A Simple Characterization of Useful Topologies in Mathematical Utility Theory

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Abstract
In this paper, we present a simple axiomatization of *useful topologies*, i.e., topologies on an arbitrary set, with respect to which every continuous total preorder admits a continuous utility representation. We introduce the concept of *weak open and closed countable chain condition* (*WOCCC*) relative to a topology, and we then show that a useful topology always satisfies this condition. The most important result in the paper shows that a completely regular topology is useful if and only if it is separable and it satisfies *WFOCCC* (a stricter version of *WOCCC*). In this way, we generalize all the previous results concerning useful topologies. We finish the paper by presenting a simple axiomatization of useful topologies under the well-known *Souslin Hypothesis*.

Keywords Useful topology · Complete separable system · Weak topology · Completely regular space · Souslin Hypothesis

1 Introduction

The problem of identifying all the *useful topologies* on a set (i.e., all the topologies on a set, such that every *continuous total preorder* has a *continuous utility representa-*

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tion, or equivalently can be represented by a continuous real-valued order-preserving function) may be considered as the most important problem in Mathematical Utility Theory. Indeed, it is of extreme interest in Mathematical Economics and Optimization Theory to make available conditions guaranteeing the existence of a continuous utility function for every continuous total preorder. Clearly, in the case of a *compact topology*, usefulness guarantees that every continuous total preorder has minimum and maximum, which can be found by "optimizing" a continuous utility representation. Continuity of a total preorder on a topological space means that the given topology is finer than the *order topology* induced by the preorder.

The concept of a useful topology was introduced by Herden [16], who also provided, in the opinion of the authors, the most interesting results in this field (see, e.g., Herden [17, 18]). This fact is not surprising at all, since it was Herden [14, 15] who provided the most general results concerning the existence of (continuous) utility functions, admitting as corollaries sparse (so to say) and famous achievements, like for example the classical theorems by Eilenberg [12] (**ET**) and Debreu [10, 11] (**DT**), according to which every continuous total preorder on a *connected and separable*, and, respectively, on a *second countable* topological space admits a continuous utility representation. Therefore, **ET** and **DT** can be viewed as sufficient conditions in order that a topology is useful. From time to time, useful topologies are referred to as *continuously representable topologies* (see, e.g., Campión et al. [8]).

Another very important result in Mathematical Utility Theory was proven by Estévez and Hervés [13], who showed that separability is a necessary condition on a metric space, in order that every continuous total preorder admits a continuous utility representation. In other words, a useful metrizable topology is necessarily separable. Since every separable metric is second countable, this latter result can be combined with **DT**, to state that a metrizable topology is useful if and only if it is second countable, or equivalently *separable*. We will refer to this latter result as Estévez–Hervés' theorem (**EHT**).

With help of the concept of a useful topology, the fundamental theorems above can be expressed as follows:

ET: *Every connected and separable topology is useful.*
DT: *Every second countable topology is useful.*
EHT: *A metrizable topology is useful if and only if it is separable, or equivalently second countable.*

More recently, Bosi and Herden [3] introduced the concept of a *complete separable system*, and then, after proving the existence of a bijective correspondence between complete separable systems on one hand, and equivalence classes of continuous total preorders on the other hand; they presented a simple characterization of useful topologies in terms of second countability of all the topologies generated by complete separable systems.

The most recent results have been presented, to the best of our knowledge, by Bosi and Zuanon [5], who, in particular, used the following three important considerations that have to be considered as crucial in the analysis of useful topologies:

1. The coarsest topology with respect to which every continuous total preorder is continuous coincides with the *weak topology of continuous functions*;

2. Since the weak topology of continuous functions is completely regular, actually it is not restrictive to limit ourselves to the consideration of completely regular topologies, when dealing with useful topologies;
3. A useful completely regular topology is necessarily separable.

In this paper, we are, therefore, primarily concerned with completely regular topologies. We introduce the concept of *weak open and closed countable chain condition* (*WOCCC*) of a topology (which generalizes the concept of *open and closed countable chain condition* (*OCCC*), introduced by Herden and Pallack [17]), to first prove that a topology is useful if and only if every subtopology generated by a *complete separable system* is separable and satisfies *WOCCC* (see, also, Bosi and Zuanon [5]).

The most important result in this paper shows that a completely regular topology is useful if and only if it is separable and it satisfies *WFOCCC* (a stricter version of *WOCCC*).

We finish the paper by incorporating the *Souslin Hypothesis* **SH** in our speeches, and therefore proving that the validity of **SH** is equivalent to the equivalence between a topology to be useful and to satisfy *WOCCC*.

2 Notation and Preliminary Results

We begin this section by presenting the classical definitions concerning binary relation.

