

On the asymptotic stability on the line of ground states of the pure power NLS with $0 \leq 2 - p \ll 1$

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Abstract

We continue our series devoted, after references [18] and [20], at proving the asymptotic stability of ground states of the pure power Nonlinear Schrödinger equation on the line. Here we assume some results on the spectrum of the linearization obtained computationally by Chang et al. [9] and then we explore the equation for exponents $p \leq 2$ sufficiently close to 2. The ensuing loss of regularity of the nonlinearity requires new arguments.

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1. Introduction

We consider the pure power focusing Nonlinear Schrödinger Equation (NLS) on the line

$$i\partial_t u + \partial_x^2 u = -f(u) \text{ where } f(u) = |u|^{p-1}u \text{ for } 0 \leq 2 - p \ll 1. \quad (1.1)$$

It is well known that equation (1.1) has solitary waves

$$u(t, x) = e^{\frac{i}{2}vx - \frac{i}{4}|v|^2t + it\omega + iy} \phi_\omega(x - vt - D) \quad (1.2)$$

where $\phi_\omega(x) = \omega^{\frac{1}{p-1}} \phi(\sqrt{\omega}x)$ with, formula (3.1) of Chang et al. [9],

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$$\phi(x) = \left(\frac{p+1}{2}\right)^{\frac{1}{p-1}} \operatorname{sech}^{\frac{2}{p-1}}\left(\frac{p-1}{2}x\right). \quad (1.3)$$

Energy $\mathbf{E}$, mass $\mathbf{Q}$ and linear momentum $\mathbf{P}$ are invariants of (1.1), where

$$\mathbf{E}(u) = \frac{1}{2} \|u'\|_{L^2(\mathbb{R})}^2 - \int_{\mathbb{R}} F(u) dx \text{ where } F(u) = \frac{|u|^{p+1}}{p+1}, \quad (1.4)$$

$$\mathbf{Q}(u) = \frac{1}{2} \|u\|_{L^2(\mathbb{R})}^2 \text{ and} \quad (1.5)$$

$$\mathbf{P}(u) = -\frac{1}{2} \langle i\partial_x u, u \rangle \text{ where } \langle f, g \rangle = \operatorname{Re} \int_{\mathbb{R}} f(x) \overline{g}(x) dx. \quad (1.6)$$

It is well known that ϕ_ω minimizes $\mathbf{E}$ under the constraint $\mathbf{Q} = \mathbf{Q}(\phi_\omega) =: \mathbf{q}(\omega)$. Notice that $\mathbf{q}(\omega) = \omega^{\frac{2}{p-1}-\frac{1}{2}} \mathbf{q}(1)$. In particular, we have $\nabla \mathbf{E}(\phi_\omega) = -\omega \nabla \mathbf{Q}(\phi_\omega)$ which reads also

$$-\phi_\omega'' + \omega \phi_\omega - \phi_\omega^p = 0. \quad (1.7)$$

The following was shown by Cazenave and Lions [8], see also Shatah [59] and Weinstein [67].

Theorem 1.1 (*Orbital stability*). *Let $p \in (1, 5)$ and let $\omega_0 > 0$. Then for any $\epsilon > 0$ there exists a $\delta > 0$ such that if $\inf\{\|u_0 - e^{i\vartheta_0} \phi_{\omega_0}(\cdot - D_0)\|_{H^1(\mathbb{R})} : \vartheta_0, x_0 \in \mathbb{R}\} \leq \delta$ then*

$$\inf\{\|u(t) - e^{i\vartheta_0} \phi_{\omega_0}(\cdot - D_0)\|_{H^1(\mathbb{R})} : \vartheta_0, x_0 \in \mathbb{R}\} \leq \epsilon \text{ for all times } t \in \mathbb{R}. \quad \square$$

Notice that, as we discuss after formula (3.26), it is possible to get $\delta \sim \epsilon$, as discussed in Stuart [65], but this observation is not essential here. In order to study the notion of asymptotic stability, like in finite dimension it is useful to have information on the *linearization* of (1.1) at ϕ_ω , which we will see later has the following form

$$\partial_t \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \mathcal{L}_\omega \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} \text{ with } \mathcal{L}_\omega := \begin{pmatrix} 0 & L_{-\omega} \\ -L_{+\omega} & 0 \end{pmatrix}, \quad (1.8)$$

where

$$L_{+\omega} := -\partial_x^2 + \omega - p\phi_\omega^{p-1} \quad (1.9)$$

$$L_{-\omega} := -\partial_x^2 + \omega - \phi_\omega^{p-1}. \quad (1.10)$$

The linearization is better seen in the context of functions in $H^1(\mathbb{R}, \mathbb{R}^2)$ rather than in $H^1(\mathbb{R}, \mathbb{C})$, because it is $\mathbb{R}$ -linear rather than $\mathbb{C}$ -linear. Defining the scaling

$$S_\omega u(x) = \omega^{1/4} u(\sqrt{\omega}x) \implies \mathcal{L}_\omega = \omega S_\omega \mathcal{L} S_\omega^{-1} \text{ for } \mathcal{L} = \mathcal{L}_1. \quad (1.11)$$

It is well known that the spectrum of $\mathcal{L}$ is symmetric with respect to real and imaginary axes and that the essential spectrum is $(-\infty, -i] \cup [i, \infty)$. Chang et al. [9] compute numerically the statements 1–5. Coles and Gustafson [11] prove rigorously 6. Krieger and Schlag [37], but notice that this has been recently been generalized by Collot et al. [13], prove rigorously 7.

Statement 1.2.

- 1 There is a value p_0 close to 1.82 such that for $p_0 < p < 3$ and for $3 < p < 5$ the linearization $\mathcal{L}$ has exactly one eigenvalue of the form $i\lambda(p)$ with $0 < \lambda(p) < 1$.
- 2 For $p_0 < p < 3$ we have $\lambda(p) > 1/2$.
- 3 There is a $p_1 \in (1, p_0)$ such that for $p_1 < p < p_0$ the linearization $\mathcal{L}$ has a second eigenvalue $i\mu(p)$ with $0 < \lambda(p) < \mu(p) < 1$.
- 4 The function $(3, 5) \ni p \mapsto \lambda(p)$ is strictly decreasing with $\lambda(3^+) = 1$, $\lambda(4) > 1/2$ and $\lambda(5^-) = 0$.
- 5 The function $(1, 3) \ni p \mapsto \lambda(p)$ is strictly increasing with $\lambda(3^-) = 1$ and $\lambda(1^+) = 0$.
- 6 For $0 < |p - 3| \ll 1$ the linearization $\mathcal{L}$ has exactly one eigenvalue of the form $i\lambda(p)$ with

$$\lambda(p) \sim 1 - \alpha_0(p - 3)^4 + o\left((p - 3)^4\right) \text{ for a constant } \alpha_0 > 0. \quad (1.12)$$

- 7 There are no eigenvalues of the form $i\mu$ with $\mu > 1$.

We have the following, which we proved in [20].

Proposition 1.3 (No threshold resonance). *Consider the following statement,*

$$\text{there is no nonzero bounded solution of } \mathcal{L}u = iu. \quad (1.13)$$

Consider the set

$$\tilde{\mathbf{F}} := \{p \in (1, +\infty) \mid \text{statement (1.13) is true}\}. \quad (1.14)$$

Then $(1, +\infty) \setminus \tilde{\mathbf{F}}$ is a discrete subset of $(1, +\infty)$. $\square$

An important ingredient in this paper is the nonlinear Fermi Golden Rule (FGR). Recall the p_0 in Statement 1.2.

Proposition 1.4. *Consider the constant $\gamma(p)$ introduced later in (5.3). Then assuming as hypotheses 1–2 in Statement 1.2 we have that $\gamma(\cdot) \in C^\omega((p_0, 3), \mathbb{R})$ and that the set*

$$\mathbf{F} := \{p \in (p_0, 3) \mid \gamma(p) \neq 0\} \quad (1.15)$$

is s.t. $(p_0, 3) \setminus \mathbf{F}$ is a discrete subset of $(p_0, 3)$.

Proof. The fact that $\gamma(\cdot) \in C^\omega((p_0, 3), \mathbb{R})$, i.e. $\gamma(p)$ is analytic in p , follows from the fact that $\mathcal{L}$ is analytic of type (A) in p , as we will see later. Then the last line of the statement follows from the fact, proved in [18, 19], that it is possible to define $\gamma(p)$ so that it is nonzero for $0 < |p - 3| \ll 1$. $\square$

In this paper we prove the following result. Notice that in the statement below $\xi_{\omega(t)}$ is an appropriately normalized generator of the eigenspace associated to the eigenvalue $\omega(t)\lambda(p)$ of the linear operator $\mathcal{L}_{\omega(t)}$ and that $z(t)\xi_{\omega(t)} + \bar{z}(t)\bar{\xi}_{\omega(t)}$ is interpreted as complex valued function (since it has values in $\mathbb{R}^2$).

Recalling the power p_0 introduced in Statement 1.2, since for $p = 2$ we have $\lambda(2) \in (1/2, 1)$, and by the continuous dependence of $\lambda(p)$ in p , we conclude that there is a $\mathbf{p}_0 \in (p_0, 2)$ such that

$$2 - p - \sqrt{1 - \lambda(p)} < 0 \text{ for all } p \in (\mathbf{p}_0, 2]. \quad (1.16)$$

Theorem 1.5. *There is a $\tilde{p}_0 \in (\mathbf{p}_0, 2)$ such that if $p \in (\tilde{p}_0, 2]$ belongs to $\tilde{\mathbf{F}} \cap \mathbf{F}$ then for any $\omega_0 > 0$, any $a > 0$ and any $\epsilon > 0$ there exists a $\delta > 0$ such that for any initial value $u_0 \in D_{H^1(\mathbb{R})}(\phi_{\omega_0}, \delta)$ there exist functions $\vartheta, \omega, v, D \in C^1(\mathbb{R}, \mathbb{R})$, $z \in C^1(\mathbb{R}, \mathbb{C})$, $\omega_+ > 0$ and $v_+ \in \mathbb{R}$ s.t. the solution of (1.1) with initial datum u_0 can be written as*

$$u(t) = e^{i\vartheta(t) - D(t)\partial_x} (\phi_{\omega(t)} + z(t)\xi_{\omega(t)} + \bar{z}(t)\bar{\xi}_{\omega(t)} + \eta(t)) \text{ with} \quad (1.17)$$

$$\int_{\mathbb{R}} \|e^{-a\langle x \rangle} \eta(t)\|_{H^1(\mathbb{R})}^2 dt < \epsilon \text{ where } \langle x \rangle := \sqrt{1 + x^2}, \quad (1.18)$$

$$\lim_{t \rightarrow +\infty} \|e^{-a\langle x \rangle} \eta(t)\|_{L^2(\mathbb{R})} = 0, \quad (1.19)$$

$$\lim_{t \rightarrow +\infty} z(t) = 0, \quad (1.20)$$

$$\lim_{t \rightarrow +\infty} \omega(t) = \omega_+ \text{ and} \quad (1.21)$$

$$\lim_{t \rightarrow +\infty} v(t) = v_+. \quad (1.22)$$

Remark 1.6. Notice that the fact that the $H^1(\mathbb{R})$ norm of η is uniformly bounded for all times, guaranteed by the orbital stability, and Theorem 1.5 imply that

$$\lim_{t \rightarrow +\infty} u(t)e^{-i\vartheta(t) + D(t)\partial_x} = \phi_{\omega_+} \text{ in } L_{\text{loc}}^\infty(\mathbb{R}). \quad (1.23)$$

Equation (1.1) is one of the most a classical Hamiltonian PDE and the asymptotic stability of its ground states has been a longstanding open problem considered in the early papers by Soffer and Weinstein [62,63]. For a rather extensive discussion of the problem, we refer to [18], where we deal with the case $p \sim 3$ but $p \neq 3$ (for $p = 3$ Theorem 1.5 is false). There are two main reasons why [18] is valid only for p near 3. The first is that everywhere in [18] we use the fact that $u \rightarrow f(u)$ is C^2 regular. The second is that the smoothing estimate is proved in [18] using the fact that the Wronskian of a certain pair of Jost functions of $\mathcal{L}$ is nonzero. This is checked for $p = 3$, where the Jost functions are explicitly known, and by continuity remains true for all the p close to 3. Obviously, [18] uses the hypothesis $p \sim 3$ also in the proof of the Fermi Golden Rule. Thanks to [20], the results in [18] extend to generic $p \in (2, q_3)$ where $q_3 \in (4, 5)$ is such that $\lambda(q_n) = 1/n$, where we assume as true the results obtained numerically by Chang et al. [9]. In [20] furthermore, if for $p \in (q_3, q_4)$ some conditions, which we believe are generically true, are satisfied then the results extend to such p 's. So [18,20] provide a theory which, either rigorously

in $(2, q_3)$ or with some degree of plausibility in (q_3, q_4) , shows that for generic $p \in (2, q_4)$ the ground states are asymptotically stable. Since it is an important and classical open problem to extend as much as possible the range of the p 's, ideally reaching the whole range $(1, 5)$, it is very important to take p as close as possible to 1. If this is true, it is very important to spell out what the status of the current knowledge is now. Theorem 1.5 provides this kind of answer.

In [20] we did not discuss explicitly the proof for generic $p \in (2, q_2)$ where $q_2 \in (4, 5)$, this assuming Chang et al. [9], because it is reasonable to state that the arguments in [18] along with the smoothing proved in [20] suffice to yield asymptotic stability. In fact in [20] we focused only on the 3rd and 4th order Fermi Golden rules. But the case $p \leq 2$ is not covered by the proof in [18]. Our main focus here is to deal with the problems entailed by taking $p \leq 2$, where $u \rightarrow f(u)$ is not C^2 regular. While our proof is similar to [18, 20], the lack of regularity complicates substantially the proof. We follow the framework of [23] which, as a special case, contains a positive answer to a question raised by Lührmann and Schlag [48] in their footnote on p. 234. Notice that this footnote shows that case $p \leq 2$ is considered difficult, along with the absence of results in the literature, except for [23].

Even the proof of the limit of $\omega(t)$ in (1.21), while similar to the analogous proof in [18], requires to be rewritten more carefully. Indeed, in our previous paper [15], on the asymptotic stability of small ground states bifurcating from a delta potential, we were not able to prove the convergence of $|z(t)|$ in the case $p \leq 2$, which corresponds to the convergence of $\omega(t)$ in this paper, and the possibility of non-convergence remained, see Theorem 1.4 of [15]. This means that based on the known literature, the statement in Theorem 1.5 is not obvious.

Another novelty in the present paper is that we drop the symmetries that in [18, 20] allowed to avoid translation. We emphasize that also Martel [50] and Rialland [58] impose such a symmetry condition. It is true that Martel [50] states the expectation of no deep additional difficulty in the non-symmetric case and that Martel [49] and Rialland [57], in the absence of internal eigenvalues, treated the non-symmetric case. But this point is a priori not obvious when there are internal eigenvalues. Even Buslaev and Perelman [3], who in the no internal eigenvalues case treated the non-symmetric case, in the presence of an internal mode considered only the symmetric case, see Buslaev and Perelman [4] and Buslaev and Sulem [5] and this restriction has remained in place in the subsequent literature which follows their framework, see the references in our survey [16]. Ideas to allow translation due to Beceanu [2] or the ones attributed to Perelman by Bambusi [1] do not work in space dimensions 1 and 2. Eventually here we show that Martel's expectation was right, and that in the framework used here and which owes much to Martel's work, translation does not create particular problems, but this was a priori not obvious to us because translation can be a source of problems. For example, in an analogous problem, we do not know how to generalize to the case of kinks with internal eigenvalues the result by Kowalczyk et al. [36], which has a more complicated virial inequality than the one we use here, even though in the case with symmetries which eliminate translations in [21] we were able to prove the stabilization in the presence of very general configurations of internal eigenvalues.

Potentially, small p is problematic because the key for the stabilization is given by the linear dispersion of the continuous mode η and by the friction exerted on the discrete mode z contained in (1.17) by the nonlinear interaction with the continuous mode η by means of the nonlinear Fermi Golden Rule (FGR) (here z is like the variable of a finite dimensional hamiltonian system which is embedded and not well isolated inside an infinite dimensional one, so it is slowly losing all its energy; the notion was introduced by Sigal [61]). The fact that dispersion occurs even with such low values of p is remarkable, in view of how difficult and central has been in the literature the problem of dealing with dispersion in the presence of a nonlinearity, which has lead to

milestones like Klainerman's vector fields and null forms, see for instance [32], Shatah's normal forms [60] and Christodoulou's conformal mapping [14], see also the discussion in Strauss [64]. Notice that a version of Theorem 1.5, with $u = \eta$, is false near 0 for $p < 5$ and a counterexample to (1.19) is given by the ground states $u(t, x) = e^{it\omega} \phi_\omega$ since $\|\phi_\omega\|_{H^1} \rightarrow 0$ as $\omega \rightarrow 0+$. For $p \geq 3$ by Ozawa [54] for any $u_+ \in L^{2,2}(\mathbb{R})$, the space defined in (2.16) below, and both for focusing and defocusing equation there exists a solution u of (1.1) with $\|u(t) - e^{iS(t,\cdot)} e^{it\partial_x^2} u_+\|_{L^2(\mathbb{R})} = O(t^{-\alpha})$ with $\alpha > 1/2$ and for an appropriate real phase $S(t, x)$, which is equal to 0 if $p > 3$. Since for $|x| \leq 1$ we have $e^{it\partial_x^2} u_+(x) = \frac{1}{\sqrt{2it}} e^{i\frac{x^2}{4t}} \widehat{u}_+(0) + O(t^{-\alpha})$ with $\alpha > 1/2$, in the generic case with $\widehat{u}_+(0) \neq 0$ we conclude that $\|u\|_{L^2(\mathbb{R}_+, L^2(|x| \leq 1))} = +\infty$. For $p \in (1, 3)$ it is known that nonzero solutions of the defocusing equation behave differently from the solutions of the linear equation and we do not know if a result like (1.19) with $u = \eta$ is false. The above considerations show that generally equation (1.1) displays better dispersive properties near the ground state than near 0. In fact, as we have seen, there is more dispersion near the soliton than near vacuum for the constant coefficient linear equation, due to the fact that the linearization at the ground state is better behaved in terms of the Kato smoothing, that is [30], in view of Proposition 1.3, while the operator $-\partial_x^2$ has a resonance.

We state here briefly the main ingredients of the proof. First of all, we use our notion of Refined Profile to find a particularly favorable ansatz for our solution. Thanks to this ansatz we obtain various convenient cancellations and good estimates for the discrete components of our solution. The ansatz is also particularly convenient to prove the FGR, both in the rather simple spectral configuration considered here, but also in much more complicated situations like for example in [22]. Next we use the first virial inequality of Kowalczyk et al. [33–36], which can be traced to the work of Martel and Merle, see for example [51] and therein, and Merle and Raphael, see for example [52] and therein, and earlier in Kato [31]. The crux is represented by the (almost) positive commutator formula (6.6) and by the remarkable pure power inequality (6.16). It is thanks to the flexibility and robustness of this tool that we can deal with low values of p . Finally we prove a low energy dispersive estimate. Here, like in [18], we modify the approach of Kowalczyk et al. [35], Martel [49,50] and Rialland [57,58] by replacing their second virial inequality with a smoothing estimate. The advantage is that we can deal with equations which are not small perturbations of the cubic NLS, contrary to Martel [49,50] and Rialland [57,58]. As we discussed at length in [18], all is needed is a smoothing estimate for the continuous mode η multiplied by a cutoff. We can derive it easily by multiplying the equation of η by a cutoff. In this way the term $f(\eta)$ once multiplied by the cutoff is easy to bound. Compared to [18], we benefit by a better smoothing estimate proved in [20].

Besides the papers involving the virial inequalities there is also a large literature which uses dispersive estimates for problems which are related but in general are different from the one we consider here, for example and without trying to give a complete list [10,25–29,53], [38–48]. In general these methods seem to require quite regular $u \rightarrow f(u)$, so they do not seem applicable here. An interesting mixing of phase space analysis and virial inequalities is in Palacios and Pusateri [55]. Germain and Collot [12] have recovered and partially generalized Martel [49] by using very different methods. Li and Luhrmann [40] for the cubic NLS partially recover the result obtained using the integrable structure by [24] with methods which do not exploit explicitly the integrable structure of the cubic NLS. A rather long list of references on the asymptotic stability of ground states of the NLS up until 2020 is in our survey [16]. While here we deal with exponents $p < 2$, it remains a fascinating classical open problem to deal with the whole range $p \in (1, 5)$.

