# Well-posedness of an asymptotic model for capillarity-driven free boundary Darcy flow in porous media in the critical Sobolev space

Stefano Scrobogna

*Departamento de Análisis Matemático & IMUS, Universidad de Sevilla, Sevilla, Spain*

A B S T R A C T

We prove that the quadratic approximation of the capillarity-driven free-boundary Darcy flow, derived in Granero-Belinchón and Scrobogna (2019), is well posed in $\dot{H}^{3/2}\left(\mathbb{S}^1\right)$, and globally well-posed if the initial datum is small in $\dot{H}^{3/2}\left(\mathbb{S}^1\right)$.

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## 1. Presentation of the problem

Fluid moving in porous media, such as sand or wood, is a common occurrence in nature. The simplest equation describing such physical phenomenon is the Darcy law

$$\frac{\mu}{\beta} u = -\nabla p - \rho g \ \mathsf{e}_2, \tag{1.1}$$

where $u$, $p$, $\rho$ and $\mu$ are the velocity, pressure, density and dynamic viscosity of the fluid, respectively. The constant $\beta$ describes a property of the porous media and it is known as the permeability. The term $\rho g \mathsf{e}_2$ stands for the acceleration due to gravity in the direction $\mathsf{e}_2 = (0,1)^\intercal$. From now on we use the renormalization $\mu/\beta \equiv 1$ so that (1.1) becomes

$$u = -\nabla p - \rho g \ \mathsf{e}_2,$$

Darcy law is valid for slow and viscous flows, and it was first derived experimentally by Henry Darcy (1856) and then derived theoretically from the Navier–Stokes equations via homogenization (cf. [1]). The free boundary Darcy flow, also known as Muskat problem (cf. [2,3]), is often used in order to model the

*E-mail address:* scrobogna@us.es.

dynamics of aquifers or oil wells. When the free-interface is the graph $\Gamma(t) = \left\{ (x, h(x)) \ \middle| \ x \in \mathbb{R} \text{ or } \mathbb{S}^1 \right\}$, which divides the space in the regions

$$\Omega^{\pm}(t) = \left\{ (x, y) \in \mathbb{R}^2 \text{ or } \mathbb{S}^1 \times \mathbb{R} \ \middle| \ y \lessgtr h(x, t) \right\},$$

with fluid of density

$$\rho(x, y, t) = \begin{cases} \rho_- & \text{if} \quad (x, y) \in \Omega^-(t) \\ \rho_+ & \text{if} \quad (x, y) \in \Omega^+(t) \end{cases},$$

(i.e. the fluid with density $\rho_+$ lies below and the fluid with density $\rho_-$ lies above), the evolution of the Muskat problem can be expressed as the contour equation

$$h_t = G[h]\left((\rho_+ - \rho_-) gh - \gamma\kappa\right), \tag{1.2}$$

where $G[h]\psi$ is the Dirichlet-to-Neumann operator (cf. [4]), $\kappa = \frac{h''}{\left(1+(h')^2\right)^{3/2}}$ is the mean curvature of the interface, $\gamma \geq 0$ is the capillarity coefficient and $g \geq 0$ is the gravitational acceleration.

The mathematical analysis of the Muskat problem has flourished in the past 20years, see [5–17] and the survey articles [18,19], but only recently due to the works of Córdoba & Lazar [20], Gancedo & Lazar [21] and Alazard & Nguyen [22,23] the problem of solvability of the Muskat problem in the critical Sobolev space $\dot{H}^{\frac{d}{2}+1}$, where $d$ is the dimension of the free interface, has been addressed. The three works [20,22,23] study the stable, two phase gravity driven Muskat problem in $\mathbb{R}^2$, i.e. $\rho_+ > \rho_-$, $g > 0$, $\gamma = 0$ and the two fluids fill the two-dimensional space $\mathbb{R}^2$. In such setting (1.2) writes in the simplified form (here $g$ is normalized to one)

$$h_t(x) = \frac{\rho_+ - \rho_-}{2\pi} \text{ p.v.} \int_{\mathbb{R}} \partial_x \arctan\left(\frac{h(x) - h(x-y)}{y}\right) \mathrm{d}y.$$