Definition 2.1 Let $\precsim$ be a binary relation on a nonempty set X (i.e., $\precsim \subset X \times X$). Then, $\precsim$ is said to be

1. *reflexive*, if $x \precsim x$, for every $x \in X$;
2. *transitive*, if $(x \precsim y)$ and $(y \precsim z)$ imply $(x \precsim z)$, for all $x, y, z \in X$;
3. *antisymmetric*, if $(x \precsim y)$ and $(y \precsim x)$ imply $x = y$, for all $x, y \in X$;
4. *total*, if $(x \precsim y)$ or $(y \precsim x)$, for all $x, y \in X$;
5. *linear* (or *complete*), if either $(x \precsim y)$ or $(y \precsim x)$, for all $x \neq y$ $(x, y \in X)$;
6. a *preorder*, if $\precsim$ is reflexive and transitive;
7. an *order*, if $\precsim$ is an antisymmetric preorder;
8. a *chain*, if $\precsim$ is a linear order.

The *strict part* (or *asymmetric part*) of a preorder $\precsim$ on X is defined as follows: for all $x, y \in X$: $x \prec y$ if and only if $(x \precsim y) and not (y \precsim x)$. Furthermore, the *symmetric part* $\sim$ of a preorder $\precsim$ on X is defined as follows for all $x, y \in X$: $x \sim y$ if and only if $(x \precsim y) and (y \precsim x)$. We have that $\sim$ is an *equivalence* on X, and we denote by $X_{|\sim}$ the *quotient set*, made up by the equivalence classes $[x] = \{z \in X | z \sim x\}$ $(x \in X)$.

If $\precsim$ is a preorder on X, then the *quotient order* $\precsim_{|\sim}$ on the quotient set $X_{|\sim}$ is defined as follows: for all $x, y \in X$: $[x] \precsim_{|\sim} [y] \Leftrightarrow x \precsim y$. If $\precsim$ is a total preorder on X, then $\preceq := \precsim_{|\sim}$ is a linear order on $X_{|\sim}$.

A subset D of a *preordered set* $(X \precsim)$ is said to be *decreasing* if $(x \in D)$ and $(z \precsim x)$ imply $z \in D$, for all $z \in X$.

If t is a *topology* on X, then a family $\mathcal{B}' \subset t$ is said to be a *subbasis* of t if the family $\mathcal{B}$ consisting of all possible intersections of finitely many elements of $\mathcal{B}'$ is a *basis* of t (i.e., every set $O \in t$ is the union of some sets of $\mathcal{B}$).

Let us summarize, in the following definition, the main classical topological concepts which will be used in this paper.

Definition 2.2 A topology t on X is said to be

(i) *second countable*, if there is a countable basis $\mathcal{B} = \{B_n | n \in \mathbb{N}^+\}$ for t;
(ii) *separable*, if there exists a countable subset D of X, such that $D \cap O \neq \emptyset$ for every $O \in t$;
(iii) *Hausdorff*, if, given any two points $x, y \in X$ with $x \neq y$, there exist two open disjoint sets U, V, such that $x \in U$ and $y \in V$;
(iv) *completely regular*, if for every $x \in X$, and every closed set $F \subseteq X$ not containing x, there exists a continuous function $f : (X, t) \to ([0, 1], t_{nat})$, such that $f(x) = 0$ and $f(y) = 1$ for $y \in F$.

Here, t_{nat} stands for the *natural* (*interval*) topology on the real line $\mathbb{R}$.

We recall that, for two topologies t', t on set X, t' is said to be *coarser* (respectively, *finer*) than t if it happens that $t' \subset t$ ($t \subset t'$). If t' is coarser than t, we also say that t' is a *subtopology* of t.

Let $(X, \precsim)$ be an arbitrarily chosen *preordered set*. We define, for every point $x \in X$, the following subsets of X:

$$d_{\precsim}(x) := \{z \in X \mid z \precsim x\}, i_{\precsim}(x) := \{z \in X \mid x \precsim z\},$$
$$l_{\precsim}(x) := \{z \in X \mid z \prec x\}, r_{\precsim}(x) := \{z \in X \mid x \prec z\}.$$

For any pair $(x, y) \in X \times X$, such that $(x, y) \in \prec$, we shall denote by $]x, y[_{\precsim}$ the (maybe empty) *open interval* defined as $]x, y[_{\precsim} := r_{\precsim}(x) \cap l_{\precsim}(y)$.

A pair $(x, y) \in \prec$ is said to be a *jump* in $(X, \precsim)$ if $]x, y[= \emptyset$.

All the previous definitions can be found, for example, in Herden [14].

Definition 2.3 A total preorder $\precsim$ on the topological space (X, t) is said to be *continuous* if the sets $l_{\precsim}(x) = \{z \in X | z \prec x\}$ and $r_{\precsim}(x) = \{z \in X | x \prec z\}$ are open subsets of X for every $x \in X$.