2. Linearization

We return to a discussion of the linearization (1.8). First of all we notice

$$\partial_\omega \left(\omega^{\frac{1}{p-1}} u(\sqrt{\omega}x) \right) = \omega^{\frac{1}{p-1}-1} (\Lambda_p u)(\sqrt{\omega}x) \text{ for } \Lambda_p = \frac{1}{2}x\partial_x + \frac{1}{p-1} \quad (2.1)$$

so that in particular $\Lambda_p u = \partial_\omega|_{\omega=1} \omega^{\frac{1}{p-1}} u(\sqrt{\omega}x)$. Weinstein [66] showed that for $1 < p < 5$ the generalized kernel $N_g(\mathcal{L}) := \cup_{j=1}^\infty \ker \mathcal{L}^j$ in $H^1(\mathbb{R}, \mathbb{C}^2)$ is

$$N_g(\mathcal{L}) = \text{span} \left\{ \begin{pmatrix} 0 \\ \phi \end{pmatrix}, \begin{pmatrix} \partial_x \phi \\ 0 \end{pmatrix}, \begin{pmatrix} \Lambda_p \phi \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x\phi \end{pmatrix} \right\}. \quad (2.2)$$

By symmetry reasons, it is known that the spectrum $\sigma(\mathcal{L}) \subseteq \mathbb{C}$ is symmetric by reflection with respect of the coordinate axes. Furthermore, by Krieger and Schlag [37, p. 909] we know that $\sigma(\mathcal{L}) \subseteq i\mathbb{R}$. By standard Analytic Fredholm theory the essential spectrum is $(-\infty i, -i] \cup [i, +\infty i)$. As we mentioned in Statement 1.2, Krieger and Schlag [37, pp. 914–915] prove that $\mathcal{L}$ has no eigenvalues $i\mu$ with $\mu > 1$.

Let us consider the orthogonal decomposition

$$L^2(\mathbb{R}, \mathbb{C}^2) = N_g(\mathcal{L}) \oplus N_g^\perp(\mathcal{L}^*). \quad (2.3)$$

We have, for $\lambda = \lambda(p, \omega)$, a further decomposition

$$N_g^\perp(\mathcal{L}^*) = \ker(\mathcal{L} - i\lambda(p)) \oplus \ker(\mathcal{L} + i\lambda(p)) \oplus X_c \text{ where} \quad (2.4)$$

$$X_c := \left(N_g(\mathcal{L}^*) \oplus \ker(\mathcal{L}^* - i\lambda(p)) \oplus \ker(\mathcal{L}^* + i\lambda_\omega) \right)^\perp. \quad (2.5)$$

We denote by P_c the projection of $L^2(\mathbb{R}, \mathbb{C}^2)$ onto $X_c(\omega)$ associated with the above decompositions. We have the following.

Lemma 2.1. *Consider the p_0 in Statement 1.2. Then for $p \in (p_0, 3)$ the algebraic and dimension of the eigenvalue $i\lambda(p)$ is one and $\ker(\mathcal{L} - i\lambda(p)) \subset H_{\text{rad}}^1(\mathbb{R}, \mathbb{C}^2)$, see Notation 2.3 below.*

Proof. First of all, Coles and Gustafson [11] show that $\dim \ker(\mathcal{L} - i\lambda(p)) = 1$ for p close to 3. Next, we show later in Lemma 8.1 that $\dim \ker(\mathcal{L} - i\lambda(p)) = N_g(\mathcal{L} - i\lambda(p))$ if $\lambda(p) \in (0, 1)$. Since $\mathcal{L}$ is analytic of type (A) in p the eigenvalue $\lambda(p)$ depends analytically on $p \in (p_0, 3)$, see [56, Ch. 12]. Since $\lambda(p)$ remains an isolated eigenvalue for $p \in (p_0, 3)$, then $\dim \ker(\mathcal{L} - i\lambda(p))$ remains constant.

Turning to the parity of the elements of $\ker(\mathcal{L} - i\lambda(p))$, for p close to 3 they are indeed even, as is shown in [18, Formula 11.8]. Then again we can conclude that this is true for all $p \in (p_0, 3)$. $\square$

The space $L^2(\mathbb{R}, \mathbb{C}^2)$ and the action of $\mathcal{L}_\omega$ on it is obtained by first identifying $L^2(\mathbb{R}, \mathbb{C}) = L^2(\mathbb{R}, \mathbb{R}^2)$ and then by extending this action to the completion of $L^2(\mathbb{R}, \mathbb{R}^2) \otimes_{\mathbb{R}} \mathbb{C}$ which is identified with $L^2(\mathbb{R}, \mathbb{C}^2)$. In $\mathbb{C}$ we consider the inner product,

$$\langle z, w \rangle_{\mathbb{C}} = \operatorname{Re}\{z\bar{w}\} = z_1 w_1 + z_2 w_2 \text{ where } a_1 = \operatorname{Re} a, \quad a_2 = \operatorname{Im} a \text{ for } a = z, w.$$

This obviously coincides with the inner product in $\mathbb{R}^2$ and expands as the standard sesquilinear $\langle X, Y \rangle_{\mathbb{C}^2} = X^\top \bar{Y}$ (row column product, vectors here are columns) form in $\mathbb{C}^2$. The operator of multiplication by i in $\mathbb{C} = \mathbb{R}^2$ extends into the linear operator $J^{-1} = -J$ where

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

For $u, v \in L^2(\mathbb{R}, \mathbb{C}^2)$ we set $\langle u, v \rangle := \int_{\mathbb{R}} \langle u(x), v(x) \rangle_{\mathbb{C}^2} dx$. We have a natural symplectic form given by $\Omega := \langle J^{-1} \cdot, \cdot \rangle$ in both $L^2(\mathbb{R}, \mathbb{C}^2)$ and $L^2(\mathbb{R}, \mathbb{R}^2) = L^2(\mathbb{R}, \mathbb{C})$, where equation (1.1) is the Hamiltonian system in $L^2(\mathbb{R}, \mathbb{C})$ with Hamiltonian the energy E in (1.4). As we mentioned we consider a generator $\xi \in \ker(\mathcal{L} - i\lambda(p))$. Then for the complex conjugate $\bar{\xi} \in \ker(\mathcal{L} + i\lambda(p))$. Notice that the well known and elementary identity $J\mathcal{L} = -\mathcal{L}^*J$ implies that $\ker(\mathcal{L}^* + i\lambda(p)) = \operatorname{span}\{J\xi\}$ and $\ker(\mathcal{L}^* - i\lambda(p)) = \operatorname{span}\{J\bar{\xi}\}$. All the above is discussed also in [18] where we recalled why we can use the normalization

$$\Omega(\xi, \xi) = -i \tag{2.6}$$

and that furthermore we can assume

$$\xi = (\xi_1, \xi_2)^\top \text{ with } \xi_1 \in \mathbb{R} \text{ and } \xi_2 \in i\mathbb{R}, \tag{2.7}$$

so that (2.6) becomes

$$\int_{\mathbb{R}} \xi_1 \operatorname{Im} \xi_2 dx = 2^{-1}. \tag{2.8}$$

Notice also that ξ is even in x as discussed in Lemma 2.1.

Lemma 2.2. *There exists $\widehat{p}_0 \in (p_0, 2)$ such that for $p \in (\widehat{p}_0, 2]$ we have the point-wise estimate*

$$|\xi(x)| \leq C_p e^{-|x|\sqrt{1-\lambda(p)}}. \tag{2.9}$$

Proof. Introducing

$$U = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}, \quad U^{-1} = \frac{1}{2} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \tag{2.10}$$

we have $U^{-1}JU = i\sigma_3$ so that

$$U^{-1}\mathcal{L}_\omega U = iH_\omega \text{ where } H_\omega = \sigma_3 \left(-\partial_x^2 + \omega \right) + V_\omega, \tag{2.11}$$

$$V_\omega := \omega M_0 \operatorname{sech}^2 \left(\sqrt{\omega} \frac{p-1}{2} x \right) \text{ with } M_0 = - \left(\frac{p+1}{2} \sigma_3 + i \frac{p-1}{2} \sigma_2 \right) \frac{p+1}{2}$$

where we recall the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (2.12)$$

Then for $H_0 = \sigma_3 (-\partial_x^2 + 1)$, $H = H_\omega|_{\omega=1}$ and $V = V_\omega|_{\omega=1}$ we have

$$(H_0 - \lambda(p) + V)w = 0 \text{ for } w := U^{-1}\xi.$$

It follows that, for $\text{diag}(a, b)$ the diagonal matrix with first a and then b on the diagonal,

$$w = -\text{diag}\left(R_{-\partial_x^2}(\lambda(p) - 1), -R_{-\partial_x^2}(-\lambda(p) - 1)\right)Vw$$

so that pointwise we have

$$|w(x)| \lesssim \int_{\mathbb{R}} e^{-|x-y|\sqrt{1-\lambda(p)}} e^{-(p-1)|y|} |w(y)| dy.$$

Since $w \in L^\infty(\mathbb{R})$ by $\sqrt{1-\lambda(p)} < 2^{-\frac{1}{2}}$ for $p > p_0$ we can pick $\widehat{p}_0$ so that $\sqrt{1-\lambda(p)} < p-1$ for $p \in (\widehat{p}_0, 2]$. Then by the triangular inequality

$$\int_{\mathbb{R}} e^{-|x-y|\sqrt{1-\lambda(p)}} e^{-(p-1)|y|} |w(y)| dy \leq e^{-|x|\sqrt{1-\lambda(p)}} \int_{\mathbb{R}} e^{(\sqrt{1-\lambda(p)}-(p-1))|y|} dy \|w\|_{L^\infty}$$

yields the following and so also (2.9),

$$|w(x)| \lesssim e^{-|x|\sqrt{1-\lambda(p)}}. \quad \square$$

Notation 2.3. We will use the following miscellanea of notations and definitions.

1. Given a Banach space X , $x_0 \in X$ and $r > 0$ we denote $D_X(x_0, r)$ the open ball in X of center $x_0 \in X$ and radius r .
2. We will set

$$\mathbf{e}(\omega) := \mathbf{E}(\phi_\omega), \mathbf{q}(\omega) := \mathbf{Q}(\phi_\omega) \text{ and } \mathbf{d}(\omega) := \mathbf{e}(\omega) + \omega \mathbf{q}(\omega). \quad (2.13)$$

3. For $z \in \mathbb{C}$ we will use $z_1 = \text{Re } z$ and $z_2 = \text{Im } z$.
4. Like in the theory of Kowalczyk et al. [35], we consider constants $A, B, \epsilon, \delta > 0$ satisfying

$$\log(\delta^{-1}) \gg \log(\epsilon^{-1}) \gg A \gg B^2 \gg B \gg 1. \quad (2.14)$$

Here we will take $A \sim B^3$, see Sect. 7 below, but in fact $A \sim B^n$ for any $n > 2$ would make no difference.

5. The notation $o_\varepsilon(1)$ means a function in ε such that

$$o_\varepsilon(1) \xrightarrow{\varepsilon \rightarrow 0^+} 0. \quad (2.15)$$

6. For $\kappa \in (0, 1)$ fixed in terms of p and small enough, we consider

$$\|\eta\|_{L^{p,s}} := \|\langle x \rangle^s \eta\|_{L^p(\mathbb{R})} \text{ where } \langle x \rangle := \sqrt{1+x^2}, \quad (2.16)$$

$$\|\eta\|_{\Sigma_A} := \left\| \operatorname{sech}\left(\frac{2}{A}x\right) \eta' \right\|_{L^2(\mathbb{R})} + A^{-1} \left\| \operatorname{sech}\left(\frac{2}{A}x\right) \eta \right\|_{L^2(\mathbb{R})} \text{ and} \quad (2.17)$$

$$\|\eta\|_{\tilde{\Sigma}} := \|\operatorname{sech}(\kappa\omega_0 x) \eta\|_{L^2(\mathbb{R})}. \quad (2.18)$$

7. We set

$$\mathbb{C}_{\pm} := \{z \in \mathbb{C} : \pm \operatorname{Im} z > 0\}. \quad (2.19)$$

8. The point $i\omega$ is a resonance for $\mathcal{L}_{\omega}$ if there exists a nonzero $v \in L^{\infty}(\mathbb{R}, \mathbb{C}^2)$ such that $\mathcal{L}_{\omega}v = i\omega v$. Notice that for $p = 3$ the point $i\omega$ is a resonance.

9. Given two Banach spaces X and Y we denote by $\mathcal{L}(X, Y)$ the space of continuous linear operators from X to Y . We write $\mathcal{L}(X) := \mathcal{L}(X, X)$.

10. We have denoted by P_c the projection on the space (2.5) associated to the spectral decomposition (2.4) of the operator $\mathcal{L}_{\omega}$. Later in (2.11) we will introduce an operator H_{ω} which is an equivalent to $\mathcal{L}_{\omega}$ and obtained from $\mathcal{L}_{\omega}$ by a simple conjugation. By an abuse of notation we will continue to denote by P_c the analogous spectral projection to the continuous spectrum component, only of H_{ω} this time.

11. We have the following elementary formulas,

$$Df(u)X = \frac{d}{dt} \left(|u + tX|^{p-1} (u + tX) \right) \Big|_{t=0} = |u|^{p-1}X + (p-1)|u|^{p-3}u \langle u, X \rangle_{\mathbb{C}} \text{ and} \quad (2.20)$$

$$\begin{aligned} D^2 f(u)X^2 &= \frac{d}{dt} Df(u + tX)X \Big|_{t=0} \\ &= 2(p-1)|u|^{p-3}X \langle u, X \rangle_{\mathbb{C}} + (p-1)|u|^{p-3}u|X|^2 + (p-1)(p-3)|u|^{p-5}u \langle u, X \rangle_{\mathbb{C}}^2. \end{aligned} \quad (2.21)$$

12. Following the framework in Kowalczyk et al. [35] we fix an even function $\chi \in C_c^{\infty}(\mathbb{R}, [0, 1])$ satisfying

$$1_{[-1,1]} \leq \chi \leq 1_{[-2,2]} \text{ and } x\chi'(x) \leq 0 \text{ and set } \chi_C := \chi(\cdot/C) \text{ for a } C > 0. \quad (2.22)$$

13. We denote by $H_{\text{rad}}^1(\mathbb{R}) = H_{\text{rad}}^1(\mathbb{R}, \mathbb{C})$ the space of even functions in $H^1(\mathbb{R}, \mathbb{C})$. $\square$

The group $e^{t\mathcal{L}_{\omega}}$ is well defined in $L^2(\mathbb{R}, \mathbb{C}^2)$, leaves invariant $L^2(\mathbb{R}, \mathbb{R}^2)$ and the terms of the direct sums in (2.3) and (2.4). The following result is an immediate consequence of a Proposition 8.1 in Krieger and Schlag [37] if $p \in \tilde{\mathbf{F}}$ in (1.15), since then $\mathcal{L}_{\omega}$ is admissible in the sense indicated in [37].

Proposition 2.4. For any fixed $s > 3/2$ there is a constant C_ω such that

$$\|P_c e^{t\mathcal{L}_\omega} : L^{2,s}(\mathbb{R}, \mathbb{C}^2) \rightarrow L^{2,-s}(\mathbb{R}, \mathbb{C}^2)\| \leq C_\omega \langle t \rangle^{-\frac{3}{2}} \text{ for all } t \in \mathbb{R}. \quad \square \quad (2.23)$$

For the proof of the following, we refer to [20].

Proposition 2.5. Let $p \in \tilde{\mathbf{F}}$. For $s > 3/2$ and $\tau > 1/2$ there exists a constant $C > 0$ such that

$$\left\| \int_0^t e^{(t-t')\mathcal{L}_\omega} P_c(\omega) g(t') dt' \right\|_{L^2(\mathbb{R}, L^{2,-s}(\mathbb{R}))} \leq C \|g\|_{L^2(\mathbb{R}, L^{2,\tau}(\mathbb{R}))} \text{ for all } g \in L^2(\mathbb{R}, L^{2,\tau}(\mathbb{R})). \quad (2.24)$$

For the proof of the following, we refer to [18].

Proposition 2.6 (Kato smoothing). Let $p \in \tilde{\mathbf{F}}$. For any ω and for any $s > 1$ there exists $c > 0$ such that

$$\|e^{it\mathcal{L}_\omega} P_c u_0\|_{L^2(\mathbb{R}, L^{2,-s}(\mathbb{R}))} \leq c \|u_0\|_{L^2(\mathbb{R})}. \quad (2.25)$$

3. Refined profile, modulation, continuation argument and proof of Theorem 1.5

It is well known, see Weinstein [66], that

$$\mathcal{S} = \left\{ e^{\frac{i}{2}vx + i\vartheta} \phi_\omega(\cdot - D) : \vartheta, D, v \in \mathbb{R}, \omega > 0 \right\} \quad (3.1)$$

is a symplectic submanifold of $L^2(\mathbb{R}, \mathbb{C})$. We set

$$\phi[z] = \phi + \tilde{\phi}[z] \text{ with } \tilde{\phi}[z] := (z\xi + \bar{z}\bar{\xi}). \quad (3.2)$$

For functions

$$\tilde{\Theta}_{\mathcal{R}}[z] = (\tilde{\vartheta}_{\mathcal{R}}[z], \tilde{\omega}_{\mathcal{R}}[z], 0, 0, \tilde{z}_{\mathcal{R}}[z]) \quad (3.3)$$

in part to be determined below we introduce

$$\begin{aligned} \tilde{z}[z] &= \tilde{z}_0[z] + \tilde{z}_{\mathcal{R}}[z] \text{ with } \tilde{z}_0[z] = i\lambda(p)z \\ \tilde{\vartheta}[z] &= 1 + \tilde{\vartheta}_{\mathcal{R}}[z] \text{ and } \tilde{\omega}[z] = \tilde{\omega}_{\mathcal{R}}[z], \quad \tilde{v}[z] = 0 \text{ and } \tilde{D}[z] = 0. \end{aligned}$$

Proposition 3.1. There exist constants $C_1, \delta_1 > 0$ and a map $\tilde{\Theta}_{\mathcal{R}}[\cdot] : D_{\mathbb{C}}(0, \delta_1) \rightarrow \mathbb{R}^4 \times \mathbb{C}$ like in (3.3) s.t. the following holds:

i we have

$$|\tilde{\Theta}_{\mathcal{R}}[z]| \lesssim |z|^2; \quad (3.4)$$

ii if we set

$$R[z] := \partial_x^2 \phi[z] + f(\phi[z]) - \tilde{\vartheta} \phi[z] + i\tilde{\omega} \Lambda_p \phi[z] + iD_z \phi[z] \tilde{z}, \quad (3.5)$$

we have for some fixed $\kappa > 0$

$$\|\cosh(\kappa x) R[z]\|_{L^2(\mathbb{R})} \lesssim |z|^2, \quad (3.6)$$

with furthermore the following orthogonality conditions, for $z_1 = \operatorname{Re} z$ and $z_2 = \operatorname{Im} z$,

$$\begin{aligned} \langle iR[z], \phi[z] \rangle &= \langle iR[z], i\Lambda_p \phi[z] \rangle = \langle iR[z], x\phi[z] \rangle = \langle iR[z], i\partial_x \phi[z] \rangle \\ &= \langle iR[z], i\partial_{z_j} \phi[z] \rangle = 0, \text{ for all } j = 1, 2; \end{aligned} \quad (3.7)$$

iii we have $\tilde{\Theta}_{\mathcal{R}}[\cdot] \in C^1(D_{\mathbb{C}}(0, \delta_1))$;

iv $\tilde{\Theta}_{\mathcal{R}}[\cdot]$ is twice differentiable in $z = 0$.

Proof. Let us set

$$\hat{R}[z] := f(\phi + \tilde{\phi}[z]) - f(\phi) - Df(\phi)\tilde{\phi}[z]. \quad (3.8)$$

From (1.7),

$$D_z \tilde{\phi}[z] \tilde{z}_0 = \mathcal{L} \tilde{\phi}[z] = -i \left(-\partial_x^2 \tilde{\phi}[z] + \tilde{\phi}[z] - Df(\phi)\tilde{\phi}[z] \right)$$

adding also $-\phi'' + \phi - f(\phi) = 0$ we obtain

$$\hat{R}[z] = \partial_x^2 \phi[z] + f(\phi[z]) - \phi[z] + iD_z \phi[z] \tilde{z}_0. \quad (3.9)$$

Claim 3.2. We have

$$|\hat{R}[z]| \lesssim |z|^2 e^{|x|(2-p-2\sqrt{1-\lambda(p)})}. \quad (3.10)$$

Proof. By Lemma 2.2 have $|\tilde{\phi}[z]| \lesssim |z| e^{-|x|\sqrt{1-\lambda(p)}}$. We distinguish two case.

