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Quantum Control for Entanglement enhancement in a dissipative environment

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Abstract

Purpose of this thesis is studying a quantum control protocol that might drive the dynamics of a two-qubit system in a dissipative environment in order to preserve or even generate entanglement between them. The aim is twofold. On one side entanglement is a precious instrument because of its many applications in modern quantum technologies, but its inner fragility with respect to dissipation makes it a very delicate tool to handle in a realistic open quantum scenario. It is thus important to have different protocols and methods to enhance entanglement formation and preservation. On the other side, the thesis addresses a peculiar feature of any open quantum dynamics; namely, although entanglement is easily destroyed by a noisy environment, it is possible to engineer particular environment correlation functions such that a purely dissipative process can create entanglement and make it persist asymptotically in time. The novelty of the work we propose in this thesis is in fact the study of quantum control protocols (known as quantum filtering and markovian feedback) able to enhance the environment capability of generating initial and asymptotic entanglement rather than a direct action on the qubit state.

Indeed, the thesis provides physical scenarios where markovian quantum feedback protocols are able to enhance the entanglement generation through dissipation.

The major technical contributions of this thesis are, one one hand, the full analysis of a quite general open quantum dynamics of two qubits independently interacting with identical bath operators, for which we were able to find the asymptotic manifold and the subsequent entanglement generation and preservation conditions. On the other hand, we investigated the effects on the previous dynamics of the measurement-based markovian quantum feedback protocol as regards the entanglement generation by modifying the correlation functions of the environment, rather than trying to stabilize directly the qubits state as it is usually done.

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Introduction

In general, the openness of quantum systems, namely the fact that the interactions with their environment cannot be neglected, is a source of dissipative and noisy effects which bring about decoherence and depletion of correlations. Yet, it has been shown that suitably engineered environments can entangle initially separable bipartite systems and even maintain the generated entanglement asymptotically in time. To put it in a nutshell, the argument of the thesis is whether quantum filtering and feedback can improve the entanglement generation and preservation capacity of a given environment. Quantum feedback and quantum filtering are particular aspects of the theory of quantum control which regards all those protocols whose aim is to externally manipulate the dynamical evolution of a quantum system. In particular, quantum filtering concerns harvesting information about an open quantum system by constantly monitoring the environment with which it interacts. Basically, the bath, via its coupling with the open system, brings information about the latter, so one can monitor the system behaviour indirectly, namely via suitable experimental access to the bath. However, the monitoring process introduces an extra degree of stochasticity in the open system dissipative dynamics that needs techniques from stochastic calculus to be harnessed. Afterwards, the information gathered through the filtering are re-inserted back into the system by means of specific procedures which go under the name of quantum feedback: the information is used to act back on the system via an external field, specifically tuned to drive the evolution of the system in an a priori chosen way.

In [5] it has been shown under which circumstances a dissipative dynamics can generate entanglement that lasts forever. This is particularly useful since on one hand entanglement, one of the most peculiar feature of quantum systems, has potentially vast and revolutionary applications in modern quantum technologies; on the other hand, as a resource, entanglement is nevertheless extremely fragile. Hence, it is of interest to understand whether and how quantum control techniques can be used to drive two qubits into an entanglement state in the short-time regime and in the long-time regime, when dissipation and noise are present and not negligible.

The structure of the thesis is as follows: in the first Chapter we will introduce the basics of classical and quantum probability. We start with a comparison of the probabilistic features of classical and quantum systems; then, we focus on the notions of joint and conditional probability, with a brief digression about Bayes theorem. In particular, conditional probabilities and the related stochastic tools known as conditional expectations are extremely important in order to appreciate the formalism and the physics behind quantum filtering and feedback. Indeed, these latter can, roughly speaking, be understood as means of conditioning via indirect measurements.

In the same chapter, by means of the paradigmatic case of qubits, we will review the inherently probabilistic structure of Quantum Mechanics, which is there independently of whether

one monitors or not the environment and acts back on a quantum systems. Finally, at the end of the Chapter, we address the issue of measurements in quantum mechanics.

In Chapter 2, we discuss open quantum systems and their dissipative dynamics. After introducing the main results relative to non-unitary quantum dynamics and their Kraus-Stinespring representations, we tackle the fundamental topic of positivity vs complete positivity, with particular focus upon quantum dynamical semigroups and the celebrated Gorini-Kossakowski-Sudarshan-Lindblad (GKSL) master equations which generate them. Basing on the master equations, we devote the last part of the chapter to the main results about their stationary and asymptotic states.

Chapter 3 is devoted to the other basic issue of the thesis and it contains one of the original results of our work. The focus is on quantum entanglement and its relations with the concept of complete positivity introduced in the previous chapter. After discussing the main feature of quantum entanglement, we introduce the main methods to witness the presence of bipartite entanglement in a quantum state, such as the Peres-Horodecki criterion and concurrence. We then address the key topic of the thesis: entanglement generation via dissipation. In order to clarify the issue at stake, we consider two qubits in interaction with an environment consisting of independent scalar fields and derive necessary and sufficient conditions for entanglement generation at short times. Finally, we study the long-time fate of the generated entanglement. As already mentioned, in this chapter our first major result is presented as we were able to find the asymptotic states and the entanglement preservation conditions for the dynamics taken under exam, which provides a not too restrictive physical scenario.

In Chapter 4 we provide a summary of those techniques from stochastic processes that are necessary for quantum filtering and for adapting quantum filtering and feedback to the entanglement dynamics scenario introduced in the previous chapter. Based on Poisson and Wiener processes as guiding examples, we first introduce the basics of classical stochastic processes and of classical stochastic calculus. Then, we move to the quantum realm and to quantum stochastic processes that are presented by means of a Bosonic representation of the noisy environment, finally providing the rudiments of quantum stochastic calculus.

With all the necessary techniques available, Chapter 5 is devoted to quantum filtering and feedback. We firstly explicitly obtain the stochastic dynamics for the observables of quantum systems immersed in a Bosonic bath, *i.e.* the so-called quantum Langevin equation. Then, we introduce the measurements one can perform on the environment in order to have a consistent quantum filtering protocol. Subsequently, using the conditioning techniques presented in Chapter 1, we derive the quantum filtering equation. Finally, in the last part of the chapter, we show how to manipulate the filtering equation in order to build a feedback protocol. Among the general feedback protocols, we choose the so-called Markovian feedback that will be applied in the last chapter.

In the last Chapter, we present all other major results achieved in the thesis. We discuss a master equation of a two-qubit system immersed in a bath of scalar fields subjected to quantum filtering and Markovian feedback; then, by restricting to a particular set of master equations, we deal with entanglement generation at short times and also in the long-time regime as we are indeed able to characterize the whole convex manifold of stationary states. In particular, we show how to control the asymptotic entanglement by properly choosing the feedback parameters.

Finally, in the Conclusions, we wrap up all the topics addressed by the thesis and its results.

Chapter 1

Basics about probability and quantum mechanics

As we briefly mentioned in the Introduction, quantum filtering concerns the process of extracting information about a quantum system: this is done by continuously monitoring the quantum system via indirect measurements.

For a better comprehension of this matter, a quick digression on probability theory is essential, not only because of the intrinsic probabilistic nature of quantum mechanics, but also because of the further injection of stochasticity operated by the monitoring.

We will start with a comparison of the differences between the peculiarities of the probabilistic aspects of quantum systems with respect to classical probability theory. Indeed, because of non-commutativity and the consequent impact of measurement processes upon quantum states, the statistical behaviour of quantum systems departs from that of classical systems, giving rise to a quantum probability theory [9, 30, 49].

Then we will introduce the concepts of joint and conditional probabilities [9], the latter being essential to understand quantum filtering. Also classically there is plenty of indirect measurements [29]: i.e. when we measure the temperature with a mercury thermometer, we are reading the length of the thermal dilatation rather then the actual temperature of the object; indeed, we are able to link the length of the dilatation of a metal to the temperature of the sample. The latter is a simple example of deterministic inference (no probability is involved); however, in quantum mechanics every measurement is inherently probabilistic. Hence, in order to perform quantum filtering, we ought to link the probability of obtaining certain measurement outcomes to the probability of inferring a value of the observable we want to indirectly monitor. This process is precisely what is called in probability theory *conditioning* [9]. We will discuss this issue both at the classical and at the quantum level.

Finally, we will introduce the concept of qubit and quantum state [27, 34], focusing then on measurement processes in quantum mechanics and on how quantum systems react when a measurement is performed on them.

1.1 Classical probability vs quantum probability

We start presenting the main differences and similarities between classical and quantum probability theory [9, 30, 49].

Definition 1.1. Classical probability theory can be described by a triplet $(\Omega, \mathcal{F}, \mathbb{P})$, where [49]:

1. Ω is the *sample set*, namely the set of all the possible outcomes (i.e. head and tail).
2. $\mathcal{F}$ is the set of *events* and it is a *σ-algebra*, namely it is a collection of the subsets of Ω with the following properties:
   (a) $\mathcal{F}$ includes the empty subset.
   (b) $\mathcal{F}$ is closed under complementarity: $\Omega \backslash A \subseteq \mathcal{F}$ if $A \subseteq \mathcal{F}$.
   (c) $\mathcal{F}$ is closed under countable unions: $\bigcup_{n=1}^{\infty} A_n \subseteq \mathcal{F}$ if $A_n \subseteq \mathcal{F}$.

   Introducing $\mathcal{F}$ has the scope to associate an algebraic structure to the sample set. In particular, every σ-algebra is an algebra, but not vice versa, indeed algebras have to satisfy all previous properties except the very last one, they have to be closed only under pairwise unions rather than countable infinite ones.
   For every $A \subseteq \mathcal{F}$, there is a function, called *random variable*, $\chi_A : \Omega \mapsto \mathbb{R}$ which associates to every element of Ω a number. By means of the notion of random variable it is possible to express the elements of $\mathcal{F}$ (the events) as functions. In the case of a discrete sample space, an event $\omega^* \in \Omega$ is identifiable with the discrete Dirac delta δ_{ω^*} such that $\delta_{\omega^*}(\omega) = 1$ if $\omega = \omega^*$, $\delta_{\omega^*}(\omega) = 1$ otherwise. An example of random variable in the discrete case is the characteristic function of the subset A: namely, for $\omega \in \Omega$:

$$\chi_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases} . \tag{1.1}$$

   The random variable $\chi_A(\omega)$ can assume a set of values according to the outcome ω, the probability of obtaining a certain value for the random variable (in other words, the probability of obtaining a certain outcome ω) is given by the probability defined in the next item.
3. $\mathbb{P}$ is a function on Ω which associates a probability, namely a real number $0 \leq p(\omega) \leq 1$, to any $\omega \in \Omega$. It can be extended to a function that provide every subset of $\mathcal{F}$ with its own probability. It can also be seen as a function that associates an expectation value to χ_A. Concretely, $\mathbb{P} : \mathcal{F} \mapsto [0, 1]$ is such that:
   (a) $\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(A_n)$ for all disjoint sets $A_n \subseteq \mathcal{F}$.
   (b) $\mathbb{P}(\Omega) = 1$, which is the condition for normalization.

   In the simplest case, the probability associated to a value of a random variable is the (limit of) ratio of the number of events in which that outcome occurred taken with respect to larger and larger numbers of events.

Summarizing: a *classical probability space* is denoted by the triplet $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω is any set, $\mathcal{F}$ is the σ-algebra of the subsets of Ω, $\mathbb{P}$ is a probability measure on the algebra [14].

Example 1.1. In the case of one coin toss, the sample set is $\Omega = \{head, tail\} \equiv \{h, t\}$, the σ-algebra associated to it is then $\mathcal{F} = \{\emptyset, h, t, \{h, t\}\}$ (in the discrete case $\mathcal{F}$ is the union of every subset of Ω plus the empty set and Ω itself). If we are interested in the event $A = \{head\}$, then the random variable $\chi_A(\omega)$ is 1 if $\omega = h$, zero if $\omega = t$. For a fair coin, the probability of having $\chi_A(\omega) = 0$ or 1 is 50%.

If the toss is repeated two times, the sample set becomes $\Omega = \{(h,h), (h,t), (t,t), (t,h)\}$. If we are interested in the events in which the first outcome is head, than we can study the random variable associated to the subset $A = \{(h,h), (h,t)\}$.

Certainly, functions over the classical probability space Ω commute. Moving to quantum mechanics, the only difference between classical and quantum probability theory is that the latter is not commutative: in other words, unlike with classical observables, the order in which two quantum observables are measured is of the utmost importance.

Definition 1.2. In quantum mechanics there is no sample space Ω, the probability theory is described by just two elements $(\mathcal{N}, \mathbb{P}_\rho)$, such that [49]:

1. $\mathcal{N}$ is a so-called *von Neumann algebra* formed by the set of all the bounded operators acting on the Hilbert space of the system. It is closed in the strong operator topology (see Appendix A.1 for a brief survey about operator algebras). Every self-adjoint element $\hat{A} \in \mathcal{N}$ is an observable, namely the quantum analogue of a classical random variable. indeed, any quantum observable has a spectrum of real values, namely its eigenvalues. In a measurement process of a given observable, its eigenvalues occur according to probability distributions determined by the state of the system on which the observable is measured (as we will see in the next bullet point).
   Every observable can be decomposed as the sum of its orthogonal spectral projectors $\hat{P} \in \mathcal{N}$ acting on the Hilbert space of the system: $\hat{A} = \sum_n a_n \hat{P}_n$, with a_n its eigenvalues (for sake of simplicity, we assume a discrete spectrum and we will discuss this decomposition in more detail in Section 1.4). Hence, the quantum events are all the projectors $\hat{P} \in \mathcal{N}$ on the subspaces of the Hilbert space.

2. In analogy with classical dynamical systems where states are probability distribution over the configuration space of the system (for instance, in classical mechanics, Gibbs states over the phase space), the kinematical description of any quantum system is completed by equipping the algebra of its observables with a reference state (or density matrix) ρ, with the properties of being positive $\rho \geq 0$ and normalized $\mathrm{tr}(\rho) = 1$. To this ρ one associates a probability measure $\mathbb{P}_\rho$, namely a linear map $\mathbb{P}_\rho : \mathcal{N} \mapsto \mathbb{C}$, with the properties of being positive, $\mathbb{P}_\rho(\hat{A}) \geq 0$ for $\hat{A} \geq 0$, and normalized, $\mathbb{P}_\rho(\mathbb{I}) = 1$. Because it is determined by the state ρ, this map is often called *state* of the system as well. The action of $\mathbb{P}_\rho$ on $\hat{A} \in \mathcal{N}$ can be written as $\mathbb{P}_\rho(\hat{A}) = \mathrm{tr}\Big(\rho\, \hat{A}\Big)$. Now, if in the place of $\hat{A}$ we put an orthogonal projector (namely an event) $\hat{P}$, then we obtain the probability for that event to take place; if we put an observable which is not a projector, we obtain its mean value. Hence $\mathbb{P}_\rho$ is an abstract

way to treat simultaneously probabilities and expectation values. To avoid misconception we will sometimes refer to the expectation value with the symbol $\langle \cdot \rangle$.

Having identified the main difference between classical and quantum probability in the commutativity and non-commutativity of their respective algebras, in the following we will introduce the main tools of classical probability theory and then we will discuss how they must be modified in order to describe quantum systems.

1.2 Joint and Conditional probability

We start by focusing on the concepts of joint and conditional probabilities. To introduce these topics we will use classical random variables but we will keep the formulation as general and abstract as possible in order to later adapt it in an easier way to quantum mechanical systems.

Given two random variables X, Y, with possible outcomes x, respectively y, we can describe the probability related to X and Y as

$$\begin{aligned} \mathbb{P}(X) &= \{\mathbb{P}(X = x) = \mathbb{P}(x) \,|\, x \in X\}, \\ \mathbb{P}(Y) &= \{\mathbb{P}(Y = y) = \mathbb{P}(y) \,|\, y \in Y\} \end{aligned} \tag{1.2}$$

and their expectation values as the sum of the outcomes times the probability of obtaining them

$$\langle X \rangle = \sum_{x \in X} x\, \mathbb{P}(x), \qquad \langle Y \rangle = \sum_{y \in Y} y\, \mathbb{P}(y). \tag{1.3}$$

Notice that random variables correspond to sets of outcomes, which we indicate by lowercase letters, the random variables being identified by uppercase letters. So, $\mathbb{P}(X)$ is a collection of probabilities, $\mathbb{P}(x)$ is the actual probability of obtaining the outcome x. Clearly $\sum_{x \in X} \mathbb{P}(x) = 1$.

The *joint probability* is the probability that two events occur simultaneously:

$$\mathbb{P}(X \cap Y) := \{\mathbb{P}(X = x, Y = y) = \mathbb{P}(x, y) \,|\, x \in X,\, y \in Y\}. \tag{1.4}$$

The random variables X and Y are statistically independent if $\mathbb{P}(x, y) = \mathbb{P}(x)\mathbb{P}(y)$ for every $x \in X$ and $y \in Y$.

From the joint probability we can recover the *marginal probabilities* in (1.2) by summing over the outcomes of the variable we want to get rid of:

$$\mathbb{P}(x) = \sum_{y \in Y} \mathbb{P}(x,\, y), \qquad \mathbb{P}(y) = \sum_{x \in X} \mathbb{P}(x,\, y). \tag{1.5}$$

The *conditional probability* of the random variable X conditioned upon the random variable Y, $\mathbb{P}(X|Y)$, is trickier: it is the probability that X assumes a certain value when a definite value has been attributed to Y. Basically, the joint probability refers to all the combinations in which $X = x$ and $Y = y$ occur together, instead the conditional probability regards what happens to X when Y has already been determined. Hence the conditional probability expresses the statistical influence that the outcomes of Y has on the outcomes of X:

$$\mathbb{P}(X|Y) := \frac{\mathbb{P}(X \cap Y)}{\mathbb{P}(Y)} := \left\{ \frac{\mathbb{P}(x,\, y)}{\mathbb{P}(y)} \,\middle|\, x \in X,\, y \in Y \right\}. \tag{1.6}$$

It is basically a joint probability in which the indeterminacy of Y has been removed.

If $\mathbb{P}(y) = 0$, the previous formula does not apply; indeed, when the probability of obtaining a certain value for Y vanishes, that outcome of Y can not condition X.

If X and Y are statistically independent, then the joint probability factorizes $\mathbb{P}(x, y) = \mathbb{P}(x)\mathbb{P}(y)$, as a consequence $\mathbb{P}(X|Y) = \mathbb{P}(X)$.

In the following subsection we will make a quick digression about Bayes relation, which aptly shows the main features and uses of conditional probability.

1.2.1 Bayes theorem

Bayes theorem is a handy way to write the conditional probability. Given two random variables X, Y and their conditional probabilities:

$$\mathbb{P}(X|Y) = \frac{\mathbb{P}(X \cap Y)}{\mathbb{P}(Y)}, \qquad \mathbb{P}(Y|X) = \frac{\mathbb{P}(Y \cap X)}{\mathbb{P}(X)}, \tag{1.7}$$

since the joint probability is symmetric $\mathbb{P}(X \cap Y) = \mathbb{P}(Y \cap X)$, we find *Bayes relation*:

$$\mathbb{P}(X|Y) = \frac{\mathbb{P}(Y|X)\mathbb{P}(X)}{\mathbb{P}(Y)}\,. \tag{1.8}$$

Sometimes $\mathbb{P}(X|Y)$ is called *posteriori* probability and $\mathbb{P}(X)$ is called *prior* probability, because the latter represents our knowledge of the random variable X without any knowledge of the random variable Y, while the former tells us how the probability of obtaining a certain value for X changes after an outcome of Y is obtained. The posteriori probability comes from the prior probability through the knowledge of the probability of Y conditioned on X, $\mathbb{P}(Y|X)$. The denominator $\mathbb{P}(Y)$ serves as normalization; for example exploiting (1.5) and (1.7):

$$\mathbb{P}(X|Y) = \frac{\mathbb{P}(Y|X)\mathbb{P}(X)}{\sum_{x\in X}\mathbb{P}(Y|x)\mathbb{P}(x)} \tag{1.9}$$

where x are all the possible outcomes of the random variable X.

Bayes relation may not look a particularly powerful expression; as a matter of fact even Bayes was not so sure about its effective significance. Nevertheless, Bayes relation is recognized as an extremely important tool because of the amount of contexts in which it can be nimbly applied (such as decision theory, artificial intelligence, even epistemology) and because of the amount of counterintuitive information it carries.

Example 1.2. A famous counterintuitive consequence of Bayes relation is the three card version of the so-called Bertrand's paradox. Given three cards, of which one, RR, has front and back both red, one, RB, red and black and one, BB, with both sides black, which is the conditional probability $\mathbb{P}(RR|R)$ that, picking one card which shows its red side also its other side is red? The intuition is that such a probability is $1/2$: indeed, there is only one card with both red sides and two cards that can be chosen showing their red side. However, Bayes relation yields

$$\mathbb{P}(RR|R) = \frac{\mathbb{P}(R|RR)\,\mathbb{P}(RR)}{\mathbb{P}(R)} = 1 \times \frac{1}{3} \times 2 = \frac{2}{3}\,.$$

Indeed, the conditional probability that one side is red if both are red is $\mathbb{P}(R|RR) = 1$, the probability of picking a card with both sides red is $\mathbb{P}(RR) = 1/3$, because there are 2 favourable events over 6, while the probability of picking a card showing its red side is $\mathbb{P}(R) = 1/2$ since there are 3 favourable events over 6.

Another counterintuitive consequence of Bayes relations is as follows.

Example 1.3. A man is tested positive to a very rare disease, the test is 99% accurate (namely 99 times over 100 it gives a positive outcome if the man tested is ill). What is the probability that the man is actually positive after a positive test? The answer is not 99% as it may seem at a first sight: Bayes relation can make it far less. Let X be the random variable that tells us if the man is ill or not $X = \{\text{ill, not ill}\}$, while $Y = \{\text{positive, negative}\}$ corresponds to the outcome of the test. We are interested in the probability of the outcome $X = ill$ when the outcome of Y is already known to be $Y = positive$, hence:

- $\mathbb{P}(X = \text{ill}|\text{Y = positive})$ is the probability that the man is actually ill after a positive test, our unknown quantity.
- $\mathbb{P}(Y = \text{positive}|\text{X = ill})$ is the probability that the test correctly finds the man positive when he is ill, namely $\mathbb{P}(Y = \text{positive}|\text{X = ill}) = \frac{99}{100}$.
- $\mathbb{P}(X = \text{ill})$ is the probability that the event X occurs spontaneously, namely the probability of contracting the disease, i.e. the percentage of population that contract the disease. For instance we can take $\mathbb{P}(X = \text{ill}) = 0.001$, since we are assuming it is a very rare disease.
- The denominator in (1.9) amounts to:
  $\mathbb{P}(\text{positive test}) = \mathbb{P}(\text{positive test}|\text{ill})\mathbb{P}(\text{ill}) + \mathbb{P}(\text{positive test}|\text{not ill})\mathbb{P}(\text{not ill})$.
  In this case, we can take $\mathbb{P}(\text{positive test}|\text{not ill}) = 1 - \mathbb{P}(\text{positive test}|\text{ill}) = 0.01$; because 1% of the times the test gives a wrong result.

So, given the data above, we easily compute

$$\mathbb{P}(X = \text{ill}|Y = \text{positive}) = \frac{0.99 \times 0.001}{0.99 \times 0.001 + 0.01 \times 0.999} = 0.09\ . \tag{1.10}$$

Then, the probability of actually being positive after a 99% accurate positive test is just 9%. Which is a consistent result because if we take 1000 people, one of which is ill and we test all of them, 1% of them, namely 10 people, will be found ill because 1% of the times the test is wrong. So, considering the really ill man, we will have 11 positive tests, but the real positive one is just 1 and $1/11 \simeq 9\%$. On the other hand, after two positive tests, the probability of being actually positive skyrockets to 91% (to recover this results it is sufficient to apply again Bayes theorem where the prior probability to be updated is this time the previous posteriori probability of 9%).

Bayes relations can thus be seen as a useful tool to refine probabilistic predictions, basing on adding some information about the event to be predicted. Indeed, it is often used when X and Y are actually not independent from each other or when the conditional (or marginal) probability is not accurate enough due to the lack of information. In the second example above, we are adding to the accuracy of the test the knowledge about the diffusion of the disease.

The previous example is also an instance of *Bayesian inference*: Bayes relation is used to update the probability of an hypothesis as soon as new related information are available. As we said the flexibility of Bayes relation allows one to exploit it in several branches of human knowledge and quantum mechanics benefits from it as well, as we will see later. We can anticipate that, since measurements in quantum mechanics always disturb the state of the system, it is evident the importance of having a tool which describes how changes in the knowledge of a random variable changes the statistics of another one.

After this brief introduction about classical probability, we are ready to move to the quantum realm. We will first review the notion of quantum state and then discuss the use of conditional probabilities in the field of open quantum systems.

1.3 The states of quantum systems

The state of a quantum system, which as anticipated also describe the statistical properties of the system, can be written as a density operator (also known as *statistical operator*), which is defined as follow:

$$\rho = \sum_k p_k \left|\psi_k\right\rangle \left\langle\psi_k\right| , \quad p_k \geq 0 , \quad \sum_k p_k = 1 , \tag{1.11}$$

where $|\psi_k\rangle$ are normalized vectors in the Hilbert space $\mathcal{H}$ of the system and p_k are the statistical weights with which the states $|\psi_k\rangle$ contribute to a given statistical ensemble associated with our imperfect knowledge of the actual state of a quantum system [27]. Notice that the density operators are generic convex combinations of not necessarily orthogonal projectors.

As a matter of fact the weights p_k simply express our subjective ignorance about the quantum state of the system and thus a classical ambiguity, while the quantum probabilistic properties of the system are embodied by the vector states $|\psi_k\rangle$. Hence, density matrices carry both classical and quantum probability properties.

In particular, it is possible to split the density matrices in two big families: *pure* states and *mixed* states.

Definition 1.3.

Pure states: they can be written as $\rho = |\psi\rangle \langle\psi|$, rank one projections, namely Hermitean and idempotent, hence

$$\rho^2 = \rho = \rho^\dagger \Leftrightarrow \mathrm{tr}\big(\rho^2\big) = \mathrm{tr}(\rho) = 1. \tag{1.12}$$

Mixed states: they are of the form (1.11) In this case

$$\mathrm{tr}\big(\rho^2\big) \leq 1 , \tag{1.13}$$

the upper bound being reached if only if the sum has only one term, in other words if the state is pure.

Pure states are a subset of mixed ones which, in turn, arise as convex mixtures of pure ones. A mixed state is said to be maximally mixed if all the coefficients in the summation are identical to $1/d$, where d is the dimension of the system: indeed in this case we have no clue about the real state of the system, since all the states would be equally probable.

A proper statistical operator ρ fulfils the following properties [27]:

Unitary trace. $\mathrm{tr}\{\rho\} = 1$

Proof.

$$\mathrm{tr}\{\rho\} = \sum_k p_k \,\mathrm{tr}\{(|\psi_k\rangle\,\langle\psi_k|)\} = \sum_k p_k = 1. \tag{1.14}$$

□

Positivity. $\rho \geq 0$.

Proof. For any $|\varphi\rangle \in \mathcal{H}$:

$$\langle\varphi|\rho|\varphi\rangle = \sum_k p_k \langle\varphi|\psi_k\rangle\langle\psi_k|\varphi\rangle = \sum_k p_k |\langle\varphi|\psi_k\rangle|^2 \geq 0. \tag{1.15}$$

□

The second property is the most physically relevant: it amounts to asking that all the eigenvalues (the spectrum) of ρ be non-negative. Together with the first condition, they make the spectrum of ρ a discrete probability distribution. If

$$\rho = \sum_j r_j \;|r_j\rangle\,\langle r_j| \;\;, \quad 0 \leq r_j \leq 1 \;, \quad \sum_j r_j = 1 \;, \quad \langle r_j|r_k\rangle = \delta_{jk} \;, \tag{1.16}$$

is the spectral decomposition, unlike for the weights p_k in the generic expression (1.11) that uses non-orthogonal projectors, the non-negative eigenvalues r_j are the probabilities of finding the system in the state $|r_j\rangle$ of the statistical ensemble associated with the spectral decomposition of ρ. Here one has one of the major departure from classical probability in that to the statistical description given by a density matrix ρ there correspond (uncountably) many different statistical ensembles, for instance $\{p_k, |\psi_k\rangle\}$ for the density matrix written as in (1.11) and $\{r_j, |r_j\rangle\}$ for the same ρ spectralized as in (1.16). Also, as already mentioned, mixedness is to the other extreme of quantum superposition: despite being expandable with respect to any orthonormal basis $\{|\phi_j\rangle\}$, the statistical ensemble associated with the vector state $|\psi\rangle = \sum_j \alpha_j\,|\phi_j\rangle$, $\sum_j |\alpha_j|^2 = 1$, is a singleton consisting of just $|\psi\rangle$ with weight 1. If the interferences due to the quantum state superposition are eliminated by some decoherence mechanism then

$$|\psi\rangle\,\langle\psi| = \sum_{k,\ell} \alpha_k \alpha_\ell^* \,|\phi_k\rangle\,\langle\phi_\ell| \longmapsto \rho = \sum_k |\alpha_k|^2 \;|\phi_k\rangle\,\langle\phi_k| \;\;,$$

relative to the statistical ensemble $\{|\alpha_k|^2, |\psi_k\rangle\}$ [39].