Notice that a preorder $\precsim$ on a topological space (X, t) is continuous if and only if t is *finer* than the *order topology* $t^{\precsim}$ on X associated with $\precsim$, which is precisely the topology *generated* by the family $\{l_{\precsim}(x) | x \in X\} \cup \{r_{\precsim}(x) | x \in X\}$ (i.e., $\{l_{\precsim}(x) | x \in X\} \cup \{r_{\precsim}(x) | x \in X\}$ is a subbasis of t). In other words, $t^{\precsim}$ is the coarsest topology on X, such that the sets $l_{\precsim}(x)$ and $r_{\precsim}(x)$ are open for every $x \in X$.

If t is a topology on X, and X' is any nonempty subset of X, then the *relativized topology* $t_{|X'}$ on X' is defined as follows: $t_{|X'} := \{O \cap X' \mid O \in t\}$.

Definition 2.4 A real-valued function u on a preordered set $(X, \precsim)$ is said to be

1. *increasing*, if, for all $x, y \in X$,

$$x \precsim y \Rightarrow u(x) \leq u(y);$$

2. *order-preserving*, if u is increasing and, for all $x, y \in X$,

$$x \prec y \Rightarrow u(x) < u(y);$$

3. a *utility function*, if $x \precsim y$ is equivalent to $u(x) \leq u(y)$, for all $x, y \in X$.

The following definition is found in Herden and Pallack [18].

Definition 2.5 A preorder $\precsim$ on a topological space (X, t) is said to be *weakly continuous* if, for every pair $(x, y) \in \prec$, there exists a continuous and increasing real-valued function u_{xy} on X, such that $u_{xy}(x) < u_{xy}(y)$.

We now recall the definition of a *complete separable system* on a topological space (X, t), which is a particularly relevant case of a *linear separable system*, as introduced by Herden [14, 15] (see Bosi and Herden [3]).

Definition 2.6 Let a topology t on X be given. A family $\mathcal{E}$ of open subsets of the topological space (X, t), such that $\bigcup_{E \in \mathcal{E}} E = X$, is said to be a *complete separable system* on (X, t) if it satisfies the following conditions:

S1: There exist sets $E_1 \in \mathcal{E}$ and $E_2 \in \mathcal{E}$, such that $\overline{E_1} \subset E_2$.
S2: For all sets $E_1 \in \mathcal{E}$ and $E_2 \in \mathcal{E}$, such that $\overline{E_1} \subset E_2$, there exists some set $E_3 \in \mathcal{E}$, such that $\overline{E_1} \subset E_3 \subset \overline{E_3} \subset E_2$.
S3: For all sets $E \in \mathcal{E}$ and $E' \in \mathcal{E}$, at least one of the following conditions $E = E'$, or $\overline{E} \subset E'$, or $\overline{E'} \subset E$ holds.

In the case when $\precsim$ is a preorder on X, a complete separable system $\mathcal{E}$ on (X, t) is said to be a *complete decreasing separable system* on $(X, \precsim, t)$ as soon as every set $E \in \mathcal{E}$ is required to be decreasing.

We just mention the fact that the requirement according to which X belongs to any complete separable system $\mathcal{E}$ on (X, t) is not restrictive at all.

The following proposition was proved by Bosi and Herden [3, Proposition 2.1].

Proposition 2.7 *Let $\mathcal{E}$ to be a complete separable system on a topological space (X, t). Then, $\mathcal{E}$ satisfies the following conditions.*

(i) $\mathcal{E}^c := \mathcal{E} \cup \{\overline{E} | E \in \mathcal{E}\}$ *is linearly ordered by set inclusion;*
(ii) $E = \bigcup_{\overline{E'} \subset E,\, E' \in \mathcal{E}} E' = \bigcup_{\overline{E'} \subset E,\, E' \in \mathcal{E}} \overline{E'}$ *for every* $E \in \mathcal{E}$*;*
(iii) $\overline{E} = \bigcap_{\overline{E} \subset E' \in \mathcal{E}} E' = \bigcap_{\overline{E} \subset E' \in \mathcal{E}} \overline{E'}$ *for every* $E \in \mathcal{E}$.

The following theorem holds, presenting different conditions all equivalent to the continuity of a total preorder on a topological space (see Bosi and Zuanon [4, Theorem 2.23]).