Case 1. Let $|\tilde{\phi}[z]| \leq |\phi|/2$. By $|\phi| \sim e^{-|x|}$ and $|f''(\phi + \tau\tilde{\phi}[z])| \sim e^{-(p-2)|x|}$, for $\tau \in (0, 1)$ we have

$$|\hat{R}[z]| \lesssim \sup_{\tau \in (0, 1)} |f''(\phi + \tau\tilde{\phi}[z])| |z|^2 e^{-2|x|\sqrt{1-\lambda(p)}} \lesssim |z|^2 e^{|x|(2-p-2\sqrt{1-\lambda(p)})}$$

Case 2. Let $|\tilde{\phi}[z]| > |\phi|/2$. Then

$$e^{-|x|} \lesssim |\tilde{\phi}[z]| \lesssim |z| e^{-|x|\sqrt{1-\lambda(p)}} \implies e^{|x|(-1+\sqrt{1-\lambda(p)})} \lesssim |z| \quad (3.11)$$

and

$$\begin{aligned}
|\widehat{R}[z]| &\lesssim |\widetilde{\phi}[z]|^p \lesssim |z|^p e^{-p|x|\sqrt{1-\lambda(p)}} \lesssim |z|^p e^{-p|x|\sqrt{1-\lambda(p)}} |z|^{2-p} e^{|x|((2-p)-(2-p)\sqrt{1-\lambda(p)})} \\
&= |z|^2 e^{|x|(2-p-2\sqrt{1-\lambda(p)})}. \quad \square
\end{aligned} \tag{3.12}$$

Now we set

$$R[z] = \widehat{R}[z] - \widetilde{\vartheta}_{\mathcal{R}}\phi[z] + i\widetilde{\omega}_{\mathcal{R}}\Lambda_p\phi[z] - i\widetilde{D}_{\mathcal{R}}\partial_x\phi[z] - \frac{\widetilde{v}_{\mathcal{R}}x}{2}\phi[z] + iD_z\phi[z]\widetilde{z}_{\mathcal{R}},$$

and impose that the orthogonality conditions (3.7) be satisfied. This yields

$$\mathbf{A}[z]\widetilde{\Theta}_{\mathcal{R}}^{\top} = \begin{pmatrix} \langle \widehat{R}[z], i\phi[z] \rangle \\ \langle \widehat{R}[z], \Lambda_p\phi[z] \rangle \\ \langle \widehat{R}[z], \partial_x\phi[z] \rangle \\ \langle \widehat{R}[z], ix\phi[z] \rangle \\ \langle \widehat{R}[z], D_z\phi \rangle \end{pmatrix} \text{ for an unknown } \widetilde{\Theta}_{\mathcal{R}} = (\widetilde{\vartheta}_{\mathcal{R}}, \widetilde{\omega}_{\mathcal{R}}, \widetilde{D}_{\mathcal{R}}, \widetilde{v}_{\mathcal{R}}, \widetilde{z}_{\mathcal{R}}) \tag{3.13}$$

where

$$\mathbf{A}[z] = \begin{pmatrix} 0 & -\langle \Lambda_p\phi[z], \phi[z] \rangle & 0 & 0 & -\langle D_z\phi[z], \phi[z] \rangle \\ \langle \phi[z], \Lambda_p\phi[z] \rangle & 0 & \cancel{\langle i\partial_x\phi[z], \Lambda_p\phi[z] \rangle} & \cancel{\langle \frac{x}{2}\phi[z], \Lambda_p\phi[z] \rangle} & -\langle iD_z\phi[z], \Lambda_p\phi[z] \rangle \\ 0 & -\langle i\Lambda_p\phi[z], \partial_x\phi[z] \rangle & 0 & -\frac{1}{4}\|\phi[z]\|_{L^2}^2 & -\langle iD_z\phi[z], \partial_x\phi[z] \rangle \\ 0 & -\langle \Lambda_p\phi[z], ix\phi[z] \rangle & -\frac{1}{2}\|\phi[z]\|_{L^2}^2 & 0 & -\langle D_z\phi[z], ix\phi[z] \rangle \\ \langle \phi[z], D_z\phi[z] \rangle & -\langle i\Lambda_p\phi[z], D_z\phi[z] \rangle & \cancel{\langle i\partial_x\phi[z], D_z\phi[z] \rangle} & \cancel{\langle \frac{x}{2}\phi[z], D_z\phi[z] \rangle} & -\langle iD_z\phi[z], D_z\phi[z] \rangle \end{pmatrix} \tag{3.14}$$

where the canceled terms are null because by (1.3) and Lemma 2.1 the function $\phi[z]$ is even in x and so

$$\begin{aligned}
\langle i\partial_x\phi[z], \Lambda_p\phi[z] \rangle &= \langle i\partial_x\phi[z], \Lambda_p\phi[z] \rangle = \langle iD_z\phi[z]\Xi, \partial_x\phi[z] \rangle \\
&= \langle iD_z\phi[z]\Xi, i\partial_x\phi[z] \rangle = 0 \text{ for any } \Xi \in \mathbb{C}.
\end{aligned}$$

In particular, near $z = 0$ we have

$$\mathbf{A}[z] = \begin{pmatrix} 0 & \frac{p-5}{p-1}q(1) & 0 & 0 & O(z) \\ \frac{5-p}{p-1}q(1) & 0 & 0 & 0 & O(z) \\ 0 & 0 & 0 & -\frac{1}{4}\|\phi[z]\|_{L^2}^2 & 0 \\ 0 & 0 & -\frac{1}{2}\|\phi[z]\|_{L^2}^2 & 0 & 0 \\ O(z) & O(z) & 0 & 0 & J \end{pmatrix} + O(z^2). \tag{3.15}$$

So we get

$$\begin{aligned}
\widetilde{\Theta}_{\mathcal{R}}[z] &= \mathbf{A}^{-1}[z]v[z] \text{ with} \\
v[z] &:= \left(\langle \widehat{R}[z], i\phi[z] \rangle, \langle \widehat{R}[z], \Lambda_p\phi[z] \rangle, \cancel{\langle \widehat{R}[z], \partial_x\phi[z] \rangle}, \cancel{\langle \widehat{R}[z], ix\phi[z] \rangle}, \langle \widehat{R}[z], D_z\phi \rangle \right)^{\top}
\end{aligned}$$

where the two canceled terms are null because $\widehat{R}[z]$ is an even function in x . This immediately implies from system (3.13) that

$$\widetilde{D}_{\mathcal{R}} = \widetilde{v}_{\mathcal{R}} \equiv 0.$$

Properties **i**, **iii** and **iv** need to be checked for $v[z]$. The inequality

$$|v[z]| \lesssim |z|^2$$

follows immediately from (3.10). By dominated convergence, $v[\cdot] \in C^0(D_{\mathbb{C}}(0, \delta_1))$. This implies (3.4) and $\widetilde{\Theta}_{\mathcal{R}}[\cdot] \in C^0(D_{\mathbb{C}}(0, \delta_1))$.

Next we consider (notice that $z \rightarrow \widetilde{\phi}[z]$ is linear)

$$D_z \widehat{R} = (f'(\phi + \widetilde{\phi}[z]) - f'(\phi)) D_z \widetilde{\phi}[z] = (f'(\phi + \widetilde{\phi}[z]) - f'(\phi)) D_z \widetilde{\phi}[0].$$

Claim 3.3. *We have the pointwise estimate*

$$|(f'(\phi + \widetilde{\phi}[z]) - f'(\phi)) D_z \widetilde{\phi}[0]| \lesssim |z| e^{|x|(2-p-2\sqrt{1-\lambda(p)})}. \quad (3.16)$$

Proof. For $|\widetilde{\phi}[z]| \leq |\phi|/2$, by $|\phi| \sim e^{-|x|}$ and $|f''(\phi + \tau \widetilde{\phi}[z])| \sim e^{-(p-2)|x|}$, for $\tau \in (0, 1)$ we have

$$\begin{aligned} |(f'(\phi + \widetilde{\phi}[z]) - f'(\phi)) D_z \widetilde{\phi}[0]| &\lesssim \sup_{\tau \in (0, 1)} |f''(\phi + \tau \widetilde{\phi}[z])| |z| e^{-2|x|\sqrt{1-\lambda(p)}} \\ &\lesssim |z| e^{|x|(2-p-2\sqrt{1-\lambda(p)})}. \end{aligned}$$

For $|\widetilde{\phi}[z]| \gtrsim |\phi|$ like in (3.11)–(3.12)

$$\begin{aligned} |(f'(\phi + \widetilde{\phi}[z]) - f'(\phi)) D_z \widetilde{\phi}[0]| &\lesssim |\widetilde{\phi}[z]|^{p-1} e^{-|x|\sqrt{1-\lambda(p)}} \lesssim |z|^{p-1} e^{-p|x|\sqrt{1-\lambda(p)}} \\ &\lesssim |z|^{p-1} e^{-p|x|\sqrt{1-\lambda(p)}} |z|^{2-p} e^{|x|((2-p)-(2-p)\sqrt{1-\lambda(p)})} \lesssim |z| e^{|x|(2-p-2\sqrt{1-\lambda(p)})}. \quad \square \end{aligned}$$

Claim 3.3 allows to conclude that we can use differentiation under the integral and that $v[\cdot] \in C^1(D_{\mathbb{C}}(0, \delta_1))$ with

$$|D_z v[z]| \lesssim |z|$$

This in turn implies $\widetilde{\Theta}_{\mathcal{R}}[\cdot] \in C^1(D_{\mathbb{C}}(0, \delta_1))$ with a similar bound.

We consider now, formally

$$D_z^2 \widehat{R}[z] = f''(\phi + \widetilde{\phi}[z]) (D_z \widetilde{\phi}[z])^2 = f''(\phi + \widetilde{\phi}[z]) (D_z \widetilde{\phi}[0])^2.$$

Now we write

$$D_z \widehat{R}[z] - D_z^2 \widehat{R}[0]z = (f'(\phi + \widetilde{\phi}[z]) - f'(\phi) - f''(\phi)\widetilde{\phi}[z]) D_z \widetilde{\phi}[0] =: I.$$

For $|\tilde{\phi}[z]| \leq |\phi|/2$, by $|f'''(\phi + \tau s \tilde{\phi}[z])| \lesssim |f'''(\phi)|$ for $s, \tau \in [0, 1]$ and $\varepsilon_0 > 0$ small,

$$\begin{aligned} |I| &\lesssim \int_{[0,1]^2} \tau |f'''(\phi + \tau s \tilde{\phi}[z])| ds d\tau |\tilde{\phi}[z]|^{1+\varepsilon_0} \frac{|\tilde{\phi}[z]|^{1-\varepsilon_0}}{\phi^{1-\varepsilon_0}} \phi^{1-\varepsilon_0} |D_z \tilde{\phi}[0]| \\ &\lesssim |z|^{1+\varepsilon_0} e^{|x|(2+\varepsilon_0-p-2\sqrt{1-\lambda(p)})}. \end{aligned}$$

For $|\tilde{\phi}[z]| \gtrsim |\phi|$ we similarly have

$$\begin{aligned} |I| &\lesssim |\tilde{\phi}[z]|^{p-1} |D_z \tilde{\phi}[0]| \lesssim |z|^{p-1} e^{-p|x|\sqrt{1-\lambda(p)}} \\ &\lesssim |z|^{p-1} e^{-p|x|\sqrt{1-\lambda(p)}} |z|^{2-p+\varepsilon_0} e^{|x|(2-p+\varepsilon_0)(1-\sqrt{1-\lambda(p)})} \leq |z|^{1+\varepsilon_0} e^{|x|(2+\varepsilon_0-p-2\sqrt{1-\lambda(p)})}. \end{aligned}$$

Now let us consider the following component of $v[z]$ (the other can be treated similarly)

$$\langle \widehat{R}[z], i\phi[z] \rangle = \langle \widehat{R}[z], i\phi \rangle + \langle \widehat{R}[z], i\tilde{\phi}[z] \rangle = II_1 + II_2.$$

Then

$$\lim_{z \rightarrow 0} \frac{\langle D_z \widehat{R}[z], i\phi \rangle - \langle D_z^2 \widehat{R}[0]z, i\phi \rangle}{|z|} = 0$$

which implies the differentiability of II_1 at $z = 0$. The same is true for II_2 , with differential 0 at $z = 0$. Since the other components of $v[z]$ can be treated similarly, this gives the 2nd order differentiability of $v[z]$, and thus also of $\widehat{\Theta}_{\mathcal{R}}[z]$, at $z = 0$. $\square$

If $(\mathcal{D}_{\sqrt{\omega}}u)(x) := \omega^{\frac{1}{p-1}}u(\sqrt{\omega}x)$ we get the following.

Lemma 3.4. Set $\phi[\omega, z](x) := \omega^{\frac{1}{p-1}}\phi[z](\sqrt{\omega}x)$. Then

$$R[\omega, z] = \partial_x^2 \phi[\omega, z] + f(\phi[\omega, z]) - \tilde{\vartheta}[\omega, z]\phi[\omega, z] + i\tilde{\omega}[\omega, z]\partial_{\omega}\phi[\omega, z] + iD_z\phi[\omega, z]\tilde{z}[\omega, z] \quad (3.17)$$

for

$$R[\omega, z](x) := \omega^{\frac{p}{p-1}}R[z](\sqrt{\omega}x), \quad (3.18)$$

$$\tilde{\vartheta}[\omega, z] := \omega\tilde{\vartheta}[z], \quad \tilde{\omega}[\omega, z] = \tilde{\omega}[z] \text{ and } \tilde{z}[\omega, z] = \omega\tilde{z}[z].$$

Furthermore, we have the following orthogonality properties,

$$\begin{aligned} \langle iR[\omega, z], \phi[\omega, z] \rangle &= \langle iR[\omega, z], i\partial_{\omega}\phi[\omega, z] \rangle \\ &= \langle iR[\omega, z], x\phi[\omega, z] \rangle = \langle iR[\omega, z], i\partial_x\phi[\omega, z] \rangle \\ &= \langle iR[\omega, z], i\partial_{z_j}\phi[\omega, z] \rangle = 0, \text{ for all } j = 1, 2. \end{aligned} \quad (3.19)$$

Proof. Notice that

$$\mathcal{D}_{\sqrt{\omega}}\partial_x = \omega^{-1/2}\partial_x\mathcal{D}_{\sqrt{\omega}} \quad , \quad \mathcal{D}_{\sqrt{\omega}}x = \omega^{1/2}x\mathcal{D}_{\sqrt{\omega}} \text{ and } \mathcal{D}_{\sqrt{\omega}}f(\phi[z]) = \omega^{-1}f(\phi[\omega, z]).$$

So that applying $\mathcal{D}_{\sqrt{\omega}}$ to equation (3.5) and using $[\mathcal{D}_{\sqrt{\omega}}, \Lambda_p] = 0$ we obtain

$$\mathcal{D}_{\sqrt{\omega}}R[z] = \omega^{-1}\partial_x^2\phi[\omega, z] + \omega^{-1}f(\phi[\omega, z]) - \tilde{\vartheta}[z]\phi[\omega, z] + i\tilde{\omega}\Lambda_p\phi[\omega, z] + iD_z\phi[\omega, z]\tilde{z}.$$

Multiplying by ω , using $\partial_\omega\phi[\omega, z] = \omega^{-1}\Lambda_p\phi[\omega, z]$ and the definitions in (3.18) we obtain (3.17).

The orthogonality properties (3.19) follow immediately from (3.7). $\square$

Lemma 3.5. Set $\phi[\omega, v, z] := e^{\frac{i}{2}vx}\phi[\omega, z]$. Then

$$\begin{aligned} R[\omega, v, z] &= \partial_x^2\phi[\omega, v, z] + f(\phi[\omega, v, z]) - \tilde{\vartheta}[\omega, v, z]\phi[\omega, v, z] + i\tilde{\omega}[\omega, v, z]\partial_\omega\phi[\omega, v, z] \\ &\quad + iD_z\phi[\omega, v, z]\tilde{z}[\omega, v, z] - iv\partial_x\phi[\omega, v, z], \end{aligned} \quad (3.20)$$

for

$$\begin{aligned} R[\omega, v, z] &:= e^{\frac{i}{2}vx}R[\omega, z], \quad \tilde{\omega}[\omega, v, z] = \tilde{\omega}_{\mathcal{R}}[\omega, v, z] \text{ with } \tilde{\omega}_{\mathcal{R}}[\omega, v, z] = \tilde{\omega}_{\mathcal{R}}[z], \\ \tilde{z}[\omega, v, z] &= \tilde{z}_0[\omega, v, z] + \tilde{z}_{\mathcal{R}}[\omega, v, z] \text{ with } \tilde{z}_0[\omega, v, z] = i\omega\lambda(p)z \text{ and } \tilde{z}_{\mathcal{R}}[\omega, v, z] = \omega\tilde{z}_{\mathcal{R}}[z], \\ \tilde{\vartheta}[\omega, v, z] &= \omega - \frac{v^2}{4} + \tilde{\vartheta}_{\mathcal{R}}[\omega, v, z] \text{ with } \tilde{\vartheta}_{\mathcal{R}}[\omega, v, z] = \omega\tilde{\vartheta}_{\mathcal{R}}[z]. \end{aligned} \quad (3.21)$$

Furthermore, we have the following orthogonality properties,

$$\begin{aligned} \langle iR[\omega, v, z], \phi[\omega, v, z] \rangle &= \langle iR[\omega, v, z], i\partial_\omega\phi[\omega, v, z] \rangle \\ &= \langle iR[\omega, v, z], x\phi[\omega, v, z] \rangle = \langle iR[\omega, v, z], i\partial_x\phi[\omega, v, z] \rangle \\ &= \langle iR[\omega, v, z], i\partial_{z_j}\phi[\omega, v, z] \rangle = 0, \text{ for all } j = 1, 2. \end{aligned} \quad (3.22)$$

Proof. We multiply (3.17) by $e^{\frac{i}{2}vx}$ and use

$$e^{\frac{i}{2}vx}\partial_x^2\phi[\omega, z] = \partial_x^2\phi[\omega, v, z] + \frac{v^2}{4}\phi[\omega, v, z] - iv\partial_x\phi[\omega, v, z]$$

obtaining the following, which yields (3.20),

$$\begin{aligned} e^{\frac{i}{2}vx}R[\omega, z] &= \partial_x^2\phi[\omega, v, z] + \frac{v^2}{4}\phi[\omega, v, z] - iv\partial_x\phi[\omega, v, z] + f(\phi[\omega, v, z]) \\ &\quad - \tilde{\vartheta}[\omega, z]\phi[\omega, v, z] + i\tilde{\omega}[\omega, z]\partial_\omega\phi[\omega, v, z] + iD_z\phi[\omega, z]\tilde{z}[\omega, v, z]. \end{aligned}$$

The orthogonality properties (3.22) follow immediately from (3.7). $\square$

Let now

$$\tilde{v}[\omega, v, z] = 0 \text{ and } \tilde{D}[\omega, v, z] = v$$

and

$$\begin{aligned} \Theta &= (\vartheta, \omega, D, v, z), \\ \tilde{\Theta}_{\mathcal{R}}[\Theta] &= (\tilde{\vartheta}_{\mathcal{R}}[\omega, v, z], \tilde{\omega}_{\mathcal{R}}[\omega, v, z], 0, 0, \tilde{z}_{\mathcal{R}}[\omega, v, z]) \text{ and} \\ \tilde{\Theta}[\Theta] &= (\tilde{\vartheta}[\omega, v, z], \tilde{\omega}[\omega, v, z], 0, v, \tilde{z}[\omega, v, z]) \text{ and} \\ \phi[\Theta] &= e^{i\vartheta - iD\partial_x} e^{\frac{i}{2}vx} \phi[\omega, z]. \end{aligned} \tag{3.23}$$

The proof of the following modulation is standard, see Stuart [65].

For $\omega, \delta \in \mathbb{R}_+ := (0, \infty)$ we consider the orbit

$$\mathcal{U}(\omega, \delta) := \bigcup_{\vartheta_0, x_0 \in \mathbb{R}} e^{i\vartheta_0} e^{-x_0\partial_x} D_{H^1(\mathbb{R})}(\phi_\omega, \delta)$$

of the ball $D_{H^1(\mathbb{R}, \mathbb{C})}(\phi_\omega, \delta)$, see Notation 2.3.

