

The lim sup is taken since, due to lack of shift-invariance (Remark 1.20), the limit of the ratio in (1.119) could not exist.

Definition 1.12. *The Alicki-Lindblad-Fannes dynamical entropy (ALF) associated to the dynamical system* $(\mathcal{A},\Theta,\omega)$ *is defined by*

$$\mathfrak{h}^\mathcal{B}_\omega(\Theta) := \sup_{\mathcal{X}\subseteq\mathcal{B}}\,\mathfrak{h}_\omega(\Theta,\mathcal{X})\,. \tag{1.120}$$

where maximization is over a reference subalgebra $\mathcal{B}\subseteq\mathcal{A}$.

In principle, the ALF entropy thus depends on the choice of a subalgebra $\mathcal{B}$ of *physically admissible OPUs*. In applications to reversible dynamical systems, e.g. for studying quantum chaos, $\mathcal{B}$ is typically assumed to be globally Θ-invariant, $\Theta(\mathcal{B}) = \mathcal{B}$, and dense in $\mathcal{A}$ [27]. We shall relax these assumptions on $\mathcal{B}$ in Chapter 5 when adapting Definition 1.12 to the framework of open quantum systems.

Remark 1.21. *A germ of the above construction can be found in [57], where the definition of the entropy was however limited to OPUs leaving the state ω invariant. As discussed in Remark 1.20, such an assumption, together with $\omega\circ\Theta = \omega$, guarantees the shift-invariance of $\omega_{\mathcal{X}}$ and the existence of* (1.119) *as a limit. However, it is generally deemed too restrictive in the quantum setting (this was noted by Lindblad himself [57], see also the discussion in [60]).*

Remark 1.22. *In the case where the algebraic setting is commutative the ALF entropy reduces to the KS one. Let $\mathcal{B}_0 \subset L^\infty_\mu(\mathfrak{X})$ the *-subalgebra made of characteristic functions of measurable subsets of $\mathcal{X}$. Given a partition $\mathcal{P} = \{P_i\}_i$ of $\mathfrak{X}$, the associated characteristic functions χ_{P_i} form a OPU $\mathcal{X}_\mathcal{P} = \{\chi_{P_i}\}_i$ in $\mathcal{B}_0$. Then, $\Theta(\chi_{P_i}) = \chi_{T^{-1}(P_i)}$ and, by performing the time-refinement, one obtains $[\rho[\mathcal{X}^{(n)}_\mathcal{P}]]_{\boldsymbol{i}\boldsymbol{j}} = \mu(P_{\boldsymbol{i}_{[0,n-1]}})\delta_{\boldsymbol{i}\boldsymbol{j}}$. The corresponding von Neumann entropy rate exists as a limit and, maximizing over $\mathcal{B}_0$, one concludes that $\mathfrak{h}^{\mathcal{B}_0}_\mu(\Theta_T) = \mathfrak{h}^{KS}_\mu(T)$. This argument can been further refined so that the KS entropy actually corresponds to the ALF one also when the optimization is not restricted to $\mathcal{B}_0$ but is allowed on the full $L^\infty_\mu(\mathfrak{X})$ [62].*

Let us recall the following useful bound that holds whenever we can express $\omega(A) = \mathrm{Tr}\,(\rho A)$ for some ρ (with ρ possibly depending on A, as for locally-normal states).

Proposition 1.12. *Let $A \in \mathcal{A}$ and $\omega(A) = \mathrm{Tr}\,(\rho A)$ for some density matrix ρ and let $\mathcal{X} = \{X_a\}_{a=1}^{|\mathcal{X}|}$ be a OPU. Then,*

$$S\,(\rho\,[\mathcal{X}]) \le S(\rho) + S(\mathbb{X}[\rho])\,. \tag{1.121}$$

Proof. As in the measurement model of Section 1.4.3, consider a dilation of ρ to the global state ρ' of system and apparatus as in (1.99). Let then $\rho'_{1(2)} = \mathrm{Tr}_{2(1)}(\rho')$ be the marginals of ρ'. Then, since ρ and ρ' have the same von-Neumann entropy, by invoking the triangle inequality $S(\rho') \ge |S(\rho'_1) - S(\rho'_2)|$ [54], one has

$$S(\rho) \ge \left|S\,(\rho[\mathcal{X}]) - S(\mathbb{X}[\rho])\right|,$$

that implies (1.121). □

Example 1.10. *Consider the finite-level dynamical system $(M_d(\mathbb{C}), \rho, \Theta)$ with $\Theta[X] = U^\dagger X U \in M_d(\mathbb{C})$. Then, the refined partition elements $X^{(n)}_{\boldsymbol{a}}$ will all belong to $M_d(\mathbb{C})$. Hence, for all $n \in \mathbb{N}$,*

$$S\left(\rho\left[\mathcal{X}^{(n)}\right]\right) \le S(\rho) + S\left(\sum_{\boldsymbol{a}} X^{(n)}_{\boldsymbol{a}}\rho X^{(n)\dagger}_{\boldsymbol{a}}\right) \le S(\rho) + \log d\,.$$

Thus, dividing by n both sides of the previous inequality and taking the limit $n \to \infty$ of the r.h.s., one deduces that the ALF entropy of a finite reversible dynamical system is zero. This represents the quantum analogue of classical dynamical systems with discrete spectrum that have zero KS entropy, as the rotations on the circle from Example 1.6.1.

Consider the shift dynamical system of Example (1.4), namely a stationary quantum spin chain described on the quasi local algebra $M_D^{\mathbb{Z}}$ generated by the single-site matrix algebra $\mathcal{A}^{(k)} = M_D(\mathbb{C})$, $k \in \mathbb{Z}$, and equipped with a shift-invariant state. One considers as a reference subalgebra that of local observables $\mathcal{B} = \mathcal{A}^{\mathrm{loc}}$, as defined in (1.13), which is dense $M_D^{\mathbb{Z}}$. The ALF entropy of the shift σ, derived in [60], is a paradigmatic result as illustrates the typical strategy for computing the ALF entropy in many concrete cases. We shall now review this result in the following

Proposition 1.13. *The ALF entropy of the shift for a quantum spin chain is given by*

$$h_{\omega}^{\mathcal{A}^{\mathrm{loc}}}(\sigma) = \mathfrak{S}_{\omega} + \log(D)\,, \tag{1.122}$$

where $\mathfrak{S}_{\omega}$ *is the mean von Neumann entropy of the chain* (1.30).

Proof. Let $\mathcal{X}$ be any partition in $\mathcal{A}^{\mathrm{loc}}$. Due to translational invariance of ω, it is no restriction to consider elements X_a to be finitely supported in $[0, j]$, $j \in \mathbb{N}$, namely $X_a^{[0,j]} \in \mathcal{A}^{[0,j]}$. The time-refinement $\mathcal{X}^{(n)}$, due to the iterated action of the shift $n-1$ times, will consist in operators $\widetilde{X}_{\boldsymbol{a}}^{[0,j+n-1]}$, thus supported by intervals $[0, j+n-1]$. Hence, the elements of the coarse-grained density matrix will read

$$\omega\left(X_{\boldsymbol{b}}^{(n)\dagger} X_{\boldsymbol{a}}^{(n)}\right) = \mathrm{Tr}\left(\rho^{[0,j+n-1]}\, \widetilde{X}_{\boldsymbol{b}}^{[0,j+n-1]\dagger} \widetilde{X}_{\boldsymbol{a}}^{0,j+n-1]}\right)\,,$$

Hence, by Proposition 1.12,

$$\begin{aligned} S\left(\rho\left[\mathcal{X}^{(n)}\right]\right) &\leq S\left(\rho^{[0,j+n-1]}\right) + S\left(\sum_{\boldsymbol{a}} \widetilde{X}_{\boldsymbol{a}}^{[0,j+n-1]} \rho^{[0,j+n-1]} \widetilde{X}_{\boldsymbol{a}}^{[0,j+n-1]\dagger}\right) \\ &\leq S\left(\rho^{[0,j+n-1]}\right) + (j+n)\log(D)\,. \end{aligned}$$

Then, dividing both sides by n and taking lim sup, one deduces $\mathfrak{h}_{\omega}^{\mathcal{A}^{\mathrm{loc}}} \leq \mathfrak{S}_{\omega} + \log(D)$. The equality is achieved by taking the partition $\widetilde{\mathcal{E}} = \left\{\widetilde{E}_{aa'}\right\}_{aa'=0}^{D-1} \subseteq M_D(\mathbb{C})$, with $\widetilde{E}_{aa'} = \dfrac{E_{aa'}}{\sqrt{D}}$ and $E_{aa'}$ being matrix units localized in $M_D^{(0)}(\mathbb{C})$. The time-refined partition consists of elements

$$\widetilde{E}_{\boldsymbol{a}\boldsymbol{a}'}^{(n)} = \sigma^{n-1}(\widetilde{E}_{a_{n-1}a'_{n-1}}^{(0)}) \dots \sigma(\widetilde{E}_{a_1 a'_1}^{(0)})\widetilde{E}_{a_0 a'_0}^{(0)} = D^{-n/2} \bigotimes_{k=0}^{n-1} E_{a_k a'_k}^{(k)}\,. \tag{1.123}$$

The entries of the corresponding coarse-grained density matrix read

$$\omega\left(\widetilde{E}_{\boldsymbol{b}\boldsymbol{b}'}^{(n)\dagger} \widetilde{E}_{\boldsymbol{a},\boldsymbol{a}'}^{(n)}\right) = D^{-n}\,\mathrm{Tr}\left(\rho^{[0,n-1]} \bigotimes_{k=0}^{n-1} E_{b'_k b_k}^{(k)} \bigotimes_{l=0}^{n-1} E_{a_l a'_l}^{(l)}\right) = D^{-n}\delta_{\boldsymbol{a}\boldsymbol{b}}\,\mathrm{Tr}\left(\rho^{[0,n-1]} \bigotimes_{k=0}^{n-1} E_{a'_k b'_k}^{(k)\dagger}\right). \tag{1.124}$$

The coarse-grained density matrix can be then written in terms of the same matrix units $E_{aa'}$,

$$\rho\left[\widetilde{\mathcal{E}}^{(n)}\right] = \sum_{\substack{\boldsymbol{a}\boldsymbol{a}' \\ \boldsymbol{b}\boldsymbol{b}'}} \omega\left(F_{\boldsymbol{b}\boldsymbol{b}'}^{(n)\dagger} F_{\boldsymbol{a}\boldsymbol{a}'}^{(n)}\right) |\boldsymbol{a}\boldsymbol{a}'\rangle\langle \boldsymbol{b}\boldsymbol{b}'|\;; \tag{1.125}$$

then, by substituting (1.124) into (1.125) ,

$$\rho\left[\widetilde{\mathcal{E}}^{(n)}\right] = D^{-n}\sum_{\boldsymbol{a}} E_{\boldsymbol{a}\boldsymbol{a}} \otimes \sum_{\boldsymbol{a}'\boldsymbol{b}'} \mathrm{Tr}\left(\bigotimes_{k=0}^{n-1} E^{(k)\dagger}_{a'_k b'_k}\rho^{[0,n-1]}\right)\bigotimes_{j=0}^{n-1} E^{(j)}_{a'_j b'_j} = \frac{\mathbb{1}_{D^n}}{D^n}\otimes\rho^{[0,n-1]}\,,$$

so that

$$S\left(\rho\left[\widetilde{\mathcal{E}}^{(n)}\right]\right) = n\log(D) + S\left(\rho^{[0,n-1]}\right)\,.$$

Dividing by n and taking the limit yields the result. □

Remark 1.23. *The term* $\log(D)$ *in the dynamical entropy of the shift is a correction of genuinely quantum origin due to the disturbance generated by the measurements. In fact, when the maximization is restricted to the quasi Abelian local algebra* $\mathcal{D}^{\mathrm{loc}}$*, generated by minimal projectors* $\{\Pi_i\}_{i=1}^D$ *(see Section 1.3.2.1), such term disappears. Consider, without loss of generality, a partition consisting of local operators* $X_a^{[0,j]} = \sum_{\boldsymbol{k}_{[0,j]}} x^a_{\boldsymbol{k}_{[0,j]}}\Pi^{[0,j]}_{\boldsymbol{k}_{[0,j]}}$*; its time-refinement under the shift will consist of operators* $\widetilde{X}_{\boldsymbol{a}}^{[0,j+n-1]} = \sum_{\boldsymbol{k}_{[0,j+n-1]}} \widetilde{x}^{\boldsymbol{a}}_{\boldsymbol{k}_{[0,j+n-1]}}\Pi^{[0,j+n-1]}_{\boldsymbol{k}_{[0,j+n-1]}}$*. Note that the OPU condition translates into*

$$\sum_{\boldsymbol{a}} \overline{\widetilde{x}^{\boldsymbol{a}}_{\boldsymbol{k}_{[0,j+n-1]}}}\,\widetilde{x}^{\boldsymbol{a}}_{\boldsymbol{k}_{[0,j+n-1]}} = 1\,. \tag{1.126}$$

Hence, we can recast

$$\begin{aligned}\rho\left[\mathcal{X}^{(n)}\right] &= \sum_{\boldsymbol{a}\boldsymbol{b}}\mathrm{Tr}\left(\rho^{[0,j+n-1]}\widetilde{X}_{\boldsymbol{b}}^{[0,j+n-1]\dagger}\widetilde{X}_{\boldsymbol{a}}^{[j+n-1]}\right)|\boldsymbol{a}\rangle\langle\boldsymbol{b}|\\ &= \sum_{\boldsymbol{a}\boldsymbol{b}}\sum_{\boldsymbol{k}_{[0,j+n-1]}}\overline{\widetilde{x}^{\boldsymbol{b}}_{\boldsymbol{k}_{[0,j+n-1]}}}\,\widetilde{x}^{\boldsymbol{a}}_{\boldsymbol{k}_{[0,j+n-1]}}\,\mathrm{Tr}\left(\rho^{[0,j+n-1]}\Pi_{\boldsymbol{k}_{[0,j+n-1]}}\right)|\boldsymbol{a}\rangle\langle\boldsymbol{b}|\\ &= \sum_{\boldsymbol{k}_{[0,j+n-1]}} p_{\boldsymbol{k}_{[0,j+n-1]}}\left|\Psi_{\boldsymbol{k}_{[0,j+n-1]}}\right\rangle\left\langle\Psi_{\boldsymbol{k}_{[0,j+n-1]}}\right|\,,\end{aligned}$$

where $\left|\Psi_{\boldsymbol{k}_{[0,j+n-1]}}\right\rangle = \sum_{\boldsymbol{a}}\widetilde{x}^{\boldsymbol{a}}_{\boldsymbol{k}_{[0,j+n-1]}}|\boldsymbol{a}\rangle$ *are normalized vectors due to* (1.126)*. Consequently, one has*

$$S\left(\rho\left[\mathcal{X}^{(n)}\right]\right) \le H\left(p_{\boldsymbol{k}_{[0,j+n-1]}}\right)\,.$$

Dividing by n *and taking the* lim sup *on both sides, and then the supremum of the l.h.s, yields*

$$\mathfrak{h}_\omega^{\mathcal{D}^{\mathrm{loc}}} \le \mathfrak{S}_\omega\,.$$

The upper bound, which is the KS entropy of the shift (see Remark 1.10), is saturated by taking the projective partition $\{P_i\}_i$ *localized at site* 0.

1.5.2 Operational interpretation of ALF entropy

If a time-invariant state $\omega\circ\Theta = \omega$ is considered, the physical interpretation of $\mathfrak{h}^{\mathcal{B}}_\omega(\Theta)$ is neatly obtained in the GNS representation as an alternating sequence of measurement processes and reversible one-step dynamics. This possibility already emerged in Remark 1.19; however, there this interpretation was only possible for the diagonal entries of $\rho\left[\mathcal{X}^{(n)}\right]$. As seen in Section 1.1.3, observables $X \in \mathcal{A}$ can conveniently be represented by bounded operators $\pi_\omega(X) \in B(\mathcal{H}_\omega)$ acting on the GNS Hilbert space $\mathcal{H}_\omega$ equipped with a cyclic state vector $|\Omega_\omega\rangle$. Moreover, the automorphism Θ is unitarily implemented by a unitary operator U_ω through (1.33). At the level of the GNS representation one then constructs a suitable pure-state dilation as in the model of measurement of Section 1.4.3,

$$\sum_{\boldsymbol{a},\boldsymbol{b}} \pi_\omega\left(X^{(n)}_{\boldsymbol{a}}\right) |\Omega_\omega\rangle\langle\Omega_\omega| \, \pi_\omega(X^{(n)}_{\boldsymbol{b}})^\dagger \otimes \left|\boldsymbol{a}_{[0,n-1]}\right\rangle\left\langle\boldsymbol{b}_{[0,n-1]}\right| \, . \tag{1.127}$$

Since the marginals of a pure state have the same entropy, the von Neumann entropy of the coarse-grained density matrix is then the same as [28, 38],

$$S\left(\rho[\mathcal{X}^{(n)}]\right) = S\left(\left(\mathbb{U}_\omega \circ \mathbb{X}_\omega\right)^n[|\Omega_\omega\rangle\langle\Omega_\omega|]\right). \tag{1.128}$$

where $\mathbb{U}^\ddagger_\omega[\pi_\omega(X)] = U^\dagger_\omega \pi_\omega(X) U_\omega = \pi_\omega(\Theta[X])$ is the unitary implementation of the dynamics in the GNS representation while $\mathbb{X}_\omega$ is the CPTP map

$$\mathbb{X}_\omega[\,\cdot\,] := \sum_a \pi_\omega(X_a) \, \cdot \, \pi_\omega(X_a)^\dagger \, . \tag{1.129}$$

The detailed derivation of (1.128) is reported in Appendix C.1 and its structure supports the interpretation of the ALF entropy, like that of the classical KS entropy, as the maximal information per time-step extracted from repeated measurements on the system intertwined with the time evolution. Only, measurements and dynamics do not commute.

Moreover, following the model of measurement process described in Section 1.4.3, (1.127) can be then interpreted as a dilation of the POVM process $\left(\mathbb{U}_\omega \circ \mathbb{X}_\omega\right)^n$ where the apparatus is explicitly taken into account. The only difference is that, now, multiple copies of the apparatus are considered due to the repeated measurements. Accordingly, (1.128) can be also seen as the entropy exchange with the apparatus due to iterated measurements.

CHAPTER 2

Open Quantum Systems

In the theory of open quantum systems, one is usually interested in the irreversible dynamics generated by the interaction of the system of interest S with an environment E. The latter, in principle, might comprehend infinitely many degrees of freedom. From now on, we shall always consider finite-level open systems described by a matrix algebra $\mathcal{A}_S = M_d(\mathbb{C})$, coupled to an environment E described by an algebra $\mathcal{A}_E$ so that the compound observables of system and environment are self-adjoint elements of the algebra

$$\mathcal{A} = \mathcal{A}_S \otimes \mathcal{A}_E \,. \tag{2.1}$$

Moreover, we shall also assume that, at $t = 0$, system and environment can be prepared independently and thus are in a tensor product state $\omega_S \otimes \omega_E$, with ω_S being described by a density matrix ρ_S. Finally, the system-environment reversible evolution will be given by a one-parameter group of *-automorphisms $\{\Theta_t\}_t$, $\Theta_t : \mathcal{A} \to \mathcal{A}$, $\Theta_t \circ \Theta_s = \Theta_{t+s}$. In the first part of the Chapter, we shall mostly focus on continuous-time groups yielding a continuous reduced dynamics. Later on, it will be convenient to pass to a discrete-time description, i.e. $t = n \in \mathbb{N}$ when dealing with collisional models (Section 2.1.4) or when focussing on the symbolic dynamics extracted by repeatedly probing the open system (Section 2.2).

2.1 Reduced dynamics

The assumption of a tensor product initial state of system and environment ensures that the so-called reduced dynamics of the system can be described by means of a linear, completely positive map

$$\Lambda_t^\ddagger[X_S] := \mathrm{id}_S \otimes \omega_E\big(\Theta_t[X_S \otimes \mathbb{1}_E]\big) \,. \tag{2.2}$$

It is obtained through the conditional expectation $\mathrm{id}_S \otimes \omega_E : \mathcal{A}_S \otimes \mathcal{A}_E \to \mathcal{A}_S$,

$$\mathrm{id}_S \otimes \omega_E(X_S \otimes X_E) = \omega_E(X_E)\, X_S \,,$$

by focussing on the observables of the open system, only. For the sake of later convenience, let us prove complete positivity of (2.2) also when Θ_t is replaced by means of a CPU map.

Proposition 2.1. *Let $\Lambda^\ddagger[X_S] = \mathrm{id}_S \otimes \omega_E\,(\Gamma\,[X_S \otimes \mathbb{1}_E])$, with Γ being a completely positive unital map on $\mathcal{A}_S \otimes \mathcal{A}_E$. Then, $\Lambda^\ddagger$ is completely positive and unital.*

Proof. Unitality is readily verified, since

$$\Lambda^{\ddagger}[\mathbb{1}_S] = \mathrm{id}_S \otimes \omega_E\left(\Gamma\left[\mathbb{1}_S \otimes \mathbb{1}_E\right]\right) = \omega(\mathbb{1}_E)\mathbb{1}_S = \mathbb{1}_S\,.$$

Complete positivity of Γ implies that, for all $X_i, Y_i \in \mathcal{A}_S$,

$$0 \leq \sum_{ij} Y_i^{\dagger}\,\Gamma\big[X_i^{\dagger}X_j \otimes \mathbb{1}_E\big]\,Y_j\,.$$

Then, since $\mathrm{id}_S \otimes \omega_E$ is a conditional expectation (see Example 1.9),

$$\begin{aligned} 0 \;\leq\; \mathrm{id}_S \otimes \omega_E\Big(\sum_{ij} Y_i^{\dagger}\Gamma\left[X_i^{\dagger}X_j \otimes \mathbb{1}_E\right]Y_j\Big) \;&=\; \sum_{ij} Y_i^{\dagger}\,\mathrm{id}_S \otimes \omega_E\left(\Gamma\left[X_i^{\dagger}X_j \otimes \mathbb{1}_E\right]\right)Y_j \\ &=\; \sum_{ij} Y_i^{\dagger}\,\Lambda^{\ddagger}[X_i^{\dagger}X_j]\,Y_j\,. \end{aligned}$$

Complete positivity of $\Lambda^{\ddagger}$ is then ensured, as discussed in Remark 1.16. □

The state evolution in the Schrödinger picture is equivalently described by a CPTP map Λ_t, acting on $d \times d$ density matrices, and obtained by duality

$$\mathrm{Tr}\left(\Lambda_t[\rho_S]X_S\right) = \omega_S\big(\Lambda_t^{\ddagger}[X_S]\big)\,. \tag{2.3}$$

The so-obtained maps Λ_t, $t \geq 0$ constitute a continuous 1-parameter family $\{\Lambda_t\}_{t\geq 0}$, whose evolution is generally governed by a highly complicated memory kernel equation

$$\dot{\Lambda}_t = \int_0^t \mathrm{d}s\, K_{t,s} \circ \Lambda_s\,, \qquad \Lambda_{t=0} = \mathrm{id}_d. \tag{2.4}$$

The latter encodes the fact that the reduced evolution is affected by memory effects, arising from the coupling with the environment. If the coupling between system and environment is particularly weak, a series of systematic approximations known as *weak coupling limit* [6, 20], leads to a much more feasible time-local *master equation*

$$\dot{\Lambda}_t = \mathcal{L} \circ \Lambda_t\,, \qquad \Lambda_{t=0} = \mathrm{id}_d\,, \tag{2.5}$$

with $\mathcal{L}$ being the *generator*. The formal solution of (2.5) is simply given by the exponential $\Lambda_t = e^{t\mathcal{L}}$ and the evolution evolves according a semigroup law, $\Lambda_t = \Lambda_{t-s} \circ \Lambda_s$ with $\Lambda_{t-s} = e^{(t-s)\mathcal{L}}$. The structure of $\mathcal{L}$ is fully determined by the celebrated theorem by Gorini, Kossakowski, Sudarshan [4] and Lindblad [5]. On the other hand, in many situations, a time-local master equation can be obtained also in the non-Markovian regime, allowing for an alternative form of the evolution law (2.4) [63].

2.1.1 Generators

Suppose that the dynamics Λ_t is algebraically invertible, i.e. there exists a linear map $\Lambda_t \circ \Lambda_t^{-1} = \Lambda_t^{-1} \circ \Lambda_t = \mathrm{id}_d$. It is then always possible to define a time-dependent generator

$$\mathcal{L}_t := \dot{\Lambda}_t \circ \Lambda_t^{-1} = \int_0^t \mathrm{d}s\, K_{t,s} \circ \Lambda_s \circ \Lambda_t^{-1}\,, \tag{2.6}$$

and recast (2.4) as a time-local master equation

$$\dot{\Lambda}_t = \mathcal{L}_t \circ \Lambda_t \,, \qquad \Lambda_{t=0} = \mathrm{id}_d \,. \tag{2.7}$$

As in Remark 1.1, fix a reference set of d^2 matrices, $F_0 = \mathbb{1}/\sqrt{d}$ and $\{F_k\}_{k=0}^{d^2-1} \subset M_d(\mathbb{C})$, $k = 1, \dots, d^2 - 1$ such that $\mathrm{Tr}\left(F_j^\dagger F_k\right) = \delta_{jk}$, thus forming an Hilbert-Schmidt orthonormal basis. Then, assuming trace and Hermiticity preservation of the evolution, the generator (2.6) can be always expressed in the canonical form

$$\mathcal{L}_t[\rho] = -i[H(t), \rho] + \mathcal{D}_t[\rho] \,, \tag{2.8}$$

with $H(t) = H^\dagger(t)$ being a time-dependent Hamiltonian, while $\mathcal{D}_t$ is the dissipator, encoding effects of noise and decoherence,

$$\mathcal{D}_t[\rho] = \sum_{i,j=1}^{n^2-1} K_{ij}(t) \left(F_i \rho F_j^\dagger - \frac{1}{2}\{F_j^\dagger F_i, \rho\} \right) , \tag{2.9}$$

where coefficients $K_{ij}(t) = \overline{K_{ji}}(t)$ form the Hermitian, time-dependent $(d^2 - 1) \times (d^2 - 1)$ matrix $K(t) = [K_{ij}(t)]$ known as *Kossakowski matrix*.

2.1.1.1 Positive and completely positive semigroups

Consider a constant generator $\mathcal{L}_t = \mathcal{L}$, yielding a semigroup composition law

$$\Lambda_t = \Lambda_{t-s} \circ \Lambda_s \,, \qquad \Lambda_{t-s} = e^{(t-s)\mathcal{L}} \,, \qquad \forall\, t \geq s \,, \tag{2.10}$$

and suppose to relax the request of complete positivity to simple positivity. The structure of generators of positive semigroups is not generally under control. Nevertheless, a partial characterization is provided by a non-commutative extension of conditions (1.66), first proved by Kossakowski in [51]. Given the usefulness of this property in Chapter 3, we include a concise proof.

Theorem 2.2 (Kossakowski). *Let $\mathcal{L} : M_d(\mathbb{C}) \to M_d(\mathbb{C})$ be a Hermiticity preserving linear map. Then, the following are equivalent*

1. *$e^{t\mathcal{L}}$ is positive and trace-preserving for all $t \geq 0$;*
2. *$\mathcal{L}$ fulfils* Kossakowski conditions:

$$\mathrm{Tr}\left(P_i \mathcal{L}[P_j]\right) \geq 0 \,, \qquad \sum_i \mathrm{Tr}\left(P_i \mathcal{L}[P_j]\right) = 0 \,, \tag{2.11}$$

for all choices of a MASA spanned by rank-one projectors $\mathcal{P} = \{P_i\}_{i=1}^d$, $\sum_i P_i = \mathbb{1}_d$,

Proof. 1. $\implies$ *2.* follows by the following expansion around $t = 0$,

$$0 \leq \mathrm{Tr}\left(P_i e^{t\mathcal{L}}[P_j]\right) = t\, \mathrm{Tr}\left(P_i \mathcal{L}[P_j]\right) + O(t^2) \,, \qquad i \neq j \,.$$

Similarly, letting $0 = \mathrm{Tr}\left(\mathcal{L}[P_j]\right) = \sum_i \mathrm{Tr}\left(P_i\mathcal{L}[P_j]\right)$ yields the second of (2.11). We now prove *2.* $\Longrightarrow$ *1.* For what concerns trace-preservation, one has for $\rho = \sum_j r_j \left|r_j\middle\rangle\middle\langle r_j\right|$

$$\mathrm{Tr}\left(\mathcal{L}[\rho]\right) = \sum_j r_j \sum_i \mathrm{Tr}\left(|r_i\rangle\langle r_i|\, \mathcal{L}[|r_j\rangle\langle r_j|]\right) = 0\,, \tag{2.12}$$

due to the second condition in (2.11). We now show that the first condition in (2.11) implies positivity of the map

$$(\mathrm{id}_d - \alpha\mathcal{L})^{-1}\,,$$

for sufficiently small $\alpha > 0$, that is,

$$0 \le Y \;\longmapsto\; A := (\mathrm{id}_d - \alpha\mathcal{L})^{-1}[Y] \ge 0\,, \qquad \alpha \text{ sufficiently small.}$$

Since $\mathcal{L}$ preserves Hermiticity, A is surely Hermitian, so that we can consider its spectral decomposition $A = \sum_{i=1}^d a_i\, |a_i\rangle\langle a_i|$, $a_i \in \mathbb{R}$. Moreover, one can always decompose it as

$$A = A^\dagger = A_+ - A_-\,, \qquad A_\pm \ge 0, \quad A_+A_- = 0\,.$$

Consider then

$$\Pi_- = \sum_{i\in I_-} |a_i\rangle\langle a_i|\,, \qquad I_- = \{i \in \mathbb{N} : 1 \le i \le n, a_i \le 0\}\,,$$

that projects onto the support of A_- and estimate

$$\begin{aligned}
0 \le\ & \mathrm{Tr}\left(\Pi_- Y\right) = \mathrm{Tr}\left(\Pi_-(\mathrm{id}_d - \alpha\mathcal{L})[A]\right)\\
&= -\mathrm{Tr}\left(A_-\right) - \alpha\,\mathrm{Tr}\left(\Pi_-\mathcal{L}[A_+]\right) + \alpha\,\mathrm{Tr}\left(\Pi_-\mathcal{L}[A_-]\right)\\
&\le -\mathrm{Tr}\left(A_-\right) + \alpha\,\mathrm{Tr}\left(\Pi_-\mathcal{L}[A_-]\right) = -\mathrm{Tr}\left(A_-\right) + \alpha \sum_{ij\in I_-} |a_i|\,\mathrm{Tr}\left(|a_j\rangle\langle a_j|\mathcal{L}[|a_i\rangle\langle a_i|]\right)\\
&\le -\mathrm{Tr}\left(A_-\right) + \alpha \sum_{\substack{ij\in I_-\\ i\neq j}} |a_i|\,\mathrm{Tr}\left(|a_j\rangle\langle a_j|\mathcal{L}[|a_i\rangle\langle a_i|]\right) - \alpha \sum_{i\in I_-}\sum_{j\neq i} |a_i|\,\mathrm{Tr}\left(|a_i\rangle\langle a_i|\mathcal{L}[|a_j\rangle\langle a_j|]\right)\\
&\le -\mathrm{Tr}\left(A_-\right) + \alpha \sum_{i\neq j\in I_-} |a_i|\,\mathrm{Tr}\left(|a_j\rangle\langle a_j|\mathcal{L}[|a_i\rangle\langle a_i|]\right)\,.
\end{aligned}$$

Setting then $L_{ji} := \mathrm{Tr}\left(|a_j\rangle\langle a_j|\mathcal{L}[|a_i\rangle\langle a_i|]\right)$, one has

$$\sum_{i\in I_-} |a_i| \left(1 - \alpha \sum_{j\neq i\in I_1} L_{ji}\right) \le 0\,. \tag{2.13}$$

Thus, it suffices to take

$$\alpha < \left(\max_{i\in I_1} \sum_{j\neq i\in I_1} L_{ji}\right)^{-1}\,,$$

so to have, from (2.13),

$$a_i = 0\,, \forall\, i \in I_- \;\Longrightarrow\; A = A_+ \ge 0\,.$$

This proves that the map $(\mathrm{id}_d - \alpha\mathcal{L})^{-1}$ is positive. Since the composition of PTP maps is PTP, by invoking the formula [31, Theorem 3.1.10]

$$e^{t\mathcal{L}} = \lim_n \left(\mathrm{id}_d - \frac{t}{n}\mathcal{L}\right)^{-n}, \tag{2.14}$$

one gets positivity ad trace-preservation of $e^{t\mathcal{L}}$ by that of $\left(\mathrm{id}_d - \frac{t}{n}\mathcal{L}\right)^{-1}$. □

Assuming that $\mathcal{L}$ is instead the generator of a completely positive map, namely a physically legitimate evolution, one can fully control its structure as proved by Gorini, Kossakowski, Sudarshan [4] for finite-level systems, and by Lindblad [5] for bounded generators on $\mathcal{B}(\mathcal{H})$.

Theorem 2.3 (GKLS). *$\mathcal{L}$ is the generator of a semigroup of completely positive and trace-preserving maps if and only if the Kossakowski matrix $K = [K_{ij}]$ is positive semidefinite.*

Remark 2.1. *Equivalently, in the Heisenberg picture, $\mathcal{L}^\ddagger$ generates a semigroup of CP unital maps if and only if*

$$\mathcal{L}^\ddagger[X] = i[H,\rho] + \phi^\ddagger[\rho] - \frac{1}{2}\{\phi^\ddagger[\mathbb{1}],\rho\} \tag{2.15}$$

for some completely positive map $\phi^\ddagger$. The structure (2.15) *for generators of CP unital maps can be further generalized to semigroups on von Neumann algebras [64, 65]. Note also that, in* (2.15), *if ϕ were only positive, then* (2.15) *would generate a semigroup of positive maps; however, this would provide only a sufficient condition for $\mathcal{L}$ to generate a positive map. Finally, note also that, for semigroups, the generator of the dual evolution corresponds to the dual of the generator, $\left(e^{t\mathcal{L}}\right)^\ddagger = \exp(t\mathcal{L}^\ddagger)$.*

2.1.1.2 Quantum divisible processes

If $\mathcal{L}_t$ carries an explicit time-dependence, the semigroup composition law (2.10) fails to hold. Nonetheless, invertibility ensures that one can uniquely define a two-parameter family of *intertwiners* $\Lambda_{t,s} = \Lambda_t \circ \Lambda_s^{-1}$, so that a two-parameter semigroup law holds,

$$\Lambda_t = \Lambda_{t,s} \circ \Lambda_s . \tag{2.16}$$

The formal solution for $\Lambda_{t,s}$ is then given by the Dyson series,

$$\Lambda_{t,s} = \mathbb{1}_d + \sum_{k=1}^{\infty} \int_s^t \mathrm{d}u_1 \int_s^{u_1} \mathrm{d}u_2 \cdots \int_s^{u_{k-1}} \mathrm{d}u_k\, \mathcal{L}_{u_1} \circ \mathcal{L}_{u_2} \circ \cdots \circ \mathcal{L}_{u_k} . \tag{2.17}$$

Remark 2.2. *Note that, in the Heisenberg picture, propagators defined in the Schrödinger picture through* (2.16) *compose in a time-reversed fashion,*

$$\Lambda_t^\ddagger = \Lambda_s^\ddagger \circ \Lambda_{t,s}^\ddagger , \qquad \Lambda_{t,s}^\ddagger = \Lambda_s^{\ddagger -1} \circ \Lambda_t^\ddagger . \tag{2.18}$$

Nothing prohibits one to define different propagators in the Heisenberg dynamics as

$$\Lambda_t^\ddagger = \widetilde{\Lambda}_{t,s}^\ddagger \circ \Lambda_s^\ddagger , \qquad \widetilde{\Lambda}_{t,s} = \Lambda_t^\ddagger \circ \Lambda_s^{\ddagger -1} .$$

The consequences of such definition and its physical interpretation were recently explored in [66]. Moreover, from the Dyson expansion one realizes that, differently to the case of semigroups, the generator of $\Lambda_t^\ddagger$ is not generally given by $\mathcal{L}_t^\ddagger$. We shall explore in depth this fact in Chapter 3.

As a consequence of the Wigner theorem, Λ_s^{-1} is never a positive map unless the evolution is unitary; accordingly, propagators $\Lambda_{t,s} = \Lambda_t \circ \Lambda_s^{-1}$ are, in general, not even positive. In analogy with one-parameter families of stochastic matrices described in Section 1.3, one calls such dynamics *divisible* and P-divisible if $\Lambda_{t,s}$ are positive maps for all $t \geq s$. On the other hand, in the non-commutative case, one can devise a much richer hierarchy of divisibility degrees of the evolution [67].

Definition 2.1. *The quantum process described by the dynamical map Λ_t is called k-divisible if $\Lambda_{t,s}$ are k-positive and trace-preserving for all $t \geq s \geq 0$. In particular, the evolution is called (C)P-divisible if intertwiners $\Lambda_{t,s}$ are (completely) positive and trace-preserving.*

Remark 2.3. *In absence of more information about the underlying system-environment interaction other than that provided by the dynamical map Λ_t, one cannot do but identify Markovian behaviour with CP-divisibility that, in this sense, represents an intrinsic definition of Markovianity [14]. It is a quantum version of the notion of classical P-divisibility (see Definition 1.8 and Section 1.3.5). However, as we shall see in Section 2.2.1, if the full multi-time statistics of the quantum process is at hand, one can refine the notion of quantum Markovianity by identifying it with the so-called Quantum Regression regime. The latter provides a non-commutative extension to Definition* (4.43).

A partial characterization of generators of P-divisible evolutions can be given by time-dependent Kossakowski conditions, generalizing Proposition 2.2.

Proposition 2.4. *A Hermiticity preserving linear map $\mathcal{L}_t : M_d(\mathbb{C}) \to M_d(\mathbb{C})$ such that* $\mathrm{Tr}\,(\mathcal{L}_t[X]) = 0$ *for all $X \in M_d(\mathbb{C})$, is the generator of a P-divisible evolution if and only if*

$$\mathrm{Tr}\left(P_i \mathcal{L}_t[P_j]\right) \geq 0\,, \tag{2.19}$$

for all choices of MASAs spanned by rank-one projectors $\mathcal{P} = \{P_i\}_{i=1}^d$, $\sum_i P_i = \mathbb{1}_d$.

Proof. The "only if" part follows again from the positivity of $\Lambda_{t,s}$, since for all $t \geq 0$

$$\mathrm{Tr}\left(P_i \mathcal{L}_t[P_j]\right) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \mathrm{Tr}\left(P_i \Lambda_{t+\epsilon,t}[P_j]\right) \geq 0\,, \qquad i \neq j. \tag{2.20}$$

For the "if" part, rewrite $\mathcal{T}_{\leftarrow} e^{\int_s^t \mathrm{d}u \mathcal{L}_u}$ by splitting the interval $[s,t]$ as

$$t \equiv t_N \geq t_{N-1} \geq \ldots \geq t_0 \equiv s\,, \quad \text{with} \quad \lim_N \sup\,(t_k - t_{k-1}) = 0\,,$$

and then invoking the time-splitting formula as done in the proof of Proposition 1.5,

$$\Lambda_{t,s} = \lim_N \prod_{k=N-1}^{0} e^{\delta t_k \mathcal{L}_{t_k}}\,, \qquad \delta t_k \equiv t_{k+1} - t_k\,. \tag{2.21}$$

Since $\mathrm{Tr}\left(P_i \mathcal{L}_{t_k}[P_j]\right) \geq 0$ for all $i \neq j$, $t \mapsto e^{(t-t_k)\mathcal{L}_{t_k}}$ is a positive map for all $t \geq t_k$, by Proposition 2.2. Positivity of $\Lambda_{t,s}$ follows from noting that the composition of positive maps is positive and that the set of positive maps is closed (for finite level systems, the closure is with respect to the norm topology on the space of linear maps of $M_d(\mathbb{C})$ onto itself) [68]. □

Remark 2.4. *It is easy to see that condition* (2.19) *is equivalent to* $\mathrm{Tr}\,(P\mathcal{L}_t[Q]) \geq 0$ *for all projectors P, Q (not necessarily of rank-1) such that $PQ = 0$.*

Note that, since $\Lambda_{t,s=0} = \Lambda_t$, P-divisibility is sufficient for simple positivity, but not for complete positivity of the evolution. General conditions for $\mathcal{L}_t$ to generate a completely positive evolution are not known. On the other hand, the stronger requirement of CP-divisibility (which is sufficient for complete positivity) is fully under control due to a straightforward time-dependent generalization of the GKLS theorem 2.3.

Proposition 2.5. *A generator $\mathcal{L}_t$ in the canonical form* (2.8) *gives rise to a CP-divisible evolution if and only if the time-dependent Kossakowski matrix $K(t)$ is positive semidefinite for all $t \geq 0$.*

2.1.2 Unital evolutions

Unitality of the evolution, namely the preservation of the identity $\Lambda_t[\mathbb{1}_d] = \mathbb{1}_d$, helps in characterizing generators of P-divisible maps. First, let us consider a necessary condition for P-divisibility.

Proposition 2.6. *Let $\Lambda_t : M_d(\mathbb{C}) \to M_d(\mathbb{C})$ be P-divisible and unital with generator as in* (2.8). *Consider a Hilbert-Schmidt basis of Hermitian operators $\{F_\alpha\}_{\alpha=0}^{d^2-1}$ such that*

$$F_0 = \frac{\mathbb{1}}{\sqrt{d}}, \qquad F_\alpha = F_\alpha^\dagger, \quad \mathrm{Tr}\left(F_\alpha F_\beta\right) = \delta_{\alpha\beta}, \qquad \alpha, \beta = 0 \ldots, d^2 - 1 .$$

Then, the $(d^2 - 1) \times (d^2 - 1)$ real symmetric matrix

$$-\widetilde{\mathcal{L}}^{\mathrm{S}}(t) = -\frac{\widetilde{\mathcal{L}}(t) + \widetilde{\mathcal{L}}^{\mathrm{T}}(t)}{2}, \quad \widetilde{\mathcal{L}}_{\alpha\beta}(t) = \mathrm{Tr}\left(F_\alpha \mathcal{L}_t[F_\beta]\right), \quad \alpha, \beta = 1, \ldots, d^2 - 1, \tag{2.22}$$

is positive semidefinite for all $t \geq 0$.

Proof. For the positive, unital intertwiners $\Lambda_{s+\epsilon,s}$, inequality (1.75) yields, for all $s, \epsilon \geq 0$,

$$\Lambda_{s+\epsilon,s}[X^2] \geq (\Lambda_{s+\epsilon,s}[X])^2 \qquad \forall\, X = X^\dagger .$$

Taking $\epsilon \ll 1$ and expanding up to order ϵ one has

$$0 \leq \Lambda_{s+\epsilon,s}[X^2] - (\Lambda_{s+\epsilon,s}[X])^2 = \epsilon(\mathcal{L}_s[X^2] - X\mathcal{L}_s[X] - \mathcal{L}_s[X]X) + O(\epsilon^2),$$

so the generators must satisfy

$$\mathcal{L}_s[X^2] - X\mathcal{L}_s[X] + \mathcal{L}_s[X]X \geq 0, \quad \forall\, X = X^\dagger, \tag{2.23}$$

for all $s \geq 0$. By taking the trace,

$$-\mathrm{Tr}\left(X\mathcal{L}_s[X]\right) = -\mathrm{Tr}\left(X\, \mathcal{L}_s^{\mathrm{sd}}[X]\right) \geq 0, \quad \forall\, X = X^\dagger, \tag{2.24}$$

where we defined the self-dual part of the generator as

$$\mathcal{L}_s^{\mathrm{sd}} := \frac{\mathcal{L}_s + \mathcal{L}_s^\ddagger}{2}, \tag{2.25}$$

$\mathcal{L}_s^\ddagger$ being the dual of $\mathcal{L}_s$. Any hermitian matrix X can be expanded along the orthonormal basis in Remark 1.1 as $X = X^\dagger = x_0 \mathbb{1}_d + \sum_{i=1}^{d^2-1} x_i F_i$, with $x_i \in \mathbb{R}, i = 0, \dots, d^2-1$. Using trace preservation and unitality, one recasts (2.24) as

$$\begin{aligned} -\operatorname{Tr}\left(X\mathcal{L}_s^{\mathrm{sd}}[X]\right) &= \sum_{i,j=1}^{d^2-1} x_i\, x_j \operatorname{Tr}\left(F_i \left(-\frac{\mathcal{L}_s + \mathcal{L}_s^\ddagger}{2}\right)[F_j]\right) \\ &= \left\langle \boldsymbol{x} \middle| -\frac{\widetilde{\mathcal{L}}(s) + \widetilde{\mathcal{L}}^{\mathrm{T}}(s)}{2} \middle| \boldsymbol{x} \right\rangle = \left\langle \boldsymbol{x} \middle| -\widetilde{\mathcal{L}}^{\mathrm{S}}(s) \middle| \boldsymbol{x} \right\rangle \geq 0\,. \end{aligned} \tag{2.26}$$

for all real vectors $\boldsymbol{x} \in \mathbb{R}^{d^2-1}$. Then (2.26) implies that the real symmetric matrix $\widetilde{\mathcal{L}}^{\mathrm{S}}(s)$ is positive semidefinite. □

Example 2.1. *For unital qubit evolutions, P-divisibility can be fully controlled in terms of the generator. We can decompose the time-dependent density matrix along the Pauli matrices,*

$$\Lambda_t[\rho] = \frac{\mathbb{1}_2 + \boldsymbol{r}_t \cdot \boldsymbol{\sigma}}{2}\,, \qquad \boldsymbol{r}_t \in \mathbb{R}^3\,,$$

where the Bloch vector $\boldsymbol{r}_t$ evolves according to a linear transformation $\widetilde{\Lambda}(t) \in M_3(\mathbb{R})$:

$$\boldsymbol{r}_t = \widetilde{\Lambda}(t)\,\boldsymbol{r}\,, \qquad \widetilde{\Lambda}_{ij}(t) := \frac{1}{2}\operatorname{Tr}\left(\sigma_i \Lambda_t[\sigma_j]\right)\,, \qquad i,j = 1,2,3\,. \tag{2.27}$$

Accordingly, the time-local master equation (2.7) *transforms into the matrix equation*

$$\dot{\widetilde{\Lambda}}(t) = \widetilde{\mathcal{L}}(t)\,\widetilde{\Lambda}(t)\,, \qquad \widetilde{\Lambda}(0) = \mathbb{1}\,. \tag{2.28}$$

whose formal solution is given by $\widetilde{\Lambda}(t) = \mathcal{T}_{\leftarrow} e^{\int_0^t \mathrm{d}s \widetilde{\mathcal{L}}(s)}$. Necessary and sufficient conditions for P-divisibility are given by (2.19). *Recalling that in $d = 2$, MASAs are spanned any rank-1 projectors and its orthogonal complement, $\mathbb{1}_2 - P$, P-divisibility is equivalent to*

$$\begin{aligned} 0 \leq \operatorname{Tr}\left((\mathbb{1}_2 - P)\mathcal{L}_t[P]\right) = -\operatorname{Tr}\left(P\mathcal{L}_t[P]\right) &= -\frac{1}{4}\sum_{ij} r_i\, r_j \operatorname{Tr}\left(\sigma_i \mathcal{L}_t[\sigma_j]\right) \\ &= -\frac{1}{4}\left\langle \boldsymbol{r} \middle| \widetilde{\mathcal{L}}(t) \middle| \boldsymbol{r} \right\rangle = \frac{1}{4}\left\langle \boldsymbol{r} \middle| -\frac{\widetilde{\mathcal{L}}(t) + \widetilde{\mathcal{L}}^{\mathrm{T}}(t)}{2} \middle| \boldsymbol{r} \right\rangle\,, \end{aligned} \tag{2.29}$$

Thus, for one qubit, condition

$$M_3(\mathbb{R}) \ni -\widetilde{\mathcal{L}}^{\mathrm{S}}(t) = -\frac{\widetilde{\mathcal{L}}(t) + \widetilde{\mathcal{L}}^{\mathrm{T}}(t)}{2} \geq 0\,, \tag{2.30}$$

is not only necessary, but also sufficient for P-divisibility. The simplest case is that of Pauli generators,

$$\mathcal{L}_t[\rho] = \frac{\eta}{2}\sum_{k=1}^{3} \gamma_t^{(k)}\,\left(\sigma_k\,\rho\,\sigma_k - \rho\right)\,, \tag{2.31}$$

with $\eta > 0$ *and* σ_k, $k = 1, 2, 3$ *the Pauli matrices. The generator is of the form* (2.8) *with no Hamiltonian contribution,* $d = 2$, $F_k = \sigma_k/\sqrt{2}$ *and diagonal, time-dependent Kossakowski matrix given by the so-called rates* $\gamma_t^{(k)}$. *The generators are diagonal along the Pauli basis at all times,*

$$\mathcal{L}_t[\sigma_i] = -\Gamma_t^{(i)}\sigma_i\,, \qquad \Gamma_t^{(i)} := \eta\left(\gamma_t^{(j)} + \gamma_t^{(k)}\right)\,, \quad j \neq k\,,\ j,k \neq i\,. \tag{2.32}$$

In particular, $\mathcal{L}_t$ *commute at different times. Then, time ordering drops and the dynamical map is given by simple exponentiation:* $\Lambda_t = e^{\int_0^t \mathrm{d}s \mathcal{L}_s}$. *Its Pauli spectrum reads, accordingly,*

$$\Lambda_t[\sigma_i] = \lambda_t^{(i)}\sigma_i\,, \qquad \lambda_t^{(i)} = e^{-\int_0^t \mathrm{d}s\, \Gamma_s^{(i)}}\,. \tag{2.33}$$

Note that $\lambda_t^{(i)} > 0$ *so that continuous-time Pauli evolutions are always invertible. The matrix representations of the map and generator are diagonal,*

$$\widetilde{\mathcal{L}}_{ij}(t) = -\Gamma_t^{(i)}\,\delta_{ij}\,, \qquad \widetilde{\Lambda}_{ij}(t) = \lambda_t^{(i)}\,\delta_{ij}\,, \tag{2.34}$$

and necessary and sufficient conditions for P-divisibility follow from (2.30) *[69]*

$$\Gamma_t^{(i)} \geq 0\,, \qquad i = 1, 2, 3\,. \tag{2.35}$$

Evidently, $\gamma_t^{(k)} \geq 0$, $k = 1, 2, 3$ *is necessary and sufficient for* Λ_t *to be CP-divisible. This condition is recast in terms of the spectrum of the generator in the well-known form*

$$\Gamma_i(t) \leq \frac{1}{2}\sum_{j=1}^{3}\Gamma_j(t)\,, \qquad i = 1, 2, 3\,. \tag{2.36}$$

In the course of the previous example, in particular, we showed that for unital qubit dynamics, P-divisibility is fully controlled by the matrix $\widetilde{\mathcal{L}}^{\mathrm{S}}(t)$.

Proposition 2.7. *Let* $\mathcal{L}_t : M_2(\mathbb{C}) \to M_2(\mathbb{C})$, $\mathcal{L}_t[\mathbb{1}_2] = 0$ *be the generator of a unital dynamics* Λ_t. *Then,* Λ_t *is P-divisible if and only if* $-\widetilde{\mathcal{L}}^{\mathrm{S}}(t) \geq 0$ *for all* $t \geq 0$.

2.1.3 Backflow of information

Classical P-divisibility has been given an operational interpretation in Proposition 1.6; namely, P-divisible evolutions contract the Kolmogorov distance between probability vectors. The non-commutative counterpart is as follows [26].

Proposition 2.8. *Let* Λ_t *be a P-divisible. Then,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda_t[X]\|_1 \leq 0\,, \qquad t \geq 0\,, \tag{2.37}$$

for all $X = X^\dagger \in M_d(\mathbb{C})$. *Similarly, if* Λ_t *is CP-divisible,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda_t \otimes \mathrm{id}_d[X]\|_1 \leq 0\,, \qquad t \geq 0\,, \tag{2.38}$$

for all $X = X^\dagger \in M_d(\mathbb{C}) \otimes M_d(\mathbb{C})$. *Moreover, if* Λ_t *is invertible,* (2.37) *and* (2.38) *are sufficient for P and CP-divisibility, respectively.*

Remark 2.5. *If Λ_t is not invertible, condition* (2.37) *does not generally imply P-divisibility of the evolution (a counterexample has been provided in [70]). On the other hand, invertibility condition can be substantially relaxed in order for* (2.38) *to imply CP-divisibility [71] and, in particular,* (2.38) *does always imply CP-divisibility for $d = 2$ [72].*

Note that every Hermitian matrix $X = X^\dagger$ can be written as

$$X = \lambda\, \Delta_\mu(\rho_S, \sigma_S), \qquad \lambda > 0\,, \tag{2.39}$$

where $\Delta_\mu(\rho_S, \sigma_S)$ is the so-called Helstrom matrix,

$$\Delta_\mu(\rho_S, \sigma_S) := \mu\rho_S - (1-\mu)\sigma_S\,, \quad \text{where} \quad 0 \le \mu \le 1, \tag{2.40}$$

and where ρ_S, σ_S are system density matrices. The physical interpretation of Proposition 2.8 then results in the so-called Breuer-Laine-Piilo approach (BLP) introduced in [24] (see also [12, 18, 73]), that we now review. One identifies the amount of information initially contained in the open system with the Helstrom distinguishability of a pair of density matrices, namely the quantity

$$D_\mu(\rho_S, \sigma_S) = \left\| \Delta_\mu(\rho_S, \sigma_S) \right\|_1 =: \mathcal{I}^q_{t=0}(\rho_S, \sigma_S; \mu)\,, \tag{2.41}$$

representing the bias in probability of distinguishing two density matrices ρ and σ in a one-shot experiment [50]. In this scheme, μ, $1-\mu$ are the a-priori probabilities in picking one state from the Helstrom ensemble:

$$\mathcal{E}_H = \{(\mu, \rho_S); (1-\mu, \sigma_S)\}\,. \tag{2.42}$$

For $\mu = 1/2$ one recovers the trace distance $D(\rho_S, \sigma_S) := D_{1/2}(\rho_S, \sigma_S) = \left\|\rho_S - \sigma_S\right\|_1/2$. The dynamics Λ_t will evolve (2.41) into the distance of time-evolving states

$$\mathcal{I}^q_t(\rho_S, \sigma_S; \mu) = \left\| \Lambda_t[\Delta_\mu(\rho_S, \sigma_S)] \right\|_1\,. \tag{2.43}$$

Therefore, (2.37) represents the monotonically decreasing distinguishability between any pair of states and is thus interpreted as an outward flow of information from the open system towards the environment. Conversely, an increase of the internal information, namely a *revival* of the two-state distinguishability, is interpreted as *backflow of information (BFI)* from the environment to the open system.

Remark 2.6. *The BFI interpretation, which involves the degrees of freedom of the environment, is also supported by the following observations [12, 74, 75].*

1. *Assume that the environment is described by a Hilbert space $\mathcal{H}_E$, and its state can be taken as a density operator ρ_E. Let then $\rho_{SE}(t) = \mathcal{U}^{SE}_t[\rho_S \otimes \rho_E]$, where $\mathcal{U}^{SE}_t$ describes the joint system-environment unitary evolution, so that the distinguishability of two system-environment states is constant and one defines accordingly an "external information" relative to two states ρ_S and σ_S of a quantum system S as*

$$\begin{aligned}\mathcal{E}^q_t(\rho_S, \sigma_S; \mu) &= \left\| \mathcal{U}^{SE}_t[\Delta_\mu(\rho_S, \sigma_S) \otimes \rho_E] \right\|_1 - \mathcal{I}^q_t(\rho_S, \sigma_S; \mu) \\ &= \left\| \Delta_\mu(\rho_S, \sigma_S) \otimes \rho_E \right\|_1 - \mathcal{I}^q_t(\rho_S, \sigma_S; \mu) = \mathcal{I}^q_0 - \mathcal{I}^q_t(\rho_S, \sigma_S; \mu) \ge 0\,. \end{aligned} \tag{2.44}$$

where, in the first equality, the unitary invariance of trace-norm was used. The time derivative on both sides of (2.44) *yields*

$$\dot{\mathcal{E}}_t^q(\rho_S, \sigma_S; \mu) = -\dot{\mathcal{I}}_t^q(\rho_S, \sigma_S; \mu)\,. \tag{2.45}$$

Hence, to a decrease (increase) of internal information, corresponds an increase (decrease) of the external one.