Theorem 2.8 *Let $(X, \precsim, t)$ be a totally preordered topological space. Then, the following conditions are equivalent:*

(i) $\precsim$ *is continuous;*
(ii) *The order topology* $t^{\precsim}$ *is coarser than* t*;*
(iii) $d_{\precsim}(x) = \{y \in X | y \precsim x\}$ *is a closed subset of* X*, and* $l_{\precsim}(x) = \{y \in X | y \prec x\}$ *is an open subset of* X*, for every point* $x \in X$*;*
(iv) $i_{\precsim}(x) = \{z \in X | x \precsim z\}$ *is a closed subset of* X*, and* $r_{\precsim}(x) = \{z \in X | x \prec z\}$ *is an open subset of* X*, for every point* $x \in X$*;*
(v) $\precsim$ *is weakly continuous;*
(vi) *For every pair* $(x, y) \in \prec$*, a complete decreasing separable system* $\mathcal{E}_{xy}$ *on* X *can be chosen, in such a way that there exist sets* $E \subset \overline{E} \subset E'$ *in* $\mathcal{E}_{xy}$*, such that* $x \in E$ *and* $y \notin E'$*.*

The following proposition is found in Herden [14, Lemma 3.2].

Proposition 2.9 *Let* $\precsim$ *be a total preorder on a set* X*. Then, the following conditions are equivalent:*

(i) *There exists a utility function* u *on* $(X, \precsim)$*, which in particular can be considered as continuous with respect to the order topology* $t^{\precsim}$ *on* X*;*
(ii) *The following conditions are verified:*
   (a) *The order topology* $t^{\precsim}$ *on* X *is separable;*
   (b) *There are only countable many jumps in* $(X_{|\sim}, \precsim_{|\sim})$*.*

The easy proof of the following propositions is left to the reader.

Proposition 2.10 *Let* $(X, \precsim)$ *be a totally preordered set. Then, the following conditions are equivalent for all* $x, y \in X$*:*

(i) *A pair* $([x], [y]) \in (X_{|\sim} \times X_{|\sim})$ *is a jump in* $(X_{|\sim}, \precsim_{|\sim})$*;*
(ii) $l_{\precsim}(y) = d_{\precsim}(x)$*.*

Proposition 2.11 *Let* (X, t) *be a topology on a set* X*, and consider a continuous total preorder* $\precsim$ *on* (X, t)*. Then, the following conditions are equivalent for all* $x, y \in X$*:*

(i) *A pair* $([x], [y]) \in (X_{|\sim} \times X_{|\sim})$ *is a jump in* $(X_{|\sim}, \precsim_{|\sim})$*;*
(ii) $l_{\precsim}(y)$ *is an open and closed set, and*

$$l_{\precsim}(y) \subsetneqq \bigcap_{l_{\precsim}(y) \subsetneqq l_{\precsim}(z)} l_{\precsim}(z).$$

Remark 2.12 According to Proposition 2.11, given a continuous total preorder $\precsim$ on a topological space (X, t), if we denote by $\mathbb{L}^J$ the family of all the strict lower sections $l_{\precsim}(y)$ associated with the right endpoints of all the jumps $([x], [y]) \in (X_{|\sim} \times X_{|\sim})$ in $(X_{|\sim}, \precsim_{|\sim})$, we have that $\mathbb{L}^J$ is a family of open and closed subsets of (X, t), satisfying the following property for every $y \in X$, such that $([x], [y]) \in (X_{|\sim} \times X_{|\sim})$ is a jump in $(X_{|\sim}, \precsim_{|\sim})$ for some point $x \in X$:

$$l_{\precsim}(y) \subsetneqq \bigcap_{l_{\precsim}(y) \subsetneqq l_{\precsim}(z),\, l_{\precsim}(y), l_{\precsim}(z) \in \mathbb{L}^J} l_{\precsim}(z).$$

3 Characterizations of Useful Topologies

The fundamental definition of a *useful topology* is originally due to Herden [16].

Definition 3.1 A topology t on X is said to be *useful* if every continuous total preorder on the topological space (X, t) has a continuous utility representation u.

To provide new and simple characterizations of useful topologies, let us introduce the definition of a *weakly isolated set* in a complete separable system.

Definition 3.2 Let $\mathcal{E}$ be a complete separable system on a topological space (X, t). Then, a set $O \in \mathcal{E}$ is said to be *weakly isolated* if

$$O \subsetneqq \bigcap_{O \subsetneqq O'', \, O'' \in \mathcal{O}} O''.$$

Definition 3.3 A chain $\mathcal{O}$ of open and closed subsets of a topological space (X, t) is said to be *weakly isolated* if it only consists of weakly isolated sets.

Definition 3.4 We say that a topology t on a nonempty set X satisfies the *weak open and closed countable chain condition* (*WOCCC*), if every weakly isolated chain $\mathcal{O}$ of open and closed subsets of X is countable.

Remark 3.5 Notice that the above condition *WOCCC* generalizes the *open and closed countable chain condition* (*OCCC*), according to which every chain $\mathcal{O}$ of open and closed subsets of a topological space (X, t) is countable, provided that it only consists of *isolated sets*, i.e., sets $O \in \mathcal{O}$ such that

$$\bigcup_{O' \subsetneqq O, \, O' \in \mathcal{O}} O' \subsetneqq O \subsetneqq \bigcap_{O \subsetneqq O'', \, O'' \in \mathcal{O}} O''.$$

The *OCCC* condition was introduced by Herden [17], and used in Bosi and Zuanon [5, Definition 3.3]

We are now ready for a first characterization of useful topologies. The following theorem improves Bosi and Herden [3, Theorem 3.1].