Lemma 3.6 (Modulation). *Let $\omega_0 > 0$. There exists an $\delta_0 > 0$ and $\Theta \in C^1(\mathcal{U}(\omega_0, \delta_0), \mathbb{R}^4 \times \mathbb{C})$ such that for any $u \in \mathcal{U}(\omega_0, \delta_0)$*

$$\eta(u) := e^{-i\vartheta(u) + D(u)\partial_x} u - \phi[\Theta(u)] \text{ satisfies} \tag{3.24}$$

$$\left\langle i\eta(u), e^{-i\vartheta(u) + D(u)\partial_x} D_{\Theta}\phi[\Theta]|_{\Theta=\Theta(u)} \Xi \right\rangle = 0, \text{ for all } \Xi \in \mathbb{R}^4 \times \mathbb{C}.$$

Furthermore we have the identities

$$\begin{aligned} \omega(\phi[\Theta]) &= \omega, \quad v(e^{i\vartheta_0 - D_0\partial_x} u) = v(u), \quad z(e^{i\vartheta_0 - D_0\partial_x} u) = z(u) \\ \vartheta(e^{i\vartheta_0 - D_0\partial_x} u) &= \vartheta(u) + \vartheta_0 \text{ and} \\ D(e^{i\vartheta_0 - D_0\partial_x} u) &= D(u) + D_0. \quad \square \end{aligned}$$

We have now the ansatz

$$u = e^{i\vartheta - D\partial_x} (\phi[\omega, v, z] + \eta) =: e^{i\vartheta - D\partial_x} e^{\frac{i}{2}vx} (\phi_\omega + r) \text{ where } r = \tilde{\phi}[\omega, z] + e^{-\frac{i}{2}vx} \eta. \tag{3.25}$$

We can prove that if $u_0 \in D_{H^1(\mathbb{R})}(\phi_{\omega_0}, \delta)$ then there exist $(\vartheta, D) = (\vartheta(t), D(t))$ such that

$$\|u - e^{i\vartheta - D\partial_x} \phi_{\omega_0}\|_{H^1} \lesssim \delta \text{ for all values of time.} \tag{3.26}$$

Notice now that

$$\mathbf{E}\left(e^{\frac{i}{2}vx}u\right) = \mathbf{E}(u) + v\mathbf{P}(u) + \frac{v^2}{4}\mathbf{Q}(u) \text{ and} \quad (3.27)$$

$$\mathbf{P}\left(e^{\frac{i}{2}vx}u\right) = \mathbf{P}(u) + \frac{v}{2}\mathbf{Q}(u). \quad (3.28)$$

In analogy to Stuart [65, pp. 67–68] we consider

$$\mathcal{H}(u) := \mathbf{E}(u) + \left(\omega + \frac{v^2(0)}{4}\right)\mathbf{Q}(u) - v(0)\mathbf{P}(u).$$

Entering the ansatz in (3.25) we get

$$\begin{aligned} \mathcal{H}(u) &= \mathcal{H}(e^{\frac{i}{2}vx}(\phi_\omega + r)) \\ &= \mathbf{E}(e^{\frac{i}{2}vx}(\phi_\omega + r)) + \left(\omega + \frac{v^2(0)}{4}\right)\mathbf{Q}(\phi_\omega + r) - v(0)\mathbf{P}(e^{\frac{i}{2}vx}(\phi_\omega + r)) \\ &= \mathbf{E}(\phi_\omega + r) + \left(\omega + \frac{v^2(0)}{4} + \frac{v^2 - 2v(0)v}{4}\right)\mathbf{Q}(\phi_\omega + r) + (v - v(0))\mathbf{P}(\phi_\omega + r) \\ &= \mathbf{E}(\phi_\omega + r) + \left(\omega + \frac{(v - v(0))^2}{4}\right)\mathbf{Q}(\phi_\omega + r) + (v - v(0))\mathbf{P}(\phi_\omega + r). \end{aligned}$$

Then expanding the above functionals around ϕ_ω we obtain

$$\begin{aligned} H(u) &= \mathbf{d}(\omega) + \frac{(v - v(0))^2}{4}\mathbf{q}(\omega) + \langle \nabla \mathbf{E}(\phi_\omega) + \omega \nabla \mathbf{Q}(\phi_\omega) + (v - v(0))\nabla \mathbf{P}(\phi_\omega), r \rangle + \\ &+ \frac{1}{2}\langle [\nabla^2 \mathbf{E}(\phi_\omega) + \omega \nabla^2 \mathbf{Q}(\phi_\omega)]r, r \rangle + \frac{(v - v(0))^2}{4}\langle \nabla \mathbf{Q}(\phi_\omega) + \frac{1}{2}\nabla^2 \mathbf{Q}(\phi_\omega)r, r \rangle + \\ &+ \frac{1}{2}(v - v(0))\langle \nabla^2 \mathbf{P}(\phi_\omega)r, r \rangle + o(\|r\|_{H^1}^2) \end{aligned}$$

where the canceled term is null by (1.7). Since $r = \tilde{\phi}[\omega, z] + \eta$, we have

$$\langle \nabla \mathbf{P}(\phi_\omega), r \rangle = -\langle i\partial_x \phi_\omega, r \rangle = -\langle i\partial_x \phi_\omega, \tilde{\phi}[\omega, z] \rangle - \langle i\partial_x \phi_\omega, \eta \rangle = O(|z|\|\eta\|_{\tilde{\Sigma}})$$

where the cancellation is due to the fact that $\partial_x \phi_\omega$ is odd and $\tilde{\phi}[\omega, z]$ is even and the last equality follows from the modulation. So

$$H(u) = d(\omega) + \frac{(v - v(0))^2}{4}q(\omega) + \frac{1}{2}\langle [\nabla^2 \mathbf{E}(\phi_\omega) + \omega \nabla^2 \mathbf{Q}(\phi_\omega)]r, r \rangle + o(\|r\|_{H^1}^2 + (v - v(0))^2)$$

where we use

$$\|r\|_{H^1}^2 \sim |z|^2 + \|\eta\|_{H^1}^2$$

which follows by modulation. Since we have

$$\begin{aligned}
\mathcal{H}(u_0) &= \mathbf{E}(u_0) + \left(\omega(t) + \frac{v^2(0)}{4} \right) \mathbf{Q}(u_0) - v(0) \mathbf{P}(u_0) \\
&= d(\omega(0)) + \frac{1}{2} \langle [\nabla^2 \mathbf{E}(\phi_{\omega(0)}) + \omega(0) \nabla^2 \mathbf{Q}(\phi_{\omega(0)})] r(0), r(0) \rangle \\
&\quad + (\omega(t) - \omega(0)) q(\omega(0)) + o(\|r(0)\|_{H^1}^2)
\end{aligned}$$

then by $\mathcal{H}(u(t)) = \mathcal{H}(u_0)$ and $\mathbf{d}'(\omega) = \mathbf{q}(\omega)$ we get

$$\begin{aligned}
\mathbf{d}(\omega) - \mathbf{d}(\omega(0)) - (\omega - \omega(0)) \mathbf{d}'(\omega(0)) &+ \frac{(v - v(0))^2}{4} \mathbf{q}(\omega) + \frac{1}{2} \langle [\nabla^2 \mathbf{E}(\phi_\omega) + \omega \nabla^2 \mathbf{Q}(\phi_\omega)] r, r \rangle \\
&\lesssim \|r(0)\|_{H^1}^2 \lesssim \delta^2.
\end{aligned}$$

Since

$$\langle [\nabla^2 \mathbf{E}(\phi_\omega) + \omega \nabla^2 \mathbf{Q}(\phi_\omega)] r, r \rangle \gtrsim \|r\|_{H^1}^2$$

and

$$\mathbf{d}(\omega) - \mathbf{d}(\omega(0)) - (\omega - \omega(0)) \mathbf{d}'(\omega(0)) = \frac{\mathbf{d}''(\omega(0))}{2} (\omega - \omega(0))^2 + o((\omega - \omega(0))^2)$$

with $\mathbf{d}''(\omega(0)) > 0$ and by $|\omega(0) - \omega_0| \lesssim \delta$ we conclude that

$$|\omega - \omega_0| + |v - v(0)| + |z| + \|\eta\|_{H^1} \lesssim \delta \text{ for all values of time.} \quad (3.29)$$

The proof of Theorem 1.5 is mainly based on the following continuation argument.

Proposition 3.7. *There exists a $\delta_0 = \delta_0(\epsilon)$ s.t. if*

$$\|\eta\|_{L^2(I, \Sigma_A)} + \|\eta\|_{L^2(I, \tilde{\Sigma})} + \|\dot{\Theta} - \tilde{\Theta}\|_{L^2(I)} + \|z^2\|_{L^2(I)} \leq \epsilon \quad (3.30)$$

holds for $I = [0, T]$ for some $T > 0$ and for $\delta \in (0, \delta_0)$ then in fact for $I = [0, T]$ inequality (3.30) holds for ϵ replaced by $o_\epsilon(1)\epsilon$.

Notice that this implies that in fact the result is true for $I = \mathbb{R}_+$. We will split the proof of Proposition 3.7 in a number of partial results obtained assuming the hypotheses of Proposition 3.7.

Proposition 3.8. *We have*

$$\|\dot{\vartheta} - \tilde{\vartheta}\|_{L^1(I)} + \|\dot{\omega} - \tilde{\omega}\|_{L^1(I)} + \|\dot{D} - v\|_{L^1(I)} + \|\dot{v}\|_{L^1(I)} \lesssim \epsilon^2, \quad (3.31)$$

$$\|\dot{z} - \tilde{z}\|_{L^2(I)} \lesssim \delta \epsilon, \quad (3.32)$$

$$\|\dot{z}\|_{L^\infty(I)} \lesssim \delta. \quad (3.33)$$

Proposition 3.9 (Fermi Golden Rule (FGR) estimate). *We have*

$$\|z^2\|_{L^2(I)} \lesssim A^{-1/2}\epsilon. \quad (3.34)$$

Proposition 3.10 (Virial inequality). *We have*

$$\|\eta\|_{L^2(I, \Sigma_A)} \lesssim A\delta + \|z^2\|_{L^2(I)} + \|\eta\|_{L^2(I, \tilde{\Sigma})} + \epsilon^2. \quad (3.35)$$

Notice that $A\delta \ll A^{-1}\epsilon^2 = o_{B^{-1}}(1)\epsilon^2$ in (3.35).

Proposition 3.11 (Smoothing inequality). *We have*

$$\|\eta\|_{L^2(I, \tilde{\Sigma})} \lesssim o_{B^{-1}}(1)\epsilon. \quad (3.36)$$

Proof of Theorem 1.5. It is straightforward that Propositions 3.8–3.11 imply Proposition 3.7 and thus the fact that we can take $I = \mathbb{R}_+$ in all the above inequalities. This in particular implies (1.20). By $z \in L^4(\mathbb{R}_+)$ and $\dot{z} \in L^\infty(\mathbb{R}_+)$ we have (1.21).

We next focus on the limit (1.19). We first rewrite our equation, entering the ansatz (3.25) in (1.1) and using (3.17) we obtain

$$\begin{aligned} \dot{\eta} + i\dot{\vartheta}\eta - \dot{D}\partial_x\eta + e^{-i\vartheta+D\partial_x}D_\Theta\phi[\Theta](\dot{\Theta} - \tilde{\Theta}) \\ = i\partial_x^2\eta + i(f(\phi[\omega, v, z] + \eta) - f(\phi[\omega, v, z])) + iR[\omega, v, z]. \end{aligned} \quad (3.37)$$

Then we can proceed like in [18] considering

$$\mathbf{a}(t) := 2^{-1}\|e^{-a\langle x \rangle}\eta(t)\|_{L^2(\mathbb{R})}^2$$

and by obvious cancellations and orbital stability getting

$$\begin{aligned} \dot{\mathbf{a}} = \frac{1}{2}\left\langle \left[e^{-2a\langle x \rangle}, i\partial_x^2 \right] \eta, \eta \right\rangle + \frac{1}{2}\dot{D}\left\langle \left[e^{-2a\langle x \rangle}, \partial_x \right] \eta, \eta \right\rangle - \left\langle e^{-i\vartheta+D\partial_x}D_\Theta\phi[\Theta](\dot{\Theta} - \tilde{\Theta}), e^{-2a\langle x \rangle}\eta \right\rangle \\ + \left\langle e^{-a\langle x \rangle} (i(f(\phi[\omega, v, z] + \eta) - f(\phi[\omega, v, z])) + iR[\omega, v, z]), e^{-a\langle x \rangle}\eta \right\rangle = O(\delta^2) \text{ for all times.} \end{aligned} \quad (3.38)$$

Since we already know from (1.20) that $\mathbf{a} \in L^1(\mathbb{R})$, we conclude that $\mathbf{a}(t) \xrightarrow{t \rightarrow +\infty} 0$. Notice that the integration by parts in (3.38) can be made rigorous considering that if $u_0 \in H^2(\mathbb{R})$ by the well known regularity result by Kato, see [6], we have $\eta \in C^0(\mathbb{R}, H^2(\mathbb{R}))$ and the above argument is correct and by a standard density argument the result can be extended to $u_0 \in H^1(\mathbb{R})$.

We prove (1.21). Here, notice that by orbital stability we can take $a > 0$ such that we have the following, which will be used below,

$$e^{-2a\langle x \rangle} \gtrsim \max\{|\phi[\omega(t), v(t), z(t)]|, |\phi[\omega(t), v(t), z(t)]|^{p-1}\} \text{ for all } t \in \mathbb{R}. \quad (3.39)$$

Since $\mathbf{Q}(\phi_\omega) = \mathbf{q}(\omega)$ is strictly monotonic in ω , it suffices to show the $\mathbf{Q}(\phi_{\omega(t)})$ converges as $t \rightarrow \infty$. From the conservation of $\mathbf{Q}$, the exponential decay of $\phi[\omega, v, z]$, (1.19) and (1.20), we have

$$\lim_{t \rightarrow \infty} (\mathbf{Q}(u_0) - \mathbf{Q}(\phi_{\omega(t)}) - \mathbf{Q}(\eta(t))) = 0. \quad (3.40)$$

Thus, our task is now to prove $\frac{d}{dt}\mathbf{Q}(\eta) \in L^1$, which is sufficient to show the convergence of $\mathbf{Q}(\eta)$. Now, from (3.37), we have

$$\begin{aligned} \frac{d}{dt}\mathbf{Q}(\eta) &= \langle \eta, \dot{\eta} \rangle = - \left\langle \eta, e^{-i\partial + D\partial_x} D_\Theta \phi[\Theta] (\dot{\Theta} - \tilde{\Theta}) \right\rangle \\ &\quad + \langle \eta, i(f(\phi[\omega, v, z] + \eta) - f(\phi[\omega, v, z])) \rangle + \langle \eta, iR[\omega, v, z] \rangle \\ &= I + II + III. \end{aligned}$$

By the bound of the 1st and the 3rd term of (3.30), we have $I \in L^1(\mathbb{R}_+)$. $III \in L^1(\mathbb{R}_+)$ follows from (3.6) and $|z|^2 \in L^2(\mathbb{R}_+)$. So the key term is II . We partition for $s \in (0, 1)$ the line where x lives as

$$\begin{aligned} \Omega_{1,t,s} &= \{x \in \mathbb{R} \mid |s\eta(t, x)| \leq 2|\phi[\omega(t), v(t), z(t)]|\} \text{ and} \\ \Omega_{2,t,s} &= \mathbb{R} \setminus \Omega_{1,t,s} = \{x \in \mathbb{R} \mid |s\eta(t, x)| > 2|\phi[\omega(t), v(t), z(t)]|\}. \end{aligned}$$

Then, we have

$$\begin{aligned} II(t) &= \sum_{j=1,2} \int_0^1 ds \int_{\Omega_{j,t,s}} \langle iDf(\phi[\omega(t), v(t), z(t)] + s\eta(t))\eta(t), \eta(t) \rangle_{\mathbb{C}} dx \\ &=: II_1(t) + II_2(t). \end{aligned}$$

For II_1 , by (2.21), we have

$$\begin{aligned} |II_1(t)| &\lesssim \int_0^1 ds \int_{\Omega_{1,t,s}} |\langle iDf(\phi[\omega(t), v(t), z(t)] + s\eta(t))\eta(t), \eta(t) \rangle_{\mathbb{C}}| dx \\ &\lesssim \int_0^1 ds \int_{\Omega_{1,t,s}} |\phi[\omega(t), v(t), z(t)] + s\eta(t)|^{p-1} |\eta(t)|^2 dx \\ &\lesssim \int_{\mathbb{R}} |\phi[\omega(t), v(t), z(t)]|^{p-1} |\eta(t)|^2 dx \\ &\lesssim \|e^{-a\langle x \rangle} \eta(t)\|_{L^2(\mathbb{R})}^2 \in L^1(\mathbb{R}_+). \end{aligned}$$

Turning to II_2 , by exploiting $\langle Df(s\eta(t))\eta(t), i\eta(t) \rangle_{\mathbb{C}} \equiv 0$, which can be easily checked from (2.20), we write

$$\begin{aligned}
II_2(t) &= - \int_{[0,1]} ds \int_{\Omega_{2,t,s}} \langle Df(\phi[\omega(t), v(t), z(t)] + s\eta(t))\eta(t) - Df(s\eta(t))\eta(t), i\eta(t) \rangle_{\mathbb{C}} dx \\
&= - \int_{[0,1]^2} d\tau ds \int_{\Omega_{2,t,s}} \left\langle D^2 f(\tau\phi[\omega(t), v(t), z(t)] + s\eta(t))(\phi[\omega(t), v(t), z(t)], \eta(t)), i\eta(t) \right\rangle_{\mathbb{C}} dx.
\end{aligned}$$

Then by $|s\eta(t, x)| > 2|\phi[\omega(t), v(t), z(t)]|$, (3.39) and orbital stability we have the following, which yields (1.21),

$$\begin{aligned}
|II_2(t)| &\lesssim \int_{[0,1]^2} d\tau ds \int_{\Omega_{2,t,s}} |s\eta(t)|^{p-2} |\phi[\omega(t), v(t), z(t)]| |\eta(t)|^2 dx \\
&\lesssim \int_{[0,1]} ds \int_{\Omega_{2,t,s}} |\phi[\omega(t), v(t), z(t)]|^{p-1} |\eta(t)|^2 dx \\
&\lesssim \|e^{-a(\cdot)} \eta(t)\|_{L^2(\mathbb{R})}^2 \in L^1(\mathbb{R}_+).
\end{aligned}$$

Finally we prove (1.22). Like in (3.40) and using (1.21) and (3.28) we have

$$0 = \lim_{t \rightarrow +\infty} \left(\mathbf{P}(u_0) - \mathbf{P}(e^{\frac{i}{2}v(t)x} \phi_{\omega(t)}) - \mathbf{P}(\eta(t)) \right) = \lim_{t \rightarrow +\infty} \left(\mathbf{P}(u_0) - \frac{v(t)}{2} \mathbf{q}(\omega_+) - \mathbf{P}(\eta(t)) \right). \quad (3.41)$$

So the key is to show $\frac{d}{dt} \mathbf{P}(\eta) \in L^1(\mathbb{R}_+)$. We have

$$\begin{aligned}
\frac{d}{dt} \mathbf{P}(\eta) &= - \langle i\partial_x \eta, \dot{\eta} \rangle = \left\langle i\partial_x \eta, e^{-i\vartheta + D\partial_x} D_{\Theta} \phi[\Theta] (\dot{\Theta} - \tilde{\Theta}) \right\rangle \\
&\quad - \langle i\partial_x \eta, i(f(\phi[\omega, v, z] + \eta) - f(\phi[\omega, v, z])) \rangle - \langle i\partial_x \eta, iR[\omega, v, z] \rangle \\
&= J_1 + J_2 + J_3.
\end{aligned}$$

Like for I and III above we have $J_1, J_3 \in L^1(\mathbb{R}_+)$ so we turn to J_2 which we write

$$\begin{aligned}
J_2 &= - \int_0^1 ds \langle f(\phi[\omega, v, z] + \eta) - f(\phi[\omega, v, z]) - Df(s\eta)\eta, \partial_x \eta \rangle \\
&= - \sum_{j=1,2} \int_0^1 ds \int_{\Omega_{j,t,s}} \langle f(\phi[\omega, v, z] + \eta) - f(\phi[\omega, v, z]) - Df(s\eta)\eta, \partial_x \eta \rangle_{\mathbb{C}} dx =: J_{21} + J_{22}.
\end{aligned}$$

We have

$$J_{21}(t) = - \int_0^1 ds \int_{\Omega_{1,t,s}} \langle [Df(\phi[\omega(t), v(t), z(t)] + s\eta(t)) - Df(s\eta(t))] \eta(t), \partial_x \eta(t) \rangle_{\mathbb{C}} dx.$$

Then

$$|J_{21}(t)| \lesssim \int_0^1 ds \int_{\Omega_{1,t,s}} |\phi[\omega(t), v(t), z(t)]|^{p-1} |\eta(t)| |\partial_x \eta(t)| dx \lesssim \|e^{-a\langle x \rangle} \eta(t)\|_{H^1(\mathbb{R})}^2 \in L^1(\mathbb{R}_+).$$

We have

$$\begin{aligned} J_{22}(t) &= - \int_{[0,1]^2} d\tau ds \int_{\Omega_{2,t,s}} \left\langle D^2 f(\tau \phi[\omega(t), v(t), z(t)] + s\eta(t)) (\eta(t), \phi[\omega(t), v(t), z(t)]), \partial_x \eta(t) \right\rangle_{\mathbb{C}} \end{aligned}$$

so that we get the following, which completes the proof of (1.22),

$$\begin{aligned} |J_{22}(t)| &\lesssim \int_{[0,1]^2} d\tau ds s^{p-2} \int_{\Omega_{2,t,s}} |\eta(t)|^{p-2} |\phi[\omega(t), v(t), z(t)]|^{p-1+2-p} |\eta(t)| |\partial_x \eta(t)| dx \\ &\lesssim \int_{\mathbb{R}} |\phi[\omega(t), v(t), z(t)]|^{p-1} |\eta(t)| |\partial_x \eta(t)| dx \lesssim \|e^{-a\langle x \rangle} \eta(t)\|_{H^1(\mathbb{R})}^2 \in L^1(\mathbb{R}_+). \quad \square \end{aligned}$$

4. Proof of Proposition 3.8

We will need the following.