As much as convex combinations of pure states are density matrices, so are convex combinations of generic density matrices: the structure of the space of states $\mathcal{S}(S)$ of a quantum system S is thus that of a convex subset of the positive operators on the Hilbert space of the system. As a concrete example of quantum states, we introduce the notion of qubit. Then, we shall focus on how measurements affect the state of the system.

1.3.1 The qubit

In the following we will consider the qubit, which is an abstract way to describe two-level quantum systems without referring to a particular physical realization.

A qubit is described by a 2×2 complex density matrix. A basis for the space of 2-dimensional complex matrices $\mathcal{M}_2(\mathbb{C})$ is $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$, where $\sigma_0 = \mathbb{I}_2$ and $\overline{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli's matrices [34]. Any density operator of a single qubit can be decomposed on the Pauli matrix basis:

$$\rho(\overline{n}) = \frac{1}{2}\left(\mathbb{I}_2 + \overline{n} \cdot \overline{\sigma}\right) = \tag{1.17}$$

$$= \frac{1}{2}\begin{pmatrix} 1+n_3 & n_1 - in_2 \\ n_1 + in_2 & 1-n_3 \end{pmatrix}, \tag{1.18}$$

where $\overline{n} = (n_1, n_2, n_3) \in \mathbb{R}^3$ is known as Block-vector.

The trace of ρ being automatically 1, the request of positivity amounts to asking that the determinant of ρ be non-negative which yields:

$$\|\overline{n}\| \leq 1. \tag{1.19}$$

As a consequence, all possible density matrices of a single qubit are identified by points of the unit three-sphere with the pure states identified by points on its surface; this feature allows us the give a geometrical representation of the convex space of qubit states by means of the so-called *Bloch sphere*.

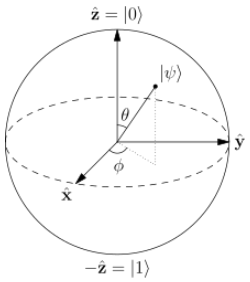

Figure 1.1: Representation of a Bloch sphere.

Clearly, on the surface of the sphere lie all density matrices relative to pure states.

1.4 Probability and measurements in quantum mechanics

Aim of this section is to show how joint and conditional probabilities work in quantum mechanics and how the measurement process affects quantum systems.

In Quantum Mechanics the status of Bayes theorem is particularly interesting because one of the characteristic quantum phenomena is exactly the disturbance that a measurement produces on their behaviour, disturbances that can always be in line of principle made negligible in classical measurement contexts.

As previously seen in Section 1.1, in quantum mechanics, the statistical properties of quantum systems are described by their, in general mixed, states ρ. Such statistical properties refer

to the fact that, when measured, the observables of quantum systems, namely any self-adjoint operator $X = X^\dagger$ acting on their Hilbert space, are distributed according to definite probability distributions characterized by statistical moments $\mathrm{Tr}(\rho\, X^n)$. For instance, expectation values are given by

$$\langle \hat{X} \rangle_\rho = \mathrm{tr}\left(\rho \hat{X}\right). \tag{1.20}$$

For sake of simplicity, we shall consider observables with discrete spectrum, that can be decomposed into a sum of orthogonal projectors. Given an observable $\hat{X}$ and calling its spectrum $spec(\hat{X})$, we write

$$\hat{X} = \sum_{x \in\, spec(\hat{X})} x\, \hat{P}_x\,, \tag{1.21}$$

where $\hat{P}_x$ is the projection over the subspace of the Hilbert space spanned by the eigenvectors of $\hat{X}$ with eigenvalue x. These projectors are orthogonal and resolve the identity:

$$\hat{P}_x\, \hat{P}_{x'} = \delta_{xx'} \hat{P}_x, \qquad \sum_{x \in\, spec(\hat{X})} \hat{P}_x = \mathbb{I}\,. \tag{1.22}$$

In a measurement of $\hat{X}$ on the system in the state ρ, the outcomes are the eigenvalues of $\hat{X}$, every x of the spectrum of $\hat{X}$ is obtained with probability:

$$\mathbb{P}_\rho(\hat{X} = x) := \mathbb{P}_\rho(x) = \mathrm{tr}\left(\rho \hat{P}_x\right). \tag{1.23}$$

Substituting (1.21) in (1.20), we find the usual result according to which the expectation value of the observable is given by the sum over all possible outcomes weighted by their probability:

$$\langle \hat{X} \rangle_\rho = \sum_{x \in\, spec(\hat{X})} x\, \mathrm{tr}\left(\rho \hat{P}_x\right) = \sum_{x \in\, spec(\hat{X})} x\, \mathbb{P}_\rho(x). \tag{1.24}$$

The mathematical description of a measurement process like the above one is obtained by considering that, after the outcome x is obtained, the state *collapses* into the posteriori state:

$$\rho_x = \frac{\hat{P}_x \rho \hat{P}_x}{\mathrm{tr}\Big\{\hat{P}_x \rho \hat{P}_x\Big\}} = \frac{\hat{P}_x \rho \hat{P}_x}{\mathrm{tr}\Big\{\hat{P}_x \rho\Big\}}, \tag{1.25}$$

which shows how the system state is conditioned by the measurement of the outcome x [9].

Indeed, if the measurement is immediately repeated, the outcome will be the same:

$$\mathbb{P}_{\rho_x}(x') = \mathrm{tr}\Big\{\rho_x \hat{P}_{x'}\Big\} = \frac{\mathrm{tr}\Big\{\hat{P}_x \rho \hat{P}_x \hat{P}_{x'}\Big\}}{\mathrm{tr}\Big\{\hat{P}_x \rho\Big\}} = \delta_{xx'}. \tag{1.26}$$

In order to complete the comparison, started in Section 1.1, of classical and quantum probability theory, let us consider a *classic* random variable X which can assume the discrete values $\{x_1, x_2\,, \cdots, x_n\}$ with probabilities $\mathbb{P}(x_1)$, $\mathbb{P}(x_2), \cdots, \mathbb{P}(x_n)$. The expectation value of the variable is given by (1.3). We can recast the classic formalism in a quantum fashion by writing sums

over the probabilities as the trace of diagonal matrices:

$$\langle X\rangle = \sum_{i=1}^{n} x_i\,\mathbb{P}(x_i) = \mathrm{tr}\left[\begin{pmatrix} x_1 & 0 & \dots & 0 \\ 0 & x_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & x_n \end{pmatrix}\begin{pmatrix} \mathbb{P}(x_1) & 0 & \dots & 0 \\ 0 & \mathbb{P}(x_2) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbb{P}(x_n) \end{pmatrix}\right] . \tag{1.27}$$

Here, the main difference between classical and quantum probability theory is that, while classical density matrices and observables can always be chosen diagonal with respect to a fixed orthonormal basis, quantum states and observables need not be simultaneously diagonal[30].

Moreover, we emphasize that if we perform a measurement related to the quantum observable $\hat{X}$, then all the functions $f(\hat{X})$ are determined, as a consequence of (1.21) [9]:

$$f(\hat{X}) = \sum_{x\in spec(\hat{X})} f(x)\,\hat{P}_x, \tag{1.28}$$

where the function has to satisfy certain regularity conditions, like being expandable as a power series around each point of the spectrum of $\hat{X}$. Then, sums and products of functions of $\hat{X}$

$$\mathcal{A}_{\hat{X}} = \{\hat{A} :\ \hat{A} = f(\hat{X}),\ f : \mathbb{R} \mapsto \mathbb{C}\} \tag{1.29}$$

constitutes a commutative algebra.

We previously stated that in quantum mechanics any system equipped with a state ρ is described by a quantum probability theory $(\mathcal{N}, \mathbb{P}_\rho)$. On the other hand, measuring an observable $\hat{X}$ makes the state of system collapse onto the various eigenprojections of $\hat{X}$. As a consequence the measurement outputs give rise to a classical probability theory $(\mathcal{A}_{\hat{X}}, \mathbb{P}_{\rho_{\hat{X}}})$, where $\mathcal{A}_{\hat{X}} \subset \mathcal{N}$ is the commutative algebra of continuous functions over the spectrum of $\hat{X}$. The measurement transforms the quantum probability setting into a classical one.

Concretely, after a projective measurement as the one we are considering, a state ρ of the system collapses into a new mixture according to the following scheme

$$\rho \mapsto \rho_X := \sum_i \hat{P}_i \rho \hat{P}_i = \sum_i \langle\psi_i|\rho|\psi_i\rangle\,\hat{P}_i\ , \quad \hat{P}_i = |\psi_i\rangle\langle\psi_i|\ . \tag{1.30}$$

Notice that the above projective measurement changes the state according to a linear map $\rho \mapsto \Lambda_X[\rho] := \rho_X$; this is the so-called Schrödinger picture. If we instead adopt the Heisenberg picture, then states stay fixed and observables change, $\hat{Y} \mapsto \Lambda^T[\hat{Y}]$, according to the so-called *dual* map Λ^* obtained by means of the *duality relation*

$$\mathrm{Tr}(\Lambda[\rho]\,\hat{Y}) = \mathrm{Tr}(\rho\,\Lambda^*[\hat{Y}])\ . \tag{1.31}$$

Then, using the cyclic of the trace-operation, $\mathrm{Tr}(\hat{X}\,\hat{Y}) - \mathrm{Tr}(\hat{Y}\,\hat{X})$, the dual of projective measurement (1.30) corresponds to the map

$$\hat{X} \mapsto \Lambda^*[\hat{X}] = \sum_i \hat{P}_i \hat{X} \hat{P}_i = \sum_i \langle\psi_i|\hat{X}|\psi_i\rangle\,\hat{P}_i\ ,$$

whose output is again a diagonal object in the commutative algebra generated by the orthogonal projectors.

Remark 1.1. Notice that, unlike for projective measurements, *POVM*[1] measurement processes whereby

$$\rho \mapsto \sum_\alpha \hat{X}_\alpha \, \rho \, \hat{X}_\alpha^\dagger \ , \quad \sum_\alpha \hat{X}^\dagger \, \hat{X}_\alpha = \mathbb{I} \ , \tag{1.32}$$

give rise to dual maps

$$\hat{Y} \mapsto \sum_\alpha \hat{X}_\alpha^\dagger \, \hat{Y} \, \hat{X}_\alpha \tag{1.33}$$

which do not map into a commutative algebra, because the operators $\hat{X}_\alpha$ need not be orthogonal projections.

The above correspondences between classical and quantum probability will become useful later on in this thesis because, as we will see, quantum filtering and feedback. being indeed based on consecutive measurement processes, will provide a bridge between these two realms.

Describing joint probabilities in quantum mechanics is not as straightforward as in the classical case, since quantum observables do not in general commute. Therefore there are pairs of observables that can not in general be measured in a single realization of the experiment, because the outcome of one may in general modify the outcome of the other. Joint probabilities can be defined only for *compatible* observables, namely commuting observables. Indeed the composition of two observables $\hat{X}\hat{Y}$ is not necessarily Hermitean and thus is not an observable itself (namely $(\hat{X}\hat{Y})^\dagger \neq \hat{X}\hat{Y}$), in other words it is not a linear combination of orthogonal projections with real coefficients, unless $\hat{X}$ and $\hat{Y}$ commute [9].

Let us consider the quantum probability space $(\mathcal{N}, \mathbb{P}_\rho)$, where $\mathcal{N}$ is a non-commutative algebra and $\mathbb{P}_\rho$ is the quantum state of the system (which defines a probability in the sense of (1.23)). We consider a commutative algebra $\mathcal{A} \subset \mathcal{N}$, so that it is always possible to find orthogonal projectors $\{\hat{P}_i\}_{i=1}^{N_a}$, with $\hat{P}_i\hat{P}_j = \delta_{ij}$, which generate $\mathcal{A}$. Let $\mathcal{A}' \subset \mathcal{N}$ be the set of all the operators of $\mathcal{N}$ that commute with every element in $\mathcal{A}$; namely,

$$\mathcal{A}' = \{\hat{A}' \in \mathcal{N} : \ \left[\hat{A}, \hat{A}'\right] = 0, \ \forall \, \hat{A} \in \mathcal{A}\} \ . \tag{1.34}$$

This linear subset is also closed under operator multiplication and thus a sub-algebra, known as the *commutant* of $\mathcal{A}$, which contains all the observables that commute and can thus be jointly measured with those of $\mathcal{A}$. Notice however that $\mathcal{A}'$ is not necessarily a commutative algebra itself and certainly, since $\mathcal{A}$ is commutative, $\mathcal{A} \subseteq \mathcal{A}'$.

Definition 1.4. The *conditional expectation* of the algebra $\mathcal{A}'$ onto the commutative algebra $\mathcal{A} \subseteq \mathcal{A}'$ generated by orthogonal projections $\{\hat{P}_i\}_{i=1}^{N_a}$, is the map $\mathbb{P}_\rho(\cdot|\mathcal{A}) : \mathcal{A}' \mapsto \mathcal{A}$ defined by

$$\mathbb{P}_\rho(\hat{A}'|\mathcal{A}) := \sum_{i=1}^{N_a} \frac{\mathrm{tr}\left(\rho \, \hat{A}' \, \hat{P}_i\right)}{\mathrm{tr}\left(\rho \, \hat{P}_i\right)} \hat{P}_i = \sum_{i=1}^{N_a} \frac{\mathbb{P}_\rho(\hat{A}' \, \hat{P}_i)}{\mathbb{P}_\rho(\hat{P}_i)} \hat{P}_i \, , \qquad \forall \, \hat{A}' \in \mathcal{A}'. \tag{1.35}$$

Such map projects operators $\hat{A}'$ in the larger algebra $\mathcal{A}'$ into operators $\mathbb{P}_\rho(\hat{A}'|\mathcal{A})$ in the commutative algebras $\mathcal{A}$, namely the map is operator-valued and thus not to be confounded with an expectation value. Rather, the conditional expectation can easily be seen to be an idempotent

[1] POVM stands for Positive Operator-Valued Measure [27, 34].

map which thus acts on the algebra $\mathcal{A}$ as a projector. Moreover, the conditional expectation is directly linked to conditional probability: if instead of $\hat{A}'$, we put one of the projectors $\hat{P}'_a$ of its spectral decomposition (1.21) $\hat{A}' = \sum_{a'} a' \hat{P}'_{a'}$, then the coefficients of the summation in (1.35) are the conditional probabilities of obtaining the outcome a' for $\hat{A}'$ when the outcome related to the projector $\hat{P}_i$ is obtained.

The main characteristic of conditional expectations is the following relation (it is easy to check that the definition given in (1.35) fulfils it):

$$\mathbb{P}_\rho(\mathbb{P}_\rho(\hat{A}'|\mathcal{A})\hat{A}) = \mathbb{P}_\rho(\hat{A}'\hat{A}) \quad \forall \hat{A}' \in \mathcal{A}' \,,\, \hat{A} \in \mathcal{A} \,, \tag{1.36}$$

where the right hand side is the expectation value of $\hat{A}'\hat{A}$ which, for self-adjoint $\hat{A}'$ and $\hat{A}$, is an observable since $\hat{A}'$ commutes with $\hat{A}$ and thus $\hat{A}'\hat{A} = (\hat{A}'\hat{A})^\dagger$. Again, if we substitute $\hat{A}'$ and $\hat{A}$, each with one of their projectors, we obtain the joint probability for the two events to simultaneously occur.

Example 1.4. A typical instance of the above setting that we will meet in the following is when we deal with coupled bipartite quantum systems $S = S_1 + S_2$ whose initially factorized state $\rho = \rho_1 \otimes \rho_2$ jointly evolves in time $\rho \mapsto \rho_t = U_t(\rho_1 \otimes \rho_2)U_t^\dagger$, where U_t is a unitary describing the dynamics up to time t. Such maps usually couple the two systems, dynamically and thus statistically. The probability at time t of obtaining a certain outcome a for an observable $\hat{A} = \sum_{a \in spec(\hat{A})} a\hat{P}_a$ of S_1 is [28]:

$$\mathbb{P}_{\rho_t}(\hat{A} = a) = \mathrm{tr}\left(U_t(\rho_1 \otimes \rho_2)U_t^\dagger(\hat{P}_a \otimes \mathbb{I})\right) \,. \tag{1.37}$$

Algebraically, measuring these observables results into the commutative algebra $\mathcal{A}$ as in (1.29), generated by the orthogonal projectors $\hat{P}_a$. We now want to know how this event affects the measurements of the observables of S_2. Since the observables of S_1 commute with those of S_2, we can condition $\hat{A}' = \sum_{a'} a' \hat{P}'_{a'}$ of S_2 with respect to the algebra $\mathcal{A}$:

$$\mathbb{P}_{\rho_t}(\hat{A}'|\mathcal{A}) = \sum_a \frac{\mathrm{tr}\left(U_t(\rho_1 \otimes \rho_2)U_t^\dagger(\hat{P}_a \otimes \hat{P}'_{a'})\right)}{\mathrm{tr}\left(U_t(\rho_1 \otimes \rho_2)U_t^\dagger(\hat{P}_a \otimes \mathbb{I})\right)} \hat{P}_a. \tag{1.38}$$

This type of conditional expectations can thus be used in the case of interacting bipartite systems, when one of the two subsystems is monitored and we want to describe how this monitoring affects the other subsystem.

After this brief overview of those notions of classical and quantum probabilities necessary to understand the topics that will be addressed in the following, in the next chapter we provide an introduction to the the dynamics of open quantum systems. Indeed, later in the thesis, we shall focus on how open quantum dynamics change when a system interacting with its environment is constantly monitored and thus several sources of stochasticity are present.

Chapter 2

Open quantum dynamics

In the previous chapter, we generally talked about probability in quantum mechanics without considering the dynamics driving the system. Here we will focus upon the equations of motion for the density matrices which generate the typical time-evolutions of quantum systems. In particular we will consider *open* quantum dynamics, namely those that drive quantum systems when immersed in an environment, usually a quantum system with (infinitely) many degrees of freedom, causing a non-unitary evolution of the smaller system.

This chapter is structured in the following way, we will firstly examine the maps describing the evolution up to a certain instant of time of a sub-system immersed in an environment, paying special attention to the properties these maps have to fulfil in order to be physically consistent. In regard to them, we will introduce the fundamental concepts of complete positivity and Kraus representation [4, 13]. In the section relative to dynamical semigroups [40], we will show the main mathematical difference of an open with respect to a closed quantum dynamics. After that, we will focus on the infinitesimal evolution of a quantum state, discussing the celebrated Gorini-Kossakowski-Sudarshan-Lindblad (GKSL) master equation [1, 4, 11, 13] which describes the reduced dynamics of an open quantum system when its coupling with the environment is sufficiently weak and possible memory effects can be neglected (*Markovian approximation*). After the derivation of the GKSL master equation by means of further approximations, generically grouped under the name of *weak-coupling limit*, we will briefly present some results concerning the asymptotic behaviour of the dynamics and its stationary states.

2.1 Quantum dynamical maps

We consider a system S described by a Hilbert space $\mathcal{H}_S$ coupled with an environment B described by $\mathcal{H}_B$. The total Hilbert space of the compound quantum system $S + B$ is given by the tensor product $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_B$. Throughout this chapter, we will distinguish states and observables related to two systems with the subscripts S and B, while we will not use any subscript when the states and observables belong to the compound system $S+B$. We will denote by $\mathcal{B}(\mathcal{H})$ the algebra of all bounded operators on $\mathcal{H}$ to which the system observables typically belong and consider Hamiltonians $\hat{H} = \hat{H}^\dagger$ of the composite system that need not be bounded,

in general:

$$\hat{H} = \hat{H}_S \otimes \hat{\mathbb{I}}_B + \hat{\mathbb{I}}_S \otimes \hat{H}_B + \lambda \hat{H}_{int} \tag{2.1}$$

where $\hat{H}_S$ and $\hat{H}_B$ are system and environment Hamiltonians, while $\hat{\mathbb{I}}$ is the identity operator and $\hat{H}_{int}$ represents the interaction between the two systems and λ is a dimensionless coupling constant (the intensity of the interaction). Clearly, if $\lambda \hat{H}_{int} = 0$, we would have separate unitary evolutions for S and B.

Remark 2.1. For finite d-level systems $\mathcal{H} = \mathbb{C}^d$, the algebra of bounded operators is the algebra of $d \times d$ complex matrices. For quantum systems with infinite dimensional Hilbert spaces the bounded operators $\hat{X} : \mathcal{H} \mapsto \mathcal{H}$ are those with finite norm

$$\|\hat{X}\| := \sup \frac{\|\hat{X}\psi\|}{\|\psi\|} < +\infty \ .$$

Such a norm fulfil the following properties

$$\|\hat{X}\,\hat{Y}\| \leq \|\hat{X}\|\,\|\hat{Y}\| \tag{2.2}$$

$$\|\hat{X}\| = \|\hat{X}^\dagger\| \tag{2.3}$$

$$\|\hat{X}\,\hat{X}^\dagger\| = \|\hat{X}^\dagger\|^2 \ . \tag{2.4}$$

Moreover, $B(\mathcal{H})$ is a Banach algebra with respect to the norm, namely all Cauchy sequences $\{\hat{X}_n\}_n \in B(\mathcal{H})$ such that $\|\hat{X}_n - \hat{X}_m\| \mapsto 0$ when $n, m \mapsto +\infty$, do converge in the algebra. The bounded operators on $\mathcal{H}$ thus constitute a so-called C^*-algebra.

The compound quantum system $S + B$ is considered as an overall closed system whose dynamics is reversible and generated by Hamiltonian $\hat{H}$ that gives rise to the usual unitary, namely reversible, Schrödinger time evolution. Assuming that the initial state at $t = 0$ is given by $\rho = \rho_S \otimes \rho_B$, we have for $t \geq 0$:

$$\rho(t) = \hat{U}_t\,\rho\,\hat{U}_t^\dagger, \qquad \hat{U}_t = e^{-i\hat{H}t}, \tag{2.5}$$

where $\hat{U}_t$ is the unitary time evolution operator, hence $\hat{U}_t^\dagger = \hat{U}_{-t}$.

To recover information about the system S alone we use the *partial trace*: we take the mean value over the degrees of freedom of B and obtain the so-called *reduced dynamics*

$$\rho_S(t) = \mathrm{tr}_B\left(\hat{U}_t(\rho_S \otimes \rho_B)\,\hat{U}_t^\dagger\right) =: \Lambda_t\left[\rho_S\right]. \tag{2.6}$$

The partial trace thus defines a *quantum dynamical map* $\Lambda_t : \rho_S \mapsto \rho_S(t)$ which retrieves the dynamics of the system S alone. We remark however that, though the reduced evolution affects only S, it actually depends on the compound dynamics in (2.5) and on the initial state of the environment. Moreover, if S and B are initially correlated, namely if the initial state of the compound system is not factorized, then the map defined by the trace in (2.6) would not be linear in general [1, 32]. The main issue at stake here is the embedding of states ρ_S of the open quantum system into the space of states ρ_{SB} of the open system plus its environment.

Remark 2.2. Given two systems S and B, we define an assignment map A such that $A[\rho_S] = \rho_{SB}$ where ρ_S is the state of S and ρ_{SB} is the state of the composite system $S+B$. The following properties for the assignment map appear to be physically necessary:

- linearity: $A[\lambda\rho_{S_1} + (1-\lambda)\rho_{S_2}] = \lambda A[\rho_{S_1}] + (1-\lambda) A[\rho_{S_2}]$, where $\lambda \in \mathbb{R}^+$,
- positivity: $A[\rho_S] \geq 0$,
- consistency: $\mathrm{tr}_B\{A[\rho_S]\} = \rho_S$.

It turns out [32] that the only assignment map that satisfies al three must be such that

$$A[\rho_S] = \rho_S \otimes \rho_B,$$

where ρ_B is a fixed state of B. This is the reason why factorized states are adopted as initial state in order to study open quantum dynamics.

The following sections are devoted to derive approximate expressions for the dynamical maps Λ_t in (2.6) that describe the reduced dynamics of open quantum systems.

2.1.1 Kraus representation

In order to have a consistent quantum dynamics, the dynamical map Λ_t has to fulfil some requirements, first of all it must map states into states. We remind that a proper state is a positive and normalized operator on the Hilbert state. We now introduce the so-called Kraus representation [4, 13] of the dynamical map, which is a remarkable result because it assures the consistency of the dynamical map, as we will show soon.

Theorem 2.1. Dynamical maps $\Lambda_t : \mathcal{B}(\mathcal{H}_S) \mapsto \mathcal{B}(\mathcal{H}_S)$ obtained as in (2.6) can always be written in the following so-called *diagonal Kraus decomposition*

$$\rho_S(t) = \Lambda_t[\rho_S] = \sum_\alpha K_\alpha(t)\, \rho_S\, K_\alpha^\dagger(t)\ , \tag{2.7}$$

where K_α are the *Kraus operators* which fulfil

$$\sum_\alpha K_\alpha^\dagger(t) K_\alpha(t) = \mathbb{I}_S. \tag{2.8}$$

Proof. Let us assume the environment B, though with infinitely many degrees of freedom, may be described by a density matrix $\rho_B = \sum_s \omega_s |\omega_s\rangle\langle\omega_s|$ with $|\omega_s\rangle$ a base of $\mathcal{H}_B$, hence

$$\begin{aligned}\Lambda_t[\rho_S] = \mathrm{tr}_B\left(\hat{U}_t(\rho_S\otimes\rho_B)\,\hat{U}_t^\dagger\right) = \sum_s \omega_s\, \mathrm{tr}_B\left(\hat{U}_t(\rho_S\otimes|\omega_s\rangle\langle\omega_s|)\,\hat{U}_t^\dagger\right) = \\ = \sum_{r,s} \omega_s \langle\omega_r|\hat{U}_t|\omega_s\rangle\, \rho_S(t)\, \langle\omega_s|\hat{U}_t^\dagger|\omega_r\rangle\end{aligned} \tag{2.9}$$

where in last step we took the partial trace over the environment eigenstates $|\omega_r\rangle$.

Since $\omega_s \geq 0$ for every s, we can write:

$$\rho_S(t) = \sum_{r,s} \hat{K}_{rs}(t)\rho_S\hat{K}_{rs}^\dagger, \qquad \hat{K}_{rs}(t) := \sqrt{\omega_s}\,\langle\omega_r|\hat{U}_t|\omega_s\rangle, \tag{2.10}$$

moreover we can condense the indexes $r, s = 1$ in just one index α.

The other statement of the theorem affirms that the Kraus operators $\hat{K}_\alpha(t)$ satisfy $\sum_\alpha \hat{K}_\alpha^\dagger(t)\hat{K}_\alpha(t) = \mathbb{I}_S$; indeed,

$$\sum_\alpha \hat{K}_\alpha^\dagger(t)\hat{K}_\alpha(t) = \sum_{r,s} \omega_s \langle\omega_s|\hat{U}_t^\dagger|\omega_r\rangle\langle\omega_r|U_t|\omega_s\rangle = \sum_s \omega_s \langle\omega_s|\hat{U}_t^\dagger\hat{U}_t|\omega_s\rangle = \mathbb{I}. \tag{2.11}$$

□

The above theorem shows the expected structure of reduced quantum dynamics of the states of an open quantum system S. We can also write it for the open system observables operators by switching to the Heisenberg picture and introducing the dual map (see (1.31)):

$$\hat{X}_S(t) = \Lambda_t^* \left[\hat{X}_S\right] = \sum_\alpha \hat{K}_\alpha^\dagger(t)\, \hat{X}_S\, \hat{K}_\alpha(t), \qquad \hat{X}_S \in \mathcal{B}(\mathcal{H}_S). \tag{2.12}$$

A consequence of (2.8) is that the dual map is *unital*; namely, it preserves the identity $\Lambda_t^* [\mathbb{I}] = \mathbb{I}$.

Having introduced the Kraus representation, we now proceed to show why this form is so ubiquitous in open quantum system theory and what is its physical significance. To appreciate it, we firstly introduce in the following section the main requirements in order to have a physically consistent reduced dynamics.

2.1.2 Positivity and complete positivity

We now introduce the notions of positivity and complete positivity of linear maps [5], which will be extremely important to understand the consequences of the Kraus decomposition and its relations with the concept of quantum entanglement, which will be addressed in the next chapter.

Definition 2.1. An operator $\hat{X} \in B(\mathcal{H})$ is said to be positive, $\hat{X} \geq 0$, if and only if all its expectation values are positive: namely, if

$$\langle\psi|\hat{X}|\psi\rangle \geq 0 \quad \forall|\psi\rangle \in \mathcal{H}\ . \tag{2.13}$$

A linear map $\Lambda : B(\mathcal{H}) \mapsto B(\mathcal{H})$ is *positive* if it sends positive operators into positive operators:

$$\Lambda[\hat{X}] \geq 0 \qquad \forall\, 0 \leq \hat{X} \in B(\mathcal{H})\ . \tag{2.14}$$

Definition 2.2. A linear map $\Lambda : B(\mathcal{H}) \mapsto B(\mathcal{H})$ is *completely positive* (CP) if and only if the composition $\Lambda \otimes \mathrm{id}_m$ of Λ with the m-dimensional identity, id_m, acting on $B(\mathcal{H}) \otimes \mathcal{M}_m(\mathbb{C})$, is positive for all $m \geq 1$.