Theorem 3.6 *The following assertions are equivalent on a topology t on set X:*

(i) *t is useful.*
(ii) *For every complete separable system $\mathcal{E}$ on (X, t), the topology $t_{\mathcal{E}}$ generated by $\mathcal{E}$ is second countable.*
(iii) *For every complete separable system $\mathcal{E}$ on (X, t), the topology $t_{\mathcal{E}}$ generated by $\mathcal{E}$ is separable and t satisfies WOCCC.*

Proof (i) $\Leftrightarrow$ (ii). See Bosi and Herden [3, Theorem 3.1].
(ii) $\Rightarrow$ (iii). Since $t_{\mathcal{E}}$ is second countable, then it is clearly separable. Furthermore, since every family chain $\mathcal{O}$ of open and closed subsets of (X, t) is a complete separable system, it is easily seen that second countability of $t_{\mathcal{O}}$ implies *WOCCC*.

(iii) $\Rightarrow$ (i). Consider any continuous total preorder $\precsim$ on (X, t). Since the topology $t_{\mathcal{E}}$ generated by the complete separable system $\mathcal{E} := \{l_{\precsim}(x)\}_{x \in X} \cup X$ on (X, t) is separable, it is easy to realize that also the topology $t^{\precsim}$ generated by $\mathcal{E} \cup \{X \setminus \overline{E} | E \in \mathcal{E}\}$ is separable. However, this latter topology is precisely the order topology on X associated with $\precsim$. Further, the condition *WOCCC* implies that there are only countably many jumps in $(X_{|\sim}, \precsim_{|\sim})$. Therefore, we have that t is useful by Propositions 2.9 and 2.11. This consideration completes the proof.

□

Lemma 3.7 *Let $\mathcal{E}$ be a complete separable system on a topological space (X, t). For every $E \in \mathcal{E}$, and for every $x \in E$, there exists a complete separable subsystem $\mathcal{E}' \subset \mathcal{E}$, such that $x \in E' \subset E$ for every $E' \in \mathcal{E}'$ ($E' \subsetneqq X$).*

Proof Given any complete separable system $\mathcal{E}$ on a topological space (X, t), consider any point $x \in E$, $E \in \mathcal{E}$. Since $\mathcal{E}$ is a complete separable system, by Proposition 2.7, condition (ii), there exists $E_1 \in \mathcal{E}$, such that $x \in \overline{E_1} \subset E$. Then, the family $\mathcal{E}' := \{E' \in \mathcal{E} \mid \overline{E_1} \subset E' \subset \overline{E'} \subset E\} \cup X$ is a complete separable system on (X, t), such that $x \in E' \subset E$ for every $E' \in \mathcal{E}'$, such that $E' \subsetneqq X$. □

We now recall the definition of the continuous total preorder, which is induced by a complete separable system $\mathcal{E}$ on a topological space (X, t).

Definition 3.8 For every complete separable system $\mathcal{E}$ on (X, t), the *continuous total preorder $\precsim_{\mathcal{E}}$ on (X, t), that is induced by $\mathcal{E}$*, is defined to be

$$\precsim_{\mathcal{E}} := \{(x, y) \in X \times X | \forall E \in \mathcal{E} \, (y \in \mathcal{E} \Rightarrow x \in \mathcal{E})\}.$$

Clearly, we shall denote by $\sim_{\mathcal{E}}$ the indifference relation associated with the total preorder $\precsim_{\mathcal{E}}$.

Lemma 3.9 *Let $\mathcal{E}$ be a complete separable system on a topological space (X, t). Then, there is a one-to-one correspondence between the set of all the open and closed subsets $O \in \mathcal{E}$, such that*

$$O \subsetneqq \bigcap_{O \subsetneqq O'',\, O'' \in \mathcal{O}} O''$$

and the set of all the jumps $([x], [y])$ in $(X_{|\sim_{\mathcal{E}}}, \precsim_{\mathcal{E}_{|\sim_{\mathcal{E}}}})$.