Lemma 4.1. For $\mathbf{v} \in \{i\phi[\omega, v, z], \partial_\omega \phi[\omega, v, z], \partial_x \phi[\omega, v, z], ix\phi[\omega, v, z]\}$ we have

$$|\langle f(\phi[\omega, v, z] + \eta) - f(\phi[\omega, v, z]) - Df(\phi[\omega, v, z])\eta, \mathbf{v} \rangle| \lesssim \|\eta\|_{\Sigma}^2 + |z|^4. \quad (4.1)$$

Furthermore we have

$$|\langle f(\phi[\omega, v, z] + \eta) - f(\phi[\omega, v, z]) - Df(\phi[\omega, v, z])\eta, \partial_{z_j} \phi[\omega, v, z] \rangle| \lesssim \|\eta\|_{\Sigma}^p. \quad (4.2)$$

Proof. We start with the following.

Claim 4.2. For $\mathbf{v} \in \{i\phi[\omega, v, z], ix\phi[\omega, v, z]\}$ we have

$$|\langle f(\phi[\omega, v, z] + \eta) - f(\phi[\omega, v, z]) - Df(\phi[\omega, v, z])\eta, \mathbf{v} \rangle| \lesssim \|\eta\|_{\Sigma}^2. \quad (4.3)$$

Proof. We start here with $\mathbf{v} = i\phi[\omega, v, z]$. Notice that the constant i does not play a particular role. Set for $s \in (0, 1)$

$$\begin{aligned} \Omega'_{1,t,s} &= \{x \in \mathbb{R} \mid 2|s\eta(t, x)| \leq |\phi[\omega(t), v(t), z(t)]|\} \text{ and} \\ \Omega'_{2,t,s} &= \mathbb{R} \setminus \Omega'_{1,t,s} = \{x \in \mathbb{R} \mid 2|s\eta(t, x)| > |\phi[\omega(t), v(t), z(t)]|\} \end{aligned}$$

and split

$$\langle f(\phi[\omega, v, z] + \eta) - f(\phi[\omega, v, z]) - Df(\phi[\omega, v, z])\eta, i\phi[\omega, v, z] \rangle = I_1(t) + I_2(t) \text{ for}$$

$$I_j(t) = \int_0^1 ds \int_{\Omega'_{j,t,s}} \langle [Df(\phi[\omega, v, z] + s\eta) - Df(\phi[\omega, v, z])] \eta, i\phi[\omega, v, z] \rangle_{\mathbb{C}} dx.$$

We have

$$I_1(t) = \int_{[0,1]^2} d\tau ds s \int_{\Omega'_{1,t,s}} \left\langle D^2 f(\phi[\omega, v, z] + s\tau\eta)(\eta, \eta), i\phi[\omega, v, z] \right\rangle_{\mathbb{C}} dx$$

with

$$|I_1(t)| \lesssim \int_{[0,1]^2} d\tau ds \int_{\Omega'_{1,t,s}} |\phi[\omega, v, z]|^{p-1} |\eta|^2 dx \lesssim \|\eta\|_{\Sigma}^2.$$

We have the following, which completes the proof of (4.1) for $\mathbf{v} = i\phi[\omega, v, z]$,

$$|I_2(t)| \lesssim \int_{[0,1]} ds \int_{\Omega'_{2,t,s}} |\phi[\omega, v, z]| |\eta|^p dx \lesssim \int_{[0,1]} ds \int_{\Omega'_{2,t,s}} |\phi[\omega, v, z]|^{p-1} |\eta|^{2-p} |\eta|^p dx \leq \|\eta\|_{\Sigma}^2.$$

For $\mathbf{v} = ix\phi[\omega, v, z]$ the argument is similar. $\square$

We continue the proof of Lemma 4.1 considering the case $\mathbf{v} = \partial_x \phi[\omega, v, z] = \frac{i}{2} v \phi[\omega, v, z] + e^{\frac{i}{2} vx} \partial_x \phi[\omega, z]$. It is enough to focus on the following terms

$$\begin{aligned} & \int_0^1 ds \left\langle [Df(\phi[\omega, v, z] + s\eta) - Df(\phi[\omega, v, z])] \eta, e^{\frac{i}{2} vx} \partial_x \phi[\omega, z] \right\rangle_{L^2(\Omega''_{1,t})} \\ & + \int_0^1 ds \left\langle [Df(\phi[\omega, v, z] + s\eta) - Df(\phi[\omega, v, z])] \eta, e^{\frac{i}{2} vx} \partial_x \phi[\omega, z] \right\rangle_{L^2(\Omega''_{2,t})} := K_1 + K_2 \end{aligned}$$

in corresponding to a partition

$$\Omega''_{1,t} = \{x \in \mathbb{R} \mid |\phi_\omega| \gg |\tilde{\phi}[\omega, z]| \text{ and } |\partial_x \phi_\omega| \gg |\partial_x \tilde{\phi}[\omega, z]|\} \text{ and } \Omega''_{2,t} = \mathbb{R} \setminus \Omega''_{1,t}.$$

Since $|\partial_x \phi_\omega| \lesssim |\phi_\omega|$ for K_1 we can repeat the arguments in Claim 4.2 replacing in the inequalities $\partial_x \phi[\omega, z]$ with $\phi[\omega, z]$. So K_1 satisfies the same bounds in (4.3). Next, we split

$$K_2 = K_3 + K_4$$

in correspondence to a partition

$$\begin{aligned}\Omega''_{2,t} &= \Omega''_{3,t} \cup \Omega''_{4,t} \text{ with} \\ \Omega''_{3,t} &= \{x \in \Omega''_{2,t} \mid |\phi_\omega| \lesssim |\tilde{\phi}[\omega, z]| \text{ and } |\partial_x \phi_\omega| \gg |\partial_x \tilde{\phi}[\omega, z]|\} \text{ and} \\ \Omega''_{4,t} &= \{x \in \Omega''_{2,t} \mid |\partial_x \phi_\omega| \lesssim |\partial_x \tilde{\phi}[\omega, z]|\}.\end{aligned}$$

Then for K_3 we can bound $|\partial_x \phi[\omega, z]| \lesssim |\partial_x \phi_\omega| \lesssim |\phi_\omega| \lesssim |\tilde{\phi}[\omega, z]|$ and we have

$$|K_3| \leq \int_0^1 ds \int_{\Omega''_{3,t}} |Df(\phi[\omega, v, z] + s\eta) - Df(\phi[\omega, v, z])| |\eta| |\tilde{\phi}[\omega, z]| dx. \quad (4.4)$$

Claim 4.3. $|K_3| \lesssim \|\eta\|_{\Sigma}^2 + |z|^4$.

Proof. We split

$$\Omega''_{1,3,t,s} = \{x \in \Omega''_{3,t} : |s\eta| \gtrsim |\phi[\omega, z]|\} \text{ and } \Omega''_{2,3,t,s} = \{x \in \Omega''_{3,t} : |s\eta| \ll |\phi[\omega, z]|\}.$$

Then we split $K_3 = K_{31} + K_{32}$

$$|K_{31}| \lesssim \int_{[0,1]} ds \int_{\Omega''_{1,3,t,s}} |\eta|^p |\tilde{\phi}[\omega, z]| ds \lesssim \int_{\mathbb{R}} e^{-|x|\sqrt{\omega}\sqrt{1-\lambda(\overline{p})}} (|z|^4 + |\eta|^{p\frac{4}{3}}) \lesssim \|\eta\|_{\Sigma}^2 + |z|^4$$

since we have $p > 3/2$. Next we have

$$\begin{aligned}|K_{32}| &\lesssim \int_{[0,1]^2} ds d\tau \int_{\Omega''_{1,3,t,s}} \left| D^2 f(\phi[\omega, v, z] + \tau s\eta) \right| |s\eta| |\eta| |\tilde{\phi}[\omega, z]| dx \\ &\lesssim \int_{[0,1]^2} ds d\tau \int_{\Omega''_{1,3,t,s}} |\phi[\omega, z]|^{p-2} |s\eta|^{p-1+2-p} |\eta| |\tilde{\phi}[\omega, z]| dx \\ &\lesssim \int_{\mathbb{R}} |\eta|^p |\tilde{\phi}[\omega, z]| \lesssim \|\eta\|_{\Sigma}^2 + |z|^4. \quad \square\end{aligned}$$

Now we consider K_4 . We have

$$|K_4| \leq \int_0^1 ds \int_{\Omega''_{4,t}} |Df(\phi[\omega, v, z] + s\eta) - Df(\phi[\omega, v, z])| |\eta| |\partial_x \tilde{\phi}[\omega, z]| dx.$$

Splitting

$$\Omega''_{1,4,t,s} = \{x \in \Omega''_{4,t} : |s\eta| \gtrsim |\phi[\omega, z]|\} \text{ and } \Omega''_{2,4,t,s} = \{x \in \Omega''_{4,t} : |s\eta| \ll |\phi[\omega, z]|\}$$

then the proof of $|K_4| \lesssim \|\eta\|_{\Sigma}^2 + |z|^4$ is similar to the proof of Claim 4.3.

Next, continuing the proof of Lemma 4.1 we consider case

$$\mathbf{v} = e^{\frac{i}{2}vx} \partial_{\omega} \left(\omega^{\frac{1}{p-1}} \phi[z](\sqrt{\omega}x) \right) = e^{\frac{i}{2}vx} \omega^{\frac{2-p}{p-1}} (\Lambda_p \phi[z])(\sqrt{\omega}x).$$

By the definition of Λ_p in (2.1) we are reduced to the previous cases.

Finally, we look at the proof of (4.2). For $\mathbf{v} = \partial_{z,j} \phi[\omega, v, 0]$ We need to bound

$$\sum_{j=1,2} \int_0^1 ds \langle (Df(\phi[\omega, v, z] + s\eta) - Df(\phi[\omega, v, z])) \eta, \mathbf{v} \rangle_{L^2(\Omega'_{j,t,s})} =: L_1 + L_2.$$

Then

$$\begin{aligned} |L_1(t)| &\lesssim \int_{[0,1]^2} d\tau ds s \int_{\Omega'_{1,t,s}} \left| \left\langle D^2 f(\phi[\omega, v, z] + s\tau\eta)(\eta, \eta), \mathbf{v} \right\rangle_{\mathbb{C}} \right| dx \\ &\lesssim \int_{[0,1]^2} d\tau ds s \int_{\Omega'_{1,t,s}} |\phi[\omega, z]|^{p-2} |s\eta|^{2-p+p-1} |\eta| e^{-|x|\sqrt{\omega}\sqrt{1-\lambda(p)}} dx \\ &\lesssim \int_{\mathbb{R}} |\eta|^p e^{-|x|\sqrt{\omega}\sqrt{1-\lambda(p)}} dx \lesssim \|\eta\|_{\Sigma}^p. \end{aligned}$$

Finally we have the following, which completes the proof of (4.2),

$$|L_2(t)| \lesssim \int_{[0,1]} ds \int_{\Omega'_{2,t,s}} |s\eta|^{p-1} |\eta| |\mathbf{v}| dx \lesssim \|\eta\|_{\Sigma}^p. \quad \square$$

Lemma 4.4. *We have the estimates*

$$|\dot{\vartheta} - \widetilde{\vartheta}| + |\dot{\omega} - \widetilde{\omega}| + |\dot{D} - v| + |\dot{v}| \lesssim |z|^4 + \|\eta\|_{\Sigma}^2 + \|\eta\|_{\Sigma_A}^2, \quad (4.5)$$

$$|\dot{z} - \widetilde{z}| \lesssim |z| \|\eta\|_{\Sigma} + \|\eta\|_{\Sigma}^p + \|\eta\|_{\Sigma_A}^2 + |z|^4. \quad (4.6)$$

Proof. Applying $\langle \cdot, ie^{-i\vartheta+D\partial_x} D_{\Theta} \phi[\Theta] \Theta \rangle$ with $\Theta \in \mathbb{R}^4 \times \mathbb{C}$ to (3.37) we have

$$\left\langle \dot{\eta}, ie^{-i\vartheta+D\partial_x} D_{\Theta} \phi[\Theta] \Theta \right\rangle + \left\langle (i\dot{\vartheta} - \dot{D}\partial_x) \eta, ie^{-i\vartheta+D\partial_x} D_{\Theta} \phi[\Theta] \Theta \right\rangle \quad (4.7)$$

$$+ \left\langle D_{\Theta} \phi[\Theta](\dot{\Theta} - \widetilde{\Theta}), iD_{\Theta} \phi[\Theta] \Theta \right\rangle \quad (4.8)$$

$$\begin{aligned} &= \left\langle \eta, \left(\partial_x^2 + Df(\phi[\omega, v, z]) \right) e^{-i\vartheta+D\partial_x} D_{\Theta} \phi[\Theta] \Theta \right\rangle \\ &- \left\langle f(\phi[\omega, v, z] + \eta) - f(\phi[\omega, v, z]) - Df(\phi[\omega, v, z])\eta, e^{-i\vartheta+D\partial_x} D_{\Theta} \phi[\Theta] \Theta \right\rangle \end{aligned} \quad (4.9)$$

where the contribution of $R[\omega, v, z]$ cancels by (3.22). Now we have

$$\left\langle \dot{\eta}, ie^{-i\vartheta+D\partial_x} D_{\Theta}\phi[\Theta]\Theta \right\rangle = - \left\langle \eta, ie^{-i\vartheta+D\partial_x} D_{\Theta}^2\phi[\Theta](\Theta, \dot{\Theta}) \right\rangle \quad (4.10)$$

$$+ \left\langle \eta, i(\dot{\vartheta} - \dot{D}\partial_x) e^{-i\vartheta+D\partial_x} D_{\Theta}\phi[\Theta]\Theta \right\rangle. \quad (4.11)$$

Setting also $R[\Theta] = e^{i\vartheta-D\partial_x} R[\omega, v, z]$, notice that equation (3.20) can be written as

$$\partial_x^2 \phi[\Theta] + f(\phi[\Theta]) + iD_{\Theta}\phi[\Theta]\tilde{\Theta} = R[\Theta].$$

Differentiating in Θ , we obtain

$$\left(\partial_x^2 + Df(\phi[\Theta]) \right) D_{\Theta}\phi[\Theta]\Theta + iD_{\Theta}^2\phi[\Theta](\tilde{\Theta}, \Theta) + iD_{\Theta}\phi[\Theta]D_{\Theta}\tilde{\Theta}\Theta = D_{\Theta}R[\Theta]\Theta.$$

Since $f(\phi[\Theta]) = e^{i\vartheta-D\partial_x} f(\phi[\omega, v, z])$ yields $e^{-i\vartheta+D\partial_x} Df(\phi[\Theta])X = Df(\phi[\omega, v, z])e^{-i\vartheta+D\partial_x} X$, we get

$$\begin{aligned} \left\langle \eta, \left(\partial_x^2 + Df(\phi[\omega, v, z]) \right) e^{-i\vartheta+D\partial_x} D_{\Theta}\phi[\Theta]\Theta \right\rangle &= \left\langle e^{i\vartheta-D\partial_x} \eta, \left(\partial_x^2 + Df(\phi[\Theta]) \right) D_{\Theta}\phi[\Theta]\Theta \right\rangle \\ &= - \left\langle e^{i\vartheta-D\partial_x} \eta, iD_{\Theta}^2\phi[\Theta](\tilde{\Theta}, \Theta) + \cancel{iD_{\Theta}\phi[\Theta]D_{\Theta}\tilde{\Theta}\Theta} - D_{\Theta}R[\Theta]\Theta \right\rangle \end{aligned}$$

where the cancellation follows by the modulation orthogonality (3.24). Entering this and (4.10) in (4.7)–(4.9) and with the cancellation of (4.11) with the second term in line (4.7) we obtain

$$\begin{aligned} &\left\langle D_{\Theta}\phi[\Theta](\dot{\Theta} - \tilde{\Theta}), iD_{\Theta}\phi[\Theta]\Theta \right\rangle - \left\langle \eta, ie^{-i\vartheta+D\partial_x} D_{\Theta}^2\phi[\Theta](\Theta, \dot{\Theta} - \tilde{\Theta}) \right\rangle \quad (4.12) \\ &\quad - \left\langle \eta, ie^{-i\vartheta+D\partial_x} \cancel{D_{\Theta}^2\phi[\Theta](\Theta, \tilde{\Theta})} \right\rangle = - \left\langle e^{i\vartheta-D\partial_x} \eta, iD_{\Theta}^2\phi[\Theta](\tilde{\Theta}, \Theta) \right\rangle + \left\langle e^{i\vartheta-D\partial_x} \eta, D_{\Theta}R[\Theta]\Theta \right\rangle \\ &\quad - \left\langle f(\phi[\omega, v, z] + \eta) - f(\phi[\omega, v, z]) - Df(\phi[\omega, v, z])\eta, e^{-i\vartheta+D\partial_x} D_{\Theta}\phi[\Theta]\Theta \right\rangle \end{aligned}$$

where the two canceled terms cancel each other. From this we get

$$\begin{aligned} &-(\dot{\omega} - \tilde{\omega}) \langle \partial_{\omega}\phi[\omega, v, z], \phi[\omega, v, z] \rangle + (\dot{\vartheta} - \tilde{\vartheta}) \langle \phi[\omega, v, z], \cancel{i\phi[\omega, v, z]} \rangle \\ &+ (\dot{D} - \tilde{D}) \langle \partial_x \phi[\omega, v, z], \phi[\omega, v, z] \rangle - (\dot{v} - \tilde{v}) \left\langle \frac{x}{2} \phi[\omega, v, z], \cancel{\phi[\omega, v, z]} \right\rangle \\ &- \langle D_z \phi[\omega, v, z](\dot{z} - \tilde{z}), \phi[\omega, v, z] \rangle + O(\|\eta\|_{\Sigma} |\dot{\Theta} - \tilde{\Theta}|) \\ &= \langle \eta, iR[\omega, v, z] \rangle - \langle f(\phi[\omega, v, z] + \eta) - f(\phi[\omega, v, z]) - Df(\phi[\omega, v, z])\eta, i\phi[\omega, v, z] \rangle \end{aligned}$$

where the canceled terms are null for elementary reasons related also for various cancellations in the sequel of this lemma also to the fact that by (1.3) and Lemma 2.1 the function $\phi[z]$ is even in x . From (3.6) and (4.1) we obtain

$$(\dot{\omega} - \tilde{\omega})q'(\omega) + O(|z| + \|\eta\|_{\Sigma})|\dot{\Theta} - \tilde{\Theta}| = O(\|\eta\|_{\Sigma}|z|^2) + O(\|\eta\|_{\Sigma}^2). \quad (4.13)$$

From (4.12) we obtain

$$\begin{aligned}
 & (\dot{\vartheta} - \tilde{\vartheta}) \langle \phi[\omega, v, z], \partial_\omega \phi[\omega, v, z] \rangle + (\dot{\omega} - \tilde{\omega}) \langle \partial_\omega \phi[\omega, v, z], \overline{i \partial_\omega \phi[\omega, v, z]} \rangle \\
 & - (\dot{D} - \tilde{D}) \langle \partial_x \phi[\omega, v, z], i \partial_\omega \phi[\omega, v, z] \rangle + (\dot{v} - \tilde{v}) \langle \overline{i \frac{x}{2} \phi[\omega, v, z]}, i \partial_\omega \phi[\omega, v, z] \rangle \\
 & + \langle D_z \phi[\omega, v, z] (\dot{z} - \tilde{z}), i \partial_\omega \phi[\omega, v, z] \rangle + O(\|\eta\|_{\Sigma} |\dot{\Theta} - \tilde{\Theta}|) \\
 & = \langle \eta, \partial_\omega R[\omega, v, z] \rangle - \langle f(\phi[\omega, v, z] + \eta) - f(\phi[\omega, v, z]) - Df(\phi[\omega, v, z])\eta, \partial_\omega \phi[\omega, v, z] \rangle
 \end{aligned}$$

where the canceled terms are zero by elementary reasons. With the canceled term null, we have

$$\begin{aligned}
 \langle \partial_x \phi[\omega, v, z], i \partial_\omega \phi[\omega, v, z] \rangle &= \left\langle \partial_x \left(e^{\frac{i}{2} vx} \phi[\omega, z] \right), i e^{\frac{i}{2} vx} \partial_\omega \phi[\omega, z] \right\rangle \\
 &= \langle \overline{\partial_x \phi[\omega, z]}, \overline{i \partial_\omega \phi[\omega, z]} \rangle + \frac{v}{2} \langle \phi[\omega, z], \partial_\omega \phi[\omega, z] \rangle = \frac{v}{2} (\mathbf{q}'(\omega) + O(|z|)).
 \end{aligned}$$