Since, when $m = 1$, $\Lambda \otimes \mathrm{id}_1 = \Lambda$, completely positive maps are positive, but not vice versa: indeed, complete positivity is much stronger a request than positivity.

The notion of complete positivity is essential in order to have consistent open dynamics: it assures that even if we couple a system S with an m-dimensional ancillary system S_m, the map still sends physical states of the compound system $S + S_m$ into physical states. This is neither a superfluous request nor a a purely mathematical one. Rather, though it may appear

obvious that, when tensorizing a positive dynamical map with the trivial identity map, the tensor product should also preserve positivity and thus be positive, this is not so in presence of entanglement. One may say that the request of complete positivity of physically consistent dynamical map is the dynamical counterpart of the presence of entangled states. In order to explain in which sense, we have to make an entanglement interlude.

Definition 2.3. A generic mixed ρ_{AB} of a bipartite quantum system described by the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ is separable if and only if it can be cast as a convex combination of tensor products of density matrices of the parties; namely if,

$$\rho_{AB} = \sum_k \lambda_k \, \rho_k^A \otimes \rho_k^B \ , \qquad \lambda_k \geq 0 \ , \ \sum_k \lambda_k = 1 \ . \tag{2.15}$$

If no such convex decomposition exists, the state ρ_{AB} is called entangled.

Example 2.1. If ρ_{AB} is a pure state, $\rho_{AB} = |\Psi_{AB}\rangle\langle\Psi_{AB}|$, then it is separable if and only $|\Psi_{AB}\rangle = |\psi_A \otimes \psi_B\rangle$. A standard example of entangled pure state for a two qubit system is the state vector

$$\left|\Psi_{sym}^{(2)}\right\rangle = \frac{1}{\sqrt{2}} \left(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle\right), \tag{2.16}$$

with $|\uparrow\rangle$ and $|\downarrow\rangle$ the eigenstates of the Pauli matrix $\hat{\sigma}_z$.

This state clearly can not be put into the tensor product of a state vector of the first qubit and one of the second qubit. Moreover, when measuring the first qubit with the spin for example in the up direction along the z-direction, then the second qubit is certainly along the down z-direction. An entangled state such as the above one is so strongly correlated that it does not permit to attribute any property to its party, pointing to the presence of highly non-classical correlations.

This peculiarity of quantum correlations can also be seen by means of the *von Neumann entropy*, defined for all density matrices $\rho \in \mathcal{S}(S)$, with S a finite n-level quantum system by

$$S(\rho) = -\operatorname{tr} \rho \log \rho = -\sum_{i=1}^{n} r_i \log r_i \ , \tag{2.17}$$

where r_i are the eigenvalues of ρ. The von Neumann entropy measures the uncertainty about S when its state is ρ: if the latter is a pure state the $S(\rho) = 0$, while if ρ is completely mixed, that is if $\rho = \dfrac{\mathbb{I}}{n}$, then $S(\rho) = \log n$ is maximal.

Consider now the associated projection and average over the other one via the partial trace, we find:

$$\rho_{sym}^{(2)} = \left|\Psi_{sym}^{(2)}\right\rangle \left\langle\Psi_{sym}^{(2)}\right| = \frac{1}{2} \left(|\uparrow\uparrow\rangle\langle\uparrow\uparrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow| \ + \ |\uparrow\uparrow\rangle\langle\downarrow\downarrow| + |\downarrow\downarrow\rangle\langle\uparrow\uparrow|\right), \tag{2.18}$$

$$\rho_A = \operatorname{tr}_B \rho_{sym}^{(2)} = \sum_{\chi=\uparrow_B,\downarrow_B} \langle\chi|\rho_{sym}^{(2)}|\chi\rangle = \frac{1}{2}(|\downarrow\rangle\langle\downarrow| + |\uparrow\rangle\langle\uparrow|) = \frac{\mathbb{I}_A}{2} \ . \tag{2.19}$$

That is, even if the composite system $A+B$ is completely known (it is in a pure state), when we focus upon one of the sub-systems alone, its statistical properties are those of a completely indeterminate state. Namely, for entangled pure states the uncertainty of the whole can be smaller

than that of the parties, something impossible for classically correlations. On the contrary, if the state were separable it would factorize into $\rho_{AB} = |\psi_A\rangle\langle\psi_A| \otimes |\psi_B\rangle\langle\psi_B|$ and its partial trace would also be a pure state.

One might guess that checking whether the von Neumann entropy of a bipartite state is smaller that that of one of the parties might reveal the entanglement of the state; unfortunately, this is true in general only for pure states while there can exist entangled mixed state whose uncertainty is nevertheless higher than that of its reduced states. However, consider the transposition map $T : \mathcal{M}_2(\mathbb{C}) \mapsto \mathcal{M}_2(\mathbb{C})$ which is positive since the transposition preserve the spectrum which is positive for positive operators. However, the so-called partial transposition $T \otimes \mathrm{id}_2$ is not positive as can be deduced by acting with the partial transposition on the above entangled state ρ_{AB}, yielding

$$T \otimes \mathrm{id}_2[\rho^{(2)}_{sym}] = \frac{1}{2}\left(|\uparrow\uparrow\rangle\langle\uparrow\uparrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow| + |\downarrow\uparrow\rangle\langle\uparrow\downarrow| + |\uparrow\downarrow\rangle\langle\downarrow\uparrow|\right) . \tag{2.20}$$

The operator on the right hand side has $\dfrac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}}$ as eigenvector with eigenvalue -1. As we shall see in Section 3.1.1, not only the von Neumann entropy is sensitive to entanglement; also positive, but not completely positive maps can witness its presence by not sending entangled states into states, since they can turn some positive eigenvalue into a negative one.

The fact that the positivity of the entangled state ρ_{AB} in the previous example is not preserved by the partial transposition is due that 1) the state cannot be written as in (2.15) and 2) that Transposition is not completely positive. If all quantum states were separable, then positivity would be enough for physical consistency; indeed, for all positive maps Λ

$$\Lambda \otimes \mathrm{id}_m[\rho_{AB}] = \sum_k \lambda_k \, \Lambda[\rho^A_k] \otimes \rho^B_k \geq 0 , \tag{2.21}$$

where B is the ancillary m-level system and $\Lambda[\rho^A_k]$ is positive because Λ is assumed to preserve positivity. One thus sees that is the presence of entanglement what forces one to impose on positive maps also the stronger request of complete positivity.

However, it can be very difficult to check the complete positivity of a map Λ, since we should assure that any positive operator is mapped into a positive one by every possible m-dimensional lifting $\Lambda \otimes \mathrm{id}_m$. Luckily, for maps on finite level systems, there exists a useful protocol given by the following theorem that is indeed based on a higher dimensional generalization of the pure state in Example 2.1, (of which we will not give a proof), called *Choi's theorem* [22].

Theorem 2.2. A linear map $\Lambda : \mathcal{M}_n(\mathbb{C}) \mapsto \mathcal{M}_n(\mathbb{C})$ is completely positive if the associated *Choi matrix* C_Λ is positive (semi-)definite, where

$$C_\Lambda := \Lambda \otimes \mathrm{id}_n[P^{(n)}_{sym}] \in \mathcal{M}_n(\mathbb{C}) \otimes \mathcal{M}_n(\mathbb{C}) = \mathcal{M}_n(\mathcal{M}_n(\mathbb{C})) , \tag{2.22}$$

and $P^{(n)}_{sym}$ is the projector $\left|\Psi^{(n)}_{sym}\right\rangle\left\langle\Psi^{(n)}_{sym}\right|$ over the maximally entangled state

$$\left|\Psi^{(n)}_{sym}\right\rangle = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} |jj\rangle \in \mathbb{C}^n \otimes \mathbb{C}^n , \tag{2.23}$$

with $\{|j\rangle\}_{j=1}^n$ any fixed orthonormal basis in $\mathbb{C}^n$.

Remark 2.3. Choi's theorem sets an isomorphism, known as Choi-Jamiołkowski isomorphism, between linear maps $\Lambda : \mathcal{M}_n(\mathbb{C}) \mapsto \mathcal{M}_n(\mathbb{C})$ and matrices $\hat{X} \in \mathcal{M}_n(\mathcal{M}_n(\mathbb{C}))$: any linear map $\Lambda : \mathcal{M}_n(\mathbb{C}) \mapsto \mathcal{M}_n(\mathbb{C})$ is associated to a unique C_Λ and any matrix $\hat{X} \in \mathcal{M}_n(\mathcal{M}_n(\mathbb{C}))$ identifies a unique map $\Lambda_X : \mathcal{M}_n(\mathbb{C}) \mapsto \mathcal{M}_n(\mathbb{C})$ whose Choi matrix is $\hat{X}$. Furthermore, the Choi matrix C_Λ allows one to check the complete positivity of a map Λ relative to an n-level system by studying the action of only one lifting, $\Lambda \otimes \mathrm{id}_n$, on just one projector.

If a transformation does not dissipate probability, it must be described by a trace-preserving map Λ^* on the convex space of states $\mathcal{S}(S)$ of a quantum system, Λ^* denoting the dual map of a map Λ on the system observables.

Definition 2.4. A linear map $\Lambda^* : \mathcal{S}(S) \mapsto \mathcal{S}(S)$ is *trace preserving* (TP) on the space of states of a quantum system if

$$\mathrm{tr}\left(\Lambda^*\left[\rho\right]\right) = \mathrm{tr}\left(\rho\right), \qquad \forall \rho \in \mathcal{S}(S) \ .$$

Guided by the previous discussion, we will then always require dynamical maps on the space of states of quantum systems Λ_t to be completely positive and trace preserving. The Kraus representation (2.7) of reduced dynamics guarantees their complete positivity.

Theorem 2.3. Every unital (identity preserving) linear map $\Lambda : \mathcal{M}_n(\mathbb{C}) \mapsto \mathcal{M}_n(\mathbb{C})$ which admits a Kraus decomposition

$$\mathcal{M}_n(\mathbb{C}) \ni \hat{X} \mapsto \Lambda[\hat{X}] = \sum_\alpha \hat{L}_\alpha^\dagger \, \hat{X} \, \hat{L}_\alpha \ , \tag{2.24}$$

where $\hat{L}_\alpha \in$ satisfy

$$\sum_\alpha \hat{L}_\alpha^\dagger \hat{L}_\alpha = \hat{\mathbb{I}} \ , \tag{2.25}$$

is completely positive as well as its dual $\Lambda^* : \mathcal{S}(S) \mapsto \mathcal{S}(S)$,

$$\mathcal{S}(S) \ni \rho \mapsto \Lambda^*[\rho] = \sum_\alpha \hat{L}_\alpha \, \rho \, \hat{L}_\alpha^\dagger \ , \tag{2.26}$$

which is trace preserving (CPTP).

Proof. Trace-preservation follows from (2.26), from the cyclic property of the trace and from (2.25):

$$\mathrm{tr}\left(\Lambda^*[\rho]\right) = \mathrm{tr}\left(\sum_\alpha \hat{L}_\alpha \, \rho \, \hat{L}_\alpha^\dagger\right) = \sum_\alpha \mathrm{tr}\left(\hat{L}_\alpha^\dagger \hat{L}_\alpha \, \rho\right) = \mathrm{tr}\left(\rho\right) \ .$$

In order to prove that a generic lifting $\Lambda \otimes \mathrm{id}_m$ preserves positivity, we need identify positive operators in the algebra $\mathcal{M}_m(\mathcal{M}_n(\mathbb{C})) := \mathcal{M}_n(\mathbb{C}) \otimes \mathcal{M}_m(\mathbb{C})$. Given an orthonormal basis $\{|j\rangle\}_{j=1}^m$ in $\mathbb{C}^m$ and the associated matrix units $\{|j\rangle\langle k|\}_{j,k=1}^m$, generic $\hat{X} \in \mathcal{M}_m(\mathcal{M}_n(\mathbb{C}))$ can be written as

$$\mathcal{M}_m(\mathcal{M}_n(\mathbb{C})) \ni \hat{X} = \sum_{j,k=1}^m \hat{X}_{jk} \otimes |j\rangle\langle k| \ , \quad \hat{X}_{jk} \in \mathcal{M}_n(\mathbb{C}) \ ,$$

that is as $m \times m$ matrices $[\hat{X}_{jk}]$ of $n \times n$ matrices $\hat{X}_{jk}$ acting on vectors of the Hilbert space $\mathbb{C}^n \otimes \mathbb{C}^m$ of the form

$$|\Psi\rangle = \sum_{j=1}^{m} |\psi_j\rangle \otimes |j\rangle \ , \quad |\psi_j\rangle \in \mathbb{C}^n \ .$$

Certainly, operators of the form $\hat{X}_{pos} = [\hat{X}_j^\dagger \hat{X}_k]$ are positive; indeed,

$$\langle \Psi | \hat{X} | \Psi \rangle = \left| \sum_{j=1}^{m} \hat{X}_j |\psi_j\rangle \right|^2 \geq 0 \ .$$

Their positivity is preserved by $\Lambda \otimes \mathrm{id}_m$ for

$$\Lambda \otimes \mathrm{id}_m[\hat{X}_{pos}] = \sum_{\alpha} \sum_{j,k=1}^{m} \hat{L}_\alpha^\dagger \hat{X}_j^\dagger \hat{X}_k \hat{L}_\alpha \otimes |j\rangle \langle k|$$

is the sum of positive operators $\hat{Y}_{pos}^{\alpha} = \left[\hat{Y}_{\alpha,j}^\dagger \hat{Y}_{\alpha,k}\right]$ with $\hat{Y}_{\alpha,k} = \hat{X}_k \hat{L}_\alpha$. The proof is completed by noticing that generic positive $\hat{X} \in \mathbb{C}^n \otimes \mathbb{C}^m$ can be written as combination of $\hat{Y}_{pos}^{\alpha}$ with positive coefficients, In fact, using the spectralization of $\hat{X} \geq 0$, one writes

$$\begin{aligned} \hat{X} &= \sum_{\alpha=1}^{mn} \lambda_\alpha \ |\Psi_\alpha\rangle \langle \Psi_\alpha| = \sum_{\alpha=1}^{mn} \sum_{j,k=1}^{m} \lambda_\alpha \ |\psi_{\alpha,j}\rangle \langle \psi_{\alpha,k}| \otimes |j\rangle \langle k| \\ &= \sum_{\alpha=1}^{mn} \sum_{j,k=1}^{m} \lambda_\alpha \, \hat{Y}_{\alpha,j}^\dagger \hat{Y}_{\alpha,k} \otimes |j\rangle \langle k| \ , \end{aligned}$$

where $\hat{Y}_{\alpha,k} = |\phi\rangle \langle \psi_{\alpha,k}|$, $|\phi\rangle \in \mathbb{C}^n$ is any normalized state vector of the system S, while $\lambda_\alpha \geq 0$ are the eigenvalues of $\hat{X}$ with $|\Psi_\alpha\rangle$ the corresponding eigenvectors. Finally, the dual map Λ^* is also completely positive since it is too of the Kraus form. □

Interestingly enough, the opposite implication with respect to the previous theorem also holds.

Theorem 2.4. Every completely positive linear map $\Lambda : \mathcal{M}_n(\mathbb{C}) \mapsto \mathcal{M}_n(\mathbb{C})$ admits a Kraus decomposition.

Proof. According to Theorem 2.2, the Choi matrix $C_\Lambda = \Lambda \otimes \mathrm{id}_n[P_{sym}^{(n)}]$ of Λ is positive semi-definite; given its spectralization and using the symmetric vector (2.23), one writes

$$\begin{aligned} \hat{C}_\Lambda &= \sum_{\alpha=1}^{mn} \lambda_\alpha \ |\Psi_\alpha\rangle \langle \Psi_\alpha| = \sum_{\alpha=1}^{mn} \sum_{j,k=1}^{m} \lambda_\alpha \ |\psi_{\alpha,j}\rangle \langle \psi_{\alpha,k}| \otimes |j\rangle \langle k| \\ &= \sum_{\alpha=1}^{mn} \hat{K}_\alpha \otimes \hat{\mathbb{I}} \, P_{sym}^{(n)} \, \hat{K}_\alpha^\dagger \otimes \hat{\mathbb{I}} \ , \end{aligned}$$

where the operators $\hat{K}_\alpha : \mathbb{C}^n \mapsto \mathbb{C}^n$ are defined on the orthonormal basis $\{|j\rangle\}_{j=1}^n$ by

$$\hat{K}_\alpha \, |j\rangle = \sqrt{n\lambda_\alpha} \ |\psi_{\alpha,j}\rangle \ .$$

The expression of C_Λ identifies as the Choi matrix of the map $\Lambda' : \mathcal{M}_n(\mathbb{C}) \mapsto \mathcal{M}_n(\mathbb{C})$ such that

$$\Lambda'[\hat{X}] = \sum_{\alpha=1}^{mn} \hat{K}_\alpha^\dagger \, \hat{X} \, \hat{K}_\alpha \ ,$$

which must thus coincide with Λ because of the Choi-Jamiołkowski isomorphism (see Remark 2.3). □

Remark 2.4. Completely positive trace-preserving linear maps are also known as *quantum channels*, a foremost example of them being the projective measurements in (1.25). Actually, the requirement of trace-preservation is not always mandatory for dynamical maps: there can indeed be operations that do not preserve the trace of the system, for instance when some information is lost during the process as for instance for decaying quantum systems. In such cases, states ρ transform as in (2.24), but the operators in it do not satisfy (2.25) rather

$$\sum_\alpha \hat{L}_\alpha^\dagger \hat{L}_\alpha \leq \hat{\mathbb{I}}$$

and complete positivity alone is sufficient to have a realistic dynamics. However, in the following we will focus on CPTP maps.

Though being perfectly physically consistent, the reduced dynamics in (2.6) is scarcely tractable as it requires the knowledge of the compound system unitary time-evolution in order to be useful. Before proceeding further with the necessary approximations that turn the dynamical maps Λ_t into a manageable semigroup with a characteristic generator, we now briefly discuss the differences of an open quantum dynamics with respect to a closed one.

2.1.3 Dynamical semigroups

We now introduce a sub-class of dynamical maps that form a so-called semigroup and aptly expose some important features of an open quantum time-evolutions.

For a closed quantum system, the unitary time-evolution $\hat{U}_t$ action on a state ρ described by the Hamiltonian $\hat{H}$ is

$$\rho_t = \mathbb{U}_t \, [\rho] := \hat{U}_t \, \rho \, \hat{U}_t^\dagger, \qquad \hat{U}_t = e^{-i \, \hat{H} \, t}. \tag{2.27}$$

Introducing the action

$$\mathbb{L}_H[\rho] := -i[\hat{H}, \rho] \ , \tag{2.28}$$

we have

$$\rho_t = \mathbb{U}_t[\rho] = e^{t\mathbb{L}_H}[\rho] = \sum_k \frac{t^k}{k!} \mathbb{L}_H \circ \mathbb{L}_H \circ \ldots \circ \mathbb{L}_H[\rho], \tag{2.29}$$

where $\circ$ is the composition law.

The unitary time evolution satisfies the following properties

i. It is invertible: $\mathbb{U}_t^{-1} = \mathbb{U}_{-t}$,

ii. It forms a group $\mathbb{U}_t \circ \mathbb{U}_s = \mathbb{U}_{t+s}$ for every $t, s \in \mathbb{R}$.

iii. It is CPTP: indeed it is in Kraus form and satisfy (2.25) with $\hat{L}_\alpha = \hat{U}_t$.

These properties, together with (2.29), show that the unitary time-evolution gives rise to a physically consistent dynamical one-parameter group of linear maps with $\mathbb{L}$ as the generator of the group.

Now, the question is: which of these properties still hold when we study the evolution of the system S immersed in an environment? Does the dynamical map Λ_t of the reduced dynamics in (2.6) satisfy them?

The third property has been abundantly discussed in what precedes and it is satisfied by the reduced dynamics (2.6). For what concerns the first property, even if it were in general possible to mathematically invert[1] the reduced dynamics Λ_t, the result is not CP, hence it has no physical meaning. Thus, the reduced dynamics is not physically invertible and there is a preferred direction in time. This result is obvious since the presence of the environment generates a dissipative evolution, moreover the operation of the partial trace, which corresponds to an average over the bath degrees of freedom, makes us irremediably lose information about the system as a whole.

Also the second property is not in general fulfilled by all dynamical maps, the family of all Λ_t lacks of the composition law, namely $\Lambda_t \circ \Lambda_s \neq \Lambda_{t+s}$. However, as we shall see, it is possible to suitably approximate the reduced dynamics in such a way that the forward-in-time composition rule $\Lambda_t \circ \Lambda_s = \Lambda_{t+s}$ is fulfilled for $t, s \geq 0$.

From a physical point of view, maps that approximate the reduced dynamics in (2.6) and fulfil the semigroup structure are the ones related to open quantum systems in which the coupling between the system and the environment is weak and whose memory effects are negligible. The procedures to eliminate the memory effects coming from the interaction with the environment in order to recover the semigroup structure are known as *Markov approximations*: we will give more details about them later in this chapter. Firstly, we focus on dynamical maps which already form a semigroup.

In order to study the actual reduced dynamics in such a case, it is more convenient to look at the differential form of the dynamical map, namely at the master equation and at the generator from which it follows.

2.2 Master equation

In the following we shall focus upon retrieving the differential expression of the dynamics, namely the *master equation* [1, 11, 13, 37]. To do so, we will firstly consider the special class of dynamical maps forming a semigroup (with the physical implications highlighted in the previous section), which allows us to derive the Gorini-Kossakowski-Sudarshan-Lindblad (GKSL) master equation. We will again base our considerations on the constraints the master equation has to fulfil in order to be physically consistent namely, complete positivity and trace-preservation. Then, we will show how, for a generic family of dynamical maps as in (2.6), without the structure of semigroup,

[1] For example, for finite-level systems the dynamical maps Λ_t can be represented by matrices that can always be inverted when restricted to the subspace orthogonal to their kernel, that is over the subpsace where they do not have vanishing eigenvalues.

it is possible to apply a specific rigorous procedure, known as weak-coupling limit, to approximate it by means of a physically consistent semigroup of dynamical maps whose generator has the GKSL form.

2.2.1 Gorini-Kossakowski-Sudarshan-Lindblad master equation

We devote this section to the derivation of the Gorini-Kossakowski-Sudarshan-Lindblad master equation, once the semigroup structure of the dynamical map is already assured. There are several different way to achieve this result, we recommend [13] for a comprehensive review of the various approaches.

We consider CPTP maps forming a semigroup, namely $\Lambda_t \circ \Lambda_s = \Lambda_{t+s}$ for $t, s \geq 0$. We want to find the generator $\mathcal{L}$ of the group such that [13]:

$$\dot{\Lambda}_t = \mathcal{L}\Lambda_t. \tag{2.30}$$

Imposing $\Lambda_0 = \mathrm{id}$ as initial condition yields

$$\Lambda_t = e^{\mathcal{L}\,t}. \tag{2.31}$$

From (2.30), one obtains the generator $\mathcal{L}$ of the dynamics by differentiating at time $t = 0$:

$$\dot{\Lambda}_t\big|_{t=0} = \mathcal{L}. \tag{2.32}$$

We assume that the semigroup be continuous with respect to the trace-norm or any other equivalent norm [40]:

$$\lim_{t\to 0} \|\Lambda_t[\rho] - \rho\|_{tr} = 0 \ , \quad \|\hat{X}\|_{tr} := \mathrm{tr}\,\sqrt{\hat{X}^\dagger \hat{X}} \ . \tag{2.33}$$

Then, we can write

$$\mathcal{L}\,[\rho_S] = \dot{\Lambda}_t\big|_{t=0}\,[\rho_S] = \lim_{t\to 0} \frac{1}{t}\left(\Lambda_t\,[\rho_S] - \Lambda_0\,[\rho_S]\right), \tag{2.34}$$

Let us consider an N-level quantum system and let $\{\hat{F}_\alpha\}_{\alpha=0}^{N^2-1} \in \mathcal{M}_N(\mathbb{C})$ be an orthonormal basis in $\mathcal{M}_N(\mathbb{C})$ with respect to the Hilbert-Schmidt scalar product

$$\ll \hat{Y}|\hat{X} \gg := \mathrm{tr}\left(\hat{Y}^\dagger\,\hat{X}\right) \ , \tag{2.35}$$

with $\hat{F}_0 = \hat{\mathbb{I}}_N/\sqrt{N}$ so that all other $\hat{F}_\alpha$ are traceless. These operators can be used to write any linear map $\Lambda : \mathcal{M}_N(\mathbb{C}) \mapsto \mathcal{M}_N(\mathbb{C})$ as follows

$$\Lambda[\hat{X}] = \sum_{\alpha,\beta=1}^{N^2} L_{\alpha\beta}\,\hat{F}_\alpha^\dagger\,\hat{X}\,\hat{F}_\beta \ , \tag{2.36}$$

where the coefficients of the expansion are given by

$$L_{\alpha\beta} = \sum_{j,k=1}^{N} \langle j|\,\hat{F}_\alpha \Lambda[|j\rangle\,\langle k|]\hat{F}_\beta^\dagger\,|k\rangle \ . \tag{2.37}$$

Such decomposition follows from the fact that the N^4 linear maps

$$\Lambda_{\alpha\beta}[\hat{X}] = \hat{F}_\alpha^\dagger \, \hat{X} \, \hat{F}_\beta \tag{2.38}$$

are orthonormal with respect to the scalar product defined on the N^4-dimensional linear space of linear maps on $\mathcal{M}_N(\mathbb{C})$ by means of their Choi matrices:

$$(\Lambda_1|\Lambda_2) := \mathrm{tr}\left(\widetilde{C}_{\Lambda_1}^\dagger \, \widetilde{C}_{\Lambda_2}\right) \ , \quad \widetilde{C}_{\Lambda_{\alpha\beta}} := N\Lambda_{\alpha\beta} \otimes \mathrm{id}_N[P_{symm}^{(N)}] \ . \tag{2.39}$$

They thus form an orthonormal basis over which one can expand any other linear map on $\mathcal{M}_N(\mathbb{C})$ and the coefficients of the expansion are obtained via the scalar product as

$$\begin{aligned} L_{\alpha\beta} &= (\Lambda_{\alpha\beta}|\Lambda) := \mathrm{tr}\left(\widetilde{C}_{\Lambda_{\alpha\beta}}^\dagger \, \widetilde{C}_\Lambda\right) = \sum_{j,k,a,b=1}^{N} \mathrm{tr}\left(\hat{F}_\beta^\dagger \, |k\rangle \langle j| \, \hat{F}_\alpha \Lambda[|a\rangle \langle b|]\right) \mathrm{tr}\left(|k\rangle \langle j|a\rangle \langle b|\right) \\ &= \sum_{j,k=1}^{N} \langle j| \, \hat{F}_\alpha \Lambda[|j\rangle \langle k|] \hat{F}_\beta^\dagger \, |k\rangle \ . \end{aligned}$$

Remark 2.5. Notice that the Kraus decomposition (2.24) is of the form (2.36) with a diagonal matrix of coefficients all equal to 1 and Kraus operators that need not be orthonormal with respect to the scalar product (2.35). If Λ is CP then expanding the Kraus operators in terms of the orthonormal matrices $\hat{F}_\alpha$ one gets the expansion (2.36) with a positive semi-definite matrix of coefficients , $[L_{\alpha\beta}] \geq 0$.

Knowing that the dynamical maps Λ_t are completely positive we can write them as

$$\Lambda_t[\rho] = \sum_{\alpha,\beta=1}^{N^2} C_{\alpha\beta}(t) \, \hat{F}_\alpha \, \rho \, \hat{F}_\beta^\dagger \ , \tag{2.40}$$

with $[C_{\alpha\beta(t)}] \geq 0$ and, since they form a semigroup, we can obtain their generator from

$$\begin{aligned} \mathcal{L}\,[\rho] &= \lim_{t\to 0} \frac{1}{t} \left(\sum_{\alpha,\beta=1}^{N^2} C_{\alpha\beta}(t) \hat{F}_\alpha \rho_S \hat{F}_\beta^\dagger - \rho \right) = \\ &= \lim_{t\to 0} \frac{1}{t} \left((C_{00}(t) - 1)\rho + \sum_{k=1}^{N^2-1} C_{0k}(t) \rho_S \hat{F}_k^\dagger + \sum_{k=1}^{N^2-1} C_{k0}(t) \hat{F}_k \rho + \sum_{k,l=1}^{N^2-1} C_{kl}(t) F_k \rho F_l^\dagger \right), \end{aligned} \tag{2.41}$$

where we absorbed the factors $\sqrt{N}$ coming from the normalization of $\hat{F}_0$ in the coefficients $C_{00}(t)$, $C_{0k}(t)$ and $C_{k0}(t)$. Now setting

$$a_{kl} := \lim_{t\to 0^+} \frac{C_{kl}(t)}{t} \ , \quad \hat{A} := \lim_{t\to 0} \frac{1}{t} \left(\frac{C_{00}(t) - 1}{2} + \sum_{k=1}^{N^2-1} C_{0k} \hat{F}_k \right) \ , \tag{2.42}$$

we obtain

$$\mathcal{L}\,[\rho] = \hat{A}\rho_S + \rho\hat{A}^\dagger + \sum_{k,l=1}^{N^2-1} a_{kl} \hat{F}_k \rho \hat{F}_l^\dagger . \tag{2.43}$$

Since $\hat{A}$ is not Hermitean, we can split it as $\hat{A} := \Gamma - i\hat{H}$ with $\hat{\Gamma}^\dagger = \hat{\Gamma}$ and $\hat{H}^\dagger = \hat{H}$.