In [20] a global well posedness result was proved for initial data $f_0 \in \dot{H}^{3/2} \cap \dot{H}^{5/2}$ with smallness assumption on $\|f_0\|_{\dot{H}^{3/2}}$ only, thus allowing initial data with arbitrarily large, albeit finite, slopes. In [22] a global well-posedness result was proved when the initial is small w.r.t. the non-homogeneous norm

$$\|u\|^2_{H^{\frac{3}{2},\frac{1}{3}}} = \int \left(1 + |\xi|^2\right)^{3/2} \left(\log\left(4 + |\xi|\right)\right)^{1/3} |\hat{u}(\xi)|^2 \mathrm{d}\xi,$$

thus allowing initial data to have infinite slope, while [20] and [21] address the problem of global solvability for initial data in $\dot{H}^{\frac{d}{2}+1} \cap \dot{W}^{1,\infty}$ with smallness assumption in $\dot{H}^{\frac{d}{2}+1}$ only.

We denote with $\mathcal{H}$ the Hilbert transform, with $\Lambda = \mathcal{H}\partial_x$ the Calderón operator on $\mathbb{S}^1$ or $\mathbb{R}$ and with $[\![A, B]\!]f = A(Bf) - B(Af)$. In [24] the equation

$$f_t + g\Lambda f + \gamma\Lambda^3 f = \partial_x [\![\mathcal{H}, f]\!] \left(g\Lambda f + \gamma\Lambda^3 f\right), \tag{1.3}$$

was derived, a thorough analysis of the case $g > 0$, $\gamma \geq 0$ was performed in [25]. Eq. (1.3) captures the dynamics of the one-phase Muskat problem (alternatively known as the Hele–Shaw problem) subject to gravity and surface tension up to quadratic order of the nonlinearity in small amplitude number regime, we refer the reader to [26–34] for further results on asymptotic models for free boundary systems. The equation we are interested to study in the present manuscript is the capillarity-driven version of (1.3), i.e. setting $(g, \gamma) = (0, 1)$ we obtain the equation[1]

$$\begin{cases} f_t + \Lambda^3 f = \partial_x [\![\mathcal{H}, f]\!] \Lambda^3 f, \\ f|_{t=0} = f_0. \end{cases} \tag{1.4}$$

[1] The author would like to mention the very recent manuscript [35] in which the authors prove local solvability for arbitrary initial data in $W^{s,p}(\mathbb{R})$, $p \in (1, 2]$, $s \in \left(1 + \frac{1}{p}, 2\right)$ for the two-phases full Muskat problem, i.e. the full system of the two-phase version of (1.4).

It is immediate to see that the transformation

$$f(x,t) \mapsto \frac{1}{\lambda} f\left(\lambda x, \lambda^3 t\right), \qquad f_0(x) \mapsto \frac{1}{\lambda} f_0(\lambda x) \tag{1.5}$$

where $\lambda > 0$ if the space domain is $\mathbb{R}$ and $\lambda \in \mathbb{N}^\star$ if the space domain is $\mathbb{S}^1$, generates a one-parameter family of solutions for (1.4), it is hence a classical consideration in the analysis of nonlinear partial differential equations to look for solutions in functional spaces whose norm is invariant w.r.t. the transformation (1.5), a simple example of such spaces is (recall that here the space dimension is one)

$$L^\infty\left(\mathbb{R}_+; \dot{H}^{3/2}\right) \cap L^2\left(\mathbb{R}_+; \dot{H}^3\right), \qquad L^4\left(\mathbb{R}_+; \dot{H}^{9/4}\right).$$

We prove in particular that for any $f_0 \in \dot{H}^{3/2}$ there exists a $T = T(f_0) > 0$ and a unique solution in $L^4\left([0,T]; \dot{H}^{9/4}\right)$ of (1.4) stemming from $f_0$, which is global if $f_0$ is small in $\dot{H}^{3/2}$. Such result is more general than any well posedness result known, up to date, for the full Muskat problem.[2] This is rather surprising since asymptotic models tend to be less regular compared to the full-models from which they derive (cf. [37–39]) lacking some fine nonlinear cancellation which is present in the full system. Such result is possible thanks to a surprising commutation property of the bilinear truncation of the Dirichlet–Neumann operator. We refer the interested reader to Lemma 3.2 for a detailed statement of the key commutation which is the fundamental tool that allows us to prove the main result of the present manuscript.

## 2. Main result and notation

We denote with $C$ a positive constant whose explicit value may vary from line to line. Given a metric space $(X, d_X)$, any $x_0 \in X$ and $r > 0$ we denote with $B_X(x_0, r)$ the open ball of center $x_0$ and radius $r$ w.r.t. the distance function $d_X$. We denote with $\mathbb{i} = \sqrt{-1}$ the imaginary unit and with $X'$ the dual space of $X$.