Proof First, consider a complete separable system $\mathcal{E}$ on (X, t), and an open and closed set $O \in \mathcal{E}$, such that $O \subsetneqq \bigcap_{O \subsetneqq O'',\, O'' \in \mathcal{O}} O''$. Then, for every point $y \in \bigcap_{O \subsetneqq O'',\, O'' \in \mathcal{O}} O'' \setminus O$, we have that $l_{\precsim_{\mathcal{E}_{|\sim_{\mathcal{E}}}}}(y) = O$, and in addition $O = d_{\precsim_{\mathcal{E}_{|\sim_{\mathcal{E}}}}}(x)$ for every $x \in O$, so that we may associate O with the jump $([x], [y])$ in $(X_{|\sim_{\mathcal{E}}}, \precsim_{\mathcal{E}_{|\sim_{\mathcal{E}}}})$ (see also the proof of [16, Lemma 2.3]). On the other hand, consider any jump $([x], [y])$ in $(X_{|\sim_{\mathcal{E}}}, \precsim_{\mathcal{E}_{|\sim_{\mathcal{E}}}})$. Then, from the definition of the continuous total preorder $\precsim_{\mathcal{E}}$ on (X, t), there exists a set $E \in \mathcal{E}$ such that $x \in E$, $y \notin E$. From the above Lemma 3.7, there exists a complete separable subsystem $\mathcal{E}' \subset \mathcal{E}$ such that $x \in E' \subset E$ for every $E' \in \mathcal{E}'$ ($E' \subsetneq X$). Since $([x], [y])$ is a jump in $(X_{|\sim\mathcal{E}}, \precsim_{\mathcal{E}_{|\sim_{\mathcal{E}}}})$, it is immediate to check that $\mathcal{E}' \backslash X$ must reduce to a singleton $E := O$, which is therefore both open and closed. The completeness of the separable system $\mathcal{E}$ implies that O is uniquely determined. To finish the proof, we just notice that we have actually defined a one-to-one correspondence between the set of all the open and closed subsets $O \in \mathcal{E}$, such that

$$O \subsetneq \bigcap_{O \subsetneq O'',\, O'' \in \mathcal{O}} O''$$

and the set of all the jumps $([x], [y])$ in $(X_{|\sim\mathcal{E}}, \precsim_{\mathcal{E}_{|\sim_{\mathcal{E}}}})$. □

The following theorem improves Bosi and Zuanon [5, Theorem 3.1].

Theorem 3.10 *Let t be a topology on a nonempty set X. Then, the following statements hold true:*

1. *If t is separable and satisfies WOCCC, then t is useful.*
2. *If t is useful, then t satisfies WOCCC.*
3. *If t is a completely regular useful topology, then t is separable.*

Proof 1. To show that t is useful, if t is separable, and satisfies *WOCCC*, consider any continuous total preorder $\precsim$ on (X, t). Since t is separable, we have that the order topology $t^{\precsim}$ ($\subset t$) is also separable. Furthermore, by Lemma 3.9, there are only countably many jumps in $(X_{|\sim}, \precsim_{|\sim})$, and the continuous total preorder $\precsim$ on (X, t) has a continuous utility representation by Proposition 2.9. Indeed, the aforementioned *Debreu Open Gap Lemma* (see Debreu [10, 11]) guarantees that a continuous total preorder, which admits a utility function, also admits a continuous utility function. This consideration completes this part of the proof.

2. To show that a useful topology t satisfies *WOCCC*, assume, by contraposition, that there exists an uncountable weakly isolated chain $\mathcal{O}$ of open and closed sets. Consider the continuous total preorder $\precsim_{\mathcal{O}}$ on (X, t), defined to be, for all $x, y \in X$,

$$x \precsim_{\mathcal{O}} y \Leftrightarrow \forall O \in \mathcal{O} (y \in O \Rightarrow x \in O).$$

A jump in $(X_{|\sim_{\mathcal{O}}}, \precsim_{|\sim_{\mathcal{O}}})$ is associated with every weakly isolated set $O \in \mathcal{O}$. Since there are uncountably many jumps in $(X_{|\sim_{\mathcal{O}}}, \precsim_{|\sim_{\mathcal{O}}})$, the continuous total preorder $\precsim_{\mathcal{O}}$ does not admit a (continuous) utility representation by Proposition 2.9. Hence, t is not useful.

3. See Bosi and Zuanon [5, Theorem 3.1, 3]. □

Remark 3.11 The above condition 3 is precisely condition 3 in Bosi and Zuanon [5, Theorem 3.1]. It has been restated in Theorem 3.10 for the ease of the exposition, and to provide a list of conditions concerning useful topologies, which are meanwhile available without invoking subtopologies.

The classical theorems by Eilenberg [12], Debreu [10, 11], and Estévez and Hervés [13] are now derived immediately as corollaries of Theorem 3.10.