Since Λ_t is trace preserving, we also have $\mathrm{tr}\left(\mathcal{L}\left[\rho\right]\right) = 0$, hence taking the trace of the above expression

$$\mathrm{tr}\left(\hat{\Gamma}\rho - i\hat{H}\rho + \rho\hat{\Gamma} + i\rho\hat{H}\right) + \sum_{k,l=1}^{N^2-1} a_{kl}\,\mathrm{tr}\left(\hat{F}_k\rho\hat{F}_l^\dagger\right) = 0, \tag{2.44}$$

and, using the cyclic property of the trace,

$$2\,\mathrm{tr}\left(\hat{\Gamma}\rho\right) = -\sum_{k,l=1}^{N^2-1} a_{kl}\,\mathrm{tr}\left(\hat{F}_l^\dagger\hat{F}_k\rho\right) \quad \forall\,\rho\in\mathcal{S}(S)\ . \tag{2.45}$$

Hence, we substitute $\hat{\Gamma} = -\frac{1}{2}\sum_{k,l=1}^{N^2-1} a_{kl}\hat{F}_l^\dagger\hat{F}_k$, finally arriving at the celebrated Gorini-Kossakowski-Sudarshan-Lindblad (GKSL) master equation

$$\mathcal{L}\left[\rho\right] = -i\left[\hat{H},\rho\right] + \sum_{k,l=1}^{N^2-1} a_{kl}\left(\hat{F}_k\rho\hat{F}_l^\dagger - \frac{1}{2}\{\hat{F}_l^\dagger\hat{F}_k\ .\rho\}\right). \tag{2.46}$$

Since we will widely talk about it in the next and in the last chapter, we emphasize the following definition.

Definition 2.5. The matrix whose entries are the coefficients a_{kl} of the dissipator in (2.46) is called *Kossakowski matrix*.

We can thus formulate the GKSL theorem in its entirety.

Theorem 2.5. A continuous one-parameter semigroup consists of CPTP maps $\Lambda_t : \mathcal{M}_N(\mathbb{C}) \mapsto \mathcal{M}_N(\mathbb{C})$ if and only if its generator has the GKSL form (2.46) with a positive semi-definite Kossakowski matrix.

Proof. Since the matrix $N\times N$ matrix $[C_{\alpha\beta}(t)]$ in (2.40) is positive definite, such is also the $(N-1)\times(N-1)$ matrix $C_{k\ell}(t)]$ and then its time-derivative at time $t=0$ in (2.42). Vice versa, if the matrix $[a_{k\ell}]\geq 0$ then the contribution

$$\rho\mapsto\mathcal{N}[\rho] := \sum_{k,\ell=1}^{N^2-1} a_{k\ell}\hat{F}_k\,\rho\,\hat{F}_\ell^\dagger$$

to the generator is a CP map, as well as the dissipative, non-trace preserving dynamics generated by the other two terms

$$\rho\mapsto\mathbb{V}_t[\rho] = \mathrm{e}^{-it(\hat{H}-i\hat{\Gamma})}\,\rho\,\mathrm{e}^{it(\hat{H}+i\hat{\Gamma})}\ .$$

Since the composition of any number of CP maps remains CP because keeps the Kraus form of the CP maps that contribute to it, all powers of $\mathbb{N}$ are CP and thus also its exponential

$$\rho\mapsto\mathbb{N}_t[\rho] = \mathrm{e}^{t\mathcal{N}}[\rho]\ .$$

Then, the Lie-Trotter formula [41]

$$\Lambda_t[\rho] = \lim_{n\to+\infty}\left(\mathbb{V}_{t/n}\circ\mathbb{N}_{t/n}\right)^n[\rho]$$

shows that Λ_t is CP. □

In the GKSL generator we distinguish two contributions,

$$\partial_t \rho_S(t) = \mathbb{H}\left[\rho_S(t)\right] + \mathbb{D}\left[\rho_S(t)\right] \ ; \tag{2.47}$$

the first term is the unitary evolution which corresponds to the commutator in the right hand side of (2.46); the last term, corresponding to the last two terms in (2.46), is known as *dissipator* and is responsible for the non-invertibility of the evolution.

Moreover, we can always write the GKSL generator in diagonal form by diagonalizing $[a_{kl}]$ with a unitary transformation U: $[a_{kl}] = U^\dagger diag(\lambda_1, \lambda_2, \cdots)U$, where the eigenvalues are positive because $[a_{kl}] \geq 0$. Hence $a_{kl} = \sum_j \lambda_j U^*_{jk} U_{jl}$, so we define $V_j := \sum_k \sqrt{\lambda_j} U^*_{jk} F_k$, obtaining

$$\mathcal{L}\left[\rho_S\right] = -i\left[H, \rho_S\right] + \sum_{k,l=1}^{N^2-1} \left(V_k \rho_S V_l^\dagger - \frac{1}{2}\{V_l^\dagger V_k, \rho_S\} \right). \tag{2.48}$$

The dual Heisenberg version of the GKSL generator acting on the system observables is:

$$\mathcal{L}^*\left[X\right] = i\left[H, X\right] + \sum_{k,l=1}^{N^2-1} a_{kl} \left(F_k^\dagger X F_l - \frac{1}{2}\{F_l^\dagger F_k, X\} \right). \tag{2.49}$$

Interestingly enough, the GKSL form (2.46) of the generator which was derived through the physical request of complete positivity emerges also from the physical coupling of the system S of interest with a suitable environment if one applies the procedure known as weak-coupling limit, as we shall see next.

Remark 2.6. The GKSL form is often used to study more general dynamics with non-negligible memory effects that are taken into account by a time-dependent Kossakowski matrix. In such cases, one can obtain families of maps, that do not constitute semigroups, but are CPTP even when the Kossakowski matrix is not positive semi-definite.

Before embarking in the presentation of the weak-coupling limit techniques, in the following example we illustrate the fact that imposing complete positivity has not minor physical consequences.

Example 2.2. Consider a qubit and its states in Bloch representation

$$\rho = \frac{1}{2}\left[\mathbb{I}_2 + \sum_{i=1}^{3} r_i \sigma_i \right] \tag{2.50}$$

which are assumed to evolve forward in time ($t \geq 0$) according to the family of maps

$$\begin{aligned} \rho \mapsto \Lambda_t[\rho] = \rho(t) &= \frac{1}{2}\left[\mathbb{I}_2 + e^{-at}\left(r_1\sigma_1 + r_2\sigma_2\right) + e^{-bt} r_3 \sigma_3 \right] \\ &= \frac{1}{2}\begin{pmatrix} 1 + e^{-bt} r_3 & e^{-at}\left(r_1 - i r_2\right) \\ e^{-at}\left(r_1 + i r_2\right) & 1 + e^{-bt} r_3 \end{pmatrix}, \end{aligned} \tag{2.51}$$

where $a, b \geq 0$ in order to keep the positivity of the state in the course of the forward in time dynamics.

Asymptotically all initial state are mapped into the completely mixed state

$$\lim_{t\to\infty}\rho(t)=\frac{1}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}. \tag{2.52}$$

the family of maps Λ_t is a semigroup because it can be written as

$$\rho(t)=\Lambda t[\rho]=e^{t\mathcal{L}}[\rho], \tag{2.53}$$

where $\mathcal{L}$ is the generator. Indeed, writing the action of the dynamical map on the Pauli matrices,

$$\begin{aligned}&\Lambda_t\left[\mathbb{I}_2\right]=\mathbb{I}_2, &&\Lambda_t\left[\sigma_1\right]=e^{-at}\sigma_1,\\&\Lambda_t\left[\sigma_2\right]=e^{-at}\sigma_2, &&\Lambda_t\left[\sigma_3\right]=e^{-bt}\sigma_3.\end{aligned} \tag{2.54}$$

As a consequence

$$\begin{aligned}&\mathcal{L}\left[\mathbb{I}_2\right]=0, &&\mathcal{L}\left[\sigma_1\right]=-a\sigma_1,\\&\mathcal{L}\left[\sigma_2\right]=-a\sigma_2, &&\mathcal{L}\left[\sigma_3\right]=-b\sigma_3.\end{aligned} \tag{2.55}$$

We can then write the action of the generator on the a generic state ρ as

$$\mathcal{L}[\rho]=\sum_{\mu=0}^{3}l_\mu\sigma_\mu\rho\,\sigma_\mu. \tag{2.56}$$

Substituting again in the previous expression the Pauli basis in place of ρ, we find the following set of relations

$$\begin{aligned}&l_0+l_1+l_2+l_3=0, &&l_0+l_1-l_2-l_3=-a,\\&l_0-l_1+l_2-l_3=-a, &&l_0-l_1-l_2+l_3=-b,\end{aligned} \tag{2.57}$$

whose solutions are

$$l_0=-\frac{2a+b}{4},\quad l_1=l_2=\frac{b}{4},\quad l_3=\frac{2a-b}{4}. \tag{2.58}$$

We substitute in (2.56)

$$\mathcal{L}[\rho]=-\frac{2a+b}{4}\rho+\frac{b}{4}\left(\sigma_1\rho\,\sigma_1+\sigma_2\rho\,\sigma_2\right)+\frac{2a-b}{4}\sigma_3\rho\,\sigma_3\ . \tag{2.59}$$

By easy algebra, the generator can be put in the GKSL form

$$\mathcal{L}[\rho]=\sum_{i,j=1}^{3}\mathcal{A}_{ij}\left(\sigma_j\rho\,\sigma_i-\frac{1}{2}\left\{\sigma_i\sigma_j,\rho\right\}\right), \tag{2.60}$$

where

$$\mathcal{A}=\begin{pmatrix}\frac{b}{4}&0&0\\0&\frac{b}{4}&0\\0&0&\frac{2a-b}{4}\end{pmatrix}. \tag{2.61}$$

As we previously seen, in order to have a completely positive dynamics, the Kossakowski matrix must be positive, hence we find the conditions

$$b \geq 0 \quad \text{and} \quad a \geq \frac{b}{2}. \tag{2.62}$$

If we had stopped just at (2.51) and at the trace preservation requirement, the only condition on the dynamics would have been $a, b \geq 0$; imposing the complete positivity restricts even more the free parameters of the dynamical map: concretely, complete positivity enforces a hierarchy between the decay constants of the irreversible qubit time-evolution.

For completeness, we show also the analysis of the Choi matrix, considering the projector on the maximally entangled state

$$\begin{aligned} P &= \frac{1}{4}\left[\mathbb{I}_2 \otimes \mathbb{I}_2 + \sigma_1 \otimes \sigma_1 - \sigma_2 \otimes \sigma_2 + \sigma_3 \otimes \sigma_3\right] = \\ &= \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \end{aligned} \tag{2.63}$$

hence we have for the Choi matrix

$$\begin{aligned} C_{\Lambda_t} = \Lambda_t \otimes \mathrm{id}\,[P] &= \frac{1}{4}\left[\mathbb{I}_2 \otimes \mathbb{I}_2 + e^{-at}\left(\sigma_1 \otimes \sigma_1 - \sigma_2 \otimes \sigma_2\right) + e^{-bt}\left(\sigma_3 \otimes \sigma_3\right)\right] = \\ &= \frac{1}{4}\begin{pmatrix} 1+e^{-bt} & 0 & 0 & 2e^{-at} \\ 0 & 1-e^{-bt} & 0 & 0 \\ 0 & 0 & 1-e^{-bt} & 0 \\ 2e^{-at} & 0 & 0 & 1+e^{-bt} \end{pmatrix}. \end{aligned} \tag{2.64}$$

The evolution is CP if the Choi matrix is positive definite, since its trace is bigger than zero, we evaluate the determinants of its sub-matrices obtaining

$$\left(1+e^{-bt}\right)^2 - 4e^{-2at} \simeq -4bt + 8at + o(t), \tag{2.65}$$

the Choi matrix is positive for small times if $a \geq \frac{b}{2}$ in accordance with the previous result. It turns out that $2a \geq b$ ensures $C_{\Lambda_t} \geq 0$ even for all $t \geq 0$, indeed

$$\left(1+e^{-bt}\right)^2 - 4e^{-2at} \geq \left(1+e^{-2at}\right)^2 - 4e^{-2at} = \left(1-e^{-2at}\right)^2 \geq 0. \tag{2.66}$$

Now, that we have the master equation we will mostly use in the following, we show the announced Markovian approximation which allows to recover the semigroup structure to the dynamical maps, hence to recover GKSL. We presented the work in this order because it is actually easier to derive GKSL without considering the most generic dynamics (the difficulties in recovering GKSL and in correctly applying the Markovian approximation will be clearer in the next section).

2.2.2 Markovian approximations

Under the name of Markovian approximations lie very different approaches to get rid of the memory effects of the dynamics of an open quantum system. Intuitively, the memory effects are negligible either when the coupling between the environment and the sub-system S is sufficiently weak or when the correlations of the environment decay in time much faster than the characteristic time-variation of the sub-system S. In the following we will make use of both the assumptions in order to recover the GKSL expression from a generic dynamics as in (2.1).

In all cases, whatever the method, in order to be sure the chosen approximations are physically reasonable, the resulting master equation has to generate a completely positive dynamics. Here we will mostly follow reference [4], however a comprehensive study of open quantum dynamics we recommend [11] as well.

First of all we will write the explicit integro-differential expression of which the dynamical maps Λ_t without approximation are formal solutions, without making any of the previous assumption; in this way we will identify the presence of memory effects and the difficulties in dealing with the global solutions.

We start from the differential expression (2.28) considering as usual the whole composite sub-system S coupled with an environment B as a closed one, described by the general Hamiltonian (2.1):

$$\partial_t \rho_{S+B}(t) = \mathbb{L}_{S+B}\left[\rho_{S+B}(t)\right]. \tag{2.67}$$

Then, in order to disentangle the system S reduced dynamics from that of $S+B$, we will make use of the so-called projection technique [26, 51] which consists in introducing on the space of states $\mathcal{S}(S+B)$ the following map:

$$P\left[\rho_{S+B}\right] := \mathrm{tr}_B\left(\rho_{S+B}\right) \otimes \rho_B. \tag{2.68}$$

Such a map is linear and behaves as a projection, $P^2 = P$. Clearly, to it there corresponds then the orthogonal projection $Q := \mathbb{I}_{S+B} - P$. Thus,

$$P\left[\rho_S \otimes \rho_B\right] = \rho_S \otimes \rho_B, \qquad Q\left[\rho_S \otimes \rho_B\right] = 0. \tag{2.69}$$

Having two projections, we can split the evolution into two coupled equations; indeed, the generator splits into the following contributions:

$$\mathbb{L}^{PP}_{S+B} := P \circ \mathbb{L}_{S+B} \circ P, \qquad \mathbb{L}^{PQ}_{S+B} := P \circ \mathbb{L}_{S+B} \circ Q, \tag{2.70}$$

$$\mathbb{L}^{QQ}_{S+B} := Q \circ \mathbb{L}_{S+B} \circ Q, \qquad \mathbb{L}^{QP}_{S+B} := Q \circ \mathbb{L}_{S+B} \circ P, \tag{2.71}$$

and the Liouville equation (2.67) into

$$\partial_t P\left[\rho_{S+B}(t)\right] = \mathbb{L}^{PP}_{S+B}\left[P\left[\rho_{S+B}(t)\right]\right] + \mathbb{L}^{PQ}_{S+B}\left[Q\left[\rho_{S+B}(t)\right]\right], \tag{2.72}$$

$$\partial_t Q\left[\rho_{S+B}(t)\right] = \mathbb{L}^{QP}_{S+B}\left[P\left[\rho_{S+B}(t)\right]\right] + \mathbb{L}^{QQ}_{S+B}\left[Q\left[\rho_{S+B}(t)\right]\right]. \tag{2.73}$$

The formal solution of (2.71) is

$$Q\left[\rho_{S+B}(t)\right] = \mathrm{e}^{t\mathbb{L}^{QQ}_{S+B}}\left[Q\left[\rho_{S+B}(0)\right]\right] + \int_0^t ds\, \mathrm{e}^{(t-s)\mathbb{L}^{QQ}_{S+B}} \circ \mathbb{L}^{QP}_{S+B}\left[P\left[\rho_{S+B}(s)\right]\right]. \tag{2.74}$$

In order to derive dynamical maps that preserve the convexity of the space of states $\mathcal{S}(S)$ [32, 1], we assume initially uncorrelated states $\rho_{S+B}(0) = \rho_S \otimes \rho_B$; then, considering the action of Q in (2.69), the first term on the right hand side of (2.74) vanishes. Further, substituting (2.74) in (2.72), we find

$$\partial_t(\rho_S(t) \otimes \rho_B) = \mathbb{L}^{PP}_{S+B}\left[\rho_S(t) \otimes \rho_B\right] + \int_0^t ds\, \mathbb{L}^{PQ}_{S+B} \circ \mathrm{e}^{(t-s)\mathrm{L}^{QQ}_{S+B}} \circ \mathbb{L}^{QP}_{S+B}\left[\rho_S(s) \otimes \rho_B\right]. \tag{2.75}$$

Now we want to get rid of the environment via partial trace. To do so, we explicitly write the interaction Hamiltonian in (2.1) as

$$\widetilde{H}_{int} = \sum_\alpha V_\alpha \otimes \widetilde{B}_\alpha, \tag{2.76}$$

where V_α and $\widetilde{B}_\alpha$ are Hermitean operators acting respectively on $\mathcal{H}_S$ and $\mathcal{H}_B$. It is convenient to define the *centred environment operators*, namely:

$$B_\alpha := \widetilde{B}_\alpha - \mathrm{tr}_B(\widetilde{B}_\alpha), \qquad \mathrm{tr}_B(B_\alpha) = 0. \tag{2.77}$$

In this way the Hamiltionian of the system S and the interaction term in (2.1) can be recast as

$$H^\lambda_S = H_S + \lambda \underbrace{\sum_\alpha V_\alpha \,\mathrm{tr}_B\left(\widetilde{B}_\alpha\right)}_{H^{(1)}_S}, \qquad H_{int} = \sum_\alpha V_\alpha \otimes B_\alpha. \tag{2.78}$$

We also split the action of the total generator:

$$\mathbb{L}_{S+B} = \mathbb{L}^\lambda_S \otimes \mathrm{id}_B + \mathrm{id}_S \otimes \mathbb{L}_B + \lambda\, \mathbb{L}_{int}, \tag{2.79}$$

where $\mathbb{L}^\lambda_S$ consists in the commutator with respect to the Hamiltonian H^λ_S in (2.78).

We assume the environment to be in equilibrium with respect to its own Hamiltonian H_B, a common assumption for bath states since the environment is always supposed to be very large

$$\mathbb{L}_B\left[\rho_B\right] = -i\left[H_B, \rho_B\right] = 0\ . \tag{2.80}$$

As a consequence of the above relation and of the last expression in (2.77), we get:

$$P \circ \left(\mathbb{L}^\lambda_S \otimes \mathrm{id}_B\right) = \left(\mathbb{L}^\lambda_S \otimes \mathrm{id}_B\right) \circ P, \qquad P \circ \mathbb{L}_{int} \circ P = 0. \tag{2.81}$$

Taking advantage of the previous relations, we are able to perform the partial trace of (2.75) over the environment B degrees of freedom that is embodied by the projection P:

$$\begin{aligned}
&\mathbb{L}^{PP}_{S+B}\left[\rho_S(t) \otimes \rho_B\right] = \left(\mathbb{L}^\lambda_S \otimes \mathbb{I}_B\right)^{PP}\left[\rho_S(t) \otimes \rho_B\right] = \mathbb{L}^\lambda_S\left[\rho_S(t)\right] \otimes \rho_B, \\
&\mathbb{L}^{PQ}_{S+B}\left[\rho_{S+B}(t)\right] = \lambda\left(\mathbb{L}_{int}\right)^{PQ}\left[\rho_{S+B}(t)\right] = \lambda\,\mathrm{tr}_B\left(\mathbb{L}_{int} \circ Q\left[\rho_{S+B}(t)\right]\right) \otimes \rho_B, \\
&\mathbb{L}^{QP}_{S+B}\left[\rho_S(t) \otimes \rho_B\right] = \lambda\left(\mathbb{L}_{int}\right)^{QP}\left[\rho_S(t) \otimes \rho_B\right] = \lambda \mathbb{L}_{int}\left[\rho_S(t) \otimes \rho_B\right]\ .
\end{aligned} \tag{2.82}$$

Finally, substituting the previous results in (2.75), we arrive at an integro-differential Master Equation with memory effects

$$\partial_t \rho_S(t) = -i \left[H_S + \lambda H_S^{(1)}, \rho_S(t) \right] - \lambda^2 \int_0^t ds \; \mathrm{tr}_B \left(\left[H_{int}, \mathrm{e}^{(t-s)\mathbb{L}_{SQ}^{S+B}} \left[H_{int}, \rho_S(s) \otimes \rho_B \right] \right] \right) . \tag{2.83}$$

The second term depends on the dynamics of the global system up to time t and incorporates the consequences on S of its interaction with the environment.

It is clear the difficulty of dealing with such an equation of motion which, in order to be explicitly solved, requires the knowledge of the unitary dynamics of $S + B$.

We now proceed to perform a series of approximations that will simplify the Master Equation turning its right hand side into a time-independent generator, without memory effects. Such approximations are valid intuitively when:

1. We are in *weak coupling* conditions, that is when the interaction between the sub-system S and the environment is weak, in other words when the dimensionless coupling constant λ in (2.1) is $\ll 1$.

2. The correlations of the environment decay in time much faster than the characteristic time-evolution of the system S. We call the typical lifetime of the bath correlations τ_B and the typical time-variation of the system τ_S and require $\tau_B/\tau_S \ll 1$. The so-called weak-coupling limit consists in sending $\tau_S \to \infty$ while τ_B remains finite. The opposite case, $\tau_B \to 0$ and τ_S finite is known as strong-coupling limit [38] and will not be considered here.

In the following section we will see how the previous assumptions allows to simplify (2.83) and eliminate the memory effects.

2.2.3 Weak coupling limit

From last expression in the previous section, (2.83), it is clear that the memory effects become relevant only on times scaling as λ^{-2}; hence, we firstly integrate (2.83)

$$\rho_S(t) = \mathrm{e}^{t\mathbb{L}_S^\lambda} \left[\rho_S(0) \right] - \lambda^2 \int_0^t dv \int_0^v du \, \mathrm{e}^{(t-v)\mathbb{L}_S^\lambda} \, \mathrm{tr}_B \left(\left[H_{int}, \mathrm{e}^{(v-u)\mathbb{L}_{S+B}^{QQ}} \left[H_{int}, \rho_S(u) \otimes \rho_B \right] \right] \right) . \tag{2.84}$$

Then, we introduce the new variable $w := v - u$ inverting the integration order and obtaining

$$\rho_S(t) = \mathrm{e}^{t\mathbb{L}_S^\lambda} \left[\rho_S(0) \right] - \lambda^2 \int_0^t du \, \mathrm{e}^{(t-u)\mathbb{L}_S^\lambda} \left\{ \int_0^{t-u} dw \, \mathrm{e}^{-w\mathbb{L}_S^\lambda} \, \mathrm{tr}_B \left(\left[H_{int}, \mathrm{e}^{w\mathbb{L}^{QQ}} \left[H_{int}, \rho_S(u) \otimes \rho_B \right] \right] \right) \right\}, \tag{2.85}$$

then we can substitute in the second integral the relevant slow time-scale $\tau = \lambda^2 \, t$, and let $\lambda \to 0$, so that

$$\rho_S(t) = \mathrm{e}^{t \, \mathbb{L}_S^\lambda} \left[\rho_S(0) \right] + \lambda^2 \int_0^t du \, \mathrm{e}^{(t-u)\mathbb{L}_S^\lambda} \circ \mathbb{D}_1 \left[\rho_S(s) \right] , \tag{2.86}$$

$$\mathbb{D}_1 \left[\rho_S \right] := - \int_0^\infty \mathrm{d}w \, \mathrm{tr}_B \left(\mathrm{e}^{-w \, \mathbb{L}_S} \left[H_{int}, \mathrm{e}^{w \, (\mathbb{L}_S + \mathbb{L}_B)} \left[H_{int}, \rho_S \otimes \rho_B \right] \right] \right) . \tag{2.87}$$

In order to better exploit the $\lambda \to 0$ limit, we rewrite the expression (2.86) as

$$\mathrm{e}^{-t\mathbb{L}_S^\lambda}\rho_S(t) = \rho_S(0) + \lambda^2 \int_0^t ds \, \left(\mathrm{e}^{-s\mathbb{L}_S^\lambda} \circ \mathbb{D}_1 \circ \mathrm{e}^{s\mathbb{L}_S^\lambda}\right) \circ \mathrm{e}^{-s\mathbb{L}_S^\lambda}\left[\rho_S(s)\right], \ . \tag{2.88}$$

Then, we introduce again the slow time scale, getting

$$\mathrm{e}^{-(\tau/\lambda^2)\mathrm{L}_S^\lambda}\rho_S\left(\tau/\lambda^2\right) = \rho_S(0) + \int_0^\tau ds \left(\mathrm{e}^{-(s/\lambda^2)\mathbb{L}_S^\lambda} \circ \mathbb{D}_1 \circ \mathrm{e}^{(s/\lambda^2)\mathbb{L}_S^\lambda}\right) \circ \mathrm{e}^{-(s/\lambda^2)\mathbb{L}_S^\lambda}\left[\rho_S\left(s/\lambda^2\right)\right]. \tag{2.89}$$

In order to manipulate the previous expression, we define

$$V_\alpha(t) := \mathrm{e}^{-t\mathbb{L}_S}\left[V_\alpha\right] = \mathrm{e}^{itH_S}V_\alpha\mathrm{e}^{-itH_S}, \tag{2.90}$$

$$B_\alpha(t) := \mathrm{e}^{-t\mathbb{L}_B}\left[B_\alpha\right] = \mathrm{e}^{itH_B}B_\alpha\mathrm{e}^{-itH_B}, \tag{2.91}$$

$$G_{\alpha\beta}(s) := \mathrm{tr}\left[\rho_B B_\alpha(s) B_\beta\right] = \mathrm{tr}\left[\rho_B B_\alpha B_\beta(-s)\right], \tag{2.92}$$

where the first two expressions represent the evolved sub-system and bath operators, the last one represents the so-called *two-point correlation functions* of the environment. They lead to rewrite the dissipator in (2.87) as

$$\mathbb{D}_1\left[\rho_S(t)\right] = -\sum_{\alpha,\beta}\int_0^\infty ds \left\{G_{\alpha\beta}(s)\left[V_\alpha(s), V_\beta\,\rho_S(t)\right] + G_{\beta\alpha}(-s)\left[\rho_S(t)V_\beta, V_\alpha(s)\right]\right\} \ . \tag{2.93}$$

Making use of (2.90), (2.91), (2.93), and by writing the Hamiltonian H_S^λ in its spectral decomposition (assuming its spectrum to be discrete for simplicity)

$$H_S^\lambda = \sum_{a=1}^n \varepsilon_a^\lambda \, P_a^\lambda \ ,$$

we obtain for the round-bracketed term in (2.88) the following expression

$$\mathrm{e}^{-(s/\lambda^2)\mathbb{L}_S^\lambda} \circ \mathbb{D}_1 \circ \mathrm{e}^{(s/\lambda^2)\mathbb{L}_S^\lambda}\left[\rho_S\right] = -\sum_{\alpha,\beta}\sum_{a,b,c,d=1}^n \int_0^\infty du \, \mathrm{e}^{is\left[\varepsilon_a^\lambda - \varepsilon_b^\lambda - \left(\varepsilon_d^\lambda - \varepsilon_c^\lambda\right)\right]/\lambda^2} \times \tag{2.94}$$

$$\times \left\{G_{\alpha\beta}(u)\left[P_a^\lambda V_\alpha(u) P_b^\lambda, P_c^\lambda V_\beta P_d^\lambda \rho_S\right] + G_{\beta\alpha}(-u)\left[\rho_S P_a^\lambda V_\beta P_b^\lambda, P_c^\lambda V_\alpha(u) P_d^\lambda\right]\right\} \ . \tag{2.95}$$

Now, when $\lambda \to 0$ only the terms with $\varepsilon_a^\lambda - \varepsilon_b^\lambda = \varepsilon_d^\lambda - \varepsilon_c^\lambda$ contribute, all the other ones rapidly oscillate once they are inserted back in (2.88), so they can be neglected. This is also called in literature *rotating wave approximation* which, in this case , is a consequence of the weak-coupling limit. Moreover, since $\lambda \to 0$, it is reasonable to state that the eigenvalues and the eigenprojectors of H_S^λ tend to the ones of H_S. These informal manipulations can be mathematically implemented by means of the following ergodic average:

$$\widetilde{\mathbb{D}} := \lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{+T} ds \, \mathrm{e}^{-s\mathbb{L}_S} \circ \mathbb{D}_1 \circ \mathrm{e}^{s\mathbb{L}_S}. \tag{2.96}$$

One thus arrives at

$$\rho_S(t) = \mathrm{e}^{t\mathbb{L}_S}\left[\rho_S(0)\right] + \lambda^2 \int_0^t ds \, \mathrm{e}^{(t-s)\mathbb{L}_S} \circ \widetilde{\mathbb{D}}\left[\rho_S(s)\right], \tag{2.97}$$

whose differential form is

$$\partial_t \rho_S(t) = \mathbb{L}_S\left[\rho_S(t)\right] + \lambda^2 \widetilde{\mathbb{D}}\left[\rho_S(t)\right]. \tag{2.98}$$

Remark 2.7. Notice that, because of the ergodic average that involves, the dissipative part of the generator commutes with the Hamiltonian part:

$$\left[\widetilde{\mathbb{D}},\, \mathbb{L}_S\right] = \widetilde{\mathbb{D}} \circ \mathbb{L}_S \,-\, \mathbb{L}_S \circ \widetilde{\mathbb{D}} = 0 \ .$$

Lack of such a property on the part of the generator of a semigroup of dynamical maps of a quantum system signals that they cannot be derived, through the weak-coupling limit techniques, from the microscopic interaction of the system with an environment.