From now on we consider the space domain on which (1.4) is defined to be the one-dimensional torus $\mathbb{S}^1$, though the computations performed in the present article can be easily adapted to the case of the one dimensional real line $\mathbb{R}$. We denote with $\mathcal{S}$ the space of Schwartz functions on $\mathbb{S}^1$ and with $\mathcal{S}_0$ the space of Schwartz functions with zero average. Let us denote with $\mathbf{e}_n(x) = e^{\mathbb{i}nx}$, $n \in \mathbb{Z}$ and let us consider a $v \in \mathcal{S}'$, the Fourier transform of $v$ is defined as

$$\hat{v}(n) = \frac{1}{2\pi} \langle v, \mathbf{e}_{-n} \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} v(x)\, e^{-\mathbb{i}nx} \mathrm{d}x, \qquad n \in \mathbb{Z},$$

since for any $n \in \mathbb{Z}$ the function $\mathbf{e}_n \in \mathcal{S}$ the above integral is well defined. For details we refer the interested reader to [40]. We define the Calderón operator $\Lambda$ as the Fourier multiplier $\widehat{\Lambda v}(n) = |n|\, \hat{v}(n)$, and for any $b \in \mathcal{C}((0,\infty); [0,\infty))$ we define the operator $\widehat{b(\Lambda) v}(n) = b(|n|)\, \hat{v}(n)$. We denote with

$$\dot{H}^s = \left\{ v \in \mathcal{S}_0' \;\middle|\; \Lambda^s v \in L^2 \right\},$$

for any $s \in \mathbb{R}$. The space $\dot{H}^s$ is endowed with the norm

$$\|v\|_{\dot{H}^s}^2 = \|\Lambda^s v\|_{L^2}^2 = \sum_n |n|^{2s} |\hat{v}(n)|^2.$$

Let us remark that the homogeneous Sobolev space $\dot{H}^s$ is the subset of the more familiar non-homogeneous Sobolev space $H^s = \left\{ v \in \mathcal{S}' \;\middle|\; (1+\Lambda)^s v \in L^2 \right\}$ with zero average. We use the abbreviated notation

$$\|v\|_s = \|v\|_{\dot{H}^s}, \qquad \|v\|_{L^p_T \dot{H}^s} = \|v\|_{L^p([0,T]; \dot{H}^s)}, \qquad \|v\|_{L^p \dot{H}^s} = \|v\|_{L^p_\infty \dot{H}^s}, \qquad T \in (0,\infty],\ p \in [1,\infty].$$

[2] The author would like to point out that shortly after the publication of the preprint version of the present manuscript T. Alazard and Q.-H. Nguyen proved in [36] that the gravity-driven two-phases 1D Muskat problem is locally well-posed in the *nonhomogeneous* critical Sobolev space $H^{3/2}$.

The main result we prove is the following one.

**Theorem 2.1.** *Given $f_0 \in \dot{H}^{3/2}$ there exists a $T = T(f_0) > 0$ such that the system (1.4) has a unique solution in the space $L^4\left([0,T];\dot{H}^{9/4}\right)$, which, in addition, belongs to the space*

$$f \in \mathcal{C}\left([0,T];\dot{H}^{3/2}\right) \cap L^2\left([0,T];\dot{H}^3\right).$$

*There exists an $\varepsilon_0 > 0$ such that if*

$$\|f_0\|_{3/2} \leq \varepsilon_0,$$

*then $T = \infty$ and the solution is global.*

In Section 3 we introduce some preliminary results which we use in Section 4 in order to prove Theorem 2.1.

## 3. Preliminaries

Let us state the *Minkowsky integral inequality* (cf. [41, Appendix A]) : let us consider $(S_1, \mu_1)$ and $(S_2, \mu_2)$ two $\sigma$–finite measure spaces and let $f : S_1 \times S_2 \to \mathbb{R}$ be measurable, $p \in [1, \infty)$, then the following inequality holds true:

$$\left[\int_{S_2}\left|\int_{S_1} f(x,y)\,\mu_1(\mathrm{d}x)\right|^p \mu_2(\mathrm{d}y)\right]^{\frac{1}{p}} \leqslant \int_{S_1}\left(\int_{S_2}|f(x,y)|^p\,\mu_2(\mathrm{d}y)\right)^{\frac{1}{p}}\mu_1(\mathrm{d}x). \tag{3.1}$$

The above equality holds as well when $p = \infty$ with obvious modifications.