Theorem 3.12 (Eilenberg theorem) *Every connected and separable topology is useful.*

Proof Just consider that t is separable, and it obviously satisfies *WOCCC*. □

Theorem 3.13 (Debreu theorem) *Every second countable topology is useful.*

Proof Clearly, every second countable topology is separable. Furthermore, it is easy to see that every second countable topology must satisfy *WOCCC*. □

Theorem 3.14 (Estévez–Hervés' theorem) *A metrizable topology is useful if and only if it is separable (or, equivalently, second countable).*

Proof If a metrizable topology t is separable, then it is second countable, and therefore, it is useful by the above Theorem 3.13. Conversely, if a topology is useful and metrizable, then it is in particular completely regular, and therefore, it must be separable by Theorem 3.10, 3. □

We recall that, when we consider the space $C(X, t, \mathbb{R})$ of all continuous real-valued function on some topological space (X, t), the *weak topology* on X, $\sigma(X, C(X, t, \mathbb{R}))$, is the coarsest topology on X satisfying the property that every continuous real-valued function on (X, t) remains being continuous.

It is well known that $(X, \sigma(X, C(X, t, \mathbb{R})))$ is a completely regular space (cf., for instance, Cigler and Reichel [9, Satz 10, p. 101], and Aliprantis and Border [1, Theorem 2.55 and Corollary 2,56]), and $(X_{|\sim_C}, \sigma(X, C(X, t, \mathbb{R}))_{|\sim_C})$ is a completely regular Hausdorff-space, where $(X_{|\sim_C}, \sigma(X, C(X, t, \mathbb{R}))_{|\sim_C})$ is the quotient space of $\sigma(X, C(X, t, \mathbb{R}))$ that is induced by the equivalence relation $\sim_C$, according to which, for all points $x, y \in X$, $x \sim_C y$ if and only if $f(x) = f(y)$ for all functions $f \in C(X, t, \mathbb{R})$.

The following lemma appears in Bosi and Zuanon [5, Lemma 3.1].

Lemma 3.15 *The coarsest topology on X satisfying the property that all continuous total preorders on (X, t) remain being continuous is $\sigma(X, C(X, t, \mathbb{R}))$ (of course, this assertion is equivalent to the statement that a total preorder $\precsim$ on (X, t) is continuous if and only if it is continuous with respect to $\sigma(X, C(X, t, \mathbb{R}))$).*

The previous considerations guarantee that complete regularity is a necessary condition for a topology t on X to be useful, when we further assume that t is the coarsest topology on X, such that every continuous preorder remains continuous.

Definition 3.16 Let $\mathcal{E}$ be a complete separable system on a topological space (X, t). Then, a set $O \in \mathcal{E}$ is said to be *weakly functionally isolated* if there exists $x_0 \in \bigcap_{O \subsetneq O'',\, O'' \in \mathcal{O}} O''$, and a continuous function $f_{x_0} : (X, t) \to ([0, 1], t_{nat})$, such that $f_{x_0}(x_0) = 0$, and $f_{x_0}(y) = 1$ for $y \in O$.

Definition 3.17 A chain $\mathcal{O}$ of open and closed subsets of a topological space (X, t) is said to be *weakly functionally isolated* if it only consists of weakly functionally isolated sets.

Definition 3.18 We say that a topology t on a nonempty set X satisfies the *weak functional open and closed countable chain condition* (*WFOCCC*), if every weakly functionally isolated chain $\mathcal{O}$ of open and closed subsets of X is countable.

The immediate proof of the following proposition is left to the reader.

Proposition 3.19 *Let (X, t) be a completely regular topological space. Then, the conditions WOCCC and WFOCCC are equivalent.*

Remark 3.20 It is clear that *WFOCCC* always implies *WOCCC*, while *WOCCC* implies *WFOCCC* in case that t is a completely regular topology.

As a consequence of the above Theorem 3.10 and Proposition 3.19, we immediately get the following characterization of useful topologies (see Bosi and Zuanon [5, Corollary 3.1].

Theorem 3.21 *Let t be a topology on a nonempty set X. The following conditions are equivalent:*

(i) *t is useful;*
(ii) *$\sigma(X, C(X, t, \mathbb{R}))$ is separable and satisfies WFOCCC.*

Clearly, the following theorem is also true.

Theorem 3.22 *Let t be a completely regular topology on a nonempty set X. Then, the following conditions are equivalent:*

(i) *t is useful;*
(ii) *The following conditions are verified:*

  (a) *t is separable;*
  (b) *t satisfies WFOCCC.*

Let us now recall the well-known definition of the *countable chain condition* (see, e.g., Bosi and Herden [2]).

Definition 3.23 A topology t on a nonempty set X satisfies the *countable chain condition* (*CCC*), if every family $\mathcal{O}$ of pairwise disjoint open subsets of X is countable.

It is clear that a separable topology satisfies *CCC*.