2. *The difference of the internal information at different times,*

$$\Delta\mathcal{I}_{t,s}^q(\rho_S, \sigma_S; \mu) := \mathcal{I}_t^q(\rho_S, \sigma_S; \mu) - \mathcal{I}_s^q(\rho_S, \sigma_S; \mu)\,, \qquad t \geq s \geq 0\,, \tag{2.46}$$

can be upper-bounded by means of trace distances involving the marginal density matrices $\rho_S(t) = \mathrm{Tr}_E\,\rho_{SE}(t)$, $\rho_E(t) = \mathrm{Tr}_S\,\rho_{SE}(t)$. *Namely,*

$$\begin{aligned}\Delta\mathcal{I}_{t,s}^q(\rho_S, \sigma_S; \mu) \;\leq\; & 2\mu\, D(\rho_{SE}(s), \rho_S(s)\otimes\rho_E(s)) + 2(1-\mu)D(\sigma_{SE}(s), \sigma_S(s)\otimes\sigma_E(s)) \\ & + 2\min\{\mu, 1-\mu\}D(\rho_E(s), \sigma_E(s))\,. \end{aligned}\tag{2.47}$$

Occurrence of BFI means that the l.h.s. must be strictly positive for some $s > 0$*; accordingly, at least one of the terms on the r.h.s. has to be strictly positive. Thus, at the time* s *when the revival occurs, information had to be stored in (1) either the system-environment correlations or (2) in the changes of the reduced state of the environment. These are sometimes referred to as "precursors of non-Markovianity" [76]. This argument supports the general interpretation that, in order for BFI to occur, information can be stored temporarily in either the environment reduced state or in the system-environment correlations and then released back to the open system at later times. Note these are just necessary conditions inferred from from the reduced dynamics. We shall argue in Chapters 4 and 5 that a more detailed description of the microscopic mechanisms underlying backflow of information cannot be achieved through the dynamical map only.*

2.1.3.1 Contractivity for unital maps

P-divisibility regulates the monotonic behaviour in time of several other distances or divergences, such as the quantum relative entropy [77] or regularized entropic divergences [78, 79]. Note that, if Λ_t is a CPTP, P-divisible unital evolution it will contract the matrix norm by Proposition (1.8). The latter is a special special case of a more general contractivity property involving the so-called Schatten p-norms,

$$\|X\|_p = \mathrm{Tr}\left(|X|^p\right)^{\frac{1}{p}}\,, \quad p \in [1, \infty)\,. \tag{2.48}$$

For $p = 1$ one retrieves the trace norm, for $p = 2$ the Hilbert-Schmidt norm and for $p \to \infty$ to the operator norm $\|X\|_\infty \equiv \|X\|$. One can then embody the Banach space of linear maps $\Lambda : M_d(\mathbb{C}) \to M_d(\mathbb{C})$ with the induced p-norms

$$\|\Lambda\|_{p-p} := \sup_{\|X\|_p = 1} \|\Lambda[X]\|_p\,. \tag{2.49}$$

Then, consider the following non-commutative Riesz-Thorin interpolation [80],

$$\|\Lambda\|_{p-p} \leq \|\Lambda\|_{1-1}\|\Lambda\|_{\infty-\infty}\,. \tag{2.50}$$

As a consequence of Proposition 1.8 and Remark 1.13, for positive, unital and trace-preserving maps $\|\Lambda\|_{1-1} = \|\Lambda\|_{\infty-\infty} = 1$; hence they also contract the p-norms, $p \in [1, \infty]$ [81, 82]. In the context of dynamical maps, one has the following

Proposition 2.9. *If Λ_t is P-divisible and unital, then for $p \in [1, \infty]$,*

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\Lambda_t[X]\|_p \leq 0\,, \quad \forall\, X \in M_d(\mathbb{C})\,, \tag{2.51}$$

and all $t \geq 0$.

Proof. The intertwiners $\Lambda_{t,s}$ are positive, trace preserving and unital maps for all $t \geq s \geq 0$, Therefore, (2.50) yields $\left\|\Lambda_{t,s}\right\|_{p-p} \leq \left\|\Lambda_{t,s}\right\|_{1-1} \left\|\Lambda_{t,s}\right\|_{\infty-\infty} = 1$, so that $\|\Lambda_t[X]\|_p = \left\|\Lambda_{t,s} \circ \Lambda_s[X]\right\|_p \leq \left\|\Lambda_{t,s}\right\|_{p-p} \|\Lambda_s[X]\|_p = \|\Lambda_s[X]\|_p$. □

Out of the Schatten p-norms one defines the quantum Rényi p-entropies

$$S_p(\rho) = \frac{1}{1-p} \log\left(\mathrm{Tr}\left(\rho^p\right)\right) = \frac{p}{1-p} \log\left(\|\rho\|_p\right), \qquad p \in (1, \infty)\,. \tag{2.52}$$

Then, we have the following [77, 83, 84]

Proposition 2.10. *For a P-divisible, unital map*

$$\frac{\mathrm{d}}{\mathrm{d}t} S_p(\Lambda_t[\rho]) \geq 0\,, \quad p \in (1, \infty). \tag{2.53}$$

In particular, the limit $p \to 1^+$ yields the monotonicity of the von Neumann entropy under a P-divisible, unital evolution.

Proof. If Λ_t is a P-divisible, unital map, then

$$\begin{aligned} S_p(\Lambda_{t,s} \circ \Lambda_s[\rho]) &= \frac{p}{1-p} \log\left(\left\|\Lambda_{t,s} \circ \Lambda_s[\rho]\right\|_p\right) \geq \frac{p}{1-p}\left[\log\left(\left\|\Lambda_{t,s}\right\|_{p-p}\right) + \log\left(\left\|\Lambda_s[\rho]\right\|_p\right)\right] \\ &\geq \frac{p}{1-p} \log\left(\left\|\Lambda_s[\rho]\right\|_p\right) = S_p(\Lambda_s[\rho])\,, \end{aligned}$$

for all $t \geq s \geq 0$, as a consequence of Proposition (2.9). □

Remark 2.7. *Among the many quantifiers that can be used to monitor information flow, one can consider the quantum mutual information between the system S and some inert ancilla A to witness lack of CP-divisibility [85]. Consider now A to be a purifying ancilla of S and evaluate the quantum mutual information*

$$\begin{aligned} \mathcal{I}(\Lambda_t \otimes \mathrm{id}_d[|\sqrt{\rho}\rangle\langle\sqrt{\rho}|]) :&= S(\Lambda_t \otimes \mathrm{id}_\mathrm{d}[|\sqrt{\rho}\rangle\langle\sqrt{\rho}|] \| \Lambda_t \otimes \mathrm{id}_d[\rho \otimes \rho]) = \\ &= S(\rho) + S(\Lambda_t[\rho]) - S\left(\Lambda_t \otimes \mathrm{id}_d[|\sqrt{\rho}\rangle\langle\sqrt{\rho}|]\right) = S(\rho) + S(\Lambda_t[\rho]) - S\left(\rho[\mathcal{X}_t]\right), \end{aligned}$$

with the notation of Remark 1.18. Evidently, such quantity is monotonically decreasing for CP-divisible evolutions. The difference $S(\Lambda_t[\rho]) - S(\rho[\mathcal{X}_t])$ appears in the literature as coherent information *of ρ through Λ_t and is used to express the quantum data processing inequality [50]. Such concept was recently rediscussed in the context of multi-time non-Markovianity in [86].*

2.1.4 Collisional models

As introduced in Section 2.1.3, memory effects are often defined and witnessed in terms of the system reduced dynamics. Nevertheless, in order to better understand their physical roots, the dynamical map is hardly a sufficient tool on its own. Practically speaking, one needs access to at least partial information about either the system–environment interaction or, as we shall see in the following Section, about the multi-time statistics of the underlying quantum stochastic process. On the other hand, using standard derivation techniques, it is far from trivial to envisage a microscopic model that (1) allows one to interpret memory effects emerging in the reduced dynamics through a few meaningful physical parameters of the bath or of the system–environment interaction, and that (2) microscopically realizes certain mathematical properties that naturally arise at the level of the dynamical map.

A very convenient way-out is offered by the so-called *collisional approach* to open quantum systems [87, 88]. It will come useful to treat collisional models in a fully algebraic fashion as discrete-time dynamical system $(\mathcal{A}_S \otimes \mathcal{A}_S, \Theta, \omega_S \otimes \omega_E)$ given by

i. a finite-level system with observables in $\mathcal{A}_S = M_d(\mathbb{C})$ endowed with a normal state $\omega_S(\,\cdot\,) = \mathrm{Tr}\,(\rho_S\,\cdot\,)$;

ii. a spin-chain environment E described by the quasi-local algebra of observables $\mathcal{A}_E$ obtained as an inductive sequence of local algebras $\mathcal{A}_E^{[-a,b]} = \bigotimes_{k=-a}^{b} \mathcal{A}_E^{(k)}$, constructed from the same single-site matrix algebra $\mathcal{A}_E^{(k)} = M_D(\mathbb{C})$ as in Section 1.1.2. The environment state is given by a locally normal state ω_E specified by a consistent set of density matrices

$$\omega_E\left(A^{[-a,b]}\right) = \mathrm{Tr}\left(\rho_E^{[-a,b]} A^{[a,b]}\right), \qquad A^{[a,b]} \in \mathcal{A}^{[-a,b]} \tag{2.54}$$

that we assume to be shift-invariant,

$$\mathrm{Tr}_{b+1}\left(\rho_E^{[-a,b+1]}\right) = \mathrm{Tr}_{-a}\left(\rho_E^{[-a,b+1]}\right) = \rho_E^{[-a,b]}\,.$$

iii. the system-environment interaction described by an automorphism $\Phi : \mathcal{A}_S \otimes \mathcal{A}_E^{(0)}$ acting non-trivially only on the system S and on a fixed site of the chain, that we conventionally identify with the (0)-th one. The action of Φ is then lifted to the full algebra $\mathcal{A}_S \otimes \mathcal{A}_E$ by extending its action to local observables $\mathcal{A}_E^{[-a,b]} \ni A^{[-a,b]} = \sum_{\boldsymbol{i}_{[-a,b]}} a_{\boldsymbol{i}_{[-a,b]}} A_{\boldsymbol{i}_{[-a,b]}}^{[-a,b]}$ where $A_{\boldsymbol{i}_{[-a,b]}}^{[-a,b]} = \bigotimes_{k=-a}^{b} A_{i_k}^{(k)}$ in the following way

$$\Phi\left[X_S \otimes A_{i_{-a}}^{(-a)} \otimes \ldots \otimes A_{i_0}^{(0)} \otimes \ldots \otimes A_{i_b}^{(b)}\right] = A_{i_{-a}}^{(-a)} \otimes \ldots \otimes \Phi\left[X_S \otimes A_{i_0}^{(0)}\right] \otimes \ldots \otimes A_{i_b}^{(b)}\,.$$

iv. the collisional dynamics is finally described by a discrete, one-parameter group of automorphisms

$$\Theta_n = \Theta^n \quad \text{with} \quad \Theta = (\mathrm{id}_S \otimes \sigma) \circ \Phi\,, \tag{2.55}$$

where σ is the right shift (1.28) on $\mathcal{A}_E$.

In Chapter 4, we shall also consider a slight modification of the above scheme, whereby the coupling automorphism Φ will be replaced by a CPU map, so as to describe a dissipative coupling between $\mathcal{A}_S$ and $\mathcal{A}_E^{(0)}$. In summary, the algebraic setting just presented accommodates a

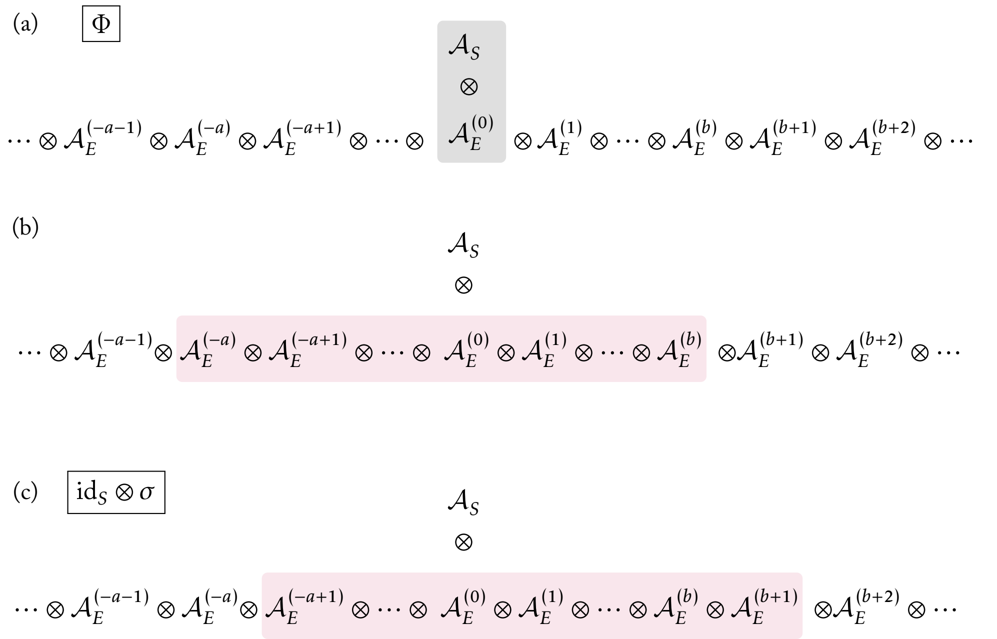

FIGURE 2.1: One step Θ of the algebraic collisional dynamics: (a) Φ acts on the algebra of the system $\mathcal{A}_S$ and on the 0-th site of the chain. (b) An operator $A^{[-a,b]}$, localized in $\mathcal{A}_E^{[-a,b]}$, is then translated to the right in (c) by the shift automorphism σ.

collisional model with a correlated multi-partite environment [89–91], where the system and the chain ancilla at site $k = 0$ may either interact reversibly or be instantaneously immersed in the same dissipative environment before the shift is applied. The algebraic framework for the collisional model described above is illustrated in Figure 2.1. The maps Θ_n in (2.55) give the dynamics of operators in the Heisenberg picture. In the Schrödinger picture, an initial state $\omega_S \otimes \omega_E$ on $\mathcal{A}_S \otimes \mathcal{A}_E$ evolves at discrete time n into

$$\omega_{SE}^{(n)} = \omega_S \otimes \omega_E \circ \Theta_n \ , \tag{2.56}$$

which is itself a locally normal state, whose restrictions to local algebras $\mathcal{A}_E^{[-a,b]}$ yield density matrices $\Omega_{S[-a,b]}^{(n)}$ such that

$$\omega_{SE}^{(n)}\left(X_S \otimes A_E^{[-a,b]}\right) = \mathrm{Tr}\left(\Omega_{S[-a,b]}^{(n)}\, X_S \otimes A_E^{[-a,b]}\right) . \tag{2.57}$$

By setting $X_S = \mathbb{1}_d$, one obtains the reduced state of the environment after n iterations as

$$\Omega_{[-a,b]}^{(n)} = \mathrm{Tr}_S\left(\Omega_{S[-a,b]}^{(n)}\right) , \tag{2.58}$$

On the other hand, the marginal obtained by restricting to the algebra $\mathcal{A}_S \otimes \mathbb{1}_E$ reads

$$\mathrm{Tr}\left(\Omega_S^{(n)} X_S\right) = \omega_S\left(\omega_E \circ \Theta_n\left(X_S \otimes \mathbb{1}_E\right)\right) = \omega_S(\Lambda_n^\ddagger[X_S]) . \tag{2.59}$$

Hence, in the collisional context, the system's reduced dynamics in the Schrödinger picture is described by a discrete, one-parameter family of CPTP maps $\{\Lambda_n\}_{n\in\mathbb{N}}$ obtained by duality, $\mathrm{Tr}\,(\Lambda_n[\rho_S]X_S) = \mathrm{Tr}\left(\rho_S\Lambda_n^\ddagger[X_S]\right)$. Proposition 2.1 ensures complete positivity of Λ_n either if the interaction Φ is taken as an automorphism or a general CPU map, due to the assumed factorization of the initial state $\omega_S \otimes \omega_E$.

Quantum non-Markovianity and experiments

In this Section, we briefly overview recent experimental applications concerning non-Markovian evolutions. Here, we do not aim to be complete; rather, we give a hint on how recent advancements in applications drive the need for a better understanding of quantum non-Markovianity and of the intricate hierarchy of concepts underlying it. On the other hand, the use of new quantum devices might also help to understand complex open dynamics and provide a platform to simulate the results of this thesis, that will be discussed in Chapters 3-5. Excellent reviews concerning the role of non-Markovianity in applications can be found in [11, 19, 92]. In the realm of quantum information theory, quantum non-Markovianity could be advantageous in certain specific tasks, essentially due to backflow of information (see for example [8, 93, 94]). Understanding non-Markovian dissipation is also of central importance due to the advancements on quantum platforms and real hardware. In particular, the role of non-Markovian noise in near-term quantum devices, especially in relation to error correction and recoverability, appears to become a promising research direction, still largely unexplored [95, 96]. Many works have been devoted to simulating open systems in the non-Markovian regime. In particular, in [97], weakly non-Markovian evolutions, a jargon for P-divisible but not CP-divisible dynamics, were experimentally realized through a fully optical setup. Such platforms could be an optimal playground for the maps introduced in the following Chapters, especially in Chapter 4, as well as for the simulation of collisional models [91].

2.2 Symbolic dynamics of Open Quantum Systems

The fact that the reduced dynamics of the open system alone is insufficient to describe the mechanisms underlying memory effects, such as backflow of information, should not come as a surprise: the dynamical map encodes only the one-time marginal of a richer stochastic process, in full analogy with classical one-time probability marginals evolved by a family of stochastic matrices, as discussed in Section 1.3.

The proper quantum extension of non-Markovian stochastic processes, including the multi-time statistics, is rooted in a long history dating back to pioneering works [1, 3, 7] and still much debated in recent years. In particular, including the multi-time statistics in the description involves repeated quantum measurements that interfere with the dynamics and generally change the quantum state [12, 27, 98]. Among the most recent proposals concerning such operational approach, particularly prominent ones include the so-called process tensor formalism [15, 99, 100], quantum channels with memory [101, 102] and quantum combs (see [98, 103] and references therein), conditional past-future independence [104, 105] and temporal entanglement (see the recent work [106]). In most of them, a shared idea is to map multi-time correlations into spatial ones encoded in a quantum many-body state. For an overview of these recent developments, see also [92].

With a similar aim, in this Section we shall obtain a symbolic description of open dynamical systems, including multi-time correlations, by exploiting the machinery developed in Section 1.5. This approach naturally leads to a non-commutative extension of the classical characterization of Markovian processes, as we shall see in Proposition 2.11.

Remark 2.8. *If one wants to generalize the construction of the coarse grained density matrix* (1.115) *beyond automorphisms, namely to replace* Θ *by a generic CPU map* Λ *on* $\mathcal{A}$*, the main problem is that the family of operators* $\{\Lambda[X_a]\}_{a=1}^{|\mathcal{X}|}$ *does not form an OPU; indeed, from the Schwartz inequality* (1.85),

$$\sum_{a=1}^{|\mathcal{X}|} \Lambda[X_a]^\dagger \Lambda[X_a] \leq \sum_{a=1}^{|\mathcal{X}|} \Lambda[X_a^\dagger X_a] = \mathbb{1} \,. \tag{2.60}$$

Nevertheless, consider the nested form (1.117) *of the coarse grained density matrix and replace the automorphism* Θ *with a CPU map* Λ*; then,*

$$\rho\left[\mathcal{X}^{(n)}\right]_{\boldsymbol{a},\boldsymbol{b}} = \omega\left(X_{b_0}^\dagger \Lambda\Big[X_{b_1}^\dagger \Lambda\Big[X_{b_2}^\dagger \cdots \Lambda\big[X_{b_{n-1}}^\dagger X_{a_{n-1}}\big] \cdots X_{a_2}\Big] X_{a_1}\Big] X_{a_0}\right) , \quad n \geq 1 \,, \tag{2.61}$$

yield a compatible family of legitimate density matrices, $\mathrm{Tr}_{n-1}\left(\rho[\mathcal{X}^{(n)}]\right) = \rho[\mathcal{X}^{(n-1)}]$*. We shall reprise* (2.61) *in Section 2.2.1, where it will emerge as the coarse-grained density matrix of an open system undergoing a Markovian dynamics, in the sense specified by the so-called Quantum Regression formula. The latter will provide a natural non-commutative extension of the Markov condition* (1.45).

The environmental degrees of freedom are typically neither accessible nor under control. Hence, information can be only gathered from the open system. In the context of the quantum symbolic dynamics introduced in Section 1.5, experimental inaccessibility of the environment translates into identifying physically admissible OPUs with those of the open quantum system:

$$\mathcal{X} \subseteq \mathcal{B} \,, \qquad \mathcal{B} \equiv \mathcal{A}_S = M_d(\mathbb{C}) \,. \tag{2.62}$$

Remark 2.9. *Differently from many applications of the ALF-entropy to quantum chaotic systems, the reference subalgebra* $\mathcal{B} = \mathcal{A}_S$ *is neither dense nor* Θ*-invariant in* $\mathcal{A}_S \otimes \mathcal{A}_E$ *(rather, we can think of it as a subalgebra of some* Θ*-invariant subalgebra* $\mathcal{B}_0 \subseteq \mathcal{A}_S \otimes \mathcal{A}_E$*).*

Let us slightly generalize the construction of Section 1.5 to allow for time-inhomogeneous families of automorphisms $\{\Theta_n\}_n$ more general than groups, namely discrete families admitting a two-parameter composition law

$$\Theta_n = \Theta_m \circ \Theta_{m,n} \,, \qquad \forall\, n \geq m \,, \tag{2.63}$$

Propagators can be defined as $\Theta_{m,n} = \Theta_m^{-1} \circ \Theta_n$, since Θ_m are invertible maps and their inverse are automorphisms. Indeed, for arbitrary A, B, by setting $A' = \Theta_m^{-1}(A)$ and $B' = \Theta_m^{-1}(B)$,

$$\Theta_m^{-1}(AB) = \Theta_m^{-1}(\Theta_m(A')\Theta_m(B')) = \Theta_m^{-1} \circ \Theta_m(A'B') = A'B' = \Theta_m^{-1}(A)\Theta_m^{-1}(B) \,.$$

Then, time-refinements of the OPU $\mathcal{X}$ are naturally introduced, keeping the same notation of Section 1.5, as

$$\mathcal{X}^{(n)} \ni X_{\boldsymbol{a}}^{(n)} = \Theta_{n-1}\left(X_{a_{n-1}}\right) \Theta_{n-2}\left(X_{a_{n-2}}\right) \dots \Theta_2\left(X_{a_2}\right) \Theta_1\left(X_{a_1}\right) X_{a_0} \,. \tag{2.64}$$

The coarse grained density matrix is defined, accordingly, from time-refinements of OPUs in $\mathcal{A}_S$ as

$$\rho_S\left[\mathcal{X}^{(n)}\right]_{\boldsymbol{a},\boldsymbol{b}} = \omega_S \otimes \omega_E\Big(X_{\boldsymbol{b}}^{(n)\dagger} X_{\boldsymbol{a}}^{(n)}\Big) = \omega_S \otimes \omega_E\Big(X_{b_0}^\dagger \otimes \mathbb{1}_E\, \Theta_1(X_{b_1}^\dagger \otimes \mathbb{1}_E) \ldots \tag{2.65}$$
$$\ldots \Theta_{n-2}(X_{b_{n-2}}^\dagger \otimes \mathbb{1}_E)\Theta_{n-1}(X_{b_{n-1}}^\dagger X_{a_{n-1}} \otimes \mathbb{1}_E)\Theta_{n-2}(X_{a_{n-2}} \otimes \mathbb{1}_E) \ldots \Theta_1(X_{a_1} \otimes \mathbb{1}_E)X_{a_0} \otimes \mathbb{1}_E\Big).$$

Using (2.63), we can further recast (2.65) in the nested form

$$\rho_S\left[\mathcal{X}^{(n)}\right]_{\boldsymbol{a},\boldsymbol{b}} = \omega_S \otimes \omega_E\Big(X_{b_0}^\dagger \otimes \mathbb{1}_E\, \Theta_1\Big(X_{b_1}^\dagger \otimes \mathbb{1}_E\, \ldots \tag{2.66}$$
$$\ldots \Theta_{n-2,n-3}\Big(X_{b_{n-2}}^\dagger \otimes \mathbb{1}_E\, \Theta_{n-1,n-2}(X_{b_{n-1}}^\dagger X_{a_{n-1}} \otimes \mathbb{1}_E)\, X_{a_{n-2}} \otimes \mathbb{1}_E\Big) \ldots X_{a_1} \otimes \mathbb{1}_E\Big) X_{a_0} \otimes \mathbb{1}_E\Big).$$

The label S in the coarse-grained density matrix $\rho_S[\mathcal{X}^{(n)}]$ stresses that we are only measuring the finite-level open system S. The dynamical entropies defined through the entropy rates of these coarse-grained density matrices will be the main object of investigation of Chapter 5.

2.2.1 Quantum Regression-Markovianity

By having access to the multi-time statistics of the system plus environment dynamics, one can substantially refine the definition of Markovian behaviour by identifying it with the so-called Quantum Regression condition (QR) [107], that we now review (see also [22, 23]). QR is a statement about multi-time correlations when measuring observables of the open system only. Let then $\{A_k\}_{k=0}^{n-1} \in \mathcal{A}_S$, $n \geq 1$ and define

$$\widetilde{A}^{(n)} := \Theta_{n-1}(A_{n-1} \otimes \mathbb{1}_E) \ldots \Theta_1(A_1 \otimes \mathbb{1}_E)A_0 \otimes \mathbb{1}_E \;\in\; \mathcal{A}_S \otimes \mathcal{A}_E\,, \tag{2.67}$$

and evaluate the multi-time correlation functions

$$\omega_S \otimes \omega_E\left(\widetilde{B}^{(n)\dagger}\, \widetilde{A}^{(n)}\right) = \tag{2.68}$$
$$= \omega_S \otimes \omega_E\Big(B_0^\dagger \otimes \mathbb{1}_E\, \Theta_1\left(B_1^\dagger \otimes \mathbb{1}_E\right) \ldots \Theta_{n-1}\left(B_{n-1}^\dagger A_{n-1} \otimes \mathbb{1}_E\right) \ldots \Theta_1\left(A_1 \otimes \mathbb{1}_E\right) A_0 \otimes \mathbb{1}_E\Big).$$

Definition 2.2. *Consider the triple* $(\mathcal{A}_S \otimes \mathcal{A}_E, \widetilde{\omega}_S \otimes \omega_E, \Theta)$, *with* $\widetilde{\omega}_S$ *being described by a generic density matrix of the system. The open dynamical system satisfies Quantum Regression (QR) if, for arbitrary* $\{A_k\}_{k=0}^{n-1}$, $\{B_k\}_{k=0}^{n-1}$ *in* $\mathcal{A}_S$, $n \geq 1$,

$$\widetilde{\omega}_S \otimes \omega_E\Big(\widetilde{B}^{(n)\dagger}\widetilde{A}^{(n)}\Big) = \widetilde{\omega}_S\left(B_0^\dagger\, \Lambda_1^\ddagger\Big[B_1^\dagger\, \Lambda_{2,1}^\ddagger\Big[\ldots\, \Lambda_{n-1,n-2}^\ddagger\big[B_{n-1}^\dagger A_{n-1}\big] \ldots\Big]A_1\Big]\, A_0\right), \tag{2.69}$$

with $\Lambda_{n,n-1}^\ddagger$, $n \in \mathbb{N}$ *completely positive, unital maps.*

As emphasized in Remark 2.3, the notion of quantum Markovianity should be regarded as context-dependent. Whenever the multi-time statistics of the process is at hand, as we shall consider in the remainder of the present Chapter and especially in Chapter 5, one can refine the notion of Markovianity in the following way.

Definition 2.3. *An open quantum dynamical system is QR-Markovian if it satisfies QR.*

Remark 2.10.

1. *The QR condition at $n = 1$ identifies Λ_1 as the reduced dynamics of S that comes from the automorphism Θ_1 on $S + E$ through the elimination of the environment degrees of freedom. Indeed, let $\widetilde{\omega}_S$ be a generic state of the system (and, in particular, generally not time-invariant). Then, setting $A_0 = B_0 = \mathbb{1}$*
$$\widetilde{\omega}_S \otimes \omega_E\Big(\Theta_1\left(B_1^\dagger A_1 \otimes \mathbb{1}_E\right)\Big) = \widetilde{\omega}_S\Big(\Lambda_1^\ddagger[B_1^\dagger A_1]\Big)\,,$$
and varying freely $\widetilde{\omega}_S$ one gets that Λ_1 describes the reduced dynamics of S in the Heisenberg picture (compare (2.2)). Moreover, let $n = 3$ and set $B_0 = A_0 = B_1 = A_1 = \mathbb{1}_S$ so that
$$\widetilde{\omega}_S \otimes \omega_E\Big(\Theta_2\left(B_2^\dagger A_2 \otimes \mathbb{1}_E\right)\Big) = \widetilde{\omega}_S\Big(\Lambda_1^\ddagger \circ \Lambda_{2,1}^\ddagger[B_2^\dagger A_2]\Big)\,.$$
Again, by comparison with (2.2), one gets the composition law $\Lambda_1^\ddagger \circ \Lambda_{2,1}^\ddagger = \Lambda_2^\ddagger$. In the same vein, one identifies the maps $\Lambda_{n,m}^\ddagger$ with completely positive and unital propagators, in the Heisenberg picture, from time-step m to time-step n. Actually, formula (2.69) makes sense only if the maps $\Lambda_{n,m}^\ddagger$ are completely positive and unital. Hence, QR-Markovianity implies CP-divisibility. We can summarize the various concepts of non-Markovianity that we have so far encountered by the following chain of implications:
$$\textit{QR-Markovianity} \implies \textit{CP-divisibility} \implies \textit{P-divisibility} \implies \textit{No BFI}\,.$$

2. *Notably, QR holds in the weak coupling limit, where $\Lambda_{n,n-1}^\ddagger = e^{\mathcal{L}^\ddagger} \equiv \Lambda^\ddagger$ is a CP unital map and $\mathcal{L}^\ddagger$ is the Davies generator in the Heisenberg picture [6]. Essentially, the QR follows from the Born-Markov approximation $\omega_E \circ \Theta^n \to \Lambda_n \otimes \omega_E$. This fact, along with the validity of QR in the singular coupling limit as well, were rigorously proved by Dümcke [21]. Interesting cases in which QR holds beyond the above mentioned limits were investigated in [108, 109].*

Let us consider the algebraic formulation of classical stochastic processes through spin chains on $D_p^{\mathbb{N}}(\mathbb{C})$, introduced in Section 1.3.2.1. QR-Markovianity then extends to the non-commutative setting the structure of classical multi-time correlation functions pertaining to Markovian processes.

Proposition 2.11. *Consider a stochastic process $\pi^{[0,n-1]} = \{p^{[0,n-1]}_{\boldsymbol{i}_{[0,n-1]}}\}$, algebraically described by a classical spin chain endowed with a locally normal state ω. The process is Markovian if and only if, for all $n \geq 0$ and all $\{A_k\}_{k=0}^{n-1}$, $A_k \in D_p(\mathbb{C})$,*
$$\omega\left(\bigotimes_{k=0}^{n-1} A_k^{(k)}\right) = \mathrm{Tr}\left(\rho^{(0)} A_0\, \Lambda_1^\ddagger\Big[A_1\, \Lambda_{2,1}^\ddagger\Big[A_2 \dots \Lambda_{n-1,n-2}^\ddagger[A_{n-1}]\Big]\Big]\right)\,, \tag{2.70}$$
where $\rho^{(0)} = \sum_i p_i^{(0)} \Pi_i$ and $\Lambda_{k,k-1}^\ddagger : D_p(\mathbb{C}) \to D_p(\mathbb{C})$ are (completely) positive unital maps.

Proof. Consider the n-point correlation function for a general collection of operators $\{A_k\}_{k=0}^{n-1}$, $A_k = \sum_i a_i^{(k)} \Pi_i$.
$$\omega\left(\bigotimes_{k=0}^{n-1} A_k^{(k)}\right) = \mathrm{Tr}\left(\rho^{[0,n-1]} \bigotimes_{k=0}^{n-1} A_k^{(k)}\right) = \sum_{\boldsymbol{i}_{[0,n-1]}} p^{[0,n-1]}_{\boldsymbol{i}_{[0,n-1]}} a^{[0,n-1]}_{\boldsymbol{i}_{[0,n-1]}}\,, \tag{2.71}$$

where $a_{\boldsymbol{i}}^{[0,n-1]} = a_{i_{n-1}}^{(n-1)} \dots a_{i_0}^{(0)}$. If the process is Markovian we can write

$$\omega\left(\bigotimes_{k=0}^{n-1} A_k^{(k)}\right) = \sum_{i_0} p_{i_0}^{(0)} a_{i_0}^{(0)} \sum_{i_1} T_{i_1 i_0}(1,0) a_{i_1}^{(1)} \dots \sum_{i_{n-1}} T_{i_{n-1} i_{n-2}}(n-1, n-2) a_{i_{n-1}}^{(n-1)} . \quad (2.72)$$

Then, set

$$A \longmapsto \Lambda_{k,k-1}^{\ddagger}[A] := \sum_j \left(\sum_i T_{ij}(k, k-1) \operatorname{Tr}(\Pi_i A) \right) \Pi_j . \quad (2.73)$$

By recursion, one checks that (2.72) is also equal to (2.70). First, let $n = 3$. Exploiting commutativity, rewrite

$$\begin{aligned}
\operatorname{Tr}\left(\rho^{(0)} A_0 \Lambda_1^{\ddagger}[A_1 \Lambda_{2,1}^{\ddagger}[A_2]]\right) &= \sum_{i_0 i_1 i_2} a_{i_0}^{(0)} a_{i_1}^{(1)} a_{i_2}^{(0)} \operatorname{Tr}\left(\rho^{(0)} \Pi_{i_0} \Lambda_1^{\ddagger}[\Pi_{i_1} \Lambda_{2,1}^{\ddagger}[\Pi_{i_2}]\Pi_{i_1}]\Pi_{i_0}\right) \\
&= \sum_{i_0 i_1 i_2} a_{i_0}^{(0)} a_{i_1}^{(1)} a_{i_2}^{(0)} \operatorname{Tr}\left(\rho^{(1)} \Pi_{i_0}\right) \operatorname{Tr}\left(\Pi_{i_0} \Lambda_1^{\ddagger}[\Pi_{i_1}]\right) \operatorname{Tr}\left(\Pi_{i_1} \Lambda_{2,1}^{\ddagger}[\Pi_{i_2}]\right) \\
&= \sum_{i_0} p_{i_0}^{(0)} a_{i_0}^{(0)} \sum_{i_1} T_{i_0 i_1}^{\top}(1,0) a_{i_1}^{(1)} \sum_{i_2} T_{i_1 i_2}^{\top}(2,1) a_{i_2}^{(0)} .
\end{aligned}$$

Now, we suppose that equality holds for $n-1$ and prove it holds for n. Letting $\widetilde{A}_{n-1} = A_{n-1} \Lambda_{n,n-1}^{\ddagger}[A_n]$, we have

$$\begin{aligned}
&\operatorname{Tr}\left(\rho^{(0)} A_0 \Lambda_1^{\ddagger}\left[A_1 \Lambda_{2,1}^{\ddagger}\left[A_2 \dots \Lambda_{n-1,n-2}^{\ddagger}[\widetilde{A}_{n-1}]\right]\right]\right) \\
&\qquad = \sum_{i_0} p_{i_0}^{(0)} a_{i_0}^{(0)} \sum_{i_1} T_{i_1 i_0}(1,0) a_{i_1}^{(1)} \dots \sum_{i_{n-1}} T_{i_{n-1} i_{n-2}}(n-1, n-2) \widetilde{a}_{i_{n-1}}^{(n-1)} \quad (2.74)
\end{aligned}$$

where

$$\begin{aligned}
\widetilde{a}_{i_{n-1}}^{(n-1)} = \operatorname{Tr}\left(\Pi_{i_{n-1}} A_{n-1} \Lambda_{n,n-1}^{\ddagger}[A_n]\right) &= \sum_{j_{n-1} i_n} a_{j_{n-1}}^{(n-1)} a_{i_n}^{(n)} \operatorname{Tr}\left(\Pi_{i_{n-1}} \Pi_{j_{n-1}} \Lambda_{n,n-1}^{\ddagger}[\Pi_{i_n}]\right) \\
&= a_{i_{n-1}}^{(n-1)} \sum_{i_n} T_{i_{n-1} i_n}^{\top}(n, n-1) .
\end{aligned}$$

By plugging the latter into (2.74), one gets the result. The converse is also true. Suppose that (2.70) holds for arbitrary A_k and a CP unital maps $\Lambda_{k,k-1}^{\ddagger}$. Define then stochastic matrices

$$T_{ij}^{\top}(k, k-1) := \operatorname{Tr}\left(\Pi_i \Lambda_{k,k-1}^{\ddagger}[\Pi_j]\right) . \quad (2.75)$$

Upon choosing $A_k = \Pi_{j_k}$, $k = 0, \dots, n-1$, one gets

$$p_{\boldsymbol{j}_{[0,n-1]}}^{[0,n-1]} = T_{j_{n-1} j_{n-2}}(n-1, n-2) \dots T_{j_1 j_0}(1,0) p_{j_0}^{(0)} ,$$

so that the probability distribution is Markovian. □

Remark 2.11. *The previous result provides an alternative characterization of classical Markov processes in terms of multi-time correlations of classical observables. For stationary processes, in particular, one can reconstruct all* n*-point multi-time correlation functions from the knowledge of the transition stochastic matrix* T *of the process.*

Note that, for quantum systems, CP-divisibility – as well as the stronger semigroup property – constitutes only a necessary condition for QR-Markovianity [3, 110, 111]. This fact represents the quantum analogue of those classical non-Markovian processes that are P-divisible and satisfy the Chapman-Kolmogorov equation (1.57), as illustrated in Example 1.7. A concrete case in which CP-divisibility occurs without QR will be discussed in Chapter 5.

Note that, in the QR formula (2.69), operators A_k and B_k are general and need not come from an OPU. We now show how to determine whether QR holds or not through a partition specific coarse-grained density matrix. In this regard, consider the OPU $\mathcal{F} \subseteq \mathcal{A}_S$, already discussed in Remark 1.18,

$$\mathcal{F} := \{F_{aa'}\}_{aa'=1}^{d}, \quad F_{aa'} = \sqrt{r_a}\, |r_a\rangle\langle r_{a'}| \ , \tag{2.76}$$

where $r_a > 0, |r_a\rangle$ are the eigenvalues, respectively, eigenvectors of a density matrix ρ_S. Recall that the associated CPTP map $\mathbb{F}[\rho] = \sum_{aa'} F_{aa'}\rho F_{aa'}^\dagger = \mathrm{Tr}\,(\rho)\,\rho_S$ leaves ρ_S invariant: $\mathbb{F}[\rho_S] = \rho_S$. Then, since we assumed $\omega_{SE} = \omega_S \otimes \omega_E$, it follows that

$$\omega_{SE}(\mathbb{F}[X_S] \otimes X_E) = \mathrm{Tr}\,(\rho_S X_S)\ \omega_E(X_E) = \omega_{SE}(X_S \otimes X_E)\ .$$

Remarkably, the structure of the coarse-grained matrices $\rho_S\big[\mathcal{F}^{(n)}\big]$ can be controlled. Moreover, they allow one to discriminate QR-Markovianity, recovering the result of Lindblad [7].

Theorem 2.12. *With respect to a time-invariant state $\omega_S \otimes \omega_E$, consider the CPU map $\mathbb{T}_n^\ddagger : M_d^{\otimes n}(\mathbb{C}) \to M_d^{\otimes n}(\mathbb{C})$, defined by the conditional expectation*[1]*:*

$$\mathbb{T}_n^\ddagger\left[\bigotimes_{k=0}^{n-1} A_k\right] := \omega_E\left(\left(\prod_{j=1}^{n}(\Theta_{j,j-1}\circ\sigma\otimes \mathrm{id}_E)\right)\left[\bigotimes_{k=0}^{n-1} A_k^{(-n+k)} \otimes \mathbb{1}_E\right]\right), \tag{2.77}$$

where (1) the operator $\bigotimes_{k=0}^{n-1} A_k$ has been embedded into the quasi local algebra $M_d^{\mathbb{Z}}(\mathbb{C})$ by localizing it within the interval $[0, n-1]$, (2) automorphisms $\Theta_{k,k-1}$ act non-trivially only on $M_d^{(0)}(\mathbb{C}) \otimes \mathcal{A}_E$ and (3) σ is the right shift on $M_d^{\mathbb{Z}}(\mathbb{C})$. Then, the system coarse-grained density matrix with respect to the OPU $\mathcal{F}$ is given, up to a rearrangement of tensor factors, by

$$\rho_S\left[\mathcal{F}^{(n+1)}\right] = \rho_S \otimes \rho_S \otimes \left(\mathbb{T}_n \otimes \mathrm{id}\left[\left|\sqrt{\rho_S^{\otimes n}}\right\rangle\left\langle\sqrt{\rho_S^{\otimes n}}\right|\right]\right), \tag{2.78}$$

where $\mathbb{T}_n$ is the CPTP map dual of $\mathbb{T}_n^\ddagger$. Furthermore, the QR condition holds if and only if

$$\mathbb{T}_n = \bigotimes_{k=1}^{n} \Lambda_{k,k-1}\ . \tag{2.79}$$

We provide an independent and self-contained proof in Appendix D.2, using the formalism of quantum spin chains.

Remark 2.12. *The interaction-shift automorphism appearing in* (2.77) *is characteristic of the collisional approach of Section 2.1.4, wherein the open system S iteratively interacts with a new site of a spin-chain environment. In* (2.77) *the emerging picture is, in a sense, dual, in that it is the same environment E that interacts with a fresh copy of the system at each time step.*

(a) $\boxed{\Theta_{j,j-1}}$

$$\cdots \otimes \mathcal{A}_S^{(-a-1)} \otimes \mathcal{A}_S^{(-a)} \otimes \mathcal{A}_S^{(-a+1)} \otimes \cdots \otimes \begin{matrix} \mathcal{A}_S^{(0)} \\ \otimes \\ \mathcal{A}_E \end{matrix} \otimes \mathcal{A}_S^{(1)} \otimes \cdots \otimes \mathcal{A}_S^{(b)} \otimes \mathcal{A}_S^{(b+1)} \otimes \mathcal{A}_S^{(b+2)} \otimes \cdots$$

(b)

$$\cdots \otimes \mathcal{A}_S^{(-a-1)} \otimes \mathcal{A}_S^{(-a)} \otimes \mathcal{A}_S^{(-a+1)} \otimes \cdots \otimes \begin{matrix} \mathcal{A}_S^{(0)} \\ \otimes \\ \mathcal{A}_E \end{matrix} \otimes \mathcal{A}_S^{(1)} \otimes \cdots \otimes \mathcal{A}_S^{(b)} \otimes \mathcal{A}_S^{(b+1)} \otimes \mathcal{A}_S^{(b+2)} \otimes \cdots$$

(c) $\boxed{\sigma \otimes \mathrm{id}_E}$

$$\cdots \otimes \mathcal{A}_S^{(-a-1)} \otimes \mathcal{A}_S^{(-a)} \otimes \mathcal{A}_S^{(-a+1)} \otimes \cdots \otimes \begin{matrix} \mathcal{A}_S^{(0)} \\ \otimes \\ \mathcal{A}_E \end{matrix} \otimes \mathcal{A}_S^{(1)} \otimes \cdots \otimes \mathcal{A}_S^{(b)} \otimes \mathcal{A}_S^{(b+1)} \otimes \mathcal{A}_S^{(b+2)} \otimes \cdots$$

FIGURE 2.2: Dynamical system underlying the construction of the map (2.77). (a) Intertwiners $\Theta_{j,j-1}$ act, at each tick of time, only on the 0-th copy of the system $\mathcal{A}_S^{(0)} = M_d^{(0)}(\mathbb{C})$ and on the algebra of the environment. (b) An operator, initially localized in $\mathcal{A}_S^{[-a,b]}$, is then translated to the right in (c) by the shift automorphism σ on the open chain. Note the similarity with the algebraic collisional model of Figure 2.1.

Remark 2.13. *The above approach to quantum non-Markovian stochastic processes is due to Lindblad, that first presented it in [7, 112], though without the concept of OPU (the latter was only introduced in a more general context [57] and then fully developed in the coarse-graining procedure discussed in Section 1.5 by Alicki and Fannes [60]). In particular, the map $\mathbb{T}_n$ emerging from suitable invariant OPUs is essentially equivalent to the so-called* process tensor *developed in [15, 99, 100], and applied successfully in several problems [111, 113, 114]. Analogously to dynamical maps, it can be tomographically reconstructed [115] and used to compute all possible multi-time correlation functions (see also Appendix D.2). In the condensed-matter literature, a similar object appears as influence matrix, see e.g. the recent work [106].*

From now on, we go back to the simpler case of one-parameter groups of automorphisms on $\mathcal{A}_S \otimes \mathcal{A}_E$, namely $\Theta_n = \Theta^n$, $\Theta_{n+m} = \Theta_n \circ \Theta_m$. In such case, the QR condition is only compatible with a semigroup reduced dynamics.

Proposition 2.13. *Suppose that the QR condition* (2.69) *holds and let $\omega_S \otimes \omega_E$ be a Θ-invariant state, with ω_S represented by a faithful density matrix $\rho_S > 0$. Then,*

$$\Lambda_{n,n-1} = \Lambda_1 \equiv \Lambda \,, \qquad \forall\, n \geq 0\,. \tag{2.80}$$

[1]There is no ambiguity in the definition of the C^*-tensor product of $M_d^{\mathbb{Z}}(\mathbb{C})$ with $\mathcal{A}_E$ since the former is the inductive limit of nuclear C^*-algebras, which is itself nuclear (see Remark 1.3)

Proof. The full proof of (2.80), carried out by induction, is reported in Appendix D.1. Here, we prove it for $n = 2$. Suppose that QR hold and consider

$$\rho_S[\mathcal{F}^{(3)}] = \omega_S \otimes \omega_E \left(F^\dagger_{a_0 a'_0}\Theta_1\left(F^\dagger_{b_1 b'_1}\,\Theta_{2,1}\left(F^\dagger_{b_2 b'_2}F_{a_2 a'_2}\right)F_{a_1 a'_1}\right)F_{a_0 a'_0}\right) \tag{2.81}$$

$$= \omega_S\left(F^\dagger_{a_0 a'_0}\Lambda^\ddagger_1\left[F^\dagger_{b_1 b'_1}\,\Lambda^\ddagger_{2,1}\left[F^\dagger_{b_2 b'_2}F_{a_2 a'_2}\right]F_{a_1 a'_1}\right]F_{a_0 a'_0}\right). \tag{2.82}$$

where $\mathcal{F}$ is the OPU in (2.76) that leaves ω_S invariant. $\rho_S[\mathcal{F}^{(3)}]$ is a density matrix acting on $\mathbb{C}^{d^2}\otimes\mathbb{C}^{d^2}\otimes\mathbb{C}^{d^2}$. Consider then the marginal density matrix obtained from by tracing over the first Hilbert space from expression (2.81). This amounts to setting $a_0 = b_0$, $a'_0 = b'_0$ and to summing over a_0 and a'_0. Due to invariance of the state under Θ and $\mathbb{F}$ (compare Remark 1.20),

$$\left(\mathrm{Tr}_I\,\rho[\mathcal{F}^{(3)}]\right)_{a_1 a'_1, b_1 b'_1} = \sum_{a_0 a'_0}\omega_S\otimes\omega_E\left(F^\dagger_{a_0 a'_0}\Theta_1\left(F^\dagger_{b_1 b'_1}\,\Theta_{2,1}\left(F^\dagger_{b_2 b'_2}F_{a_2 a'_2}\right)F_{a_1 a'_1}\right)F_{a_0 a'_0}\right)$$

$$= \omega_S\otimes\omega_E\left(F^\dagger_{b_1 b'_1}\,\Theta_{2,1}\left(F^\dagger_{b_2 b'_2}F_{a_2 a'_2}\right)F_{a_1 a'_1}\right) = \omega_S\otimes\omega_E(F^\dagger_{b_1 b'_1}\,\Theta_1\left(F^\dagger_{b_2 b'_2}F_{a_2 a'_2}\right)F_{a_1 a'_1}).$$

where, in the last equality, we used the group assumption: $\Theta_{2,1} = \Theta_1 = \Theta$. Thus, we have $\mathrm{Tr}_I\left(\rho[\mathcal{F}^{(3)}]\right) = \rho[\mathcal{F}^{(2)}]$. Taking instead the partial trace over the first Hilbert space from expression (2.82), we have

$$\left(\mathrm{Tr}_I\,\rho[\mathcal{F}^{(3)}]\right)_{a_1 a'_1, b_1 b'_1} = \sum_{a_0, a'_0}\mathrm{Tr}\left(\rho_S\,F^\dagger_{a_0, a'_0}\Lambda^\ddagger_1\left[F^\dagger_{b_1, b'_1}\,\Lambda^\ddagger_{2,1}\left[F^\dagger_{b_2 b'_2}F_{a_2 a'_2}\right]F_{a_1 a'_1}\right]F_{a_0 a'_0}\right)$$

$$= \mathrm{Tr}\left(\Lambda_1\circ\mathbb{F}[\rho_S]\,F^\dagger_{b_1 b'_1}\,\Lambda^\ddagger_{2,1}\left[F^\dagger_{b_2 b'_2}F_{a_2 a'_2}\right]F_{a_1 a'_1}\right) = \mathrm{Tr}\left(\rho_S\,F^\dagger_{b_1, b'_1}\,\Lambda^\ddagger_{2,1}\left[F^\dagger_{b_2 b'_2}F_{a_2 a'_2}\right]F_{a_1 a'_1}\right),$$

where, in the last equality, we used invariance of ρ_S under $\mathbb{F}$ and Λ_1. Then, $\mathrm{Tr}_I\left(\rho\left[\mathcal{F}^{(3)}\right]\right) = \rho\left[\mathcal{F}^{(2)}\right]$ yields

$$\mathrm{Tr}\left(\rho_S\,F^\dagger_{b_1 b'_1}\,\Lambda^\ddagger_{2,1}\left[F^\dagger_{b_2 b'_2}F_{a_2 a'_2}\right]F_{a_1 a'_1}\right) = \mathrm{Tr}\left(\rho_S\,F^\dagger_{b_1 b'_1}\,\Lambda^\ddagger_1\left[F^\dagger_{b_2 b'_2}F_{a_2 a'_2}\right]F_{a_1 a'_1}\right).$$

Taking $a_2 = b_2$, $a'_1 = b'_1$, yields, in particular

$$r_{a_2} r_{a'_1}\sqrt{r_{a_1} r_{b_1}}\,\langle r_{a'_2}|\,\Lambda_{2,1}[\,|r_{a_1}\rangle\langle r_{b_1}|\,]\,|r_{b'_2}\rangle = r_{a_2} r_{a'_1}\sqrt{r_{a_1} r_{b_1}}\,\langle r_{a'_2}|\,\Lambda_{2,1}[\,|r_{a_1}\rangle\langle r_{b_1}|\,]\,|r_{b'_2}\rangle\,,$$

which implies $\Lambda_{2,1} = \Lambda_1$. □

Part II

Results

CHAPTER 3

Classical reduction of quantum dynamical maps

In Chapter 2, we discussed the standard interpretation of distinguishability revivals in open quantum systems as a backflow of information. The latter, though, is not a quantum phenomenon *per se*: an analogous interpretation can be carried out for classical systems in terms of revivals of Kolmogorov distinguishability (see Proposition 1.6). In fact, the focus of this Chapter will be on classical memory effects arising from suitable reductions of legitimate quantum evolutions. In the following Sections, we shall concentrate on the MASAs of a matrix algebra $\mathcal{A} = M_d(\mathbb{C})$, selected by projective OPUs, namely sets $\mathcal{P} = \{P_i\}_{i=1}^d$ consisting of minimal rank-1 projectors P_i, with $P_j P_k = \delta_{jk}$ and $\sum_{j=1}^d P_j = \mathbb{1}_d$. For the sake of simplicity, we shall identify with the same symbol $\mathcal{P}$ both the OPU and the associated MASA. Consider then a quantum dynamical map on $M_d(\mathbb{C})$; a classical stochastic process can be naturally extracted out of it by restricting the quantum evolution to a MASA $\mathcal{P}$. We will investigate the properties inherited by such *classical reduction*; in particular, P-divisibility of the quantum evolution might be lost in the reduction procedure, thereby yielding a classical backflow of information arising from a quantum dynamics that does not exhibit it on its own.

3.1 Classical reductions: dynamics vs. generators

Let us consider a quantum evolution described by a one-parameter family of CPTP maps $\{\Lambda_t\}_{t\geq 0}$, the latter being governed by a time-local master-equation as in (2.7). In order to extract a classical dynamics out of a quantum one, one can follow two natural prescriptions:

i. Reduce the generator of the quantum evolution $\mathcal{L}_t = \dot{\Lambda}_t \circ \Lambda_t^{-1}$ by considering the $d \times d$ real matrix

$$K_{ij}(t) := \mathrm{Tr}\left(P_i \mathcal{L}_t[P_j]\right) \tag{3.1}$$

and study the one-parameter family of $d\times d$ stochastic matrices $\{D(t)\}_{t\geq 0}$ solving the classical master equation

$$\dot{D}(t) = K(t)\, D(t)\,. \tag{3.2}$$

ii. Reduce the quantum dynamics itself by considering the stochastic matrix $T(t)$ with entries

$$T_{ij}(t) = \mathrm{Tr}\left(P_i \Lambda_t[P_j]\right) . \tag{3.3}$$

If the so-obtained matrix $T(t)$ is invertible, one can then write the master equation

$$\dot{T}(t) = L(t)\,T(t)\,, \tag{3.4}$$

with time-local generator $L(t) := \dot{T}(t)\,T^{-1}(t)$.

The two different points of view yield different classical dynamics that are discussed in the following. In particular, regarding the property of P-divisibility, the following question naturally emerges:

Given a P-divisible dynamics $\{\Lambda_t\}_{t\geq 0}$, *are their classical reductions P-divisible?* (Q)

By prescription *i.*, namely by reducing the quantum generator, the answer to the previous Question is straightforward, in that quantum P-divisibility is equivalent to classical P-divisibility, as a consequence of Proposition 2.2.

Corollary 3.1. *Let* $\mathcal{L}_t$ *be the generator of a CPTP dynamical map* $\{\Lambda_t\}_{t\geq 0}$. *Then* $\{\Lambda_t\}_{t\geq 0}$ *is P-divisible if and only if the* $d \times d$ *matrix defined by* $K_{ij}(t) = \mathrm{Tr}\left(P_i \mathcal{L}_t[P_j]\right)$ *generates a classical P-divisible dynamics for any choice of MASA* $\mathcal{P} = \{P_i\}_{i=1}^{d}$ *of* $M_d(\mathbb{C})$.

Conversely, as we shall see, by reducing the dynamical map one has the peculiar effect that a P-divisible quantum dynamics need not give rise to classically reduced P-divisible dynamics.

Remark 3.1. *The reduction procedure ii., namely the reduction of the dynamical map, is somewhat dual to the so-called embeddability problem of a given classical stochastic matrix into a Markovian quantum evolution, that may give rise to some quantum advantage [116, 117].*

3.1.1 Dynamics vs. generators: physical interpretation

The physical interpretation of the stochastic dynamics $\{T(t)\}_{t\geq 0}$, defined through (3.3), becomes clearer by considering the map

$$\mathbb{P}[X] = \sum_{i=1}^{d} P_i X P_i\,, \tag{3.5}$$

which, as discussed in Remark 1.17, can be seen as a conditional expectation from $M_d(\mathbb{C})$ to the MASA generated by $\mathcal{P}$. Given any state $\omega(\,\cdot\,) = \mathrm{Tr}\,(\rho\,\cdot)$ on $M_d(\mathbb{C})$, consider

$$\omega_t^{\mathbb{P}} := \omega \circ \mathbb{P} \circ \Lambda_t^{\ddagger} \circ \mathbb{P} =: \omega \circ \left(\Lambda_t^{\mathbb{P}}\right)^{\ddagger}, \tag{3.6}$$

which acts initially on observables in $\mathcal{P}$, then lets them evolve with $\Lambda_t^{\ddagger}$, and finally projects them again on $\mathcal{P}$ with the left-most $\mathbb{P}$. Equivalently, moving by duality to the Schrödinger picture, we have

$$\omega_t^{\mathbb{P}}(X) = \mathrm{Tr}\left(\Lambda_t^{\mathbb{P}}[\rho] X\right), \qquad \Lambda_t^{\mathbb{P}} := \mathbb{P} \circ \Lambda_t \circ \mathbb{P}\,, \tag{3.7}$$

where $\Lambda_t^{\mathbb{P}}$ acts initially on diagonal matrices, turns them into non-diagonal ones at time t and, finally, the developed coherences are suppressed by the left-most $\mathbb{P}$. Explicitly, one gets:

$$\Lambda_t^{\mathbb{P}}[\rho] = \sum_{j,k=1}^{d} T_{jk}(t)\ \mathrm{Tr}\,(P_k \rho)\ P_j\,, \tag{3.8}$$

with $T(t)$ as in (3.3). Practically speaking, in this approach, at each instant of time, the quantum dynamics is preceded and followed by two projective measurements specified by the OPU $\mathcal{P}$.