We set $\omega = \varepsilon_\alpha - \varepsilon_\beta$, where ε_α runs over the spectrum of H_S; considering the eigenprojectors of the latter, we define

$$V_\alpha(\omega) := \sum_{\varepsilon_a - \varepsilon_b = \omega} P_a V_\alpha P_b, \qquad V_\beta(-\omega) := \sum_{\varepsilon_d - \varepsilon_c = \omega} P_c V_\beta P_d = V_\beta^\dagger(\omega), \tag{2.99}$$

hence from (2.95) and (2.96):

$$\widetilde{\mathbb{D}}\left[\rho_S\right] = \sum_{\alpha,\beta}\sum_{\omega}\int_0^\infty du\, \left\{ \mathrm{e}^{iu\omega} G_{\alpha\beta}(u) \left[V_\alpha(\omega), V_\beta^\dagger(\omega)\rho_S\right] + \mathrm{e}^{-iu\omega} G_{\beta\alpha}(-u) \left[\rho_S V_\beta(\omega), V_\alpha^\dagger(\omega)\right]\right\}. \tag{2.100}$$

From usual Fourier analysis, we have

$$\int_0^\infty dt\, \mathrm{e}^{it\omega} G_{\alpha\beta}(t) = \frac{h_{\alpha\beta}(\omega)}{2} + i s_{\alpha\beta}(\omega) \ , \tag{2.101}$$

$$h_{\alpha\beta}(\omega) := \int_{-\infty}^{+\infty} dt\, \mathrm{e}^{it\omega} G_{\alpha\beta}(t) = h^*_{\beta\alpha}(\omega) \ , \tag{2.102}$$

$$s_{\alpha\beta}(\omega) := \frac{1}{2\pi}\mathcal{P}\int_{-\infty}^{+\infty} \mathrm{d}w \frac{h_{\alpha\beta}(\omega)}{w-\omega} = s^*_{\beta\alpha}(\omega), \tag{2.103}$$

where $\mathcal{P}$ stands for principal value. Making use of the previous expressions, we get for (2.100)

$$\begin{aligned}\widetilde{\mathbb{D}}\left[\rho_S\right] = &- i \Big[\underbrace{\sum_{\alpha,\beta}\sum_{\omega} s_{\alpha\beta}(\omega) V_\alpha(\omega) V_\beta^\dagger(\omega)}_{H_B^{(2)}}, \rho_S \Big] + \\ &+ \underbrace{\sum_{\alpha,\beta}\sum_{\omega} h_{\alpha\beta}(\omega) \left(V_\beta^\dagger(\omega)\rho_S V_\alpha(\omega) - \frac{1}{2}\left\{ V_\alpha(\omega) V_\beta^\dagger(\omega), \rho_S \right\}\right)}_{\mathbb{D}},\end{aligned} \tag{2.104}$$

where $\mathbb{D}$ is now precisely in the same form of the GKSL dissipator in (2.46).

Finally substituting the previous expression and (2.78) in (2.98)

$$\partial_t \rho_S(t) = -i \left[H_S + \lambda^2 H_B^{(2)} \,,\, \rho_S(t)\right] + \lambda^2 \mathbb{D}\left[\rho_S(t)\right]. \tag{2.105}$$

The dissipative corrections to the unitary time-evolution of the system S are both of order λ^2 as they should be; the first one is a Lamb-shift correction to the system Hamiltonian, while the second one embodies the dissipative and noisy effects due to the presence of the environment. Dissipation is described by the anti-commutator; indeed as shown in the derivation of

the GKSL generators before Theorem 2.5, the latter is connected with the anti-Hermitian part Γ of a non Hermitian Hamiltonian, thus describing decay of probability. That the remaining contribution can be associated with a quantum noise term follows from the fact that, for each ω the Kossakowski matrix $[h_{\alpha\beta}(\omega)]$ is positive semi-definite and thus, because of Theorem 2.5, the generator is in GKSL form and the generated dynamics CPTP.

The positive semi-definiteness of the Kossakowski matrix $h(\omega) = [h_{\alpha\beta}(\omega)]$ can be seen as follows; inserting (2.91), (2.92) into (2.102) and using the common spectralization of ρ_B and $\hat{H}_B$ of the commuting bath state and Hamiltonian, $\rho_B = \sum_\alpha r_\alpha |\alpha\rangle\langle\alpha|$, $\hat{H}_B = \sum_\ell h_\ell |\ell\rangle\langle\ell|$, one gets

$$\begin{aligned}
\langle\psi| h(\omega) |\psi\rangle &= \sum_{\alpha\beta} \psi^*_\alpha \psi_\beta \int_{-\infty}^{+\infty} dt \ \mathrm{tr}\left[\rho_B \mathrm{e}^{it\hat{H}_B} \hat{B}_\alpha \mathrm{e}^{-it\hat{H}_B} \hat{B}_\beta\right] \\
&= \sum_{\ell,k} r_\ell \int_{-\infty}^{+\infty} dt \ \mathrm{e}^{it(\omega+h_\ell-h_k)} \langle\ell| \big(\sum_\alpha \psi^*_\alpha \hat{B}_\alpha |k\rangle\big) |k\rangle \ \langle k| \big(\sum_\beta \psi_\beta \hat{B}_\beta\big) |\ell\rangle \\
&= 2\pi \sum_{\ell,k} r_\ell \delta(\omega - h_\ell + h_k) \left|\langle\ell| \big(\sum_\alpha \psi^*_\alpha \hat{B}_\alpha\big) |k\rangle\right|^2 \geq 0 \ ,
\end{aligned}$$

the last inequality following from the fact that the Dirac delta distribution is positive, namely it transforms positive test functions into positive numbers.

Given the master equation governing the dissipative dynamics of an open quantum system, it is in some cases possible to study concretely the manifold of its stationary states and if and how the system approaches them asymptotically in time.

2.3 Asymptotic states

If the infinite time limit of a state of an open quantum system, $\rho_\infty := \lim_{t\to+\infty} \Lambda_t[\rho]$, evolving according to the reduced dynamics Λ_t does exist, ρ_∞ is called an *asymptotic state*. On the other hand, the stationary states of the reduced dynamics are those that stay invariant under Λ_t, $\Lambda_t[\rho] = \rho$. Under a continuous semigroup, asymptotic states are stationary; indeed,

$$\Lambda_t[\rho_\infty] = \Lambda_t[\lim_{s\to+\infty} \Lambda_s[\rho]] = \lim_{s\to+\infty} \Lambda_{t+s}[\rho] = \rho_\infty \ . \tag{2.106}$$

However, stationary states need not be asymptotic states: for instance, the eigen-projections $|E_\alpha\rangle$ of a Hamiltonian $\hat{H}$, $\hat{H}|E_\alpha\rangle = E_\alpha |E_\alpha\rangle$, are of course invariant under the Schrödinger time-evolution generated by that Hamiltonian, but cannot be reached asymptotically from any other initial state. Indeed, the dynamics of a generic initial state ρ is either trivial, if ρ is an eigen-projector of $\hat{H}$, or is oscillatory in time:

$$\mathrm{e}^{-it\hat{H}} \rho \, \mathrm{e}^{it\hat{H}} = \sum_{\alpha,\beta} \mathrm{e}^{-it(E_\alpha - E_\beta)} \ \langle E_\alpha| \rho |E_\beta\rangle \ |E_\alpha\rangle \langle E_\beta| \ .$$

In this setting, the stationary states are identified by their belonging to the kernel of the generator: ρ is stationary if and only if $\mathcal{L}[\rho] = 0$.

The fundamental questions to ask about the long time behaviour of an open quantum system concern the structure of its stationary states and which of them an initial state of the system evolves into, if it tends to a stationary state at all.

In the case of a finite dimensional Hilbert space (as the ones we will study in this work) at least one stationary state always exists. This is due to the fact that in finite dimension the ergodic average of the action of a one-parameter semigroup always provides a stationary state.

If the system admits more the one possible stationary state, it is possible to find all the other ones starting from a single *faithful* stationary state, i.e. a stationary state without negative eigenvalues. Hence if we find a faithful stationary state, we can build the whole manifold of possible stationary states. We will now present a result regarding under which circumstances an initial state subjected to a dynamics consisting of a semigroup of CPTP is mapped into to the asymptotic manifold.

We consider the dual map $\Lambda_t^* : \mathcal{M}_n(\mathbb{C}) \mapsto \mathcal{M}_n(\mathbb{C})$ and we define the ergodic average of $\hat{X} \in \mathcal{M}_n(\mathbb{C})$ as

$$E^*[\hat{X}] := \lim_{T\to\infty} \frac{1}{T} \int_0^\infty \Lambda_t^*[\hat{X}]\, dt. \tag{2.107}$$

The ergodic average of an initial state ρ,

$$E[\rho] := \lim_{T\to\infty} \frac{1}{T} \int_0^\infty \Lambda_t[\hat{X}]\, dt. \tag{2.108}$$

is obtained from (2.107) by the duality relation.

Interestingly, the ergodic average has the same properties of the conditional expectation defined in Section 1.4: given a state ρ and the von Neumann algebra $\mathcal{M}$ onto which the dynamical map Λ_t^* acts, we define the set of all the operators invariant under the action of Λ_t^* (namely the fixed points of the dynamics)

$$\mathcal{M}(\Lambda^*) := \left\{ \hat{X} \in \mathcal{M} \mid \Lambda_t^*[\hat{X}] = \hat{X},\, \forall t \geq 0 \right\}. \tag{2.109}$$

Then, assuming that a faithful invariant state exists, we have [15]

1. $\mathrm{tr}\Big(E[\rho]\hat{X}\Big) = \mathrm{tr}\left(\rho E^*[\hat{X}]\right)$.
2. $E^*\left[\hat{X}_1\hat{Y}\hat{X}_2\right] = \hat{X}_1 E^*[\hat{Y}]\hat{X}_2$ for all $\hat{X}_{1,2} \in \mathcal{M}(\Lambda^*)$, $\hat{Y} \in \mathcal{M}_n(\mathbb{C})$.

Indeed the Λ_t^*-invariant operators constitute a von Neumann algebra [15]: if $\hat{X} \in \mathcal{M}(\Lambda^*)$, then also $\hat{X}^\dagger$ and $\hat{X}^\dagger\hat{X}$ are Λ_t^*-invariant. This follows by considering the expression

$$D_t(\hat{X},\hat{Y}) := \Lambda_t^*\left[\hat{X}^\dagger\hat{Y}\right] - \Lambda_t^*\left[\hat{X}^\dagger\right]\Lambda_t^*\left[\hat{Y}\right] \ . \tag{2.110}$$

From the complete positivity of Λ_t^* it follows that $D_t(\hat{X},\hat{X}) \geq 0$ and that $\Lambda_t^*[\hat{X}\hat{Y}] = \hat{X}\,\Lambda_t^*[\hat{Y}]$ whenever $\hat{X} \in \mathcal{M}(\Lambda^*)$ (see Appendix B).

Remark 2.8. Notice that, even for a finite d-level open quantum system, the existence of a faithful stationary state strongly depends on the generator of the dissipative dynamics. For instance, given a complete set of orthogonal projections $\{|\psi_i\rangle\langle\psi_i|\}_{i=1}^d$, the purely dissipative master equation

$$\partial_t\rho_t = \lambda \sum_{i=1}^{d} |\psi_0\rangle\langle\psi_i|\rho_t|\psi_i\rangle\langle\psi_0| - \lambda\,\rho_t = \lambda\,\mathrm{Tr}(\rho_t)\,|\psi_0\rangle\langle\psi_0| - \lambda\rho_t = \lambda\,|\psi_0\rangle\langle\psi_0| - \lambda\rho_t$$

is easily checked to be solved by

$$\rho_t = \mathrm{e}^{-\lambda t}\,\rho + \left(1 - \mathrm{e}^{-\lambda t}\right)\,|\psi_0\rangle\langle\psi_0|\ .$$

One thus sees that all initial states tend asymptotically to $|\psi_0\rangle\langle\psi_0|$ which is certainly not faithful.

Let us consider the so-called *dissipation function* from $\mathcal{M}\times\mathcal{M}$ into $\mathcal{M}$

$$\dot{D}_t(\hat{X},\hat{X})_{|_{t=0}} = \mathcal{L}^*\left[\hat{X}^\dagger\hat{X}\right] - \mathcal{L}^*\left[\hat{X}^\dagger\right]\hat{X} - \hat{X}^\dagger\mathcal{L}^*[\hat{X}]\ . \tag{2.111}$$

Now, the so-called kernel of $D_t(\hat{X},\hat{X})$, namely the subset

$$\mathcal{N}(\Lambda^*) := \left\{\hat{X}\in\mathcal{M}\mid D_t(\hat{X},\hat{X}) = 0,\,\forall t\geq 0\right\} \tag{2.112}$$

is such that

$$\mathcal{M}(\Lambda^*)\subseteq\mathcal{N}(\Lambda^*). \tag{2.113}$$

Indeed, if $\hat{X}$ is invariant under Λ_t^*, then, as mentioned above, $\Lambda_t^*[\hat{X}] = \hat{X}$, $\Lambda_t^*[\hat{X}^\dagger\hat{X}] = \hat{X}^\dagger\hat{X}$ and thus $D_t(\hat{X},\hat{X}) = 0$, but not vice versa.

By substituting in (2.111) the generator $\mathcal{L}^*$ written in its diagonal form (2.48), we get

$$\dot{D}_t(\hat{X},\hat{X})_{|_{t=0}} = \sum_i \left[\hat{X},\hat{V}_i\right]^\dagger\left[\hat{X},\hat{V}_i\right] = 0, \tag{2.114}$$

whenever $\hat{X}\in\mathcal{N}(\Lambda^*)$. Thus we must have $\left[\hat{X},\hat{V}_i\right] = 0$ for all i. This means that the $\mathcal{N}(\Lambda^*)$ commutes with all Kraus operators appearing in the generator $\mathcal{L}$. Denoting by $\{\hat{V}_i\}'$ the commutant algebra of the Kraus operators, namely the algebra of all the operators that commute with the Kraus operators, it follows that $\mathcal{N}(\Lambda^*)\subseteq\{V_i\}'$.

We have just shown that if $\hat{X}$ is a fixed point of the evolution, it also commutes with the dissipation operator; the opposite inclusion is not however true because $\mathcal{N}(\Lambda^*)$ is in general larger than $\mathcal{M}(\Lambda^*)$. However, if $\hat{X}$ commutes not only with the Kraus operators $\{V_i\}$, but also with the Hamiltonian in the unitary part of the evolution in (2.48), namely if $\hat{X}\in\{H,V_i\}'$, then it follows $\mathcal{L}^*\left[\hat{X}\right] = 0$. This implies that $\{\hat{H},\hat{V}_i\}'\subseteq\mathcal{M}(\Lambda^*)$. As a consequence, *if* $\{\hat{V}_i\}'\subseteq\{\hat{H},\hat{V}_i\}'$, then

$$\{V_i\}' = \{\hat{H},V_i\}'\subseteq\mathcal{M}(\Lambda^*)\subseteq\mathcal{N}(\Lambda^*)\subseteq\{V_i\}'\quad\text{so that}\quad\mathcal{M}(\Lambda^*) = \mathcal{N}(\Lambda^*)\ . \tag{2.115}$$

It can be proven [15] that if the previous equality (2.115) is satisfied, namely if

1. the commutant of the Kraus operators, $\{V_i\}_i$, equals the commutant of the Kraus operators plus the Hamiltonian operator $\{V_i,H\}'$ (in general $\{V_i,H\}'\subseteq\{V_i\}_i$;
2. there exists a faithful stationary state $\hat{\rho}$, namely a density matrix without zero eigenvalues, not necessarily unique,

then every initial state ρ is mapped by the dynamics into an element ρ_∞ of the asymptotic manifold, which is given by the ergodic average

$$\lim_{t\to\infty}\Lambda_t\left[\rho\right] = \rho_\infty = E\left[\rho\right]\ . \tag{2.116}$$

In general, different initial states can tend to different stationary states: however, if the commutant $\{H, V_i\}'$ is trivial, namely it contains only multiples of the identity, the system has a unique stationary state. Furthermore, though the condition $\mathcal{M}(\Lambda^*) = \mathcal{N}(\Lambda^*)$ is only sufficient for the tendency to equilibrium of any initial state, none the less one expects that generically, if it fails, there might be initial states which do not tend to stationary states in the long-time regime.

In order to illustrate the previous theoretical framework, we restrict to a specific setting that will allow us to analytically deal with the stationary states of a concrete open qubit dynamics that will be discussed in the next Chapter where we will focus on the concept of quantum entanglement and its behaviour under a dissipative dynamics. We shall thus assume the stationary algebra $\mathcal{M}(\Lambda^*)$ to be commutative and generated by a finite set of orthogonal projections $\{P_i\}_{i=1}^d$ such that

$$P_i P_j = \delta_{ij} P_j \ , \qquad \sum_{i=1}^{d} P_i = \mathbb{I} \ . \tag{2.117}$$

These projections are Λ_t^*-invariant, $\mathcal{L}^*[P_i] = 0$, and commute with the Kraus operators V_j so that they also commute with H; hence

$$\mathcal{L}[P_i\, \hat{\rho}\, P_i] = P_i\, \mathcal{L}[\hat{\rho}]\, P_i = 0 \ , \tag{2.118}$$

where $\hat{\rho}$ is a faithful stationary state, $\mathcal{L}[\hat{\rho}] = 0$. Indeed, given any state ρ, from $[H, P_i] = 0$ and $[V_j, P_i] = 0$ one derives

$$\begin{aligned} [H, P_i\, \rho\, P_i] &= P_i\, H\, \rho\, P_, - \ P_i\, \rho\, H\, P_i = P_i\ [H, \rho]\ P_i \ , \\ V_j\, P_i\, \rho\, P_i\, V_j^\dagger - \frac{1}{2}\left\{V_j^\dagger\, V_j\ ,\ P_i\, \rho\, P_i\right\} &= P_i \left(V_j\, \rho\, V_j^\dagger \ - \ \frac{1}{2}\left\{V_j^\dagger\, V_j\ ,\ \rho\right\}\right) P_i \ . \end{aligned}$$

After normalization, the positive operators $P_i \hat{\rho}\, P_i$ provides extremal invariant states of which all stationary states are convex combinations:

$$\rho_{st} = \sum_{k=1}^{d} \lambda_k\, \frac{P_k\, \hat{\rho}\, P_k}{\mathrm{Tr}(\hat{\rho}\, P_k)} \ , \quad \lambda_k \geq 0 \ , \quad \sum_{k=1}^{d} \lambda_k = 1 \ . \tag{2.119}$$

Notice that, though $\mathcal{L}^*[P_i] = 0$, the projections P_i themselves are not Λ_t-invariant states; indeed the action of the generator

$$\mathcal{L}[P_i] = \frac{1}{2}\left(\sum_j \left[V_j\ ,\ V_j^\dagger\right]\right) P_i \tag{2.120}$$

does not vanish, in general.

Suppose an initial state ρ tends to a stationary state ρ_{st} as in (2.119), asymptotically in time; then, using the duality relation and the Λ_t^*-invariance one relates the weights λ_k of the convex expansion to the initial state as follows:

$$\lambda_k = \mathrm{Tr}\left(P_k\, \rho_{st}\right) = \lim_{t \to +\infty} \mathrm{Tr}\left(P_k\, \Lambda_t[\rho]\right) = \lim_{t \to +\infty} \mathrm{Tr}\left(\Lambda_t^*[P_k]\, \rho\right) = \mathrm{Tr}\left(P_k\, \rho\right) \ . \tag{2.121}$$

Chapter 3

Quantum entanglement

In this chapter we focus on one of the main topics tackled in this thesis: entanglement against dissipation. In the previous Chapter we showed how the state of an open quantum system evolves in presence of noise and dissipation due to an environment with which it weakly interacts; in the following, we will focus on the behaviour of the entangled states of a compound open quantum system. In particular, we will study under which circumstances an open system of two qubits can develop entanglement or preserve the initial one, even asymptotically in time.

This is a very interesting matter: entanglement is one the main peculiarities of quantum systems with tons of applications in modern quantum technologies; however, it is a very fragile feature, it is easily formed by interactions, but also destroyed, above all by the decoherence brought about by noise and dissipation. In this Chapter we derive the requirements for an environment described by scalar fields to be able, despite the noise and dissipation of which it is a source, not only to preserve the entanglement of a two-qubit system, but also to generate it.

Before doing so, we will explore in more detail some of the properties of quantum entanglement already introduced in Section 2.1.2 and how it can be detected and quantified. Then we will analyse the dynamics of a two-qubit system in a bath described by scalar quantum fields, giving the criteria for entanglement generation at short times. After that, we will study the long time behaviour of the system states, finding a condition for entanglement preservation also for the asymptotic states.

In particular, at the end of the chapter we present our first major result, namely the analytic construction of the convex manifold of stationary states and thus the derivation of the entanglement preservation conditions for the dynamics under exam.

3.1 Entanglement witnesses

Among the methods to create entanglement between particles in a laboratory, one may let particles to suitably interact among themselves, but also engineer an interaction not directly between the two particles, rather with a third system, or one may perform an appropriate measurement of a non-local operator, namely an observable common to the two systems, whose eigenstates are entangled. The last two possibilities suggest that the common idea that entanglement is always related to the interaction between systems is not entirely appropriate.

In fact, one of the aim of this Chapter is to provide the conditions under which a dissipative environment is by itself able to generate entanglement between two qubits, without taking into account any measurements.

Ascertaining whether a generic mixed state of a bipartite quantum system is entangled or separable is the more difficult the higher the dimension of the Hilbert space of the compound system. In Section 2.1.2 we discussed the relations of entanglement with the complete positivity of linear maps on one of the two parties. Moreover, in Example 2.1 we saw that partial transposition, exactly because it is positive but lacks complete positivity, is able to witness the entanglement of a state as $P^{(2)}_{sym}$ in (2.16). In the following we briefly consider two techniques that have been formulated one for checking whether a state is entangled, the so-called *Positivity under Partial Transposition* (PPT) criterion [22, 33], the other to quantify the amount of entanglement present in a bipartite state, the so-called *Concurrence* [45].

Under the name of *entanglement witnesses* lie all those methods which allow to check wether a state is entangled or not.

In general a proper entanglement witness E should satisfy the following properties, which reflect the behaviour of entanglement itself [2]:

1. $E(\rho_{AB}) = 0 \Leftrightarrow \rho_{AB}$ *is separable*
2. $E(U_A \otimes U_B\ \rho_{AB}\ U_A^\dagger \otimes U_B^\dagger) = E(\rho_{AB})$, where U is a unitary transformation.
3. $E\left(\Lambda_{LOCC}(\rho_{AB})\right) = E(\rho_{AB})$, where $LOCC$[1] stands for local operation and classical communication.
4. $E\left(\sum_k p_k \rho_k\right) \leq \sum_k p_k E(\rho_k)$, mixing states can not increase the entanglement, it just adds classic uncertainty. This propriety is also called *convexity*.
5. $E(\rho^{\otimes n}) = nE(\rho)$, where $\rho^{\otimes n} = \underbrace{\rho \otimes \rho \otimes \cdots \otimes \rho}_{n\ times}$.

Now we show some criteria to determine the entanglement of the system.

3.1.1 Peres-Horodecki criterion (PPT)

The PPT, also known as Peres-Horodecki, criterion [22] gives a necessary condition for the separability of generic bipartite states as well. In bipartite systems of 2×2 or 2×3 dimension the condition is also sufficient.

In the following we shall be mainly focussing on systems S consisting of two qubits. Let us reconsider the partial transposition, already introduced in Example 2.1 and its action on the second party of a generic state, written with respect to the tensor product of the standard basis state vectors $\{|0\rangle, |1\rangle\}$:

$$\rho_{AB} = \sum_{i,j,k,l=0,1} p_{ijkl} \left|i_A\right\rangle \left\langle j_A\right| \otimes \left|k_B\right\rangle \left\langle l_B\right|. \tag{3.1}$$

[1] We will not linger over $LOCC$ formalism and mathematical properties, we suggest [2] for an introduction about this topic.

Performing the partial transposition by means of the operator $\mathbb{I} \otimes T$ yields

$$\rho_{AB}^{\mathbb{I}\otimes T} = (\mathbb{I} \otimes T)[\rho_{AB}] = \sum_{i,j,k,l=0,1} p_{ijlk} \left|i_A\right\rangle \left\langle j_A\right| \otimes \left|k_B\right\rangle \left\langle l_B\right|. \tag{3.2}$$

By writing ρ_{AB} as a 4 block-matrix

$$\rho_{AB} = \sum_{i,j=0,1} \left|i_A\right\rangle \left\langle j_A\right| \otimes \hat{X}_{ij} = \begin{pmatrix} \hat{X}_{11} & \hat{X}_{12} \\ \hat{X}_{21} & \hat{X}_{22} \end{pmatrix} = \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} \\ \rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} \\ \rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} \end{pmatrix},$$

the partial transposition on B means transposing the elements inside every block of ρ_{AB}:

$$\rho_{AB}^{\mathbb{I}\otimes T} = \begin{pmatrix} \hat{X}_{11}^T & \hat{X}_{12}^T \\ \hat{X}_{21}^T & \hat{X}_{22}^T \end{pmatrix} = \begin{pmatrix} \rho_{11} & \rho_{21} & \rho_{13} & \rho_{23} \\ \rho_{12} & \rho_{22} & \rho_{14} & \rho_{24} \\ \rho_{31} & \rho_{41} & \rho_{33} & \rho_{43} \\ \rho_{32} & \rho_{42} & \rho_{34} & \rho_{44} \end{pmatrix}. \tag{3.3}$$

Theorem 3.1 (PPT criterion)**.** A state ρ_{AB} of two qubits is separable if and only if it remains positive under partial transposition.

Proof. Because of the discussion after Example 2.1, we already know that if ρ_{AB} is not entangled then it is PPT. The other implication follows from two results. The first one by Woronowicz [46] states that all positive maps Λ from one qubit to one qubit are *decomposable*, namely of the form $\Lambda = \Lambda_{CP}^{(1)} + \Lambda_{CP}^{(2)} \circ T$ where $\Lambda_{CP}^{(1)}$ and $\Lambda_{CP}^{(2)}$ are completely positive maps and T is the transposition. The second result by the Horodecki [22] states that any bipartite state ρ_{AB} (not only of two qubits) is entangled if and only if there exists a positive but not completely positive map Λ from one system to the other such that $\mathrm{id}_A \otimes \Lambda[\rho_{AB}]$, equivalently $\Lambda \otimes \mathrm{id}_B[\rho_{AB}]$, is not positive while $\mathrm{id}_A \otimes \Lambda[\rho_{sep}] \geq 0$, equivalently $\Lambda \otimes \mathrm{id}_B[\rho_{sep}] \geq 0$, for all separable states ρ_{sep}. Then, if ρ_{AB} is PPT one has $\mathrm{id}_A \otimes T[\rho_{AB}] \geq 0$; hence, since $\Lambda_{CP}^{(1)}$ and $\Lambda_{CP}^{(2)}$ are CP,

$$\mathrm{id}_A \otimes \Lambda[\rho_{AB}] = \mathrm{id}_A \otimes \Lambda_{CP}^{(1)}[\rho_{AB}] + \mathrm{id}_A \otimes \Lambda_{CP}^{(2)} \circ \mathrm{id}_A \otimes T[\rho_{AB}] \geq 0$$

for all positive Λ and cannot thus be entangled. □

3.1.2 Concurrence

The concurrence [22, 45] is a function which possesses all the properties requested to a quantity whose aim is to measure the "amount" of entanglement of a mixed 2-qubit state; namely, it is a non-negative function which satisfies all the requirement listed in Section 3.1. Given a bipartite two qubit state ρ, its concurrence $C(\rho)$ is defined as

$$C(\rho) := \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}, \tag{3.4}$$

where λ_i are the square roots, listed in decreasing order, of the eigenvalues of the positive matrix

$$R := \rho\,(\sigma_2 \otimes \sigma_2)\rho^*(\sigma_2 \otimes \sigma_2), \tag{3.5}$$

where ρ^* is ρ with complex-conjugated entries.

The state is entangled if $C(\rho) > 0$ and the greater $C(\rho)$, the more ρ is entangled.