The proof of Theorem 2.1 relies on a fixed point argument, in particular the fixed point theorem we rely on is the following one (see [42, Lemma 5.5 p. 207])

**Lemma 3.1.** *Let $X$ be a Banach space, $\mathcal{B}$ a continuous bilinear map form $X \times X$ to $X$ and $r > 0$ such that*

$$r < \frac{1}{4\|\mathcal{B}\|}, \qquad \|\mathcal{B}\| = \sup_{u,v\in B_X(0,1)} \|\mathcal{B}(u,v)\|_X.$$

*For any $x_0$ in the ball $B_X(0,r) = \{y \in X \ : \ \|y\|_X < r\}$, there exists a unique $x \in B_X(0,2r)$ such that*

$$x = x_0 + \mathcal{B}(x,x).$$

The next result we need is a particular commutation property which is specific to the nonlinearity of Eq. (1.4)

**Lemma 3.2.** *Let $s, \sigma \geq 0,\ \alpha \in [0,\sigma]$ and $\phi \in \dot{H}^{s+\frac{1}{4}+\alpha},\ \psi \in \dot{H}^{\sigma+\frac{1}{4}-\alpha}$ then we have that*

$$\left\|\Lambda^s\left([\![\mathcal{H},\phi]\!]\Lambda^\sigma\psi\right)\right\|_{L^2} \leq C\|\phi\|_{s+\frac{1}{4}+\alpha}\|\psi\|_{\sigma+\frac{1}{4}-\alpha}.$$

**Proof.** Let us remark that

$$\widehat{[\![\mathcal{H},\phi]\!]\Lambda^\sigma\psi}(n) = \mathrm{i}\sum_{k\in\mathbb{Z}}\left(-\mathrm{sgn}(n) + \mathrm{sgn}(n-k)\right)|n-k|^\sigma\,\hat{\phi}(k)\,\hat{\psi}(n-k).$$

Now we have that

$$-\mathrm{sgn}(n) + \mathrm{sgn}(n-k) \neq 0 \qquad \Leftrightarrow \qquad (n>0 \wedge n-k<0) \vee (n<0 \wedge n-k>0) \qquad \Leftrightarrow \qquad 0<|n|<|k|,$$

as a consequence we obtain that

$$|n| < |k| , \qquad \text{and} \qquad |n-k| < |k| . \tag{3.2}$$

Using the monotonicity property (3.2) we obtain that

$$\begin{aligned}
\left\| \Lambda^s \left( [\![\mathcal{H}, \phi]\!] \phi \Lambda^\sigma{}_\psi \right) \right\|_{L^2}^2 &\leq C \sum_n |n|^{2s} \left| \sum_k |n-k|^\sigma \left| \hat{\phi}(k) \right| \left| \hat{\psi}(n-k) \right| \right|^2 , \\
&\leq C \sum_n \left| \sum_k |k|^{s+\alpha} |n-k|^{\sigma-\alpha} \left| \hat{\phi}(k) \right| \left| \hat{\psi}(n-k) \right| \right|^2 , \\
&\leq C \left\| \Lambda^{s+\alpha} \Phi \ \Lambda^{\sigma-\alpha} \Psi \right\|_{L^2}^2 ,
\end{aligned}$$

where

$$\hat{\Phi}(n) = \left| \hat{\phi}(n) \right| , \qquad \hat{\Psi}(n) = \left| \hat{\psi}(n) \right| .$$

We use now the Hölder inequality and the embedding $\dot{H}^{1/4} \hookrightarrow L^4$

$$\left\| \Lambda^{s+\alpha} \Phi \ \Lambda^{\sigma-\alpha} \Psi \right\|_{L^2} \leq C \left\| \Phi \right\|_{s+\frac{1}{4}+\alpha} \left\| \Psi \right\|_{\sigma+\frac{1}{4}-\alpha} = C \left\| \phi \right\|_{s+\frac{1}{4}+\alpha} \left\| \psi \right\|_{\sigma+\frac{1}{4}-\alpha} ,$$

concluding the proof. ☐

Let $\phi, \psi$ be given functions, let us denote with $U = U(\phi, \psi)$ the solution to the linear fractional-diffusion equation with forcing

$$\begin{cases} U(\phi,\psi)_t + \Lambda^3 U(\phi,\psi) = \partial_x [\![\mathcal{H}, \phi]\!] \Lambda^3 \psi, \\ U(\phi,\psi)|_{t=0} = 0. \end{cases} \tag{3.3}$$