Remark 3.2. *Note the similarity between the construction* (3.6) *for the classical reduction* $\Lambda_t^{\mathbb{P}}$ *and that for the reduced dynamics* Λ_t *of an open system in* (2.2). *Indeed, one defines* $\Lambda_t^{\ddagger}$ *in the Heisenberg picture by three steps; namely, (1) by restricting to the algebra of the system, then (2) by letting evolve an operator in* $\mathcal{A}_S$ *under the automorphism* Θ_t *that does not leave* $\mathcal{A}_S$ *invariant and (3) finally, by acting with the conditional expectation* ω_E *from* $\mathcal{A}_S \otimes \mathcal{A}_E \to \mathcal{A}_S$. *For the classical reduction, instead, (1) one first restricts to the MASA* $\mathcal{P}$ *via the right-most conditional expectation* $\mathbb{P}$ *in* (3.6), *then (2) lets the system evolve under the CPU evolution* Λ_t, *that does not leave* $\mathcal{P}$ *invariant, and finally, (3) applies the conditional expectation* $\mathbb{P} : M_d(\mathbb{C}) \to \mathcal{P}$ *to obtain the classical reduction* $\left(\Lambda_t^{\mathbb{P}}\right)^{\ddagger}$.

To compare the classical reduction (3.7) with the one obtained from (3.1), for sake of simplicity we restrict to the case when the quantum dynamics Λ_t is a semigroup with time-independent generator $\mathcal{L}$: $\Lambda_t = e^{t\mathcal{L}}$. Then, let $N \gg 1$ and consider

$$\omega_t^{\mathbb{P},N} := \omega \circ \left(\mathbb{P} \circ \Lambda_{t/N}^{\ddagger} \circ \mathbb{P}\right)^N = \omega \circ \mathbb{P} \circ \left(\mathrm{id} + \frac{t}{N}\left(\mathcal{L}^{\mathbb{P}}\right)^{\ddagger}\right)^N \circ \mathbb{P} + O\left(\left(\frac{t}{N}\right)^2\right), \tag{3.9}$$

$$\overset{N\gg 1}{\simeq} \omega \circ \mathbb{P} \circ \left(e^{t\mathcal{L}^{\mathbb{P}}}\right)^{\ddagger} \circ \mathbb{P}, \qquad \left(\mathcal{L}^{\mathbb{P}}\right)^{\ddagger} := \mathbb{P} \circ \mathcal{L}^{\ddagger} \circ \mathbb{P}. \tag{3.10}$$

Passing again to the dual in the Schrödinger picture, one has

$$\mathcal{L}^{\mathbb{P}}[\rho] = \mathbb{P} \circ \mathcal{L} \circ \mathbb{P}[\rho] = \sum_{j,k=1}^{d} K_{jk}\ \mathrm{Tr}\,(P_k\,\rho)\ P_j\,, \tag{3.11}$$

so that

$$\mathbb{P} \circ e^{t\mathcal{L}^{\mathbb{P}}} \circ \mathbb{P}[\rho] = \sum_{j,k=1}^{d} D_{jk}(t)\ \mathrm{Tr}\,(P_k\,\rho)\ P_j\,. \tag{3.12}$$

where $K(t) = K$ is the matrix defined in (3.1) and $D(t) = e^{tK}$. From the previous argument, therefore, the classical reduction of the quantum generator physically amounts to letting the quantum dynamics to act in between iterated projective measurements separated by smaller and smaller time intervals.

Remark 3.3. *Notice that, for* $N \geq 1$ *and* $\epsilon = t/N$,

$$\left(\mathbb{P} \circ \Lambda_\epsilon \circ \mathbb{P}\right)^N[\rho] = \sum_{i,j_N,\dots,j_1} T_{ij_N}(\epsilon)\cdots T_{j_2j_1}(\epsilon)\ \mathrm{Tr}\left(\rho P_{j_1}\right)\ P_i = \sum_{i,j}\left[T(\epsilon)^N\right]_{ij}\ \mathrm{Tr}\left(\rho P_j\right)\ P_i\,,$$

On the other hand, in the semi-group case, from (3.1) *and* (3.3), *with* $\mathcal{L}_t = \mathcal{L}$ *and* $K(t) = K$, *it follows that* $T(\epsilon) \simeq \mathbb{1} + \epsilon K \simeq D(\epsilon)$; *therefore,*

$$\lim_{N\to+\infty} T\left(\frac{t}{N}\right)^N = D(t) \neq T(t)\,.$$

The origin of the discrepancy is the fact that the family of matrices $T(t)$ *is not a semigroup; indeed, if* $T(t)$ *is forced to be a semigroup, so that* $T(t+\epsilon) = T(\epsilon)T(t)$, *then one has* $T(t) = D(t)$.

3.2 Classical reductions and P-divisibility

Differently from what occurs when reducing generators, when the reduction via the dynamical map (3.3) is adopted, the relation between P-divisibility of Λ_t and that of its classical reduction $T(t)$ becomes non-trivial and, in particular, classical P-divisibility may be lost. The rest of this Chapter is devoted to classical processes of the type (3.3), analysing in detail instances when P-divisibility is either preserved or lost through the reduction procedure. The physical meaning of such a loss of P-divisibility by classical reductions is clarified through the following Section 3.2.1, where we develop a general interpretation of the effect in terms of BFI.

Remark 3.4. *Notice that, from* (3.3)*, when the matrices* $T(t)$ *are invertible, the classical intertwiners are given by* $T(t,s) = T(t)T(s)^{-1}$*. In general, they have no simple connection to the quantum intertwiners* $\Lambda_{t,s} = \Lambda_t \circ \Lambda_s^{-1}$*. A particular case is when the dynamics* Λ_t *is of the kind* $\Lambda_{t,s}\,\mathbb{P} = \mathbb{P}\,\Lambda_{t,s}\,\mathbb{P}$*, with* $\mathbb{P}$ *as in* (3.5)*, so that one can identify a stochastic propagator as* $T_{ik}(t,s) = \mathrm{Tr}\left(P_i \Lambda_{t,s}[P_k]\right)$*. Such maps are a subset of the so-called non-coherence-generating-and-detecting dynamics considered in [98] to characterize the classicality of a Markovian process.*

3.2.1 Coherence-assisted backflow of information

The goal of this Section is to interpret memory effects emerging in the stochastic dynamics (3.3) obtained by classically reducing a P-divisible quantum evolution Λ_t. In the BLP approach described in Section 2.1.3, the Holevo-Helstrom distinguishability is interpreted as "internal information" of a quantum system as embodied by two states ρ and σ under the dynamics Λ_t:

$$\mathcal{I}_t^q(\rho,\sigma;\mu) = \left\|\Lambda_t[\Delta_\mu(\rho,\sigma)]\right\|_1\,, \quad \text{where} \quad \Delta_\mu(\rho,\sigma) = \mu\rho - (1-\mu)\sigma\,. \tag{3.13}$$

Restricting to the classical subalgebra, the Helstrom matrix encodes the vector $\boldsymbol{\delta}_\mu(\boldsymbol{p},\boldsymbol{q}) = \mu\boldsymbol{p} - (1-\mu)\boldsymbol{q}$, with $\boldsymbol{p},\boldsymbol{q}$ probability vectors, and its trace norm reduces to the Kolmogorov distance (1.70). Let us now consider a Helstrom matrix $\Delta_\mu(\rho_{\boldsymbol{p}},\sigma_{\boldsymbol{q}})$ within the MASA $\mathcal{P}$, namely with $\rho_{\boldsymbol{p}} = \sum_i p_i P_i$ and $\sigma_{\boldsymbol{q}} = \sum_i q_i P_i$. Although the chosen density matrices $\rho_{\boldsymbol{p}}, \sigma_{\boldsymbol{q}}$ commute and are incoherent classical states with respect to $\mathcal{P}$, the dynamics Λ_t takes them out of the commutative algebra $\mathcal{P}$ and generates non-vanishing coherences, while the classical reduction to $\mathcal{P}$ eliminates them. Accordingly, the classical internal information of the classical reduction to $\mathcal{P}$ of the quantum dynamics reads

$$\mathcal{I}_t^{cl}(\boldsymbol{p},\boldsymbol{q};\mu) = \left\|T(t)\boldsymbol{\delta}_\mu(\boldsymbol{p},\boldsymbol{q})\right\|_1 = \left\|\mathbb{P}\circ\Lambda_t[\Delta_\mu(\rho_{\boldsymbol{p}},\sigma_{\boldsymbol{q}})]\right\|_1\,. \tag{3.14}$$

Let us then introduce the difference between the quantum and classical internal information relative to two classical states:

$$\begin{aligned}\mathcal{C}_t(\boldsymbol{p},\boldsymbol{q};\mu) &:= \mathcal{I}_t^q(\rho_{\boldsymbol{p}},\rho_{\boldsymbol{q}};\mu) - \mathcal{I}_t^{cl}(\boldsymbol{p},\boldsymbol{q};\mu)\\ &= \left\|\Lambda_t[\Delta_\mu(\rho_{\boldsymbol{p}},\sigma_{\boldsymbol{q}})]\right\|_1 - \left\|\mathbb{P}\circ\Lambda_t[\Delta_\mu(\rho_{\boldsymbol{p}},\sigma_{\boldsymbol{q}})]\right\|_1\,.\end{aligned} \tag{3.15}$$

Since $\mathbb{P}$ is a contraction, the quantity $\mathcal{C}_t(\boldsymbol{p},\boldsymbol{q};\mu)$ is positive; moreover, it allows to decompose the quantum internal information of two quantumly evolving classical states as sum of two contributions:

$$\mathcal{I}_t^q(\rho_{\boldsymbol{p}},\rho_{\boldsymbol{q}};\mu) = \mathcal{I}_t^{cl}(\boldsymbol{p},\boldsymbol{q};\mu) + \mathcal{C}_t(\boldsymbol{p},\boldsymbol{q};\mu)\,. \tag{3.16}$$

Note the analogy with definition (2.44) of "external information" in the open system setting. It is thus appropriate to name $\mathcal{C}_t(\boldsymbol{p},\boldsymbol{q};\mu)$ *coherent internal information*. P-divisibility of Λ_t implies that $\mathcal{I}_t^q(\rho_{\boldsymbol{p}}, \rho_{\boldsymbol{q}}; \mu)$ monotonically decreases in time, namely the information contained in the system leaks towards the environment and never comes back. Hence, for $t \geq s \geq 0$,

$$\mathcal{I}_t^{cl}(\boldsymbol{p}, \boldsymbol{q}; \mu) + \mathcal{C}_t(\boldsymbol{p}, \boldsymbol{q}; \mu) \leq \mathcal{I}_s^{cl}(\boldsymbol{p}, \boldsymbol{q}; \mu) + \mathcal{C}_s(\boldsymbol{p}, \boldsymbol{q}; \mu) . \tag{3.17}$$

Then, letting

$$\Delta\mathcal{I}_{t,s}^{cl}(\boldsymbol{p}, \boldsymbol{q}; \mu) := \mathcal{I}_t^{cl}(\boldsymbol{p}, \boldsymbol{q}; \mu) - \mathcal{I}_s^{cl}(\boldsymbol{p}, \boldsymbol{q}; \mu) , \tag{3.18}$$

be the variation of the classical internal information (3.17) can be equivalently recast as

$$\Delta\mathcal{I}_{t,s}^{cl}(\boldsymbol{p}, \boldsymbol{q}; \mu) \leq \mathcal{C}_s(\boldsymbol{p}, \boldsymbol{q}; \mu) - \mathcal{C}_t(\boldsymbol{p}, \boldsymbol{q}; \mu) . \tag{3.19}$$

We are then ready to state the following

Proposition 3.1. *Let Λ_t be P-divisible and consider the variation of the classical internal information* (3.18) *between times s and $t \geq s$. The latter can be upper-bounded by*

$$\Delta\mathcal{I}_{t,s}^{cl}(\boldsymbol{p}, \boldsymbol{q}; \mu) \leq \mu\, C_1^{\mathcal{P}}(\Lambda_s[\rho_{\boldsymbol{p}}]) + (1-\mu)\, C_1^{\mathcal{P}}(\Lambda_s[\rho_{\boldsymbol{q}}]) , \tag{3.20}$$

where $C_1^{\mathcal{P}}$ denotes the so-called ℓ_1-norm of coherence of a density matrix [118, 119],

$$C_1^{\mathcal{P}}(\rho) := \sum_{i \neq j} \left| \langle i|\rho|j\rangle \right| , \tag{3.21}$$

with respect to the MASA generated by $\mathcal{P} = \{|i\rangle\langle i|_i\}_{i=1}^d$.

Proof. From (3.19), since $\mathcal{C}_t(\boldsymbol{p}, \boldsymbol{q}; \mu) \geq 0$, we have

$$\Delta\mathcal{I}_{t,s}^{cl}(\boldsymbol{p}, \boldsymbol{q}; \mu) \leq \mathcal{C}_s(\boldsymbol{p}, \boldsymbol{q}; \mu) .$$

Let then $\mathbb{P}^\perp := \mathrm{id} - \mathbb{P}$ and apply twice the triangle inequality, so to get

$$\begin{aligned} \mathcal{C}_s(\boldsymbol{p}, \boldsymbol{q}; \mu) &\leq \|\mathbb{P}^\perp \Lambda_s[\Delta_\mu(\boldsymbol{p}, \boldsymbol{q})]\|_1 \\ &\leq \mu\, \|\mathbb{P}^\perp \Lambda_s[\rho_{\boldsymbol{p}}]\|_1 + (1-\mu)\, \|\mathbb{P}^\perp \Lambda_s[\rho_{\boldsymbol{q}}]\|_1 . \end{aligned} \tag{3.22}$$

Noting that $\left\| |i\rangle\langle j| \right\|_1 = 1$, with $|i\rangle, |j\rangle$ belonging to the reference orthonormal basis, one applies the triangle inequality once again to the r.h.s. of (3.22) so to obtain

$$\begin{aligned} \Delta\mathcal{I}_{t,s}^{cl}(\boldsymbol{p}, \boldsymbol{q}; \mu) &\leq \mu\, \|\mathbb{P}^\perp \Lambda_s[\rho_{\boldsymbol{p}}]\|_1 + (1-\mu)\, \|\mathbb{P}^\perp \Lambda_s[\rho_{\boldsymbol{q}}]\|_1 \\ &\leq \mu\, C_1^{\mathcal{P}}(\Lambda_s[\rho_{\boldsymbol{p}}]) + (1-\mu)\, C_1^{\mathcal{P}}(\Lambda_s[\rho_{\boldsymbol{q}}]) . \end{aligned}$$

□

Remark 3.5. *The quantities $C_1^{\mathcal{P}}(\Lambda_s[\rho_{\boldsymbol{p}}])$ and $C_1^{\mathcal{P}}(\Lambda_s[\rho_{\boldsymbol{q}}])$ measure the amount of coherence produced by the dynamics at time s acting on the diagonal states $\rho_{\boldsymbol{p}}$ and $\rho_{\boldsymbol{q}}$. In analogy with* (2.47)*, from* (3.20)*, we can thus interpret the revival of the classical internal information, $\Delta\mathcal{I}_{t,s}^{cl}(\boldsymbol{p}, \boldsymbol{q}; \mu) > 0$, between times s and $t \geq s$ as a classical BFI, which can occur only if a certain degree of quantum coherence has been built up to time s in the quantumly evolving classical states. Hence, in the classical reduction of the quantum evolution of pairs of commuting quantum states, quantum coherences play an information storing role as the environment does in the quantum scenario, as emerges by comparing the r.h.s of* (3.20) *and* (2.47).

3.3 Unital qubit dynamics

We now delve into the question (Q), namely whether P-divisibility of the quantum process is inherited by its classical reduction obtained through (3.3). In particular, for time-local evolutions as in (2.8), we shall argue that this problem becomes particularly relevant for purely dissipative dynamics, namely those not containing commutators with Hamiltonians in their generators, which would typically provide non-P-divisible classical processes.

In what follows, we will mostly focus on unital qubit dynamics, namely $\Lambda_t[\mathbb{1}_2] = \mathbb{1}_2$, for which the P-divisibility of both Λ_t and its classical reduction $T(t)$ is analytically tractable. Indeed, as shown in Proposition 2.7, P-divisibility of Λ_t is equivalent to the condition

$$-\widetilde{\mathcal{L}}^{\mathrm{s}}(t) = -\frac{\widetilde{\mathcal{L}}(t) + \widetilde{\mathcal{L}}^{\mathrm{T}}(t)}{2} \geq 0\,, \tag{3.23}$$

where $\widetilde{\mathcal{L}}_{ij}(t) = \mathrm{Tr}\left(\sigma_i \mathcal{L}_t[\sigma_j]\right)$ is the 3×3 Bloch representation of the generator $\mathcal{L}_t$ (see Example 2.1).

On the other hand, let P_0, $P_1 = \mathbb{1}_2 - P_0$ be two orthogonal projectors generating a maximally Abelian subalgebra $\mathcal{P} \subset M_2(\mathbb{C})$. The 2×2 stochastic matrix $T(t)$ obtained from Λ_t by means of (3.3) then reads

$$T(t) = \begin{pmatrix} T_{00}(t) & 1 - T_{00}(t) \\ 1 - T_{00}(t) & T_{00}(t) \end{pmatrix}, \qquad T_{00}(t) = \mathrm{Tr}\left(P_0 \Lambda_t[P_0]\right)\,, \tag{3.24}$$

so that it is fully determined by one real function $0 \leq T_{00}(t) \leq 1$. In the Bloch representation, P_0 is identified by means of a unit real vector $\boldsymbol{r} \in \mathbb{R}^3$, $P_{\boldsymbol{r}} = (\mathbb{1} + \boldsymbol{r} \cdot \boldsymbol{\sigma})/2$. Then, the bistochasticity condition is recast as

$$0 \leq T_{00}(t) = \mathrm{Tr}\left(P_0 \Lambda_t[P_0]\right) = \frac{1}{2} + \frac{1}{2} \langle \boldsymbol{r} | \widetilde{\Lambda}(t) | \boldsymbol{r} \rangle \leq 1, \tag{3.25}$$

where $\widetilde{\Lambda}(t)$ is the 3×3 Bloch representation of Λ_t. If $T_{00}(t) \neq 1/2$, $T(t)$ is invertible and one computes the classical generator as

$$L(t) = \dot{T}(t) T(t)^{-1} = \frac{\dot{T}_{00}(t)}{2T_{00}(t) - 1} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \tag{3.26}$$

By means of Kolmogorov conditions found in Proposition 1.5, $L(t)$ generates a P-divisible classical process if and only if

$$f_t := \frac{\dot{T}_{00}(t)}{2T_{00}(t) - 1} \leq 0\,, \qquad \forall\, t \geq 0\,. \tag{3.27}$$

Taking the derivative of (3.25) and recalling that $\dot{\widetilde{\Lambda}}(t) = \widetilde{\mathcal{L}}(t)\widetilde{\Lambda}(t)$, one recasts (3.27) as

$$f_t = \frac{1}{2} \frac{\langle \boldsymbol{r} | \widetilde{\mathcal{L}}(t) \widetilde{\Lambda}(t) | \boldsymbol{r} \rangle}{\langle \boldsymbol{r} | \widetilde{\Lambda}(t) | \boldsymbol{r} \rangle} \leq 0\,, \qquad \forall\, t \geq 0\,. \tag{3.28}$$

Remark 3.6. *Let $d = 2$ and suppose that $T(t)$ satisfies a time-local master equation with generator $L(t)$. Consider the classical 2-Renyi entropy*

$$H_2(t) = -\log \langle \boldsymbol{p} | T^{\mathrm{T}}(t) T(t) | \boldsymbol{p} \rangle\,, \tag{3.29}$$

where $\boldsymbol{p}$ is a probability vector. Then, $H_2(t)$ grows monotonically if and only if

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \boldsymbol{p}|T^{\mathsf{T}}(t)T(t)|\boldsymbol{p}\rangle = 2\langle \boldsymbol{p}|T^{\mathsf{T}}(t)L^{\mathsf{S}}(t)T(t)|\boldsymbol{p}\rangle \leq 0\,,$$

where $L^{\mathsf{S}}(t) = (L(t) + L^{\mathsf{T}}(t))/2$. In particular, if $T(t)$ is bistochastic, $L^{\mathsf{S}}(t) = L(t)$ and $L_{00}(t) = L_{11}(t) = -L_{01} = -L_{10}(t)$ so that

$$2\langle \boldsymbol{p}|T^{\mathsf{T}}(t)L^{\mathsf{S}}(t)T(t)|\boldsymbol{p}\rangle = L_{00}(t)(2T_{00}(t)-1)^2(p_0-p_1)^2\,.$$

Therefore, if $T(t)$ is P-divisible, $L_{00}(t)$ is necessarily negative, so that $H_2(t)$ grows monotonically.

Example 3.1. *Further characterization of P-divisibility of classical reductions of qubit unital evolutions can be done as follows. Consider two basis of $\mathbb{C}^2$ related by*

$$|\tilde{\alpha}\rangle = \frac{1}{\sqrt{2}}\sum_j (-1)^{-j\alpha}|j\rangle\,, \qquad |\langle\tilde{\alpha}|j\rangle|^2 = \frac{1}{2}\,, \tag{3.30}$$

and let $\widetilde{\mathbb{P}}$ be the map that causes full decoherence with respect to basis $\{|\tilde{\alpha}\rangle\}_\alpha$. Moreover, choose ρ to be diagonal in such basis,

$$\rho = \tilde{p}_0|\tilde{0}\rangle\langle\tilde{0}| + \tilde{p}_1|\tilde{1}\rangle\langle\tilde{1}| = \frac{\tilde{p}_0+\tilde{p}_1}{2}\mathbb{1}_2 + \frac{\tilde{p}_0-\tilde{p}_1}{2}(|0\rangle\langle 1| + |1\rangle\langle 0|)\,.$$

Then, consider the evolution

$$\begin{aligned}\widetilde{\mathbb{P}}\circ\Lambda_t\circ\widetilde{\mathbb{P}}[\rho] &= \sum_\alpha\left(\frac{1}{2}\sum_{ij}\langle i|\Lambda_t[\rho]|j\rangle(-1)^{-\alpha(i-j)}\right)|\tilde{\alpha}\rangle\langle\tilde{\alpha}| \\ &= \tilde{p}_0(t)|\tilde{0}\rangle\langle\tilde{0}| + \tilde{p}_1(t)|\tilde{1}\rangle\langle\tilde{1}|\,,\end{aligned} \tag{3.31}$$

where,

$$\tilde{p}_0(t) = \frac{1+c_t}{2}\,,\quad \tilde{p}_1(t) = \frac{1-c_t}{2}\,,\qquad c_t := 2\,\mathrm{Re}\langle 0|\Lambda_t[\rho]|1\rangle\,. \tag{3.32}$$

On the other hand, from the l.h.s. of (3.31) *one has*

$$\tilde{\boldsymbol{p}}(t) = \widetilde{T}(t)\tilde{\boldsymbol{p}}\,,\qquad \widetilde{T}_{\alpha\beta}(t) = \langle\tilde{\alpha}|\Lambda_t\left[|\tilde{\beta}\rangle\langle\tilde{\beta}|\right]|\tilde{\alpha}\rangle\,. \tag{3.33}$$

Comparing (3.33) *and* (3.32),

$$c_t = \tilde{p}_0(t) - \tilde{p}_1(t) = (2\widetilde{T}_{00}(t)-1)c_0 \implies \dot{c}_t = 2\dot{\widetilde{T}}_{00}(t)c_0\,.$$

Assuming now that $\widetilde{T}(t)$ is invertible, we have

$$f_t = 2\frac{\mathrm{d}}{\mathrm{d}t}\log|c_t|\,.$$

In particular, the sign of f_t is controlled by the monotonicity of $|c_t|$ The latter, from (3.32), *is a function of the coherences of the state with respect to the $\{|i\rangle\}_i$ basis.*

We now study the behaviour of f_t under two paradigmatic examples of oppositely behaving dynamics. The first one is a purely unitary qubit rotation which cannot give rise to a classically P-divisible process, while the second is that of a purely dissipative Pauli dynamics for which P-divisibility of the dynamics is equivalent to that of its classical reduction.

Unitary qubit dynamics Consider the qubit Hamiltonian $H = 1/2\boldsymbol{\omega}\cdot\boldsymbol{\sigma}$, $\boldsymbol{\omega}\in\mathbb{R}^3$, and the evolution $\mathcal{U}_t[\rho] = U_t\,\rho\,U_t^\dagger$, $U_t = e^{-itH}$. The classical reduction of its generator $\mathcal{L}[\cdot] = -i[H,\cdot]$ to any commutative subalgebra $\mathcal{P}\subset M_2(\mathbb{C})$ vanishes. Indeed, $\mathrm{Tr}\,(PHQ) = \mathrm{Tr}\,(PQH) = 0$. However, the Bloch representation of $\mathcal{U}_t$ acts as 3×3 rotation matrix $\widetilde{\mathcal{U}}(t) = e^{t\widetilde{\mathcal{L}}}$ with generator

$$\widetilde{\mathcal{L}} = \omega\begin{pmatrix} 0 & -u_3 & u_2\\ u_3 & 0 & -u_1\\ -u_2 & u_1 & 0\end{pmatrix}, \tag{3.34}$$

where $\omega = \|\boldsymbol{\omega}\|$ and $\boldsymbol{u} = (\omega_1,\omega_2,\omega_3)/\omega\in\mathbb{R}^3$ is a unit vector. Since $\widetilde{\mathcal{L}}^3 = -\omega^2\widetilde{\mathcal{L}} = \omega^2\widetilde{\mathcal{L}}^{\mathsf{T}}$, one rewrites

$$\widetilde{\mathcal{U}}(t) = e^{t\widetilde{\mathcal{L}}} = \mathbb{1} + \frac{\sin(\omega t)}{\omega}\,\widetilde{\mathcal{L}} + \frac{1-\cos(\omega t)}{\omega^2}\,\widetilde{\mathcal{L}}^2\,. \tag{3.35}$$

Also, $\langle \boldsymbol{r}|\widetilde{\mathcal{L}}|\boldsymbol{r}\rangle = 0$, $\langle \boldsymbol{r}|\widetilde{\mathcal{L}}^2|\boldsymbol{r}\rangle = -\|\widetilde{\mathcal{L}}\,|\boldsymbol{r}\rangle\,\|^2$. Given a projector with Bloch vector $\boldsymbol{r}$, with θ the angle between the latter and $\boldsymbol{u}$, one has $\|\widetilde{\mathcal{L}}\,|\boldsymbol{r}\rangle\,\| = \omega\sin(\theta)$ and

$$T_{00}(t) = \mathrm{Tr}\left(P_0\,\mathcal{U}_t[P_0]\right) = \frac{1}{2}\left(1 + \langle \boldsymbol{r}|\widetilde{\mathcal{U}}_t|\boldsymbol{r}\rangle\right) = \frac{1}{2}(1+\cos^2(\theta)+\cos(\omega t)\sin^2(\theta))\,,$$

so that (3.28) is rewritten as follows,

$$f_t = -\frac{\omega}{2}\frac{\sin(\omega t)\sin^2(\theta)}{\cos^2(\theta)+\cos(\omega t)\sin^2(\theta)}\,. \tag{3.36}$$

Thus, the sign of f_t changes for all $\theta\in(0,\pi)$, unless $\theta = 0,\pi$ (when $|\boldsymbol{r}\rangle$ corresponds to an eigenstate of H) yielding $f_t = 0$. Notice that the denominator in f_t is the determinant of the classical stochastic matrix $T(t)$ which is thus invertible if $\cos(2\theta) > 0$, namely, if $\theta\in(0,\pi/4)\cup(3\pi/4,\pi)$. In this case, classical P-divisibility breaks when $\dot{T}_{00}(t)$ becomes positive. As described in Section 3.2.1, such loss of P-divisibility for an invertible classical reduction $T(t)$ can be interpreted in terms of coherence-assisted BFI.

Pauli dynamics Consider the Pauli generator

$$\mathcal{L}_t^P[\rho] = \frac{1}{2}\sum_{k=1}^{3}\gamma_k(t)(\sigma_k\rho\sigma_k - \rho). \tag{3.37}$$

Notice that $\left[\mathcal{L}_t^P,\mathcal{L}_s^P\right] = 0$, yielding the exponential solution $\Lambda_t = e^{\int_0^t \mathrm{d}s\mathcal{L}_s^P}$. As seen in Example 2.1, the Bloch representation is a diagonal matrix

$$\widetilde{\mathcal{L}}_{ij}^P(t)[\rho] = -\Gamma_t^{(i)}\,\delta_{ij}\,,\qquad \Gamma_t^{(i)} = \sum_{k\neq i}\gamma_t^{(k)}\,, \tag{3.38}$$

so that necessary and sufficient conditions for P-divisibility easily follow from (3.23),

$$\Gamma_t^{(k)}\geq 0\,,\quad k = 1,2,3\,. \tag{3.39}$$

$\widetilde{\Lambda}(t)$ is also diagonal, with strictly positive eigenvalues

$$\widetilde{\Lambda}_{ij}(t) = \lambda_t^{(i)}\,\delta_{ij}\,,\qquad \lambda_t^{(i)} = e^{-\int_0^t \mathrm{d}s\Gamma_i(t)}\,.$$

so that

$$f_t = \frac{1}{2}\frac{\langle \boldsymbol{r}|\widetilde{\mathcal{L}}(t)\widetilde{\Lambda}(t)|\boldsymbol{r}\rangle}{\langle \boldsymbol{r}|\widetilde{\Lambda}(t)|\boldsymbol{r}\rangle} = -\frac{1}{2}\frac{\sum_i \Gamma_i(t)\lambda_i(t)n_i^2}{\sum_j \lambda_j(t)n_j^2} \leq 0 \tag{3.40}$$

for all $\boldsymbol{r}$, $\|\boldsymbol{r}\| = 1$. Thus, Λ_t is P-divisible if and only if $T(t)$ is P-divisible for all choices of the reference MASA. Adding a Hamiltonian to a Pauli generator $\mathcal{L}_t^P$ as in (3.37) may generally lead to a positive f_t, so that the process $T(t)$ is not P-divisible. Indeed, consider

$$\mathcal{L}_t[\rho] = -i[\sigma_z, \rho] + \mathcal{L}_t^P[\rho], \tag{3.41}$$

and take, for the sake of simplicity, $\Gamma_1(t) = \Gamma_2(t) \equiv \Gamma(t)$, so that the Hamiltonian generator commutes with the Pauli one. Hence, the evolution will have the form

$$\Lambda_t = \mathcal{U}_t \circ \Lambda_t^P \implies \widetilde{\Lambda}(t) = \widetilde{\mathcal{U}}(t)\,\widetilde{\Lambda}^P(t)\,. \tag{3.42}$$

Using (3.28) and (3.35), it follows that $f_t = f_t^P + f_t^H$, where f_t^P is as in (3.40) and always negative, while f_t^H is as in (3.36) and generally oscillates between positive and negative values. In particular, P-divisibility is lost due to the divergence of the f_t^H for some $\boldsymbol{r}$. On the other hand, for those $\boldsymbol{r}$ that lead to an invertible $T(t)$, f_t^H is bounded, so that f_t can stay negative for all times for large enough Pauli rates $\Gamma_t^{(i)}$.

The previous examples suggest that the presence of the commutator in the generator is in general responsible for the loss of P-divisibility for the classically reduced process defined by (3.3), while the classical reduction of Pauli dynamics preserves P-divisibility. Therefore, in order to study which P-divisible quantum dynamics Λ_t give certainly rise to classically divisible processes defined by (3.3), one better focus upon purely dissipative generators. Let us express the canonical form (2.9) of the dissipator of a qubit evolution in the Pauli basis, with dissipative part

$$\mathcal{D}_t[\rho] = \frac{1}{2}\sum_{i,j=1}^{3} K_{ij}(t)\left(\sigma_i\rho\sigma_j - \frac{1}{2}\{\sigma_j\,\sigma_i, \rho\}\right)\,. \tag{3.43}$$

Moreover, imposing that the generated dynamics is unital helps in controlling the purely dissipative nature of the evolution.

Lemma 3.2. *Consider a qubit dynamics with generator $\mathcal{L}_t$ with dissipative part as in* (3.43). *Then, the dynamics is unital if and only if the Kossakowski matrix $K(t)$ is real and symmetric, so that $\mathcal{D}_t$ can be written in the diagonal form*

$$\mathcal{D}_t[\rho] = \frac{1}{2}\sum_i \gamma_i(t)\,(\sigma_i(t)\,\rho\,\sigma_i(t) - \rho)\,, \tag{3.44}$$

with respect to a set of (generally time dependent) Pauli matrices $\{\sigma_i(t)\}_{i=1}^3$. Moreover, the unital dynamics is purely dissipative, namely $\mathcal{L}_t = \mathcal{D}_t$, if and only if the Bloch-representation of the generator is a real symmetric matrix, $\mathcal{L}(t) = \mathcal{L}^{\mathsf{T}}(t) = \mathcal{L}^{\mathsf{S}}(t)$.

Proof. Let us recall that Λ_t is unital if and only if $\mathcal{L}_t[\mathbb{1}] = 0$ for all $t \geq 0$. Indeed, if Λ_t is unital, $0 = \dot{\Lambda}_t[\mathbb{1}] = \mathcal{L}_t \circ \Lambda_t[\mathbb{1}] = \mathcal{L}_t[\mathbb{1}]$; conversely, if the generator kills the identity at all times, unitality follows from the Dyson series (2.17). Hence, in the qubit case, plugging the identity into (3.43) Λ_t is unital iff

$$\sum_{i,j=1}^{3} K_{ij}(t)[\sigma_i, \sigma_j] = 0, \tag{3.45}$$

which is satisfied iff $K_{ij}(t) = K_{ji}(t) = \overline{K_{ij}}(t)$. Then, the real symmetric Kossakowski matrix $K(t)$ can be diagonalized by means of an orthogonal transformation,

$$K(t) = O(t)\gamma(t)O(t)^T , \qquad \gamma(t) = \mathrm{diag}\{\gamma_1(t), \gamma_2(t), \gamma_3(t)\} .$$

Therefore, $O(t)O^T(t) = O^T(t)O(t) = \mathbb{1}$ guarantees that the matrices $\sigma_i(t) = \sum_j O_{ji}(t)\sigma_j$ satisfy the Pauli algebra. Moreover, the Bloch representation of the dissipative part $\mathcal{D}_t$ of the generator reads

$$\widetilde{\mathcal{D}}_{ij}(t) = \frac{1}{2}\sum_k \gamma_k(t)\left[\mathrm{Tr}\left(\sigma_i\sigma_k(t)\sigma_j\sigma_k(t)\right) - 2\delta_{ij}\right] = \widetilde{\mathcal{D}}_{ji}(t) , \tag{3.46}$$

where the second equality follows from the cyclicity of the trace. Thus, an anti-symmetric contribution to the matrix representation of the full generator $\mathcal{L}(t)$ can only come from the commutator with the Hamiltonian. It follows that the unital dynamics is purely dissipative, namely that $\widetilde{\mathcal{L}}(t) = \widetilde{\mathcal{D}}(t)$, if and only if $\widetilde{\mathcal{L}}(t) = \widetilde{\mathcal{L}}^{\mathsf{T}}(t)$. □

Remark 3.7.

1. *Λ_t is called self-dual if $\Lambda_t = \Lambda_t^\ddagger$, $\Lambda_t^\ddagger$ being its dual in the Heisenberg picture obtained through* (2.3). *Note that a self-dual evolution is necessarily unital, while the reverse is obviously not true. When Λ_t is a qubit dynamical map, going to the Bloch representation, one then has that self-duality is equivalent to $\widetilde{\Lambda}(t) = \widetilde{\Lambda}^{\mathsf{T}}(t)$. On the other hand, if the generator $\mathcal{L}_t$ of the qubit dynamics Λ_t is self-dual, $\mathcal{L}_t = \mathcal{L}_t^\ddagger$, it must be purely dissipative, $\mathcal{L}_t = \mathcal{D}_t$, and Λ_t unital. Indeed, $\mathcal{L}_t[\mathbb{1}] = \mathcal{L}_t^\ddagger[\mathbb{1}] = 0$ yields a symmetric Kossakowski matrix which in turn implies the absence of the commutator with a Hamiltonian. The unitality of Λ_t then follows from its Dyson expansion. Moreover, in the qubit case, Lemma 3.2 implies that a purely dissipative generator of a unital dynamics is necessarily self-dual. In the following Section 3.4, we shall provide a concrete qubit construction for a non self-dual unital dynamics arising from a self-dual generator.*

2. *If the solution is of the exponential form $\Lambda_t = e^{\int_0^t \mathrm{d}s\mathcal{L}_s}$, self-duality of the generator implies that of Λ_t. In fact,*

$$\Lambda_t^\ddagger = \left(e^{\int_0^t \mathrm{d}s\mathcal{L}_s}\right)^\ddagger = \left(\sum_{k=0}^{\infty}\frac{1}{k!}\left(\int_0^t \mathrm{d}s\,\mathcal{L}_s\right)^k\right)^\ddagger = \sum_{k=0}^{\infty}\frac{1}{k!}\int_0^t \mathrm{d}s_1 \cdots \int_0^t \mathrm{d}s_k\left(\mathcal{L}_{s_1}\circ\cdots\circ\mathcal{L}_{s_k}\right)^\ddagger$$
$$= \sum_{k=0}^{\infty}\frac{1}{k!}\int_0^t \mathrm{d}s_k\,\mathcal{L}_{s_k}^\ddagger\circ\cdots\circ\int_0^t \mathrm{d}s_1\,\mathcal{L}_{s_1}^\ddagger = \sum_{k=0}^{\infty}\frac{1}{k!}\left(\int_0^t \mathrm{d}s\,\mathcal{L}_s^\ddagger\right)^k = e^{\int_0^t \mathrm{d}s\mathcal{L}_s^\ddagger} .$$

From Lemma 3.2 follows that if Λ_t is unital and purely dissipative, $\widetilde{\mathcal{L}}(t)$ is real symmetric and then, from (3.23), P-divisibility of Λ_t becomes equivalent to $-\widetilde{\mathcal{L}}(t) \geq 0$. Having noted that, the fact that the P-divisibility of Pauli dynamics is preserved by their classical reductions can be generalized to self-dual qubit dynamics with self-dual generators.

Proposition 3.3. *Let $\Lambda_t = \Lambda_t^\ddagger$ be a self-dual, purely dissipative, invertible qubit dynamics. Then, Λ_t is P-divisible if and only if all its classical reductions are P-divisible.*

Proof. One has to check when (3.28) holds. Since $\Lambda_t = \Lambda_t^\ddagger$, this implies that their Bloch representations satisfy

$$\widetilde{\Lambda}(t) = \widetilde{\Lambda}^{\mathrm{T}}(t)\,. \tag{3.47}$$

Also, the assumed pure dissipativeness of Λ_t entails that the generator itself is self-dual, $\mathcal{L}_t = \mathcal{L}_t^\ddagger$, so that $\widetilde{\mathcal{L}}^{\mathrm{T}}(t) = \widetilde{\mathcal{L}}(t)$ (see Lemma 3.2). Then, taking the time derivative of both members of (3.47), one gets

$$\dot{\widetilde{\Lambda}}(t) = \widetilde{\mathcal{L}}(t)\widetilde{\Lambda}(t) = \widetilde{\Lambda}(t)\widetilde{\mathcal{L}}(t)\,, \tag{3.48}$$

or, equivalently, $[\widetilde{\Lambda}(t), \dot{\widetilde{\Lambda}}(t)] = 0$. The invertibility of Λ_t means that none of the real eigenvalues of $\widetilde{\Lambda}(t) = \widetilde{\Lambda}^{\mathrm{T}}(t)$ can change sign with varying t. Since, $\widetilde{\Lambda}(t = 0) = \mathbb{1}$ all of them must remain positive. Therefore, one can consistently express the generator as a logarithmic derivative:

$$\widetilde{\mathcal{L}}(t) = \dot{\widetilde{\Lambda}}(t)\widetilde{\Lambda}(t)^{-1} = \widetilde{\Lambda}(t)^{-1}\dot{\widetilde{\Lambda}}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\log\widetilde{\Lambda}(t)\,. \tag{3.49}$$

Thus, $\widetilde{\Lambda}(t) = e^{\int_0^t \mathrm{d}s\widetilde{\mathcal{L}}(s)}$ without time-ordering; furthermore,

$$f_t = -\frac{1}{2}\frac{\left\langle \boldsymbol{n}\middle|\sqrt{\widetilde{\Lambda}(t)}\left(-\widetilde{\mathcal{L}}(t)\right)\sqrt{\widetilde{\Lambda}(t)}\middle|\boldsymbol{n}\right\rangle}{\left\langle \boldsymbol{n}\middle|\widetilde{\Lambda}(t)\middle|\boldsymbol{n}\right\rangle} \le 0\,. \tag{3.50}$$

Then, from Proposition 2.7, the dynamics Λ_t is P-divisible iff $-\widetilde{\mathcal{L}}(t) \geq 0$, that is equivalent to $f_t \leq 0$ and so to P-divisibility of $T(t)$. □

Remark 3.8. *The assumptions in Proposition 3.3 lead to a proper exponential solution* $\widetilde{\Lambda}(t) = e^{\int_0^t \mathrm{d}s\widetilde{\mathcal{L}}(s)}$ *without asking for commuting generators at different times as for Pauli maps. Indeed, a sufficient condition for the exponential to solve* $\dot{\widetilde{\Lambda}}(t) = \widetilde{\mathcal{L}}(t)\widetilde{\Lambda}(t)$ *is that* $\left[\widetilde{\mathcal{L}}(t)\,,\,\int_0^t \mathrm{d}s\,\widetilde{\mathcal{L}}(s)\right] = 0$ *for all* $t \geq 0$ *(notice that, from the power series of the exponential,* (3.48) *also follows). As an example, consider a generator of a unital dynamics of the form*

$$\widetilde{\mathcal{L}}(t) = \begin{cases} \begin{pmatrix} -1 & \cos t & 0 \\ \cos t & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix} & 0 \leq t \leq \pi \\ -\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \pi \leq t \end{cases}\,.$$

$\widetilde{\mathcal{L}}(t)$ *and* $\widetilde{\mathcal{L}}(s)$ *do not commute when* $0 \leq s \leq \pi$ *and* $t > \pi$, *while they do when either* $0 \leq s, t \leq \pi$ *or* $s, t > \pi$; *on the other hand, since*

$$\int_0^t \mathrm{d}s\,\widetilde{\mathcal{L}}(s) = \begin{pmatrix} -t & \sin t & 0 \\ \sin t & -t & 0 \\ 0 & 0 & -2t \end{pmatrix}$$

when $0 \leq t \leq \pi$ *and, when* $t \geq \pi$,

$$\int_0^t \mathrm{d}s\, \widetilde{\mathcal{L}}(s) = -\pi \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} - (t-\pi) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

the condition $\left[\widetilde{\mathcal{L}}(t), \int_0^t \mathrm{d}s\, \widetilde{\mathcal{L}}(s)\right] = 0$*, for all* $t \geq 0$*, that avoids time-ordering is fulfilled. Moreover,* $\widetilde{\mathcal{L}}(t)$ *is symmetric (self-dual generator), so that the dynamics is purely dissipative. The exponential solution then entails the self-duality of the map.* $-\widetilde{\mathcal{L}}(t) \geq 0$ *finally assures P-divisibility of the generated dynamics, so that all the hypothesis of Proposition 3.3 are matched.*

3.4 A class of orthogonally covariant dynamics

In the previous Section, we saw that quantum P-divisibility is not preserved under classical reductions of qubit unitary evolutions with a constant Hamiltonian, whereas it is preserved when classically reducing some paradigmatic examples of purely dissipative generators. On the other hand, in Section 3.2.1, we showed that, in order for classical BFI to emerge, it is necessary that the quantum P-divisible evolution builds up sufficient coherence. We shall now see that this mechanism can also be driven by pure dissipation: namely, P-divisibility of purely dissipative quantum evolutions can be lost in the reduction procedure, thus triggering classical BFI. In the context of qubit unital maps, this scenario will translate into the mathematical condition of a self-dual generator, giving rise to a non-self dual dynamics with non-trivial time ordering. To find such instance, it proves convenient to study the following family of maps,

$$\Phi^{(A,\lambda,\mu)}[\rho] = \sum_{i,j=0}^{1} A_{ij} E_{ij} \rho E_{ji} + \lambda E_{00} \rho E_{11} + \overline{\lambda} E_{11} \rho E_{00} + \mu E_{11} \rho^{\top} E_{00} + \overline{\mu} E_{00} \rho^{\top} E_{11} \,, \quad (3.51)$$

where $E_{ij} = |i\rangle\langle j|$ are the matrix units in the eigenbasis of σ_3, $\sigma_3 |0\rangle = |0\rangle, \sigma_3 |1\rangle = -|1\rangle$. These maps depend parametrically on a triple (A, λ, μ), given by a 2×2 matrix $A = [A_{ij}]$ and coefficients $\lambda, \mu \in \mathbb{C}$. They were studied in the d-dimensional case in [120]. In particular, the following group composition law holds,

$$\Phi^{(A,\lambda,\mu)} \circ \Phi^{(A',\lambda',\mu')} = \Phi^{(AA',\, \lambda\lambda' + \mu\overline{\mu'},\, \lambda\mu + \mu\overline{\lambda'})} \,. \quad (3.52)$$

Moreover, if $\mu = 0$, maps $\Phi^{(A,\lambda,0)}$ satisfy the diagonal unitary-covariance property

$$\Phi^{(A,\lambda,0)}[UXU^{\dagger}] = U\Phi^{(A,\lambda,0)}[X]U^{\dagger} \,, \quad U = \sum_i e^{i\theta_i} E_{ii} \,.$$

Conversely, if $\lambda = 0$, the maps $\Phi^{(A,0,\mu)}$ are conjugate diagonal unitary-covariant, namely

$$\Phi^{(A,0,\mu)}[UXU^{\dagger}] = \overline{U}\Phi^{(A,0,\mu)}[X]U^{\top} \,.$$

Lastly, if both λ, μ are different from zero, the only symmetry left is with respect to rotations around the z axis, corresponding to the diagonal orthogonal covariance

$$\Phi^{(A,\lambda,\mu)}[OXO^{\top}] = O\Phi^{(A,\lambda,\mu)}[X]O^{\top} \,,$$

where $O = \sum_i o_i E_{ii}$, $o_i \in \{-1, 1\}$. Consider a time-dependent family of Hermiticity and trace-preserving maps within the above class, $\Lambda_t = \Phi^{(A(t),\lambda_t,\mu_t)}$. Then, $A(t)$ is to be taken real and of the form $A_{ij}(t) \in \mathbb{R}$ and $\sum_i A_{ij}(t) = 1$, so that

$$A(t) = \begin{pmatrix} a_t & 1-b_t \\ 1-a_t & b_t \end{pmatrix}, \quad a_t, b_t \in \mathbb{R}\,. \tag{3.53}$$

Asking that $\Lambda_{t=0} = \mathrm{id}_2$ yields $\lambda_0 = 1, \mu_0 = 0$ and $a_0 = 1$; whereas positivity can be proved to be equivalent to [120]

$$A_{ij}(t) \geq 0\,, \qquad |\lambda_t| + |\mu_t| \leq \sqrt{a_t b_t} + \sqrt{(1-a_t)(1-b_t)}\,. \tag{3.54}$$

so that $A(t)$ has to be a stochastic matrix. On the other hand, the conditions for complete positivity can be obtained by imposing positivity of the associated Choi matrix (1.93), leading to the stronger necessary and sufficient conditions

$$|\lambda_t| \leq \sqrt{a_t b_t}\,, \quad |\mu_t| \leq \sqrt{(1-a_t)(1-b_t)}\,. \tag{3.55}$$

Recently, Schwartz-positivity (1.85) for maps within this class has also been fully characterized [121]. Sufficient conditions for complete positivity in terms of the generator for this class of maps were also investigated in [122]. We shall now characterize the divisibility properties of the dynamics within class (3.51). Due to the composition property (3.52), the generator $\mathcal{L}_t = \dot{\Lambda}_t \circ \Lambda_t^{-1}$ will itself be of the type $\mathcal{L}_t = \Phi^{(B(t),\ell_t,m_t)}$, with $B(t)$ of the form

$$B(t) = \dot{A}(t)A(t)^{-1} = \begin{pmatrix} -\gamma_-(t) & \gamma_+(t) \\ \gamma_-(t) & -\gamma_+(t) \end{pmatrix}, \tag{3.56}$$

where $\gamma_-(t)$, $\gamma_+(t)$ are related to a_t, b_t via

$$\gamma_-(t) = -\frac{\dot{a}_t(1-b_t) + \dot{b}_t a_t}{a_t + b_t - 1}, \qquad \gamma_+(t) = -\frac{\dot{a}_t b_t + \dot{b}_t(1-a_t)}{a_t + b_t - 1}\,,$$

and l_t, m_t are related to λ_t, μ_t via

$$l_t = \frac{\dot{\lambda}_t\overline{\lambda_t} - \dot{\mu}_t\overline{\mu_t}}{|\lambda_t|^2 - |\mu_t|^2}\,, \qquad m_t = \frac{\dot{\mu}_t\lambda_t - \dot{\lambda}_t\mu_t}{|\lambda_t|^2 - |\mu_t|^2}\,. \tag{3.57}$$

It is convenient to introduce time-dependent "transversal" and "longitudinal" rates,

$$\Gamma_T(t) := -\mathrm{Re}(l_t)\,, \qquad \Gamma_L(t) := \gamma_+(t) + \gamma_-(t)\,, \tag{3.58}$$

and let also $\omega(t) := -\mathrm{Im}(l_t)$. Notice that $\Gamma_T(t)$ only depends on the absolute values of λ_t and μ_t,

$$\Gamma_T(t) = -\frac{1}{2}\frac{\partial_t|\lambda_t|^2 - \partial_t|\mu_t|^2}{|\lambda_t|^2 - |\mu_t|^2}\,. \tag{3.59}$$

(C)P-divisibility of the dynamics can be then fully characterized from the generator as follows.

Proposition 3.4. *Let $\mathcal{L}_t = \Phi^{(B(t),l_t,m_t)}$ be the generator of an evolution $\Lambda_t = \Phi^{(A(t),\lambda_t,\mu_t)}$. Then,*

1. *Λ_t is P-divisible if and only if*

$$\gamma_\pm(t) \geq 0\,, \tag{3.60}$$

$$\Gamma_T(t) - \frac{\Gamma_L(t)}{2} + \sqrt{\gamma_+(t)\gamma_-(t)} \geq |m_t|\,; \tag{3.61}$$

2. *Λ_t is CP-divisible if and only if*

$$\gamma_\pm(t) \geq 0\,, \qquad \gamma_+(t)\gamma_-(t) \geq |m_t|^2\,, \tag{3.62}$$

$$\Gamma_T(t) \geq \frac{\Gamma_L(t)}{2}\,. \tag{3.63}$$

Though in a different form, this result has already been obtained in [123]; in Appendix A.1 we report the proof, since it is somewhat simpler than in the quoted reference. Notice that condition (3.63) is the time-dependent version of the celebrated constraint between relaxation rates already discussed in seminal paper [4], which has been recently proved to be universal for all completely positive dynamical semigroups [124, 125].

Remark 3.9. *The Bloch representation of Λ_t leads to the following dynamics of the Bloch vector:*

$$\boldsymbol{n}_t = \widetilde{\Lambda}(t)\boldsymbol{n} + \boldsymbol{u}_t\,, \qquad \boldsymbol{u}_t = (0, 0, a_t - b_t)\,, \tag{3.64}$$

with

$$\widetilde{\Lambda}(t) = \begin{pmatrix} \mathrm{Re}(\lambda_t + \mu_t) & \mathrm{Im}(\lambda_t - \mu_t) & 0 \\ -\mathrm{Im}(\lambda_t + \mu_t) & \mathrm{Re}(\lambda_t - \mu_t) & 0 \\ 0 & 0 & a_t + b_t - 1 \end{pmatrix}.$$

The block diagonal structure of $\widetilde{\Lambda}(t)$ makes evident the orthogonal covariance of the map with respect to orthogonal transformations diagonal in the σ_3 basis. The generator can also be rewritten in GKLS form as in (3.43), *with Hamiltonian*

$$H(t) = \frac{\omega(t)}{2}\sigma_3\,. \tag{3.65}$$

Furthermore, letting $\kappa(t) = -\mathrm{Re}(m_t)$, $\eta(t) = -\mathrm{Im}(m_t)$ and $\delta(t) = (\gamma_+(t) - \gamma_-(t))/2$, the Kossakowski matrix can be written as

$$K(t) = \begin{pmatrix} \frac{\Gamma_L(t)}{2} - \kappa(t) & \eta(t) + i\delta(t) & 0 \\ \eta(t) - i\delta(t) & \frac{\Gamma_L(t)}{2} + \kappa(t) & 0 \\ 0 & 0 & \Gamma_T(t) - \frac{\Gamma_L(t)}{2} \end{pmatrix}.$$

Notice that CP-divisibility conditions can be then read off asking for $K(t) \geq 0$. If Λ_t is also unital, $A(t)$ has to be bistochastic, thus $a_t = b_t$ and

$$\gamma_+(t) = \gamma_-(t) = \frac{\Gamma_L(t)}{2} = \frac{\dot{a}_t}{2a_t - 1}\,,$$

so that the P-divisibility conditions simplify to

$$\Gamma_L(t) \geq 0\,, \qquad \Gamma_T \geq |m_t|\,. \tag{3.66}$$

Indeed, the generator $\widetilde{\mathcal{L}}(t)$ will also be block-diagonal, and the conditions (3.66) *follow from $-(\widetilde{\mathcal{L}}(t) + \widetilde{\mathcal{L}}^{\mathsf{T}}(t)) \geq 0$. On the other hand, the conditions for CP-divisibility in the unital case simplify to*

$$\Gamma_T(t) \geq \frac{\Gamma_L(t)}{2} \geq |m_t|\,. \tag{3.67}$$

3.4.1 Non-self dual dynamics from self-dual generator

We now turn to the construction of a class of maps that fulfils the requirements listed at the beginning of Section 3.4; namely, we want a unital map which (1) is P-divisible, (2) arises from a self-dual generator but (3) is not itself self-dual. These requirements are not compatible with the exponential form, as discussed in Remark 3.7. The construction will then serve to a construct a purely dissipative evolution whose classical reductions are not P-divisible.

Step 1: Non Self-dual dynamics. From the general form (3.51), the dual of $\Phi^{(A,\lambda,\mu)}$ acts on a matrix $X \in M_2(\mathbb{C})$ as

$$(\Phi^{(A,\lambda,\mu)})^{\ddagger}[X] = \sum_{ij} A_{ji} E_{ij} X E_{ji} + \overline{\lambda} E_{00} X E_{11} + \lambda E_{11} X E_{00} + \mu E_{00} X^{\mathsf{T}} E_{11} + \overline{\mu} E_{11} X^{\mathsf{T}} E_{00} \,.$$

Then, self-duality corresponds to $A = A^{\mathsf{T}}$ and $\lambda \in \mathbb{R}$. For imposing lack of self-duality of the map Λ_t, it is sufficient to choose $\Lambda_t = \Phi^{(A(t),\lambda_t,\mu_t)}$, with $\lambda_t = |\lambda_t| e^{i\varphi_t}$ $\mu_t = |\mu_t| e^{i\theta_t}$, $\varphi_t, \theta_t \in \mathbb{R}$ and $\varphi_t \neq 0$.