We have

$$\begin{aligned}
 \langle \eta, \partial_\omega R[\omega, v, z] \rangle &= \omega^{\frac{1}{p-1}} \left\langle e^{\frac{i}{2} vx} \eta, \left[\left(\frac{p}{p-1} + \frac{x}{2} \partial_x \right) R[z] \right] (\sqrt{\omega x}) \right\rangle \\
 &= \omega^{\frac{1}{p-1} + \frac{1}{2}} \left\langle \left(\frac{p}{p-1} + \frac{x}{2} \partial_x \right)^* \left(\left(e^{\frac{i}{2} vx} \eta \right) \left(\omega^{-\frac{1}{2} x} \right) \right), R[z] \right\rangle
 \end{aligned}$$

so that it is elementary to conclude that

$$|\langle \eta, \partial_\omega R[\omega, v, z] \rangle| \lesssim (\|\eta\|_{\Sigma_A} + \|\eta\|_{\Sigma}) |z|^2.$$

We obtain

$$\begin{aligned}
 & \left((\dot{\vartheta} - \tilde{\vartheta}) - \frac{v}{2} (\dot{D} - \tilde{D}) \right) \mathbf{q}'(\omega) + O((|z| + \|\eta\|_{\Sigma}) |\dot{\Theta} - \tilde{\Theta}|) \\
 & = O(\|\eta\|_{\Sigma}^2 + |z|^4).
 \end{aligned} \tag{4.14}$$

From (4.12) we have

$$\begin{aligned}
 & (\dot{\vartheta} - \tilde{\vartheta}) \langle \phi[\omega, v, z], \overline{\partial_x \phi[\omega, v, z]} \rangle + (\dot{\omega} - \tilde{\omega}) \langle \partial_\omega \phi[\omega, v, z], i \partial_x \phi[\omega, v, z] \rangle \\
 & - (\dot{D} - \tilde{D}) \langle \overline{\partial_x \phi[\omega, v, z]}, i \partial_\omega \phi[\omega, v, z] \rangle - (\dot{v} - \tilde{v}) \left(\frac{\mathbf{q}(\omega)}{2} + O(z) \right) \\
 & + \langle D_z \phi[\omega, v, z] (\dot{z} - \tilde{z}), i \partial_\omega \phi[\omega, v, z] \rangle + O(\|\eta\|_{\Sigma} |\dot{\Theta} - \tilde{\Theta}|) \\
 & = \langle \eta, \partial_x R[\omega, v, z] \rangle - \langle f(\phi[\omega, v, z] + \eta) - f(\phi[\omega, v, z]) - Df(\phi[\omega, v, z])\eta, \partial_x \phi[\omega, v, z] \rangle,
 \end{aligned}$$

from which we derive

$$\begin{aligned}
& -\frac{v}{2}(\dot{\omega} - \tilde{\omega})\mathbf{q}'(\omega) - (\dot{v} - \tilde{v})\frac{\mathbf{q}(\omega)}{2} \\
& + O\left((|z| + \|\eta\|_{\tilde{\Sigma}})|\dot{\Theta} - \tilde{\Theta}|\right) = O\left(\|\eta\|_{\tilde{\Sigma}_A}^2 + \|\eta\|_{\tilde{\Sigma}}^2 + |z|^4\right).
\end{aligned} \tag{4.15}$$

From (4.12) we have

$$\begin{aligned}
& -(\dot{v} - \tilde{v})\langle \dot{\mathbf{i}}\phi[\omega, v, z], \cancel{x\phi[\omega, v, z]} \rangle - (\dot{\omega} - \tilde{\omega})\langle \partial_{\omega}\phi[\omega, v, z], \cancel{x\phi[\omega, v, z]} \rangle \\
& + (\dot{D} - \tilde{D})\langle \partial_x\phi[\omega, v, z], x\phi[\omega, v, z] \rangle - (\dot{v} - \tilde{v})\left\langle \frac{\dot{\mathbf{i}}}{2}x\phi[\omega, v, z], \cancel{x\phi[\omega, v, z]} \right\rangle \\
& - \langle D_z\phi[\omega, v, z](\dot{z} - \tilde{z}), x\phi[\omega, v, z] \rangle + O\left(\|\eta\|_{\tilde{\Sigma}}|\dot{\Theta} - \tilde{\Theta}|\right) \\
& = \langle \eta, \partial_v R[\omega, v, z] \rangle - \langle f(\phi[\omega, v, z] + \eta) - f(\phi[\omega, v, z]) - Df(\phi[\omega, v, z])\eta, x\phi[\omega, v, z] \rangle
\end{aligned}$$

where the canceled terms are null for elementary reasons. This yields

$$(\dot{D} - \tilde{D})\frac{q(\omega)}{2} = O\left((|z| + \|\eta\|_{\tilde{\Sigma}})|\dot{\Theta} - \tilde{\Theta}|\right) + O\left(\|\eta\|_{\tilde{\Sigma}_A}^2 + \|\eta\|_{\tilde{\Sigma}}^2 + |z|^4\right) \tag{4.16}$$

where we used

$$\langle \eta, \partial_v R[\omega, v, z] \rangle = \left\langle \eta, \frac{\dot{\mathbf{i}}}{2}x e^{\frac{\dot{\mathbf{i}}}{2}xv} R[\omega, z] \right\rangle = O\left(\|\eta\|_{\tilde{\Sigma}}|z|^2\right)$$

and Lemma 4.1. From (4.12) we have

$$\begin{aligned}
& \langle D_z\phi[\omega, v, z](\dot{z} - \tilde{z}), \dot{\mathbf{i}}\partial_{z_j}\phi[\omega, v, z] \rangle + O\left((|z| + \|\eta\|_{\tilde{\Sigma}})|\dot{\Theta} - \tilde{\Theta}|\right) \\
& = \langle \eta, \partial_{z_j} R[\omega, v, z] \rangle \\
& - \langle f(\phi[\omega, v, z] + \eta) - f(\phi[\omega, v, z]) - Df(\phi[\omega, v, z])\eta, \partial_{z_j}\phi[\omega, v, z] \rangle.
\end{aligned}$$

Claim 4.5. *We have*

$$|\langle \eta, \partial_{z_j} R[\omega, v, z] \rangle| \lesssim \|\eta\|_{\tilde{\Sigma}} |z|. \tag{4.17}$$

Proof. From (3.20) we can write

$$\begin{aligned}
R[\omega, v, z] &= \widehat{R}[\omega, v, z] - \tilde{\vartheta}\mathcal{R}[\omega, v, z]\phi[\omega, v, z] + \dot{\mathbf{i}}\tilde{\omega}\mathcal{R}[\omega, v, z]\partial_{\omega}\phi[\omega, v, z] \\
&+ \dot{\mathbf{i}}D_z\phi[\omega, v, z]\tilde{z}\mathcal{R}[\omega, v, z]
\end{aligned}$$

where $\widehat{R}[\omega, v, z] = e^{\frac{\dot{\mathbf{i}}}{2}vx}\omega^{\frac{p}{p-1}}\widehat{R}[z](\sqrt{\psi}x)$. Notice that by elementary computations we have

$$\widehat{R}[\omega, v, z] = f(\phi[\omega, v] + \tilde{\phi}[\omega, v, z]) - f(\phi[\omega, v]) - Df(\phi[\omega, v])\tilde{\phi}[\omega, v, z]$$

where $\phi[\omega, v] := e^{\frac{\dot{\mathbf{i}}}{2}vx}\phi_{\omega}$. By Proposition 3.1 we have

$$\begin{aligned} & |\langle \eta, \partial_{z_j} (-\tilde{\vartheta} \mathcal{R}[\omega, v, z] \phi[\omega, v, z] + i\tilde{\omega} \mathcal{R}[\omega, v, z] \partial_\omega \phi[\omega, v, z] + iD_z \phi[\omega, v, z] \tilde{z} \mathcal{R}[\omega, v, z]) \rangle| \\ & \lesssim \|\eta\|_{\tilde{\Sigma}} |z|. \end{aligned}$$

Finally, we have

$$\begin{aligned} & |\langle \eta, \partial_{z_j} \widehat{R}[\omega, v, z] \rangle| = |\langle \eta, \partial_{z_j} (f(\phi[\omega, v]) + \tilde{\phi}[\omega, v, z]) - f(\phi[\omega, v]) - Df(\phi[\omega, v]) \tilde{\phi}[\omega, v, z] \rangle| \\ & = |\langle \eta, (Df(\phi[\omega, v]) + \tilde{\phi}[\omega, v, z]) - Df(\phi[\omega, v]) \partial_{z_j} \tilde{\phi}[\omega, v, z] \rangle| \\ & \lesssim \int_{\mathbb{R}} |\eta| |z| e^{|x|(2-p-2\sqrt{1-\lambda(p)})} \lesssim \|\eta\|_{\tilde{\Sigma}} |z|. \quad \square \end{aligned}$$

From Claim 4.5 and from (4.2) we derive

$$\begin{aligned} & \langle D_z \phi[\omega, v, z] (\dot{z} - \tilde{z}), i\partial_{z_j} \phi[\omega, v, z] \rangle + O((|z| + \|\eta\|_{\tilde{\Sigma}}) |\dot{\Theta} - \tilde{\Theta}|) \\ & = O(\|\eta\|_{\tilde{\Sigma}} |z|) + O\left(\|\eta\|_{\tilde{\Sigma}}^{\frac{p}{2}}\right). \end{aligned}$$

From (3.15) the above implies

$$|\dot{z} - \tilde{z}| \lesssim O((|z| + \|\eta\|_{\tilde{\Sigma}}) |\dot{\Theta} - \tilde{\Theta}|) + O(\|\eta\|_{\tilde{\Sigma}} |z|) + O\left(\|\eta\|_{\tilde{\Sigma}}^{\frac{p}{2}}\right) \quad (4.18)$$

and so also

$$|z| |\dot{z} - \tilde{z}| \lesssim O(|z| (|z| + \|\eta\|_{\tilde{\Sigma}}) |\dot{\Theta} - \tilde{\Theta}|) + O\left(\|\eta\|_{\tilde{\Sigma}}^2 + |z|^4\right). \quad (4.19)$$

Then from (4.14), (4.13), (4.15), (4.16) and (4.19) we derive

$$|\dot{\vartheta} - \tilde{\vartheta}| + |\dot{\omega} - \tilde{\omega}| + |\dot{D} - \tilde{D}| + |\dot{v} - \tilde{v}| + |z| |\dot{z} - \tilde{z}| \lesssim |z|^4 + \|\eta\|_{\tilde{\Sigma}}^2 + \|\eta\|_{\tilde{\Sigma}_A}^2$$

which yields (4.5) and entered in (4.18) yields (4.6). $\square$

Proof of Proposition 3.8. Lemma 4.4 and (3.30) imply immediately (3.31)–(3.32). Entering this, (3.4) and (3.29) in (4.18) we obtain (3.33).

5. The Fermi Golden Rule: proof of Proposition 3.9

Like in Kowalczyk and Martel [33] and Martel [50] and exactly like in [18] we consider the functional

$$\mathcal{J}_{\text{FGR}} := \left\langle J\eta, \chi_A \left(z^2 g^{(\omega)} + \bar{z}^2 \bar{g}^{(\omega)} \right) \right\rangle, \quad (5.1)$$

with a nonzero $g^{(\omega)} \in L^\infty(\mathbb{R}, \mathbb{C}^2)$ satisfying

$$\mathcal{L}_\omega g^{(\omega)} = 2i\lambda(p, \omega) g^{(\omega)} \quad (5.2)$$

where $\lambda(p, \omega) = \omega\lambda(p)$. Here $g^{(\omega)}(x) := g(\sqrt{\omega}x)$ where g solves (5.2) for $\omega = 1$. That $g^{(\omega)}$ exists is known since Krieger and Schlag [37], see Lemma 6.3, or earlier Buslaev and Perelman [3].

We define the FGR constant $\gamma(p)$, for $g = (g_1, g_2)^\top$, by

$$\gamma(p) := \left\langle \phi^{p-2} \left(p\xi_1^2 + \xi_2^2 \right), g_1 \right\rangle + 2 \left\langle \phi^{p-2} \xi_1 \xi_2, g_2 \right\rangle. \quad (5.3)$$

Here notice that we will prove Proposition 3.9 later in section 8. In the next lemma we will need the following reformulation of equation (3.37), where we identify $J = -i$,

$$\begin{aligned} \dot{\eta} = & \mathcal{L}_\omega \eta - J(\tilde{\vartheta}_{\mathcal{R}} + \tilde{\vartheta} - \dot{\vartheta})\eta + \dot{D}\partial_x \eta - e^{J\vartheta + D\partial_x} D_\Theta \phi[\Theta](\dot{\Theta} - \tilde{\Theta}) \\ & + J(Df(\phi[\omega, v, z]) - Df(\phi_\omega))\eta \\ & - J(f(\phi[\omega, v, z] + \eta) - f(\phi[\omega, v, z]) - Df(\phi[\omega, v, z])\eta) - JR[\omega, v, z]. \end{aligned} \quad (5.4)$$

We have the following.

Lemma 5.1. *There is a C^1 in time function $\mathcal{I}_{\text{FGR}}$, which satisfies $|\mathcal{I}_{\text{FGR}}| \lesssim \sqrt{\delta}$ such that*

$$\left| \dot{\mathcal{J}}_{\text{FGR}} + \dot{\mathcal{I}}_{\text{FGR}} - \omega^{\frac{p+1}{2(p-1)}} \gamma(p) |z|^4 \right| \lesssim A^{-1} \left(|z|^4 + \|\eta\|_{\Sigma_A}^2 + \|\eta\|_{\Sigma}^2 \right). \quad (5.5)$$

Proof. Differentiating $\mathcal{J}_{\text{FGR}}$, we have

$$\begin{aligned} \dot{\mathcal{J}}_{\text{FGR}} = & \left\langle J\dot{\eta}, \chi_A \left(z^2 g^{(\omega)} + \bar{z}^2 \bar{g}^{(\omega)} \right) \right\rangle + \left\langle J\eta, \chi_A \left(2z\tilde{z}g^{(\omega)} + 2\bar{z}\tilde{\bar{z}}\bar{g}^{(\omega)} \right) \right\rangle \\ & + \left\langle J\eta, \chi_A \left(2z(\dot{z} - \tilde{z})g^{(\omega)} + 2\bar{z}(\dot{\bar{z}} - \tilde{\bar{z}})\bar{g}^{(\omega)} \right) \right\rangle \\ & + \left\langle J\eta, \chi_A \left(z^2 \partial_\omega g^{(\omega)} + \bar{z}^2 \partial_\omega \bar{g}^{(\omega)} \right) \right\rangle (\dot{\omega} - \tilde{\omega}) + \left\langle J\eta, \chi_A \left(z^2 \partial_\omega g^{(\omega)} + \bar{z}^2 \partial_\omega \bar{g}^{(\omega)} \right) \right\rangle \tilde{\omega} \\ =: & A_1 + A_2 + A_3 + A_4 + A_5. \end{aligned}$$

We consider first the last three terms, the simplest ones. By (4.12) we have

$$\begin{aligned} |A_3| & \lesssim \|\eta\chi_A\|_{L^1} |z| |\dot{z} - \tilde{z}| \lesssim A^{\frac{3}{2}} A^{-1} \|\text{sech}\left(\frac{2}{A}x\right)\eta\|_{L^2} |\dot{z} - \tilde{z}| \\ & \lesssim A^{\frac{1}{2}} \delta \left(\|\eta\|_{\Sigma_A}^2 + \|\eta\|_{\Sigma}^2 + |z|^4 \right). \end{aligned}$$

Since $\|\partial_\omega g^{(\omega)}\|_{L^\infty} \lesssim \langle x \rangle$, using also (4.5) we have

$$|A_4| \lesssim A \|\eta\chi_A\|_{L^1} |z|^4 |\dot{\omega} - \tilde{\omega}| \lesssim A^{\frac{5}{2}} |z|^4 \|\eta\|_{\Sigma_A} |\dot{\omega} - \tilde{\omega}| \lesssim \delta A^{\frac{5}{2}} |z|^4.$$

Finally, using (3.4) we have

$$|A_5| \lesssim A \|\eta\chi_A\|_{L^1} |z|^2 |\tilde{\omega}| \lesssim A^{\frac{5}{2}} |z|^4 \|\eta\|_{\Sigma_A} \lesssim \delta A^{\frac{5}{2}} |z|^4.$$

Turning to the main terms, we have

$$A_2 = \omega \left\langle J\eta, 2i\lambda(p)\chi_A \left(z^2 g^{(\omega)} - \bar{z}^2 \bar{g}^{(\omega)} \right) \right\rangle + \left\langle J\eta, \chi_A \left(2z\tilde{z}\mathcal{R}g^{(\omega)} + 2\bar{z}\tilde{z}\mathcal{R}\bar{g}^{(\omega)} \right) \right\rangle = A_{21} + A_{22}.$$

By (3.4) proceeding like for A_3 ,

$$|A_{22}| \lesssim \|\eta\chi_A\|_{L^1} |z|^3 \lesssim \delta A^{\frac{3}{2}} \|\eta\|_{\Sigma_A} |z|^2 \lesssim \delta A^{\frac{3}{2}} \left(\|\eta\|_{\Sigma_A}^2 + |z|^4 \right).$$

By (5.4) and by (5.2) for the cancellation, we have

$$\begin{aligned} A_1 + A_{21} = & - \left\langle J\eta, \chi_A \mathcal{L}_\omega \left(z^2 g^{(\omega)} + \bar{z}^2 \bar{g}^{(\omega)} \right) \right\rangle + \cancel{A_{21}} + (\tilde{\vartheta}_{\mathcal{R}} + \tilde{\vartheta} - \dot{\vartheta}) \left\langle \eta, \chi_A (z^2 g^{(\omega)} + \bar{z}^2 \bar{g}^{(\omega)}) \right\rangle \\ & + \dot{D} \left\langle \partial_x \eta, \chi_A (z^2 g^{(\omega)} + \bar{z}^2 \bar{g}^{(\omega)}) \right\rangle - \left\langle J e^{J\tilde{\vartheta} + D\partial_x} D_\Theta \phi[\Theta](\dot{\Theta} - \tilde{\Theta}), \chi_A (z^2 g^{(\omega)} + \bar{z}^2 \bar{g}^{(\omega)}) \right\rangle \\ & + \left\langle (2\chi'_A \partial_x + \chi''_A) \eta, z^2 g^{(\omega)} + \bar{z}^2 \bar{g}^{(\omega)} \right\rangle + \left\langle (Df(\phi[\omega, v, z]) - Df(\phi[\omega, v])) \eta, \chi_A (z^2 g^{(\omega)} + \bar{z}^2 \bar{g}^{(\omega)}) \right\rangle \\ & + \left\langle f(\phi[\omega, v, z] + \eta) - f(\phi[\omega, v, z]) - Df(\phi[\omega, v]) \eta, \chi_A (z^2 g^{(\omega)} + \bar{z}^2 \bar{g}^{(\omega)}) \right\rangle \\ & + \left\langle R[\omega, v, z], \chi_A (z^2 g^{(\omega)} + \bar{z}^2 \bar{g}^{(\omega)}) \right\rangle = A_{11} + A_{12} + A_{13} + A_{14} + A_{15} + A_{16}. \end{aligned}$$

It is easy to see the following

$$\sum_{j=1}^3 |A_{1j}| \lesssim \delta A^{\frac{3}{2}} \left(|z|^4 + \|\eta\|_{\Sigma_A}^2 + \|\eta\|_{\Sigma}^2 \right).$$

We have

$$\begin{aligned} A_{14} = & \left\langle (Df(\phi[\omega, v, z]) - Df(\phi[\omega, v])) \eta, \chi_A (z^2 g^{(\omega)} + \bar{z}^2 \bar{g}^{(\omega)}) \right\rangle \\ & + \left\langle (Df(\phi[\omega, v]) - Df(\phi_\omega)) \eta, \chi_A (z^2 g^{(\omega)} + \bar{z}^2 \bar{g}^{(\omega)}) \right\rangle =: A_{141} + A_{142} \end{aligned}$$