**Lemma 3.3.** *For any $\phi \in L^4_T \dot{H}^{s+\frac{3}{4}}$, $s \geq 1/2$ and $\psi \in L^4_T \dot{H}^{9/4}$ the following inequality holds true*

$$\left\| U(\phi,\psi) \right\|_{L^4_T \dot{H}^{s+\frac{3}{4}}} \leq C \left\| \phi \right\|_{L^4_T \dot{H}^{s+\frac{3}{4}}} \left\| \psi \right\|_{L^4_T \dot{H}^{9/4}} .$$

**Proof.** We write $U$ instead of $U(\phi,\psi)$ for sake of simplicity. A standard energy estimate combined with Lemma 3.2 gives that

$$\begin{aligned}
\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \left\| U \right\|_s^2 + \left\| U \right\|_{s+\frac{3}{2}}^2 &= \int \Lambda^s \left( \partial_x [\![\mathcal{H}, \phi]\!] \Lambda^3 \psi \right) \Lambda^s U \mathrm{d}x, \\
&\leq \frac{1}{2} \left\| U \right\|_{s+\frac{3}{2}}^2 + C \left\| \Lambda^{s-\frac{1}{2}} \left( [\![\mathcal{H}, \phi]\!] \Lambda^3 \psi \right) \right\|_{L^2}^2 , \\
&\leq \frac{1}{2} \left\| U \right\|_{s+\frac{3}{2}}^2 + C \left\| \phi \right\|_{s+\frac{3}{4}}^2 \left\| \psi \right\|_{9/4}^2 ,
\end{aligned}$$

hence, integrating in time, we obtain the bound

$$\left\| U \right\|_{L^\infty_T \dot{H}^s}^2 + \left\| U \right\|_{L^2_T \dot{H}^{s+\frac{3}{2}}}^2 \leq C \left\| \phi \right\|_{L^4_T \dot{H}^{s+\frac{3}{4}}}^2 \left\| \psi \right\|_{L^4_T \dot{H}^{9/4}}^2 .$$

The interpolation estimate

$$\left\| U \right\|_{L^4_T \dot{H}^{s+\frac{3}{4}}} \leq \left\| U \right\|_{L^\infty_T \dot{H}^s}^{1/2} \left\| U \right\|_{L^2_T \dot{H}^{s+\frac{3}{2}}}^{1/2} \leq C \left( \left\| U \right\|_{L^\infty_T \dot{H}^s} + \left\| U \right\|_{L^2_T \dot{H}^{s+\frac{3}{2}}} \right) ,$$

combined with the Young inequality $ab \leq \frac{1}{2} \left( a^2 + b^2 \right)$ provides the desired control. ☐

## 4. Proof of Theorem 2.1

Solving the Cauchy problem (1.4) is equivalent to find a solution to the integral equation

$$f(x,t) = e^{-t\Lambda^3} f_0(x) + \int_0^t e^{-(t-t')\Lambda^3} \partial_x [\![ \mathcal{H}, f ]\!] \Lambda^3 f(x,t')\, \mathrm{d}t', \qquad (x,t) \in \mathbb{S}^1 \times [0,T], \tag{4.1}$$

we hence prove that the map

$$f \longmapsto e^{-t\Lambda^3} f_0 + U(f,f), \tag{4.2}$$

where $U(f,f)$ is defined as the solution of (3.3), has a unique fixed point in the space $L^4_T \dot{H}^{9/4}$. Indeed for any $T \in (0,\infty]$ and $\phi, \psi \in L^4_T \dot{H}^{9/4}$ we invoke Lemma 3.3 obtaining that

$$\left\| U(\phi,\psi) \right\|_{L^4_T \dot{H}^{9/4}} \leq C \left\| \phi \right\|_{L^4_T \dot{H}^{9/4}} \left\| \psi \right\|_{L^4_T \dot{H}^{9/4}},$$

thus proving that $U$ is a continuous bilinear map form $\left(L^4_T \dot{H}^{9/4}\right)^2$ onto $L^4_T \dot{H}^{9/4}$. What remains to prove in order to apply Lemma 3.1 to the functional equality (4.2) is that for any $f_0 \in \dot{H}^{3/2}$ here exists a $T = T(f_0) \in (0,\infty]$ such that $\left\| e^{-\bullet \Lambda^3} f_0 \right\|_{L^4_T \dot{H}^{9/4}}$ can be made arbitrarily small. We treat at first the case of small initial datum: we use (3.1) in order to compute