Step 2: Pure dissipativity. For qubit unital dynamics, purely dissipative generator is equivalent to $\mathcal{L}_t = \mathcal{L}_t^{\ddagger}$, as shown in Lemma 3.2. The generator belongs itself to the class (3.51), $\mathcal{L}_t = \Phi^{(B(t),l_t,m_t)}$. Hence, $\mathcal{L}_t = \mathcal{L}_t^{\ddagger}$ if and only if $\gamma_+(t) = \gamma_-(t)$, implying unitality of the generated dynamics, and

$$l_t = -\Gamma_T(t) \,, \qquad \omega(t) = 0 \,, \tag{3.68}$$

namely, l_t is real. Then, (3.57) and (3.68) imply the following relation between the phases and the moduli of λ_t and μ_t of $\Lambda_t = \Phi^{(A(t),\lambda_t,\mu_t)}$,

$$\dot{\varphi}_t |\lambda_t|^2 = \dot{\theta}_t |\mu_t|^2 \,. \tag{3.69}$$

Notice that this fixes $\dot{\varphi}_{t=0} = 0$.

Step 3: Quantum P-divisibility. We now further impose P-divisibility of the generated dynamics. Let us define

$$g_t := |\lambda_t| + |\mu_t| \,, \qquad h_t := |\lambda_t| - |\mu_t| \,, \tag{3.70}$$

with $0 < h_t \leq g_t$, $g_0 = h_0 = 1$. Then, (3.59) can be recast as

$$\Gamma_T(t) = -\frac{1}{2}\frac{\partial_t(g_t h_t)}{g_t h_t} = -\frac{\dot{g}_t}{2g_t} - \frac{\dot{h}_t}{2h_t} \,, \tag{3.71}$$

which must be positive for P-divisibility. Substituting (3.70) and the dissipativity (3.69) into the second of (3.57), one recasts $|m_t|$ as

$$|m_t| = \left| \frac{\dot{g}_t}{2g_t} - \frac{\dot{h}_t}{2h_t} + i\,\dot{\theta}_t \frac{g_t - h_t}{g_t + h_t} \right| .$$

Taking the square of the inequality $\Gamma_T(t) \geq |m_t|$, the following necessary and sufficient conditions for P-divisibility can be finally found,

$$|\dot{\theta}_t|^2 \left(\frac{g_t - h_t}{g_t + h_t} \right)^2 \leq \frac{\dot{g}_t}{g_t} \frac{\dot{h}_t}{h_t} \,. \tag{3.72}$$

From (3.71) and (3.72) one deduces, in particular, that g_t, h_t must be monotonically decreasing.

Example 3.2. *Let $\beta \geq 0$ and define $\Lambda_t = \Phi^{(A(t),\lambda_t,\mu_t)}$ by*

$$|\lambda_t| = e^{-2t}\cosh(t)\,, \qquad \varphi_t = \beta\tanh^3(t)\,, \tag{3.73}$$

$$|\mu_t| = e^{-2t}\sinh(t)\,, \qquad \theta_t = 3\beta\tanh(t)\,, \tag{3.74}$$

$$a_t = b_t = e^{-t}\ \cosh(t)\,, \tag{3.75}$$

corresponding to the positive and monotonically decreasing functions $h_t = e^{-3t}$, $g_t = e^{-t}$. As one easily checks, (3.55) *are satisfied and the map is thus completely positive. On the other hand, the self-dual generator is given by*

$$l_t = -\Gamma_T(t) = -2\,, \qquad m_t = \sqrt{1 + r_t^2}\, e^{i(\theta_t+\varphi_t)}\,, \tag{3.76}$$

$$\gamma_+(t) = \gamma_-(t) = \frac{\Gamma_L(t)}{2} = 1\,, \tag{3.77}$$

where $r_t = 3\beta(1 - \tanh^2(t))\tanh(t)$. Notice that

$$|m_t| = \sqrt{1 + r_t^2} \geq 1 = \frac{\Gamma_L(t)}{2}\,,$$

so the dynamics is CP-divisible iff $\beta = 0$. On the other hand, the P-divisibility condition (3.72) *reduces to $r_t^2 \leq 3$. Since r_t reaches a maximum of $2\beta/\sqrt{3}$, the dynamics is P-divisible iff $\beta \leq 3/2$.*

3.4.2 Loss of classical P-divisibility out of pure dissipation

The construction of the previous Section provides a good candidate to investigate the presence of coherence-assisted BFI in its classical reductions. For the class $\Lambda_t = \Phi^{(A(t),\lambda_t,\mu_t)}$, clearly, there exists a preferred basis to perform the classical reduction, namely $\{E_{00}, E_{11}\}$. For this choice of MASA, the classical map $T(t)$ coincides with $A(t)$, whose P-divisibility is necessary to ensure that of Λ_t (see (3.60)).

We shall now see that choosing a different basis generally breaks P-divisibility of $T(t)$, even for purely dissipative generators built in Section 3.4.1. Moreover, for a suitable basis, this can occur with $T(t)$ being invertible for all $t \geq 0$. A classical reduction of $\Lambda_t = \Phi^{(A(t),\lambda_t,\mu_t)}$ builds upon fixing a rank-1 projector $P_{\boldsymbol{n}} = \sum_{i,j=0,1} P_{\boldsymbol{n}}^{(ij)} E_{ij}$ and considering

$$\begin{aligned} T_{00}(t) &= \mathrm{Tr}\,(P_{\boldsymbol{n}}\Lambda_t[P_{\boldsymbol{n}}]) \\ &= \sum_{ij} A_{ij}(t) P_{\boldsymbol{n}}^{(jj)} P_{\boldsymbol{n}}^{(ii)} + 2\,\mathrm{Re}(\lambda_t)|P_{\boldsymbol{n}}^{(01)}|^2 + 2\,\mathrm{Re}\left(\mu_t (P_{\boldsymbol{n}}^{(10)})^2\right)\,, \end{aligned} \tag{3.78}$$

which is sufficient to construct $T(t)$ in the unital case. Expressing $\boldsymbol{n}$ in polar coordinates, $\boldsymbol{n} = (\sin(\chi)\cos(\xi), \sin(\chi)\sin(\xi), \cos(\chi))$, (3.78) can be then recast as

$$T_{00}(t) = \frac{1}{2}\big(1 + (2a_t - 1)\cos^2(\chi) + |\lambda_t|\cos(\varphi_t)\sin^2(\chi) + |\mu_t|\sin(\chi)\cos(\theta_t + 2\xi)\big)\,. \tag{3.79}$$

As we already noted, for $\chi = 0$, $P = E_{00}$, $T_{00}(t) = a_t$, yielding a P-divisible process. Considering instead $\chi = \pi/2$, one has

$$T_{00}(t) = \frac{1}{2}\big(1 + |\lambda_t|\cos(\varphi_t) + |\mu_t|\cos(\theta_t + 2\xi)\big)\,. \tag{3.80}$$

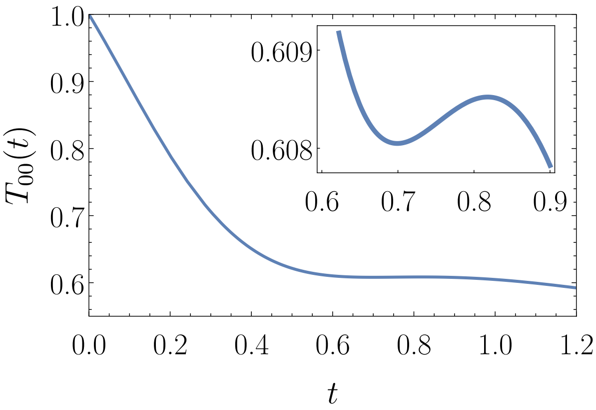

FIGURE 3.1: Plot of $T_{00}(t)$ from Example 3.2, with $\beta = 3/2$ and $\xi = \pi/4$ corresponding to the Bloch vector $(\sqrt{2}/2, \sqrt{2}/2, 0)$ which defines the reference classical basis.

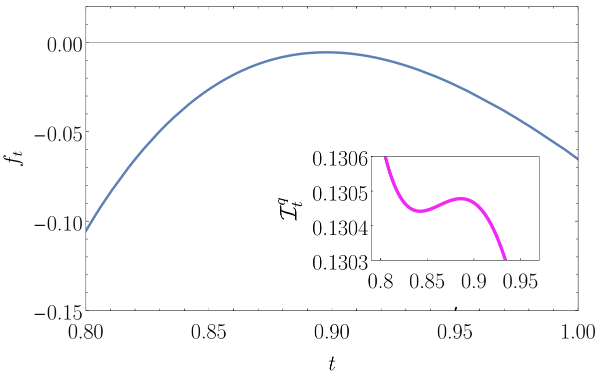

FIGURE 3.2: Global maximum of f_t as defined in (3.27), for the classical reduction of the map of Example 3.2, with $\beta = 1.64$ and reference classical algebra defined by the projector $P_{\boldsymbol{n}}$ with $\boldsymbol{n} = (\cos(\xi), \sin(\xi), 0)$ and $\xi = \pi/8$. Numerically, it is checked that f_t reaches a maximum of -0.006, so that $f_t < 0$ for all t and $T(t)$ is P-divisible. In the inset, the non-monotonic behaviour of $\mathcal{I}_t^q(P_{\boldsymbol{n}}, P_{-\boldsymbol{n}}; 1/2)$ is displayed, corresponding to BFI that thus involves only the coherences w.r.t. such basis.

Recalling that P-divisibility of the classical reduction amounts to checking condition (3.27), let $\xi = \pi/4$ and consider the map of Example 3.2,

$$2T_{00}(t) - 1 = e^{-2t}\cosh(t)\cos(\beta\tanh^3(t)) - e^{-2t}\sinh(t)\sin(3\beta\tanh(t)) . \tag{3.81}$$

Choosing $\beta = 3/2$, (3.72) saturates for some t. Noting that $0 \leq \varphi \leq \pi/2$ and $0 \leq \theta \leq 3\pi/2$, so that $\sin(\theta_t) < 0 \;\; \forall\, t > \text{artanh}(2\pi/9)$, one can verify that $2T_{00}(t) - 1 > 0$, $\forall t \geq 0$. Thus, $T(t)$ is invertible. Nevertheless, as displayed in Fig. 3.1, it is not monotonically decreasing, since $\dot{T}_{00}(t)$ can become positive. Hence, f_t also becomes positive implying that $T(t)$ is not P-divisible.

Remark 3.10. *Our previous considerations focussed on checking when P-divisibility of a purely dissipative qubit dynamical map is inherited by its classical reduction and found that in some cases it can be lost, namely the classical reduction can become non P-divisible. We now like to comment that the* *contrary can also occur; namely, the classical reduction of a non P-divisible purely dissipative qubit dynamics can become P-divisible. In Fig. 3.2, an instance of such behaviour is shown by the maps of Example 3.2 for a suitable* $\kappa > 3/2$ *for which* Λ_t *is not P-divisible. Interestingly, the corresponding backflow of information can be witnessed by suitable orthogonal projections* $P_{\pm\boldsymbol{n}}$ *such that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|\Lambda_t[P_{\boldsymbol{n}} - P_{-\boldsymbol{n}}]\|_1 = \frac{\mathrm{d}}{\mathrm{d}t}\|\widetilde{\Lambda}(t)\boldsymbol{n}\| > 0\,,$$

for some $t > 0$*, as shown by the non-monotonic behaviour of* $\mathcal{I}_t^q(P_{\boldsymbol{n}}, P_{-\boldsymbol{n}}; 1/2) = \|\widetilde{\Lambda}(t)\boldsymbol{n}\|$ *in the inset of Fig. 3.2. Nevertheless, the stochastic process* $T(t)$ *defined through the subalgebra* $\mathcal{P}_{\boldsymbol{n}}$ *generated by them is P-divisible. This means that, unlike the quantum dynamics it originates from, such a* $T(t)$ *cannot exhibit classical backflow of information. The orthogonal projections* $P_{\pm\boldsymbol{n}}$ *provide classical probabilities and, concerning information backflow, they are concrete instances of the following behaviour. Suppose that there exist* $\rho_{\boldsymbol{p}}, \rho_{\boldsymbol{q}}$ *in a commutative subalgebra* $\mathcal{P}$*, that is classical probability distributions, and* $t > s > 0$ *such that*

$$\begin{aligned}&\mathcal{I}_t^q(\rho_{\boldsymbol{p}}, \rho_{\boldsymbol{q}}; \mu) > \mathcal{I}_s^q(\rho_{\boldsymbol{p}}, \rho_{\boldsymbol{q}}; \mu)\\ &\qquad\Longleftrightarrow\ \mathcal{I}_t^{cl}(\boldsymbol{p}, \boldsymbol{q}; \mu) + \mathcal{C}_t(\boldsymbol{p}, \boldsymbol{q}; \mu) > \mathcal{I}_s^{cl}(\boldsymbol{p}, \boldsymbol{q}; \mu) + \mathcal{C}_s(\boldsymbol{p}, \boldsymbol{q}; \mu)\,. \qquad (3.82)\end{aligned}$$

In addition, suppose that the classical dynamics $T(t)$*, obtained by restricting the quantum dynamics* Λ_t *on the same subalgebra* $\mathcal{P}$*, is P-divisible, yielding thus*

$$\mathcal{C}_t(\boldsymbol{p}, \boldsymbol{q}; \mu) - \mathcal{C}_s(\boldsymbol{p}, \boldsymbol{q}; \mu) > \mathcal{I}_s^{cl}(\boldsymbol{p}, \boldsymbol{q}; \mu) - \mathcal{I}_t^{cl}(\boldsymbol{p}, \boldsymbol{q}; \mu) \geq 0\,. \qquad (3.83)$$

Therefore, the backflow of information from the environment to the open system only affects the coherent contribution to $\mathcal{I}_t^q$ *and cannot be witnessed by the classical reduced map.*

This Chapter was devoted to studying whether and how the divisibility properties of time-dependent, non-Markovian qubit quantum dynamics are inherited by their classical reductions. The investigation was carried out by classically reducing the dynamics and then studying the generator of the classical reduction. In Section 3.2.1, we have argued that the lack of P-divisibility in classical reductions can be ascribed to information being created by the quantum dynamics, stored in the quantum coherences, and then released back into the classical component of the dynamics. In addition, for P-divisible and unital qubit dynamics we showed that (1) the Hamiltonian contributions to the time-dependent quantum generators generally give rise to qubit quantum dynamics with non-P-divisible classical reductions; (2) purely dissipative, self-dual dynamics always have P-divisible classical reductions, while (3) purely dissipative, non-self-dual dynamics may give rise to non-P-divisible classical reductions. Though this behaviour is somewhat typical of unitary quantum dynamics, it can also emerge from purely dissipative quantum evolutions due to the presence of a non-trivially time-ordered Dyson expansion.

CHAPTER 4

Superactivation of memory effects

In the previous Chapter, we investigated the classical backflow of information arising from a quantum P-divisible evolution. Throughout this Chapter, instead, we shall consider memory effects that have no classical counterpart as those emerging from a P-divisible but not CP-divisible dynamical map Λ_t, sometimes referred to as *weakly non-Markovian* [97]. From Proposition 2.8 and the subsequent discussion, P-divisible maps display no BFI – that is, no distinguishability revivals – when considered independently. On the other hand, lack of CP-divisibility implies that this behaviour is not stable under tensorization. It is particularly interesting to investigate whether memory effects can activate in non-trivial dilations of Λ_t through tensor products of the type $\Lambda_t \otimes \widetilde{\Lambda}_t$. Rather than tensoring with an inert (and uncontrollable) ancilla, such dynamics represent the clear physical scenario of two copies of the system independently evolving in their own environments. Of course, the two subsystems are dynamically independent but, in general, statistically coupled due to possible correlations – be they classical or quantum – in the initial state. We are thus interested in the situation in which each subsystem, taken individually, exhibits no memory effects, such as BFI, at the level of its dynamical map, whereas the bipartite system does. This phenomenon has no classical analogue. Indeed, consider a P-divisible stochastic evolution $T(t)$ on d-dimensional probability vectors; it does not display revivals of the ℓ_1-norm as proved in Proposition 1.6, namely

$$\|T(t)\boldsymbol{x}\|_1 \leq \|T(s)\boldsymbol{x}\|_1\,, \quad \forall\, t \geq s \geq 0\,, \forall\, \boldsymbol{x} \in \mathbb{R}^d.$$

Moreover, the tensor product of two P-divisible stochastic evolutions $T(t) \otimes \widetilde{T}(t)$ is always P-divisible, since positivity and complete positivity coincide for mappings on commutative algebras (Proposition 1.9). Explicitly, under such dynamics, the ℓ_1-norm of a time-evolving vector $\boldsymbol{x} = \{x_{ij}\} \in \mathbb{R}^d \times \mathbb{R}^d$ cannot increase in time. Indeed, since for all $t \geq s \geq 0$ we can define stochastic propagators $T(t,s)$

$$\begin{aligned} \left\|T(t) \otimes \widetilde{T}(t)\,\boldsymbol{x}\right\|_1 &= \sum_{ij} \left| \sum_{kl} T_{ik}(t,s)\,\widetilde{T}_{jl}(t,s) x_{kl}(s) \right| \qquad (4.1) \\ &\leq \sum_{kl} \left(\sum_i T_{ik}(t,s) \right) \left(\sum_j \widetilde{T}_{jl}(t,s) \right) |x_{kl}(s)| \quad \leq \sum_{kl} |x_{kl}(s)| = \left\|T(s) \otimes \widetilde{T}(s)\,\boldsymbol{x}\right\|_1 , \end{aligned}$$

where, in the first inequality, the triangle inequality was used together with the fact that the propagator is positive entry-wise, whereas in the second inequality, conservation of probability was used.

4.1 Superactivation of Backflow of Information

Central to our study will be the case of two dynamically independent parties evolving in identical copies of the same environment. Mathematically, this corresponds to the second tensor power $\Lambda_t \otimes \Lambda_t$. In the continuous-time setting, necessary and sufficient conditions for its P-divisibility were derived in [126].

Theorem 4.1. *Let $\{\Lambda_t\}_{t\geq 0}$ be a one-parameter family of dynamical maps obeying the time-local master equation* (2.7)*. Then, $\Lambda_t \otimes \Lambda_t$ is P-divisible if and only if Λ_t is CP-divisible.*

Remark 4.1. *The previous result, crucially, relies on time-continuity and on the existence of a time-local generator. In fact, the second tensor power of a positive map may be positive even when the map itself is not completely positive. As noted in Remark 1.14, the transposition provides such an instance. In Section 4.2 we shall see that, actually, Theorem 4.1 does not hold for the discrete-time dynamics arising from a concrete collisional model.*

The rest of this Chapter is devoted to the study of memory effects *superactivating* whenever Λ_t is only P but not CP-divisible. In fact, a straightforward consequence of Theorem (4.1) and Proposition 2.8 is the following,

$$\begin{array}{l}\Lambda_t \text{ P-divisible,} \\ \text{not CP-divisible}\end{array} \Rightarrow \begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\left\|\Lambda_t[\Delta_\mu(\rho_S,\sigma_S)]\right\|_1 \leq 0, \qquad \forall\, t\geq 0,\ \forall\, \Delta_\mu(\rho_S,\sigma_S) \\ \\ \exists\, t>0,\ \Delta_\mu(\rho_{S+S},\sigma_{S+S}):\ \dfrac{\mathrm{d}}{\mathrm{d}t}\left\|\Lambda_t\otimes\Lambda_t[\Delta_\mu(\rho_{S+S},\sigma_{S+S})]\right\|_1 > 0, \end{cases} \tag{4.2}$$

where $\Delta_\mu(\cdot,\cdot)$ indicates the Helstrom matrix introduced in (2.40). We shall refer to such revivals, occurring only for the bipartite system, as *superactivation of backflow of information (SBFI)*. As evident from (4.1), this effect has no classical counterpart.

Example 4.1. *As an example of P but not CP-divisible evolution, consider the Pauli dynamics* (2.31) *and pick time-dependent rates*

$$\gamma_t^{(1)} = \gamma_t^{(2)} = 1\,, \qquad \gamma_t^{(3)} = \sin(\omega\, t)\,, \qquad \omega\in\mathbb{R}\,, \qquad \eta>0\,. \tag{4.3}$$

Since $\gamma_t^{(3)}$ becomes negative, the evolution cannot be CP-divisible (Proposition 2.5). The spectrum of the generator is then given by

$$\Gamma_t^{(1)} = \Gamma_t^{(2)} = \eta(1+\sin(\omega t)) =: \Gamma(t)\,, \qquad \Gamma_t^{(3)} = 2\eta\,, \tag{4.4}$$

which are positive functions of time, thus guaranteeing P-divisibility. Since the map is P-divisible but not CP-divisible, it exhibits SBFI when the second-tensor power is considered. The spectrum of the map is obtained through (2.32) *and reads*

$$\lambda_t^{(1)} = \lambda_t^{(2)} = e^{-\eta\left(t+\frac{1-\cos(\omega t)}{\omega}\right)} =: \lambda_t\,, \qquad \lambda_t^{(3)} = e^{-2\eta t} := \mu_t\,. \tag{4.5}$$

Notice that (4.4) *ensures that the dynamics is positive. To check whether the maps* Λ_t *represent a physically legitimate evolution – namely, whether they are completely positive – note that they can be recast as*

$$\Lambda_t[\rho] = \frac{1+\mu_t+2\lambda_t}{4}\rho + \frac{1-\mu_t}{4}(\sigma_1\rho\sigma_1 + \sigma_2\rho\sigma_2) + \frac{1+\mu_t-2\lambda_t}{4}\sigma_3\rho\sigma_3\,, \qquad (4.6)$$

Being conveniently expressed in Kraus-Stinespring form, Λ_t *is completely positive iff all the coefficients of* (4.6) *are positive, so the only relevant condition is*

$$1+\mu_t \ \geq\ 2\lambda_t \iff e^{\eta t} + e^{-\eta t} \geq 2\, e^{-\frac{\eta}{\omega}(1-\cos(\omega t))}\,. \qquad (4.7)$$

If $\omega \geq 0$, *the right-hand side of the second inequality above is always smaller than or equal to 2, and thus complete positivity is guaranteed. Instead, for* $\omega < 0$, *expanding both sides around* $t = 0$ *shows that* (4.7) *is violated when* $\eta < |\omega|$. *At fixed* $|\omega|$, *complete positivity is ensured for* η *sufficiently large.*

Determining conditions for P-divisibility of tensor products of different maps is, perhaps surprisingly, much more difficult than in the case of tensor powers. A detailed discussion of this topic can be found in [127]. Here, we restrict ourselves to the (albeit instructive) case of two different Pauli evolutions. The next result provides necessary and sufficient conditions for P-divisibility of general tensor products $\Lambda_{1,t}\otimes\Lambda_{2,t}$ of two Pauli maps, generated by (2.31). The proof can be found in [127].

Proposition 4.2. *Let* $\Lambda_{\alpha,t}$, $\alpha = 1,2$ *be Pauli CPTP maps with time-local generators* $\mathcal{L}_{\alpha,t}$ *of the form* (2.31), *specified by time dependent rates* $\{\gamma_{\alpha,t}^{(j)}\}_{j=1}^3$, $\alpha = 1,2$. *Their tensor product,* $\Lambda_{1,t}\otimes\Lambda_{2,t}$, *is P-divisible if and only if (1) both* $\Lambda_{1,t}$ *and* $\Lambda_{2,t}$ *are P-divisible, namely*

$$\gamma_{1,t}^{(i)} + \gamma_{1,t}^{(j)} \geq 0\,, \qquad \forall t \geq 0,\ \forall\, i \neq j\,, \qquad (4.8)$$

and (2)

$$\gamma_{1,t}^{(i)} + \gamma_{2,t}^{(j)} \geq 0\,, \qquad \forall t \geq 0,\ \forall\, i,j = 1,2,3\,. \qquad (4.9)$$

Remark 4.2. *If* $\Lambda_{1,t}$ *is not CP-divisible,* $\Lambda_{1,t}\otimes\Lambda_{1,t}$ *is not P-divisible and* (4.9) *further implies that one cannot restore P-divisibility of* $\Lambda_{1,t}\otimes\Lambda_{2,t}$ *with* $\Lambda_{2,t}$ *being a small perturbation of* $\Lambda_{1,t}$. *In fact, If* $\Lambda_{1,t}$ *is not CP divisible,* $\gamma_{1,t}^{(k)} < 0$ *for some* k *and* $t > 0$. *Then, letting* $\gamma_{2,t}^{(k)} = \gamma_{1,t}^{(k)} + \epsilon\,\delta_t^{(k)}$, $\epsilon \ll 1$, *yields*

$$\gamma_{1,t}^{(k)} + \gamma_{2,t}^{(k)} = 2\,\gamma_{1,t}^{(k)} + \epsilon\,\delta_t^{(k)} < 0\,,$$

so (4.9) *is not satisfied. In practice, this means that to cancel SBFI affecting* $\Lambda_{1t}\otimes\Lambda_{1t}$ *by changing the dynamics of the second party, one in general has to seek a generator of the second-party evolution which is sufficiently strong (this holds generally; see [127] for further details on the* d*-dimensional case).*

The following example illustrates how it is possible to restore global P-divisibility with a suitable variation of the second dynamical map.

Example 4.2. *Take* $\Lambda_{\alpha,t}$, $\alpha = 1,2$ *as in Example 4.1, where* $\gamma_{\alpha,t}^{(3)} = \eta\sin(\omega_\alpha t)$, *and* $\omega_2 = -\omega_1$. *Both dynamics are P-divisible but not CP-divisible for* $\gamma_{\alpha,t}^{(3)}(t)$ *become negative. Nevertheless,* $\gamma_3^{(1)}(t) + \gamma_3^{(2)}(t) = 0$. *Both conditions* (4.8) *and* (4.9) *are satisfied, so that* $\Lambda_{1,t}\otimes\Lambda_{2,t}$ *is P-divisible.*

4.1.1 Quantum signature of SBFI

SBFI, defined as in (4.2) is a memory effect with no classical counterpart (compare (4.1)). On the other hand, as we shall argue in the following, its quantum character is somewhat hidden, in that the states belonging to the Helstrom ensemble needed not be entangled in order to witness the effect. The goal of this Section is to expose the quantumness of SBFI by means of a necessary condition, derived in the form of an inequality in the same vein of (2.47) and (3.20). SBFI was defined in terms of revivals of the bipartite Helstrom distinguishability, namely there exists bipartite states ρ_{S+S}, σ_{S+S}, $0 \leq \mu \leq 1$ such that

$$\Delta D_\mu(\rho_{S+S}, \sigma_{S+S}; t, s) := \left\|\Lambda_t \otimes \Lambda_t[\Delta_\mu(\rho_{S+S}, \sigma_{S+S})]\right\|_1 - \left\|\Lambda_s \otimes \Lambda_s[\Delta_\mu(\rho_{S+S}, \sigma_{S+S})]\right\|_1 \quad (4.10)$$

assumes a strictly positive value at some $t > s > 0$. The quantum character of SBFI can be assessed by a suitable measure of the quantum correlations present in the Helstrom ensemble:

$$\mathcal{E}_H(t) := \{(\mu; \rho_{S+S}(t)), (1-\mu; \sigma_{S+S}(t))\}, \qquad \rho_{S+S}(t) = \Lambda_t \otimes \Lambda_t[\rho_{S+S}].$$

As we shall see in Example 4.3, $\rho_{S+S}(t)$ and $\sigma_{S+S}(t)$, need not be entangled for SBFI to occur.

The quantumness of a single-party ensemble $\mathcal{E} = \{(\mu_i, \rho_i)\}$ has been identified with the possibility of simultaneously diagonalizing it [128, 129]. Equivalently, the ensemble can be encoded into a quantum-classical state

$$\chi^{\mathcal{E}} = \sum_i \mu_i \rho_i \otimes |i\rangle\langle i| \quad (4.11)$$

and one can measure the ensemble quantumness in terms of the quantum correlations as left-sided quantum discord in $\chi^{\mathcal{E}}$ [129, 130]. Among the variety of available discord measures [131], we shall consider the so-called measurement induced geometric measure of quantum correlations defined in the trace norm by

$$\mathcal{Q}_{\{\mathbb{P}\}}(\rho) := \min_{\mathbb{P}} D(\rho, \mathbb{P} \otimes \mathrm{id}[\rho]),$$

where $\mathbb{P}[X] = \sum_i P_i X P_i$ is a projective measurement associated to the OPU $\mathcal{P} = \{P_i\}_i, P_i = |i\rangle\langle i|$ being orthonormal rank-1 projectors. If $\mathcal{E}$ is an ensemble of bipartite states, one rather focuses on finding a simultaneous diagonalization on a set of rank-1 projections of the type $\{P_i^1 \otimes P_j^2\}_{ij}$. Accordingly, in [132] the following measure of bipartite "ensemble quantumness of correlations" was introduced:

$$\mathcal{Q}_{\{\mathbb{P}_1 \otimes \mathbb{P}_2\}}(\chi^{\mathcal{E}}) = \min_{\mathbb{P}_1 \otimes \mathbb{P}_2} \sum_i \mu_i \, D(\rho_i, \mathbb{P}_1 \otimes \mathbb{P}_2[\rho_i]), \quad (4.12)$$

with the density matrix $\chi^{\mathcal{E}}$ now encoding the bipartite ensemble by means of an additional classical register.

The next result connects the bipartite quantumness of correlations of the Helstrom ensemble, as defined by (4.12), to the quantity (4.10) exposing SBFI.

Proposition 4.3. *Given a dynamics $\Lambda_t \otimes \Lambda_t$ with Λ_t P-divisible, the variation of the Helstrom distinguishability can be bounded as follows*

$$\Delta D_\mu(t, s) \leq 2 \left\|\Lambda_{t,s}\right\|_\diamond^2 \, \mathcal{Q}_{\{\mathbb{P}_1 \otimes \mathbb{P}_2\}}\left(\chi^{\mathcal{E}_H}(s)\right), \quad (4.13)$$

where

$$\chi^{\mathcal{E}_H}(s) = \mu\, \Lambda_s \otimes \Lambda_s[\rho_{S+S}] \otimes |0\rangle\langle 0| + (1-\mu)\, \Lambda_s \otimes \Lambda_s[\sigma_{S+S}] \otimes |1\rangle\langle 1|\ ,$$

while $\|\cdot\|_\diamond$ *denotes the diamond norm of a map,* $\|\Lambda\|_\diamond = \|\Lambda \otimes \mathrm{id}_d\|_{1-1}$.

Proof. Let us fix $\{|p_\alpha\rangle\}_\alpha$ with $|p_\alpha\rangle = |p_i^1 \otimes p_j^2\rangle$, $\{|p_i^1\rangle\}_i$, $\{|p_j^2\rangle\}_j$ being arbitrary local orthonormal basis, from which one has a corresponding orthonormal set of rank-1 projectors $\{P_i^1 \otimes P_j^2\}_{ij}$. Accordingly, a completely decohering map with respect to such basis is described by a (bi)local projective measurement:

$$\mathbb{P}_1 \otimes \mathbb{P}_2[X] = \sum_{ij} P_i^1 \otimes P_j^2\ X\ P_i^1 \otimes P_j^2\ .$$

Then, for $t > s > 0$ both in discrete and continuous time, considering the Helstrom matrix at time t, $\Lambda_t \otimes \Lambda_t[\Delta_\mu(\rho_{S+S}, \sigma_{S+S})]$, via the triangle inequality and the contractivity of $\Lambda_{t,s}$ and $\mathbb{P}_1 \otimes \mathbb{P}_2$, one estimates

$$\begin{aligned}\left\|(\Lambda_{t,s} \otimes \Lambda_{t,s}) \circ (\mathbb{P}_1 \otimes \mathbb{P}_2)[\Delta_\mu(s)]\right\|_1 &\leq \sum_{ij} \left|\delta_\mu^{ij}(s)\right| \left\|\Lambda_{t,s}[P_i^1]\right\|_1 \left\|\Lambda_{t,s}[P_j^2]\right\|_1 \\ &\leq \left\|\mathbb{P}_1 \otimes \mathbb{P}_2[\Delta_\mu(s)]\right\|_1 \leq \left\|\Delta_\mu(s)\right\|_1\ , \end{aligned} \tag{4.14}$$

where $\delta_\mu^{ij}(s) := \langle p_i^1 \otimes p_j^2 | \Delta_\mu(s) | p_i^1 \otimes p_j^2\rangle$. Recalling the induced trace norm of a map defined in (2.49), the Helstrom matrix norm can be upper-bounded as follows

$$\begin{aligned}&\left\|\Lambda_{t,s} \otimes \Lambda_{t,s}[\Delta_\mu(s)]\right\|_1 \\ &= \big\|\mu\, \Lambda_{t,s} \otimes \Lambda_{t,s}\big[\rho_{S+S}(s) - \mathbb{P}_1 \otimes \mathbb{P}_2[\rho_{S+S}(s)]\big] \\ &\quad - (1-\mu)\Lambda_{t,s} \otimes \Lambda_{t,s}\big[\sigma_{S+S}(s) - \mathbb{P}_1 \otimes \mathbb{P}_2[\sigma_{S+S}(s)]\big] + \Lambda_{t,s} \otimes \Lambda_{t,s} \circ \mathbb{P}_1 \otimes \mathbb{P}_2[\Delta_\mu(s)]\big\|_1 \\ &\leq\ \mu \left\|\Lambda_{t,s} \otimes \Lambda_{t,s}\big[\rho_{S+S}(s) - \mathbb{P}_1 \otimes \mathbb{P}_2[\rho_{S+S}(s)]\big]\right\|_1 \\ &\quad + (1-\mu) \left\|\Lambda_{t,s} \otimes \Lambda_{t,s}\big[\sigma_{S+S}(s) - \mathbb{P}_1 \otimes \mathbb{P}_2[\sigma_{S+S}(s)]\big]\right\|_1 + \left\|\Lambda_{t,s} \otimes \Lambda_{t,s} \circ \mathbb{P}_1 \otimes \mathbb{P}_2[\Delta_\mu(s)]\right\|_1 .\end{aligned}$$

Using (4.14) and the fact that $\|\Lambda \otimes \Lambda\|_1 \leq \|\Lambda\|_\diamond^2$, we then have

$$\begin{aligned}\Delta D_\mu\left(\rho_{S+S}, \sigma_{S+S}; t, s\right) \leq\ &\left\|\Lambda_{t,s}\right\|_\diamond^2 \Big(\mu \left\|\rho_{S+S}(s) - \mathbb{P}_1 \otimes \mathbb{P}_2[\rho_{S+S}(s)]\right\|_1 \\ &+ (1-\mu)\|\sigma_{S+S}(s) - \mathbb{P}_1 \otimes \mathbb{P}_2[\sigma_{S+S}(s)]\|_1\Big).\end{aligned}$$

Since $\mathbb{P}_{1,2}$ are arbitrary, one can tighten the latter inequality by minimizing over the projective measurements. One then finally obtains the following upper-bound for $\Delta D_\mu(t,s)$,

$$\Delta D_\mu(t,s)\ \leq\ 2 \left\|\Lambda_{t,s}\right\|_\diamond^2\ \mathcal{Q}_{\{\mathbb{P}^1 \otimes \mathbb{P}^2\}}(\chi^{\mathcal{E}}(s))\ ,$$

where the quantumness of the Helstrom ensemble $\mathcal{E}_H(s) = \{(\mu; \rho_{S+S}(s)), (1-\mu; \sigma_{S+S}(s))\}$, encoded in the quantum-classical state

$$\chi^{\mathcal{E}}(s) = \mu\, \rho_{S+S}(s)\, |0\rangle\langle 0| + (1-\mu)\, \sigma_{S+S}(s) \otimes |1\rangle\langle 1|\ ,$$

is measured by the (left-sided) quantum correlations of $\chi^{\mathcal{E}_H}(s)$ [132]:

$$\begin{aligned}\mathcal{Q}_{\{\mathbb{P}_1 \otimes \mathbb{P}_2\}}(\chi^{\mathcal{E}_H}(s)) = \frac{1}{2} \min_{\mathbb{P}_1 \otimes \mathbb{P}_2} \Big(&\mu \left\|\rho_{S+S}(s) - \mathbb{P}_1 \otimes \mathbb{P}_2[\rho_{S+S}(s)]\right\|_1 \\ &+ (1-\mu)\|\sigma_{S+S}(s) - \mathbb{P}_1 \otimes \mathbb{P}_2[\sigma_{S+S}(s)]\|_1\Big).\end{aligned}$$ □

If SBFI triggers at time s, i.e. $\Delta D_\mu(s+\epsilon, s) > 0$, then the quantumness of correlations of the ensemble $\mathcal{E}_H(s) = \{(\mu, \rho_{S+S}(s)); (1-\mu, \sigma_{S+S}(s))\}$ has to be strictly positive, that is, the state $\chi^{\mathcal{E}_H}(s)$ must possess non zero quantum discord. Thus, with the same reasoning as in (2.47) and (3.20), the quantumness of the Helstrom ensemble represents a precursor of SBFI [76]. In particular, the condition $\mathcal{Q}_{\{\mathbb{P}_1 \otimes \mathbb{P}_2\}}(\chi^{\mathcal{E}_H}(t)) > 0$ does not imply that the states are entangled, as we explicitly show in the following and similarly noted in [133, 134].

Example 4.3. *Consider again a P-divisible Pauli evolution as in Example 4.1 fully specified by the spectrum of its generator*

$$\mathcal{L}_t[\sigma_\alpha] = -\Gamma_\alpha(t)\sigma_\alpha\,, \qquad \alpha = 1, 2, 3\,.$$

where $\Gamma_\alpha(t) \geq 0$ *ensures P-divisibility. Further, let* $\Gamma_1(t) = \Gamma_2(t) = \Gamma(t)$ *so that* $\lambda_t^{(1)} = \lambda_t^{(2)} \equiv \lambda_t = e^{-\int_0^t \mathrm{d}s\Gamma(s)}$ *and* $\lambda_t^{(3)} \equiv \mu_t = e^{-\int_0^t \mathrm{d}s\Gamma_3(s)}$. *CP-divisibility amounts to the further constraint*

$$2\Gamma(t) \geq \Gamma_3(t)\,. \tag{4.15}$$

In particular, for Pauli maps, lack of P-divisibility of $\Lambda_{t,s}\otimes\Lambda_{t,s}$ *can be witnessed by the maximally entangled state* $P_+^{(2)}$

$$\Lambda_{s+\epsilon,s}\otimes\Lambda_{s+\epsilon,s}[P_+^{(2)}] = P_+^{(2)} - \frac{\epsilon\Gamma(s)}{2}\left(\sigma_1\otimes\sigma_1 - \sigma_2\otimes\sigma_2\right) - \frac{\epsilon\Gamma_3(s)}{2}\sigma_3\otimes\sigma_3 \; + O(\epsilon^2)\,, \tag{4.16}$$

which has, for $\epsilon \ll 1$, *only one negative eigenvalue equal to* $x_-(s) := (2\Gamma(s) - \Gamma_3(s))/2$. *Then,*

$$\left\|\Lambda_{s+\epsilon,s}\otimes\Lambda_{s+\epsilon,s}[P_+^{(2)}]\right\|_1 = \mathrm{Tr}\left(P_+^{(2)}\right) + 2|x_-(s)| = 1 + |2\Gamma(s) - \Gamma_3(s)| \; > 1 = \left\|P_+^{(2)}\right\|_1.$$

Now, we argue that there exists a Helstrom matrix

$$\Delta_\mu = \mu\rho_1 - (1-\mu)\rho_2, \tag{4.17}$$

with ρ_1, ρ_2 *separable, such that* $\Lambda_s\otimes\Lambda_s[\Delta_\mu] = \alpha P_+^{(2)}$ *so that, for some* $\epsilon > 0$

$$\left\|\Lambda_{s+\epsilon}\otimes\Lambda_{s+\epsilon}[\Delta_\mu]\right\|_1 = \alpha\left\|\Lambda_{s+\epsilon,s}\otimes\Lambda_{s+\epsilon,s}[P_+^{(2)}]\right\|_1 > \alpha\left\|P_+^{(2)}\right\|_1 = \left\|\Lambda_s\otimes\Lambda_s[\Delta_\mu]\right\|_1. \tag{4.18}$$

Consider the isotropic state

$$\rho_a = (1-a)\frac{\mathbb{1}_4}{4} + aP_+^{(2)}, \qquad 0 \leq a \leq 1\,. \tag{4.19}$$

By inspection of the spectrum of its partial transpose, ρ_a *is separable iff* $a \leq 1/3$. *The preimage of* $P_+^{(2)}$ *can be then rewritten as*

$$\Lambda_s^{-1}\otimes\Lambda_s^{-1}[P_+^{(2)}] = \frac{1}{a}\Lambda_s^{-1}\otimes\Lambda_s^{-1}[\rho_a] - \frac{1-a}{a}\frac{\mathbb{1}_4}{4}\,. \tag{4.20}$$

Recall that $\Lambda_s^{-1}\otimes\Lambda_s^{-1}$ *only guarantees Hermiticity, but not, in general, positivity preservation. Nevertheless, for sufficiently small* a, $\Lambda_s^{-1}\otimes\Lambda_s^{-1}[\rho_a]$ *is a separable state by being sufficiently close to the separable state* $\mathbb{1}_4/4$. *Explicitly, by expressing the isotropic state as* $\rho_a = \frac{1}{4}\left(\mathbb{1}_4 + a\sum_{i=1}^3 \sigma_i\otimes\sigma_i^{\mathsf{T}}\right)$, *one deduces that its preimage under* $\Lambda_s^{-1} \otimes \Lambda_s^{-1}$ *is*

$$\rho_a^0 := \Lambda_s^{-1} \otimes \Lambda_s^{-1}[\rho_a] = \frac{1}{4}\left(\mathbb{1}_4 + a\lambda_s^{-2}(\sigma_1 \otimes \sigma_1 - \sigma_2 \otimes \sigma_2) + \mu_s^{-2}\sigma_3 \otimes \sigma_3\right) \tag{4.21}$$

$$= \begin{pmatrix} \frac{1}{4}\left(\frac{a}{\mu_s^2} + 1\right) & 0 & 0 & \frac{a}{2\lambda_s^2} \\ 0 & \frac{1}{4}\left(1 - \frac{a}{\mu_s^2}\right) & 0 & 0 \\ 0 & 0 & \frac{1}{4}\left(1 - \frac{a}{\mu_s^2}\right) & 0 \\ \frac{a}{2\lambda_s^2} & 0 & 0 & \frac{1}{4}\left(\frac{a}{\mu_s^2} + 1\right) \end{pmatrix},$$

For bipartite systems of two-qubits, a state is separable if and only if its partial transposition is positive [68]. The partial transpose of (4.21) *then reads:*

$$\mathsf{T} \otimes \mathrm{id}_d[\rho_a^0] = \begin{pmatrix} \frac{1}{4}\left(\frac{a}{\mu_s^2} + 1\right) & 0 & 0 & 0 \\ 0 & \frac{1}{4}\left(1 - \frac{a}{\mu_s^2}\right) & \frac{a}{2\lambda_s^2} & 0 \\ 0 & \frac{a}{2\lambda_s^2} & \frac{1}{4}\left(1 - \frac{a}{\mu_s^2}\right) & 0 \\ 0 & 0 & 0 & \frac{1}{4}\left(\frac{a}{\mu_s^2} + 1\right) \end{pmatrix}, \tag{4.22}$$

Then, ρ_a^0 *is a separable state if both* (4.21) *and* (4.22) *are positive; namely, if*

$$1 \pm \frac{2a}{\mu_s^2} + \left(\frac{2a}{\mu_s^2}\right)^2 \left(\frac{1}{4} - \frac{\mu_s^4}{\lambda_s^4}\right) \geq 0 \qquad a \leq \mu_s^2 \,. \tag{4.23}$$

Hence, it suffices to suitably choose $s > 0$ *and, accordingly,* $a \ll \mu_s^2/2$ *so that* (4.23) *is satisfied and* ρ_a^0 *is a separable state. In such case, SBFI is witnessed by the image at time* s *of the initial Helstrom matrix* $\Delta_{\mu_a} = \mu_a \rho_a^0 - (1-\mu_a)\frac{\mathbb{1}_4}{4}$*, with* $\mu_a = 1/(2-a)$*. Its constituent states remain separable throughout the entire evolution due to factorization of the dynamics. SBFI occurs only if in the Helstrom ensemble the quantity* (4.12) *is positive at time* s*. Since* $\mathbb{1}_4/4$ *is fully incoherent with respect to any basis, the ensemble quantumness of correlations at time* s *reduces to*

$$\mathcal{Q}_{\{\mathbb{P}_1 \otimes \mathbb{P}_2\}}\left(\chi^{\mathcal{E}_H}(s)\right) = \mu(a) \min_{\mathbb{P}^1 \otimes \mathbb{P}^2} \left\|\rho_a - \mathbb{P}_1 \otimes \mathbb{P}_2[\rho_a]\right\|_1 > 0\,,$$

corresponding to a geometric measure of quantum discord in the isotropic state [131].

4.1.2 Entropic SBFI

For unital evolutions, it makes sense to study the superactivation phenomenon from the point of view of Renyi p-entropies (2.52). We could in fact expect to find a P-divisible but not CP-divisible dynamics for which the entropy of one party always grows monotonically (see Proposition 2.10), while the entropy of two identically-evolving copies experiences a decrease in time. In other words, for Λ_t P-divisible and unital, the SBFI effect we seek is rephrased, in entropic terms, as:

$$\frac{\mathrm{d}}{\mathrm{d}t} S_p(\Lambda_t[\rho_S]) \geq 0\,,\ \forall t \geq 0\,, \forall\, \rho_S\,, \qquad \exists\, t^* > 0\,, \rho_{S+S} : \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=t^*} S_p(\Lambda_t \otimes \Lambda_t[\rho_{S+S}]) < 0\,. \tag{4.24}$$

First, we show that (4.24) is never the case for $p = 2$, corresponding to $S_2(\rho) = -\log \mathrm{Tr}\left(\rho^2\right)$, which is a standard measure of the purity of ρ.

Proposition 4.4. *Let* $\Lambda_t : M_d(\mathbb{C}) \to M_d(\mathbb{C})$ *be a P-divisible unital map. Then for all* $X \in M_d(\mathbb{C})$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda_t \otimes \Lambda_t[X]\|_2^2 \leq 0\,, \qquad t \geq 0\,. \tag{4.25}$$

Proof. Consider first the case of Λ_t. For any $X \in M_d(\mathbb{C})$, and setting $X_t \equiv \Lambda_t[X]$, one has

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda_t[X]\|_2^2 &= \mathrm{Tr}\left(\mathcal{L}_t[X_t^\dagger]X_t\right) + \mathrm{Tr}\left(X_t^\dagger \mathcal{L}_t[X_t]\right) \\ &= \mathrm{Tr}\left((\mathcal{L}_t + \mathcal{L}_t^\ddagger)[X_t^\dagger]X_t\right) = 2\,\mathrm{Tr}\left(\mathcal{L}_t^{\mathrm{sd}}[X_t^\dagger]X_t\right)\,,\end{aligned} \tag{4.26}$$

where $\mathcal{L}^{\mathrm{sd}}$ is the self-dual part of the generator $\mathcal{L}_s^{\mathrm{sd}} = (\mathcal{L}_t + \mathcal{L}_t^\ddagger)/2$. Recall that, for second tensor powers,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda_t \otimes \Lambda_t = \left(\mathcal{L}_t \otimes \mathrm{id}_d + \mathrm{id}_d \otimes \mathcal{L}_t\right) \circ \left(\Lambda_t \otimes \Lambda_t\right),$$

so that, for $Y \in M_d(\mathbb{C}) \otimes M_d(\mathbb{C})$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda_t \otimes \Lambda_t[Y]\|_2^2 = 2\,\mathrm{Tr}\left(\mathcal{L}_t^{\mathrm{sd}} \otimes \mathrm{id}_d \circ \Lambda_t \otimes \Lambda_t[Y]\right) + 2\,\mathrm{Tr}\left(\mathrm{id}_d \otimes \mathcal{L}_t^{\mathrm{sd}} \circ \Lambda_t \otimes \Lambda_t[Y]\right)\,,$$

Then, setting $Y_t := \Lambda_t \otimes \Lambda_t[Y]$, one can decompose Y_t along the orthonormal Hermitian basis of matrices $F_\alpha \otimes F_\beta$, (see Remark 1.1): $Y_t = \sum_{\alpha,\beta=0}^{3} y_t^{\alpha\beta} F_\alpha \otimes F_\beta$. Plugging the latter into (4.26) we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda_t \otimes \Lambda_t[Y]\|_2^2 = 2\sum_{\beta=0}^{d^2-1}\sum_{\alpha,\mu=1}^{d^2-1} \overline{y}_t^{\alpha\beta} y_t^{\mu\beta}\,\mathrm{Tr}\left(F_\mu \mathcal{L}_t^{\mathrm{sd}}[F_\alpha]\right) \\ + 2\sum_{\alpha=0}^{d^2-1}\sum_{\beta,\nu=1}^{d^2-1} \overline{y}_t^{\alpha\beta} y_t^{\alpha\nu}\,\mathrm{Tr}\left(F_\beta \mathcal{L}_t^{\mathrm{sd}}[F_\nu]\right)\,,\end{aligned} \tag{4.27}$$

where the inner sums run from 1 to $d^2 - 1$ since, by assumption, $\mathcal{L}_t[\mathbb{1}_d] = 0$ and $\mathrm{Tr}\left(\mathcal{L}_t[F_\alpha]\right) = 0$. Then, recalling the matrix representation (2.22) of $\mathcal{L}_t^{\mathrm{sd}}$ and defining vectors $\left|y_t^{(2,\beta)}\right\rangle = \left(y_t^{\beta 1}, y_t^{\beta 2}, \dots, y_t^{\beta d^2-1}\right) \in \mathbb{C}^{d^2-1}$, one recasts (4.27) as

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda_t \otimes \Lambda_t[X]\|_2^2 = 2\sum_{\beta=0}^{d^2-1}\left(\left\langle y_t^{(1,\beta)}\middle|\widetilde{\mathcal{L}}^{\mathrm{s}}(t)\middle|y_t^{(1,\beta)}\right\rangle + \left\langle y_t^{(2,\beta)}\middle|\widetilde{\mathcal{L}}^{\mathrm{s}}(t)\middle|y_t^{(2,\beta)}\right\rangle\right) \leq 0\,,$$

since, by Proposition 2.6, P-divisibility of Λ_t implies positive semi-definiteness of $-\widetilde{\mathcal{L}}^{\mathrm{s}}(t)$. □

The previous result also extends to higher tensor powers $\Lambda_t^{\otimes k}$, $k \in \mathbb{N}$. Hence, SBFI never manifests in revivals of the purity. Conversely, as we argue in the following Section, it is possible to detect it through the von Neumann entropy.

4.1.3 von Neumann entropy and SBFI for Pauli dynamics

We shall now construct a class of qubit evolutions exhibiting SBFI as witnessed by (4.24) in the limit $p \to 1^+$, corresponding to the von Neumann entropy, $S(\rho) = -\operatorname{Tr}(\rho \log(\rho))$.

Consider again the Pauli map from Example 4.3 and the isotropic state (4.19). First, let us investigate the action of $\Lambda_t \otimes \mathrm{id}_2$ on ρ_a:

$$\begin{aligned}\Lambda_t \otimes \mathrm{id}_2[\rho_a] &= \frac{1}{4}\left(\mathbb{1}_4 + a\,\lambda_t\,(\sigma_1\otimes\sigma_1 - \sigma_2\otimes\sigma_2) + a\,\mu_t\,\sigma_3\otimes\sigma_3\right) \\ &= \frac{1}{4}\begin{pmatrix} 1+a\mu_t & 0 & 0 & 2a\lambda_t \\ 0 & 1-a\mu_t & 0 & 0 \\ 0 & 0 & 1-a\mu_t & 0 \\ 2a\lambda_t & 0 & 0 & 1+a\mu_t \end{pmatrix}. \end{aligned} \tag{4.28}$$

Accordingly, the eigenvalues read

$$\ell_3^a(t) := \frac{1-a\mu_t}{4}\,, \qquad \ell_\pm^a(t) := \frac{1+a(\mu_t \pm 2\lambda_t)}{4}\,, \tag{4.29}$$

so that

$$\begin{aligned} S(\Lambda_t \otimes \mathrm{id}_d[\rho_a]) = -\frac{1-a\mu_t}{2}\log\!\left(\frac{1-a\mu_t}{4}\right) - \frac{1+a(\mu_t-2\lambda_t)}{4}\log\!\left(\frac{1+a(\mu_t-2\lambda_t)}{4}\right) \\ - \frac{1+a(\mu_t+2\lambda_t)}{4}\log\!\left(\frac{1+a(\mu_t+2\lambda_t)}{4}\right). \end{aligned} \tag{4.30}$$

Its derivative is

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} S(\Lambda_t \otimes \mathrm{id}_d[\rho_a]) = \frac{a\,\Gamma_3(t)\mu_t}{4}\log\frac{(1+a(\mu_t-2\lambda_t))(1+a(\mu_t+2\lambda_t))}{(1-a\mu_t)^2} \\ + \frac{a\,\Gamma(t)\lambda_t}{2}\log\frac{1+a(\mu_t+2\lambda_t)}{1+a(\mu_t-2\lambda_t)}\,. \end{aligned} \tag{4.31}$$

Assume that $\Gamma(t) = 0$ and $\Gamma_3(t) > 0$ for some $t = t^* > 0$. Notice that such an assumption is compatible with a P-divisible evolution but not with a CP-divisible one due to (4.15). Then, (4.31) becomes negative if

$$(1+a\mu_{t^*})^2 - 4a^2\lambda_{t^*}^2 < (1-a\mu_{t^*})^2\,, \tag{4.32}$$

which yields a sufficient condition for entropy SBFI

$$\Gamma(t^*) = 0\,, \qquad a > \frac{\mu_{t^*}}{\lambda_{t^*}^2} \implies \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=t^*} S(\Lambda_t \otimes \mathrm{id_d}[\rho_a]) < 0\,. \tag{4.33}$$

One can find a similar condition for the second tensor power $\Lambda_t \otimes \Lambda_t$. The von Neumann entropy in this case reads

$$\begin{aligned} S(\Lambda_t \otimes \Lambda_t[\rho_a]) = -\frac{1-a\mu_t^2}{2}\log\!\left(\frac{1-a\mu_t^2}{4}\right) - \frac{1+a(\mu_t^2-2\lambda_t^2)}{4}\log\!\left(\frac{1+a(\mu_t^2-2\lambda_t^2)}{4}\right) \\ - \frac{1+a(\mu_t^2+2\lambda_t^2)}{4}\log\!\left(\frac{1+a(\mu_t^2+2\lambda_t^2)}{4}\right), \end{aligned} \tag{4.34}$$

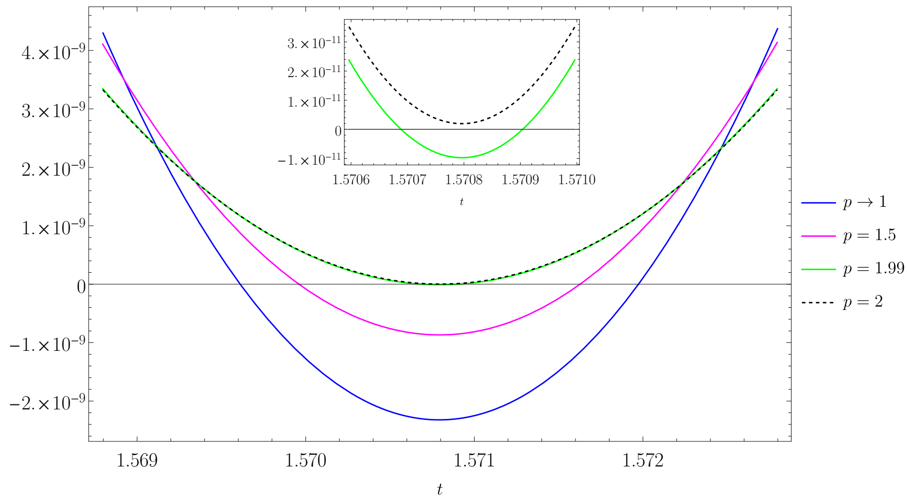

FIGURE 4.1: First derivative in time of the Renyi entropy $S_p(\Lambda_t \otimes \Lambda_t[\rho_a])$ for $1 < p \leq 2$. The initial isotropic state ρ_a is considered for $a = 1/5 < 1/3$, so that $\Lambda_t \otimes \Lambda_t[\rho_a]$ is separable for all $t \geq 0$. The evolution Λ_t is that of Examples 4.1 and 4.4, with $\eta = 2$, $\omega = -1$. The neighbourhood of $t^* = \pi/2$, at which the generator has a zero in the spectrum, $\Gamma(t^* = \pi/2) = 0$, is displayed. SBFI is indicated by a negative derivative of the p-entropy, which is never observed for $p = 2$. Instead values of $p < 2$, including $p \to 1$ corresponding to the von Neumann entropy, can exhibit entropy SBFI while their one-qubit counterparts monotonically increase.