On the support of χ_A we can assume $|\phi[\omega, v]| \gg |\tilde{\phi}[\omega, v, z]|$. We can write

$$A_{141} = \int_0^1 \left\langle D^2 f(\phi[\omega, v] + s\tilde{\phi}[\omega, v, z]) (\tilde{\phi}[\omega, v, z], \eta), \chi_A (z^2 g^{(\omega)} + \bar{z}^2 \bar{g}^{(\omega)}) \right\rangle$$

with

$$|A_{141}| \lesssim |z|^2 \left\langle \phi_\omega^{p-2} |\tilde{\phi}[\omega, v, z]| |\eta|, \chi_A \right\rangle \lesssim |z|^3 \int_{\mathbb{R}} e^{|x|\sqrt{\omega}(2-p-\sqrt{1-\lambda(p)})} |\eta| dx \lesssim |z|^3 \|\eta\|_{\Sigma}.$$

We have

$$A_{142} = \left\langle \left(e^{\frac{i}{2}vx} Df(\phi_\omega) e^{-\frac{i}{2}vx} - Df(\phi_\omega) \right) \eta, \chi_A(z^2 g^{(\omega)} + \bar{z}^2 \bar{g}) \right\rangle.$$

So from this we get

$$|A_{142}| \lesssim |z|^2 |v| \left\langle |x| \phi_\omega^{p-1}, |\eta| \right\rangle \lesssim A^{3/2} |v| |z|^2 \|\eta\|_{\tilde{\Sigma}} \lesssim \delta A^{3/2} \left(|z|^4 + \|\eta\|_{\tilde{\Sigma}}^2 \right).$$

Since we can assume $|\phi[\omega, v]| \gg |\tilde{\phi}[\omega, v, z]| + |\eta|$ in $\text{supp } \chi_A$ we write

$$A_{15} = \int_{[0,1]^2} \left\langle D^2 f(\phi[\omega, v] + \tau(\tilde{\phi}[\omega, v, z] + s\eta)) (\eta, \tilde{\phi}[\omega, v, z] + s\eta), \chi_A(z^2 g^{(\omega)} + \bar{z}^2 \bar{g}^{(\omega)}) \right\rangle ds d\tau$$

so that

$$\begin{aligned} |A_{15}| &\lesssim |z|^2 \int_{\mathbb{R}} \chi_A |\phi_\omega|^{p-2} |\eta| (|\tilde{\phi}[\omega, v, z]| + |\eta|) dx \\ &\lesssim |z|^3 \int_{\mathbb{R}} e^{|x|\sqrt{\omega}(2-p-\sqrt{1-\lambda(p)})} |\eta| dx + |z|^2 e^{2A(2-p)} A^2 \|\eta\|_{\Sigma_A}^2 \end{aligned}$$

We have

$$\begin{aligned} A_{16} &= \left\langle R[\omega, z], \chi_A(z^2 g^{(\omega)} + \bar{z}^2 \bar{g}^{(\omega)}) \right\rangle \\ &\quad + \left\langle \left(e^{\frac{i}{2}vx} - 1 \right) R[\omega, z], \chi_A(z^2 g^{(\omega)} + \bar{z}^2 \bar{g}^{(\omega)}) \right\rangle =: A_{161} + A_{162}. \end{aligned}$$

Then we have

$$|A_{162}| \lesssim |v| |A| |z|^2 \|R[\omega, z]\|_{L^1(\mathbb{R})} \lesssim |v| |A| |z|^4 \lesssim \delta A |z|^4.$$

By (3.18) and the line below (5.2) we have

$$A_{161} = \omega^{\frac{p+1}{2(p-1)}} \left\langle R[z], \chi_{A\sqrt{\omega}}(z^2 g + \bar{z}^2 \bar{g}) \right\rangle. \quad (5.6)$$

Omitting the harmless factor $\omega^{\frac{p+1}{2(p-1)}}$ we have

$$\begin{aligned} A_{161} &= \left\langle f(\phi[z]) - f(\phi) - Df(\phi) \tilde{\phi}[z], \chi_{A\sqrt{\omega}}(z^2 g + \bar{z}^2 \bar{g}) \right\rangle \\ &\quad + \left\langle -\tilde{\vartheta}_{\mathcal{R}} \phi[z] + i\tilde{\omega}_{\mathcal{R}} \Lambda_p \phi[z] - i\tilde{D}_{\mathcal{R}} \partial_x \phi[z] - \frac{\tilde{v}_{\mathcal{R}} x}{2} \phi[z] + iD_z \phi[z] \tilde{z}_{\mathcal{R}}, \chi_{A\sqrt{\omega}}(z^2 g + \bar{z}^2 \bar{g}) \right\rangle \\ &=: A_{1611} + A_{1612}. \end{aligned}$$

We claim

$$|A_{1612}| \lesssim A^{-1} |z|^4. \quad (5.7)$$

Indeed, for example we have

$$\begin{aligned} \tilde{\vartheta}_{\mathcal{R}} \left\langle \phi[z], \chi_{A\sqrt{\omega}}(z^2 g + \bar{z}^2 \bar{g}) \right\rangle &= \tilde{\vartheta}_{\mathcal{R}} \left\langle \phi + \tilde{\phi}[z], z^2 g + \bar{z}^2 \bar{g} \right\rangle \\ &- \tilde{\vartheta}_{\mathcal{R}} \left\langle \phi[z], (1 - \chi_{A\sqrt{\omega}})(z^2 g + \bar{z}^2 \bar{g}) \right\rangle = O(z^5) + O(e^{-\kappa\omega_0 A} z^4) \end{aligned}$$

where we used the orthogonality (2.3)–(2.4), the bound (3.4) and the exponential decay of ξ . The other terms forming A_{1612} can be bounded similarly, yielding (5.7).

Exploiting $\phi \gg |\tilde{\phi}[z]|$ on $\text{supp } \chi_{A\sqrt{\omega}}$ we write the Taylor formula

$$\begin{aligned} A_{1611} &= 2^{-1} \left\langle D^2 f(\phi) (\tilde{\phi}[z])^2, \chi_{A\sqrt{\omega}}(z^2 g + \bar{z}^2 \bar{g}) \right\rangle \\ &+ \int_0^1 \frac{(1-s)^2}{2} \left\langle D^3 f(\phi + s\tilde{\phi}[z]) (\tilde{\phi}[z])^3, \chi_{A\sqrt{\omega}}(z^2 g + \bar{z}^2 \bar{g}) \right\rangle ds =: A_{16111} + A_{16112}. \end{aligned}$$

Then

$$\begin{aligned} |A_{16112}| &\lesssim |z|^2 \int_{|x| \leq 2A\sqrt{\omega}} |\phi|^{p-3} |\tilde{\phi}[z]|^3 dx \lesssim C_{A,\omega} |z|^5 \\ \text{where } C_{A,\omega} &= \int_{|x| \leq 2A\sqrt{\omega}} e^{|x|(3-p-3\sqrt{1-\lambda(p)})} dx. \end{aligned}$$

Here we take $\delta > 0$ so that

$$|A_{16112}| \lesssim C_{A,\omega} \delta |z|^4 \leq A^{-1} |z|^4.$$

We write

$$\begin{aligned} A_{16111} &= 2^{-1} \left\langle D^2 f(\phi) (\tilde{\phi}[z])^2, z^2 g + \bar{z}^2 \bar{g} \right\rangle \\ &2^{-1} \left\langle D^2 f(\phi) (\tilde{\phi}[z])^2, (1 - \chi_{A\sqrt{\omega}})(z^2 g + \bar{z}^2 \bar{g}) \right\rangle =: A_{161111} + A_{161112}. \end{aligned}$$

Then

$$|A_{161112}| \lesssim |z|^4 \int_{|x| \geq \frac{A\sqrt{\omega}}{2}} e^{|x|(2-p-2\sqrt{1-\lambda(p)})} dx \lesssim A^{-1} |z|^4.$$

By (2.21) for $\xi = (\xi_1, \xi_2)^\top$, $X = \tilde{\phi}[z] = (z\xi_1 + \bar{z}\bar{\xi}_1) + i(z\xi_2 + \bar{z}\bar{\xi}_2)$, $u = \phi$ and identifying $\mathbb{C} = \mathbb{R}^2$, we have

$$D^2 f(\phi) (\tilde{\phi}[z])^2 = (p-1)\phi^{p-2} \begin{pmatrix} p(z\xi_1 + \bar{z}\bar{\xi}_1)^2 + (z\xi_2 + \bar{z}\bar{\xi}_2)^2 \\ 2(z\xi_1 + \bar{z}\bar{\xi}_1)(z\xi_2 + \bar{z}\bar{\xi}_2) \end{pmatrix}. \quad (5.8)$$

Then we have

$$\begin{aligned}
 A_{161111} &= (p-1)|z|^4\gamma(p) \\
 &+ 2(p-1)|z|^2 \left(\left\langle \phi_\omega^{p-2} \left(p|\xi_1|^2 + |\xi_2|^2 \right), z^2 g_1^{(\omega)} \right\rangle + 2 \left\langle \phi_\omega^{p-2} (\xi_1 \bar{\xi}_2 + \bar{\xi}_1 \xi_2), z^2 g_2^{(\omega)} \right\rangle \right) \\
 &+ (p-1) \left(\left\langle \phi_\omega^{p-2} \left(p\bar{\xi}_1^2 + \bar{\xi}_2^2 \right), z^4 g_1^{(\omega)} \right\rangle + 2 \left\langle \phi_\omega^{p-2} \bar{\xi}_1 \bar{\xi}_2, z^4 g_2^{(\omega)} \right\rangle \right) \\
 &=: A_{1611111} + A_{1611112} + A_{1611113}.
 \end{aligned}$$

We claim that there exists a function $\mathcal{I}_{\text{FGR}}$ like in the statement such that

$$\left| -\frac{d}{dt} \mathcal{I}_{\text{FGR}} + \sum_{j=2}^3 A_{1611111j} \right| \lesssim |z|^5 + |z|^3 |\dot{\Theta} - \tilde{\Theta}|. \quad (5.9)$$

To $n = 2, 4$ write

$$\frac{d}{dt} z^n = n \tilde{z} z^{n-1} + n (\dot{z} - \tilde{z}) z^{n-1} = n i \lambda z^n + n \tilde{z} \mathcal{R} z^{n-1} + n (\dot{z} - \tilde{z}) z^{n-1}. \quad (5.10)$$

Then for example

$$A_{1611113} = 2 \left\langle \phi^{p-2} \bar{\xi}_1 \bar{\xi}_2, z^4 g_2 \right\rangle = 2 \left\langle \phi^{p-2} \bar{\xi}_1 \bar{\xi}_2, \frac{1}{4i\lambda(p)} \left(\frac{d}{dt} z^4 - 4\tilde{z} \mathcal{R} z^3 - 4(\dot{z} - \tilde{z}) z^3 \right) g_2 \right\rangle$$

and applying the Leibnitz rule for the time derivative, it is easy to obtain the claim. Summing up and recalling also that we have to reintroduce the factor $\omega^{\frac{p+1}{2(p-1)}}$ of (5.6) which we have omitted for simplicity, we obtain (5.5). $\square$

Proof of Proposition 3.9. Integrating (5.5) we obtain $\|z^2\|_{L^2(I)}^2 \lesssim \sqrt{\delta} + A^{-1}\epsilon^2$ yielding (3.34). $\square$

6. High energies: proof of Proposition 3.10

Within the framework in Kowalczyk et al. [35] and using χ in (2.22) we consider

$$\zeta_A(x) := \exp\left(-\frac{|x|}{A}(1 - \chi(x))\right) \text{ and } \varphi_A(x) := \int_0^x \zeta_A^2(y) dy \quad (6.1)$$

and the vector field

$$S_A := \varphi'_A + 2\varphi_A \partial_x. \quad (6.2)$$

Next, following Martel [49] we introduce

$$\mathcal{I} := 2^{-1} \langle i\xi, S_A \xi \rangle \text{ where } \xi := \exp\left(-\frac{i}{2} x v\right) \eta. \quad (6.3)$$

Lemma 6.1. *There exists a fixed constant $C > 0$ s.t.*

$$\|\eta\|_{\Sigma_A}^2 \leq C \left[\dot{\mathcal{I}} + \|\eta\|_{\Sigma}^2 + |\dot{\Theta} - \tilde{\Theta}|^2 + |z|^4 \right]. \quad (6.4)$$

Proof. We have

$$\dot{\mathcal{I}} = -\langle \dot{\xi}, iS_A \xi \rangle = -\left\langle \dot{\eta} - \frac{i}{2} \dot{v} x \eta, iS_A \eta + v \varphi_A \eta \right\rangle$$

so that

$$\dot{\mathcal{I}} = -\langle \dot{\eta}, iS_A \eta \rangle - \frac{\dot{v}}{4} \langle [S_A, x] \eta, \eta \rangle - v \langle \dot{\eta}, \varphi_A \eta \rangle. \quad (6.5)$$

Notice that if, as we can do here, we assume for the initial value $u_0 \in H^2(\mathbb{R})$, then it is known that $u \in C^0(\mathbb{R}, H^2(\mathbb{R}))$, see Cazenave [6, Ch. 5.3], which in turn implies that equation (3.37) holds in a distributional sense in $\mathcal{D}'(\mathbb{R}, L^2(\mathbb{R}))$, see [7, Ch. 1] and in particular $\eta \in C^1(\mathbb{R}, L^2(\mathbb{R}))$. Then (6.5) can be obtained not just formally but rigorously, by taking a sequence $\eta_n \in C^1(\mathbb{R}, L^{2,1}(\mathbb{R}))$ such that $\eta_n \xrightarrow{n \rightarrow +\infty} \eta$ locally uniformly in $C^1(\mathbb{R}, L^{2,1}(\mathbb{R}))$, since the above computations are rigorous for any η_n and by obtaining (6.5) for η thanks to a limit argument.

From (3.37), which is valid in $\mathcal{D}'(\mathbb{R}, L^2(\mathbb{R}))$, we obtain

$$\begin{aligned} -\langle \dot{\eta}, iS_A \eta \rangle &= -\left\langle \partial_x^2 \eta, S_A \eta \right\rangle - \langle R[\omega, v, z], S_A \eta \rangle - \frac{1}{2} \dot{D} \langle \eta, i[\partial_x, S_A] \eta \rangle \\ &= \langle f(\phi[\omega, v, z] + \eta) - f(\phi[\omega, v, z]), S_A \eta \rangle + O(|\dot{\Theta} - \tilde{\Theta}| \|\eta\|_{\Sigma}) \end{aligned}$$

and

$$\begin{aligned} -v \langle \dot{\eta}, \varphi_A \eta \rangle &= -v \left\langle i \partial_x^2 \eta, \varphi_A \eta \right\rangle - v \dot{D} \langle \eta', \varphi_A \eta \rangle - v \langle iR[\omega, v, z], \varphi_A \eta \rangle \\ &= -v \langle i(f(\phi[\omega, v, z] + \eta) - f(\phi[\omega, v, z])), \varphi_A \eta \rangle. \end{aligned}$$

From [49] we have

$$-\left\langle \partial_x^2 \eta, S_A \eta \right\rangle \geq 2\|(\zeta_A \eta)'\|_{L^2}^2 - \frac{C}{A} \|\eta\|_{\Sigma}^2. \quad (6.6)$$

We have

$$-v \dot{D} \langle \eta', \varphi_A \eta \rangle = \frac{v}{2} \dot{D} \|\zeta_A \eta\|_{L^2}^2 = \frac{v^2}{2} \|\zeta_A \eta\|_{L^2}^2 + \frac{v}{2} (\dot{D} - v) \|\zeta_A \eta\|_{L^2}^2 \geq \frac{v}{2} (\dot{D} - v) \|\zeta_A \eta\|_{L^2}^2.$$

So we obtain

$$2\|(\zeta_A \eta)'\|_{L^2}^2 \leq \dot{I} + \frac{1}{2} \dot{D} \langle \eta, i[\partial_x, S_A] \eta \rangle + v \langle i\partial_x^2 \eta, \varphi_A \eta \rangle \quad (6.7)$$

$$+ \langle R[\omega, v, z], S_A \eta \rangle + v \langle iR[\omega, v, z], \varphi_A \eta \rangle \quad (6.8)$$

$$+ \langle f(\phi[\omega, v, z] + \eta) - f(\phi[\omega, v, z]), S_A \eta \rangle \quad (6.9)$$

$$+ v \langle i(f(\phi[\omega, v, z] + \eta) - f(\phi[\omega, v, z])), \varphi_A \eta \rangle \quad (6.10)$$

$$+ \frac{\dot{v}}{4} \langle [S_A, x] \eta, \eta \rangle - \frac{v}{2} (\dot{D} - v) \|\zeta_A \eta\|_{L^2}^2 + O(|\dot{\Theta} - \tilde{\Theta}| \|\eta\|_{\tilde{\Sigma}}). \quad (6.11)$$

We have

$$\frac{1}{2} \dot{D} \langle \eta, i[\partial_x, S_A] \eta \rangle = \dot{D} \langle \eta, i[\partial_x, \varphi_A] \eta' \rangle = \dot{D} \langle \eta, i\zeta_A^2 \eta' \rangle$$

and

$$v \langle i\partial_x^2 \eta, \varphi_A \eta \rangle = -v \langle i\eta', \zeta_A^2 \eta \rangle$$

and so by (3.29) and (4.5)

$$\left| \frac{1}{2} \dot{D} \langle \eta, i[\partial_x, S_A] \eta \rangle + v \langle i\partial_x^2 \eta, \varphi_A \eta \rangle \right| = |(\dot{D} - v) \langle \eta, i\zeta_A^2 \eta' \rangle| \lesssim \delta^2 (|z|^4 + \|\eta\|_{\tilde{\Sigma}}^2 + \|\eta\|_{\Sigma_A}^2). \quad (6.12)$$

By (3.6), for a preassigned small constant $\delta_2 > 0$

$$|\langle R[\omega, v, z], S_A \eta \rangle| + |v \langle iR[\omega, v, z], \varphi_A \eta \rangle| \lesssim |z|^4 + \|\eta\|_{\tilde{\Sigma}}^2 + \delta_2 \|\eta\|_{\Sigma_A}^2. \quad (6.13)$$

By (3.29) and (4.5) we have

$$\left| \frac{\dot{v}}{4} \langle [S_A, x] \eta, \eta \rangle \right| \lesssim A |\dot{v}| \|\eta\|_{L^2}^2 \lesssim A \delta^2 (|z|^4 + \|\eta\|_{\tilde{\Sigma}}^2 + \|\eta\|_{\Sigma_A}^2) \quad (6.14)$$

and

$$\left| \frac{v}{2} (\dot{D} - v) \|\zeta_A \eta\|_{L^2}^2 \right| \lesssim \delta^4 (|z|^4 + \|\eta\|_{\tilde{\Sigma}}^2 + \|\eta\|_{\Sigma_A}^2). \quad (6.15)$$

We write

$$\langle f(\phi[\omega, v, z] + \eta) - f(\phi[\omega, v, z]) - f(\eta), S_A \eta \rangle + \langle f(\eta), S_A \eta \rangle =: B_1 + B_2.$$

Then

$$B_2 = \left\langle f(\eta) \bar{\eta} - F(\eta), \zeta_A^2 \right\rangle = \frac{p}{p+1} \int_{\mathbb{R}} |\eta|^{p+1} \zeta_A^2 dx$$

and using the crucial estimate by Kowalczyk et al. [35]

$$|B_2| \lesssim A^2 \|\eta\|_{L^\infty(\mathbb{R})}^{p-1} \|(\zeta_A \eta)'\|_{L^2(\mathbb{R})}^2 \lesssim A^{-1} \|(\zeta_A \eta)'\|_{L^2(\mathbb{R})}^2. \quad (6.16)$$

Claim 6.2. *We have for any small preassigned $\delta_2 > 0$*

$$|B_1| \lesssim \|\eta\|_{\Sigma}^2 + \delta_2 \|(\zeta_A \eta)'\|_{L^2}^2. \quad (6.17)$$

Proof. We consider

$$\Omega_{1,t} = \{x \in \mathbb{R} \mid |\eta(t, x)| \leq 2|\phi[\omega(t), v(t), z(t)]|\} \text{ and} \quad (6.18)$$

$$\Omega_{2,t} = \mathbb{R} \setminus \Omega_{1,t} = \{x \in \mathbb{R} \mid |\eta(t, x)| > 2|\phi[\omega(t), v(t), z(t)]|\},$$

and split accordingly

$$B_1 = \sum_{j=1,2} \langle f(\phi[\omega, v, z] + \eta) - f(\phi[\omega, v, z]) - f(\eta), S_A \eta \rangle_{L^2(\Omega_{j,t})} =: B_{11} + B_{12}.$$