In general, the concurrence is a difficult quantity to evaluate analytically; in particular, only for a few dynamics its behaviour in time can be analytically controlled. Luckily, there exists a class of states for which the evaluation of the concurrence is particularly easy, the so-called X-states, which are of the following form in the standard basis (see (3.1))

$$\rho_X = a\,|00\rangle\langle 00| + w\,|00\rangle\langle 11| + b\,|01\rangle\langle 01| + z\,|01\rangle\langle 10| \tag{3.6}$$

$$+ z^*\,|10\rangle\langle 01| + c\,|10\rangle\langle 10| + w^*\,|11\rangle\langle 00| + d\,|11\rangle\langle 11| = \begin{pmatrix} a & 0 & 0 & w \\ 0 & b & z & 0 \\ 0 & z^* & c & 0 \\ w^* & 0 & 0 & d \end{pmatrix}. \tag{3.7}$$

Positivity asks that the matrices

$$\begin{pmatrix} a & w \\ w^* & d \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} b & z \\ z^* & c \end{pmatrix}$$

be both positive semi-definite which in turn requires

$$a \geq 0\ ,\ d \geq 0\ ,\ ad - |w|^2 \geq 0\ ; \qquad b \geq 0\ ,\ c \geq 0\ ,\ bc - |z|^2 \geq 0\ , \tag{3.8}$$

while normalization asks for

$$a + b + c + d = 1\ . \tag{3.9}$$

The partial trasposition $\rho_X^{\mathbb{I}\otimes T}$ of the previous state gives

$$\rho_X^{\mathbb{I}\otimes T} = \begin{pmatrix} a & 0 & 0 & z \\ 0 & b & w & 0 \\ 0 & w^* & c & 0 \\ z^* & 0 & 0 & d \end{pmatrix}. \tag{3.10}$$

Therefore, according to Theorem 3.1, ρ_X is separable if and only if $\rho_X^{\mathbb{I}\otimes T} \geq 0$. Again, this is so if and only if

$$a \geq 0\ ,\ d \geq 0\ ,\ ad - |z|^2 \geq 0\ ; \qquad b \geq 0\ ,\ c \geq 0\ ,\ bc - |w|^2 \geq 0\ . \tag{3.11}$$

Indeed, it can be proven [47] that for an X-state the concurrence is directly related to determinants of the external and internal sub-matrices of (3.10):

$$C(\rho_X) = 2\ \max\{|z| - \sqrt{ad}, |w| - \sqrt{bc}, 0\}. \tag{3.12}$$

3.2 Two-qubit entanglement generation via dissipation

In this section we will focus upon an open two qubit system in interaction with an environment described by a set of uncorrelated scalar fields. Concretely, given the usual Hamiltonian system S plus bath B

$$H = H_S \otimes \mathbb{I}_B + \mathbb{I}_S \otimes H_B + H_{int}\ , \tag{3.13}$$

H_S is the following two qubit Hamiltonian:

$$H_S = H_S^{(1)} + H_S^{(2)}, \quad H_S^{(1)} = \frac{\omega}{2}\sum_{i=1}^{3} n_i \left(\sigma_i \otimes \mathbb{I}\right), \quad H_S^{(2)} = \frac{\omega}{2}\sum_{i=1}^{3} n_i \left(\mathbb{I} \otimes \sigma_i\right), \tag{3.14}$$

while we consider an interaction of the following kind

$$H_{int} = \sum_{i=1}^{3} \left[(\sigma_i \otimes \mathbb{I}) \otimes \mathbf{\Phi}_i(x) + (\mathbb{I} \otimes \sigma_i) \otimes \mathbf{\Psi}_i(x)\right], \tag{3.15}$$

where $\mathbf{\Psi}_i(x)$ and $\mathbf{\Phi}_i(x)$ are statistically independent massless scalar fields and x represents the point of the space time where the qubit and the field do interact. This interaction term is just an example of a concrete dynamics, the resulting Kossakowski matrix in the master equation we are about to determine can be achieved with different set ups, not necessarily involving scalar fields. Moreover, we highlight that the Hamiltonians of the two qubits have the same form and that we have absorbed into the expressions of the fields the coupling constant $\lambda \ll 1$.

To recover the GKSL master equation from the above Hamiltonian, we follow precisely the same steps illustrated in Section 2.2.2. In particular we start from (2.67) and we consider the state of the qubit-system and the bath to be initially separable. Then we trace out the degrees of freedom of the bath and putting ourselves in weak coupling limit conditions, we find a master equation of the kind

$$\frac{\partial \rho(t)}{dt} = \mathbb{H}_S\left[\rho(t)\right] + \widetilde{\mathbb{D}}\left[\rho(t)\right], \tag{3.16}$$

where $\mathbb{H}_S\left[\rho(t)\right] = -i\left[H_S, \rho(t)\right]$ and $\widetilde{\mathbb{D}}$ is given by the ergodic average

$$\widetilde{\mathbb{D}} = \lim_{T\to+\infty} \frac{1}{T}\int_0^T ds\, \mathrm{e}^{-s\mathbb{L}_S} \circ \mathbb{D}_1 \circ \mathrm{e}^{s\mathbb{L}_S} \tag{3.17}$$

where

$$\mathbb{D}_1\left[\rho(t)\right] = -\int_0^\infty \mathrm{d}w\, \mathrm{tr}_B\left(\mathrm{e}^{-w\,\mathbb{L}_S}\left[H_{int}, \mathrm{e}^{w\,(\mathbb{L}_S+\mathbb{L}_B)}\left[H_{int}, \rho_S \otimes \rho_B\right]\right]\right). \tag{3.18}$$

We remind that the introduction of the ergodic average in needed in order to obtain a consistent completely positive open dynamics.

Using the explicit forms (3.14) and (3.15), we find

$$\begin{aligned}\widetilde{\mathbb{D}}\left[\rho_S(t)\right] = & \\ = -\lim_{T\to+\infty}\frac{1}{T}\int_0^T ds \int_0^\infty dw \sum_{\alpha,\beta=1}^{2}\sum_{i,j=1}^{2} \Big\{ & G_{ij}(w)\left[\sigma_i^{(\alpha)}(w+s), \sigma_j^{(\beta)}(s)\,\rho_S(t)\right] + \\ & + G_{ji}(-w)\left[\rho_S(t)\sigma_j^{(\alpha)}(s), \sigma_i^{(\beta)}(w+s)\right]\Big\}\ ,\end{aligned} \tag{3.19}$$

where

$$\begin{aligned}&\sigma_i^{(\alpha)} := \sigma_i \otimes \mathbb{I} \quad \text{for } \alpha = 1, \qquad \sigma_i^{(\alpha)} := \mathbb{I} \otimes \sigma_i \ \text{ for } \alpha = 2; \\ &\sigma_i^{(\alpha)}(t) := \mathrm{e}^{-t\mathbb{L}_S}\left[\sigma_i^{(\alpha)}\right] = \mathrm{e}^{itH_S}\sigma_i^{(\alpha)}\mathrm{e}^{-itH_S},\end{aligned} \tag{3.20}$$

and, setting $B_i^{(\alpha)} = \mathbf{\Phi}_i$ for $\alpha = 1$ and $B_i^{(\alpha)} = \mathbf{\Psi}_i$ for $\alpha = 2$, we have

$$\begin{aligned} G_{ij}^{(\alpha\beta)}(s) &:= \mathrm{tr}\left[\rho_B B_i^{(\alpha)}(s) B_j^{(\beta)}\right], \\ B_i^{(\alpha)}(s) &:= \mathrm{e}^{-s\mathbb{L}_B}\left[B_i^{(\alpha)}(s)\right] = \mathrm{e}^{isH_B} B_i^{(\alpha)}\, \mathrm{e}^{-isH_B}, \end{aligned} \tag{3.21}$$

the first row represents the correlation functions of the environment.

Now, in order to simplify the expression (3.19), we introduce the projector operators over the eigenstates of the free Hamiltonian of the single qubit with eigenvalues $\pm\omega/2$:

$$P_\pm = \frac{1 + \vec{n}\vec{\sigma}}{2}. \tag{3.22}$$

With this tool we can write the auxiliary matrices $\sigma(\xi)_i$ with $\xi = 0, \pm$:

$$\sigma_i(0) = P_+\sigma_i P_+ + P_-\sigma_i P_-, \quad \sigma_i(\pm) = P_\pm \sigma_i P_\mp \tag{3.23}$$

and we can write the time-evolved Pauli matrices as

$$\sigma_i(t) = e^{iH_S t}\sigma_i e^{-iH_S t} = \sum_{\xi=0,\pm} e^{i\xi\omega t}\sigma_i(\xi). \tag{3.24}$$

This substitution allows us to perform the limit in (3.19). Moreover, by defining the following Fourier and Hilbert transforms

$$\alpha_{ij}^{(\alpha\beta)}(\omega,\xi) = \int_{-\infty}^{\infty} dt e^{i\xi\omega t}\langle B_i^{(\alpha)}(t) B_j^{(\beta)}\rangle, \tag{3.25}$$

$$\beta_{ij}^{(\alpha\beta)}(\omega,\xi) = \int_{-\infty}^{\infty} dt\, \mathrm{sgn}(t)\, e^{i\xi\omega t}\langle B_i^{(\alpha)}(t) B_j^{(\beta)}\rangle, \tag{3.26}$$

and by expanding the $\sigma_i(\xi)$ matrices on the Pauli basis [4]

$$\sigma_0(\xi) = \delta_{\xi 0}\sigma_0, \quad \sigma_i(\xi) = \sum_{j=1}^{3} \psi_{ij}^{(\xi)}\sigma_j, \tag{3.27}$$

where

$$\psi_{ij}^{(0)} = n_i n_j, \quad \psi_{ij}^{(\pm)} = \frac{1}{2}\left(\delta_{ij} - n_i n_j \pm i\epsilon_{ijk}n_k\right), \tag{3.28}$$

we arrive to the celebrated GKSL expression

$$\frac{\partial\rho(t)}{\partial t} = -i\left[H_{\mathrm{eff}}, \rho(t)\right] + \mathbb{D}[\rho(t)], \tag{3.29}$$

where the effective Hamiltonian is

$$H_{\mathrm{eff}} = H_S - \frac{i}{2}\sum_{ij}\left(H_{ij}^{(11)}(\sigma_i\sigma_j \otimes \mathbb{I}) + H_{ij}^{(22)}(\mathbb{I}\otimes\sigma_i\sigma_j) + H_{ij}^{(12)}(\sigma_i\otimes\sigma_j)\right) \tag{3.30}$$

in particular

$$H_{ij}^{(\alpha\beta)} = \sum_{\xi=0,\pm}\sum_{k,l=1}^{3} \beta_{kl}^{(\alpha\beta)}(\omega,\xi)\,\psi_{ki}^{(\xi)}\psi_{lj}^{(-\xi)}. \tag{3.31}$$

We highlight that $H^{(11)}$ and $H^{(22)}$ are the Lamb corrections for the two qubits free evolution separately, while $H^{(12)}$ is a coupling terms between the qubits due to the interaction with the environment. Here is clear how a common interaction with an environment may entangle two qubits, we want to know if this entanglement is then destroyed by the dissipator in (3.29) or if the latter may lead to entanglement preservation or may be even responsible of entanglement generation.

Focusing hence on the dissipator, it can be written as

$$\mathbb{D}[\rho] = \sum_{\alpha,\beta=1}^{6} K_{\alpha\beta} \left[\mathcal{F}_\beta \rho \mathcal{F}_\alpha - \frac{1}{2} \{ \mathcal{F}_\alpha \mathcal{F}_\beta, \rho \} \right], \tag{3.32}$$

where

$$\begin{aligned} \mathcal{F}_\alpha &:= \sigma_\alpha \otimes \mathbb{I} \quad \text{for } \alpha = 1,2,3, \\ \mathcal{F}_\alpha &:= \mathbb{I} \otimes \sigma_{\alpha-3} \quad \text{for } \alpha = 4,5,6, \end{aligned} \tag{3.33}$$

and $K_{\alpha\beta}$ is the 6×6 Kossakowski matrix, which we require to be positive.

The coefficients $K_{\alpha\beta}$ are determined by the Fourier transform of the field correlations [6]:

$$K_{\alpha\beta} = \sum_{\xi=0,\pm} \sum_{k,l=1}^{3} \alpha_{kl}^{(\alpha\beta)}(\omega,\xi)\, \psi_{l\beta}^{(-\xi)}. \tag{3.34}$$

We write the latter as the composition of 3×3 block matrices $\mathcal{A} = \mathcal{A}^\dagger$, $\mathcal{C} = \mathcal{C}^\dagger$ and $\mathcal{B}$:

$$K = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{B}^\dagger & \mathcal{C} \end{pmatrix}. \tag{3.35}$$

Given that, we can explicitly write the dissipator $\mathbb{D}[\rho]$ as

$$\begin{aligned} \mathbb{D}[\rho] = \sum_{i,j=1}^{3} \Bigg(& \mathcal{A}_{ij} \left[(\sigma_j \otimes \mathbb{I})\, \rho\, (\sigma_i \otimes \mathbb{I}) - \frac{1}{2} \{ (\sigma_i \sigma_j \otimes \mathbb{I})\, \rho \} \right] \\ & + \mathcal{C}_{ij} \left[(\mathbb{I} \otimes \sigma_j)\, \rho\, (\mathbb{I} \otimes \sigma_i) - \frac{1}{2} \{ (\mathbb{I} \otimes \sigma_i \sigma_j)\, , \rho \} \right] \\ & + \mathcal{B}_{ij} \left[(\sigma_j \otimes \mathbb{I})\, \rho\, (\mathbb{I} \otimes \sigma_i) - \frac{1}{2} \{ (\sigma_j \otimes \sigma_i)\, , \rho \} \right] \\ & + \mathcal{B}_{ji}^* \left[(\mathbb{I} \otimes \sigma_j)\, \rho\, (\sigma_i \otimes \mathbb{I}) - \frac{1}{2} \{ (\sigma_i \otimes \sigma_j)\, , \rho \} \right] \Bigg). \end{aligned} \tag{3.36}$$

Here we notice that the first two rows of the previous dissipator represent the dissipation for the two qubits separately from each other, while the last two rows are dissipating terms which also may contribute with the coupling between the two qubits. Clearly there can not be entanglement formation due to dissipation if $\mathcal{B} = 0$. The question now is if it is possible to find some conditions on the dynamics such that the coupling behaviour overcomes the dissipation. We will tackle this problem both at short time regime and for the asymptotic state. Surely if there is no entanglement generation at short time, there will not be any at infinite time, when

the action of the dissipation has completely being injected. On the other hand, if there is entanglement formation at short times, it will be interesting to see whether it is preserved even in the asymptotic state or if the dissipator alone may even contribute to its generation without the help of the coupling terms in the Hamiltonian evolution.

To study the entanglement formation, we consider the following initial two-qubit state, we can take it to be pure without loss of generality, since entanglement properties given in Section 3.1, if the environment is not able to create entanglement for pure states, it certainly can not for mixed ones,

$$\rho(0) = |\varphi\rangle\langle\varphi| \otimes |\psi\rangle\langle\psi|. \tag{3.37}$$

We apply the method of partial transposition (described in Section 3.1.1) to the master equation in (3.32), getting a master equation for the partially transposed state $\widetilde{\rho}(t)$, which keeps the same structural form as (3.29), namely

$$\partial_t \widetilde{\rho}(t) = -i\left[\widetilde{H}_{\text{eff}}, \widetilde{\rho}(t)\right] + \widetilde{\mathbb{D}}[\widetilde{\rho}(t)], \tag{3.38}$$

with the difference that in the new effective Hamiltonian there are also terms belonging to the dissipation:

$$\widetilde{H}_{\text{eff}} = \sum_{i=1}^{3} H_i^{(1)} (\sigma_i \otimes \mathbb{I}) + \sum_{ij=1}^{3} H_i^{(2)} E_{ij} (\mathbb{I} \otimes \sigma_j) + \sum_{ij=1}^{3} \mathcal{I}m(\mathcal{B} \cdot E)_{ij} (\sigma_i \otimes \sigma_j) \tag{3.39}$$

$$E = \text{diag}(-1, 1, -1); \tag{3.40}$$

and the dissipator is the same as in (3.32) with the following substitution for the Kossakowski matrix $K \mapsto \mathcal{E}\widetilde{K}\mathcal{E}$:

$$\widetilde{K} = \begin{pmatrix} \mathcal{A} & \mathcal{R}e(\mathcal{B}) + iH^{(12)} \\ \mathcal{R}e\left(\mathcal{B}^T\right) - iH^{(12)T} & \mathcal{C}^T \end{pmatrix}, \tag{3.41}$$

$$\mathcal{E} = \begin{pmatrix} \mathbb{I}_3 & 0 \\ 0 & E \end{pmatrix}, \tag{3.42}$$

where the upperscript T stands for transposition.

Now, the master equation in (3.38) needs not to be completely positive nor positive, since it is the result of a partial transposition which is not a positive map. In particular $\widetilde{K}$ may have negative eigenvalues, we can tell from its expression in (3.41) that it can have negative eigenvalues thanks only to $H^{(12)}$ and $\mathcal{B}$, in complete accordance with what we stated above: those are the only contributions that can actually couple the two qubits, the first one from the Hamiltonian corrections, the second one from pure dissipation. Moreover, we can get negative eigenvalue for $\widetilde{\rho}(t)$ also via the $\mathcal{B}$-dependent term in $\widetilde{H}_{\text{eff}}$.

In order to see if the state $\widetilde{\rho}(t)$ generates negative eigenvalues, it proves convenient to introduce the following quantity, given any 4-dimensional vector χ

$$\mathcal{Q}(t) := \langle\chi|\widetilde{\rho}(t)|\chi\rangle. \tag{3.43}$$

The state is entangled if the previous quantity becomes negative for a certain χ at a certain time t. We specify that in order to have a negative $\mathcal{Q}$ (hence entanglement), χ has to be entangled,

as a matter of fact $\mathcal{Q}(t)$ can be rewritten and manipulated in the following way

$$\mathcal{Q}(t) = \mathrm{tr}\left(|\chi\rangle\langle\chi|\widetilde{\rho}(t)\right) = \mathrm{tr}\left(|\chi\rangle\langle\chi|(T\otimes \mathrm{Id})\Lambda_t\left[\rho\right]\right) = \mathrm{tr}\left((T\otimes \mathrm{Id})|\chi\rangle\langle\chi|\Lambda_t\left[\rho\right]\right), \tag{3.44}$$

since Λ_t is a CP dynamical map, the only way to have negative values is that χ is entangled as well, since the partial transposition $(T\otimes \mathrm{Id})$ acts on it now.

Since there is a time in which $\mathcal{Q}(t)$ may become negative, there is also a time at which $\mathcal{Q}(t)$ is exactly zero, when there is the switch from positive values to negative ones. We can always recast this time in order to have $\mathcal{Q}(t) = 0$ precisely at $t = 0$. So the state is entangled if it exists an entangled vector χ such that

$$\mathcal{Q}(0) = 0 \quad \text{and} \quad \partial_t\mathcal{Q}(0) < 0, \tag{3.45}$$

which give sufficient condition for entangled generation for short times.

Rather than (3.45), we can find an even more manageable condition for entangled formation at finite time, by suitably manipulating $\partial_t\mathcal{Q}(0)$: we consider the 2-dimensional Hilbert spaces of the two qubits, then we consider the base of the first Hilbert space given by the state φ in (3.37) and its orthonormal vector $\widetilde{\varphi}$; we do the same for the second qubit. Hence, we have two basis $\{|\varphi\rangle, |\tilde{\varphi}\rangle\}, \{|\psi\rangle, |\tilde{\psi}\rangle\}$. All the basis elements can be obtained via an appropriate rotation of the standard basis $\{|0\rangle, |1\rangle\}$ of eigenvectors of the Pauli matrix σ_3:

$$|\varphi\rangle = U|1\rangle, \quad |\tilde{\varphi}\rangle = U|0\rangle, \tag{3.46}$$

$$|\psi\rangle = V|1\rangle, \quad |\tilde{\psi}\rangle = V|0\rangle, \tag{3.47}$$

where U and V are unitary transformation which induce orthogonal transformations $\mathcal{U}$ and $\mathcal{V}$ on the Pauli matrices

$$U^\dagger\sigma_i U = \sum_{j=1}^{3}\mathcal{U}_{ij}\sigma_j, \quad V^\dagger\sigma_i V = \sum_{j=1}^{3}\mathcal{V}_{ij}\sigma_j. \tag{3.48}$$

Having said that, we can rewrite $\partial_t\mathcal{Q}(0)$ as a quadratic expression of $|\chi\rangle$, as a consequence $\partial_t\mathcal{Q}(0) < 0$ if its corresponding discriminant is negative, whose formal expression is

$$\langle u|\mathcal{A}|u\rangle\left\langle v\left|\mathcal{C}^T\right|v\right\rangle < \left|\left\langle u\left|\left(\mathcal{R}e(\mathcal{B}) + iH^{(12)}\right)\right|v\right\rangle\right|^2. \tag{3.49}$$

The 3-dimensional vectors $|u\rangle$ and $|v\rangle$ depend on the initial state (3.37) of the system and their components can be expressed as

$$u_i = \sum_{j=1}^{3}\mathcal{U}_{ij}\left\langle 0\left|\sigma_j\right|1\right\rangle, \quad v_i = \sum_{j=1}^{3}\mathcal{V}_{ij}\left\langle 1\left|\sigma_j\right|0\right\rangle. \tag{3.50}$$

Hence, by means of (3.49) we were able to express a condition for entanglement formation depending on the structure of the Kossakowski matrix and the initial state of the system. As expected, the entanglement formation depends on $\mathcal{B}$ and $iH^{(12)}$. Indeed, these are the terms that couple different qubits. Also from (3.49), we can tell which contribution between the pure dissipation and the Hamiltonian coupling contributes the most to the entanglement formation.

A simpler expression can be obtained when 1.) there are no Hamiltonian coupling terms, $H^{(12)} = 0$ and 2.) the external fields in (3.15) are identical $\mathbf{\Psi}_i = \mathbf{\Phi}_i$, (it is like having two qubits interacting with the same environment). In this case, the Kossakowski matrix has identical block entries $\mathcal{A}_{ij} = \mathcal{B}_{ij} = \mathcal{C}_{ij}$, generally given by a symmetric and an anti-symmetric part:

$$\mathcal{A}_{ij} = \mathcal{B}_{ij} = \mathcal{C}_{ij} = A_{ij} + i \sum_{k=1}^{3} \varepsilon_{ijk} B_k, \tag{3.51}$$

where n_i are unit vectors.

If we also choose $u_i = v_i$ (which also means $|\psi\rangle = |\tilde{\varphi}\rangle$) the expression (3.49) becomes even simpler:

$$|\langle u|\mathcal{I}m(\mathcal{A})|u\rangle|^2 > 0. \tag{3.52}$$

This expression makes sense because as long as $\mathcal{A}$ has an imaginary part, the parameter B in (3.51) does not vanish.

3.3 Entanglement in the asymptotic state

Given the condition for entanglement generation via dissipation at finite time, a natural question arises: is it possible to find some condition for it to be maintained even in asymptotic state? To answer to this question we have first to investigate the dynamical long-term regime.

To do so we write the time-evolving density matrix describing the two qubits in the Pauli matrices decomposition

$$\rho(t) = \frac{1}{4} \left[\mathbb{I} \otimes \mathbb{I} + \sum_{i=1}^{3} \rho_{0i}(t) \mathbb{I} \otimes \sigma_i + \sum_{i=1}^{3} \rho_{i0}(t) \sigma_i \otimes \mathbb{I} + \sum_{i,j=1}^{3} \rho_{ij}(t) \sigma_i \otimes \sigma_j \right]. \tag{3.53}$$

Moreover, we introduce the following operators

$$\Sigma_i := \sigma_i \otimes \mathbb{I} + \mathbb{I} \otimes \sigma_i, \quad i = 1, 2, 3 \tag{3.54}$$

$$S_{ij} := \sigma_i \otimes \sigma_j + \sigma_j \otimes \sigma_i, \quad i, j = 1, 2, 3, \tag{3.55}$$

which are symmetric under qubit exchange and constitute an algebra as shown in Appendix A.2. They are most useful; in order to see why, we start by considering a simplified setting without any Hamiltonian term, so that the generator reduces to a purely dissipative contribution $\mathbb{D}$. Further, we specialize it to the case when $\mathcal{A} = \mathcal{B} = \mathcal{C}$, and the expression in (3.32) can be rewritten in the compact form

$$\frac{\partial \rho(t)}{\partial t} = \mathbb{D}[\rho(t)] = \sum_{i,j=1}^{3} \mathcal{A}_{ij} \left[\Sigma_j \rho(t) \Sigma_i - \frac{1}{2} \{ \Sigma_i \Sigma_j, \rho(t) \} \right]. \tag{3.56}$$

Substituting (3.51) and (3.53) in the previous expression, we find the following evolution equa-

tions for the components of $\rho(t)$ [7]:

$$\begin{aligned}
\frac{\partial\rho_{0i}(t)}{\partial t} =& -2A\rho_{0i}(t) + 2\sum_{k=1}^{3}\left[A_{ik}\rho_{0k}(t) - \rho_{ik}(t)B_k\right] + 2(2+\tau)B_i,\\
\frac{\partial\rho_{i0}(t)}{\partial t} =& -2A\rho_{i0}(t) + 2\sum_{k=1}^{3}\left[A_{ik}\rho_{k0}(t) - \rho_{ki}(t)B_k\right] + 2(2+\tau)B_i,\\
\frac{\partial\rho_{ij}(t)}{\partial t} =& -4A\left[\rho_{ij}(t)+\rho_{ji}(t)\right] + 2\sum_{k=1}^{3}\left[A_{ik}\rho_{kj}(t) + A_{jk}\rho_{ik}(t)\right] - 4A_{ij}\tau\\
&+ 4\sum_{k=1}^{3}\left[A_{ik}\rho_{jk}(t) + A_{jk}\rho_{ki}(t)\right] + 4\left[A\tau - \sum_{k,l=1}^{3}A_{kl}\rho_{lk}(t)\right]\delta_{ij}\\
&+ 2\left[B_i\rho_{j0}(t) + B_j\rho_{0i}(t)\right] + 4\left[B_i\rho_{0j}(t) + B_j\rho_{i0}(t)\right]\\
&- 2\sum_{k=1}^{3}B_k\left[\rho_{0k}(t) + \rho_{k0}(t)\right]\delta_{ij},
\end{aligned} \tag{3.57}$$

where

$$\tau = \sum_{i=1}^{3}\rho_{ii} \tag{3.58}$$

is the trace of the coefficients matrix in last term of (3.53). Interestingly enough, this quantity is a constant of motion. Indeed, setting $i = j$ in (3.57) and summing from $i = 1$ to $i = 3$ yields $\dfrac{\mathrm{d}\tau}{\mathrm{d}t} = 0$. Moreover, the positivity of the initial state, $\rho(0) \geq 0$, requires that $-3 \leq \tau < 1$.

The 15 evolution equations (3.57) can be separated into two independent sets of 9, respectively 6 differential equations that involve only symmetric, respectively anti-symmetric variables: $\rho_{(0i)} := \rho_{0i} + \rho_{i0}$, $\rho_{(ij)} = \rho_{ij} + \rho_{ji}$, respectively $\rho_{[0i]} := \rho_{0i} - \rho_{i0}$, $\rho_{[ij]} = \rho_{ij} - \rho_{ji}$. By inspecting the differential equations satisfied by the antisymmetric variables, one sees that they exponentially decay in time. As a consequence, one can seek stationary states among those of the form

$$\hat{\rho} = \frac{1}{4}\left[\mathbb{I}\otimes\mathbb{I} + \sum_{i=1}^{3}\hat{\rho}_i\Sigma_i + \sum_{i,j=1}^{3}\hat{\rho}_{ij}S_{ij}\right], \tag{3.59}$$

where $\hat{\rho}_{ij} = \hat{\rho}_{ji}$.

A more direct and enlightening way to show that one can restrict to asymptotic states as in the previous expression is as follows. Let us introduce the flip operator V such that

$$V(|\psi\rangle\otimes|\phi\rangle) := |\phi\rangle\otimes|\psi\rangle, \tag{3.60}$$

$$V^\dagger(A\otimes B)V = B\otimes A, \tag{3.61}$$

where $|\psi\rangle$ and $|\phi\rangle$ are vectors of a 2-dimensional Hilbert space, while A and B are operators. Clearly, V fulfils $V = V^\dagger = V^{-1}$. In 4×4 dimension, the flip operator can be written as

$$V = \sum_{i,j=1}^{2}|i\rangle\langle j|\otimes|j\rangle\langle i|. \tag{3.62}$$

Thus we have the identities:

$$V^\dagger \Sigma_k V = \Sigma_k, \quad V^\dagger S_{ij} V = S_{ij}, \tag{3.63}$$

Consider the (completely positive) map $X \mapsto F[X] := V^\dagger X V$: since V is unitary, it satisfies $F^2 = F$ and $F[X]F[Y] = F[X]F[Y]$. Since the operators which appears in the evolution (3.56) are invariant under the flip operator, the dissipator is symmetric under qubit exchange:

$$\mathbb{D} \circ F = F \circ \mathbb{D} \tag{3.64}$$

Then, if $\mathbb{D}[\rho] = 0$, also $\mathbb{D}[F[\rho]] = 0$. Due to the structure of the map F, if ρ is an asymptotic state, such is also $F[\rho]$. Therefore $\rho_{sym} := \dfrac{\rho + F[\rho]}{2}$ is a state too, symmetric under qubit exchange: $F[\rho_{sym}] = \rho_{sym}$. Moreover, because of the linearity of the dissipator, it is stationary: $\mathbb{D}\left[\dfrac{\rho + F[\rho]}{2}\right] = 0$.