Taking the derivative, one finds as sufficient condition

$$\Gamma(t^*) = 0\,, \qquad a > \frac{\mu_{t^*}^2}{\lambda_{t^*}^4} \implies \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=t^*} S(\Lambda_t \otimes \Lambda_t[\rho_a]) < 0\,. \tag{4.35}$$

Example 4.4. *Consider the Pauli dynamics of Example 4.1 which is P-divisible and unital so that $S(\Lambda_t[\rho])$ increases monotonically. On the other hand, the map is not CP-divisible. Moreover, we choose $\omega < 0$ so that,*

$$\Gamma(t) = \eta(1 - \sin(|\omega| t)\,, \qquad \Gamma\left(t^* = \frac{\pi}{2|\omega|}\right) = 0\,.$$

and

$$\frac{\mu_{t^*}}{\lambda_{t^*}^2} = \frac{e^{-2\eta t^*}}{e^{-2\eta t^* - 2\frac{\eta}{\omega}(1-\cos(\omega t^*))}} = e^{-2\frac{\eta}{|\omega|}} < 1\,.$$

Thus, in order to witness SBFI by means of the von Neumann entropy, it suffices to choose the initial isotropic state ρ_a with

$$e^{-2\frac{\eta}{|\omega|}} < a \leq 1\,.$$

Recall that complete positivity depends on the ratio $\eta/|\omega|$ (see (4.7)). In particular, let $\eta = 2$ and $\omega = -1$, so that complete positivity of Λ_t is guaranteed (condition (4.7) is satisfied) and

$$\frac{\mu_{t^*}}{\lambda_{t^*}^2} = e^{-2\frac{\eta}{|\omega|}} = e^{-4} \approx 0.018\,.$$

Hence, it is sufficient to select $a > e^{-4}$ for the von Neumann entropy to exhibit SBFI. Notice that the effect can still be observed even for $a \leq 1/3$, that is, when the initial isotropic state ρ_a is separable and its image under $\Lambda_t \otimes \Lambda_t$ remains separable for all times.

In Fig. 4.1 we compare the 2-entropy, measuring the purity, and the von Neumann entropy for the case of the two-qubit tensor power $\Lambda_t \otimes \Lambda_t$ for the evolution in Example 4.1 along with other Renyi p-entropies.

4.2 SBFI in a classical Markov environment

Up until now, we have studied SBFI from the phenomenological point of view of the dynamical map. In order to gain a better understanding of the microscopic origin of this effect, we shall now analyse its emergence through a concrete collisional model. In this context, we shall investigate the SBFI effect by directly examining the properties of the bipartite system-environment state, and, in particular of its correlations. In Chapter 5, we shall revisit the same collisional model and interpret the activation and superactivation of memory effects through a more refined approach.

In particular, we seek for a microscopic realization in which P and CP-divisibility of the emergent dynamical map can be controlled by some meaningful parameter of the model. In order to achieve this, we shall study in detail the evolution of a system coupled to a suitably parametrized classical spin chain environment. Later, we will double the system, so to consider two statistically coupled qubits $S = S_1 + S_2$, each independently interacting with its own collisional classical environment, with compound reduced dynamics $\Lambda_n \otimes \Lambda_n$. Then, we will be able to evaluate their bipartite system-environment correlations through their mutual information.

Let us consider the algebraic rendering of collisional models introduced in Section 2.1.4, where we now choose the environment E as a commutative chain with at each site the same commutative algebra $\mathcal{A} = D_D(\mathbb{C})$ spanned by 1-dimensional orthogonal projections $\{\Pi_i\}_{i=0}^{D-1}$, $\sum_{k=0}^{D-1} \Pi_k = \mathbb{1}$, as in the coarse-grained dynamics of Section 1.3.2.1. Then, any shift invariant state is described by locally normal states

$$\omega_E(A_E^{[-a,b]}) = \mathrm{Tr}\left(\rho_E^{[-a,b]} A_E^{[-a,b]}\right), \qquad \rho_E^{[-a,b]} = \sum_{\boldsymbol{i}_{[-a,b]}} p_{\boldsymbol{i}_{[-a,b]}} \Pi_{\boldsymbol{i}_{[-a,b]}}^{[-a,b]} \in \mathcal{A}_E^{[-a,b]}, \tag{4.36}$$

where the projections $\Pi_{\boldsymbol{i}_{[-a,b]}}^{\otimes[-a,b]} = \bigotimes_{k=-a}^{b} \Pi_{i_k}^{(k)}$ generate the commutative subalgebras $\mathcal{A}_E^{[-a,b]}$, and the probabilities $p_{\boldsymbol{i}_{[-a,b]}}$ satisfy $\sum_{i_{-a}} p_{\boldsymbol{i}_{[-a,b+1]}} = p_{\boldsymbol{i}_{[-a,b]}} = \sum_{i_{b+1}} p_{\boldsymbol{i}_{[-a,b+1]}}$ due to the assumed stationarity. Choose also the coupling map Φ to be

$$\Phi[X_S \otimes A_{i_0}^{(0)}] = \sum_{i=0}^{D-1} \phi_i[X_S] \otimes \Pi_i A_{i_0}^{(0)} \Pi_i, \tag{4.37}$$

the maps ϕ_i being completely positive and unital, $\phi_i[\mathbb{1}] = \mathbb{1}$. Then, its extension to the whole tensor product $\mathcal{A}_S \otimes \mathcal{A}_E$ gives the step-1 dynamics

$$\Theta_1[X_S \otimes A_{i_0}^{(0)}] = (\mathrm{id}_S \otimes \sigma) \circ \Phi[X_S \otimes A_{i_0}^{(0)}] = \sum_{i=0}^{D-1} \phi_i[X_S] \otimes \Pi_i^{(1)} A_{i_0}^{(1)} \Pi_i^{(1)}. \tag{4.38}$$

Let us now evaluate the marginals (2.59) and (2.58) of S and E, respectively. First, note that $\Phi[\mathbb{1}_S \otimes A_{i_0}^{(0)}] = \mathbb{1}_S \otimes A_{i_0}^{(0)}$. As a consequence, the environment is stationary,

$$\omega_S \otimes \omega_E(\Theta_n[\mathbb{1}_S \otimes A_E^{[-a,b]}]) = \omega_S \otimes \omega_E(\mathbb{1}_S \otimes A_E^{[-a,b]}) \implies \Omega^{(n)}_{[-a,b]} = \rho_E^{[-a,b]}. \tag{4.39}$$

The reduced state of S instead reads $\Omega_S^{(n)} = \Lambda_n[\rho_S]$, where the discrete-time dynamical map Λ_n is as follows.

Proposition 4.5. *The reduced dynamics arising by collisional coupling* (4.38) *of the system S to a classical spin chain in a state specified by* (4.36) *consists of a discrete-time family of CPTP maps,*

$$\Lambda_n[\rho_S] := \sum_{\boldsymbol{i}_{[1,n]}} p_{\boldsymbol{i}_{[1,n]}} \phi^{\ddagger}_{\boldsymbol{i}_{[1,n]}}[\rho_S] = \Omega_S^{(n)} , \qquad \phi^{\ddagger}_{\boldsymbol{i}_{[1,n]}} = \phi^{\ddagger}_{i_n} \cdots \phi^{\ddagger}_{i_1} . \tag{4.40}$$

with $\phi_i^\ddagger$ the CPTP map dual to the CPU map ϕ_i in (4.37), $\mathrm{Tr}\left(\rho_S \phi_i[X_S]\right) = \mathrm{Tr}\left(\phi_i^\ddagger[\rho_s] X_S\right)$. *On the other hand,*

$$\Omega^{(n)}_{[-a,b]} = \rho_E^{[-a,b]} , \quad n \geq 1 \tag{4.41}$$

namely, the environment state is stationary.

A detailed proof is reported in Appendix B.1. A factorized state $\omega_{SE} = \omega_S \otimes \omega_E$ on $\mathcal{A}_S \otimes \mathcal{A}_E$ is represented on $\mathcal{A}_S \otimes \mathcal{A}_E^{[-a,b]}$ by a factorized density matrix $\Omega_{S[-a,b]} = \rho_S \otimes \rho_E^{[-a,b]}$ and shows no correlations between system and collisional environment. Evidently, due to the dynamical coupling (4.37), correlations will develop between S and E under the action of Θ_n. Given a bipartite density matrix ρ_{AB}, a general measure of the correlations – be them classical or quantum – between A and B is provided by the *mutual information*

$$I_{AB} = S(\rho_A) + S(\rho_B) - S(\rho_{AB}) \geq 0 , \tag{4.42}$$

where $S(\rho) = -\mathrm{Tr}\,\rho \log \rho$ is the von Neumann entropy and $\rho_{A,B}$ are the marginals of ρ_{AB}.

Remark 4.3. $I_{AB} \geq 0$ *since* $S(\rho_{AB}) \leq S(\rho_A) + S(\rho_B)$, *while equality holds if only if* $\rho_{AB} = \rho_A \otimes \rho_B$. *Moreover, in the commutative setting, the von Neumann entropy becomes the Shannon entropy and* I_{AB} *reduces to the classical mutual information.*

For the rest of the Chapter, we shall take the environment E as classical Markov chain: namely, we assume that the probability distribution $p_{\boldsymbol{i}_{[-a,b]}}$ satisfies

$$p_{\boldsymbol{i}_{[-a,b]}} = T_{i_b i_{b-1}} \, T_{i_{b-1} i_{b-2}} \cdots T_{i_{-a+1} i_a} \, p_{i_{-a}} \tag{4.43}$$

for some stochastic matrix $T = [T_{ij}]$, $T_{ij} \geq 0$, $\sum_{i=1}^d T_{ij} = 1$ with stationary probability vector $\boldsymbol{p}$, $p_i \geq 0$, $\sum_{i=1}^d p_i = 1$, $T\boldsymbol{p} = \boldsymbol{p}$ so that $\mathrm{Tr}_a \rho_E^{[a,b]} = \rho_E^{[a+1,b]} = \rho_E^{[a,b-1]}$ and shift-invariance of the environment state is ensured, that is $\rho_E^{[a,b]} = \rho_E^{[a+n,b+n]}$ for all $n \in \mathbb{N}$.

4.2.1 Pauli dynamics

To ultimately be able to control the divisibility properties of the dynamical map Λ_n, let the subsystem be a qubit ($d = 2$), and choose the dimension of the classical spins as $D = d^2 = 4$ dimensional. Furthermore, we parametrize the Markov transition matrix T in (4.43) as

$$T = \begin{pmatrix} p_0 & p_0 & p_0 & p_0 \\ p & p+\Delta & p-\Delta & p \\ p & p-\Delta & p+\Delta & p \\ r & r & r & r \end{pmatrix} \tag{4.44}$$

with positive parameters such that

$$0 \leq \Delta \leq p \leq \frac{1}{2}\,, \qquad p_0 + 2p + r = 1\,, \tag{4.45}$$

and with invariant probability vector $\boldsymbol{p} = (p_0, p, p, r)$. Furthermore, the four maps $\phi_i, i = 0, \ldots, 3$ are taken as Pauli maps:

$$\phi_k[\sigma_j] = \mu_k^{(j)}\,\sigma_j\,, \qquad \mu_0^{(j)} = \mu_k^{(0)} = 1\,, \qquad \mu_k^{(j)} = \varphi^{1-\delta_{jk}}\,, \tag{4.46}$$

for $j \neq 0, k \neq 0$ with φ a real parameter. From (4.40), the unital one-qubit evolution Λ_n results in a Pauli map itself, with

$$\Lambda_n[\sigma_j] = \lambda_n^{(j)}\sigma_j\,, \qquad \lambda_n^{(j)} = \sum_{\boldsymbol{i}_{[1,n]}} p_{\boldsymbol{i}_{[1,n]}}\,\mu^{(j)}_{\boldsymbol{i}_{[1,n]}}\,, \tag{4.47}$$

where $\mu^{(j)}_{\boldsymbol{i}_{[1,n]}} \equiv \prod_{k=1}^n \mu^{(j)}_{i_k}$. Then, we set $\Lambda_n = \Lambda_{n,n-1} \circ \Lambda_{n-1}$ with interwiners $\Lambda_{n,n-1} = \Lambda_n \circ \Lambda_{n-1}^{-1}$ and

$$\Lambda_{n,n-1}[\sigma_j] = \frac{\lambda_n^{(j)}}{\lambda_{n-1}^{(j)}}\,\sigma_j\,. \tag{4.48}$$

Accordingly, when $\Delta = 0$, it follows that $T_{ij} = p_i$, for all j so that the probabilities factorize, $p_{\boldsymbol{i}_{[-a,b]}} = \prod_{k=-a}^b p_{i_k}$, and

$$\rho_E^{[-a,b]} = \rho_E^{(-a)} \otimes \cdots \otimes \rho_E^{(b)}\,, \qquad \rho_E^{(j)} = \sum_{i=0}^{3} p_i\,\Pi_i^{(j)}\,.$$

and, from (4.40), it also follows that such an uncorrelated environment yields a CPTP discrete-time semigroup $\Lambda_n = \Lambda^n$, where $\Lambda[\rho_S] = \sum_{i=0}^3 p_i\,\phi_i^\ddagger[\rho_S]$.

On the contrary, if $\Delta > 0$, the mutual information in (4.42) with $\rho_A = \rho_E^{(k)}$, $\rho_B = \rho_E^{(k+1)}$ and $\rho_{AB} = \rho_E^{[k,k+1]}$ yields

$$I_{k,k+1} = 4p^2\left(\log 2 - h\left(\frac{1+Q}{2}\right)\right)\,, \qquad Q \equiv \frac{\Delta}{p}\,,$$

and $h(x) = -x\log x - (1-x)\log(1-x)$ decreases for $1/2 \leq x \leq 1$. Due to the stationarity of the Markov process, $I_{k,k+1}$ is site independent and the correlations between any two successive environment sites increase with $0 \leq \Delta \leq p$. Furthermore, for $\Delta > 0$ the dynamical map Λ_n is no longer a semigroup and the evolution is governed by the following

Proposition 4.6. *Choosing the maps* ϕ_k *as in* (4.46) *and the transition matrix as in* (4.44), *the spectrum of the dynamics* Λ_n,

$$\Lambda_n[\sigma_j] = \lambda_n^{(j)}\sigma_j\,, \qquad \lambda_n^{(j)} = \sum_{\boldsymbol{i}_{[1,n]}} p_{\boldsymbol{i}_{[1,n]}}\,\mu^{(j)}_{\boldsymbol{i}_{[1,n]}}\,, \qquad j = 0,1,2,3\,,$$

satisfies the following recurrences

$$\lambda_n^{(1,2)} =: \lambda_n = [1-(p+r)(1-\varphi)]\lambda_{n-1} + p\,\Delta(1-\varphi)^2\sum_{j=0}^{n-2}\lambda_j\,[(1+\varphi)\Delta]^{n-j-2}, \tag{4.49}$$

$$\lambda_n^{(3)} = [1-2p\,(1-\varphi)]\,\lambda_{n-1}^{(3)}\,. \tag{4.50}$$

Proof. Due to the form of the transition matrix,

$$T = \begin{pmatrix} p_0 & p_0 & p_0 & p_0\\ p & p & p & p\\ p & p & p & p\\ r & r & r & r\end{pmatrix} + \Delta\begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 1 & -1 & 0\\ 0 & -1 & 1 & 0\\ 0 & 0 & 0 & 0\end{pmatrix},$$

summing over the index i_n in (4.47) yields

$$\lambda_n^{(j)} = A_{n-1}^{(j)}\,\lambda_{n-1}^{(j)} + \Delta\Big(\mu_1^{(j)}-\mu_2^{(j)}\Big)B_{n-1}^{(j)} \qquad \forall j = 0,1,2,3\,, \tag{4.51}$$

where, for $n \geq 1$

$$A_{n-1}^{(j)} = p_0 + p\left(\mu_1^{(j)}+\mu_2^{(j)}\right) + r\mu_3^{(j)}\,,\quad B_{n-1}^{(j)} = \sum_{\boldsymbol{i}_{[1,n-2]}}\Big(T_{1i_{n-2}}\mu_1^{(j)} - T_{2i_{n-2}}\mu_2^{(j)}\Big)p_{\boldsymbol{i}_{[1,n-2]}}\,\mu^{(j)}_{\boldsymbol{i}_{[1,n-2]}}\,, \tag{4.52}$$

with $p_{\boldsymbol{i}^0} = 1$, $T_{i0} = p_i$ and $B_0^{(j)} = 0$. Then, summing over i_{n-2} in the expression for $B_{n-1}^{(j)}$, one gets

$$B_{n-1}^{(j)} = p\left(\mu_1^{(j)}-\mu_2^{(j)}\right)\lambda_{n-2}^{(j)} + \Delta\left(\mu_1^{(j)}+\mu_2^{(j)}\right)B_{n-2}^{(j)}\,, \tag{4.53}$$

and, iterating,

$$\begin{aligned} B_{n-1}^{(j)} &= p\left(\mu_1^{(j)}-\mu_2^{(j)}\right)\left(\lambda_{n-2}^{(j)} + \Delta\left(\mu_1^{(j)}+\mu_2^{(j)}\right)\lambda_{n-3}^{(j)}\right) + \Delta^2\left(\mu_1^{(j)}+\mu_2^{(j)}\right)^2 B_{n-3}^{(j)}\\ &= p\left(\mu_1^{(j)}-\mu_2^{(j)}\right)\sum_{k=0}^{n-2}\lambda_k^{(j)}\,\Delta^{n-k-2}\left(\mu_1^{(j)}+\mu_2^{(j)}\right)^{n-k-2}\,, \end{aligned} \tag{4.54}$$

where we set $\lambda_0^{(j)} = 1$. Since $\mu_j^{(0)} = 1$ for $j = 0,1,2,3$, from (4.45) it follows that $A_{n-1}^{(j)} = p_0 + 2p + r = 1$ and $B_{n-1}^{(j)} = 0$ so that $\lambda_n^{(0)} = 1$ for all $n \in \mathbb{N}$. On the other hand, the choice of the other coefficients $\mu_k^{(j)}$ in (4.46) gives

$$A_{n-1}^{(1,2)} = p_0 + p(1+\varphi) + r\varphi\,,\qquad B_{n-1}^{(1,2)} = \pm p(1-\varphi)\sum_{k=0}^{n-2}\lambda_k^{(1,2)}\Delta^{n-k-2}(1+\varphi)^{n-k-2}, \tag{4.55}$$

$$A_{n-1}^{(3)} = p_0 + 2p\varphi + r\,,\qquad B_{n-1}^{(3)} = 0\,. \tag{4.56}$$

Since $p_0 + 2p + r = 1$ the expressions in (4.49) and (4.50) follow. □

We shall now study the model for two distinct choices of φ in (4.46), corresponding respectively to (1) a unitary coupling, discussed in Section 4.2.2, for which the solution of (4.49) can be analytically computed and (2) a dissipative coupling, presented in Section 4.2.3, for which the natural stroboscopic limit of collisional models [87, 89, 135] is analytically available and allows one to compare the continuous-time scenario with the discrete-time one.

4.2.2 Unitary coupling

Set $\varphi = -1$; then, $\phi_k[X] = \sigma_k X \sigma_k$ and the map (4.37) becomes a "controlled-unitary" typical of collisional models [136, 137]. In this scenario, the interaction between the system S and the environment E is described by means of a unitary matrix $U_\tau = e^{-ig\tau \sum_k \sigma_k \otimes \Pi_k}$ for a duration $\tau = \pi/2g$: $\Phi[X] = U^\dagger_{\pi/2g}\, X\, U_{\pi/2g}$. Only $j = n-2$ contributes to the sum in (4.49), namely

$$\lambda_n^{(1,2)} = (1 - 2(p+r))\, \lambda_{n-1}^{(1,2)} + 4\, p\, \Delta\, \lambda_{n-2}^{(1,2)}\,, \tag{4.57}$$

and in Appendix B.2, the recurrence relations (4.57) and (4.50) are shown to yield

$$\begin{aligned} \lambda_n := \lambda_n^{(1,2)} &= \left(\frac{B_{p,r,\Delta} + A_{p,r}}{2B_{p,r,\Delta}}\right)\left(\frac{B_{p,r,\Delta} + A_{p,r}}{2}\right)^n + \left(\frac{B_{p,r,\Delta} - A_{p,r}}{2B_{p,r,\Delta}}\right)\left(\frac{A_{p,r} - B_{p,r,\Delta}}{2}\right)^n\,, \\ \lambda_n^{(3)} &= (1-4p)^n\,, \end{aligned} \tag{4.58}$$

where we set

$$A_{p,r} \equiv 1 - 2(p+r)\,, \qquad B_{p,r,\Delta} \equiv \sqrt{A_{p,r}^2 + 16\, p\, \Delta}\,. \tag{4.59}$$

Let $A_{p,r} > 0$ so that $\lambda_n > 0$. The type of divisibility of the reduced dynamics depends on the environment correlations as follows.

Proposition 4.7.

i. Λ_n *is CP-divisible if and only if*

$$\frac{\Delta}{A_{p,r}} \leq \frac{r}{2p}\,, \tag{4.60}$$

ii. Λ_n *is P-divisible if and only if*

$$\frac{\Delta}{A_{p,r}} \leq \frac{r}{2p} + \frac{1}{2}\,, \tag{4.61}$$

iii. $\Lambda_n \otimes \Lambda_n$ *is P-divisible if and only if*

$$\frac{\Delta}{A_{p,r}} \leq \frac{r}{2p} + \frac{1}{2} - \frac{1 - \sqrt{1 - 4p(1-2p)}}{4p}\,. \tag{4.62}$$

For the proof, see Appendix B.2. A plot of the above conditions for fixed values of r is displayed in Figure 4.2. Notice that the strength of the environmental correlations Δ governs the divisibility degree of the reduced dynamics, in that $i. \Rightarrow iii. \Rightarrow ii.$; on the other hand, $iii. \not\Rightarrow i.$ (see Remark 4.4 below). To illustrate how the intensity of the environmental correlations relates to the emergence of SBFI, consider $r = 0$ so that $2\Delta \leq A_{p,0}$ and, by $ii.$, Λ_n is guaranteed to

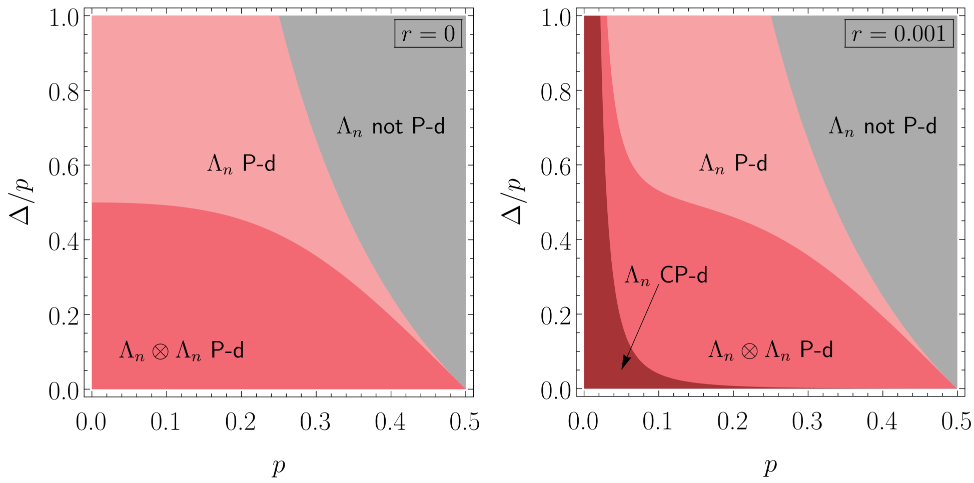

FIGURE 4.2: Divisibility regions for the discrete-time Pauli dynamics Λ_n obtained from the unitary coupling with a Markov chain. The regions are determined by (4.60), (4.61) and (4.62); they are displayed for fixed values of $r = 0$ and $r = 0.001$ as a function of $p \in [0, 1/1]$ and $\Delta/p \in [0, 1]$.

be P-divisible. Then, the discrete-time intertwiners $\Lambda_{n,m}$ are contractive and forbid BFI for a single qubit.

We now consider $p \ll 1$ and proceed with a perturbative analysis. Given any $X = X^\dagger \in M_2(\mathbb{C})$, one has that (see Appendix B.2 for details) up to second order in p,

$$\left\|\Lambda_{n,n-1}[X]\right\|_1 - \|X\|_1 = -K_1\, p + K_2(\Delta)\, p^2 + o(p^2)\,,$$

with $K_1 \geq 0$ and $K_2(\Delta) > 0$ and no discrete-time dependence. Therefore, possible environment correlations ($\Delta \neq 0$) contribute with a positive second order term in the small parameter p; this latter cannot counteract the negative, correlation independent first order term which then makes the maps $\Lambda_{n,n-1}$ contractive for all time-steps n in the regime $0 \leq \Delta \leq p \ll 1$, thus concretely showing why there cannot be BFI for one qubit: the single qubit state distinguishability can never increase in time.

On the other hand, considering now two qubits, again setting $r = 0$, at leading order in $0 \leq p \ll 1$, condition (4.62) implies $\Delta/p \equiv Q \leq 1/2$. Therefore, if $Q > 1/2$, $\Lambda_{n,n-1} \otimes \Lambda_{n,n-1}$ cannot be positive and is thus not contractive. Moreover, being $\Lambda_n \otimes \Lambda_n$ invertible, the collisional dynamics of two qubits certainly exhibits SBFI. Also, the lack of positivity of $\Lambda_{n,n-1} \otimes \Lambda_{n,n-1}$ for $Q > 1/2$ is easily seen by acting on totally symmetric projector $P_+^{(2)}$. Indeed, as shown in Appendix B.2,

$$\left\|\Lambda_{n,n-1} \otimes \Lambda_{n,n-1}[P_+^{(2)}]\right\|_1 - \left\|P_+^{(2)}\right\|_1 = 4p^2\,(2\,Q - 1) > 0\,,$$

hence $\Lambda_{n,n-1} \otimes \Lambda_{n,n-1}$ is non-contractive.

Remark 4.4. *Unlike $\Lambda_t \otimes \Lambda_t$ in continuous time, in discrete time $\Lambda_n \otimes \Lambda_n$ can be P-divisible even if Λ_n is not CP-divisible. Indeed, Theorem 4.1 relies upon the existence of time-local generators. Thus, even if Λ_n is not CP-divisible, $\Lambda_n \otimes \Lambda_n$ need not automatically display SBFI. However, as we saw above, in our case SBFI is triggered by sufficiently strong environment correlations that help to violate the inequality* (4.62).

4.2.2.1 Two-qubit System–Environment correlations: unitary case

We now study the single and two qubit information flows from and into the collisional environment by means of the system-environment correlations as quantified by the mutual information. As we are interested in the discrete-time behaviour of correlations between open system S and subalgebras $\mathcal{A}_E^{[-a,b]}$, we focus upon the following time-dependent mutual information:

$$I^{(n)}_{S[-a,b]} = S\Big(\Omega^{(n)}_S\Big) + S\Big(\Omega^{(n)}_{[-a,b]}\Big) - S\Big(\Omega^{(n)}_{S[-a,b]}\Big)\,, \tag{4.63}$$

with the notation of Section 2.1.4. We consider the latter as a faithful quantifier of the system-chain correlations: an increase/decrease with n of $I^{(n)}_{S[-a,b]}$ would signal increasing/decreasing correlations between system and environment.

First, we restrict the system-environment state at discrete-time n, $\omega^{(n)}_{SE}$, on a local observable $X_S \otimes A_E^{[-a+1,b]}$, $a, b \in \mathbb{N}$. One thus retrieves the evolved local system-environment density matrix given by (see (B.27) in Appendix B.3)

$$\Omega^{(n)}_{S[-a,b]} = \sum_{\boldsymbol{k}_{[-n+1,b]}} p_{\boldsymbol{k}_{[-a,b]}}\, \phi^{\ddagger}_{\boldsymbol{k}_{[-n+1,0]}}[\rho_S] \otimes \Pi^{[-a,b]}_{\boldsymbol{k}_{[-n+1,b]}}\,. \tag{4.64}$$

In Appendix B.3, it is shown that the mutual information (4.63) for the above density matrix (4.64) takes the form

$$I^{(n)}_{S[-a,b]} = S(\Lambda_n[\rho_S]) - \sum_{\boldsymbol{k}_{[1,n]}} p_{\boldsymbol{k}_{[1,n]}}\, S\left(\phi^{\ddagger}_{\boldsymbol{k}_{[1,n]}}[\rho_S]\right)\,.$$

Note that the previous expression depends only on n and not on the size of the portion of the chain considered. Taking now into account two independent qubits coupled to identical chains, the maximal mutual information of their local density matrix reads

$$I^{(n)}_{(S+S)E} = S(\Lambda_n \otimes \Lambda_n[\rho_{S+S}]) \; - \sum_{\boldsymbol{i}_{[1,n]},\boldsymbol{k}_{[1,n]}} p_{\boldsymbol{i}_{[1,n]}}\, p_{\boldsymbol{k}_{[1,n]}}\, S\left(\phi^{\ddagger}_{\boldsymbol{i}_{[1,n]}} \otimes \phi^{\ddagger}_{\boldsymbol{k}_{[1,n]}}[\rho_{S+S}]\right)\,. \tag{4.65}$$

In the case under consideration, the unital maps ϕ_i are unitary; thus (4.65) yields

$$I^{(n)}_{(S+S)E} = S(\Lambda_n \otimes \Lambda_n[\rho_{S+S}]) - S(\rho_{S+S})\,; \tag{4.66}$$

in particular, the variation of the mutual information between two discrete-times $n \geq m$ reduces to checking the behaviour of two-qubit entropy:

$$\Delta I^{(n,m)}_{(S+S)E} \equiv S(\Lambda_n \otimes \Lambda_n[\rho_{S+S}]) - S(\Lambda_m \otimes \Lambda_m[\rho_{S+S}])\,. \tag{4.67}$$

Let us recall that the von Neumann entropy increases under PTP unital maps; thus, when the unital single-qubit reduced dynamics is P-divisible, $\Delta I^{(n,m)}_{SE} \geq 0$. On the other hand, moving to two qubits, choose as a concrete instance

$$r = 0\,, \quad \frac{1}{4} \leq p \leq \frac{1}{2}\,, \quad \Delta = \frac{1-2p}{2} \leq p \leq \frac{1}{2}\,, \tag{4.68}$$

so that Λ_n is P-divisible with (4.61) being saturated and the Pauli eigenvalues (4.58) at the first two successive discrete-time steps satisfy $\lambda_1 = \lambda_2 = A_{p,0} = 1 - 2p$. Further, choosing

$p = 1/4 + \epsilon$, $\epsilon \ll 1$, one can perform a perturbative study and show that the two-qubit completely symmetric projector $\rho_{S+S} = P_+^{(2)}$ witnesses a decrease of the two-qubit von Neumann entropy (details can be found in Appendix B.5),

$$\Delta I^{(2,1)}_{(S+S)E} = -4\log\left(\frac{4}{3}\right)\epsilon^2 < 0\,, \tag{4.69}$$

hence a decrease of system-environment correlations between the first and the second collision.

4.2.3 Dissipative coupling: stroboscopic limit

Let us now take $\varphi = e^{-2\gamma\tau}$, $\gamma, \tau > 0$ so that $\phi_0 = \mathrm{id}_2$ and

$$\phi_k = e^{\tau\mathcal{L}_k}\,, \qquad \mathcal{L}_k[X] = \gamma\,(\sigma_k X \sigma_k - X)\,, \quad k = 1, 2, 3\,. \tag{4.70}$$

In such case, our model is analogous to a collisional model in which the qubit $\mathcal{A}_S$ and the ancilla $\mathcal{A}_E^{(0)}$ undergo a joint dissipative evolution $X_S \otimes A_E^{(0)} \mapsto e^{\tau\mathbb{L}}[X_S \otimes A_E^{(0)}]$ for a time τ, before the shift on the chain is applied (the form of the the GKLS generator $\mathbb{L}$ is reported in Appendix B.4). The Markov chain correlations then contribute with memory effects on top of this Markovian semigroup dynamics.

This choice proves particularly convenient for retrieving a continuous-time dissipative dynamics and thereby for comparing BFI and SBFI within such a continuous framework. The technique employed is the so-called stroboscopic limit defined by $\tau \to 0$, $n \to \infty$, $n\tau \to t$ [89, 135]. Choosing $\Delta = e^{-\kappa\tau}/2$, $p \to 1/2$ and, straightforwardly, $\lambda_t^{(3)} = e^{-2\gamma t}$, while the other two Pauli eigenvalues are both equal to the solution λ_t of the integro-differential equation

$$\dot{\lambda}_t = -\gamma\,\lambda_t + \gamma^2 \int_0^t \mathrm{d}s\, e^{-(\kappa+\gamma)(t-s)}\,\lambda_s\,, \tag{4.71}$$

which yields (see Appendix B.4)

$$\lambda_t = e^{-(\gamma+\frac{\kappa}{2})t}\left[\cosh(Kt) + \frac{\kappa}{2K}\sinh(Kt)\right], \tag{4.72}$$

where $K \equiv \sqrt{\kappa^2 + 4\gamma^2}/2$. We thus obtain a family of P-divisible Pauli dynamical maps, with generator $\mathcal{L}_t[\rho] = \frac{1}{2}\sum_{i=1}^3 \gamma_t^{(i)}(\sigma_i\rho\sigma_i - \rho)$ and rates

$$\gamma_t^{(1)} = \gamma_t^{(2)} = \gamma\,, \tag{4.73}$$

$$\gamma_t^{(3)} = -\frac{2\gamma^2}{\sqrt{\kappa^2+4\gamma^2}\coth\left(\frac{1}{2}t\sqrt{\kappa^2+4\gamma^2}\right)+\kappa}\,, \tag{4.74}$$

with $\gamma_t^{(3)}$ being negative at all times.

4.2.3.1 Two-qubit System–Environment correlations: dissipative case

Let us consider the case $\Delta = 1/2$ and $p = 1/2$. Notice that such case corresponds to $\kappa = 0$ and $\gamma_t^{(3)} = -\gamma\tanh(\gamma t)$, namely to the well known "eternally" non-Markovian evolution firstly

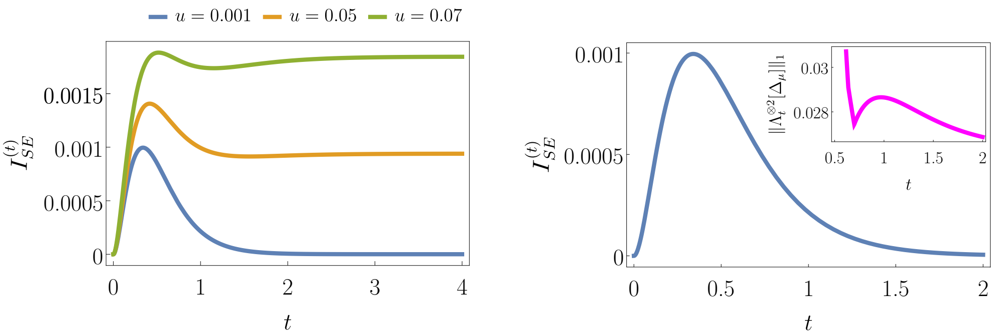

FIGURE 4.3: System-chain mutual information for the state $\rho_X^{(1)}$ with fixed $\mu_1 = \mu_2 = \nu = 1/4, v = i/8$ at different values of real u. For $u = 0.001$, the behaviour is compared with that of the trace norm of the Helstrom matrix between $\Lambda_t \otimes \Lambda_t[\rho_X^{(1)}]$ and $\Lambda_t \otimes \Lambda_t[\rho_X^{(3)}]$, with bias $\mu = 0.52$, $\rho_X^{(3)}$ being in the form (4.76) w.r.t.the computational basis and defined by parameters $\mu_1' = \mu_2' = 1/2, \nu' = 0, u' = 1/8, v' = 0$. One easily sees that the marginals of $\rho_X^{(1)}$ are the maximally mixed state $\mathbb{1}_2/2$.

discussed in [138]. In such case, only two sequences $\boldsymbol{i}_{[1,n]}$ have non-vanishing probabilities and thus contribute to (4.40), namely $\mathbf{1} = 111\ldots$ and $2 = 222\ldots$ with probabilities $p_\mathbf{1} = p_\mathbf{2} = 1/2$. Accordingly, the continuous-time limit of (4.65) reads

$$I_{(S+S)E}^{(t)} = S(\Lambda_t \otimes \Lambda_t[\rho_{S+S}]) - \frac{1}{4} \sum_{i,j=1,2} S(e^{t\mathcal{L}_i} \otimes e^{t\mathcal{L}_j}[\rho_{S+S}]). \tag{4.75}$$

Notice that, unlike in the unitary case, each of the entropies in the second term now grows in time due to the joint unital dissipative evolution of $\mathcal{A}_S$ and $\mathcal{A}_E^{(0)}$ that mixes them. We study $I_{(S+S)E}^{(t)}$ picking ρ_{S+S} of X-shape with respect to the eigenvectors of the matrix $\sigma_1 \otimes \sigma_1$:

$$\rho_X^{(1)} = \begin{pmatrix} \mu_1 & 0 & 0 & u \\ 0 & \nu & v & 0 \\ 0 & \bar{v} & 1-(\mu_1+\mu_2+\nu) & 0 \\ \bar{u} & 0 & 0 & \mu_2 \end{pmatrix}. \tag{4.76}$$

In Appendix B.5, its decomposition in terms of the Pauli matrix tensor products $\{\sigma_i \otimes \sigma_j\}_{ij}$ is reported, from which the time-evolving states entering (4.75) can be easily inferred. In Figure 4.3, we display the system-chain mutual information when the system is initialized in a state of the class (4.76), which displays a growth and collapse of correlations. We also compare such behaviour with that of $\left\|\Lambda_t \otimes \Lambda_t[\Delta_\mu(\rho_X^{(1)}, \rho_X^{(3)})]\right\|_1$, where $\rho_X^{(3)}$ has X shape in the computational basis.

Thus, the system-chain correlations can undergo a decrease for a certain time interval, despite the stationarity of the environment.

Remark 4.5. *As discussed in Remark 2.6.2, the information lost by the system and subjected to BFI is generally thought to be stored either in system-environment correlations or in changes of the environmental state (notice that in our Example, the environment is stationary* (4.39)*) [12, 75]. In Fig.4.3 the mutual information $I_{SE}^{(t)}$ of* (4.75) *is plotted for X states with $\mu_{1,2} = \nu = 1/4$. As for the maximally entangled state state $P_+^{(2)}$ considered in* (4.69)*, these states have maximally mixed marginals.*

For a state ρ_{S+S} *with maximally mixed marginals, using trace preservation and factorization, one shows that*

$$\mathrm{Tr}_{1(2)}(\Lambda_t \otimes \Lambda_t[\rho_{S+S}]) = \Lambda_t[\mathrm{Tr}_{1(2)}(\rho_{S+S})] = \frac{\mathbb{1}_2}{2}\,.$$

Similarly, one checks that the one-qubit local density matrix (4.64)*, obtained by tracing over one of the two open systems together with its own environment, reduces to*

$$\Omega^{(t)}_{S[-a,b]} = \frac{\mathbb{1}_2}{2} \otimes \rho_E^{[-a,b]} \implies I^{(t)}_{SE} = 0. \tag{4.77}$$

For such states, the bipartite correlations have a non-monotonic behaviour in time, while the qubit-chain marginals are uncorrelated at all times due to (4.77)*. Thus, in such case, the information is temporarily stored non-locally in the system-environment correlations.*

In this Chapter, we studied several aspects of SBFI both from the phenomenological perspective of the dynamical map and from the point of view of a concrete collisional model. In Section 4.1, we discussed SBFI in terms of entropies. All Renyi p-entropies decrease under P-divisible evolutions for $p \in (1, \infty)$. Nevertheless P-divisible, not CP-divisible dynamics may display a decrease in the Renyi p-entropies when coupled to an identical copy of themselves, since the compound dynamics necessarily loses P-divisibility. Alas, this is never the case for the Renyi 2-entropy, a standard measure of the purity of a state. On the other hand, we constructed a class of Pauli dynamical maps for which SBFI is witnessed by the Renyi p-entropies with $1 \leq p < 2$ and by the von Neumann entropy ($p \to 1^+$). Interestingly, a sufficient condition for that is to consider time-dependent Pauli generators with a zero eigenvalue in their spectrum at some given instant of time so to be at the border of P-divisibility. Furthermore, no entanglement is needed at any time of the evolution in order to witness the effect.

In Section 4.2 we studied SBFI both in the discrete and continuous-time regimes for an open system of two qubits, each coupled to a classical Markov chain. The emergence of bipartite memory effects has been investigated by means of the system-chain mutual information of local density matrices obtained through the algebraic approach. Growths and collapses of correlations have been detected for both unitary and dissipative collisions: in the former case, the mutual information is simply the systems entropy up to a constant. Then, one proceeds similarly as done in Section 4.1.3. In the dissipative case, instead, the mutual information has the form of a Jensen-Shannon divergence. The non-monotonicity of the aforementioned quantities provides a clear-cut physical interpretation in terms of system-environment correlations. Interestingly, although information might be stored in and released through classical correlations, SBFI has no classical counterpart. Its quantum character, while not manifest through entanglement of the states involved, is exposed by the quantumness of the bipartite Helstrom ensemble.

CHAPTER 5

Quantum dynamical entropy and non-Markovianity

In Chapter 1, we emphasized the interpretation of the ALF entropy as the information gained per time step about the dynamical system through iterated measurements, in analogy with its classical counterpart, the KS entropy. Despite the earlier approach by Lindblad concerned irreversible processes [7], the vast majority of applications of ALF entropy have been developed in the context of reversible systems displaying chaotic behaviour [27]. After having adapted the framework of symbolic models to the study of open systems in Section 2.2, we now turn to the discussion of their dynamical entropy. Since the ALF entropy encodes information about the full multi-time statistics of the process, we aim to exploit it for the study of the system–environment exchange of information, with the goal of better understanding the mechanisms underlying memory effects. Moreover, these results can be compared with properties of the reduced dynamics, such as divisibility and information revivals, which were the focus of Chapters 3 and 4. In particular, the concrete example of a collisional model presented in Section 4.2 allows for an exact computation of the open-system ALF entropy and its interpretation in terms of the parameters of the model. Finally, we move to investigate the dynamics in the GNS representation: in this context, further connections emerge between the ALF entropy and the superactivation of memory effects in the reduced dynamics.

5.1 Open-system ALF entropy

Consider the triple $(\mathcal{A}_S \otimes \mathcal{A}_E, \omega_S \otimes \omega_E, \Theta)$, where the joint system-environment evolution consists in a discrete one-parameter group of automorphisms that leave $\omega_S \otimes \omega_E$ invariant, with ρ_S a faithful density matrix. As in Section 2.2, by means of repeated measurements, the Alicki-Fannes procedure allows to extract a symbolic model of the dynamics which, in the open-system setting, takes the form of a coarse-grained density matrix of the type

$$\begin{aligned}\rho_S\left[\mathcal{X}^{(n)}\right]_{\boldsymbol{a},\boldsymbol{b}} &= \omega_S \otimes \omega_E\left(X_{\boldsymbol{b}}^{(n)\dagger} X_{\boldsymbol{a}}^{(n)}\right) \qquad (5.1)\\ &= \omega_S \otimes \omega_E\Big(X_{b_0}^\dagger \otimes \mathbb{1}_E\, \Theta(X_{b_1}^\dagger \otimes \mathbb{1}_E) \dots \Theta^{n-1}(X_{b_{n-1}}^\dagger X_{a_{n-1}} \otimes \mathbb{1}_E) \dots \Theta(X_{a_1} \otimes \mathbb{1}_E) X_{a_0} \otimes \mathbb{1}_E\Big).\end{aligned}$$

where the OPU $\mathcal{X}$ is constrained to be a subset of $\mathcal{A}_S$ due to the inaccessibility of the environment. In other words, the entries of (5.1) are the multi-time correlation functions obtained from measuring the open system, only. As in Section 1.5, to the density matrix (5.1) there naturally correspond entropy rates

$$\mathfrak{h}_S(\Theta, \mathcal{X}) := \mathfrak{h}_{\omega_{SE}}(\Theta, \mathcal{X}) = \limsup_n \frac{1}{n} S\left(\rho_S\left[\mathcal{X}^{(n)}\right]\right), \qquad \mathcal{X} \subseteq \mathcal{A}_S. \tag{5.2}$$

Then, one maximizes over all physically admissible OPUs, namely, over the POVM measurements of the open system ($\mathcal{B} = \mathcal{A}_S$) obtaining the *open-system ALF entropy*:

$$\mathfrak{h}_S(\Theta) := \sup_{\mathcal{X} \subseteq \mathcal{A}_S} \mathfrak{h}_S(\Theta, \mathcal{X}). \tag{5.3}$$

When the state ω_S corresponds to a faithful density matrix $\rho_S > 0$, the GNS representation associated with the state $\omega_{SE} = \omega_S \otimes \omega_E$ is the tensor product $(\mathcal{H}_S \otimes \mathcal{H}_E, \pi_S \otimes \pi_E, |\sqrt{\rho_S}\rangle \otimes |\Omega_E\rangle)$ of the GNS representations of S and E, respectively (see Proposition 1.3). Here, $|\sqrt{\rho_S}\rangle$ is the canonical purification of ρ_S as in Example 1.3. If $\omega_S \otimes \omega_E$ is also invariant under the global automorphism Θ, the latter can be implemented unitarily

$$\begin{aligned} U_\Theta^\dagger \pi_S \otimes \pi_E(X_S \otimes X_E)\left|\sqrt{\rho_S} \otimes \Omega_E\right\rangle &:= \pi_S \otimes \pi_E(\Theta(X_S \otimes X_E))\left|\sqrt{\rho_S} \otimes \Omega_E\right\rangle, \\ \mathbb{U}_\Theta^\ddagger[\pi_S \otimes \pi_E(X_S \otimes X_E)] &:= U_\Theta^\dagger \pi_S \otimes \pi_E(X_S \otimes X_E) U_\Theta = \pi_S \otimes \pi_E(\Theta(X_S \otimes X_E)), \\ U_\Theta\left|\sqrt{\rho_S} \otimes \Omega_E\right\rangle &= \left|\sqrt{\rho_S} \otimes \Omega_E\right\rangle. \end{aligned} \tag{5.4}$$

Then, by (1.128), the entropy of $\rho_S[\mathcal{X}^{(n)}]$ can be equivalently computed in the GNS representation as the entropy of

$$(\mathbb{U}_\Theta \circ (\mathbb{X} \otimes \mathrm{id}_S \otimes \mathrm{id}_E))^n \left[\, |\sqrt{\rho_S}\rangle\langle\sqrt{\rho_S}| \otimes |\Omega_E\rangle\langle\Omega_E| \,\right] \tag{5.5}$$

$$\text{where} \quad M_d(\mathbb{C}) \ni Y \longmapsto \mathbb{X}[Y] = \sum_{a=1}^{|\mathcal{X}|} X_a\, Y\, X_a^\dagger$$

is the CPTP map associated to the OPU $\mathcal{X}$ as in Section 1.4.3 and Appendix C.1. Notice that, in (5.5), measurements act non-trivially only on the first party of the purified open system.

Remark 5.1. *Following the general discussion of Section 1.5, we can then interpret $\mathfrak{h}_S(\Theta)$ as the entropy production rate due to the discrete semigroup evolution* (5.5), *that describes repeated measurements on S intertwining the S + E unitary evolution: it quantifies the maximal rate of information about the dynamics that is extractable by measurements on the open system S only.*

5.1.1 ALF entropy in the Quantum Regression regime

Let us first consider the open system symbolic dynamics in the Quantum Regression regime. Since we are assuming a group automorphism $\Theta_n = \Theta^n$ and an invariant state $\omega_S \otimes \omega_E$, ω_S being described by a faithful density matrix $\rho_S > 0$, Proposition 2.13 establishes that the entries of the coarse-grained density matrix (5.1) in the QR regime become, for all $n \geq 1$,

$$\rho_S\left[\mathcal{X}^{(n)}\right]_{\boldsymbol{ab}} = \omega_S\left(X_{b_0}^\dagger\, \Lambda^\ddagger\left[X_{b_1}^\dagger\, \Lambda^\ddagger\left[\dots\, \Lambda^\ddagger\left[X_{b_{n-1}}^\dagger X_{a_{n-1}}\right] \dots\right] X_{a_1}\right] X_{a_0}\right), \tag{5.6}$$

where $\Lambda \equiv \Lambda_1$. In this scenario, it is instructive to compare the behaviour of the ALF entropy under reversible and non-reversible dynamics.

Finite, closed system. Let S be a closed system not interacting with its environment, namely

$$\Theta(X_S \otimes X_E) = \Theta_S(X_S) \otimes \Theta_E(X_E)\,,$$

with $\Theta_S(X_S) = U_S^\dagger X_S U_S$, $U_S^\dagger U_S = U_S U_S^\dagger = \mathbb{1}_d$ and Θ_E being automorphisms of $\mathcal{A}_S$ and $\mathcal{A}_E$, respectively. The coarse-grained density matrix (5.1) then becomes

$$\rho_S[\mathcal{X}^{(n)}]_{\boldsymbol{a},\boldsymbol{b}} = \mathrm{Tr}\left(\rho_S\, X_{b_0}^\dagger\, \Theta_S(X_{b_1}^\dagger\, \dots \Theta_S(X_{b_n}^\dagger X_{a_n})\, \dots\, X_{a_1})\, X_{a_0}\right). \tag{5.7}$$

so that QR holds trivially, but for a reversible dynamics. Then, as explained in Example 1.10,

$$\mathfrak{h}_S(\Theta) = 0 \qquad \text{for a finite-level closed system } S. \tag{5.8}$$

Finite, open system. One can determine whether the QR property holds from partition-specific entropy rates, as already noted in [7]. We refer again to the OPU introduced in Remark 1.18 and used in Theorem 2.12:

$$\mathcal{F} := \{F_{a,a'}\}_{a,a'=1}^d\,, \quad F_{a,a'} = \sqrt{r_a}\,|r_a\rangle\langle r_{a'}| \in \mathcal{A}_S\,, \tag{5.9}$$

where r_a and $|r_a\rangle$ are the eigenvalues, respectively, eigenvectors of ρ_S; the above OPU thus leaves $\omega_S \otimes \omega_E$ invariant.

Corollary 5.1. *The entropy rate associated with the OPU $\mathcal{F}$ is bounded from above by:*

$$\mathfrak{h}_S(\Theta, \mathcal{F}) \;\leq\; S\left(\Lambda \otimes \mathrm{id}_d\left[\,|\sqrt{\rho_S}\rangle\langle\sqrt{\rho_S}|\,\right]\right). \tag{5.10}$$

Furthermore, the inequality saturates to an equality if and only if $\mathbb{T}_n = \bigotimes_{k=1}^n \Lambda$, where $\mathbb{T}_n$ was defined in (2.78)*, or, equivalently, if and only if QR condition holds.*

The proof is reported in Appendix D.3. Suppose that QR-Markovianity holds. Then, (5.10) saturates to an equality and

$$\mathfrak{h}_S(\Theta) \;\geq\; \mathfrak{h}_S(\Theta, \mathcal{F}) \;=\; S\left(\Lambda \otimes \mathrm{id}_d\left[\,|\sqrt{\rho_S}\rangle\langle\sqrt{\rho_S}|\,\right]\right), \tag{5.11}$$

which are strictly positive quantities unless Λ is an isometry, $\Lambda[X] = VXV^\dagger, V^\dagger V = \mathbb{1}_d$. Thus, for finite quantum systems, the dynamical entropy vanishes in all reversible dynamical settings, while it is strictly positive for an irreversible QR-Markovian one.

5.2 Dynamical entropy and collisional models

We now aim to investigate the behaviour of the open-system ALF entropy beyond the QR regime, in which memory effects are expected to play a significant role in the dynamics. We shall refer again to the collisional framework introduced in Section 2.1.4. Within this scheme, we first show that the open-system dynamical entropy $\mathfrak{h}_S(\Theta)$ cannot exceed the dynamical entropy of the shift on the quantum spin chain $\mathcal{A}_E$, which provides the environment of the finite open quantum system S and was computed in Example 1.13.

Proposition 5.1. *Let $\omega_S \otimes \omega_E$ be a Θ-invariant state. Then,*

$$\mathfrak{h}_S(\Theta) \;\leq\; \mathfrak{S}_{\omega_E} + \log(D)\,, \tag{5.12}$$

where $\mathfrak{S}_{\omega_E}$ is the mean von Neumann entropy of the chain (1.30).

Proof. With respect to a Θ-invariant state, one proceeds as in Example 1.13. After n steps of the dynamics the algebra of the system is mapped into the local algebras $\mathcal{A}_S \otimes \mathcal{A}_E^{[1,n]}$, due to the action of the shift,

$$\Theta^n(\mathcal{A}_S \otimes \mathbb{1}_E) \subseteq \mathcal{A}_S \otimes \mathcal{A}_E^{[1,n]}. \tag{5.13}$$

Accordingly, time-refinements of OPU elements initially pertaining only to the system, $X_a \in \mathcal{A}_S$ consist of operators

$$\widetilde{X}_{\boldsymbol{a}}^{[1,n]} := \Theta^n(X_{a_n} \otimes \mathbb{1}_E) \dots \Theta(X_{a_1} \otimes \mathbb{1}_E) X_{a_0} \otimes \mathbb{1}_E \,,$$

which are localized $\mathcal{A}_S \otimes \mathcal{A}_E^{[1,n]}$. Accordingly, the correlation matrix reads:

$$\rho\left[\mathcal{X}^{(n+1)}\right]_{\boldsymbol{a},\boldsymbol{b}} = \omega_{SE}\left(\widetilde{X}_{\boldsymbol{b}}^{[1,n]\dagger}\,\widetilde{X}_{\boldsymbol{a}}^{[1,n]}\right) = \mathrm{Tr}\left(\Omega_{S[1,n]}\,\widetilde{X}_{\boldsymbol{b}}^{[1,n]\dagger}\,\widetilde{X}_{\boldsymbol{a}}^{[1,n]}\right),$$

with $\Omega_{S[1,n]} = \rho_S \otimes \rho_E^{[1,n]} \in M_d(\mathbb{C}) \otimes M_D^{[1,n]}(\mathbb{C})$. Hence, from Proposition 1.12, together with the fact that $S(\Omega_{S[1,n]}) = S(\rho_S) + S(\rho_E^{[1,n]})$ and

$$S\left(\sum_{\boldsymbol{a}} \widetilde{X}_{\boldsymbol{a}}^{[1,n]}\,\Omega_{S[1,n]}\,\widetilde{X}_{\boldsymbol{a}}^{[1,n]\dagger}\right) \le \log(dD^n)\,,$$

the following upper bound ensues:

$$S\left(\rho\left[\mathcal{X}^{(n)}\right]\right) \le S(\rho_S) + S\left(\rho_E^{[1,n]}\right) \le \log(d) + n\log(D)\,.$$

Dividing by n and taking the $\limsup$ on both sides, and maximizing over all OPUs on the left-hand side, yields inequality (5.12). □

Remark 5.2. *Consider the quantum mutual information*

$$I_n(\omega_E) := S(\rho_E^{[1,n]}) + S(\rho_E^{\{1\}}) - S(\rho_E^{[1,n+1]}) = S_1 - (S_{n+1} - S_n)\,,$$

quantifying the correlations between n sites of the spin chain and subsequent one. From (1.30)*, the large n limit yields*

$$\mathfrak{I}_{\omega_E} := \lim_n I_n(\omega_E) = S_1 - \mathfrak{S}_{\omega_E} \ge 0, \tag{5.14}$$

where the inequality saturates to an equality for a Bernoulli source. Thus, the stronger the correlations in (5.14)*, namely the lower the mean entropy of the environment, the tighter the bound in* (5.12) *on the maximal entropy production of the open system S.*

5.2.1 Classical coupling

In Chapter 4, we discussed a particular case of a collisional model with a classical environment, described by an Abelian spin chain algebra endowed with a shift-invariant state, and repeatedly interacting with the system through a CP unital map Φ taken as a "classical control" (4.37). We shall now focus again on this class of collisional models, whereby the coupling map is given by the automorphism

$$\Phi[X_S \otimes A_E^{[-a,b]}] = \sum_{k=1}^{D} \phi_k[X_S] \otimes \Pi_k^{(0)} A_E^{[-a,b]} \Pi_k^{(0)}, \quad \phi_k[X_S] = U_k^\dagger X_S U_k\,,\ U_k U_k^\dagger = U_k^\dagger U_k = \mathbb{1}_d. \tag{5.15}$$

with $X_S \in \mathcal{A}_S$ and $A_E^{[-a,b]} \in \mathcal{A}^{[-a,b]}$. In Proposition 4.5, we showed that the reduced dynamics of the system after n collisions is governed by the CPTP evolution

$$\Lambda_n = \sum_{\boldsymbol{i}_{[1,n]}} p_{\boldsymbol{i}_{[1,n]}} \, \phi^{\ddagger}_{\boldsymbol{i}_{[1,n]}} \,, \tag{5.16}$$

with $\phi^{\ddagger}_{\boldsymbol{i}_{[1,n]}} = \phi^{\ddagger}_{i_n} \dots \phi^{\ddagger}_{i_1}$. We first show that the coarse-grained density matrices obey a convex decomposition similar to that of the dynamical map (5.16).