We have

$$\begin{aligned} |B_{11}| &\leq \int_0^1 \int_{\Omega_{1,t}} | \langle (Df(\phi[\omega, v, z] + s\eta) - Df(s\eta)) \eta, S_A \eta \rangle_{\mathbb{C}} | dx ds \\ &\lesssim \int_{\Omega_{1,t}} \zeta_A^2 |\phi[\omega, z]|^{p-1} |\eta|^2 dx + \int_{\Omega_{1,t}} \zeta_A^{-1} |\varphi_A| |\phi[\omega, z]|^{p-1} |\eta| |(\zeta_A \eta)' - \zeta'_A \eta| dx =: B_{111} + B_{112}. \end{aligned}$$

Then

$$B_{111} \lesssim \|\eta\|_{\Sigma}^2$$

and, for a fixed small and preassigned $\delta_2 > 0$,

$$B_{112} \lesssim \int_{\Omega_{1,t}} \zeta_A^{-1} |\psi_A| |\phi[\omega, z]|^{p-1} |\eta| \left(|(\zeta_A \eta)'| + \frac{1}{A} \zeta_A |\eta| \right) dx \lesssim \|\eta\|_{\Sigma}^2 + \delta_2 \|(\zeta_A \eta)'\|_{L^2}^2.$$

We have

$$\begin{aligned} |B_{12}| &\leq \int_0^1 \int_{\Omega_{2,t}} | \langle (Df(s\phi[\omega, v, z] + \eta) - Df(s\phi[\omega, v, z])) \phi[\omega, v, z], S_A \eta \rangle_{\mathbb{C}} | dx ds \\ &\lesssim \int_{\Omega_{2,t}} \zeta_A^2 |\eta|^p |\phi[\omega, z]| dx + \int_{\Omega_{2,t}} \zeta_A^{-1} |\varphi_A| |\eta|^{p-1} |\phi[\omega, z]| |(\zeta_A \eta)' - \zeta'_A \eta| dx =: B_{121} + B_{122}. \end{aligned}$$

We have

$$B_{121} \lesssim \int_{\Omega_{2,t}} \zeta_A^2 |\eta|^p |\phi[\omega, z]|^{2-p+p-1} dx \leq \int_{\mathbb{R}} |\eta|^2 |\phi[\omega, z]|^{p-1} dx \lesssim \|\eta\|_{\Sigma}^2.$$

We have similarly, for a fixed small and preassigned $\delta_2 > 0$ and completing the proof of Claim 6.2,

$$B_{122} \lesssim \int_{\Omega_{2,t}} \zeta_A^{-1} |\psi_A| |\eta| |\phi[\omega, z]|^{p-1} \left(|(\zeta_A \eta)'| + \frac{1}{A} \zeta_A |\eta| \right) dx \lesssim \|\eta\|_{\Sigma}^2 + \delta_2 \|(\zeta_A \eta)'\|_{L^2}^2. \quad \square$$

We consider the term in (6.10) which equals

$$\sum_{j=1,2} v \langle i(f(\phi[\omega, v, z] + \eta) - f(\phi[\omega, v, z]) - f(\eta)), \varphi_A \eta \rangle_{L^2(\Omega_{j,t})} =: C_1 + C_2$$

and then we proceed like in Claim 6.2. We have

$$\begin{aligned} |C_1| &\leq |v| \int_0^1 \int_{\Omega_{1,t}} |i(Df(\phi[\omega, v, z] + s\eta) - Df(s\eta)) \eta, \varphi_A \eta|_{\mathbb{C}} dx ds \\ &\lesssim |v| \int_{\Omega_{1,t}} |\varphi_A| |\phi[\omega, z]|^{p-1} |\eta|^2 dx \lesssim \delta \|\eta\|_{\Sigma}^2. \end{aligned}$$

We have

$$\begin{aligned} |C_2| &\leq |v| \int_0^1 \int_{\Omega_{2,t}} |\langle (Df(s\phi[\omega, v, z] + \eta) - Df(s\phi[\omega, v, z])) \phi[\omega, v, z], \varphi_A \eta \rangle_{\mathbb{C}}| dx ds \\ &\lesssim |v| \int_{\Omega_{2,t}} |\varphi_A| |\eta|^p |\phi[\omega, z]|^{2-p+p-1} dx \leq |v| \int_{\mathbb{R}} |\eta|^2 |\phi[\omega, z]|^{p-1} dx \lesssim \delta \|\eta\|_{\Sigma}^2. \end{aligned}$$

Collecting together (6.7)–(6.11) and the estimates we get for a small constant $\delta_2 > 0$

$$2\|(\zeta_A \eta)'\|_{L^2}^2 \leq \dot{I} + \delta_2 \|(\zeta_A \eta)'\|_{L^2}^2 + O(|z|^4 + \|\eta\|_{\Sigma}^2)$$

which yields a bound of $\|(\zeta_A \eta)'\|_{L^2}^2$. Finally to get (6.4) we use the following, see Lemma 6.2 [22],

$$\|\operatorname{sech}\left(\frac{2}{A}x\right) \eta'\|_{L^2}^2 + A^{-2} \|\operatorname{sech}\left(\frac{2}{A}x\right) \eta\|_{L^2}^2 \lesssim \|(\zeta_A \eta)'\|_{L^2(\mathbb{R})}^2 + A^{-1} \|\eta\|_{\Sigma}^2. \quad \square$$

Proof of Proposition 3.10. Integrating inequality 6.1 we obtain (3.35). $\square$

7. Low energies: proof of Proposition 3.11

Applying U^{-1} to equation (5.4) we get

$$\begin{aligned} \partial_t(U^{-1}\eta) &= iH_\omega U^{-1}\eta - i\sigma_3(\tilde{\vartheta}\mathcal{R} + \tilde{\vartheta} - \dot{\vartheta})U^{-1}\eta + \dot{D}\partial_x U^{-1}\eta \\ &\quad - e^{i\sigma_3\tilde{\vartheta} + D\partial_x} U^{-1} D_\Theta \phi[\Theta](\dot{\Theta} - \tilde{\Theta}) + i\sigma_3 U^{-1} (Df(\phi[\omega, v, z]) - Df(\phi_\omega)) \eta \\ &\quad - i\sigma_3 U^{-1} (f(\phi[\omega, v, z] + \eta) - f(\phi[\omega, v, z]) - Df(\phi[\omega, v, z])\eta) - i\sigma_3 U^{-1} R[\omega, v, z]. \end{aligned} \quad (7.1)$$

Set $v := \chi_B U^{-1}\eta$ for B like in (2.14). We denote $P_d(\omega)$ the discrete spectrum projection and $P_c(\omega)$ the continuous spectrum projection associated to H_ω , which are closely related to the corresponding projections for $\mathcal{L}_\omega$, indeed $U P_a(H_\omega) U^{-1} = P_a(\mathcal{L}_\omega)$ for $a = c, d$. Then we write

$$v = P_c(\omega_0)v + P_d(\omega_0)v \text{ where we have} \quad (7.2)$$

$$P_d(\omega_0)v = P_d(\mathcal{L}_\omega)\eta + U^{-1} (P_d(\mathcal{L}_{\omega_0}) - P_d(\mathcal{L}_\omega)) \eta - P_d(\omega_0) (1 - \chi_B) U^{-1}\eta$$

It is easy to check that

$$\begin{aligned} &\|P_d(\omega_0) (1 - \chi_B) U^{-1}\eta\|_{L^{2,s}(\mathbb{R})} + \|U^{-1} (P_d(\mathcal{L}_{\omega_0}) - P_d(\mathcal{L}_\omega)) \eta\|_{L^{2,s}(\mathbb{R})} \\ &\leq o_{B^{-1}}(1) \|\eta\|_{\tilde{\Sigma}} \text{ for any } s \in \mathbb{R}. \end{aligned}$$

Finally, with an argument similar to one in [18, §7] by means of the orthogonality in (3.24) it is easy to see that

$$\|U^{-1} P_d(\mathcal{L}_\omega)\eta\|_{L^{2,s}(\mathbb{R})} \lesssim |z| \|\eta\|_{\tilde{\Sigma}} \text{ for any } s \in \mathbb{R}.$$

So we conclude that we have

$$\|P_d(\omega_0)v\|_{L^{2,s}(\mathbb{R})} = o_{B^{-1}}(1) \|\eta\|_{\tilde{\Sigma}} \text{ for any } s \in \mathbb{R}. \quad (7.3)$$

Setting $w = P_c(\omega_0)v$, we have

$$\partial_t w - iH_{\omega_0} w = -i\varpi P_c(\omega_0)\sigma_3 w + iP_c(\omega_0)\sigma_3 (2\chi'_B \partial_x + \chi''_B) U^{-1}\eta + \dot{D}P_c(\omega_0)\chi_B \partial_x U^{-1}\eta \quad (7.4)$$

$$- i\varpi P_c(\omega_0)\sigma_3 P_d(\omega_0)v + iP_c(V_{\omega_0} - V_\omega)w + iP_c(V_{\omega_0} - V_\omega)P_d(\omega_0)v \quad (7.5)$$

$$- P_c(\omega_0)\chi_B e^{i\sigma_3\tilde{\vartheta} + D\partial_x} U^{-1} D_\Theta \phi[\Theta](\dot{\Theta} - \tilde{\Theta}) + iP_c(\omega_0)\sigma_3 \chi_B U^{-1} (Df(\phi[\omega, v, z]) - Df(\phi_\omega)) \eta \quad (7.6)$$

$$\begin{aligned} &- iP_c(\omega_0)\chi_B \sigma_3 U^{-1} (f(\phi[\omega, v, z] + \eta) - f(\phi[\omega, v, z]) - Df(\phi[\omega, v, z])\eta) \\ &- iP_c(\omega_0)\sigma_3 \chi_B U^{-1} R[\omega, v, z] \end{aligned} \quad (7.7)$$

where

$$\varpi := \tilde{\vartheta}\mathcal{R} + \tilde{\vartheta} - \dot{\vartheta} + \omega - \omega_0. \quad (7.8)$$

Now we have

$$\begin{aligned}
 w &= e^{itH_{\omega_0}} w(0) - i \int_0^t e^{-i(t-s)H_{\omega_0}} \varpi(s) P_c(\omega_0) \sigma_3 w(s) ds \\
 &\quad + i \int_0^t e^{-i(t-s)H_{\omega_0}} P_c(\omega_0) \sigma_3 (2\chi_B' \partial_x + \chi_B'') U^{-1} \eta(s) ds \\
 &\quad + \int_0^t e^{-i(t-s)H_{\omega_0}} \dot{D}(s) P_c(\omega_0) \chi_B \partial_x U^{-1} \eta(s) ds \\
 &\quad + \int_0^t e^{-i(t-s)H_{\omega_0}} \text{lines (7.5)–(7.7)} ds.
 \end{aligned} \tag{7.9}$$

The following is a simplified version of [18, Lemma 7.2].

Lemma 7.1. *For $S > 3/2$ we have*

$$\|w\|_{L^2(I, L^{2, -S}(\mathbb{R}))} \leq o_{B^{-1}}(1) \epsilon. \tag{7.10}$$

Proof. By the analogue for H_ω of (2.25), see §9 [18], we have

$$\|e^{itH_{\omega_0}} w(0)\|_{L^2(\mathbb{R}, L^{2, -S}(\mathbb{R}))} \lesssim \|w(0)\|_{L^2(\mathbb{R})} \lesssim \|\eta(0)\|_{L^2(\mathbb{R})} \lesssim \delta (= o_{B^{-1}}(1) \epsilon).$$

Using the version for H_ω of Proposition 2.4 (notice incidentally that in Krieger and Schlag [37] the operators are written as H_ω) we have

$$\begin{aligned}
 &\left\| \int_0^t e^{-i(t-s)H_{\omega_0}} \varpi(s) P_c(\omega_0) \sigma_3 w(s) ds \right\|_{L^2(I, L^{2, -S}(\mathbb{R}))} \lesssim \|\varpi\|_{L^\infty(I)} \|w\|_{L^2(I, L^{2, S}(\mathbb{R}))} \\
 &\lesssim \delta \left(\|\chi_B U^{-1} \eta\|_{L^2(I, L^{2, S}(\mathbb{R}))} + \|[P_d(\omega_0), \chi_B] U^{-1} \eta\|_{L^2(I, L^{2, S}(\mathbb{R}))} \right) \\
 &\lesssim \delta \left(B^S A A^{-1} \|\text{sech}\left(\frac{2}{A} x\right) \eta\|_{L^2(I, L^2(\mathbb{R}))} + o_{B^{-1}}(1) \|\eta\|_{L^2(I, \tilde{\Sigma})} \right) \\
 &= o_{B^{-1}}(1) \left(\|\eta\|_{L^2(I, \Sigma_A)} + \|\eta\|_{L^2(I, \tilde{\Sigma})} \right) = o_{B^{-1}}(1) \epsilon.
 \end{aligned}$$

Using the analogue for H_ω of (2.24) we consider

$$\left\| \int_0^t e^{-i(t-s)H_{\omega_0}} P_c(\omega_0) \sigma_3 (2\chi_B' \partial_x + \chi_B'') U^{-1} \eta(s) ds \right\|_{L^2(I, L^{2, -S}(\mathbb{R}))}$$

$$\begin{aligned}
&\lesssim \| (2\chi_B' \partial_x + \chi_B'') \eta \|_{L^2(I, L^{2,\tau}(\mathbb{R}))} \lesssim B^{\tau-1} \|\operatorname{sech}\left(\frac{2}{A}x\right) \eta'\|_{L^2(I, L^2(\mathbb{R}))} \\
&+ B^{\tau-2} \|1_{B \leq |x| \leq 2B} \operatorname{sech}\left(\frac{2}{A}x\right) \eta\|_{L^2(I, L^2(\mathbb{R}))} \lesssim B^{\tau-1} \|\eta\|_{L^2(I, \Sigma_A)} \\
&+ B^{\tau-1} \left(\left\| \left(\operatorname{sech}\left(\frac{2}{A}x\right) \eta \right)' \right\|_{L^2(I, L^2(\mathbb{R}))} + \|\eta\|_{L^2(I, \tilde{\Sigma})} \right) = o_{B^{-1}}(1)\epsilon,
\end{aligned}$$

where we used $\tau \in (1/2, 1)$ and, see Merle and Raphael [52, Appendix C],

$$\|u\|_{L^2(|x| \leq 2B)} \lesssim B \left(\|u'\|_{L^2(\mathbb{R})} + \|u\|_{\tilde{\Sigma}} \right).$$

Next, using again the version for H_ω of Proposition 2.4

$$\begin{aligned}
&\left\| \int_0^t e^{-i(t-s)H_{\omega_0}} \dot{D}(s) P_c(\omega_0) \chi_B \partial_x U^{-1} \eta(s) ds \right\|_{L^2(I, L^{2,-s}(\mathbb{R}))} \lesssim \|\dot{D}\|_{L^\infty(I)} \|\chi_B \partial_x \eta\|_{L^2(I, L^{2,s}(\mathbb{R}))} \\
&\lesssim \delta B^S \|\operatorname{sech}\left(\frac{2}{A}x\right) \eta'\|_{L^2(I, L^2(\mathbb{R}))} \leq o_{B^{-1}}(1) \|\eta\|_{L^2(I, \Sigma_A)} = o_{B^{-1}}(1)\epsilon.
\end{aligned}$$

Finally the terms in line (7.9) can be similarly bounded using in particular the analogue for H_ω of Proposition 2.4. The estimates are elementary and similar to [17, Sect. 8]. $\square$

Proof of Proposition 3.11. From (7.2), (7.3) and (7.10) we have $\|v\|_{L^2(I, \tilde{\Sigma})} \lesssim o_{B^{-1}}(1)\epsilon$. Next, from $v = \chi_B U^{-1} \eta$, and thanks to the relation $A \sim B^3$ set in (2.14) we have

$$\begin{aligned}
\|\eta\|_{\tilde{\Sigma}} &\lesssim \|v\|_{\tilde{\Sigma}} + \|(1 - \chi_B)\eta\|_{\tilde{\Sigma}} \lesssim \|v\|_{\tilde{\Sigma}} + A^{-2} \|\operatorname{sech}\left(\frac{2}{A}x\right) \eta\|_{L^2} \\
&\lesssim \|v\|_{\tilde{\Sigma}} + A^{-1} \|\eta\|_{\Sigma_A}.
\end{aligned} \tag{7.11}$$

So by (3.35) and (3.34) we get the following, which implies (3.36),

$$\begin{aligned}
\|\eta\|_{L^2(I, \tilde{\Sigma})} &\lesssim \|v\|_{L^2(I, \tilde{\Sigma})} + A^{-1} \|\eta\|_{L^2(I, \Sigma_A)} \\
&\lesssim o_{B^{-1}}(1)\epsilon + A^{-1} \left(\|z^2\|_{L^2(I)} + \|\eta\|_{L^2(I, \tilde{\Sigma})} \right) \lesssim o_{B^{-1}}(1)\epsilon + A^{-1} \|\eta\|_{L^2(I, \tilde{\Sigma})}. \quad \square
\end{aligned}$$

8. Proof of Proposition 1.4

Proof of Proposition 1.4. Here from (2.11)

$$\begin{aligned}
H &= \sigma_3 \left(-\partial_x^2 + 1 \right) + V(p, x), \\
V(p, x) &:= M_0 \operatorname{sech}^2 \left(\frac{p-1}{2} x \right) \text{ with } M_0 = - \left(\frac{p+1}{2} \sigma_3 + i \frac{p-1}{2} \sigma_2 \right) \frac{p+1}{2}.
\end{aligned}$$

From this we see that H is analytic of type (A) in p and this implies that $\lambda(p)$ depends analytically on $p \in (p_0, 3)$, see [56, Ch. 12]. By Lemma 2.1 we already know that $\lambda(p)$ has (algebraic and geometric) multiplicity 1 as an eigenvalue of H for $p \in (p_0, 3)$. From analyticity in the sense of Kato, we conclude by [56, Theorem XII.8] that it is possible to define ξ dependent analytically in p . Proceeding like in §7 [20] it is possible to define the g in (5.2) to dependent analytically in p . It follows that the function $\gamma(p)$ in (5.3) is analytic in $p \in (p_0, 3)$. Finally, since it is possible to arrange things so that for $0 < |p - 3| \ll 1$ we have $\gamma(p) \neq 0$ as shown in [18, 19], we conclude that $(p_0, 3) \setminus \{p \in (p_0, 3) : \gamma(p) = 0\}$ is a discrete subset of $(p_0, 3)$, proving Proposition 1.4. $\square$

Finally we consider the following lemma, which was used in the proof of Lemma 2.1 and where all the vector spaces are considered with corresponding scalar field $\mathbb{C}$.

Lemma 8.1. *For any $p > 1$ we have $\ker(\mathcal{L} - i\lambda) = N_g(\mathcal{L} - i\lambda)$ if $\lambda \in (0, 1)$.*

Proof. Recall that $\mathcal{L} = JK$ where $K = \text{diag}(L_+, L_-)$. The operator K has a single and simple negative eigenvalue and $\dim \ker K = 2$, see Weinstein [66]. Consider now the Hermitian bilinear form $B(u) = \langle Ku, u \rangle$ using the Hermitian form $\langle \cdot, \cdot \rangle$ in $L^2(\mathbb{R}, \mathbb{C})$ introduced in §2. Then, if we consider a beginning of spectral decomposition of K we have $L^2(\mathbb{R}, \mathbb{C}^2) = \mathcal{N} \oplus \ker K \oplus \mathcal{P}$ with $\dim \mathcal{N} = 1$ and $B > 0$ in $\mathcal{P}$. It is elementary to see that if X is a vector subspace with $\dim_{\mathbb{C}} X > 3$ we have $X \cap \mathcal{P} \neq \emptyset$. Suppose by contradiction that $\ker(\mathcal{L} - i\lambda) \subsetneq N_g(\mathcal{L} - i\lambda)$. Then there exist nontrivial $u \in \ker(\mathcal{L} - i\lambda)$ and $v \in N_g(\mathcal{L} - i\lambda)$ with $u = (\mathcal{L} - i\lambda)^n v$ for some $n \geq 1$. Then (by well known commutation properties of $\mathcal{L}$)

$$\langle Ku, u \rangle = i\lambda \langle J^{-1}u, u \rangle = i\lambda \langle J^{-1}(\mathcal{L} - i\lambda)^n v, u \rangle = i\lambda \langle J^{-1}v, (\mathcal{L} - i\lambda)^n u \rangle = 0.$$

So $B(u) = 0$. Notice that we have also $B(\bar{u}) = 0$ where for the complex conjugate we have $\bar{u} \in \ker(\mathcal{L} + i\lambda)$. Now consider the vector space

$$X = \text{Span}_{\mathbb{C}}\{u, \bar{u}\} \oplus \ker K.$$

We have $\dim_{\mathbb{C}} X = 4$ with $B \leq 0$ in X , which is a contradiction. $\square$

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No data was used for the research described in the article.

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