$$\begin{aligned}
\left\| e^{-\bullet \Lambda^3} f_0 \right\|_{L^4_T \dot{H}^{9/4}} &= \left( \int_0^T \left( \sum_n e^{-2t|n|^3} |n|^{9/2} \left| \hat{f}_0(n) \right|^2 \right)^2 \mathrm{d}t \right)^{1/4}, \\
&\leq \left( \sum_n \left( \int_0^T e^{-4t|n|^3} |n|^9 \left| \hat{f}_0(n) \right|^4 \mathrm{d}t \right)^{1/2} \right)^{1/2}, \\
&\leq \frac{1}{\sqrt{2}} \left\| f_0 \right\|_{3/2}.
\end{aligned} \tag{4.3}$$

The bound derived above is independent of $T > 0$, hence

$$\left\| e^{-\bullet \Lambda^3} f_0 \right\|_{L^4 \dot{H}^{9/4}} \leq \frac{1}{\sqrt{2}} \left\| f_0 \right\|_{3/2}.$$

The above inequality proves that the application $f_0 \mapsto e^{-\bullet \Lambda^3} f_0$ is continuous in zero as an application from $\dot{H}^{3/2}$ to $L^4 \dot{H}^{9/4}$, hence there exists a $\varepsilon_0 > 0$ s.t. if $\left\| f_0 \right\|_{3/2} \leq \varepsilon_0$ the conditions of Lemma 3.1 are satisfied and there exists a unique solution of (1.4) in the space $L^4\left(\mathbb{R}_+; \dot{H}^{9/4}\right)$.

We consider now the case of large initial data in $\dot{H}^{3/2}$. Let us set $\rho > 0$ and define $f_0 = \bar{f}_0 + \underline{f_0}$ where $\hat{\bar{f}}_0(n) = \mathbf{1}_{\{|n| > \rho\}}(n) \hat{f}_0(n)$. Since $\bar{f}_0$ is localized on high-frequencies, we exploit (4.3) in order to obtain that

$$\left\| e^{-\bullet \Lambda^3} \bar{f}_0 \right\|_{L^4_T \dot{H}^{9/4}} \leq C \left\| \bar{f}_0 \right\|_{3/2} = o(1) \quad \text{as} \quad \rho \to \infty,$$

and thus can be made arbitrarily small letting $\rho = \rho(f_0)$ be large enough. Next we use the localization in the Fourier space in order to argue that

$$\left\| e^{-\bullet \Lambda^3} \underline{f_0} \right\|_{L^4_T \dot{H}^{9/4}} \leq \rho^{3/4} \left\| e^{-\bullet \Lambda^3} \underline{f_0} \right\|_{L^4_T \dot{H}^{3/2}},$$

while using again (3.1) we obtain that

$$\left\| e^{-\bullet \Lambda^3} \underline{f_0} \right\|_{L^4_T \dot{H}^{3/2}} \leq \qquad \left( \sum_n |n|^3 \sqrt{\frac{1 - e^{-4T|n|^3}}{4|n|^3}} \left| \underline{\hat{f_0}}(n) \right|^2 \right)^{1/2} \leq C \sqrt[4]{T} \left\| f_0 \right\|_{3/2},$$

which in turn implies that
$$\left\| e^{-\bullet \Lambda^3} \underline{f_0} \right\|_{L^4_T \dot{H}^{9/4}} \leq C \sqrt[4]{\rho^3 T} \left\| f_0 \right\|_{3/2},$$

thus proving that if $T \ll 1/\rho^3$ we can again apply Lemma 3.1 proving the existence part of the statement of Theorem 2.1 for arbitrary data in $\dot{H}^{3/2}$. The fact that

$$f \in L^\infty \left( [0,T] ; \dot{H}^{3/2} \right) \cap L^2 \left( [0,T] ; \dot{H}^3 \right),$$

follows by a $\dot{H}^{3/2}$ energy estimate on Eq. (1.4), while using the Duhamel formulation (4.1) we obtain that, fixed $n \in \mathbb{Z}$, the application $t \mapsto \hat{f}(n,t)$ is continuous over $[0,T]$, the Lebesgue dominated convergence allows us to conclude that $f \in \mathcal{C}\left( [0,T] ; \dot{H}^{3/2} \right)$, concluding the proof of Theorem 2.1. $\square$


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