Structure of the coarse-grained density matrix Basing on the collisional model with a classical environment of Section 4.2, the coarse-grained density matrix takes the form

$$\rho_S\left[\mathcal{X}^{(n+1)}\right] = \sum_{\boldsymbol{i}_{[1,n]}} p_{\boldsymbol{i}_{[1,n]}} \, \rho_S\left[\mathcal{X}_{\boldsymbol{i}_{[1,n]}}\right] , \tag{5.17}$$

where $\boldsymbol{i}_{[1,n]} = i_1 \dots i_n$, and indexed coarse-grained density matrices

$$\begin{aligned}
\rho_S\left[\mathcal{X}_{\boldsymbol{i}_{[1,n]}}\right]_{\boldsymbol{a},\boldsymbol{b}} &= \omega_S\Big(X^\dagger_{b_0}\phi_{\boldsymbol{i}_{\{1\}}}[X^\dagger_{b_1}]\phi_{\boldsymbol{i}_{[1,2]}}[X^\dagger_{b_2}]\dots\phi_{\boldsymbol{i}_{[1,n]}}[X^\dagger_{b_n}X_{a_n}]\dots\phi_{\boldsymbol{i}_{[1,2]}}[X_{a_1}]\phi_{\boldsymbol{i}_{\{1\}}}[X_{a_1}]X_{a_0}\Big), \\
&= \omega_S\Big(X^\dagger_{b_0}\phi_{i_1}\Big[X^\dagger_{b_1}\phi_{i_2}\Big[X_{b_2^\dagger}\dots\phi_{i_n}\Big[X^\dagger_{b_n}X_{a_n}\Big]\dots X_{a_2}\Big]X_{a_1}\Big]X_{a_0}\Big)
\end{aligned} \tag{5.18}$$

with respect to the automorphisms $\phi_{\boldsymbol{i}_{[1,k]}} = \phi_{i_1} \circ \cdots \circ \phi_{i_k}$ and with

$$\mathcal{X}_{\boldsymbol{i}_{[1,n]}} = \left\{\left(X_{\boldsymbol{i}_{[1,n]}}\right)_{\boldsymbol{a}}\right\}_{\boldsymbol{a}} , \qquad \left(X_{\boldsymbol{i}_{[1,n]}}\right)_{\boldsymbol{a}} := \phi_{\boldsymbol{i}_{[1,n]}}[X_{a_n}]\cdots\phi_{\boldsymbol{i}_1}[X_{a_1}]X_{a_0} \,,$$

forming a OPU in $\mathcal{A}_S$.

Proof. Consider $X_{a_1} \in \mathcal{A}_S$; then,

$$\begin{aligned}
\Theta[X_{a_1} \otimes \mathbb{1}_E] &= \sum_{k_1} \phi_{k_1}[X_{a_1}] \otimes \Pi^{(1)}_{k_1} \,, \\
\Theta^2[X_{a_2} \otimes \mathbb{1}_E] &= \sum_{k_1,k_2} \phi_{k_1} \circ \phi_{k_2}[X_{a_2}] \otimes \Pi^{(1)}_{k_1} \otimes \Pi^{(2)}_{k_2} \,,
\end{aligned}$$

so that the refined partition after the first two time steps after consists of operators

$$\Theta^2[X_{a_2} \otimes \mathbb{1}_E]\Theta[X_{a_1} \otimes \mathbb{1}_E]X_{a_0} \otimes \mathbb{1}_E = \sum_{k_1,k_2} \phi_{k_1} \circ \phi_{k_2}[X_{a_2}]\phi_{k_1}[X_{a_1}]X_{a_0} \otimes \Pi^{(1)}_{k_1} \otimes \Pi^{(2)}_{k_2} \,.$$

Thus, one has

$$\begin{aligned}
X^{(n+1)}_{\boldsymbol{a}} &= \Theta^n[X_{a_n} \otimes \mathbb{1}_E]\dots\Theta[X_{a_1} \otimes \mathbb{1}_E]X_{a_0} \otimes \mathbb{1}_E \\
&= \sum_{\boldsymbol{k}_{[1,n]}} \phi_{\boldsymbol{k}_{[1,n]}}[X_{a_n}]\phi_{\boldsymbol{k}_{[1,n-1]}}[X_{a_{n-1}}]\dots\phi_{\boldsymbol{k}_{\{1\}}}[X_{a_1}]X_{a_0} \otimes \bigotimes_{j=1}^{n} \Pi^{(j)}_{k_j} \,.
\end{aligned} \tag{5.19}$$

The elements of the coarse-grained density matrix are

$$\omega_{SE}\left(X_{\boldsymbol{b}}^{(n+1)\dagger}X_{\boldsymbol{a}}^{(n+1)}\right) = \sum_{\boldsymbol{k}_{[1,n]}} p_{\boldsymbol{k}_{[1,n]}}\,\omega_S\Big(X_{b_0}^\dagger\phi_{k_1}[X_{b_1}^\dagger]\dots\phi_{\boldsymbol{k}_{[1,n]}}[X_{b_n}^\dagger X_{a_n}]\dots\phi_{k_1}[X_{a_1}]X_{a_0}\Big).$$

Notice that for fixed $\boldsymbol{i}_{[1,n]}$ the set $\left\{X_{\boldsymbol{i}_{[1,n]},\boldsymbol{a}}\right\}_{\boldsymbol{a}}$, with

$$M_d(\mathbb{C}) \ni X_{\boldsymbol{i}_{[1,k]},\boldsymbol{a}} := \phi_{\boldsymbol{i}_{[1,k]}}[X_{a_k}]\dots\phi_{\boldsymbol{i}_{[1,2]}}[X_{a_2}]\phi_{i_1}[X_{a_1}]X_{a_0}\,,$$

forms an OPU. Indeed, the maps $\phi_{\boldsymbol{i}_{[1,k]}}$ are automorphisms: $\phi_{\boldsymbol{i}_{[1,k]}}[X^\dagger] = (\phi_{\boldsymbol{i}_{[1,k]}}[X])^\dagger$ and $\phi_{\boldsymbol{i}_{[1,k]}}[XY] = \phi_{\boldsymbol{i}_{[1,k]}}[X]\phi_{\boldsymbol{i}_{[1,k]}}[Y]$. In the case $k=2$, for example,

$$\begin{aligned}\sum_{\boldsymbol{a}} X_{\boldsymbol{i}_{[1,2]},\boldsymbol{a}}^\dagger X_{\boldsymbol{i}_{[1,2]},\boldsymbol{a}} &= \sum_{a_0,a_1} X_{a_0}^\dagger\phi_{i_1}[X_{a_1}^\dagger]\phi_{\boldsymbol{i}_{[1,2]}}\left[\sum_{a_2}X_{a_2}^\dagger X_{a_2}\right]\phi_{i_1}[X_{a_1}]X_{a_0}\\ &= \sum_{a_0}X_{a_0}^\dagger\phi_{i_1}\left[\sum_{a_1}X_{a_1}^\dagger X_{a_1}\right]X_{a_0} = \sum_{a_0}X_{a_0}^\dagger X_{a_0} = \mathbb{1}_S\,. \qquad \square\end{aligned}$$

Remark 5.3. *If the classical chain represents a Bernoulli process,* $p_{\boldsymbol{k}_{[1,n]}} = \prod_{j=1}^n p_{k_j}$ *, the collisional environment is fully uncorrelated and the reduced dynamics is a semigroup,* $\Lambda_n = \Lambda_1^n$*,* $\Lambda_1 = \sum_k p_k\,\phi_k$*. Furthermore, QR holds. For example, after two iterations of the dynamics,*

$$\begin{aligned}\rho_S[\mathcal{X}^{(3)}]_{\boldsymbol{a},\boldsymbol{b}} &= \sum_{k_2,k_1} p_{k_1}p_{k_2}\,\omega_S\left(X_{b_0}^\dagger\phi_{k_1}\left[X_{b_1}^\dagger\phi_{k_2}[X_{b_2}^\dagger X_{a_2}]X_{a_1}\right]X_{a_0}\right)\\ &= \omega_S\left(X_{b_0}^\dagger\sum_{k_1}p_{k_1}\phi_{k_1}\left[X_{b_1}^\dagger\sum_{k_2}p_{k_2}\phi_{k_2}\left[X_{b_2}^\dagger X_{a_2}\right]X_{a_1}\right]\right)\\ &= \omega_S\left(X_{b_0}^\dagger\Lambda_1\left[X_{b_1}^\dagger\Lambda_1\left[X_{b_2}^\dagger X_{a_2}\right]X_{a_1}\right]X_{a_0}\right)\,,\end{aligned}$$

so that, by just knowing Λ_1*, one can compute the full coarse-grained density matrix.*

Upper and lower bounds to the ALF entropy Given the convex combination (5.17), the von Neumann entropy of the mixture (5.17) can be then bounded from above by

$$\begin{aligned}S\left(\rho_S\left[\mathcal{X}^{(n+1)}\right]\right) &\le H\left(\pi^{[1,n]}\right) + \sum_{\boldsymbol{i}_{[1,n]}} p_{\boldsymbol{i}_{[1,n]}} S\left(\rho_S\left[\mathcal{X}_{\boldsymbol{i}_{[1,n]}}\right]\right)\\ &\le H\left(\pi^{[1,n]}\right) + 2\,\log(d)\,,\end{aligned} \tag{5.20}$$

where Proposition 1.12 was used. Dividing by n and taking the limsup of both sides of (5.20), and taking the supremum on the l.h.s., one gets that the dynamical entropy of the open system is upper-bounded by the KS entropy of the chain,

$$\mathfrak{h}_S(\Theta) \le \lim_n \frac{1}{n} H\left(\pi^{[1,n]}\right) = \mathfrak{S}_{\omega_E}\,. \tag{5.21}$$

We now show that the latter inequality can be saturated.

Proposition 5.2. *Consider a d-level system collisionally interacting with a classical stationary spin chain through* (5.15)*. With respect to the invariant state $\mathbb{1}_d/d\otimes\omega_E$ and the OPU $\mathcal{F}$ defined in* (5.9)*, the coarse-grained density matrix takes the form*

$$\rho_S\left[\mathcal{F}^{(n+1)}\right]=\frac{\mathbb{1}_d}{d}\otimes\frac{\mathbb{1}_d}{d}\otimes\left(\mathbb{T}_n\otimes\mathrm{id}_d^{\otimes n}\left[\left|\sqrt{\frac{\mathbb{1}_d}{d}}^{\otimes n}\right\rangle\!\left\langle\sqrt{\frac{\mathbb{1}_d}{d}}^{\otimes n}\right|\right]\right), \tag{5.22}$$

$$\textit{with}\quad \mathbb{T}_n=\sum_{\boldsymbol{i}_{[1,n]}}p_{\boldsymbol{i}_{[1,n]}}\,\phi_{\boldsymbol{i}_{[1,n]}}^{\otimes[1,n]\ddagger},\qquad \phi_{\boldsymbol{i}_{[1,n]}}^{\otimes[1,n]}:=\bigotimes_{k=1}^{n}\phi_{i_k},\quad \phi_{i_k}[\,\cdot\,]=U_{i_k}^\dagger\cdot U_{i_k}\,.$$

Moreover, if the unitaries in (5.15) *are such that* $\mathrm{Tr}\left(U_j^\dagger U_k\right)=d\delta_{jk}$,

$$\mathfrak{h}_S(\Theta)=\mathfrak{h}_S(\Theta,\mathcal{F})=\lim_n\frac{1}{n}H\left(\pi^{[1,n]}\right)=\mathfrak{S}_{\omega_E}\,, \tag{5.23}$$

that is, the ALF entropy of the system is equal to the mean Shannon entropy of the chain.

Proof. From its definition (2.78) in Theorem 2.12, the map $\mathbb{T}_n^\ddagger$ reads

$$\mathbb{T}_n^\ddagger\left[\bigotimes_{k=0}^{n-1}A_k^{(k-n)}\right]=\omega_E\Big(\left(\Theta\circ\sigma_S\otimes\mathrm{id}_E\right)^n\left[\bigotimes_{k=0}^{n-1}A_k^{(k-n)}\otimes\mathbb{1}_E\right]\Big),\quad\text{with}\quad \Theta=\mathrm{id}_S\otimes\sigma_E\circ\Phi\,.$$

Note that in the one-step automorphism $\Theta\circ\sigma_S\otimes\mathrm{id}_E$ the shift appears twice: the right-most one, σ_S, acts on the chain $M_d^{\mathbb{Z}}(\mathbb{C})$ of infinite copies of the open system (as in Theorem (2.12)), while σ_E is the collisional shift on the spin chain environment. Then, let us first evaluate

$$\left(\Theta\circ\sigma_S\otimes\mathrm{id}_E\right)\left[\bigotimes_{k=0}^{n-1}A_k^{(k-n)}\otimes\mathbb{1}_E\right]=\bigotimes_{k=0}^{n-2}A_k^{(k-n+1)}\otimes\sum_{i_n}\phi_{i_n}\left[A_{n-1}^{(0)}\right]\otimes\Pi_{i_n}^{(1)}\,,$$

where Π_{i_n} is localized in the first site of the environment chain due to the action of σ_E. The second iteration leads to

$$\begin{aligned}\left(\Theta\circ\sigma_S\otimes\mathrm{id}_E\right)^2&\left[\bigotimes_{k=0}^{n-1}A_k^{(k-n)}\otimes\mathbb{1}_E\right]=\\&=\bigotimes_{k=0}^{n-3}A_k^{(k-n+2)}\otimes\sum_{i_{n-1}i_n}\phi_{i_{n-1}}\otimes\phi_{i_n}\left[A_{n-2}^{(0)}\otimes A_{n-1}^{(1)}\right]\otimes\Pi_{i_{n-1}}^{(1)}\otimes\Pi_{i_n}^{(2)}\,.\end{aligned}$$

Hence, after n iterations:

$$\left(\Theta\circ\sigma_S\otimes\mathrm{id}_E\right)^n\left[\bigotimes_{k=0}^{n-1}A_k^{(k-n)}\otimes\mathbb{1}_E\right]=\sum_{\boldsymbol{i}_{[1,n]}}\phi_{\boldsymbol{i}_{[1,n]}}^{\otimes[1,n]}\left[\bigotimes_{k=0}^{n-1}A_k^{(k)}\right]\otimes\Pi_{\boldsymbol{i}_{[1,n]}}^{[1,n]}\,. \tag{5.24}$$

Acting on the latter with the conditional expectation ω_E, we get

$$\mathbb{T}_n^\ddagger\left[\bigotimes_{k=0}^{n-1}A_k^{(k)}\right]=\sum_{\boldsymbol{i}_{[1,n]}}\phi_{\boldsymbol{i}_{[1,n]}}^{\otimes[1,n]}\left[\bigotimes_{k=0}^{n-1}A_k^{(k)}\right]\omega_E\left(\Pi_{\boldsymbol{i}_{[1,n]}}^{[1,n]}\right)=\sum_{\boldsymbol{i}_{[1,n]}}p_{\boldsymbol{i}_{[1,n]}}\,\phi_{\boldsymbol{i}_{[1,n]}}^{\otimes[1,n]}\left[\bigotimes_{k=0}^{n-1}A_k^{(k)}\right].$$

With respect to the tracial state $\rho_S = \mathbb{1}_d/d$, the coarse-grained density matrix then reads:

$$\rho_S\left[\mathcal{F}^{(n+1)}\right] = \frac{\mathbb{1}_d}{d} \otimes \frac{\mathbb{1}_d}{d} \otimes \left(\mathbb{T}_n \otimes \mathrm{id}_d^{\otimes n}\left[\left|\psi_+^{[1,n]}\right\rangle\!\left\langle\psi_+^{[1,n]}\right|\right]\right), \tag{5.25}$$

$$\text{where } \left|\psi_+^{[1,n]}\right\rangle = \left|\sqrt{\frac{\mathbb{1}}{d}}^{\otimes n}\right\rangle = d^{-\frac{n}{2}} \sum_{\boldsymbol{k}_{[1,n]}} \left|\boldsymbol{k}_{[1,n]} \otimes \boldsymbol{k}_{[1,n]}\right\rangle$$

and where $\mathbb{T}_n$ is the dual of $\mathbb{T}_n^\ddagger$ in the Schrödinger picture, namely the map

$$\mathbb{T}_n = \sum_{\boldsymbol{i}_{[1,n]}} p_{\boldsymbol{i}_{[1,n]}}\, \phi_{\boldsymbol{i}_{[1,n]}}^{\otimes[1,n]\ddagger}, \qquad \phi_{\boldsymbol{i}_{[1,n]}}^{\otimes[1,n]\ddagger} = \bigotimes_{k=1}^{n} \phi_{i_k}^\ddagger, \quad \phi_{i_k}^\ddagger[\,\cdot\,] = U_k\ \cdot\ U_k^\dagger. \tag{5.26}$$

We now prove (5.23). First, by setting $\left|\psi_{\boldsymbol{i}_{[1,n]}}\right\rangle = \left(\bigotimes_{k=1}^n U_{i_k}\right) \otimes \mathbb{1}_d^{\otimes n} \left|\psi_+^{[1,n]}\right\rangle$ one has

$$\mathbb{T}_n \otimes \mathrm{id}_d^{\otimes n}\left[\left|\psi_+^{[1,n]}\right\rangle\!\left\langle\psi_+^{[1,n]}\right|\right] = \sum_{\boldsymbol{i}_{[1,n]}} p_{\boldsymbol{i}_{[1,n]}} \left|\psi_{\boldsymbol{i}_{[1,n]}}\right\rangle\!\left\langle\psi_{\boldsymbol{i}_{[1,n]}}\right|, \tag{5.27}$$

In particular, since the vector states $\left|\psi_{\boldsymbol{i}_{[1,n]}}\right\rangle$ are normalized,

$$S\left(\rho\left[\mathcal{F}^{(n+1)}\right]\right) = 2\,S(\rho_S) + S\left(\mathbb{T}_n \otimes \mathrm{id}_d^{\otimes n}\left[\left|\psi_+^{[1,n]}\right\rangle\!\left\langle\psi_+^{[1,n]}\right|\right]\right) \ \leq\ 2\,S(\rho_S) + H\left(\pi^{[1,n]}\right), \tag{5.28}$$

with equality holding if and only if $\left|\psi_{\boldsymbol{i}_{[1,n]}}\right\rangle$ form an orthonormal basis in $\mathbb{C}_d^{\otimes n}$, that is, when (5.27) is the spectral decomposition of $\mathbb{T}_n \otimes \mathrm{id}_d^{\otimes n}\left[\left|\psi_+^{[1,n]}\right\rangle\!\left\langle\psi_+^{[1,n]}\right|\right]$. Equivalently, the equality holds if and only if the unitary operators U_k satisfy the orthogonality relations

$$\mathrm{Tr}\left(U_j^\dagger U_k\right) = d\,\delta_{jk}\,. \tag{5.29}$$

In such case, dividing both sides of (5.28) by $n+1$ and taking the limit, one gets

$$\mathfrak{h}_S(\Theta, \mathcal{F}) = \lim_n \frac{1}{n} H\left(\pi^{[1,n]}\right) = \mathfrak{S}_{\omega_E}, \tag{5.30}$$

that, along with (5.21), implies (5.23). □

Markov chain environment Choose $\pi^{[1,n]} = \{p_{\boldsymbol{i}_{[1,n]}}\}_{\boldsymbol{i}_{[1,n]}}$ as the stationary Markov distribution,

$$p_{\boldsymbol{i}_{[1,n]}} = \prod_{k=2}^{n} T_{i_k\, i_{k-1}}\, p_{i_1}, \tag{5.31}$$

defined by the stochastic matrix $T_{ij} \geq 0$, with $\sum_i T_{ij} = 1$, $\sum_j T_{ij}\, p_j = p_i$. The mean entropy rate of the Markov source is equal to the two-site conditional entropy of the chain,

$$\mathfrak{S}_{\omega_E} = -\sum_{ij} p_j\, T_{ij} \log\left(T_{ij}\right) = H(\pi^{(1)}) - I(\pi^{(1)}; \pi^{(2)}), \tag{5.32}$$

where

$$I(\pi^{(1)};\pi^{(2)}) = H(\pi^{(1)}) + H(\pi^{(2)}) - H(\pi^{[1,2]}), \tag{5.33}$$

denotes the mutual information between the first two subsequent sites of the classical chain. Because of stationarity, the latter quantity does not depend on where the chosen pair is located along the chain and measures the correlations between any two subsequent sites. Then, one sees that the entropy rate decreases with increasing correlations between subsequent sites. As we investigate in the following example, one can also easily construct a process with U_k as in (5.29) and for which $T_{ij} \in \{0,1\}$, so that $\mathfrak{h}_S(\Theta) = \mathfrak{S}_{\omega_E} = 0$, as one would have for a reversible evolution.

Example 5.1. *Consider the Pauli evolution of Section 4.2.2, obtained by taking $\phi_k[\,\cdot\,] = \sigma_k \cdot \sigma_k$ and the environment as a stationary Markov chain with transition matrix T as in* (4.44)*. The dynamical map Λ_n, through its Pauli spectrum* (4.58)*, depends parametrically on the triple (p, r, Δ). By tuning Δ and keeping fixed the other parameters, conditions* (4.60)–(4.62) *found in Proposition 4.7 establish that one can change the divisibility degree of the dynamics by increasing the correlations of the environment. Indeed, by increasing Δ, one first loses CP-divisibility, then the P-divisibility of the second tensor power $\Lambda_n \otimes \Lambda_n$ and finally the P-divisibility of the one qubit dynamics Λ_n. Notice that, for the single qubit, only in the latter case one can detect BFI as distinguishability revivals for one qubit.*

In this context, the map $\mathbb{T}_n$ in (5.22) *becomes a n-qubit Pauli channel,*

$$\mathbb{T}_n[X] = \sum_{\boldsymbol{i}_{[1,n]}} p_{\boldsymbol{i}_{[1,n]}}\, \sigma^{[1,n]}_{\boldsymbol{i}_{[1,n]}}\, X\, \sigma^{[1,n]}_{\boldsymbol{i}_{[1,n]}}, \qquad \sigma^{[1,n]}_{\boldsymbol{i}_{[1,n]}} := \bigotimes_{k=1}^{n} \sigma_{i_k}. \tag{5.34}$$

Notice that it is self dual: $\mathbb{T}_n = \mathbb{T}_n^\ddagger$. Vectors

$$|\psi_k\rangle = \sigma_k \otimes \mathbb{1}_2 |\psi_+^{(d)}\rangle, \quad k = 0, \dots, 3, \tag{5.35}$$

form the Bell basis, so that, from their orthogonality, the dynamical entropy of the system equals the entropy rate of the Markov source. With T parametrized as in (4.44)*, one gets*

$$\mathfrak{h}_S(\Theta) = \eta(p_0) + \eta(r) + 2\left(2p\,\frac{\eta(p+\Delta)+\eta(p-\Delta)}{2} + (1-2p)\eta(p)\right), \tag{5.36}$$

where $\eta(x) = -x\log x$, $\eta(0) := 0$. The associated two-site mutual information reads

$$I(\pi^{(1)};\pi^{(2)}) = 2p\Big(2\eta(p) - \eta(p+\Delta) - \eta(p-\Delta)\Big) = 4p^2\left(\log 2 - h\left(\frac{1}{2} + \frac{\Delta}{2p}\right)\right), \tag{5.37}$$

where $h(x) := \eta(x) + \eta(1-x)$, $0 \le x \le 1$ is the Shannon binary entropy, which is a decreasing function of x for $1/2 \le x \le 1$. Therefore, the two-site correlations increase with Δ and are maximal at $p = \Delta$. Correspondingly, the system's dynamical entropy (5.36) *is a monotonically decreasing function of the chain correlations Δ. Consider $\Delta = p = 1/2$, namely the parameters for which the two site correlations* (5.37) *are maximal. From* (5.36)*, then, since $r = p_0 = 0$, one has*

$$p = \Delta = \frac{1}{2} \quad \Longrightarrow \quad \mathfrak{h}_S(\Theta) = 0, \tag{5.38}$$

as one would have for a closed finite system. It is instructive to study the dynamical map Λ_n in the same regime. By setting $p = \Delta = 1/2$ in (4.59) *one gets $A_{\frac{1}{2},0} = 0$, $B_{\frac{1}{2},0,\frac{1}{2}} = 2$. Accordingly, the spectrum of Λ_n follows from* (4.58)*:*

$$\lambda_n = \frac{1+(-1)^n}{2}, \qquad \lambda_n^{(3)} = (-1)^n, \tag{5.39}$$

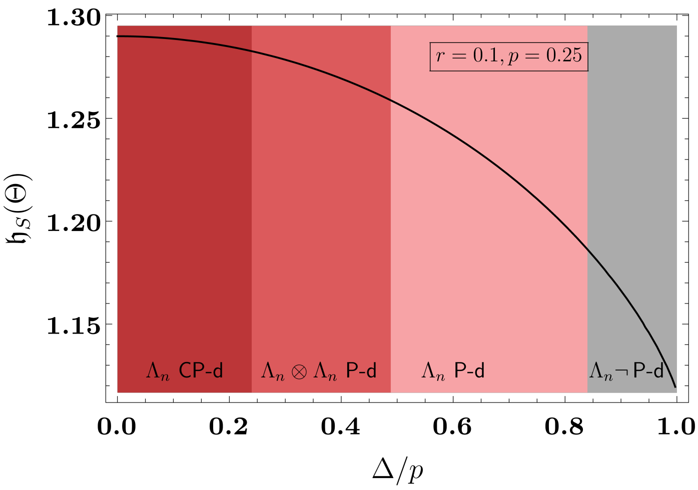

FIGURE 5.1: Open-system dynamical entropy (5.36) as a function of Δ/p for fixed values $r = 0.1$ and $p = 0.25$. The colored regions correspond to different divisibility degrees of the reduced dynamics Λ_n. For $\Delta/p \to 1$, Λ_n is not P-divisible and shows distinguishability revivals; in the pink region, Λ_n becomes P-divisible and does not show revivals, though they can superactivate for $\Lambda_n \otimes \Lambda_n$. The latter becomes P-divisible in the red region, while in the dark-red region $\Delta/p \ll 1$, CP-divisibility is achieved.

corresponding to the dynamical map

$$\Lambda_n = \begin{cases} \mathrm{id}_2 & n \text{ even}\,, \\ \Lambda_1 & n \text{ odd}\,, \end{cases} \qquad \Lambda_1 = \frac{\phi_1 + \phi_2}{2}\,. \tag{5.40}$$

Note that the dynamics is not algebraically invertible since $\lambda_{2m+1} = 0$. Nevertheless, we can explicitly study the contractivity of the trace norm under Λ_n. In particular, the trace norm is always contractive between step $2m$ and $2m+1$,

$$\|\Lambda_{2m+1}[X]\|_1 - \|\Lambda_{2m}[X]\|_1 = \|\Lambda_1[X]\|_1 - \|X\|_1 \leq 0\,, \tag{5.41}$$

since Λ_1 is CPTP. Between steps $2m-1$ and $2m$, instead, we have the converse inequality:

$$\|\Lambda_{2m}[X]\|_1 - \|\Lambda_{2m-1}[X]\|_1 = \|X\|_1 - \|\Lambda_1[X]\|_1 \geq 0\,, \tag{5.42}$$

so that the trace norm always revives at odd times, exposing backflow of information.

Remark 5.4.

1. *Remarkably, the dynamical map* (5.40) *experiences an extreme backflow of information since the trace norm revives after each odd time, while the dynamical entropy is zero, as one would have for a reversible evolution (see* (5.8)*). This means that asymptotically, no new information can be gathered by further probing the open system. The occurrence of a low, possibly zero, entropy production for a dissipative system is compatible with the interpretation of information having flown back to the open system. Such an interpretation can be generalized through inequality* (5.12)*: for a strongly correlated environment, we expect the system dynamics to be affected by strong memory effects. In Figure 5.1, we compare the behaviour of the open-system dynamical entropy as a function of the strength of the chain correlations with the divisibility degree of the reduced dynamics.*

2. *When the classical environment is a Markov process we can write the coarse-grained density matrix as (compare with Remark 5.3)*

$$\rho_S[\mathcal{X}^{(3)}]_{\boldsymbol{a},\boldsymbol{b}} = \sum_{i_2 i_1} T_{i_2 i_1}\, p_{i_1}\, \omega_S\left(X^\dagger_{b_0}\phi_{i_1}[X^\dagger_{b_1}]\phi_{i_1}\phi_{i_2}[X^\dagger_{b_2}X_{a_2}]\phi_{i_1}[X_{a_1}]X_{a_0}\right).$$

For the concrete case of $T_{i_2 i_1}$ *as in* (4.44), *we have*

$$\begin{aligned}\rho_S[\mathcal{X}^{(3)}]_{\boldsymbol{a},\boldsymbol{b}} &= \omega_S\Big(X^\dagger_{b_0}\Lambda_1\left[X^\dagger_{b_1}\Lambda_1\left[X^\dagger_{b_2}X_{a_2}\right]X_{a_1}\right]X_{a_0}\Big)\\ &+p\,\Delta\Big(\omega_S(X^\dagger_{b_0}\phi_1[X^\dagger_{b_1}](\mathrm{id}_2-\phi_3)[X^\dagger_{b_2}X_{a_2}]\phi_1[X_{a_1}]X_{a_0})\\ &\quad+\omega_S(X^\dagger_{b_0}\phi_2[X^\dagger_{b_1}](\mathrm{id}_2-\phi_3)[X^\dagger_{b_2}X_{a_2}]\phi_2[X_{a_1}]X_{a_0})\Big).\end{aligned}$$

Thus Δ *signals the deviation of the evolution from the QR (and semigroup) regime. In particular, the reduced dynamics can be P or even CP-divisible for* $\Delta > 0$ *(compare Proposition 4.7) while QR does not hold.*

3. *A natural generalization of Example 5.1 to a* d*-level system is obtained by choosing* $D = d^2$ *and unitaries in* (5.15) *as the discrete Weyl operators,*

$$U_{a,b} = \sum_{k=0}^{d-1}\omega^{kb}\,|a+k\rangle\langle k|\ , \qquad \omega = e^{2\pi i/d}\ , \tag{5.43}$$

which are unitary and satisfy $\mathrm{Tr}\left(U^\dagger_{a,b}U_{c,d}\right) = d\,\delta_{ac}\delta_{bd}$.

5.2.2 Reduced Dynamics in the GNS representation

As already emphasized in Section 1.5.2, the ALF entropy has a natural interpretation in the GNS representation. We shall now explicitly consider the GNS construction for the collisional model of Section 5.2.1. With respect to the product invariant state $\omega_{SE} = \mathbb{1}/d\otimes\omega_E$, the GNS representation can be taken as a tensor product $\pi_S\otimes\pi_E$. For the finite-level system S, as illustrated in Example 1.3, the representation is given by

$$\pi_S(A) = A\otimes\mathbb{1}_d\ , \qquad \pi'_S(A) = \mathbb{1}_d\otimes\overline{A}\ , \qquad A\in M_d(\mathbb{C})\,. \tag{5.44}$$

and $|\Omega_S\rangle \equiv \left|\psi^{(d)}_+\right\rangle = \sum_{i=1}^d |i\otimes i\rangle/\sqrt{d}$ (the conjugation in (5.44) is with respect to the standard basis). The tensor shift on the chain is implemented by the unitary operator defined by

$$U^\dagger_\sigma\,\pi_E\left(\Pi^{[a,b]}_{\boldsymbol{i}^{[a,b]}}\right)|\Omega_E\rangle := \pi_E\left(\Pi^{[a+1,b+1]}_{\boldsymbol{i}^{[a,b]}}\right)|\Omega_E\rangle\ , \qquad U^\dagger_\sigma\,|\Omega_E\rangle = U_\sigma\,|\Omega_E\rangle = |\Omega_E\rangle\ . \tag{5.45}$$

We then implement unitarily the automorphism Θ by the operator

$$\begin{aligned}&U^\dagger_\Theta\,\pi_S(A)\pi'_S(B)\otimes\pi_E\left(\Pi^{[a,b]}_{\boldsymbol{i}^{[a,b]}}\right)\left|\psi^{(d)}_+\otimes\Omega_E\right\rangle\\ &\qquad := \sum_k U^\dagger_k A\otimes\overline{U}^\dagger_k\overline{B}\otimes U^\dagger_\sigma\pi_E\left(\Pi^{(0)}_k\Pi^{[a,b]}_{\boldsymbol{i}^{[a,b]}}\right)\left|\psi^{(d)}_+\otimes\Omega_E\right\rangle\ .\end{aligned} \tag{5.46}$$

Notice that in (5.46), the action on $\pi'_S(\mathcal{A}_S)$ is such that the GNS cyclic vector is invariant; namely,

$$U_\Theta^\dagger \left|\psi_+^{(d)} \otimes \Omega_E\right\rangle = \sum_k U_k^\dagger \otimes \overline{U_k}^\dagger \left|\psi_+^{(d)}\right\rangle \otimes U_\sigma^\dagger \pi_E\left(\Pi_k^{(0)}\right)|\Omega_E\rangle$$
$$= \left|\psi_+^{(d)}\right\rangle \otimes U_\sigma^\dagger \pi_E \left(\sum_k \Pi_k^{(0)}\right)|\Omega_E\rangle = \left|\psi_+^{(d)} \otimes \Omega_E\right\rangle , \tag{5.47}$$

where we used the fact that, along with (5.45), $V \otimes \overline{V}\left|\psi_+^{(d)}\right\rangle = \left|\psi_+^{(d)}\right\rangle$ for any unitary $V \in M_d(\mathbb{C})$. Also, it suffices to define U_Θ only on $\pi_E(\mathcal{A}_E)$. Indeed, because the environment is a classical chain, this latter dense subalgebra is contained in its commutant $\pi'_E(\mathcal{A}_E)$. By taking the adjoint with respect to the scalar product in the GNS Hilbert space, the action of U_Θ can be also inferred,

$$U_\Theta\, \pi_S(A)\pi'_S(B) \otimes \pi_E\left(\Pi^{[a,b]}_{\boldsymbol{i}_{[a,b]}}\right)\left|\psi_+^{(d)} \otimes \Omega_E\right\rangle$$
$$:= \sum_k U_k A \otimes \overline{U_k}\, \overline{B} \otimes \pi_E\left(\Pi_k^{(0)}\right) U_\sigma\, \pi_E\left(\Pi^{[a,b]}_{\boldsymbol{i}_{[a,b]}}\right)\left|\psi_+^{(d)} \otimes \Omega_E\right\rangle \tag{5.48}$$

from which it also follows that $U_\Theta\left|\psi_+^{(d)} \otimes \Omega_E\right\rangle = \left|\psi_+^{(d)} \otimes \Omega_E\right\rangle$. By acting on local operators as in (5.46) and (5.48), one checks explicitly that $U_\Theta U_\Theta^\dagger = U_\Theta^\dagger U_\Theta = \mathbb{1}$. Moreover, the automorphism Θ is correctly implemented. Indeed, from (5.46),

$$U_\Theta^\dagger\, \pi_S(A) \otimes \pi_E\left(\Pi^{[a,b]}_{\boldsymbol{i}_{[a,b]}}\right) U_\Theta \left|\psi_+^{(d)} \otimes \Omega_E\right\rangle$$
$$= \sum_{k,l} U_k^\dagger A U_l \otimes \overline{U_k}^\dagger \overline{U_l} \otimes U_\sigma^\dagger\, \pi_E\left(\Pi_k^{(0)}\Pi^{[a,b]}_{\boldsymbol{i}_{[a,b]}}\Pi_l^{(0)}\right)\mathbb{1}_d \otimes \mathbb{1}_d \otimes U_\sigma \left|\psi_+^{(d)} \otimes \Omega_E\right\rangle$$
$$= \mathbb{1}_d \otimes \mathbb{1}_d \otimes U_\sigma^\dagger \left(\sum_k U_k^\dagger A U_k \otimes \mathbb{1}_d \otimes \pi_E\left(\Pi_k^{(0)}\Pi^{[a,b]}_{\boldsymbol{i}_{[a,b]}}\Pi_k^{(0)}\right)\right)\mathbb{1}_d \otimes \mathbb{1}_d \otimes U_\sigma \left|\psi_+^{(d)} \otimes \Omega_E\right\rangle$$
$$= \mathbb{1}_{d^2} \otimes U_\sigma^\dagger \pi_S \otimes \pi_E\left(\Phi\left(A \otimes \Pi^{[a,b]}_{\boldsymbol{i}_{[a,b]}}\right)\right)\mathbb{1}_{d^2} \otimes U_\sigma \left|\psi_+^{(d)} \otimes \Omega_E\right\rangle$$
$$= \pi_S \otimes \pi_E\left(\sigma \circ \Phi\left(A \otimes \Pi^{[a,b]}_{\boldsymbol{i}_{[a,b]}}\right)\right)\left|\psi_+^{(d)} \otimes \Omega_E\right\rangle = \pi_S \otimes \pi_E\left(\Theta\left(A \otimes \Pi^{[a,b]}_{\boldsymbol{i}_{[a,b]}}\right)\right)\left|\psi_+^{(d)} \otimes \Omega_E\right\rangle .$$

We now consider the following expectation,

$$\left\langle\psi_+^{(d)} \otimes \Omega_E\right| \pi_S(Y)U_\Theta^\dagger \pi_S(A)\pi'_S(B)U_\Theta \pi_S(X)\left|\psi_+^{(d)} \otimes \Omega_E\right\rangle .$$

By applying (5.48), we have

$$\left\langle\psi_+^{(d)} \otimes \Omega_E\right| \pi_S(Y)U_\Theta^\dagger \pi_S(A)\pi'_S(B)U_\Theta \pi_S(X)\left|\psi_+^{(d)} \otimes \Omega_E\right\rangle$$
$$= \sum_{k,l} \langle\psi_+^{(d)}|\, Y U_l^\dagger A U_k X \otimes \overline{U_l}^\dagger\, \overline{B}\, \overline{U_k}\left|\psi_+^{(d)}\right\rangle \langle\Omega_E|\pi_E\left(\Pi_l^{(0)}\Pi_k^{(0)}\right)\Omega_E\rangle$$
$$= \sum_k \mathrm{Tr}\left(\phi_k \otimes \overline{\phi}_k[A \otimes \overline{B}]X \otimes \mathbb{1}_d P_+^{(d)} Y \otimes \mathbb{1}_d\right)\langle\Omega_E|\pi_E\left(\Pi_k^{(0)}\right)\Omega_E\rangle$$
$$= \mathrm{Tr}\left(A \otimes \overline{B}\left(\sum_k p_k\, \phi_k^\ddagger \otimes \overline{\phi}_k^\ddagger\right)[X \otimes \mathbb{1}_d\, P_+^{(d)}\, Y \otimes \mathbb{1}_d]\right) . \tag{5.49}$$

Similarly,

$$\begin{aligned}
&\left\langle\psi_+^{(d)}\otimes\Omega_E\right|\pi_S(Y_1)U_\Theta^\ddagger\pi_S(Y_2)U_\Theta^\ddagger\pi_S(A)\pi_S'(B)U_\Theta\pi_S(X_2)U_\Theta\pi_S(X_1)\left|\psi_+^{(d)}\otimes\Omega_E\right\rangle\\
&=\sum_{k_1,k_2,l_1,l_2}\left\langle\psi_+^{(d)}\right|Y_1U_{l_1}^\dagger Y_2U_{l_2}^\dagger AU_{k_2}X_2U_{k_1}X_1\otimes\overline{U}_{l_1}^\dagger\,\overline{U}_{l_2}^\dagger\,\overline{B}\,\overline{U}_{k_2}\,\overline{U}_{k_1}\left|\psi_+^{(d)}\right\rangle\\
&\qquad\qquad\langle\Omega_E|\,U_\sigma^\dagger\pi_E(\Pi_{l_1}^{(0)})U_\sigma^\dagger\pi_E(\Pi_{l_2}^{(0)}\Pi_{k_2}^{(0)})U_\sigma\pi_E(\Pi_{k_1}^{(0)})U_\sigma\,|\Omega_E\rangle\\
&=\sum_{k_1k_2}p_{k_1k_2}\,\mathrm{Tr}\Big(A\otimes\overline{B}\,\phi_{k_2}^\ddagger\otimes\overline{\phi}_{k_2}^\ddagger\Big[X_2\,\phi_{k_1}^\ddagger\otimes\overline{\phi}_{k_1}^\ddagger\,[X_1\,P_+^{(d)}\,Y_1]\,Y_2\Big]\Big)
\end{aligned}$$

where we used that

$$\begin{aligned}
\langle\Omega_E|U_\sigma^\dagger\pi_E(\Pi_{l_1}^{(0)})U_\sigma^\dagger\pi_E(\Pi_{l_2}^{(0)}\Pi_{k_2}^{(0)})U_\sigma\pi_E(\Pi_{k_1}^{(0)})U_\sigma\Omega_E\rangle&=\pi_E\Big(\sigma\Big(\Pi_{l_1}^{(0)}\sigma\Big(\Pi_{l_2}^{(0)}\Pi_{k_2}^{(0)}\Big)\Pi_{k_1}^{(0)}\Big)\Big)\\
&=\pi_E(\Pi_{k_1}^{(1)}\Pi_{k_2}^{(2)})\delta_{l_1,k_1}\delta_{l_2,k_2}=p_{k_1k_2}\delta_{l_1,k_1}\delta_{l_2,k_2}\,.
\end{aligned}$$

Therefore, when we consider the evolution intertwined by measurements on the open system as appears in the coarse-grained density matrix, we get

$$\begin{aligned}
&\left\langle\psi_+^{(d)}\otimes\Omega_E\right|\Big((\mathbb{X}^\ddagger\otimes\mathrm{id}_d\otimes\mathrm{id}_E)\circ\mathbb{U}_\Theta^\ddagger\Big)^n[\pi_S(A)\pi_S'(B)]\left|\psi_+^{(d)}\otimes\Omega_E\right\rangle \qquad (5.50)\\
&=\sum_{k_1\dots k_n}p_{k_1\dots k_n}\,\mathrm{Tr}\Big(A\otimes\overline{B}\,\phi_{k_n}^\ddagger\otimes\overline{\phi}_{k_n}^\ddagger\circ\mathbb{X}\otimes\mathrm{id}_d\circ\cdots\circ\phi_{k_1}^\ddagger\otimes\overline{\phi}_{k_1}^\ddagger\circ\mathbb{X}\otimes\mathrm{id}_d[P_+^{(d)}]\Big)\,.
\end{aligned}$$

Accordingly, the reduced state of the GNS projector $P_+^{(d)}$, evolving under the unitary evolution and iterated measurements, is described by the following CPTP map $\Gamma_n^{\mathbb{X}}:M_d(\mathbb{C})\otimes M_d(\mathbb{C})\to M_d(\mathbb{C})\otimes M_d(\mathbb{C})$,

$$\Gamma_n^{\mathbb{X}}:=\sum_{k_1\dots k_n}p_{k_1\dots k_n}\,(\phi_{k_n}^\ddagger\circ\mathbb{X})\otimes\overline{\phi}_{k_n}^\ddagger\circ\cdots\circ(\phi_{k_1}^\ddagger\circ\mathbb{X})\otimes\overline{\phi}_{k_1}^\ddagger\,. \qquad (5.51)$$

By choosing $\mathbb{X}=\mathrm{id}_d$, one is left with

$$\Gamma_n^{\mathrm{id}}=\sum_{\boldsymbol{k}_{[1,n]}}p_{\boldsymbol{k}_{[1,n]}}\,\phi_{\boldsymbol{k}_{[1,n]}}^\ddagger\otimes\overline{\phi}_{\boldsymbol{k}_{[1,n]}}^\ddagger\,,\qquad\phi_{\boldsymbol{k}_{[1,n]}}=\phi_{k_1}\circ\cdots\circ\phi_{k_n}\,, \qquad (5.52)$$

which is the intrinsic reduced evolution in the GNS space, having traced away the environment degrees of freedom. Formally, (5.52) corresponds to the joint evolution of two subsystems evolving in a common environment. Notice that the partial trace over the commutant gives,

$$\mathrm{Tr}_{II}\left(\Gamma_n^{\mathrm{id}}[X\otimes Y]\right)=\sum_{\boldsymbol{k}_{[1,n]}}p_{\boldsymbol{k}_{[1,n]}}\,\phi_{\boldsymbol{k}_{[1,n]}}^\ddagger[X]\,\mathrm{Tr}\,(Y)=\Lambda_n[\mathrm{Tr}\,(Y)\;X]=\Lambda_n[\mathrm{Tr}_{II}(X\otimes Y)]\,.$$

Hence, consistently, Γ_n^{id} is a CP dilation of the reduced dynamics of the system Λ_n.

Remark 5.5. *Notice that, in general, the existence of $\Gamma_n^{\mathbb{X}}$, defined through*

$$\begin{aligned}
&\left\langle\sqrt{\rho_S}\otimes\Omega_E\right|\pi_S(X)^\dagger\Big((\mathbb{X}^\ddagger\otimes\mathrm{id}_d\otimes\mathrm{id}_E)\circ\mathbb{U}_\Theta^\ddagger\Big)^n[\pi_S(A)\pi_S'(B)]\,\pi_S(X)\left|\sqrt{\rho_S}\otimes\Omega_E\right\rangle\\
&\qquad=\mathrm{Tr}\Big(\Gamma_n^{\mathbb{X}}[X\otimes\mathbb{1}_d\left|\sqrt{\rho_S}\right\rangle\!\left\langle\sqrt{\rho_S}\right|X^\dagger\otimes\mathbb{1}_d]A\otimes B\Big) \qquad (5.53)
\end{aligned}$$

is guaranteed by the choice of a reference state of the form $\omega_S \otimes \omega_E$. As follows from Proposition 1.3, indeed, the GNS triple can be identified with $(\pi_S \otimes \pi_E, \mathcal{H}_S \otimes \mathcal{H}_E, |\sqrt{\rho_S} \otimes \Omega_E\rangle)$. Then, factorization of the S + E cyclic vector, along with complete positivity of $(\mathbb{X}^\ddagger \otimes \mathrm{id}_d \otimes \mathrm{id}_E) \circ \mathbb{U}^\ddagger_\Theta$, ensures complete positivity of $\Gamma^{\mathbb{X}}_n$.

5.2.3 Pauli GNS reduced dynamics

We now consider, as in Section 4.2.1 and Example 5.1, the environment to be a classical Markov chain

$$p_{\boldsymbol{k}_{[1,n]}} = T_{k_n k_n-1} \dots T_{k_2 k_1} p_{k_1} ,$$

and take the open quantum system to be a qubit, $d = 2$, with the environment consisting of a one-dimensional lattice of classical four-level systems, $D = d^2 = 4$. Furthermore, we shall choose the stochastic matrix T as in (4.44), while setting $U_k = \sigma_k$, $k = 0, \dots, 3$. Accordingly, $\Gamma^{\mathrm{id}}_n : M_4(\mathbb{C}) \to M_4(\mathbb{C})$ defined in (5.52) becomes a two-qubit Pauli map, namely for all $n \geq 0$ it is diagonal in the Pauli matrices

$$\Gamma^{\mathrm{id}}_n[\sigma_\alpha \otimes \sigma_\beta] = \gamma^{(\alpha,\beta)}_n \sigma_\alpha \otimes \sigma_\beta , \qquad \alpha, \beta = 0, \dots, 3 . \tag{5.54}$$

The spectrum of Γ^{id}_n follows a similar rule than that of Λ_n

$$\gamma^{(\alpha,\beta)}_n = \left(p_0 + p(\mu^{(\alpha,\beta)}_1 + \mu^{(\alpha,\beta)}_2) + r\mu^{(\alpha,\beta)}_3\right)\gamma^{(\alpha,\beta)}_{n-1} + \Delta p \left(\mu^{(\alpha,\beta)}_1 - \mu^{(\alpha,\beta)}_2\right)^2 \gamma^{(\alpha,\beta)}_{n-2} , \tag{5.55}$$

where $\phi_k \otimes \phi_k[\sigma_\alpha \otimes \sigma_\beta] = \mu^{(\alpha,\beta)}_k \sigma_\alpha \otimes \sigma_\beta$, $\mu^{(\alpha,\beta)}_k := \mu^{(\alpha)}_k \mu^{(\beta)}_k$. The formal solution of (5.55) is then the same as that of (4.57). Accordingly, the 16 eigenvalues $\gamma^{(\alpha,\beta)}_n$ are associated to their respective eigenvectors $\sigma_\alpha \otimes \sigma_\beta$ according to the following 4×4 matrix

$$G_n := \left[\gamma^{(\alpha,\beta)}_n\right] = \begin{pmatrix} 1 & \lambda_n & \lambda_n & \lambda^{(3)}_n \\ \lambda_n & 1 & \lambda^{(3)}_n & \lambda_n \\ \lambda_n & \lambda^{(3)}_n & 1 & \lambda_n \\ \lambda^{(3)}_n & \lambda_n & \lambda_n & 1 \end{pmatrix} . \tag{5.56}$$

Note that all the Bell-diagonal states, which are linear combinations of $\sigma_\alpha \otimes \sigma_\alpha$, as well as the GNS reduced initial state $P^{(d)}_+$, are invariant under Γ^{id}_n. The spectrum of the intertwining propagators $\Gamma^{\mathrm{id}}_{n,n-1} = \Gamma^{\mathrm{id}}_n \circ \left(\Gamma^{\mathrm{id}}_{n-1}\right)^{-1}$ is determined multiplicatively,

$$\Gamma^{\mathrm{id}}_{n,n-1}[\sigma_\alpha \otimes \sigma_\beta] = \frac{\gamma^{(\alpha,\beta)}_n}{\gamma^{(\alpha,\beta)}_{n-1}} \sigma_\alpha \otimes \sigma_\beta , \tag{5.57}$$

and given by

$$G_{n,n-1} := \left[\lambda^{(\alpha,\beta)}_{n,n-1}\right] = \begin{pmatrix} 1 & \lambda_{n,n-1} & \lambda_{n,n-1} & \lambda^{(3)}_{n,n-1} \\ \lambda_{n,n-1} & 1 & \lambda^{(3)}_{n,n-1} & \lambda_{n,n-1} \\ \lambda_{n,n-1} & \lambda^{(3)}_{n,n-1} & 1 & \lambda_{n,n-1} \\ \lambda^{(3)}_{n,n-1} & \lambda_{n,n-1} & \lambda_{n,n-1} & 1 \end{pmatrix}, \tag{5.58}$$

where $\lambda_{n,n-1}$, $\lambda^{(3)}_{n,n-1}$ are the eigenvalues of $\Lambda_{n,n-1}$. Then, as derived in Appendix D.4, the action of Γ^{id}_n can be recast as

$$\Gamma^{\mathrm{id}}_n[X] = \sum_\alpha q^{(\alpha)}_n \,\sigma_\alpha \otimes \sigma_\alpha \, X \, \sigma_\alpha \otimes \sigma_\alpha \,, \qquad X \in M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \,, \tag{5.59}$$

with

$$q^{(0)}_n = \frac{1}{4}\left(1 + 2\lambda_n + \lambda^{(3)}_n\right), \qquad q^{(1)}_n = q^{(2)}_n = \frac{1}{4}\left(1 - \lambda^{(3)}_n\right), \qquad q^{(3)}_n = \frac{1}{4}\left(1 + 2\lambda_n - \lambda^{(3)}_n\right).$$

We now aim at characterizing the divisibility properties of the GNS discrete dynamics Γ^{id}_n.

Proposition 5.3. *Suppose that* $\Gamma_n : M_4(\mathbb{C}) \to M_4(\mathbb{C})$ *is a Pauli map acting as*

$$\Gamma_n[X] = \sum_{\alpha=0}^{3} q^{(\alpha)}_n \,\sigma_\alpha \otimes \sigma_\alpha \, X \, \sigma_\alpha \otimes \sigma_\alpha \,, \quad X \in M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \,, \tag{5.60}$$

and let

$$\Lambda_n[Z] = \sum_{\alpha=0}^{3} q^{(\alpha)}_n \sigma_\alpha \, Z \, \sigma_\alpha \,, \quad Z \in M_2(\mathbb{C}) \,, \tag{5.61}$$

so that $\mathrm{Tr}_{II} \circ \Gamma_n = \Lambda_n \circ \mathrm{Tr}_{II}$. *Then, following conditions are equivalent*

i. Λ_n *is CP-divisible;*

ii. Γ_n *is P-divisible;*

iii. Γ_n *is CP-divisible.*

Proof. Take $X \in M_4(\mathbb{C})$ and let $Y = \Gamma_n[X]$. The partial trace of X over the second qubit reads,

$$\mathrm{Tr}_{II}(X) = \mathrm{Tr}_{II}(\Gamma^{-1}_n[Y]) \tag{5.62}$$

but also

$$\mathrm{Tr}_{II}(X) = \Lambda^{-1}_n \circ \Lambda_n \, \mathrm{Tr}_{II}(X) = \Lambda^{-1}_n \, \mathrm{Tr}_{II}(\Gamma_n[X]) = \Lambda^{-1}_n \, \mathrm{Tr}_{II}(Y), \tag{5.63}$$

Hence, for all $Y \in M_4(\mathbb{C})$

$$\Lambda^{-1}_n \, \mathrm{Tr}_{II}(Y) = \mathrm{Tr}_{II}(\Gamma^{-1}_n Y) \tag{5.64}$$

and, applying Λ_m, $m \geq n$, to both sides, yields

$$\Lambda_{m,n} \, \mathrm{Tr}_{II}(Y) = \mathrm{Tr}_{II}(\Gamma_{m,n}[Y]) \,, \qquad m \geq n \,. \tag{5.65}$$

Intertwiners of Pauli maps are Pauli maps, so that

$$\Gamma_{n,n-1}[X] = \sum_{\alpha=0}^{3} q^{(\alpha)}_{n,n-1} \sigma_\alpha \otimes \sigma_\alpha X \sigma_\alpha \otimes \sigma_\alpha \,, \tag{5.66}$$

and

$$\mathrm{Tr}_{II}\left(\Gamma_{n,n-1}[X]\right) = \Lambda_{n,n-1}\left[\mathrm{Tr}_{II} X\right], \qquad \forall\, n \geq 1 \,, \tag{5.67}$$

with

$$M_2(\mathbb{C}) \ni Z \longmapsto \Lambda_{n,n-1}[Z] = \sum_{\alpha=0}^{3} q_{n,n-1}^{(\alpha)} \sigma_\alpha \, Z \, \sigma_\alpha \,. \tag{5.68}$$

Positivity of $\Gamma_{n,n-1}$, $n \geq 1$, is expressed by the condition

$$\mathrm{Tr}\left(Q \, \Gamma_{n,n-1}[P]\right) \geq 0 \,, \qquad \forall \, P, Q \geq 0 \,, \tag{5.69}$$

where P, Q can be chosen as projectors in $M_4(\mathbb{C})$. Pick separable projectors

$$P = Q = |+\rangle\langle+| \otimes |0\rangle\langle 0| \,, \qquad |\pm\rangle = \frac{|0\rangle \pm |1\rangle}{\sqrt{2}} \,, \tag{5.70}$$

with $\sigma_3 \,|0\rangle = |0\rangle$, $\sigma_3 \,|1\rangle = -\,|1\rangle$. One readily finds

$$\mathrm{Tr}\left(Q \, \Gamma_{n,n-1}[P]\right) = q_{n,n-1}^{(0)} \geq 0 \,. \tag{5.71}$$

The choice

$$P = |+\rangle\langle+| \otimes |0\rangle\langle 0| \,, \qquad Q = |-\rangle\langle-| \otimes |0\rangle\langle 0| \,, \quad PQ = 0 \,, \tag{5.72}$$

yields instead the condition

$$\mathrm{Tr}\left(Q \, \Gamma_{n,n-1}[P]\right) = q_{n,n-1}^{(3)} \geq 0 \,. \tag{5.73}$$

Similarly, one finds $q_{n,n-1}^{(\alpha)} \geq 0$, $\alpha = 0, \dots, 3$, so that $ii. \Rightarrow i \Rightarrow iii$. Clearly, $iii. \Rightarrow ii.$ □

Remark 5.6. *The map Γ_n^{id} represents the open dynamics in the GNS representation, involving the commutant of the system. It provides a dilation of the one-qubit evolution Λ_n that exhibits memory effects not detectable at the level of Λ_n itself. This is reminiscent of the superactivation phenomenon studied in Chapter 4 through dilations of the form $\Lambda_n \otimes \Lambda_n$, although here the two parties are dynamically coupled. In fact, suppose to select Δ for which Λ_n is P-divisible but not CP-divisible. Then, the evolution of the system does not display backflow of information in the sense of revivals of the trace norm. On the other hand, in the same regime, the two-qubit dilation Γ_n^{id} cannot be P-divisible due to Proposition 5.3. Hence, there exist $X \in M_4(\mathbb{C})$ and $n > m \in \mathbb{N}$ such that*

$$\left\|\Gamma_n^{\mathrm{id}}[X]\right\|_1 > \left\|\Gamma_m^{\mathrm{id}}[X]\right\|_1 . \tag{5.74}$$

Note that the commutant is involved in evaluating the entropy production rate of (5.5) *obtained through the GNS construction.*

In this Chapter, we computed the ALF entropy for a finite open system coupled to a stationary classical chain yielding a random unitary reduced dynamics. The open-system entropy equals the mean entropy rate of the environment. This result has a simple but clear information-theoretical interpretation: the ALF entropy measures the average amount of information that can be extracted by probing the open system alone. Such an information rate decreases as the environment becomes more correlated, namely, the measurements become progressively less informative. Such decrease of entropy production is compatible with the interpretation of information having flown back to the open system. In the qubit case, this physical effect has been compared to the memory effects present in the qubit reduced dynamics, which are likewise governed by the strength of environmental correlations. In a more general scenario of a system coupled to a quantum spin chain, the above interpretation is supported by an upper bound to the ALF entropy of the open system by the mean entropy of the environment (plus a quantum correction). For a stationary chain, the more the chain is correlated, the tighter is such bound.