We finish this paper by including the *Soudlin hypothesis* in our considerations concerning useful topologies. The reader may recall that a chain $(Z, <)$ satisfies *CCC* (i.e., the *countable chain condition*) if every family of pairwise disjoint open intervals of $(Z, <)$ is countable, or, equivalently, if the order topology $t^{<}$ that is induced by $<$ satisfies *CCC*. The Souslin Hypothesis (**SH**) states that every chain $(Z, <)$ that satisfies *CCC* and only has countably many jumps can be order-embedded into the real line. **SH** was posed by M. Souslin (1894–1919) in the only paper that he published during his life (see Souslin [20]). Since the late 60s, it is known that **SH** is independent of **ZFC** (i.e., **ZFC** (Zermelo–Fraenkel + Axiom of Choice). **SH** has been applied by Vohra [21] to prove in **ZFC+SH** a general continuous utility representation theorem. Other applications to utility theory are found in Bosi and Herden [2].

The following theorem characterizes the Souslin hypothesis in terms of a simple equivalence concerning useful topologies, and involving the above introduced *WOCCC* condition.

Theorem 3.24 *The following assertions are equivalent:*

(i) **SH** *holds.*
(ii) *For every set X and any topology t on X, the concepts t to satisfy WOCCC and t to be useful are equivalent.*

Proof (i) $\Rightarrow$ (ii): Let X be an arbitrary set and let t be topology on X satisfying *WOCCC*. To show that t is useful, we consider some continuous total preorder $\precsim$ on X. Then, restricting our considerations to equivalence classes, a routine and well-known argument allows us to assume that $\precsim$, actually, is an order on X. The validity of **SH** implies that assertion (ii) will follow, if we are able to prove that $(X, \precsim)$ satisfies *CCC* and only has countably many jumps (see Proposition 2.9). Therefore, we choose the complete separable system $\mathcal{E} := \{l_{\precsim}(x) \mid x \in X\}$. Since $\mathcal{E}$ only contains countably many weakly isolated sets, $(X, \precsim)$ only has countably many jumps by Lemma 3.9, and it remains to verify that $(X, \precsim)$ satisfies *CCC*. Let us, thus, assume in contrast that $(X, \precsim)$ does not satisfy *CCC*. Then, there exists an uncountable family $\{]x_i, y_i[\}_{i \in I}$ of pairwise disjoint nonempty open intervals of $(X, \precsim)$. However, this means that the complete separable system $\mathcal{E}' := \{l_{\precsim}(y_i) \mid i \in I\}$ contains uncountably many weakly isolated sets. Indeed, it follows from the disjointness of the open intervals $]x_i, y_i[$ of $(X, \precsim)$ that every set $l_{\precsim}(y_i) \in \mathcal{E}'$ is weakly isolated. This contradiction implies that $(X, \precsim)$ satisfies *CCC* and, thus, finishes the proof of assertion (ii).

(ii) $\Rightarrow$ (i): This implication will be proved by contraposition. Let us, therefore, assume that **SH** does not hold. Then, there exists some linearly ordered set $(Z, <)$ which satisfies *CCC*, only has countably many jumps and is not representable by a real-valued order-preserving function. Therefore, we set $X := Z$ and consider the linearly ordered topology $t := t^{\mathcal{E}}$ on X that is induced by the complete separable system $\mathcal{E} := \{l_{\precsim}(x) \mid x \in X\}$. To now verify the desired implication, it suffices to show that the complete separable system $\mathcal{E}$ on (X, t) only contains countably many weakly isolated sets. Let some set $E \in \mathcal{E}$ be arbitrarily chosen. We choose the continuous total (pre)order $\precsim_{\mathcal{E}}$ on X that is induced by $\mathcal{E}$. The definition of t implies that for all points $x, y \in X$, the implication "$x \prec_{\mathcal{E}} y \Rightarrow x < y$" holds. If we, thus, assume that $\mathcal{E}$ contains uncountably many isolated sets, then it follows that $(Z, <)$ does not satisfy *CCC* or has uncountably many jumps. This contradiction proves the validity of the implication "(ii) $\Rightarrow$ (i)" and, therefore, finishes the proof of the theorem. □

4 Conclusions

In this paper, we have presented a very simple and attractive characterization of *useful topologies*, i.e., topologies on a set, such that every continuous total preorder is representable by a continuous utility function. Our main result shows that a topology is useful if and only if the weak topology of continuous functions is separable, and it satisfies the *weak open and closed countable chain condition* (*WOCCC*), which is newly introduced in this paper. All the famous utility representations' theorems are easy corollaries of our results. We finish the paper by incorporating the *Souslin Hypothesis* **SH**, and therefore proving that the validity of **SH** is equivalent to the equivalence between a topology to be useful and to satisfy *WOCCC*.

It could be interesting to consider further properties of a utility function, besides continuity. This is the case, for example, of *quasi-concavity* (see, e.g., Soleimani-damaneh et al. [19]). In addition, the general framework of the Conjoint Measurement Problem could be, so to say, incorporated (see, e.g., Bouyssou and Marchant [6, 7]). We shall study such extensions in the future.

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