Therefore, in order to seek a faithful stationary state one can restrict the search to the kernel of the set of 9 differential equations for the symmetric variables. In order to have a chance of finding an analytic expression for the faithful stationary state, we further simplify the Kossakowski matrix and consider the decomposition in (3.51) with $A = [\lambda_i \delta_{ij}]$ diagonal and choosing $B_1 = B_2 = 0$, $B_3 \neq 0$.

Notice that, while both simplified structures are singly always achievable by suitable unitary rotations of the matrices A and B, they can be implemented together by a same change of basis only for very special A and B.

Since $\mathcal{A}$ is not negative in order to guarantee the complete positivity of the evolution, we can introduce the operators $\mathcal{V}_i = \sum_{j=1}^3 (\sqrt{\mathcal{A}})_{ij} \Sigma_j$, by means of the square root of $\mathcal{A}$. One can then write the dissipator in diagonal form

$$\mathbb{D}[\rho] = \sum_{i,j=1}^{3} \left[\mathcal{V}_j \rho \mathcal{V}_i^\dagger - \frac{1}{2} \left\{ \mathcal{V}_i^\dagger \mathcal{V}_j, \rho \right\} \right]. \tag{3.65}$$

We denote by $\{\mathcal{V}_j\}'$ the commutant of the set $\{\mathcal{V}_i\}$, that is the set of operators commuting with all the Kraus operators $\mathcal{V}_i$, $\mathcal{V}_i^\dagger$[2]. In Appendix A.2 it is shown that $\{\mathcal{V}_j\}'$ contains the operator $S := \sum_{i=1}^2 S_{ii}$ and the identity and is thus commutative. Because of this, there are several invariant states, actually a 2-dimensional convex manifold of them. In order to find them, we identify the extremal invariant states by means of the two mutually orthogonal projectors constructed with the elements of $\mathcal{M}$:

$$P = \frac{1}{4} \left[\mathbb{I} \otimes \mathbb{I} - \frac{S}{2} \right], \qquad Q = 1 - P. \tag{3.66}$$

Then any initial state is mapped by the evolution in the following equilibrium state

$$\rho(0) \to \hat{\rho} = \frac{P \hat{\rho}_0 P}{\mathrm{tr}\,[P \hat{\rho}_0 P]} \,\mathrm{tr}\,[P\rho(0)] + \frac{Q \hat{\rho}_0 Q}{\mathrm{tr}\,[Q \hat{\rho}_0 Q]} \,\mathrm{tr}[Q\rho(0)], \tag{3.67}$$

[2]The commutant of $\mathcal{V}_i$ corresponds to the commutant of Σ_i; indeed, it is always possible to express Σ_i as a linear combination of $\mathcal{V}_i$ due to the invertibility of $\mathcal{A}$.

where $\hat{\rho}_0$ is a stationary state with all eigenvalues different from zero, thanks to which we can construct all the family of equilibrium states. We call such state *faithful state*.

From (3.67), it becomes clear why τ is a constant of motion; indeed, it is even clearer why τ is a constant of motion, as a matter of fact P commutes with all the Kraus operators of the evolution (since P is precisely build from the commutant algebra of the latter); hence P itself is a constant of motion, from which we recover

$$\mathrm{tr}\left(\Lambda_t[\rho(0)]P\right) = \mathrm{tr}\left(\rho(0)\Lambda_t^*[P]\right) = \mathrm{tr}(\rho(0)P) = \frac{1}{4}(1-\tau) \qquad \forall t,\, \forall \rho(0), \tag{3.68}$$

where Λ_t is the dynamical map related to the differential evolution we are currently analysing.

For the dynamics under exam, one faithful state is the following[3]

$$\hat{\rho}_0 = \frac{1}{4}\left[\mathbb{I}\otimes\mathbb{I} + M\Sigma_3 - N\left(S_{11} - S_{22}\right) + RS_{33}\right], \tag{3.69}$$

where

$$\begin{aligned} M &= \frac{2B}{\lambda_1+\lambda_2}, \\ N &= \frac{(\lambda_1-\lambda_2)\,B^2}{2\left(\lambda_1+\lambda_2\right)\left(\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3\right)}, \\ R &= \frac{(\lambda_1+\lambda_2+4\lambda_3)\,B^2}{2\left(\lambda_1+\lambda_2\right)\left(\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3\right)}\ , \end{aligned} \tag{3.70}$$

where $\lambda_{1,2,3}$ are the eigenvalues of the symmetric part A of $\mathcal{A}$, while B is the only non-vanishing entry of its antisymmetric part. We notice that the state in (3.69) is an X-state; indeed, it amounts to a particular form of the Fano decomposition of X-states (see equation (C.11) in Appendix C).

By inserting (3.69) into (3.67), all stationary states result to be of the symmetric form (3.59) with contributions

$$\begin{aligned} \hat{\rho}_3 &= \frac{3+\tau}{3+2R}M, \\ \hat{\rho}_{11} &= \frac{(1+2N)\tau + 2(3N-R)}{2(3+2R)}, \\ \hat{\rho}_{22} &= \frac{(1-2N)\tau - 2(3N+R)}{2(3+2R)}, \\ \hat{\rho}_{33} &= \frac{4R+(1+2R)\tau}{2(3+2R)}. \end{aligned} \tag{3.71}$$

Again then, these non-null components in the Fano decomposition identify the stationary states as X-states (C.11).

It is therefore particularly simple to evaluate the concurrence of such family, as illustrated in Section 3.1.2:

$$\mathcal{C}[\hat{\rho}] = \max\left\{\frac{(2+\Delta)}{2(3+2R)}\left[\frac{4R-3\Delta}{2+\Delta} - \tau\right], 0\right\}, \tag{3.72}$$

[3]Here we do not spend too much time in showing how the asymptotic manifold has been found, in the last Chapter of this thesis it is given the full analysis of a more general dynamics which can be easily reduced to the one of this Chapter.

where

$$\Delta = \left[(1-2R)^2 + 4\left(2R - M^2\right)\right]^{1/2}. \tag{3.73}$$

Therefore, the upper bound

$$\tau < \frac{5R^2 - 3}{3 - R^2}, \tag{3.74}$$

guarantees that initial entanglement is preserved in the long-time regime; indeed, τ is a constant of motion that depends on the initial state only, while R brings information about the dissipative dynamics.

In the preceding analysis we have only considered a purely dissipative generator without Hamiltonian contributions; during our work we were able to demonstrate that the previous preliminary result holds true even if we add a weak-coupling limit consistent Hamiltonian term; namely, the asymptotic manifold does not change with such an insertion. It is a property of this particular dynamics, as we are about to show.

Let us firstly consider the following theorem:

Theorem 3.2. Given a dynamics in GKSL form

$$\mathbb{L} := \mathbb{H} + \mathbb{D}, \tag{3.75}$$

where $\mathbb{H}$ is the unitary evolution and $\mathbb{D}$ is the dissipator; given a state ρ_∞ such that $\mathbb{D}[\rho_\infty] = 0$; *if* both of the following properties are true

- $[\mathbb{H}, \mathbb{D}] = 0$,
- $\dim(\ker \mathbb{D}) = 1$,

then ρ_∞ is an asymptotic state of the full dynamics $\mathbb{L}[\rho_\infty] = 0$.

Proof. We have

$$\mathbb{D}[\mathbb{H}[\rho_\infty]] = 0 \tag{3.76}$$

because $[\mathbb{H}, \mathbb{D}] = 0$. If the kernel of the dissipator is 1-dimensional, then the action of $\mathbb{H}$ on ρ_∞ has to be proportional to ρ_∞ itself, namely $\mathbb{H}[\rho_\infty] = \lambda\rho_\infty$. Given

$$\mathrm{tr}[\mathbb{H}[\rho_\infty]] = \mathrm{tr}(\lambda\rho_\infty) = \lambda\,\mathrm{tr}(\rho_\infty) = \lambda \tag{3.77}$$

and considering that the trace of a commutator is always null, $\mathrm{tr}[\mathbb{H}[\rho_\infty]] = 0$, we recover $\lambda = 0$; thus, $\mathbb{L}[\rho_\infty] = 0$. □

Our dynamics surely fulfils the first bullet point in the previous theorem. Indeed, as seen in Remark 2.7, the Hamiltonian contribution must commute with the dissipator if the Hamiltonian part is obtained through a weak coupling limit, which is our case. An example of a proper Hamiltonian evolution it is given by $\mathbb{H}[\,\cdot\,] = -i\,[\Sigma_3, \cdot\,]$ (the most general unitary part commuting with the dissipator is given in Appendix C.1).

For what concerns the dimension of the kernel of the dissipator, we need to give a more detailed insight of how the faithful state in (3.69) has been found[4].

[4] Again, the full analysis can be found in the last Chapter of the thesis.

To do so, it is useful to introduce the Bell states:

$$|\psi_1\rangle := \frac{|00\rangle + |11\rangle}{\sqrt{2}}, \quad |\psi_2\rangle := \frac{|01\rangle + |10\rangle}{\sqrt{2}}, \quad |\psi_3\rangle := \frac{|00\rangle - |11\rangle}{\sqrt{2}}, \quad |\psi_4\rangle := \frac{|01\rangle - |10\rangle}{\sqrt{2}}. \tag{3.78}$$

Considering the projection introduced in (3.66), we notice the following correspondence

$$P = \frac{1}{4}\left(\mathbb{I} - \frac{S}{2}\right) = |\psi_4\rangle\langle\psi_4|. \tag{3.79}$$

Now, considering (3.67), we use a faithful state as initial state $\rho(0) = \hat{\rho}_0$ (we are considering faithful state with generic matrix entries ρ_{ij}, not the one in (3.69)), we find:

$$\hat{\rho} = P\hat{\rho}_0 P + Q\hat{\rho}_0 Q = \hat{\rho}_0. \tag{3.80}$$

On the Bell basis, we have $P = diag\{0,0,0,1\}$ and $Q = diag\{1,1,1,0\}$, hence:

$$Q\hat{\rho}_0 Q = \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} & 0 \\ \rho_{12}^* & \rho_{22} & \rho_{23} & 0 \\ \rho_{13}^* & \rho_{23}^* & \rho_{33} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad P\hat{\rho}_0 P = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho_{44} \end{pmatrix}. \tag{3.81}$$

As a consequence, the faithful state takes following shape:

$$\hat{\rho}_0 = \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} & 0 \\ \rho_{12}^* & \rho_{22} & \rho_{23} & 0 \\ \rho_{13}^* & \rho_{23}^* & \rho_{33} & 0 \\ 0 & 0 & 0 & \rho_{44} \end{pmatrix} = \begin{pmatrix} & & & 0 \\ & \rho^Q & & 0 \\ & & & 0 \\ 0 & 0 & 0 & \rho_{44} \end{pmatrix}. \tag{3.82}$$

As a consequence, we can reduce the investigation of the dynamics just to the 3×3 subspace characterized by the projection Q, because the matrix element ρ_{44} will be then uniquely determined by the normalization of the state.

Moreover, in this 3×3 subspace, we can restrict the action of the operators $\{\Sigma_i,\, S_{ij}\}$ onto the subspace spanned by the Bell states $\{\psi_1, \psi_2, \psi_3\}$ which is indeed mapped into itself by $\{\Sigma_i,\, S_{ij}\}$. We shall denote by $\{\Sigma_i^Q,\, S_{ij}^Q\}$ these restrictions.

We can thus also restrict the master equation (3.56) to states supported by the same subspace, where it becomes:

$$\frac{\partial \rho_Q(t)}{\partial t} = \sum_{i,j=1}^{3} \mathcal{A}_{ij}\left[\Sigma_j^Q\, \rho_Q(t)\, \Sigma_i^Q - \frac{1}{2}\{\Sigma_i^Q \Sigma_j^Q,\, \rho_Q(t)\}\right], \tag{3.83}$$

where ρ_Q is the 3×3 sub-matrix in (3.82).

Direct computation shows that it is precisely in this 3×3 subspace that the kernel of the dissipator is 1-dimensional. Then, if we add a proper Hamiltonian evolution reduced to the this subspace, i.e. $\mathbb{H}^Q\left[\rho_Q(t)\right] = -i\left[\Sigma_3^Q, \rho_Q(t)\right]$, the 3×3 faithful state will not change. As we before mentioned, the ρ_{44} component of the full faithful state in univocally determined by the $\rho_Q(t)$ submatrix.

We have analysed the behaviour of quantum entanglement against dissipation, our aim now is to adopt a quantum control protocol in order to drive the evolution of the two-qubit system into an entangled asymptotic state. To do so, some other notions and middle steps have to be addressed, such as stochastic processes and quantum filtering. We will discuss stochastic processes (classic and quantum both) in the next Chapter, which will be fundamental to understand the next topics of quantum filtering and quantum feedback.

Chapter 4

Stochastic processes

In Chapter 1 we illustrated some basic notions of classical probability theory in the case of discrete random variables; now we are interested in stochastic processes consisting of one-parameter (typically a discrete or continuous time-parameter) families of random variables. In the following, we will focus upon *continuous random variables* and introduce two particularly relevant classical stochastic processes, namely counting (or Poisson) process and Brownian motion (or Wiener process). They will be used in the subsequent Chapter which deals with quantum control protocols whereby monitoring processes inject classic stochasticity into the system.

As an introduction the mathematical techniques of classical stochastic processes we will also provide some basic notions of stochastic calculus [3, 14, 16].

In the second part of the Chapter we will switch to the quantum realm introducing quantum stochastic processes [17, 30] and the corresponding quantum stochastic calculus, concentrating upon the quantum counterparts of Poisson and Wiener processes.

4.1 Classical stochastic processes

Given the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a *continuous* random variable X with outcomes $x \in \mathbb{R}$ (it can also be a vector function, rather than a simple function), the expressions for mean values and variances become:

1. The expectation value: $\langle X \rangle = \int_\Omega X \, d\mathbb{P}$.
2. The variance: $V(X) = \int_\Omega |X - \langle X \rangle|^2 \, d\mathbb{P} = \langle |X - \langle X \rangle|^2 \rangle = \langle |X|^2 \rangle - |\langle X \rangle|^2$.

In the case of discrete random variables, the integration measures are Dirac deltas and the integrations become sums.

We consider an absolutely continuous probability distribution $d\mathbb{P}$ that can be written as $f_X(x)dx$, where $f_X : \mathbb{R} \mapsto \mathbb{R}^+$ is the so-called *probability density* associated to the stochastic variable X. Namely, given a subset $A \subset \Omega$, the probability that X takes its values in A is given by:

$$\mathbb{P}(X \in A) = \int_A f_X(x) \, dx \ , \tag{4.1}$$

whereby

$$\langle X\rangle = \int_{\mathbb{R}} x\, f_X(x)\, dx\,, \qquad V(X) = \int_{\mathbb{R}} |x|^2\, f_X(x)\, dx\ - |\langle X\rangle|^2. \tag{4.2}$$

We can now introduce the notion of stochastic process.

A collection of random variables $\{X(t)|t \geq 0\}$ is a *classical stochastic process* and for each point $\omega \in \Omega$, the mapping $t \mapsto X(t,\omega)$ is the corresponding *sample path*, t being possibly a time-parameter. The sample path will of course be different for every realization of the experiment (for example, the motion of a speck of dust on a pond).

To be more general and in order to introduce some mathematical concepts that will turn out useful when discussing filtering protocols, let us define a classical stochastic process in the following way [3]:

Definition 4.1. A classical stochastic process is a collection $(\Omega, \mathcal{F}, \mathcal{F}(t), X(t), \mathbb{P})_{t\in T}$, where:

i. $(\Omega, \mathcal{F}, \mathbb{P})$ is a classical probability space as defined in Section 1.1.

ii. T is the total time, hence a subset of $\mathbb{R}^+$.

iii. $\mathcal{F}(t)$ is called *filtration* and amounts to an increasing family of sub-σ-algebras of $\mathcal{F}$ such that $\mathcal{F}(s) \subset \mathcal{F}(t)$ for $t \geq s$. It basically models the amount of known information up to a time $t \leq T$; in other words, $\mathcal{F}(t)$ contains all the random events occurred up to time t.

iv. $X(t)$ is a collection of random variables (for example the position of a particles driven by Brownian motion) with the property of being a *non-anticipating* or *adapted* function, which means that $X(t)$ is $\mathcal{F}(t)$-measurable. This requirement is needed to define the stochastic integration (We will see the operative meaning of this statement later).

To better understand the concept of filtration, suppose we want to study the behaviour of $X(t)$. We can evaluate the probability of obtaining a value for $X(t)$ in a certain set via (4.1). Or we may need to evaluate the conditional probability with respect to another outcome as showed in Section 1.2. Or, since we are considering a stochastic processes (i.e. a suitable dynamics) rather than a single random variable (i.e. a coin toss), we might want to study the time-evolution of $X(t)$. It is then more efficient to gather a string of outcomes occurred before time t with respect to which we can evaluate the conditional probability of obtaining a certain value for $X(t)$. So the filtration is the algebra of all the events that have already happened until the time t which may affect the outcome of $X(t)$. As shown for the quantum scenario, also in classical probability theory we can indeed condition with respect to a whole algebra.

The notion of filtration will become useful when discussing quantum filtering; indeed, the main idea of such a procedure is to study the evolution of an observable $\hat{X}(t)$ of the system driven by noise when we constantly collect information about the system by measuring commuting observables of a suitable environment. The commutative algebra of the environmental operators one measures corresponds to a filtration (it is an increasing family because we perform successive measurements in time); then we evaluate the expectation value of the system observable $\hat{X}(t)$ conditioned by the whole algebra related to the measurements one has performed on the environment.

What stated for classical stochastic processes extends to quantum stochastic processes. The only condition being that in order to have a correct quantum filtration the conditioning sub-algebra $\mathcal{F}(t)$ must be commutative, otherwise conditional expectations can not be consistently constructed. We will see how this commutative algebra arises from the string of measurements in the section devoted to quantum filtering.

We now introduce two typical classical stochastic processes, namely the counting process (or Poisson process) and the Brownian motion (or Wiener process), and then outline the basic concepts of classical stochastic calculus. After that we will do the same for quantum stochastic processes and quantum noise, for which we will introduce the quantum Wiener process and the quantum Poisson process, which provide common settings for quantum filtering.

4.1.1 Counting process

A counting process is a continuous time stochastic process consisting of discrete random variables, usually denoted as $\{N(t)|t \geq 0\}$, where $N(t)$ corresponds to the number of independent events that have occurred up to time t. Its values are non-decreasing positive integers, namely $N(t) \geq N(s) \geq 0$ for $t \geq s$.

Definition 4.2. A counting process satisfies:

1. $N(0) = 0$.
2. The process has independent increments; namely, for all times $0 < t_1 < t_2 < \cdots < t_n$ the random variables $N(t_2) - N(t_1), \cdots, N(t_n) - N(t_{n-1})$ are statistically independent from each other.
3. The probability of having a certain number n of events at the time t is given by the Poisson distribution

$$\mathbb{P}(N(t) = n) = \frac{(\lambda t)^n}{n!} e^{-\lambda t} \,. \tag{4.3}$$

Due to the last property, counting processes are often referred to as Poisson processes. Hence, we have:

Properties 4.1.

$$i.)\ \langle N(t)\rangle = \lambda t \ , \quad ii.)\ \langle N^2(t)\rangle = \langle N(t)\rangle^2 + \langle N(t)\rangle \ , \quad iii.)\ \langle dN^2(t)\rangle = \langle dN(t)\rangle \ .$$

Notice that the differential equality has to be understood at leading order and that it is often written as $dN^2(t) = dN(t)$ without mean values.

Proof.

i.

$$\langle N(t)\rangle = \sum_{n=0}^{\infty} n \frac{(\lambda t)^n}{n!} e^{-\lambda t} = \sum_{n=1}^{\infty} \frac{(\lambda t)^n}{(n-1)!} e^{-\lambda t} = \lambda t\, e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} = \lambda t \,.$$

ii.

$$\langle N^2(t)\rangle = \sum_{n=0}^{\infty} n^2 \frac{(\lambda t)^n}{n!} e^{-\lambda t} = \underbrace{\sum_{n=0}^{\infty} n \frac{(\lambda t)^n}{(n)!} e^{-\lambda t}}_{\lambda t} + \underbrace{\sum_{n=0}^{\infty} n(n-1) \frac{(\lambda t)^n}{(n)!} e^{-\lambda t}}_{(\lambda t)^2} .$$

iii. taking the mean value of $dN^2(t) = N^2(t+dt) - N^2(t)$ one gets

$$\langle N^2(t+dt) - N^2(t)\rangle = \lambda(t+dt) + \lambda^2(t+dt)^2 - \lambda t - (\lambda t)^2 \simeq \lambda dt = \langle dN(t)\rangle ,$$

at leading order with respect to dt.

□

4.1.2 Classical Wiener process

The Wiener process $\{W(t) : t \geq 0\}$ is a continuous-time stochastic process consisting of continuous random variables: typically the positions at subsequent times of a particle subjected to Brownian motion. It is defined as follows [14].

Definition 4.3. A Wiener process $W(t)$ is defined by

1. $W(0) = 0$.
2. The process has independent increments, for all times $0 < t_1 < t_2 < \cdots < t_n$ the random variables $W(t_1)$, $W(t_2) - W(t_1)$, $W(t_n) - W(t_{n-1})$ are statistically independent from each other.
3. The probability density associated to $W(t)$ is a Gaussian centred in zero

$$f_{W(t)}(x) = \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} , \qquad \mathbb{P}(a \leq W(t) \leq b) = \frac{1}{\sqrt{2\pi t}} \int_a^b e^{-\frac{x^2}{2t}} dx .$$

Properties 4.2. A Wiener process is characterized by the following properties:

$$i.)\ \langle W(t)\rangle = 0 ,\ \langle W^2(t)\rangle = t , \quad ii.)\ \langle W(t)W(s)\rangle = \min\{t,s\} , \quad iii.)\ \langle dW^2\rangle = dt .$$

Also this latter expression has to be understood at leading order and the mean value is usually omitted. The quantity dW is usually called *Wiener increment*, it is the infinitesimal increment $W(t+dt) - W(t)$ and its meaning is better understood analysing its role in the stochastic integration, which we will address in the following section.

Proof.

i. From Gaussian integration:

$$\begin{aligned}
\langle W(t)\rangle &= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} x\, e^{-\frac{x^2}{2t}}\, dx = 0, \\
\langle W^2(t)\rangle &= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} x^2 e^{-\frac{x^2}{2t}}\, dx = \sqrt{\frac{2}{\pi t}} \int_0^{+\infty} x^2 e^{-\frac{x^2}{2t}}\, dx \\
&= \frac{2t}{\sqrt{\pi}} \int_0^{+\infty} \sqrt{u}\, e^{-u}\, du = \frac{2t}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}\right) = t,
\end{aligned}$$

where $\Gamma(z)$ is the Euler function. Notice that the variance of $W(t)$ satisfies $V(W(t)) = \langle W^2(t)\rangle - \langle W(t)\rangle^2 = \langle W^2(t)\rangle$.

$$\begin{aligned}\langle W(t)W(s)\rangle &= \langle (W(s) + W(t) - W(s))W(s)\rangle = \\ &= \langle W^2(s)\rangle + \langle W(t) - W(s)\rangle \langle W(s)\rangle = \quad \text{due to independent increments} \\ &= s + \underbrace{\langle W(t) - W(s)\rangle}_{0} \underbrace{\langle W(s)\rangle}_{0} = s\,.\end{aligned}$$

ii. This property can be proved in different ways (see [16] for a rigorous proof): notice that

$$\begin{aligned}dt = \langle (W(t+dt))^2 - (W(t))^2\rangle &= \langle (W(t) + dW(t))^2 - (W(t))^2\rangle = \\ &= \langle W(t)dW(t) + (dW(t))^2\rangle \langle (dW(t))^2\rangle\,. \end{aligned} \tag{4.4}$$

Indeed, $\langle W(t)dW(t)\rangle = \langle W(t)\rangle\langle dW(t)\rangle = \langle W(t)\rangle = 0$.

The latter relation shows the connection between the square of the average infinitesimal Wiener increment and the time differential.

□

Wiener processes are also related to stochasticity given by *white noise*: the white noise is an ideal random signal $\xi(t)$ having constant amplitude for every frequency and uncorrelated values at all different times, apart from equal times where correlations diverge:

$$\langle \xi(t)\rangle = 0, \qquad \langle \xi(t)\xi(t')\rangle = \delta(t - t'). \tag{4.5}$$

These features follow from setting $\xi(t) := dW/dt$, as can be demonstrated using Definition 4.3.

4.2 Classical stochastic calculus

Classical stochastic calculus is usually built in relation to Brownian motion, namely for dynamics driven by white noise. In general there can be other random behaviours, such as the random jumps described by the Poisson distribution, as we mentioned in Section 4.1.1. Here we will focus on the so-called *Itô's* and *Stratonovich stochastic calculus* [3], two different mathematical formalisms to approach the differentiability issue regarding stochastic processes.

At the beginning of the present Chapter, we introduced stochastic processes as collections of time-dependent random variables. The dynamics of a stochastic process $X(t)$ driven by white noise is written as [16]

$$X(r) := X(s) + \int_s^r F(X(t),\, t)\, dt + \int_s^r G(X(t),\, t)\, dW\,, \tag{4.6}$$

where $F(X(t),\, t)$ and $G(X(t),\, t)$ are in general continuous functions of the stochastic process $X(t)$ and of time. Moreover, they have to be non-anticipating function (namely they are statistically independent from the increment $W(t') - W(t)$ for $t' \geq t$); dW is a Wiener increment (as introduced in Section 4.1.2). The stochasticity of the evolution is given by the very last term, which is a stochastic integral (in general there can be a summation over different time integrals and stochastic integrals).

Remark 4.1. In general, stochastic processes are vectorial (multi-variate) with more components, each one driven by a different statistically independent white-noise contributions. In this case, one deals with a vector of stochastic variables, $\boldsymbol{X}(t) \in \mathbb{R}^n$ that obeys

$$\boldsymbol{X}(r) = \boldsymbol{X}(s) + \int_s^r \boldsymbol{F}(\boldsymbol{X}(t),\, t)\, dt + \int_s^r G(\boldsymbol{X}(t),\, t)\, d\boldsymbol{W}\,, \tag{4.7}$$

where $\boldsymbol{F}(\boldsymbol{X}(t),\, t) : \mathbb{R}^n \mapsto \mathbb{R}^n$ is a smooth vector field; $G(\boldsymbol{X}(t),\, t) : \mathbb{R}^n \mapsto \mathcal{M}_{n,m}$, where $\mathcal{M}_{n,m}$ is the space of the $n \times m$ matrices; $\boldsymbol{W}(t)$ is the m-dimensional Wiener process.

In the following we will reduce our analysis to one-component stochastic processes.

We define the corresponding stochastic differential equation (SDE) of (4.6) as:

$$dX(t) = F(X(t),\, t)\, dt + G(X(t),\, t)\, dW\,. \tag{4.8}$$

Stochastic processes described by the two previous relations are called *Itô's stochastic processes*, in the sense that Itô's rules have to be adopted (we will introduce Stratonovich processes later in Section 4.2.1).

We will now see how to formally handle dW-integrals as the one in (4.6), after that we will see the stochastic differentiation rules.

4.2.1 Stochastic integrals

Let's see how to treat expressions as [14]:

$$\int_0^T G(t)\, dW\,, \tag{4.9}$$

where dW is a Wiener increment.

This is not an ordinary integral; according to the characteristics of the function $G(t)$, there are different properties that can be exploited to recover an ordinary integral. In stochastic calculus it is very common to evaluate mean values in order to get rid of the noise that causes the stochasticity. In the following we will see how to evaluate the expectation value of the previous integral and how to write it as an ordinary time-integration when $G(t)$ is a deterministic continuous function and when $G(t)$ is a non-anticipating stochastic process itself.