The ALF entropy, obtained from the open-system multi-time correlation functions, is neatly interpreted when going to the GNS representation. For quantum systems, this involves a purifying ancillary system. Accordingly, we studied the GNS reduced dynamics obtained by tracing away the environment for the model of Section 5.2.1. Interestingly, the GNS dilation is affected by memory effects that are not present in the reduced dynamics. In particular, it can show superactivation of backflow of information, as it typically occurs when a tensor product dilation of the reduced dynamics is considered.

Outlook

Non-Markovian evolutions could be intuitively placed in between two opposite points down a line: that of a reversible, unitary dynamics and that of an irreversible, Markovian one. In the latter case, in fact, the noisy and decohering effects – that are, in most situations, detrimental to any practical application – account for information being irretrievably dissipated outside the degrees of freedom of the system of interest. On the other hand, typical features of non-Markovian open dynamics, such as the possibility of increasing coherence and distinguishability of states, typically interpreted as backflow of information, are in marked contrast with this behaviour. In the body of the thesis, we discussed several such features which are typically emergent in the reduced dynamics of the open system. In at least two cases, as we shall now review, we could also explicitly mimic some features typical of unitary evolutions through pure non-Markovian noise.

In Chapter 3, we studied how the property of P-divisibility can be lost when classically reducing a quantum dynamical map, namely when restricting it to an algebra of commutative observables. In such a case, classical backflow of information arises, provided that enough coherence has been developed by the quantum dynamics. This effect typically occurs when classically reducing unitary evolutions; however, the same effect can be retrieved by non-Markovian noise describing pure dissipation. The usual role of the environment in the open system context, here, is instead taken by coherences of the quantum evolution with respect to a given MASA, as emerged from the discussion of Section 3.2.1. There, we could clearly relate the loss of classical P-divisibility with the gain in coherence with respect to a suitable basis. The dynamics that can achieve such an effect through pure dissipation are, in the context of a single qubit, rather non-trivial and involve several constraints. We could fulfil them all through the characterization of a large class of orthogonally covariant dynamics. A natural extension of the work presented in Chapter 3 would involve reductions of quantum dynamical maps onto Abelian subalgebras that are though not maximally Abelian. The product of such reductions would result in maps on block-diagonal matrices, that will thus retain some quantum character due to the partial loss of coherence. Note that these kinds of hybrid dynamics have recently attracted, especially in the Markovian regime, a certain interest in quite different contexts [139, 140] (see also the recent work [141]).

Much of the thesis was devoted to the study of the SBFI effect as prototype of a memory effect with no classical counterpart though, as in our case, it can be assisted by a classical memory as in the coupling with a classical Markov chain. In Chapter 4, we adopted a "brute force" approach in explicitly evaluating the bipartite system-environment correlations in both discrete and continuous time in order to trace back SBFI to the features of a concrete interaction model. There, we saw that system-environment correlations undergo a growth and collapse between subsequent collisions. This result has a clear-cut information theoretical interpretation in our model. Of course, the assumption of the controlled-type of interaction much simplifies the form of the system-environment mutual information. The interpretation of the memory effects arising from such class of interactions is still actively debated (see [142, 143] and, more recently, [144]).

Another instance of a non-Markovian dynamics mimicking the behaviour of a reversible one has been encountered in Chapter 5. There, we showed how the dynamical entropy of an open system might decrease with increasing correlations in the underlying environment. As a limiting case, we have also seen that one can achieve zero entropy production, as one would have for a closed system, without the dynamics being reversible (see Example 5.1). Though the dynamical entropy involves the asymptotic behaviour of the evolution, while revivals characterizing BFI are usually a transient, the result in our model is fully compatible with a return of information from the environment to the open system. Indeed, in our model, the information per time step gained about the dynamics is decreasing with increasing correlations. In other words, the measurements become less and less informative. A natural direction for further investigation would be to generalize our results to the more physically palatable case of a genuinely quantum spin chain environment. Certainly, such task looks more challenging as far as an analytic treatment of the coarse-grained dynamics is concerned. In this regard, the upper bound (5.12) provides a preliminary result, in that the maximal entropy production rate of the system is upper-bounded by the mean entropy of the environment, which is itself a decreasing function of the correlations. Symbolic models for open quantum systems, similarly to other existing approaches [15, 98], naturally lead to a description of the multi-time statistics of the system of interest as a quantum many-body state. Characterizing the correlations present in this emergent state would be relevant, for instance, in view of recent efforts to identify genuinely quantum memories [145–147].

In Chapter 5, moreover, a signature of SBFI has been seen through the joint-system environment GNS evolution. It would then also be interesting to check whether Proposition 5.3 holds in more general situations. Another direction for further investigation would avoid the GNS representation of the environment and rather ask for the covariant implementation of dynamical maps, within the GNS setting, as a one parameter family of contractions [148]. Indeed, such operators could not contract between intermediate stages of the evolution.

As a further outlook of this thesis, it could be very interesting to investigate potential applications of SBFI in a quantum information framework. Indeed, the presence of backflow of information could be helpful in some applications to contrast the detrimental effects of dissipation. SBFI then suggests that coupling systems might increase the non-Markovianity and, accordingly, the related beneficial effects (see also the recent work [149]) . Weak non-Markovian dynamics, of the kind responsible for the SBFI effect, were reproduced in a fully optical setup in [97]. Quantum optical frameworks are also optimal to simulate collisional models with correlated environments, as those treated in Chapters 4 and 5 (see [87, 91]).

The main goal of this thesis was to investigate concepts such as P-divisibility and backflow of information which, together with the unifying notion of dynamical entropy, arise quite naturally in both classical and quantum dynamical systems. Although based on a rather specific example, the concrete analysis of the collisional model of Section 4.2 – from the perspective of the dynamical map (Chapter 4) as well as that of multi-time statistics (Chapter 5) – allowed us to compare these two viewpoints and thereby better appreciate the context-dependent nature of quantum non-Markovianity. In particular, the open-system dynamical entropy appears to be a promising tool for gaining more physically grounded insights into the system-environment exchange of information.

APPENDIX

APPENDIX A

Orthogonally covariant qubit maps

A.1 Proof of Proposition 3.4

Let $P = |\psi\rangle\langle\psi|$ be a generic 2×2 projector, with $|\psi\rangle = (w_1, w_2) = (w_1, |w_2|e^{-i\frac{\Omega}{2}})$, $w_1, \Omega \in \mathbb{R}$, $|w_1|^2 + |w_2|^2 = 1$ and let Q be its orthogonal complement $Q = \mathbb{1}_2 - P$. Letting also $m_t = |m_t|e^{i\chi_t}$, one has

$$\begin{aligned}\mathrm{Tr}\,(Q\mathcal{L}_t[P]) &= \sum_{ij} B_{ij}(t)Q_{ii}P_{jj} + l_tP_{01}Q_{10} + \overline{l_t}P_{10}Q_{01} + m_tP_{10}Q_{10} + \overline{m_t}P_{01}Q_{01} \\ &= \gamma_-(t)|w_1|^4 + \gamma_+(t)|w_2|^4 + 2(\Gamma_T(t) - \frac{\Gamma_L(t)}{2})|w_1|^2|w_2|^2 \\ &\qquad - 2|m_t||w_1|^2|w_2|^2\cos(\chi_t - \Omega). \qquad (A.1)\end{aligned}$$

For the sufficiency part, let $\gamma_\pm(t) \geq 0$ and

$$\mathcal{G}(t) := \Gamma_T(t) - \frac{\Gamma_L(t)}{2} + \sqrt{\gamma_+(t)\gamma_-(t)} - |m_t| \geq 0\,.$$

Then,

$$\begin{aligned}\mathrm{Tr}\,(Q\mathcal{L}_t[P]) &= \left(\sqrt{\gamma_-(t)}|w_1|^2 - \sqrt{\gamma_+(t)}|w_2|^2\right)^2 + 2\big(\Gamma_t(t) - \frac{\Gamma_L(t)}{2} + \sqrt{\gamma_+(t)\gamma_-(t)})|w_1|^2|w_2|^2 \\ &\qquad - 2\,|m_t|\cos(\chi_t - \Omega)\big)|w_1|^2|w_2|^2 \\ &\geq \left(\sqrt{\gamma_-(t)}|w_1|^2 - \sqrt{\gamma_+(t)}|w_2|^2\right)^2 + 2\mathcal{G}(t)|w_1|^2|w_2|^2 \geq 0\,.\end{aligned}$$

so that P-divisibility follows from Proposition 2.4 and Remark 2.4. To prove the necessity part, first notice that, asking for $\mathrm{Tr}\,(Q\mathcal{L}_t[P]) \geq 0$ the choices $w_1 = 1$, and $w_2 = 1$ imply that $\gamma_\pm(t) \geq 0$. Choose instead, for fixed $t \geq 0$,

$$|w_1|^2 = \frac{\sqrt{\gamma_+(t)}}{\sqrt{\gamma_+(t)} + \sqrt{\gamma_-(t)}}\,, \qquad |w_2|^2 = \frac{\sqrt{\gamma_-(t)}}{\sqrt{\gamma_+(t)} + \sqrt{\gamma_-(t)}}\,,$$

and $\Omega = \chi_t$ so that, from (A.1),

$$0 \leq \mathrm{Tr}\left(Q\mathcal{L}_t[P]\right) = 2\mathcal{G}(t)|w_1|^2|w_2|^2,$$

which implies (3.61). For what concerns CP-divisibility, a necessary and sufficient condition for $\mathcal{L}_t$ to be in the GKSL form for all $t \geq 0$ is [150, 151]

$$Y_t \equiv (\mathbb{1}_4 - P_2^+)\mathcal{L}_t \otimes \mathrm{id}_2[P_2^+](\mathbb{1}_4 - P_2^+) \geq 0\,, \tag{A.2}$$

Noting that

$$Y_t = \begin{pmatrix} \frac{2\Gamma_T(t)-\Gamma_L(t)}{4} & 0 & 0 & -\frac{2\Gamma_T(t)-\Gamma_L(t)}{4} \\ 0 & \gamma_+(t) & m_t & 0 \\ 0 & \overline{m_t} & \gamma_-(t) & 0 \\ -\frac{2\Gamma_T(t)-\Gamma_L(t)}{4} & 0 & 0 & \frac{2\Gamma_T(t)-\Gamma_L(t)}{4} \end{pmatrix}.$$

which is positive iff conditions (3.62) and (3.63) are verified. □

APPENDIX B

Collisional model with Markov chain environment

B.1 Reduced dynamics

It suffices to consider tensor product elements of the local algebra $\mathcal{A}_E^{[-a,b]}$, supported by the interval of integers $-a \leq j \leq b$, that are denoted by multi-indices $\boldsymbol{i}_{[-a,b]} = i_{-a} i_{-a+1} \cdots i_b$ as follows:

$$A_{\boldsymbol{i}_{[-a,b]}}^{\otimes[-a,b]} = A_{i_{-a}}^{(-a)} \otimes A_{i_{-a+1}}^{(-a+1)} \otimes \cdots \otimes A_{i_b}^{(b)} = \bigotimes_{k=-a}^{b} A_{i_k}^{(k)} ,$$

where the upper index in $A_{i_k}^{(k)}$ indicates the site k at which the operator A_{i_k} is located. Recall the CPU map $\Theta_n = (\sigma \circ \Phi)^n$ describing the collisional dynamics, with Φ as in (4.37). Then, Θ_1 acts on the algebra $\mathcal{A}_S \otimes \mathcal{A}_E^{[-a,b]}$ as follows:

$$\Theta_1\Big[X_S \otimes A_{\boldsymbol{i}_{[-a,-1]}}^{\otimes[-a,-1]} \otimes A_{i_0}^{(0)} \otimes A_{\boldsymbol{i}_{[1,b]}}^{\otimes[1,b]}\Big] = \sum_{k=0}^{d-1} \phi_k[X_S] \otimes A_{\boldsymbol{i}_{[-a,-1]}}^{\otimes[-a+1,0]} \otimes \Pi_k^{(1)} A_{i_0}^{(1)} \Pi_k^{(1)} \otimes A_{\boldsymbol{i}_{[1,b]}}^{\otimes[2,b+1]} . \tag{B.1}$$

Iterating, one gets

$$\Theta_n\Big[X_S \otimes A_{\boldsymbol{i}_{[-a,b]}}^{\otimes[-a,b]}\Big] = \sum_{\boldsymbol{k}_{[1,n]}} \phi_{\boldsymbol{k}_{[1,n]}}[X_S] \otimes A_{\boldsymbol{i}_{[-a,-n]}}^{\otimes[-a+n,0]} \otimes \Pi_{k_1}^{(1)} A_{i_{-n+1}}^{(1)} \Pi_{k_1}^{(1)} \otimes \cdots \otimes \Pi_{k_n}^{(n)} A_{i_0}^{(n)} \Pi_{k_n}^{(n)} \otimes A_{i_{[1,b]}}^{\otimes[n+1,b+n]} , \tag{B.2}$$

where we set $\phi_{\boldsymbol{k}_{[1,n]}} \equiv \phi_{k_1} \circ \phi_{k_2} \circ \cdots \circ \phi_{k_n}$.

B.1.1 System S reduced dynamics

The reduced dynamics Λ_n of the states of the open system S at discrete time n in (2.59) is obtained by restricting the compound state $\omega_S \otimes \omega_E \circ \Theta_n$ to the system S algebra $\mathcal{A}_S \otimes \mathbb{1}_E$.

Using (B.2) one gets

$$\Theta_n[X_S \otimes \mathbb{1}_E] = \sum_{\boldsymbol{k}_{[1,n]}} \phi_{\boldsymbol{k}_{[1,n]}}[X_S] \otimes \bigotimes_{j=1}^{n} \Pi^{(j)}_{k_j} . \tag{B.3}$$

Then, by evaluating the latter on the state $\omega_S \otimes \omega_E$, where ω_S is represented by a density matrix ρ_S and ω_E is as in (4.36)

$$\omega_S \otimes \omega_E \left(\Theta_n[X_S \otimes \mathbb{1}_E]\right) = \mathrm{Tr}\left(\rho_S \sum_{\boldsymbol{k}_{[1,n]}} p_{\boldsymbol{k}_{[1,n]}} \phi_{\boldsymbol{k}_{[1,n]}}[X_S]\right) = \mathrm{Tr}\left(\sum_{\boldsymbol{k}_{[1,n]}} p_{\boldsymbol{k}_{[1,n]}} \phi^{\ddagger}_{\boldsymbol{k}_{[1,n]}}[\rho_S] X_S\right) ,$$

where $\phi^{\ddagger}_{\boldsymbol{k}_{[1,n]}} = \phi^{\ddagger}_{k_n} \circ \cdots \circ \phi^{\ddagger}_{k_1}$ with $\phi^{\ddagger}_{k_i}$ the dual map of ϕ_{k_i}. Hence, in the Schrödinger picture, the dynamical map reads

$$\Lambda_n = \sum_{\boldsymbol{i}_{[1,n]}} p_{\boldsymbol{i}_{[1,n]}} \phi^{\ddagger}_{\boldsymbol{i}_{[1,n]}} . \tag{B.4}$$

B.1.2 Environment E reduced dynamics

The single site operators A_i belong to the commutative algebra generated by the orthogonal projectors $\Pi^{(k)}_j$; then, $\sum_{k=0}^{d-1} \Pi_k A_i \Pi_k = A_i$. Therefore, due to the assumed unitality of the CP maps ϕ_k, from (B.2) it follows that

$$\begin{aligned}
\Theta_n\Big[\mathbb{1}_S \otimes A^{\otimes[-a,b]}_{\boldsymbol{i}_{[-a,b]}}\Big] &= \sum_{\boldsymbol{k}_{[1,n]}} \mathbb{1}_S \otimes A^{\otimes[-a+n,0]}_{\boldsymbol{i}_{[-a,-n]}} \otimes \Pi^{(1)}_{k_1} A^{(1)}_{i_{-n+1}} \Pi^{(1)}_{k_1} \otimes \cdots \otimes \Pi^{(n)}_{k_n} A^{(n)}_{i_0} \Pi^{(n)}_{k_n} \otimes A^{\otimes[n+1,b+n]}_{\boldsymbol{i}_{[1,b]}} \\
&= \mathbb{1}_S \otimes A^{\otimes[-a+n,0]}_{\boldsymbol{i}_{[-a,-n]}} \otimes A^{(1)}_{i_{-n+1}} \otimes \cdots \otimes A^{(n)}_{i_0} \otimes A^{\otimes[n+1,b+n]}_{\boldsymbol{i}_{[1,b]}} \\
&= \mathbb{1}_S \otimes \sigma^n\Big[A^{\otimes[-a,b]}_{\boldsymbol{i}_{[-a,b]}}\Big] .
\end{aligned} \tag{B.5}$$

Since the environment state is shift-invariant by construction, it follows that the environment state is stationary:

$$\omega_S \otimes \omega_E \circ \Theta_n\Big(\mathbb{1}_S \otimes \mathcal{A}_E\Big) = \omega_S \otimes \omega_E\Big(\mathbb{1}_S \otimes \mathcal{A}_E\Big) = \omega_E\Big(\mathcal{A}_E\Big) .$$

B.2 Reduced dynamics in the unitary case

The unitary case correspond to choosing $\varphi = -1$ in (4.46). Then, only $j = n-2$ contributes to the sum in (4.49) so that:

$$\lambda^{(\ell)}_n = (1 - 2(p+r))\, \lambda^{(\ell)}_{n-1} + 4\, p\, \Delta\, \lambda^{(\ell)}_{n-2} , \quad \ell = 1, 2 . \tag{B.6}$$

The general solutions of (B.6) can be found with the ansatz $\lambda^{(\ell)}_n = x \lambda^{(\ell)}_{n-1}$ for all $n \geq 2$, by means of the roots $x^{\pm}$ of [152]

$$P(x) = x^2 - A_{p,r}\, x + 4p\Delta , \qquad A_{p,r} = 1 - 2(p+r) . \tag{B.7}$$

The general solution will thus have the form $\lambda_n^{(\ell)} = c_+ x_+^n + c_- x_-^n$, with the constants $c_\pm$ fixed by the initial conditions $\lambda_0^{(\ell)} = 1$ and $\lambda_1^{(\ell)} = A_{p,r}$. The eigenvalues $\lambda_n^{(1,2,3)}$ then read

$$\lambda_n^{(\ell)} = \frac{\beta + A_{p,r}}{2B_{p,r,\Delta}} \left(\frac{A_{p,r} + B_{p,r,\Delta}}{2}\right)^n + \frac{B_{p,r,\Delta} - A_{p,r}}{2B_{p,r,\Delta}} \left(\frac{A_{p,r} - B_{p,r,\Delta}}{2}\right)^n =: \lambda_n , \quad \ell = 1,2 , \quad \text{(B.8)}$$

$$\lambda_n^{(3)} = (1-4p)^n . \quad \text{(B.9)}$$

where we set $B_{p,r,\Delta} = \sqrt{A_{p,r}^2 + 16\, p\Delta}$,. From the multiplicative action of the Pauli maps Λ_n on the Pauli matrices, one deduces that Λ_n is a convex combination of two discrete-time semi-groups:

$$\Lambda_n = \frac{B_{p,r,\Delta} + A_{p,r}}{2B_{p,r,\Delta}} \Psi_+^n + \frac{B_{p,r,\Delta} - A_{p,r}}{2B_{p,r,\Delta}} \Psi_-^n , \qquad \Psi_\pm[X] = \sum_{i=0}^{3} \psi_\pm^{(i)} \operatorname{Tr}(\sigma_i X)\, \sigma_i , \qquad \text{where}$$

$$\psi_\pm^{(1,2)} = \frac{A_{p,r} \pm B_{p,r,\Delta}}{2} , \qquad \psi_\pm^{(0)} = 1 , \ \psi_\pm^{(3)} = 1 - 4p . \quad \text{(B.10)}$$

It will be sufficient to consider the case $A_{p,r} > 0$, namely $r < 1/2 - p$. If $p \neq 1/4$, then $\lambda_n^{(j)}$ and Λ_n. We can thus compute the intertwining maps $\Lambda_{n,n-1} = \Lambda_n \circ \Lambda_{n-1}^{-1}$ between two subsequent collisions. Setting $\gamma := \frac{B_{p,r,\Delta} + A_{p,r}}{2} > \frac{B_{p,r,\Delta} - A_{p,r}}{2} =: \delta > 0$, these maps are of Pauli type with eigenvalues

$$\lambda_{n,n-1} := \lambda_{n,n-1}^{(1)} = \lambda_{n,n-1}^{(2)} = \frac{\lambda_n}{\lambda_{n-1}} = \frac{\gamma^{n+1} + (-1)^n \delta^{n+1}}{\gamma^n + (-1)^{n-1} \delta^n} , \qquad \lambda_{n,n-1}^{(3)} = 1 - 4p . \quad \text{(B.11)}$$

The P-divisibility of the discrete family of Pauli maps Λ_n, that is the contractivity of the intertwining maps $\Lambda_{n.n-1}$, is equivalent to asking that $\left|\lambda_{n,n-1}^{(i)}\right| \leq 1$, $i = 1,2,3$. In order to show this, we first prove that

$$\lambda_{2,1} > \lambda_{n,n-1} \qquad \forall n > 2 . \quad \text{(B.12)}$$

To see this, let $[0,1] \ni x \equiv \delta/\gamma$. For even $n = 2k > 2$,

$$\lambda_{n,n-1} = \gamma \frac{1 + x^{n+1}}{1 - x^n} , \quad \text{(B.13)}$$

monotonically decreases with n. Instead, for odd $n = 2k+1 > 2$,

$$\lambda_{n,n-1} = \gamma \frac{1 - x^{n+1}}{1 + x^n} , \quad \text{(B.14)}$$

increases with n; nevertheless,

$$\frac{\lambda_{n,n-1}}{\lambda_{2,1}} = \frac{1 - x^{n+1}}{1 + x^n} \frac{1 - x^2}{1 + x^3} \leq 1 . \quad \text{(B.15)}$$

Notice that $p_0 = 1 - r - 2p \geq 0$ and $r \geq 0$ imply $0 \leq p \leq 1/2$, so that $\left|\lambda_{n,n-1}^{(3)}\right| = |1 - 4p| \leq 1$. From (B.11) and the previous discussion, one checks when $|\lambda_{2,1}| \leq 1$:

$$0 \leq \lambda_{2,1} = 1 - 2(p+r) + \frac{4\Delta p}{1 - 2(p+r)} \leq 1 \iff \frac{\Delta}{A_{p,r}} \leq \frac{r}{2p} + \frac{1}{2} . \quad \text{(B.16)}$$

Conditions for the complete positivity of $\Lambda_{n,n-1}$ are obtained by asking for the positivity Choi matrix $\Lambda_{n,n-1}\otimes\mathrm{id}_2[P_+^{(2)}]$.The eigenvalues are easily computed to be: $E_1(n) = p$, twice degenerate and

$$E_2(n) = \frac{1}{4}(1 + \lambda^{(3)}_{n,n-1} + 2\lambda_{n,n-1}) = \frac{1-2p}{2} + \frac{\lambda_{n,n-1}}{2} \,, \tag{B.17}$$

$$E_3(n) = \frac{1}{4}(1 + \lambda^{(3)}_{n,n-1} - 2\lambda_{n,n-1}) = \frac{1-2p}{2} - \frac{\lambda_{n,n-1}}{2} \,. \tag{B.18}$$

From $0 \le p \le 1/2$ and $\lambda_{n,n-1} \ge 0$ it follows that $E_1(n) \ge 0$. Further, (B.12) implies $E_3(n) \ge E_3(2)$; then the positivity of $E_3(n)$ is ensured by

$$1 + \lambda^{(3)}_{2,1} - 2\lambda_{2,1} = 4r - \frac{8\,p\,\Delta}{A_{p,r}} \ge 0 \iff \frac{\Delta}{A_{p,r}} \le \frac{r}{2p} \,. \tag{B.19}$$

We now consider the positivity of $\Lambda_{n,n-1}\otimes\Lambda_{n,n-1}$. Since $\Lambda_{n,n-1}$ are Pauli maps, then $\Lambda_{n,n-1}\otimes\Lambda_{n,n-1}$ is positive if and only if $\Lambda^2_{n,n-1}$ is completely positive [153], that is if and only if the Choi matrix $\Lambda^2_{n,n-1}\otimes\mathbb{1}[P_+^{(2)}] \ge 0$. Recasting

$$P_+^{(2)} = \frac{1}{4}\Big(\mathbb{1}\otimes\mathbb{1} + \sigma_1\otimes\sigma_1 - \sigma_2\otimes\sigma_2 + \sigma_3\otimes\sigma_3\Big) = \frac{1}{4}\begin{pmatrix} \mathbb{1}+\sigma_3 & \sigma_1 + i\sigma_2 \\ \sigma_1 - i\sigma_2 & \mathbb{1}-\sigma_3 \end{pmatrix}$$

yields

$$\Lambda_{n,n-1}\otimes\Lambda_{n,n-1}[P_+^{(2)}] = \Lambda^2_{n,n-1}\otimes\mathbb{1}[P_+^{(2)}]$$
$$= \frac{1}{4}\begin{pmatrix} 1+(\lambda^{(3)}_{n,n-1})^2 & 0 & 0 & 2\lambda^2_{n,n-1} \\ 0 & 1-(\lambda^{(3)}_{n,n-1})^2 & 0 & 0 \\ 0 & 0 & 1-(\lambda^{3}_{n,n-1})^2 & 0 \\ 2\lambda^2_{n,n-1} & 0 & 0 & 1+(\lambda^{(3)}_{n,n-1})^2 \end{pmatrix} . \tag{B.20}$$

Then, from $0 \le p \le 1/2$ it follows that $\Lambda_{n,n-1}\otimes\Lambda_{n,n-1}$ is completely positive iff

$$1 + (1-4p)^2 - 2\big(\lambda_{n,n-1}\big)^2 \ge 0 \,. \tag{B.21}$$

Moreover, since $\lambda_{n,n-1} \ge \lambda_{2,1} \ge 0$, we get the inequality

$$4\,p^2\left(\frac{\Delta}{A_{p,r}}\right)^2 + 2\,p\,A_{p,r}\left(\frac{\Delta}{A_{p,r}}\right) - (p^2 + r\,p_0) \le 0 \,, \tag{B.22}$$

where we recall that $p_0 = 1-2p-r$. Equation (B.22) then gives the condition for P-divisibility of $\Lambda_n\otimes\Lambda_n$,

$$\frac{\Delta}{A_{p,r}} \le \frac{r}{2p} + \frac{1}{2} - \frac{1-\sqrt{1-4p(1-2p)}}{4p} \,. \tag{B.23}$$

To see explicitly how the environmental correlations relate to the lack of BFI for one qubit and super-activation of BFI for two qubits, let us consider $r = 0$ and $p \ll 1$ and let $\Delta = Q\,p$. From (B.8) and (B.9), one sees that

$$\lambda_{n,n-1} = 1 - 2p + 4Qp^2 + O(p^3)\,, \qquad \lambda^{(3)}_{n,n-1} = 1 - 4p\,, \tag{B.24}$$

for all $n \geq 2$. Notice that the eigenvalues of $X = X^\dagger = x_0 + \sum_{i=1}^{3} x_i\sigma_i$ in $M_2(\mathbb{C})$ are $x_0 \pm \|\boldsymbol{x}\|$. Then, $\|X\|_1 = 2x_0 = \mathrm{Tr}X$ if $x_0 \geq \|\boldsymbol{x}\|$, otherwise $\|X\|_1 = 2\|\boldsymbol{x}\|$. Let us assume $\mathrm{Tr}\,(X) = 2x_0 \geq 0$ and set $Y = \Lambda_{n,n-1}[X]$, its eigenvalues being $x_0 \pm \|\boldsymbol{y}\|$, with $\boldsymbol{y} = (\lambda_{n,n-1}x_1, \lambda_{n,n-1}x_2, \lambda^{(3)}_{n,n-1}x_3)$. Thus, $\|Y\|_1 = \|X\|_1 = 2x_0$ if $x_0 > \|\boldsymbol{y}\|$, otherwise $\|Y\|_1 = 2\|\boldsymbol{y}\|$. Then, expanding up to the second order in p one finds

$$\|\boldsymbol{y}\|^2 = \|\boldsymbol{x}\|^2 - 2p\left(2(x_1^2+x_2^2)+x_3^2\right) + 4p^2\left(Q(x_1^2+x_2^2)+x_3^2\right) + O(p^3)\,. \tag{B.25}$$

Therefore, for $x_0 \leq \|\boldsymbol{y}\|$, $x_0 \leq \|\boldsymbol{x}\|$ so that $\|Y\|_1 \leq \|\boldsymbol{x}\| = \|X\|_1$ and contractivity ensues. Indeed that a Q-dependent, positive contribution in (B.25) only appears at second order in p and is dominated by a strictly negative contribution, thus preventing BFI for one qubit. Also, notice that up to second order in p there is no dependence on the successive discrete-time steps n and $n-1$.

Instead, let us consider the case of two qubits and consider the trace norm of $Z := \Lambda_{n,n-1}\otimes\Lambda_{n,n-1}[P_+^{(2)}]$ in the same small p regime. From (B.20) one sees that the eigenvalues of Z are $1-(\lambda^{(3)})^2 \geq 0$ twice degenerate and

$$1+(\lambda^{(3)})^2 + 2\lambda^2_{n,n-1} \geq 0\,, \qquad 1+(\lambda^{(3)})^2 - 2\lambda^2_{n,n-1}\,.$$

If the latter is positive it follows that $\|Z\|_1 = 1 = \|P_2^+\|_1$; otherwise, if $2\lambda^2_{n,n-1} > 1+(\lambda^{(3)}_{n,n-1})^2$, which for small p occurs whenever $Q > 1/2$,

$$\|Z\|_1 = 2\left(1-(\lambda^{(3)})^2\right) + 4\lambda^2_{n.n-1} \simeq 1 + 4p^2(2Q-1)$$

becomes larger than 1 for $Q > 1/2$. Therefore, unlike for a single qubit, for two qubits the leading correction is a term of order 2 in p; this becomes positive for sufficiently correlated sites in the Markov chain environment in which case $\Lambda_{n,n-1}\otimes\Lambda_{n,n-1}$ ceases to be contractive.

B.3 Local system-chain density matrices and mutual information

Let us consider again the local algebra $\mathcal{A}_E^{[-a,b]}$ supported by the integers $0 \leq a \leq b$ whose elements are linear combinations of tensor products $A^{\otimes[-a,b]}_{\boldsymbol{i}_{[-a,b]}}$. Each single-site operator belongs to the commutative algebra $\mathcal{A} = D_d(\mathbb{C})$ generated by the orthogonal projections Π_k, $0 \leq k \leq d-1$ and is thus of the form $A^{(l)}_{i_l} = \sum_{k_l=0}^{d-1} a^{k_l}_{i_l}\Pi^{(l)}_{k_l}$. Then,

$$A^{\otimes[-a,b]}_{\boldsymbol{i}_{[-a,b]}} = \sum_{\boldsymbol{k}_{[-a,b]}} a^{\boldsymbol{k}_{[-a,b]}}_{\boldsymbol{i}_{[-a,b]}}\Pi^{[-a,b]}_{\boldsymbol{k}_{[-a,b]}}\,,\quad \Pi^{[-a,b]}_{\boldsymbol{k}_{[-a,b]}} \equiv \bigotimes_{l=-a}^{b}\Pi^{(l)}_{k_l}\,,\quad a^{\boldsymbol{k}_{[-a,b]}}_{\boldsymbol{i}_{[1-a,b]}} \equiv \prod_{l=-a}^{b} a^{k_l}_{i_l}\,.$$

The dynamics (B.2) thus gives

$$\Theta_n\left[X_S\otimes A^{\otimes[-a,b]}_{\boldsymbol{i}_{[-a,b]}}\right] = \begin{cases} \sum_{\boldsymbol{k}_{[-a,b]}} a^{\boldsymbol{k}_{[-a,b]}}_{\boldsymbol{i}_{[-a,b]}}\phi_{\boldsymbol{k}_{[-n+1,0]}}[X_S]\otimes\Pi^{[-a+n,n+b]}_{\boldsymbol{k}_{[-a,b]}} & \dots\quad 0\leq n\leq a \\ \sum_{\boldsymbol{k}_{[-n+1,b]}} a^{\boldsymbol{k}_{[-a,b]}}_{\boldsymbol{i}_{[-a,b]}}\phi_{\boldsymbol{k}_{[-n+1,0]}}[X_S]\otimes\Pi^{[1,n+b]}_{\boldsymbol{k}_{[-n+1,b]}} & \dots\quad n>a \end{cases}\,, \tag{B.26}$$

where $\phi_{\boldsymbol{k}_{[-n+1,0]}} \equiv \phi_{k_{-n+1}} \circ \cdots \circ \phi_{k_0}$.

Let us now consider the discrete-time evolution of local density matrices that is obtained by duality:

$$\omega_S \otimes \omega_E \circ \Theta_n \left[X_S \otimes A_E^{[-a,b]} \right] = \mathrm{Tr}\left(\Omega^{(n)}_{S[-a,b]} \, X_S \otimes A_E^{[-a,b]} \right) .$$

Using the shift invariance of the environment state ω_E one gets:

$$\omega_S \otimes \omega_E \circ \Theta_n \left[X_S \otimes A^{\otimes[-a,b]}_{\boldsymbol{i}_{[-a,b]}} \right] = \begin{cases} \sum_{\boldsymbol{k}_{[-a,b]}} p_{\boldsymbol{k}_{[-a,b]}} a^{\boldsymbol{k}_{[-a,b]}}_{\boldsymbol{i}_{[-a,b]}} \, \mathrm{Tr}\left(\phi^{\ddagger}_{\boldsymbol{k}_{[-n+1,0]}}[\rho_S] \, X_S \right) & n \leq a \, , \\ \sum_{\boldsymbol{k}_{[-n+1,b]}} p_{\boldsymbol{k}_{[-n+1,b]}} a^{\boldsymbol{k}_{[-a,b]}}_{\boldsymbol{i}_{[-a,b]}} \, \mathrm{Tr}\left(\phi^{\ddagger}_{\boldsymbol{k}_{[-n+1,0]}}[\rho_S] \, X_S \right) & n > a \, . \end{cases}$$

Therefore, the local density matrices at discrete time-step n, $\Omega^{(n)}_{S[-a,b]}$, read

$$\Omega^{(n)}_{S[-a,b]} = \begin{cases} \sum_{\boldsymbol{k}_{[-a,b]}} p_{\boldsymbol{k}_{[-a,b]}} \phi^{\ddagger}_{\boldsymbol{k}_{[-n+1,0]}}[\rho_S] \otimes \Pi^{[-a,b]}_{\boldsymbol{k}_{[-a,b]}} & n \leq a \, , \\ \sum_{\boldsymbol{k}_{[-n+1,b]}} p_{\boldsymbol{k}_{[-n+1,b]}} \phi^{\ddagger}_{\boldsymbol{k}_{[-n+1,0]}}[\rho_S] \otimes \Pi^{[-n+1,b]}_{\boldsymbol{k}_{[-n+1,b]}} & n > a \, . \end{cases} \tag{B.27}$$

To quantify the system-chain correlations, we compute the mutual information

$$I\left(\Omega^{(n)}_{S[-a,b]}\right) = S\left(\Omega^{(n)}_S\right) + S\left(\Omega^{(n)}_{[-a,b]}\right) - S\left(\Omega^{(n)}_{S[-a,b]}\right) \, ; \tag{B.28}$$

relative to the evolved local density matrices $\Omega^{(n)}_{S[-a,b]}$ (B.27) and their marginals $\Omega^{(n)}_S$, respectively $\Omega^{(n)}_{[-a,b]}$ that are obtained by performing the trace over $\mathcal{A}_S$, respectively $\mathcal{A}_E$. Using the notation in (4.36), they read

$$\Omega^{(n)}_S = \sum_{\boldsymbol{k}_{[-n+1,0]}} p_{\boldsymbol{k}_{[-n+1,0]}} \, \phi^{\ddagger}_{\boldsymbol{k}_{[-n+1,0]}}[\rho_S] \, , \tag{B.29}$$

$$\Omega^{(n)}_{[-a,b]} = \begin{cases} \sum_{\boldsymbol{k}_{[-a,b]}} p_{\boldsymbol{k}_{[-a,b]}} \, \Pi^{[-a,b]}_{\boldsymbol{k}_{[-a,b]}} = \rho_E^{[-a,b]} & n \leq a \, , \\ \sum_{\boldsymbol{k}_{[-n+1,b]}} p_{\boldsymbol{k}_{[-n+1,b]}} \, \Pi^{[1-n,b]}_{\boldsymbol{k}_{[-n+1,b]}} = \rho_E^{[-n+1,b]} & n \geq a \, . \end{cases} \tag{B.30}$$

Notice that (B.29) follows since

$$a \geq n \Rightarrow \sum_{\boldsymbol{k}_{[-a,-n]}} \sum_{\boldsymbol{k}_{[1,b]}} p_{\boldsymbol{k}_{[-a,b]}} = p_{\boldsymbol{k}_{[-n+1,0]}} \, .$$

Furthermore, by relabelling the indices in the right-hand side of (B.29), one obtains $\Omega^{(n)}_S = \Lambda_n[\rho_S]$ with Λ_n as in (B.4). Since the contributing operators in (B.27) are all orthogonal, one gets

$$S\left(\Omega^{(n)}_{S[-a,b]}\right) = \begin{cases} S\left(\rho_E^{[-a,b]}\right) + \sum_{\boldsymbol{k}_{[-n+1,0]}} p_{\boldsymbol{k}_{[-n+1,0]}} \, S\left(\phi^{\ddagger}_{\boldsymbol{k}_{[-n+1,0]}}\right) & n \leq a \, , \\ S\left(\rho_E^{[-n+1,b]}\right) + \sum_{\boldsymbol{k}_{[-n+1,0]}} p_{\boldsymbol{k}_{[-n+1,0]}} \, S\left(\phi^{\ddagger}_{\boldsymbol{k}_{[-n+1,0]}}[\rho_S]\right) & n > a \, . \end{cases} \tag{B.31}$$

Therefore, by relabelling the summation indices, the mutual information simplifies to

$$I\left(\Omega^{(n)}_{S[-a+1,b]}\right) = S(\Lambda_n[\rho_S]) - \sum_{\boldsymbol{k}_{[1,n]}} p_{\boldsymbol{k}[1,n]}\, S\left(\Phi^{\ddagger}_{\boldsymbol{k}_{[1,n]}}[\rho_S]\right) \equiv I^{(n)}_{SE}\,. \tag{B.32}$$

Notice that the right-hand side of (B.32) only depends on n and not on the specific local subalgebra $\mathcal{A}_E^{[-a,b]}$. The previous result can be easily extended to two qubits, each evolving in the same collisional environment, and equipped with initial state $\omega_{S+S} \otimes \omega_{E+E}$ where ω_{S+S} corresponds to a two-qubit density matrix ρ_{S+S}, while $\omega_{E+E} = \omega_E \otimes \omega_E$ since the two environments are independent. One then analogously derives a qubit-qubit-environment mutual information as

$$I^{(n)}_{(S+S)(E+E)} := S(\Lambda_n \otimes \Lambda_n[\rho_{S+S}]) - \sum_{\boldsymbol{l}_{[1,n]},\boldsymbol{k}_{[1,n]}} p_{\boldsymbol{l}_{[1,n]}} p_{\boldsymbol{k}_{[1,n]}}\, S\left(\phi^{\ddagger}_{\boldsymbol{l}_{[1,n]}} \otimes \phi^{\ddagger}_{\boldsymbol{k}_{[1,n]}}[\rho_{S+S}]\right)\,.$$

B.4 Stroboscopic limit

Let us consider Pauli maps as in (4.70) of the form $\phi_k = e^{\tau\mathcal{L}_k}$ and $\varphi = e^{-2\gamma\tau}$. This choice corresponds to the case in which the system, identified by $\mathcal{A}_S$, and the chain ancilla at site (0), described by $\mathcal{A}_E^{(0)}$, are dissipatively coupled for a time τ through the following GKLS generator,

$$\mathbb{L}[O_S \otimes O_E^{(0)}] = \gamma \sum_{i=0}^{3} \left((\sigma_i \otimes \Pi_i)\, O_S \otimes O_E^{(0)}\, (\sigma_i \otimes \Pi_i) - \frac{1}{2}\left\{ \mathbb{1}_2 \otimes \Pi_i, O_S \otimes O_E^{(0)} \right\} \right)\,, \tag{B.33}$$

which satisfies $\mathbb{L}[O_S \otimes \Pi_j] = \mathcal{L}_j[O_S] \otimes \Pi_j$. The reduced dynamics will be of the Pauli type, with spectrum $\lambda_n^{(i)}$ obeying the recurrences (4.49) and (4.50). In the so-called stroboscopic limit typical of collision models, one takes

$$\tau \to 0\,, \qquad n \to \infty, \qquad n\tau \to t\,, \tag{B.34}$$

and expands (4.50) at first order in τ obtaining

$$\frac{\lambda_n^{(3)} - \lambda_{n-1}^{(3)}}{\tau} = -2\gamma\lambda_{n-1}^{(3)} \implies \dot{\lambda}_t^{(3)} = -2\gamma\lambda_t^{(3)} \Rightarrow \lambda_t^{(3)} = e^{-2\gamma t}\,. \tag{B.35}$$

On the other hand, denoting by λ_n the other two equal Pauli eigenvalues and expanding (4.49) up to order τ yield the following finite-difference equation:

$$\frac{\lambda_n - \lambda_{n-1}}{\tau} = -2(p+r)\gamma\lambda_{n-1} + 2p\,\gamma^2 \sum_{j=0}^{n-2} \tau\,(2\Delta)^{n-j-1}(1-\gamma\tau)^{n-j-2}\lambda_j\,. \tag{B.36}$$

Choosing $\Delta = \dfrac{e^{-\kappa\tau}}{2}$, the stroboscopic limit (B.34) and the constraints (4.45) yield $p \to 1/2, r \to 0$ and turn (B.36) into the integro-differential equation

$$\dot{\lambda}_t = -\gamma\,\lambda_t + \gamma^2 \int_0^t \mathrm{d}s\, e^{-(\kappa+\gamma)(t-s)}\lambda_s\,. \tag{B.37}$$

The latter is readily solvable through its Laplace transform $\widetilde{\lambda}_z = \int_0^{+\infty} \mathrm{d}t\, e^{-zt} \lambda_t$, with the initial condition $\lambda_{t=0} = 1$, yielding:

$$\widetilde{\lambda}_z = \frac{z + \kappa + \gamma}{z^2 + z(\kappa + 2\gamma) + \kappa\gamma} \quad \text{with simple poles at} \quad z_\pm = \frac{-(\kappa + 2\gamma) \pm \sqrt{\kappa^2 + 4\gamma^2}}{2} \leq 0\,. \tag{B.38}$$

Therefore, for $a \geq z_+$, one gets

$$\lambda_t = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \mathrm{d}z\, e^{zt} \widetilde{\lambda}_z = e^{-(\gamma + \frac{\kappa}{2})t} \left[\cosh\left(\frac{1}{2} t \sqrt{\kappa^2 + 4\gamma^2}\right) + \frac{\kappa \sinh\left(\frac{1}{2} t \sqrt{\kappa^2 + 4\gamma^2}\right)}{\sqrt{\kappa^2 + 4\gamma^2}} \right]. \tag{B.39}$$

By inspection of the Choi matrix of Λ_t, namely $\Lambda_t \otimes \mathrm{id}_2[P_+^{(2)}]$, one realizes that complete positivity requires

$$1 + \lambda_t^{(3)} - 2\lambda_t \geq 0\,, \tag{B.40}$$

which is checked to be always satisfied.

B.5 Details about the Mutual information

In the case of a unitary coupling between system and collisional environment, the variation of the system-chain mutual information reduces to the variation of the von Neumann entropy in discrete time as in (4.67). As considered in the main text, the choice (4.68) together with $p = 1/4 + \epsilon$ yield the following Pauli eigenvalues at discrete-time steps 1, respectively 2: $\lambda_1 = \lambda_2 = \frac{1}{2} - 2\epsilon$, respectively $\lambda_1^{(3)} = -4\epsilon$, $\lambda_2^{(3)} = 16\epsilon^2$.

Since ϵ is taken as a small perturbative parameter, it follows that the intertwiner $\Lambda_{2,1}$ is a positive map. Indeed, the corresponding Pauli eigenvalues satisfy

$$\lambda_{2,1} = \frac{\lambda_2}{\lambda_1} = 1\,, \quad \left|\lambda_{2,1}^{(3)}\right| = \left|\frac{\lambda_2^{(3)}}{\lambda_1^{(3)}}\right| = 4\epsilon < 1\,.$$

Then, consider the first two time-step dynamics of two-qubit totally symmetric state $P_+^{(2)}$:

$$\Lambda_1 \otimes \Lambda_1[P_+^{(2)}] = \begin{pmatrix} \frac{1}{4} + 4\epsilon^2 & \cdot & \cdot & \frac{1}{8}(1-4\epsilon)^2 \\ \cdot & \frac{1}{4} - 4\epsilon^2 & \cdot & \cdot \\ \cdot & \cdot & \frac{1}{4} - 4\epsilon^2 & \cdot \\ \frac{1}{8}(1-4\epsilon)^2 & \cdot & \cdot & \frac{1}{4} + 4\epsilon^2 \end{pmatrix} \tag{B.41}$$

$$\Lambda_2 \otimes \Lambda_2[P_+^{(2)}] = \begin{pmatrix} \frac{1}{4} + 64\epsilon^4 & \cdot & \cdot & \frac{1}{8}(1-4\epsilon)^2 \\ \cdot & \frac{1}{4} - 64\epsilon^4 & \cdot & \cdot \\ \cdot & \cdot & \frac{1}{4} - 64\epsilon^4 & \cdot \\ \frac{1}{8}(1-4\epsilon)^2 & \cdot & \cdot & \frac{1}{4} + 64\epsilon^4 \end{pmatrix}. \tag{B.42}$$

By evaluating the spectrum of the two states and expanding the von Neumann entropies of the two above states (B.41) up to second order in ϵ, one gets

$$S(\Lambda_1 \otimes \Lambda_1[P_+^{(2)}]) = \frac{20\log(2) - 3\log(3)}{8} + \log(3)\,\epsilon + \left(8\log(2) - 6\log(3) - \frac{16}{3}\right)\epsilon^2 + O(\epsilon^3)\,,$$

$$S(\Lambda_2 \otimes \Lambda_2[P_+^{(2)}]) = \frac{20\log(2) - 3\log(3)}{8} + \log(3)\,\epsilon - \left(2\log(3) + \frac{16}{3}\right)\epsilon^2 + O(\epsilon^3)\,.$$

Their difference coincides with the variation of the system-chain correlations and is given, up to order ϵ^2, by

$$\Delta I_{SE}^{(2,1)}(P_+^{(2)}) = S(\Lambda_2 \otimes \Lambda_2[P_+^{(2)}]) - S(\Lambda_1 \otimes \Lambda_1[P_+^{(2)}]) = -4\log\left(\frac{4}{3}\right)\epsilon^2 + O(\epsilon^3) < 0\,.$$

Let us now compute the quantum mutual information for the case $p = 1/2$, $\Delta = 1/2$, corresponding to master equation rates $\gamma_t = 1$, $\gamma_t^{(3)} = -\tanh(t)$ and Pauli eigenvalues $\lambda_t = e^{-t}\cosh(t)$, $\lambda_t^{(3)} = e^{-2t}$. Notice that, with these choices, the stochastic matrix T in (4.44) takes a particularly simple form:

$$T = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{B.43}$$

so that the only non-zero probabilities $p_{\boldsymbol{i}_{[1,n]}}$ correspond to the sequences $\boldsymbol{i}_{[1,n]} = 111\cdots \equiv \mathbf{1}$ and $\boldsymbol{i}_{[1,n]} = 222\cdots \equiv \mathbf{2}$. Accordingly, only two CPTP unital semigroups $\phi_{\boldsymbol{i}_{[1,n]}}$ contribute in (4.40),

$$\Lambda_t = \frac{\phi_{\mathbf{1}} + \phi_{\mathbf{2}}}{2}\,, \tag{B.44}$$

with equal, time-independent weights $p_{\mathbf{1}} = p_{\mathbf{2}} = 1/2$. In the continuous-time limit, $\phi_{\mathbf{1}}$ and $\phi_{\mathbf{2}}$ are the Pauli maps defined by

$$\begin{aligned} &\phi_{\mathbf{1}}[\sigma_1] = 1\,, \qquad &&\phi_{\mathbf{1}}[\sigma_2] = e^{-2t}\,, \qquad &&\phi_{\mathbf{1}}[\sigma_3] = e^{-2t}\,, \\ &\phi_{\mathbf{2}}[\sigma_1] = e^{-2t}\,, \qquad &&\phi_{\mathbf{2}}[\sigma_2] = 1\,, \qquad &&\phi_{\mathbf{2}}[\sigma_3] = e^{-2t}\,. \end{aligned} \tag{B.45}$$

Notice that for other choices of T, in the stroboscopic limit, the weights $p_{\boldsymbol{i}_{[1,n]}}$ would generally become functions of time as well. In the special case of (B.43), the mutual information as function of t reads

$$I_{SE}^{(t)}(\rho_S) = S(\Lambda_t \otimes \Lambda_t)[\rho_{S+S}]) - \frac{1}{4}\sum_{i,j=\mathbf{1},\mathbf{2}} S(\phi_i \otimes \phi_j[\rho_{S+S}])\,. \tag{B.46}$$

The non-monotonic behaviour of the system-chain mutual information (B.46) has been inspected numerically by means of the following family of X states,

$$\begin{aligned} \rho_X^{(1)} = \frac{1}{4}\big\{&\mathbb{1}_4 - (1 - 2(\mu_1 + \nu))\sigma_1 \otimes \mathbb{1}_2 + (1 - 2(\mu_2 + \nu))\mathbb{1}_2 \otimes \sigma_1 \\ &- (1 - 2(\mu_1 + \mu_2))\sigma_1 \otimes \sigma_1 - 2\,\mathrm{Re}(u - v)\sigma_2 \otimes \sigma_2 \\ &+ 2\,\mathrm{Re}(u + v)\sigma_3 \otimes \sigma_3 + 2\,\mathrm{Im}(u + v)\sigma_2 \otimes \sigma_3 + 2\,\mathrm{Im}(u - v)\sigma_3 \otimes \sigma_2\big\}\,, \end{aligned} \tag{B.47}$$

having the X shape when written in the basis of $\sigma_1 \otimes \sigma_1$, which can be obtained from the standard one by applying the matrix $V \otimes V$, $V = (\sigma_1 + \sigma_3)/\sqrt{2}$. The positivity condition are then readily obtained and read

$$0 \le \mu_1, \mu_2 \le 1 \qquad\qquad 0 \le \nu \le 1 - (\mu_1 + \mu_2)\,, \tag{B.48}$$

$$|u| \le \sqrt{\mu_1 \mu_2}\,, \qquad\qquad |v| \le \sqrt{\nu(1 - \mu_1 - \mu_2 - \nu)}\,. \tag{B.49}$$

Setting $\alpha_t = e^{-t}\cosh(t)$ and $\beta_t = e^{-2t}$, the states in (B.46) read

$$\begin{aligned}
\Lambda_t \otimes \Lambda_t[\rho_X^{(1)}] \;=\; & \frac{1}{4}\big\{\mathbb{1}_4 - (1-2(\mu_1+\nu))\alpha_t\sigma_1\otimes\mathbb{1}_2 + (1-2(\mu_2+\nu))\alpha_t\mathbb{1}_2\otimes\sigma_1 \\
& - (1-2(\mu_1+\mu_2))\alpha_t^2\sigma_1\otimes\sigma_1 - 2\,\mathrm{Re}(u-v)\alpha_t^2\sigma_2\otimes\sigma_2 \\
& + 2\,\mathrm{Re}(u+v)\beta_t^2\sigma_3\otimes\sigma_3 + 2\,\mathrm{Im}(u+v)\alpha_t\beta_t\sigma_2\otimes\sigma_3 + 2\,\mathrm{Im}(u-v)\alpha_t\beta_t\sigma_3\otimes\sigma_2\big\}\,, \\
\Phi_{\mathbf{1}}\otimes\Phi_{\mathbf{1}}[\rho_X^{(1)}] = & \frac{1}{4}\big\{\mathbb{1}_4 - (1-2(\mu_1+\nu))\sigma_1\otimes\mathbb{1}_2 + (1-2(\mu_2+\nu))\mathbb{1}_2\otimes\sigma_1 \\
& - (1-2(\mu_1+\mu_2))\sigma_1\otimes\sigma_1 - 2\,\mathrm{Re}(u-v)\beta_t^2\sigma_2\otimes\sigma_2 \\
& - \quad +2\,\mathrm{Re}(u+v)\beta_t^2\sigma_3\otimes\sigma_3 + 2\,\mathrm{Im}(u+v)\beta_t^2\sigma_2\otimes\sigma_3 + 2\,\mathrm{Im}(u-v)\beta_t^2\sigma_3\otimes\sigma_2\big\}\,, \\
\Phi_2\otimes\Phi_2[\rho_X^{(1)}] = & \frac{1}{4}\big\{\mathbb{1}_4 - (1-2(\mu_1+\nu))\beta_t\sigma_1\otimes\mathbb{1}_2 + (1-2(\mu_2+\nu))\beta_t\mathbb{1}_2\otimes\sigma_1 \\
& - (1-2(\mu_1+\mu_2))\beta_t^2\sigma_1\otimes\sigma_1 - 2\,\mathrm{Re}(u-v)\sigma_2\otimes\sigma_2 \\
& + 2\,\mathrm{Re}(u+v)\beta_t^2\sigma_3\otimes\sigma_3 + 2\,\mathrm{Im}(u+v)\beta_t\sigma_2\otimes\sigma_3 + 2\,\mathrm{Im}(u-v)\beta_t\sigma_3\otimes\sigma_2\big\}\,, \\
\Phi_{\mathbf{1}}\otimes\Phi_2[\rho_X^{(1)}] = & \frac{1}{4}\big\{\mathbb{1}_4 - (1-2(\mu_1+\nu))\sigma_1\otimes\mathbb{1}_2 + (1-2(\mu_2+\nu))\beta_t\mathbb{1}_2\otimes\sigma_1 \\
& - (1-2(\mu_1+\mu_2))\beta_t^2\sigma_1\otimes\sigma_1 - 2\,\mathrm{Re}(u-v)\beta_t\sigma_2\otimes\sigma_2 \\
& + 2\,\mathrm{Re}(u+v)\beta_t^2\sigma_3\otimes\sigma_3 + 2\,\mathrm{Im}(u+v)\beta_t^2\sigma_2\otimes\sigma_3 + 2\,\mathrm{Im}(u-v)\beta_t\sigma_3\otimes\sigma_2\big\}\,, \\
\Phi_2\otimes\Phi_{\mathbf{1}}[\rho_X^{(1)}] = & \frac{1}{4}\big\{\mathbb{1}_4 - (1-2(\mu_1+\nu))\beta_t\sigma_1\otimes\mathbb{1}_2 + (1-2(\mu_2+\nu))\mathbb{1}_2\otimes\sigma_1 \\
& - (1-2(\mu_1+\mu_2))\beta_t\sigma_1\otimes\sigma_1 - 2\,\mathrm{Re}(u-v)\beta_t\sigma_2\otimes\sigma_2 \\
& + 2\,\mathrm{Re}(u+v)\beta_t^2\sigma_3\otimes\sigma_3 + 2\,\mathrm{Im}(u+v)\beta_t\sigma_2\otimes\sigma_3 + 2\,\mathrm{Im}(u-v)\beta_t^2\sigma_3\otimes\sigma_2\big\}\,.
\end{aligned}$$

APPENDIX C

Coarse-graining and ALF entropy

C.1 Entropy exchange and the GNS construction

We now derive equation (1.128). For a generic partition $\mathcal{X}$, let $\{|a\rangle\}_{a=1}^{|\mathcal{X}|}$ be a basis of $\mathbb{C}^{|\mathcal{X}|}$ and define, similarly as in the model of measurement of Section 1.4.3, an isometry $V : \mathcal{H}_\omega \otimes |0\rangle \longrightarrow \mathcal{H}_\omega \otimes \mathbb{C}^{|\mathcal{X}|}$

$$V\,|\Omega_\omega\rangle \otimes |0\rangle := \sum_{a=1}^{|\mathcal{X}|} \pi_\omega(X_a)\,|\Omega_\omega\rangle \otimes |a\rangle\ . \tag{C.1}$$

The projector onto $|\Omega_\omega\rangle \otimes |0\rangle$ is then mapped into the pure state

$$\left|\psi\middle\rangle\middle\langle\psi\right| := V\,|\Omega_\omega\rangle\langle\Omega_\omega| \otimes |0\rangle\langle 0|\,V^\dagger = \sum_{ab=1}^{|\mathcal{X}|} \pi_\omega(X_a)\,|\Omega_\omega\rangle\langle\Omega_\omega|\,\pi_\omega(X_b)^\dagger \otimes |a\rangle\langle b|\ . \tag{C.2}$$

The marginals of $\left|\psi\middle\rangle\middle\langle\psi\right|$ yield

$$\mathrm{Tr}_I(\left|\psi\middle\rangle\middle\langle\psi\right|) = \sum_{a,b} \omega\left(X_b^\dagger X_a\right)|a\rangle\langle b| = \rho[\mathcal{X}]\,, \tag{C.3}$$

and

$$\mathrm{Tr}_{II}(\left|\psi\middle\rangle\middle\langle\psi\right|) = \sum_a \pi_\omega(X_a)\,|\Omega_\omega\rangle\langle\Omega_\omega|\,\pi(X_a)^\dagger =: \mathbb{X}_\omega[|\Omega_\omega\rangle\langle\Omega_\omega|]\,. \tag{C.4}$$

The marginals (C.3) and (C.4) of the pure state $\left|\psi\middle\rangle\middle\langle\psi\right|$ have the same spectrum, multiplicities included, a part from the zero eigenvalue; thus, they have the same von Neumann entropy,

$$S(\rho[\mathcal{X}]) = S\left(\mathbb{X}_\omega[|\Omega_\omega\rangle\langle\Omega_\omega|]\right)\ .$$

For example, take $\mathcal{X} \subseteq \mathcal{A} = M_d(\mathbb{C})$, so that $\omega(\,\cdot\,) = \mathrm{Tr}\,(\rho\,\cdot\,)$ and the GNS representation is achieved through $|\Omega_\omega\rangle = \left|\sqrt{\rho}\right\rangle$ and $\pi_\omega(X) = X \otimes \mathbb{1}_d$ (see Example 1.3). Then,

$$\mathbb{X}_\omega = \mathbb{X} \otimes \mathrm{id}_d\,, \qquad \text{where} \quad \mathbb{X}[\rho] = \sum_{a=1}^{|\mathcal{X}|} X_a \rho X_a^\dagger\,,$$

so that

$$S(\rho[\mathcal{X}]) = S\big(\mathbb{X} \otimes \mathrm{id}_d\left[|\sqrt{\rho}\rangle\langle\sqrt{\rho}|\right]\big)\,. \tag{C.5}$$

From the latter expression, it is also evident how the entropy exchange discussed in Section 1.4.3 does depend on the chosen OPU, but not on its specific Kraus representation.

C.2 ALF entropy as multi-time entropy exchange

Repeating the same steps for the time-refined OPU $\mathcal{X}^{(n)}$, one dilates to a pure state

$$\sum_{\boldsymbol{a}} \pi_\omega(X_{\boldsymbol{a}}^{(n)})\,|\Omega_\omega\rangle \otimes \left|\boldsymbol{a}_{[0,n-1]}\right\rangle\,,$$

and considers the corresponding projector

$$\sum_{\boldsymbol{a},\boldsymbol{b}} \pi_\omega(X_{\boldsymbol{a}}^{(n)})\,|\Omega_\omega\rangle\langle\Omega_\omega|\,\pi_\omega(X_{\boldsymbol{b}}^{(n)})^\dagger \otimes \left|\boldsymbol{a}_{[0,n-1]}\right\rangle\left\langle\boldsymbol{b}_{[0,n-1]}\right|\,. \tag{C.6}$$

Its marginals have thus the same entropy:

$$S\big(\rho[\mathcal{X}^{(n)}]\big) = S\left(\sum_{\boldsymbol{a}} \pi_\omega(X_{\boldsymbol{a}}^{(n)})\,|\Omega_\omega\rangle\langle\Omega_\omega|\,\pi_\omega(X_{\boldsymbol{a}}^{(n)})^\dagger\right)\,. \tag{C.7}$$

From (1.113),

$$\begin{aligned}
\pi_\omega\left(X_{\boldsymbol{a}}^{(n)}\right)|\Omega_\omega\rangle &= \pi_\omega(\Theta^{n-1}(X_{a_{n-1}}))\,\ldots\,\pi_\omega(\Theta(X_{a_1}))\pi_\omega(X_{a_0})\,|\Omega_\omega\rangle \\
&= (U_\omega^\dagger)^{n-1}\pi_\omega(X_{a_{n-1}})U_\omega^{n-1}\,(U_\omega^\dagger)^{n-2}\ldots U_\omega^\dagger\pi_\omega(X_{a_1})U\pi_\omega(X_{a_0})\,|\Omega_\omega\rangle \\
&= (U_\omega^\dagger)^n\left(U_\omega\pi_\omega(X_{a_{n-1}})\ldots U_\omega\pi_\omega(X_{a_0})\right)|\Omega_\omega\rangle\,,
\end{aligned}$$

so that

$$\sum_{\boldsymbol{a}} \pi_\omega(X_{\boldsymbol{a}}^{(n)})\,|\Omega_\omega\rangle\langle\Omega_\omega|\,\pi_\omega(X_{\boldsymbol{a}}^{(n)})^\dagger = \left(\mathbb{U}_\omega^\ddagger\right)^n \circ (\mathbb{U}_\omega \circ \mathbb{X}_\omega)^n\left[|\Omega_\omega\rangle\langle\Omega_\omega|\right]. \tag{C.8}$$

From (C.7), by the invariance of the von Neumann entropy under unitary maps, one has

$$S\left(\rho\big[\mathcal{X}^{(n)}\big]\right) = S\left((\mathbb{U}_\omega \circ \mathbb{X}_\omega)^n\left[\,|\Omega_\omega\rangle\langle\Omega_\omega|\,\right]\right). \tag{C.9}$$

APPENDIX D

Symbolic models for Open Systems

D.1 Proof of Proposition 2.13.

We need to prove that

$$\Lambda_{k,k-1} = \Lambda_1 =: \Lambda\,, \qquad \forall\, k \geq 1\,. \tag{D.1}$$

In the main text, we showed that the latter holds for $k = 2$: $\Lambda_{2,1} = \Lambda$. Now, suppose (D.1) holds for $k = n-1$. Choose again the OPU, made of operators

$$\mathcal{F} = \{F_{a,a'}\}_{aa'=1}^{d}\,, \qquad F_{aa'} = \sqrt{r_a}\,|r_a\rangle\langle r_{a'}| \in \mathcal{A}_S\,, \tag{D.2}$$

where $0 < r_a \leq 1$, $|r_a\rangle$, $a = 1, \dots, d$ are the eigenvalues, respectively, the eigenvectors of ρ_S, so that this OPU leaves ω_S invariant. Due to the assumed invariance of the $\omega_S \otimes \omega_E$ under both Θ and $\mathbb{F}$, the state $\omega_{\mathcal{X}}$ on the half-spin chain defined through the coarse-grained density matrices $\rho_S\big[\mathcal{F}^{(n+1)}\big]$, $n \geq 1$ is invariant under the shift to the right. On the other hand, explicit evaluation of $\rho\left[\mathcal{F}^{(n+1)}\right]$ yields

$$\rho\big[\mathcal{F}^{(n+1)}\big] = \sum_{\substack{a_0 b_0 \dots a_n b_n \\ a'_0 b'_0 \dots a'_n b'_n}} \delta_{b_n a_n} r_{a_n} \delta_{b'_0 a'_0} r_{a'_0} \prod_{k=1}^{n} \sqrt{r_{a_{k-1}} r_{b_{k-1}}}\, \langle r_{a'_k}|\, \Lambda_{k,k-1}\left[\, |r_{a_{k-1}}\rangle\langle r_{b_{k-1}}|\,\right] |r_{b'_k}\rangle$$
$$\big|r_{a_0} r_{a'_0} \dots r_{a_n} r_{a'_n}\big\rangle\big\langle r_{b_0} r_{b'_0} \dots r_{b_n} r_{b'_n}\big|\,. \tag{D.3}$$

Shift invariance means that $\mathrm{Tr}_I\, \rho\left[\mathcal{F}^{(n+1)}\right] = \rho\left[\mathcal{F}^{(n)}\right]$. Here, Tr_I denotes the trace on the first two tensor factors in (D.3), this amounts to setting $a_0 = b_0$ and $a'_0 = b'_0$ and then summing over a_0, a'_0. Consider thus the entries

$$\mathrm{Tr}_I\left(\rho[\mathcal{F}^{(n+1)}]\right)_{a_1a_1'a_na_n'\ldots b_1a_1'a_nb_n'} = r_{a_n}\sum_{a_0'} r_{a_0'}\left\langle r_{a_1'}\right|\Lambda_1\left[\rho_S\right]\left|r_{b_1'}\right\rangle$$

$$\prod_{k=2}^{n}\sqrt{r_{a_{k-1}}r_{b_{k-1}}}\,\langle r_{a_k'}|\,\Lambda_{k,k-1}\left[\,|r_{a_{k-1}}\rangle\langle r_{b_{k-1}}|\right]\,|r_{b_k'}\rangle$$

$$= r_{a_n}r_{a_1'}\prod_{k=2}^{n-1}\sqrt{r_{a_{k-1}}r_{b_{k-1}}}\,\langle r_{a_k'}|\,\Lambda\left[\,|r_{a_{k-1}}\rangle\langle r_{b_{k-1}}|\right]\,|r_{b_k'}\rangle\,\langle r_{a_n'}|\,\Lambda_{n,n-1}\left[\,|r_{a_{n-1}}\rangle\langle r_{b_{n-1}}|\right]\,|r_{b_n'}\rangle\,.$$

The latter must be equal to

$$\rho[\mathcal{F}^{(n)}]_{a_1a_1'a_na_n'\ldots b_1a_1'a_nb_n'} = r_{a_n}r_{a_1'}\prod_{k=2}^{n}\sqrt{r_{a_{k-1}}r_{b_{k-1}}}\,\langle r_{a_k'}|\,\Lambda\left[\,|r_{a_{k-1}}\rangle\langle r_{b_{k-1}}|\right]\,|r_{b_k'}\rangle\,,$$

which yields

$$\langle r_{a_n'}|\,\Lambda_{n,n-1}\left[\,|r_{a_{n-1}}\rangle\langle r_{b_{n-1}}|\right]\,|r_{b_n'}\rangle = \langle r_{a_n'}|\,\Lambda\left[\,|r_{a_{n-1}}\rangle\langle r_{b_{n-1}}|\right]\,|r_{b_n'}\rangle\,.$$

and consequently, $\Lambda_{n,n-1} = \Lambda$.

D.2 Proof of Theorem 2.12.

D.2.1 Proof of (2.78)

We need to recast system coarse-grained matrix $\rho_S\left[\mathcal{F}^{(n+1)}\right]$ for the OPU $\mathcal{F}$ (2.76) (see also Remark 1.18) in the form

$$\rho_S\left[\mathcal{F}^{(n+1)}\right] = \rho_S\otimes\rho_S\otimes\left(\mathbb{T}_n\otimes\mathrm{id}\left[\left|\sqrt{\rho_S^{\otimes n}}\right\rangle\left\langle\sqrt{\rho_S^{\otimes n}}\right|\right]\right)\,.$$

By definition,

$$\mathbb{T}_n^{\ddagger}\left[\bigotimes_{k=0}^{n-1}A_k\right] = \omega_E\left(\widetilde{\Theta}_n^{\mathbb{Z}E}\left(\bigotimes_{k=0}^{n-1}A_k^{(k-n)}\otimes\mathbb{1}_E\right)\right) \tag{D.4}$$

$$= \omega_E\left(\Theta_1\left(A_0^{(0)}\otimes\Theta_{2,1}\left(A_1^{(1)}\otimes\ldots\Theta_{n,n-1}\left(A_{n-1}^{(n-1)}\otimes\mathbb{1}_E\right)\right)\right)\right). \tag{D.5}$$

where we set

$$\widetilde{\Theta}_n^{\mathbb{Z}E} := \prod_{j=1}^{n}\left(\Theta_{j,j-1}\circ\sigma\otimes\mathrm{id}_E\right)\,, \tag{D.6}$$

and σ denotes the shift on $M_d^{\mathbb{Z}}(\mathbb{C})$. From (D.4), $\mathbb{T}_n$ is manifestly completely positive and unital. Notice that $\mathbb{T}_1[A_0] = \omega_E\left(\Theta_1\left(A_0^{(0)}\otimes\mathbb{1}_E\right)\right) = \Lambda_1[A_0]$. Moreover, for sake of brevity, the positioning (upper indices) of system operators along the S-chain in (D.5) has been chosen in accordance with how many times the shift has acted after the automorphism Θ made the 0-th site system operators interact with the environment; indeed,

$$\Theta_{n,n-1}\circ(\sigma\otimes\mathrm{id}_E)\Big[A_0^{(-n)}\otimes\ldots\otimes A_{n-1}^{(-1)}\Big]:=A_0^{(-n+1)}\otimes\ldots\otimes\Theta_{n,n-1}\Big(A_{n-1}^{(0)}\otimes\mathbb{1}_E\Big).$$

Then, the action of the automorphism, $\Theta_{n,n-1}\Big(A_{n-1}^{(0)}\otimes\mathbb{1}_E\Big)=\widetilde{A}^{(0)}\otimes B_E$, is such that

$$\begin{aligned}\sigma\otimes\mathrm{id}_E\Big[A_0^{(-n+1)}\otimes\cdots\otimes\Theta_{n,n-1}\Big(A_{n-1}^{(0)}\otimes\mathbb{1}_E\Big)\Big)\Big]&=A_0^{(-n+2)}\otimes\cdots\otimes A_{n-2}^{(0)}\otimes\widetilde{A}_{n-1}^{(1)}\otimes B_E\\&=A_0^{(-n+2)}\otimes\cdots\otimes A_{n-2}^{(0)}\otimes\Theta_{n,n-1}\Big(A_{n-1}^{(1)}\otimes\mathrm{id}_E\Big).\end{aligned}$$

One thus gets

$$\begin{aligned}&\Big(\Theta_{n-1,n-2}\circ(\sigma\otimes\mathrm{id}_E)\Big)\circ\Big(\Theta_{n,n-1}\circ(\sigma\otimes\mathrm{id}_E)\Big)\Big[A_0^{(-n)}\otimes\ldots\otimes A_{n-1}^{(-1)}\Big]\\&\qquad=A_0^{(-n+2)}\otimes\cdots\otimes\Theta_{n-1,n-2}\Big(A_{n-2}^{(0)}\otimes\Theta_{n,n-1}\Big(A_{n-1}^{(1)}\otimes\mathrm{id}_E\Big)\Big).\end{aligned}$$

The elements of the coarse-grained density matrix with respect to the partition (5.9) read

$$\begin{aligned}&\rho_S\left[\mathcal{F}^{(n+1)}\right]_{\boldsymbol{a},\boldsymbol{b}}\\&=\omega_S\otimes\omega_E\Big(F^\dagger_{b_0b_0'}\otimes\mathbb{1}_E\,\Theta_1\Big(F^\dagger_{b_1b_1'}\otimes\mathbb{1}_E\ldots\Theta_{n,n-1}\big(F^\dagger_{b_nb_n'}F_{a_na_n'}\otimes\mathbb{1}_E\big)\ldots F_{a_1a_1'}\otimes\mathbb{1}_E\Big)F_{a_0a_0'}\otimes\mathbb{1}_E\Big)\\&=r_{a_n}\delta_{a_nb_n}r_{a_0'}\delta_{a_0'b_0'}\left(\prod_{m=0}^{n-1}\sqrt{r_{a_m}r_{b_m}}\right)\mathrm{Tr}\Big(R_{a_0b_0}\,\omega_E\Big(\Theta_1\Big(R^\dagger_{b_1b_1'}\otimes\mathbb{1}_E\ldots\\&\qquad\ldots\Theta_{n,n-1}\big(R^\dagger_{a_n'b_n'}\otimes\mathbb{1}_E\big)\ldots R_{a_1a_1'}\otimes\mathbb{1}_E\Big)\Big)\Big),\end{aligned}\tag{D.7}$$

with multi-indices $\boldsymbol{a}=(a_0a_0',\ldots a_na_n')$, $\boldsymbol{b}=(b_0b_0',\ldots,b_bb_n')$ and where $R_{ab}=|r_a\rangle\langle r_b|$ are the matrix units corresponding to the eigenbasis of ρ_S; they satisfy

$$R^\dagger_{ab}=R_{ba}\,,\qquad R_{ab}\,R_{cd}=\delta_{bc}\,R_{ad}\,.\tag{D.8}$$

We shall now obtain another expression for (D.7) in terms of the CPU map (D.4). To this end, we split the proof in several steps.

Step 1. Firstly, we introduce a state functional $\Omega_S^{\mathbb{Z}\mathbb{Z}}:M_d^{\mathbb{Z}}(\mathbb{C})\otimes M_d^{\mathbb{Z}}(\mathbb{C})\to\mathbb{C}$ on the doubled S-chain, whose local restrictions are the purified vector states corresponding to the tensor products of the system S state; namely,

$$\begin{aligned}&\Omega_S^{\mathbb{Z}\mathbb{Z}}\left(X^{[a,a+n]}\right)=\left\langle\sqrt{\rho_S^{\otimes n}}\right|X^{[a,a+n]}\left|\sqrt{\rho_S^{\otimes n}}\right\rangle\,,\qquad X^{[a,a+n]}\in M_d^{[a,n]}(\mathbb{C})\otimes M_d^{[a,n]}(\mathbb{C})\,,\\&\left|\sqrt{\rho_S^{\otimes n}}\right\rangle=\sum_{\boldsymbol{a}_{[1,n]}}\sqrt{r_{\boldsymbol{a}_{[1,n]}}}\left|r_{\boldsymbol{a}_{[1,n]}}\otimes r_{\boldsymbol{a}_{[1,n]}}\right\rangle\,,\qquad r_{\boldsymbol{a}_{[1,n]}}=\prod_{k=1}^n r_{a_k}\,,\end{aligned}\tag{D.9}$$

and consider the compound dynamical system

$$\Big(M_d^{\mathbb{Z}}(\mathbb{C})\otimes M_d^{\mathbb{Z}}(\mathbb{C})\otimes\mathcal{A}_E,\,\Omega_S^{\mathbb{Z}\mathbb{Z}}\otimes\omega_E,\,\widetilde{\Theta}_n^{\mathbb{Z}\mathbb{Z}E}\Big),\tag{D.10}$$

$$\text{where}\qquad\widetilde{\Theta}_n^{\mathbb{Z}\mathbb{Z}E}:=\prod_{k=1}^n\big(\Theta_{k,k-1}\otimes\mathrm{id}_{\mathbb{Z}}\circ\sigma\otimes\sigma\otimes\mathrm{id}_E\big)$$

Introducing the operators

$$Z_{LR}^{[-n,-1]} \otimes \mathbb{1}_E\,,\quad Z_{LR}^{[-n,-1]} := Z_L^{[-n,-1]} \otimes Z_R^{[-n,-1]}\,, \tag{D.11}$$

$$Z_L^{[-n,-1]} := \bigotimes_{k=1}^{n} R^{\dagger(k-n-1)}_{a'_k b'_k}\,,\quad Z_R^{[-n,-1]} := \bigotimes_{l=0}^{n-1} R^{\dagger(l-n)}_{a_l b_l}\,, \tag{D.12}$$

the definition in (D.5) of the CPU map $\mathbb{T}_n^\ddagger$ yields

$$\Omega_S^{\mathbb{Z}\mathbb{Z}} \otimes \omega_E\left(\widetilde{\Theta}_n^{\mathbb{Z}\mathbb{Z}E}\left(Z_{LR}^{[-n,-1]} \otimes \mathbb{1}_E\right)\right) = \Omega_S^{\mathbb{Z}\mathbb{Z}}\left(\mathbb{T}_n^\ddagger\left[Z_L^{[-n,-1]}\right] \otimes Z_R^{[0,n-1]}\right)\,. \tag{D.13}$$

Step 2. Since $Z_{LR}^{[-n,-1]}$ is supported by the double chain sites $-n \le j \le -1$, the action of the n-th power of the shift $\sigma \otimes \sigma$, makes the operator $\mathbb{T}_n^\ddagger\left[Z_L^{[-n,-1]}\right] \otimes Z_R^{[0,n-1]}$ supported by the interval $[0, n-1]$. Then, in the Schrödinger picture, using the dual $\mathbb{T}_n$, one can rewrite the expectation in the right hand side of (D.13) as

$$\begin{aligned}&\Omega_S^{\mathbb{Z}\mathbb{Z}}\left(\mathbb{T}_n^\ddagger\left[Z_L^{[-n,-1]}\right] \otimes Z_R^{[0,n-1]}\right) = \\ &= \mathrm{Tr}_{[-n,-1]}\left(\mathbb{T}_n \otimes \mathrm{id}_d^{\otimes n}\left[\left|\sqrt{\rho_S^{\otimes[0,n-1]}}\right\rangle\!\left\langle\sqrt{\rho_S^{\otimes[0,n-1]}}\right|\right] Z_R^{[-n,-1]} \otimes Z_R^{[0,n-1]}\right).\end{aligned} \tag{D.14}$$

On the other hand, the left hand side of (D.13) reads

$$\begin{aligned}\Omega_S^{\mathbb{Z}\mathbb{Z}} \otimes \omega_E\left(\widetilde{\Theta}_n^{\mathbb{Z}\mathbb{Z}E}\left(Z_{LR}^{[-n,-1]} \otimes \mathbb{1}_E\right)\right) &= \Omega_S^{\mathbb{Z}\mathbb{Z}} \otimes \omega_E\left(\Theta_n^{\mathbb{Z}E}\left(Z_L^{[-n,-1]} \otimes \mathbb{1}_E\right) \otimes Z_R^{[0,n-1]}\right) \\ &= \left\langle\sqrt{\rho_S^{\otimes[0,n-1]}}\right|\omega_E\left(\widetilde{\Theta}_n^{\mathbb{Z}E}\left(Z_L^{[-n,-1]} \otimes \mathbb{1}_E\right) \otimes Z_R^{[0,n-1]}\right)\left|\sqrt{\rho_S^{\otimes[0,n-1]}}\right\rangle\,.\end{aligned}$$

By explicitly using the spectral form of the vector state in (B.7), one gets

$$\begin{aligned}&\Omega_S^{\mathbb{Z}\mathbb{Z}} \otimes \omega_E\left(\widetilde{\Theta}_n^{\mathbb{Z}\mathbb{Z}E}\left(Z_{LR}^{[-n,-1]} \otimes \mathbb{1}_E\right)\right) = \\ &= \sum_{\substack{\boldsymbol{i}_{[0,n-1]}\\ \boldsymbol{j}_{[0,n-1]}}} \prod_{m=0}^{n-1}\sqrt{r_{i_m} r_{j_m}}\ \mathrm{Tr}\left(\bigotimes_{q=0}^{n-1} R^{(q)}_{i_q j_q}\ \omega_E\left(\widetilde{\Theta}_n^{\mathbb{Z}E}\left(Z_L^{[-n,-1]} \otimes \mathbb{1}_E\right)\right)\right)\mathrm{Tr}\left(\bigotimes_{p=0}^{n-1} R^{(p)}_{i_p j_p}\ Z_R^{[0,n-1]}\right)\,.\end{aligned}$$

Using the expression for $Z_R^{[0,n-1]}$ in (D.12) yields $\mathrm{Tr}\left(\bigotimes_{p=0}^{n-1} R^{(p)}_{i_p j_p}\ Z_R^{[0,n-1]}\right) = \prod_{p=0}^{n-1}\delta_{i_p a_p}\,\delta_{j_p b_p}$, so that

$$\begin{aligned}&\Omega_S^{\mathbb{Z}\mathbb{Z}} \otimes \omega_E\left(\widetilde{\Theta}_n^{\mathbb{Z}\mathbb{Z}E}\left(Z_{LR}^{[-n,-1]} \otimes \mathbb{1}_E\right)\right) = \left(\prod_{m=0}^{n-1}\sqrt{r_{a_m} r_{b_m}}\right)\times \\ &\times \mathrm{Tr}_{[0,n-1]}\left(\bigotimes_{l=0}^{n-1} R^{(l)}_{a_l b_l}\omega_E\left(\Theta_1\left(R^{\dagger(0)}_{a'_1,b'_1} \otimes \Theta_{2,1}\left(R^{\dagger(1)}_{a'_2,b'_2} \ldots \otimes \Theta_{n,n-1}\left(R^{\dagger(n-1)}_{a'_n b'_n} \otimes \mathbb{1}_E\right)\right)\right)\right)\right)\,.\end{aligned} \tag{D.15}$$

Step 3. The trace over the local sub-algebras $M_d^{\otimes n}(\mathbb{C})$ in (D.15) can be converted into a trace over the single algebra $M_d(\mathbb{C})$ at time-step 0; namely,

$$\Omega_S^{\mathbb{Z}\mathbb{Z}} \otimes \omega_E\left(\widetilde{\Theta}_n^{\mathbb{Z}\mathbb{Z}E}\left(Z_{LR}^{[-n,-1]} \otimes \mathbb{1}_E\right)\right) = \tag{D.16}$$

$$= \left(\prod_{m=0}^{n-1} \sqrt{r_{a_m} r_{b_m}}\right) \mathrm{Tr}\left(R_{a_0 b_0} \omega_E\left(\Theta_1\left(R^\dagger_{b_1 b_1'} \otimes \mathbb{1}_E\, \Theta_{2,1}\left(\cdots\right.\right.\right.\right.$$

$$\left.\left.\left.\left.\cdots \Theta_{n-1,n-2}\left(R^\dagger_{b_{n-1} b_{n-1}'} \otimes \mathbb{1}_E\, \Theta_{n,n-1}\left(R^\dagger_{a_n' b_n'} \otimes \mathbb{1}_E\right) R_{a_{n-1} a_{n-1}'} \otimes \mathbb{1}_E\right) \cdots\right) R_{a_1 a_1'} \otimes \mathbb{1}_E\right)\right)\right).$$

This a fact is a consequence of the following equality,

$$\mathrm{Tr}_{[0,n-1]}\left(\bigotimes_{l=0}^{n-1} R^{(l)}_{a_l b_l} \Theta_1\left(R^{\dagger(0)}_{a_1',b_1'} \otimes \Theta_{2,1}\left(R^{\dagger(1)}_{a_2',b_2'} \ldots \otimes \Theta_{n,n-1}\left(R^{\dagger(n-1)}_{a_n' b_n'} \otimes E\right)\right)\right)\right) =$$

$$= \mathrm{Tr}\left(R_{a_0 b_0} \Theta_1\left(R^\dagger_{b_1 b_1'} \otimes \mathbb{1}_E\, \Theta_{2,1}\left(R^\dagger_{b_2 b_2'} \otimes \mathbb{1}_E \Theta_{3,2}\left(\cdots\right.\right.\right.\right.$$

$$\left.\left.\left.\left.\cdots \Theta_{n-1,n-2}\left(R^\dagger_{b_{n-1} b_{n-1}'} \otimes \mathbb{1}_E \Theta_{n,n-1}\left(R^\dagger_{a_n' b_n'} \otimes E\right) R_{a_{n-1} a_{n-1}'} \otimes \mathbb{1}_E\right) \cdots R_{a_2 a_2'} \otimes \mathbb{1}_E\right) R_{a_1 a_1'} \otimes \mathbb{1}_E\right)\right)\right), \tag{D.17}$$

that holds for any environment operator E and which can be proved by recursion. Indeed, the equality is true for $n = 1$; assume it holds for up to $n = k$ and consider its left hand side for $n = k + 1$:

$$\mathrm{Tr}_{[0,k]}\left(\bigotimes_{l=0}^{k} R^{(l)}_{a_l b_l} \Theta_1\left(R^{\dagger(0)}_{a_1',b_1'} \otimes \Theta_{2,1}\left(R^{\dagger(1)}_{a_2',b_2'} \ldots \otimes \Theta_{k,k-1}\left(R^{\dagger(k)}_{a_k' b_k'} \otimes \Theta_{k+1,k}\left(R^{\dagger(k)}_{a_{k+1}' b_{k+1}'} \otimes E\right)\right)\right)\right)\right). \tag{D.18}$$

Note that, expanding with respect to the system matrix units, one can always cast the action of the automorphism $\Theta_{k+1,k}$ on $M_d^{(0)}(\mathbb{C}) \otimes \mathcal{A}_E$ as

$$\Theta_{k+1,k}\left(R^\dagger_{a_{k+1}' b_{k+1}'} \otimes E\right) = \sum_{ij} R_{ij} \otimes E_{ij}^{a_{k+1}' b_{k+1}'}, \tag{D.19}$$

with suitable environment operators $E_{ij}^{a_{k+1}' b_{k+1}'}$ and then shift k times to the right the system operators, so that:

$$\Theta_{k+1,k}\left(R^{\dagger(k)}_{a_{k+1}' b_{k+1}'} \otimes E\right) = \sum_{ij} R^{(k)}_{ij} \otimes E_{ij}^{a_{k+1}' b_{k+1}'}. \tag{D.20}$$

Tracing over the site k then yields

$$\mathrm{Tr}_{\{k\}}\left(R^{(k)}_{a_k b_k} R^{\dagger(k)}_{a_{k+1}' b_{k+1}'} \otimes E\right) = E_{b_k a_k}^{a_{k+1}' b_{k+1}'}.$$

Plugging this result in the expression (D.18), it becomes

$$\mathrm{Tr}_{[0,k-1]}\left(\bigotimes_{l=0}^{k} R^{(l)}_{a_l b_l}\Theta_1\left(R^{\dagger(0)}_{a_1',b_1'}\otimes\Theta_{2,1}\left(R^{\dagger(1)}_{a_2',b_2'}\ldots\otimes\Theta_{k,k-1}\left(R^{\dagger(k)}_{a_k' b_k'}\otimes E^{a_{k+1}' b_{k+1}'}_{b_k a_k}\right)\right)\right)\right). \tag{D.21}$$

By the recursion assumption, this expression equals

$$\mathrm{Tr}\Bigg(R_{a_0 b_0}\Theta_1\Big(R^{\dagger}_{b_1 b_1'}\otimes\mathbb{1}_E\,\Theta_{2,1}\Big(\cdots \tag{D.22}$$

$$\cdots\Theta_{k-1,k-2}\Big(R^{\dagger}_{b_{k-1}b_{k-1}'}\otimes\mathbb{1}_E\Theta_{k,k-1}\Big(R^{\dagger}_{a_k' b_k'}\otimes E^{a_{k+1}' b_{k+1}'}_{b_k a_k}\Big)R_{a_{k-1}a_{k-1}'}\otimes\mathbb{1}_E\Big)\cdots\Big)R_{a_1 a_1'}\otimes\mathbb{1}_E\Big)\Bigg).$$

The proof of equality (D.17) is concluded by using (D.19) with $E=\mathbb{1}_E$ and by observing that

$$R^{\dagger}_{a_k' b_k'}\otimes E^{a_{k+1}' b_{k+1}'}_{b_k a_k} = R^{\dagger}_{b_k b_k'}\otimes\mathbb{1}_E\,\Theta_{k+1,k}\Big(R^{\dagger}_{a_{k+1}' b_{k+1}'}\otimes\mathbb{1}_E\Big)R_{a_k a_k'}\otimes\mathbb{1}_E\,. \tag{D.23}$$

Step 4. Putting together the equalities (D.13), (D.14) and (D.16) one gets

$$\left(\prod_{m=0}^{n-1}\sqrt{r_{a_m}r_{b_m}}\right)\mathrm{Tr}\Bigg(R_{a_0 b_0}\omega_E\Big(\Theta_1\Big(R^{\dagger}_{b_1 b_1'}\otimes\mathbb{1}_E\,\Theta_{2,1}\Big(R^{\dagger}_{b_2 b_2'}\otimes\mathbb{1}_E\Theta_{3,2}\Big(\cdots$$

$$\cdots\Theta_{n-1,n-2}\Big(R^{\dagger}_{b_{n-1}b_{n-1}'}\otimes\mathbb{1}_E\,\Theta_{n,n-1}\Big(R^{\dagger}_{a_n' b_n'}\otimes\mathbb{1}_E\Big)R_{a_{n-1}a_{n-1}'}\otimes\mathbb{1}_E\Big)\cdots R_{a_2 a_2'}\otimes\mathbb{1}_E\Big)R_{a_1 a_1'}\otimes\mathbb{1}_E\Big)\Big)\Bigg)$$

$$=\mathrm{Tr}_{[-n,-1]}\left(\mathbb{T}_n\otimes\mathrm{id}_d^{\otimes n}\left[\left|\sqrt{\rho_S^{\otimes[0,n-1]}}\right\rangle\!\left\langle\sqrt{\rho_S^{\otimes[0,n-1]}}\right|\right]Z_R^{[-n,-1]}\otimes Z_R^{[0,n-1]}\right). \tag{D.24}$$

Plugging this relation in (D.7), one gets

$$\rho_S\left[\mathcal{F}^{(n+1)}\right]_{\boldsymbol{a},\boldsymbol{b}} = r_{a_0'}\delta_{a_0' b_0'}\,r_{a_n}\delta_{a_n b_n}\;\mathrm{Tr}_{[-n,-1]}\Bigg(\bigotimes_{k=1}^{n}R^{\dagger(k-n-1)}_{a_k' b_k'}\otimes\bigotimes_{l=0}^{n-1}R^{\dagger(l-n)}_{a_l b_l}$$

$$\mathbb{T}_n\otimes\mathrm{id}_d^{\otimes n}\left[\left|\sqrt{\rho_S^{\otimes[0,n-1]}}\right\rangle\!\left\langle\sqrt{\rho_S^{\otimes[0,n-1]}}\right|\right]\Bigg). \tag{D.25}$$

Finally, a more convenient spectrally equivalent expression for the coarse-grained matrix $\rho_S\left[\mathcal{F}^{(n+1)}\right]$ can be obtained by unitarily swapping factors so that

$$\rho_S\left[\mathcal{F}^{(n+1)}\right] = \sum_{\boldsymbol{a}\,\boldsymbol{b}}\rho_S\left[\mathcal{F}^{(n+1)}\right]_{\boldsymbol{a},\boldsymbol{b}}\;R_{a_0' b_0'}\otimes R_{a_n b_n}\otimes\bigotimes_{k=1}^{n}R^{(k-n-1)}_{a_k' b_k'}\otimes\bigotimes_{l=0}^{n-1}R^{(l-n)}_{a_l b_l}\,, \tag{D.26}$$

so that, by substituting (D.25) into the latter, one obtains

$$\rho_S\left[\mathcal{F}^{(n+1)}\right] = \rho_S\otimes\rho_S\otimes\left(\mathbb{T}_n\otimes\mathrm{id}_d^{\otimes n}\left[\left|\sqrt{\rho_S^{\otimes n}}\right\rangle\!\left\langle\sqrt{\rho_S^{\otimes n}}\right|\right]\right).$$

D.2.2 QR regime.

Finally, we prove that

$$\mathrm{QR} \iff \mathbb{T}_n = \bigotimes_{k=1}^{n} \Lambda_{k,k-1}\,.$$

Consider generic operators $\{A_k\}_{k=0}^n$ and $\{B_k\}_{k=0}^n$ in $M_d(\mathbb{C})$ and evaluate

$$\sum_{\boldsymbol{i}_{[1,n-1]}\,,\,\boldsymbol{j}_{[1,n-1]}} \mathrm{Tr}_{[1,n-1]}\Bigg(\bigotimes_{k=1}^{n-1} R^{\dagger(k)}_{i_k j_k}\, \mathbb{T}_n^{\ddagger}\Big[B_1^\dagger\, R_{i_1 j_1}\, A_1 \otimes B_2^\dagger\, R_{i_2 j_2}\, A_2 \otimes \dots$$
$$\dots \otimes B_{n-1}^\dagger\, R_{i_{n-1} j_{n-1}}\, A_{n-1} \otimes B_n^\dagger A_n\Big]\Bigg) \tag{D.27}$$
$$= \sum_{\boldsymbol{i}_{[1,n-1]}\,,\,\boldsymbol{j}_{[1,n-1]}} \mathrm{Tr}_{[1,n-1]}\Bigg(\bigotimes_{k=1}^{n-1} R^{\dagger(k)}_{i_k j_k}\, \omega_E\Big(\Theta_1\Big(B_1^\dagger R^{(0)}_{i_1 j_1} A_1 \otimes \Theta_{2,1}\Big(B_2^\dagger R^{(1)}_{i_2 j_2} A_2 \dots$$
$$\dots \Theta_{n-1,n-2}\big(B_{n-1}^\dagger R^{(n-2)}_{i_{n-1} j_{n-1}} A_{n-1} \otimes \Theta_{n,n-1}(B_n^\dagger A_n^{(n-1)} \otimes \mathbb{1}_E))\Big)\Big)\Bigg)$$
$$= \omega_E\Big(\Theta_1\Big(B_1^\dagger \otimes \mathbb{1}_E\, \Theta_{2,1}\Big(B_2^\dagger \otimes \mathbb{1}_E \dots \Theta_{n,n-1}\left(B_n^\dagger A_n \otimes \mathbb{1}_E\right) \dots A_2 \otimes \mathbb{1}_E\Big) A_1 \otimes \mathbb{1}_E\Big)\Big). \tag{D.28}$$

Similarly to (D.17), we used the fact that, for a generic operator $Z^{(k)}_{SE} = \sum_{ij} R^{(k)}_{ij} \otimes Z^E_{ab}$, with the system part localized in $M_d^{(k)}(\mathbb{C})$,

$$\sum_{i_k j_k} B_k^\dagger R^{(k-1)}_{i_k j_k} A_k \otimes \mathrm{Tr}_k\left(R^{\dagger(k)}_{i_k j_k}\, Z^{(k)}_{SE}\right) = \sum_{i_k j_k} B_k^\dagger R^{(k-1)}_{i_k j_k} A_k \otimes Z^E_{i_k j_k}$$
$$= B_k^\dagger \otimes \mathbb{1}_E \left(\sum_{i_k j_k} R^{(k-1)}_{i_k j_k} \otimes Z^E_{i_k j_k}\right) A_k \otimes \mathbb{1}_E = B_k^\dagger \otimes \mathbb{1}_E\; Z^{(k-1)}_{SE}\; A_k \otimes \mathbb{1}_E\,.$$

In particular, if $\mathbb{T}_n = \bigotimes_k \Lambda_{k,k-1}$, (D.27) is also equal to

$$\sum_{\boldsymbol{i}_{[1,n-1]},\,\boldsymbol{j}_{[1,n-1]}} \mathrm{Tr}_{[1,n-1]}\Bigg(\bigotimes_{k=1}^{n-1} R^{\dagger(k)}_{i_k j_k}\, \Lambda_1^\ddagger\left[B_1^\dagger R^{(0)}_{i_1 j_1} A_1\right] \otimes \Lambda^\ddagger_{2,1}\left[B_2^\dagger R^{(1)}_{i_2 j_2} A_2\right] \otimes \dots$$
$$\dots \otimes \Lambda^\ddagger_{n-1,n-2}\left[B_{n-1}^\dagger R^{(n-2)}_{i_{n-1} j_{n-1}} A_{n-1}\right] \otimes \Lambda^\ddagger_{n,n-1}\left[B_n^\dagger A_n^{(n-1)}\right]\Bigg) \tag{D.29}$$
$$= \Lambda_1^\ddagger\Big[B_1^\dagger\, \Lambda^\ddagger_{2,1}\Big[B_2^\dagger \dots \Lambda^\ddagger_{n-1,n-2}\Big[B_{n-1}^\dagger\, \Lambda^\ddagger_{n,n-1}\big[B_n^\dagger A_n\big]\, A_{n-1}\Big] \dots A_2\Big] A_1\Big], \tag{D.30}$$

so that, by equating (D.28) and (D.30), one obtains the QR condition. Conversely, choose $B_k = R^\dagger_{b_k b'_k}$ and $A_k = R_{a_k a'_k}$, $k = 1, \dots, n$ in (D.28) and (D.27). Then, consider

$$\begin{aligned}\mathrm{Tr}\Bigg(R_{a_0 b_0}\Theta_1\Big(R^\dagger_{b_1 b'_1}\otimes \mathbb{1}_E\,\Theta_{2,1}\Big(R^\dagger_{b_2 b'_2}\otimes \mathbb{1}_E \cdots \Theta_{n,n-1}\Big(R^\dagger_{b_n b'_n}R_{a_n a_n}\otimes \mathbb{1}_E\Big)\cdots \\ \cdots R_{a_2 a'_2}\otimes \mathbb{1}_E\Big)R_{a_1 a'_1}\otimes \mathbb{1}_E\Big)\Bigg) \\ = \mathrm{Tr}_{[0,n-1]}\left(\bigotimes_{k=0}^{n-1} R^{\dagger(k)}_{b_k a_k}\ \mathbb{T}^\ddagger_n\left[\bigotimes_{l=1}^{n} R_{b_l a'_l}\right]\right)\,,\end{aligned} \tag{D.31}$$

and, choosing the same operators in (D.30) and (D.29),

$$\begin{aligned}\mathrm{Tr}\Bigg(R_{a_0 b_0}\Lambda_1\Big[R^\dagger_{b_1 b'_1}\Lambda_{2,1}\Big[R^\dagger_{b_2 b'_2}\cdots \Lambda_{n,n-1}\Big[R^\dagger_{b_n b'_n}R_{a_n a_n}\Big]\cdots R_{a_2 a'_2}\Big]R_{a_1 a'_1}\Big]\Bigg) \\ = \mathrm{Tr}_{[0,n-1]}\left(\bigotimes_{k=0}^{n-1} R^{\dagger(k)}_{b_k a_k}\ \bigotimes_{j=1}^{n}\Lambda^\ddagger_{k,k-1}\left[\bigotimes_{l=1}^{n} R_{b_l a'_l}\right]\right).\end{aligned} \tag{D.32}$$

If QR holds, (D.31) and (D.32) are equal. This implies that $\mathbb{T}_n = \bigotimes_{j=1}^n \Lambda_{k,k-1}$. □

D.3 Proof of Corollary 5.1

Consider again the dynamical system (D.10) as in the Proof of Theorem 2.78, now for a group of automorphisms $\Theta_n = \Theta^n$. Note that setting $A_{n-1} = \mathbb{1}_d$ in (D.5),

$$\mathbb{T}^\ddagger_n[A_0\otimes\ldots\otimes A_{n-2}\otimes\mathbb{1}_d] = \mathbb{T}^\ddagger_{n-1}[A_0\otimes\ldots\otimes A_{n-2}]\otimes\mathbb{1}_d\,, \tag{D.33}$$

yields

$$\begin{aligned}&\Omega^{\mathbb{Z}\mathbb{Z}}_S\otimes\omega_E\left(\mathbb{T}^\ddagger_n\otimes \mathrm{id}_d^{\otimes n}[A_0\otimes\ldots\otimes A_{n-2}\otimes\mathbb{1}_d\otimes A'_0\otimes\ldots\otimes A'_{n-2}\otimes\mathbb{1}_d]\right) \\ &= \mathrm{Tr}_{[-n,-2]}\left(\mathrm{Tr}_{\{n-1\}}\left(\mathbb{T}_n\otimes \mathrm{id}_d^{\otimes n}\left[\left|\sqrt{\rho_S^{[0,n-1]}}\right\rangle\!\left\langle\sqrt{\rho_S^{[0,n-1]}}\right|\right]\right)\bigotimes_{k=0}^{n-2}A_k\bigotimes_{l=0}^{n-2}A'_l\right) \\ &= \mathrm{Tr}_{[-n,-2]}\left(\mathbb{T}_{n-1}\otimes \mathrm{id}_d^{\otimes n-1}\left[\left|\sqrt{\rho_S^{[0,n-2]}}\right\rangle\!\left\langle\sqrt{\rho_S^{[0,n-2]}}\right|\right]\bigotimes_{k=0}^{n-2}A_k\bigotimes_{l=0}^{n-2}A'_l\right)\end{aligned}$$

from which

$$\mathrm{Tr}_{\{-1\}}\left(\mathbb{T}_n\otimes \mathrm{id}_d^{\otimes n}\left|\sqrt{\rho_S^{[0,n-1]}}\right\rangle\!\left\langle\sqrt{\rho_S^{[0,n-1]}}\right|\right) = \mathbb{T}_{n-1}\otimes \mathrm{id}_d^{\otimes n-1}\left|\sqrt{\rho_S^{[0,n-2]}}\right\rangle\!\left\langle\sqrt{\rho_S^{[0,n-2]}}\right|. \tag{D.34}$$

On the other hand, one also notices that

$$\begin{aligned}\mathrm{Tr}_{\{0\}}\left(\rho_S\,\mathbb{T}_n^{\ddagger}\left[\mathbb{1}_d\otimes\bigotimes_{k=1}^{n-1}A_k\right]\right)&=\omega_S\otimes\omega_E\left(\Theta\left(\mathbb{1}_d^{(0)}\otimes\Theta\left(A_1^{(1)}\otimes\ldots\Theta\left(A_{n-1}^{(n-1)}\otimes\mathbb{1}_E\right)\right)\right)\right)\\&=\omega_E\left(\Theta\left(A_1^{(1)}\otimes\ldots\Theta\left(A_{n-1}^{(n-1)}\otimes\mathbb{1}_E\right)\right)\right)=\mathbb{T}_{n-1}^{\ddagger}[A_1\otimes\ldots\otimes A_n]\,.\end{aligned}$$

Hence,

$$\begin{aligned}&\Omega_S^{\mathbb{ZZ}}\otimes\omega_E\left(\mathbb{T}_n^{\ddagger}\otimes\mathrm{id}_d^{\otimes n}[\mathbb{1}_d\otimes A_1\ldots\otimes A_{n-1}\otimes\mathbb{1}_d\otimes A_1'\otimes\ldots\otimes A_{n-1}']\right)\\&=\mathrm{Tr}_{[1,n-1]}\left(\left|\sqrt{\rho_S^{[1,n-1]}}\right\rangle\!\left\langle\sqrt{\rho_S^{[1,n-1]}}\right|\mathrm{Tr}_{\{0\}}\left(\rho_S\,\mathbb{T}_n^{\ddagger}\left[\mathbb{1}_d\otimes\bigotimes_{k=1}^{n-1}A_k\right]\right)\otimes\bigotimes_{l=1}^{n-1}A_l'\right)\\&=\mathrm{Tr}_{[1,n-1]}\left(\left|\sqrt{\rho_S^{[1,n-1]}}\right\rangle\!\left\langle\sqrt{\rho_S^{[1,n-1]}}\right|\mathbb{T}_{n-1}^{\ddagger}\otimes\mathrm{id}_d^{\otimes n-1}[A_1\otimes\ldots\otimes A_n\otimes A_1'\otimes\ldots\otimes A_n']\right)\\&=\mathrm{Tr}_{[-n+1,-1]}\left(\mathbb{T}_{n-1}\otimes\mathrm{id}_d^{\otimes n-1}\left[\left|\sqrt{\rho_S^{[1,n-1]}}\right\rangle\!\left\langle\sqrt{\rho_S^{[1,n-1]}}\right|\right]A_1\otimes\ldots\otimes A_n\otimes A_1'\otimes\ldots\otimes A_n'\right),\end{aligned}$$

From which, one has

$$\mathrm{Tr}_{\{-n\}}\left(\mathbb{T}_n\otimes\mathrm{id}_d^{\otimes n}\left[\left|\sqrt{\rho_S^{\otimes[0,n-1]}}\right\rangle\!\left\langle\sqrt{\rho_S^{\otimes[0,n-1]}}\right|\right]\right)=\mathbb{T}_{n-1}\otimes\mathrm{id}_d^{\otimes n-1}\left[\left|\sqrt{\rho_S^{\otimes[1,n-1]}}\right\rangle\!\left\langle\sqrt{\rho_S^{\otimes[1,n-1]}}\right|\right].\tag{D.35}$$

We can now prove (5.10). By exploiting subadditivity of the von Neumann entropy along with (D.34) and (D.35),

$$\begin{aligned}S(\rho[\mathcal{F}^{(n+1)}])&=2\,S(\rho_S)+S\left(\mathbb{T}_n\otimes\mathrm{id}_d^{\otimes n}\left[\left|\sqrt{\rho_S^{\otimes n}}\right\rangle\!\left\langle\sqrt{\rho_S^{\otimes n}}\right|\right]\right)\\&\leq 2\,S(\rho_S)+n\,S(\mathbb{T}_1\otimes\mathrm{id}_d[|\sqrt{\rho_S}\rangle\langle\sqrt{\rho_S}|])\\&=2\,S(\rho_S)+n\,S(\Lambda\otimes\mathrm{id}_d[|\sqrt{\rho_S}\rangle\langle\sqrt{\rho_S}|])\,.\end{aligned}\tag{D.36}$$

so that dividing both sides of the inequality by n and taking the limit,

$$\lim_n\frac{1}{n+1}S(\rho_S\left[\mathcal{F}^{(n+1)}\right])\leq S(\Lambda\otimes\mathrm{id}_d[|\sqrt{\rho_S}\rangle\langle\sqrt{\rho_S}|])\,.\tag{D.37}$$

Equality is achieved in (D.36), and consequently in (D.37), if and only if

$$\mathbb{T}_n=\bigotimes_{k=1}^{n}\mathbb{T}_1=\bigotimes_{k=1}^{n}\Lambda\,,\tag{D.38}$$

corresponding to the QR regime by Theorem 2.12 and Proposition 2.13. □

D.4 Two-qubit Pauli maps

A one qubit Pauli map has a spectral decomposition

$$\Lambda[X]=\frac{1}{2}\sum_{\alpha=0}^{3}\lambda^{(\alpha)}\,\mathrm{Tr}\,(\sigma_\alpha X)\,\sigma_\alpha\,,\tag{D.39}$$

and can be equivalently rewritten as

$$\Lambda[X] = \sum_{\alpha=0}^{3} q^{(\alpha)} \sigma_\alpha X \sigma_\alpha \,, \qquad \sum_{\alpha=0}^{3} q^{(\alpha)} = 1 \,. \tag{D.40}$$

The spectrum $\lambda^{(\alpha)}$ and the coefficients $q^{(\alpha)}$ are related by a linear transformation

$$q^{(\alpha)} = \frac{1}{4} \sum_{\alpha\beta=0}^{3} H_{\alpha\beta} \lambda^{(\beta)} \,, \tag{D.41}$$

given by an Hadamard matrix H, $H_{\alpha\beta} = \mathrm{Tr}\left(\sigma_\alpha \sigma_\beta \sigma_\alpha \sigma_\beta\right)$. Complete positivity of Λ is equivalent to $q^{(\alpha)} \geq 0$. Similarly, a two-qubit Pauli map is defined by its spectral decomposition

$$\Gamma[X] = \frac{1}{4} \sum_{\alpha\beta=0}^{3} \lambda^{(\alpha,\beta)}\, \mathrm{Tr}\left(\sigma_\alpha \otimes \sigma_\beta X\right) \sigma_\alpha \otimes \sigma_\beta \,, \tag{D.42}$$

and can be recast in the form

$$\Gamma[X] = \sum_{\alpha\beta=0}^{3} q^{(\alpha,\beta)} \sigma_\alpha \otimes \sigma_\beta X \sigma_\alpha \otimes \sigma_\beta \,, \quad \sum_{\alpha\beta=0}^{3} q^{(\alpha,\beta)} = 1 \,, \tag{D.43}$$

where the matrices describing the spectrum $G := [\gamma^{(\alpha,\beta)}]$, respectively the coefficients $Q := [q^{(\alpha,\beta)}]$ are similar and related through the Hadamard transformation,

$$Q = \frac{H\,G\,H}{16} \,. \tag{D.44}$$

Suppose now that the spectrum has the special form

$$G = \begin{pmatrix} 1 & \lambda & \lambda & \lambda^{(3)} \\ \lambda & 1 & \lambda^{(3)} & \lambda \\ \lambda & \lambda^{(3)} & 1 & \lambda \\ \lambda^{(3)} & \lambda & \lambda & 1 \end{pmatrix} , \tag{D.45}$$

Then, using (D.44), the matrix coefficients is diagonal,

$$Q := [q^{(\alpha,\beta)}] = \frac{1}{4} \begin{pmatrix} 1 + \lambda^{(3)} + 2\lambda & 0 & 0 & 0 \\ 0 & 1 - \lambda^{(3)} & 0 & 0 \\ 0 & 0 & 1 - \lambda^{(3)} & 0 \\ 0 & 0 & 0 & 1 + \lambda^{(3)} - 2\lambda \end{pmatrix} , \tag{D.46}$$

whence (D.43) has the form

$$\Gamma[X] = \sum_{\alpha=0}^{3} q^{(\alpha,\alpha)} \sigma_\alpha \otimes \sigma_\alpha X \sigma_\alpha \otimes \sigma_\alpha \,.$$

List of publications

The work presented in this thesis has led to the following publications:

1. F. Benatti and G. Nichele. “Open Quantum Dynamics: Memory Effects and Superactivation of Backflow of Information”. *Mathematics* 12.1 (2024), p. 37. DOI: 10.3390/math12010037

2. F. Benatti, D. Chruściński, and G. Nichele. “Quantum versus classical P-divisibility”. *Physical Review A* 110.5 (2024), p. 052212. DOI: 10.1103/PhysRevA.110.052212

3. F. Benatti and G. Nichele. “Superactivation of memory effects in a classical Markov environment”. *Physica Scripta* 100.6 (2025), p. 065115. DOI: 10.1088/1402-4896/add57e

4. G. Nichele and F. Benatti. “Entropic Superactivation of Backflow of Information”. *Quantum Economics and Finance* 2.2 (2025), pp. 90–99. DOI: 10.1177/29767032251361881

Two further manuscripts are currently under preparation:

5. G. Nichele and F. Benatti. “Quantum Dynamical Entropy and non-Markovianity: a collisional model perspective”. *In preparation.*

6. G. Nichele and F. Benatti. “Dissipative Information Flows and Quantum Dynamical Entropy”. *In preparation.*

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