$G(t)$ is a deterministic and continuously differentiable function

Suppose $G(t) :\, [0,1] \mapsto \mathbb{R}$, with $G(0) = G(1) = 0$, G is a deterministic time-dependent function, not a stochastic process, then:

$$1.)\ \int_0^1 G(t)\, dW = -\int_0^1 G'(t) W(t)\, dt\ , \quad 2.)\ \langle \int_0^1 G(t)\, dW \rangle = 0\ ,$$
$$3.)\ \Big\langle \left(\int_0^1 G(t)\, dW \right)^2 \Big\rangle = \int_0^1 G^2(t)\, dt\ .$$

Proof. :

1. The first property is just the result of an integration by part and a consequence of the assumptions $G(0) = G(1) = 0$:
   $\int_0^1 G(t)\, dW = \int_0^1 G(t)\, \frac{dW}{dt}\, dt = -\int_0^1 G'(t)\, W(t)\, dt$.
2. $\langle \int_0^1 G(t)\, dW \rangle = -\int_0^1 G'(t) \langle W(t) \rangle\, dt = 0 \quad$ because $\langle W(t) \rangle = 0$.
3. $\left\langle \left(\int_0^1 G(t) dW \right)^2 \right\rangle = \langle \int_0^1 G'(t) W(t)\, dt \int_0^1 G'(s) W(s)\, ds \rangle = \int_0^1 G'(t) G'(s) \underbrace{\langle W(t) W(s) \rangle}_{\min\{s,t\}}\, dt\, ds =$
   $= \int_0^1 G'(t) \left(\int_0^t s\, G'(s)\, ds + \int_t^1 t\, G'(s)\, ds \right)\, dt = \int_0^1 G'(t) \left(t\, G(t) - \int_0^t G(s)\, ds - t\, G(t) \right)\, dt =$
   $= \int_0^1 G'(t) \left(-\int_0^t G(s)\, ds \right)\, dt = \int_0^1 G^2(t)\, dt.$

□

$G(t)$ is a stochastic process

Let us first consider the case in which $G(t)$ is a *stochastic step process*, namely, it there exists a partition $P = \{0 = t_0 < t_1 < \cdots < t_m = T\}$, such that

$$G(t) = G_k \quad \text{for } t_k \le t < t_{k+1} \quad (k = 0, \cdots, m-1)\,. \tag{4.10}$$

Since we required $G(t)$ to be a non-anticipating function, we can define a stochastic integral as

$$\int_0^T G(t)\, dW = \sum_{k=0}^{m-1} G_k \left(W(t_{k+1}) - W(t_k) \right)\,. \tag{4.11}$$

This is also called *Itô's integral*. It has the following properties:

1. Linearity: $\int_0^T (a\, G(t) + b\, H(t))\, dW = a \int_0^T G(t)\, dW + b \int_0^T H(t)\, dW$, where $G(t)$ and $H(t)$ are two stochastic step processes.
2. $\langle \int_0^T G(t)\, dW \rangle = 0$.
3. $\left\langle \left(\int_0^T G(t)\, dW \right)^2 \right\rangle = \langle \int_0^T G^2(t)\, dt \rangle$.
4. $\langle \int_0^T G(t)\, dW \int_0^T H(t)\, dW \rangle = \langle \int_0^T G(t) H(t)\, dt \rangle$.

To extend these results to non-step function, we can approximate any process $G(t)$ with a step function G_n, such that

$$\left\langle \int_0^T |G(t) - G_n|^2\, dt \right\rangle \mapsto 0\,, \tag{4.12}$$

hence:

$$\int_0^t G(t)\, dW = \lim_{n\to\infty} \int_0^T G_n\, dW\,. \tag{4.13}$$

The previous list of properties hold as well for non-step functions.

Rather than as in (4.11), there is another way to define the stochastic integral for a stochastic variable driven by white noise, which is called *Stratonovich integral* (it is usually specified with

the symbol $\circ$ inside the integral or with **(S)** before the integral; while Itô's integration has sometimes the symbol **(I)** before the integral):

$$\int_0^T G(t) \circ dW = \sum_{k=0}^{m-1} \frac{1}{2}(G_k + G_{k+1}) \left(W(t_{k+1}) - W(t_k)\right) . \tag{4.14}$$

The main difference with Itô's integral is that the integrand function is not independent from the increment, therefore when we take the mean value of the integral, the average product does not factorize.

One may choose either Itô's or Stratonovich formalism: while the latter follows the usual differentiation rules, but it is less easy in evaluating mean values, the contrary holds for Itô's calculus.

Summarizing, (4.6) is a Stratonovich stochastic process if the dW-integral is defined in Stratonovich sense rather than in Itô's one.

It is possible to switch from one to the other formalism. Indeed, they differ for just one term: we will see how to achieve that after the discussion about stochastic differential equations.

In the following we will concentrate on Itô's differentiation rule.

4.2.2 Stochastic differential equations

We will now devote our attention to the differential formulation of a stochastic process, introducing the main differentiation rules.

As we mentioned in the introduction to this section, this kind of stochastic process is also called Itô's process.

Since we showed in Section 4.1.2 that $dW \sim \sqrt{dt}$, the differentiation of a stochastic variable is not as in the common calculus. As a matter of fact to perform the differentiation all the contributes up to second order in the Wiener increments must be kept and at the end the following *Itô's rules* have to be applied, otherwise we risk to lose the dW^2 contribute which are actually of the first order of time:

$$dW^2 = dt\,, \quad dW\,dt = 0. \tag{4.15}$$

We remind that the previous are not real equalities (the first one is true only for mean values, the second one by neglecting terms of the second order of time), they are just rules to apply for stochastic differentiation.

Chain rule Given

$$dX(t) = F(X(t),\, t)\, dt + G(X(t),\, t)\, dW\,, \tag{4.16}$$

then we use Taylor expansion to evaluate the differential of the function $f(X(t))$

$$\begin{aligned} df(X(t)) &= f(X(t) + dX(t)) - f(X(t)) = \\ &= f'(X(t))\, dX(t) + \frac{1}{2} f''(X(t))\, dX^2(t) + \cdots = \\ &= f'(X(t))\, F(X(t),\, t)\, dt + f'(X(t))\, G(X(t),\, t)\, dW + \frac{1}{2} f''(X(t))\, G^2(X(t),\, t)\, dt. \end{aligned} \tag{4.17}$$

As we did before, we kept all the terms until the second order of time and then we applied Itô's rule.

Product rule Given two stochastic processes (we omit the time-dependence for sake of simplicity)

$$dX = F\,dt + G\,dW\,, \quad dY = H\,dt + L\,dW\,, \tag{4.18}$$

it follows that

$$\begin{aligned} d(XY) &= Y\,dX + X\,dY + dX\,dY = \\ &= Y\,dX + X\,dY + G\,L\,dt\,, \end{aligned} \tag{4.19}$$

where the last term, which is of second order in dW and would not be present in standard differentiation, is called Itô's term.

From Itô to Stratonovich

$$\textbf{(S)} \int_0^T G(X(t),t)\,dW = \textbf{(I)} \int_0^T G(X(t),t)\,dW + \frac{1}{2}\int_0^T \frac{\partial G(X(t),t)}{\partial X(t)}\,dt\,. \tag{4.20}$$

Where $X(t)$ satisfies (4.8). So

$$\text{Itô's SDE:} \qquad dX = F\,dt + G\,dW, \tag{4.21}$$

$$\text{Stratonovich SDE:} \qquad dX = \left(F - \frac{1}{2}G\frac{\partial G}{\partial X}\right)\,dt + G\,dW\,. \tag{4.22}$$

Proof. We consider $X(t) = F(X(t),t)\,dt + G(X(t),t)\,dW$, then the Stratonovich integral of the function $\beta(X(t))$ is:

$$\textbf{(S)} \int_0^T \beta(X(t))\,dW(t) = \sum_k \frac{\beta(X(t_{k+1})) + \beta(X(t_k))}{2}\left(W(t_{k+1}) - X(t_k)\right)\,. \tag{4.23}$$

Now

$$\beta(X(t_{k+1})) = \beta(X(t_k)) + \Delta\beta(X(t_k))\,. \tag{4.24}$$

Just to simplify the notation and to shorten the expressions, we consider $\Delta\beta$ as the infinitesimal $d\beta$ and we apply Itô's chain rule (4.17) to $d\beta$ (if we had kept the finite interval $\Delta\beta$ we would have find Δt and ΔW in the following expression):

$$d\beta(X(t)) = \frac{\partial\beta(X(t))}{\partial X(t)}\,F(X(t),t)\,dt + \frac{\partial\beta(X(t))}{\partial X(t)}\,G(X(t),t)\,dW + \frac{1}{2}\frac{\partial^2\beta(X(t))}{\partial X^2(t)}\,G^2(X(t),t)\,dt\,. \tag{4.25}$$

The first and last terms in the right hand side of the previous expression will not contribute once they are inserted in (4.23) because they will give rise to infinitesimals $dt\,dW$ and $dt\,dt$ that can be neglected. The only term remaining is the one depending only on dW, we write it in

its finite form ΔW and we substitute it in (4.24) and then in (4.23). In this way we find the Itô-Stratonovich correspondence:

$$\begin{aligned}
\textbf{(S)} \int_0^T \beta(X(t))\, dW(t) = \\
&= \sum_k \left(\beta(X(t_k)) + \frac{1}{2} \frac{\partial \beta(X(t))}{\partial X(t)} G(X(t_k), t_k) \left(W(t_{k+1}) - W(t_k) \right) \right) \left(W(t_{k+1}) - X(t_k) \right) = \\
&= \textbf{(I)} \int_0^T \beta(X(t))\, dW(t) + \frac{1}{2} \int_0^T \frac{\partial \beta(X(t))}{\partial X(t)} G(X(t), t)\, dt. \qquad (4.26)
\end{aligned}$$

□

4.3 Quantum stochastic processes

Stochastic processes in quantum mechanics typically concern quantum systems S in (weak) interaction with in an environment B which usually acts as a source of decoherence through dissipation and noise, irreversibility affecting the evolution of the observables of the system S.

The environment is usually assumed to be a thermal bath consisting of Bosons for which we will firstly introduce the symmetrized Fock space upon which we will base a short introduction to quantum stochastic processes and quantum stochastic calculus. Then, we will return to the dynamics of an open quantum system S immersed in a noisy environment, finally introducing the quantum stochastic master equation, which will be more thoroughly investigated in the next chapter.

4.3.1 Boson Fock space

The *Fock space* is the Hilbert space used to describe quantum systems with an undefined number of particles. If the particles are indistinguishable Bosons, then we restrict the Fock space to be the sub-space spanned by symmetrized states, namely the states which do not vary if particles are exchanged.

Quantum stochastic calculus is based on the following construction. One considers the Hilbert space of a single particle, $\mathcal{H} = L^2(\mathbb{R}^+)$, to consist of all the square integrable functions $f(t) \in \mathbb{C}$ (where $t \in \mathbb{R}^+$ is interpreted as time). Then, the Boson Fock space is the direct sum

$$\Gamma(\mathcal{H}) = \mathbb{C} \oplus \mathcal{H} \oplus \mathcal{S}\,\mathcal{H}^{\otimes 2} \oplus \cdots \oplus \mathcal{S}\,\mathcal{H}^{\otimes r} \oplus \cdots , \qquad (4.27)$$

where $\mathcal{S}$ is the operator which symmetrizes tensor products of single particle state vectors so that they remain invariant under particle exchange. Notice that the orthogonal subspace in the direct sum are n-Boson sectors and the $n = 0$ one corresponds to the one dimensional subspace ($\mathbb{C}$ spanned by the so-called vacuum vector $|\Omega_0\rangle$.

Given an orthonormal basis $\{|j\rangle\}_j$ in the single particle Hilbert space $\mathcal{H}$, one introduces creation and annihilation operators $b_k^\dagger, b_k$ such that

$$b_k|\Omega_0\rangle = 0 \ , \qquad b_k^\dagger|\Omega_0\rangle = |k\rangle \qquad \forall k$$

and satisfying the canonical commutation relations

$$\left[b_k, b_\ell^\dagger\right] = \delta_{k\ell} \ . \tag{4.28}$$

If the state $|n_1, n_2, \cdots, n_n\rangle$ describes a state with n_ℓ Bosons in the state $|\ell\rangle$, the action of $b_k, b_k^\dagger$ respectively removes and adds a particle in the k^{th} state:

$$b_k \, |n_1, n_2, \cdots, n_k, \cdots n_n\rangle = \sqrt{n_k - 1} \, |n_1, n_2, \cdots, n_k - 1, \cdots, n_n\rangle \tag{4.29}$$

$$b_k^\dagger \, |n_1, n_2, \cdots, n_k, \cdots, n_n\rangle = \sqrt{n_k} \, |n_1, n_2, \cdots, n_k + 1, \cdots n_n\rangle \, . \tag{4.30}$$

It proves convenient to pass from the discrete orthonormal basis to the continuous one [10, 31] provided by pseudo orthogonal vectors $|t\rangle$, $|s\rangle$ such that

$$\langle t|s\rangle = \delta(t-s) \ , \quad \int_0^{+\infty} \mathrm{d}t \, |t\rangle\langle t| = 1 \ , \tag{4.31}$$

where $\delta(t)$ denotes the Dirac delta, thus the corresponding annihilation and creation operators satisfying

$$b_t|\Omega_0\rangle = 0 \ , \ b_t^\dagger|\Omega_0\rangle = |t\rangle \quad \forall t \geq 0 \ , \qquad \left[b_t, b_s^\dagger\right] = \delta(t-s) \ . \tag{4.32}$$

Notice that, given a square-integrable function $|\alpha\rangle \in \mathcal{H}$ with discrete, respectively continuous Fourier coefficients $\alpha_j = \langle j|\alpha\rangle$, respectively $\alpha(t) = \langle t|\alpha\rangle$, the corresponding annihilation and creation operators are given by

$$b(\alpha) = \sum_j \alpha_j^* \, b_j = \int_0^{+\infty} \mathrm{d}t \, \alpha^*(t) \, b(t) \ , \quad \alpha^\dagger(f) = \sum_j \alpha_j \, b_j^\dagger = \int_0^{+\infty} \mathrm{d}t \, \alpha(t) \, b^\dagger(t) \ . \tag{4.33}$$

These relations indeed follow by acting on the vacuum state with the creation operators, while the canonical commutation relations can be generalized to

$$[b(\alpha), \, b^\dagger(\beta)] = \langle \alpha|\beta\rangle \ . \tag{4.34}$$

Some of the considerations in the following will make use of *exponential vectors*, namely non-normalized coherent states defined by

$$|\exp(\alpha)\rangle = \mathrm{e}^{b^\dagger(\alpha)} \, \mathrm{e}^{-b(\alpha)} |\Omega_0\rangle = \sum_{k=0}^{\infty} \frac{(b^\dagger(\alpha))^k}{k!} \, |\Omega_0\rangle \ . \tag{4.35}$$

The canonical commutation relations then yield

$$\langle \exp(\beta)|\exp(\alpha)\rangle = e^{\langle \beta|\alpha\rangle} \, , \quad b(t)|\exp(\beta)\rangle = \beta_t|\exp(\beta)\rangle \ . \tag{4.36}$$

When the Boson Fock space is used to model the environment in which a quantum system is immersed, the coupling of the latter to the former through $b(t)$ and $b^\dagger(t)$ will make them a source of *quantum white noise*, as we shall see.

In order to proceed further, we define the integrated version of $b(t)$ and $b^\dagger(t)$ and call them *annihilation and creation processes*:

$$B(t) = \int_0^t b(\tau) \, d\tau, \qquad B^\dagger(t) = \int_0^t b^\dagger(\tau) \, d\tau; \tag{4.37}$$

$$dB = b(t) \, dt, \qquad dB(t)^\dagger = b^\dagger(t) \, dt. \tag{4.38}$$

On the same footing, we also introduce the so-called *number process*:

$$\Lambda(t) = \int_0^t b^\dagger(\tau)\, b(\tau)\, d\tau \ , \qquad d\Lambda(t) = b^\dagger(t)\, b(t)\, dt. \tag{4.39}$$

All the previous ones are quantum stochastic processes: we will see that $\{B(t)|t \geq 0\}$ and $\{B^\dagger(t)|t \geq 0\}$ are the quantum counterpart of a Wiener process, while $\{\Lambda(t)|t \geq 0\}$ is the quantum version of a Poisson process.

Remark 4.2. In the multivariate case of a vectorial Wiener process $\mathbf{W}(t) = \{W_j(t)\}_{j=1}^n$, to each component stochastic process $W_j(t)$ one associates a single particle Hilbert space $\mathcal{H} = L^2(\mathbb{R}^+)$ and creation and annihilation operators $b_j(t)$, $b_j^\dagger(t)$. In practice, one can consider a unique quantum environment described by a Fock space over a single particle Hilbert space $L^2(\mathbb{R}^+) \otimes \mathbb{C}^n$ with n internal degrees of freedom.

4.3.2 Quantum stochastic evolution

Let's consider a quantum system S immersed in a Bosonic environment with Boson Fock space $\mathcal{H}_B = \Gamma\left(L^2(\mathbb{R}^+) \otimes \mathbb{C}\right)$. Given a "time" $t \geq 0$, we adopt the following time decomposition of the Hilbert space $\mathcal{H}$ of the compound system $S + B$:

$$\begin{aligned} &\mathcal{H}([0,t[) = \mathcal{H}_S \otimes \Gamma\left(L^2([0,t[) \otimes \mathbb{C}\right), \\ &\mathcal{H}([t,\infty[) = \Gamma\left(L^2([t,\infty[) \otimes \mathbb{C}\right). \end{aligned} \tag{4.40}$$

An *adaptive* process consists of a family of operators on the total Hilbert space $\mathcal{H}$ that can always be written as [31]

$$X(t) = X_t \otimes \mathbb{I}_{[t} \ , \tag{4.41}$$

where X_t acts on $\mathcal{H}([0,t[)$ and $\mathbb{I}_{[t}$ is the identity operator on $\mathcal{H}([t,\infty[)$. We can read the previous object as an operator that acts trivially in the future (after the time t) [20].

Then the family $\{X(t)|t \geq 0\}$ is a *quantum stochastic process* if $X(t)$ can be written as [31]:

$$X(t) = X(0) + \int_0^t \sum_{k=1}^n \left(D_k(s)\, d\Lambda_k(s) + E_k(s)\, dB^\dagger(s)_k + F_k(s)\, dB_k(s) + G_k(s)\, ds \right) , \tag{4.42}$$

$$X(0) = X_0 \otimes \mathbb{I}_B \, , \tag{4.43}$$

where $X_0 \in \mathcal{H}_S$ and $\mathbb{I}_B$ is the identity in the bath Hilbert space; the integrands $D_k(s)$, $E_k(s)$, $F_k(s)$, $G_k(s)$ are all adaptive processes; $dB_k^\dagger(s)$, $dB_k(s)$ are quantum Wiener processes obeying the quantum Itô's rules to be introduced and discussed later in this Chapter; while $d\Lambda_k(s)$ is a quantum Poisson process. Clearly, $dB_k^\dagger(s)\, , dB_k(s)$ and $d\Lambda_k(s)$ are the responsible of the stochasticity of the evolution.

In the following section we will see how to handle the previous quantum stochastic integrals.

4.3.3 Quantum stochastic calculus

As for the classical case, there are two main formalisms that can be used to construct a quantum stochastic calculus, depending on how the integration is defined [50].

- Itô's integration:

$$\textbf{(I)} \quad \int_0^t f(s)\, dB(s) = \lim_{n\to\infty} \sum_{i=0}^{n} f(t_i)\left[B(t_{i+1}) - B(t_i)\right] . \tag{4.44}$$

- Stratonovich integration:

$$\textbf{(S)} \quad \int_0^t f(s)\, dB(s) = \lim_{n\to\infty} \sum_{i=0}^{n} \frac{1}{2}\left[f(t_{i+1}) + f(t_i)\right]\left[B(t_{i+1}) - B(t_i)\right] . \tag{4.45}$$

In the above expressions, $f(t)$ is a *non-anticipating* or *adapted* stochastic process that depends only on $B(s)$ if $s < t$. In other words, $f(t_i)$ is statistically independent from the values taken by $B(t)$ in the future interval $[t_i, t_{i+1}]$ [14]. Hence, the main difference between the two integrations is that in Itô $f(t_i)$ is independent from the increment $B(t_{i+1}) - B(t_i)$, while in the Stratonovich formulation $f(t_{i+1}) + f(t_i)$ is not. We emphasize that the two approaches differ for a contribution $f(t_{i+1})$ that in Itô is absent.

However, it is to be remarked that the above ones are not ordinary integrals, so most of the time one needs to perform their averages rather than evaluate them. In the classical case there are some tricks to transform stochastic integrals to regular ones.

To show some examples of the different properties of the two formulations, assume the environment operators to act on the vacuum state, $|0\rangle$, such that $b(t)|\Omega_0\rangle = 0$, $\forall\, t \geq 0$, and consider the vacuum expectation values $\langle b(t)\rangle_0 := \langle\Omega_0|b(t)|\Omega_0\rangle$. Then, one obtains the conditions for having a quantum white noise:

$$\langle b(t)b^\dagger(t')\rangle_0 = \delta(t-t') , \quad \langle b^\dagger(t')b(t)\rangle_0 = 0 . \tag{4.46}$$

Also, from the previous relations it follows a useful tool to distinguish between Itô's and Stratonovich integration:

$$\langle\left[B(t) - B(t')\right]\left[B^\dagger(t) - B^\dagger(t')\right]\rangle_0 = |t-t'| , \tag{4.47}$$

$$\langle B(t) - B(t')\rangle_0 = \langle\left[B^\dagger(t) - B^\dagger(t')\right]^2\rangle_0 = \langle\left[B(t) - B(t')\right]^2\rangle_0 = 0 . \tag{4.48}$$

Proof. Let us begin with (4.47),

$$\langle\left[B(t) - B(t')\right]\left[B^\dagger(t) - B^\dagger(t')\right]\rangle_0 = \langle B(t)B^\dagger(t) - B(t)B^\dagger(t') - B(t')B^\dagger(t) + B(t')B^\dagger(t')\rangle_0 . \tag{4.49}$$

Let us consider the quantity $\langle B(t')B^\dagger(t)\rangle_0$:

$$\begin{aligned}\langle B(t')B^\dagger(t)\rangle_0 &= \Big\langle \int_0^{t'} b(\tau)\, d\tau \int_0^t b^\dagger(s)\, ds \Big\rangle_0 = \int_0^{t'}\int_0^t \langle b(\tau)b^\dagger(s)\rangle_0\, d\tau\, ds = \\ &= \int_0^{t'}\int_0^t \delta(\tau - s)\, d\tau\, ds ,\end{aligned} \tag{4.50}$$

We firstly evaluate the $d\tau$ integral (where we intend $\delta(\tau - s)$ in the sense of a distribution and we omit the test functions for simplicity):

$$f(s,t') := \int_0^{t'} \delta(\tau - s)\, d\tau = \begin{cases} 1 & \text{if } s \in [0,t'] \\ 0 & \text{otherwise} \end{cases} \quad . \tag{4.51}$$

hence:

$$\langle B(t')B^\dagger(t)\rangle_0 = \int_0^t f(s,t')\, ds = \begin{cases} \int_0^t 1 \cdot\, ds = t & \text{if } t \le t' \\ \int_0^{t'} 1 \cdot ds + \int_{t'}^t 0 \cdot ds = t' & \text{if } t > t' \end{cases} \quad . \tag{4.52}$$

Following the same procedure for all the terms in (4.49), we find:

$$\langle \left[B(t) - B(t')\right] \left[B^\dagger(t) - B^\dagger(t')\right] \rangle_0 = \begin{cases} t - t' & \text{if } t > t' \\ t' - t & \text{if } t \le t' \end{cases} = |t - t'| \,. \tag{4.53}$$

The demonstration of (4.48) works exactly as the previous one, thus we omit it. □

Remark 4.3. Notice that (4.47) and (4.48) change if the average is taken with respect to a state different from the vacuum.

Given them, the following characteristic difference between the two integrations arise:

$$\textbf{(I)}\ \langle \int_0^t dB(s)B^\dagger(s)\rangle_0 = \int_0^t \langle dB(s)\rangle_0 \langle B^\dagger(s)\rangle_0 = 0\,, \tag{4.54}$$

$$\textbf{(S)}\ \langle \int_0^t dB(s)B^\dagger(s)\rangle_0 = \frac{1}{2}|t|\,. \tag{4.55}$$

Itô result follows from the fact that the objects inside the integral are independent, therefore we can factorize the mean values (remarkably, the mean value of an Itô integral is always zero). Instead, Stratonovich result follows directly from the integration definition and the fact we are taking averages over the vacuum state.

We notice again that Itô and Stratonovich formulations usually differ for a term. It is in fact possible to find a relation between the two[1], we will show how in the following chapter with an explicit example.

We pass now to the discussion of the differentiation rules:

- Stratonovich differentiation obeys the conventional rules

$$\textbf{(S)}\ d(B(t)B^\dagger(t)) = dB(t)B^\dagger(t) + B(t)dB^\dagger(t)\,. \tag{4.56}$$

  That it is so follows from showing that the above rule is compatible with the Stratonovich integration rule. Indeed, if we integrate the previous expression and we use (4.55), we find:

$$B(t)B^\dagger(t')\big|_0^t = \textbf{(S)} \int_0^t dB(s)B^\dagger(s) + \textbf{(S)} \int_0^t B(s)dB^\dagger(s)\,, \tag{4.57}$$

  taking the averages over the vacuum:

$$|t| = \frac{1}{2}|t| + \frac{1}{2}|t|, \tag{4.58}$$

  as it should be.

[1] According to [19] they are related by Wick ordering.

- Itô differential is trickier, we use the usual differentiation $dX(t) = X(t+dt) - X(t)$ and we keep up to second order terms:

$$\textbf{(I)}\ d(X(t)\,Y(t)) = (X(t)+dX(t))(Y(t)+dY(t)) - XY = \\ = dX(t)\,Y(t) + X(t)\,dY(t) + dX(t)\,dY(t)\,, \tag{4.59}$$

where $X(t)$, $Y(t)$ are quantum stochastic processes (therefore they are function of $B(t)$, $B^\dagger(t)$ and the order of the differentials is relevant, unlike in the classical case). Then we have to apply the following *quantum Itô's rules* (which we will demonstrate at the end of this bullet point) to recover the correct differential:

$\times$	dt	dB	$dB^\dagger$	$d\Lambda$
dt	0	0	0	0
dB	0	0	dt	dB
$dB^\dagger$	0	0	0	0
$d\Lambda$	0	0	$dB^\dagger$	$d\Lambda$

So, if $X(t) = B(t)$ and $Y(t) = B^\dagger(t)$, we find:

$$\textbf{(I)}\ d(B(t)B^\dagger(t)) = dB(t)B^\dagger(t) + B(t)dB^\dagger(t) + dt\,. \tag{4.60}$$

Again, we show that this differentiation rule is in accordance with Itô integration by integrating the previous expression

$$B(t)B^\dagger(t)\big|_0^t = \int_0^t dB(s)B^\dagger(s) + \int_0^t B(s)dB^\dagger(s) + \int_0^t ds\,. \tag{4.61}$$

now taking the average over the vacuum, which for Itô it means that the first two integrals give zeri, and using (4.54), we find the consistent equality $|t| = |t|$.

We now proceed with the derivation of some of the quantum Itô's rules shown in the previous table.

Proof. The Itô's rules for the quantum Wiener process dB, $dB^\dagger$ are obtained by inspecting the matrix elements of the various products of stochastic differentials with respect to the exponential states in (4.36). For instance, the second property in (4.36) yields [20]:

$$\langle \exp(\alpha)|dB^\dagger(t)dB(t)|\exp(\beta)\rangle = \langle \exp(\alpha)|\exp(\beta)\rangle\,\alpha^*(t)\beta(t)(dt)^2. \tag{4.62}$$

Thus, $(dt)^2 = 0$ gives $dB^\dagger dB = 0$. Furthermore, from the following commutator (whose value can be recovered from (4.47))

$$\Big[B(t) - B(s), B^\dagger(t) - B^\dagger(s)\Big] = t - s\ , \quad \Rightarrow \quad \Delta B\,\Delta B^\dagger = \Delta B^\dagger \Delta B + \Delta t\,, \tag{4.63}$$

differentiating with respect to t and applying the previously proven $dB^\dagger dB = 0$, it follows that $dB(t)\,dB^\dagger(t) = dt$.

The multiplication of the quantum Wiener process with the infinitesimal dt can be neglected because euristically $dB, dB^\dagger \sim \sqrt{dt}$.

This proof shows an important feature; namely, in order to really have a quantum stochastic evolution, the state of the bath must be specified, because we recover Itô's rule from the action of the bath operators on the state [17].

To recover the Itô's rule for the Poisson increment $d\Lambda$, we can euristically use the following expression [17]:

$$d\Lambda = \frac{dB^\dagger dB}{dt}, \tag{4.64}$$

whose meaning can be understood from the definition in (4.39) and (4.38). Hence, using the other Itô's rules:

$$d\Lambda^2 = \frac{dB^\dagger dB\, dB^\dagger dB}{dt\, dt} = d\Lambda \tag{4.65}$$

□

We now have all the main tools necessary to introduce the quantum stochastic master equations. These will describe the evolution of a quantum system S immersed in a noisy environment, without the need of tracing the environment out as usually done in order to derive the master equations of open quantum systems, as shown in Section 2.2.

In the next Chapter we will make use of all the information gave in this Chapter to finally introduce quantum filtering and quantum feedback.

Chapter 5

Quantum control

In the previous chapter we provided some of the main tools needed to quantum stochastic calculus. In this chapter we will use the rules of quantum stochastic calculus with the ultimate goal of building the master equation of an open quantum system subject to measurement-based quantum feedback. The scope of this chapter is to provide the reader with the main techniques relative to measurement-based quantum feedback. We will devote the whole last chapter of this thesis to our original contributions regarding the use of this matter in dissipative entanglement generation and persistence.

We subdivide the presentation of measurement-based quantum feedback into three steps:

1. We will describe the quantum stochastic master equation (QSME) [17, 31], which is a master equation in which the degrees of freedom of the bath have not been traced out and are actually responsible for the stochastic behaviour, originated by quantum noise. In doing this, we will also introduce the quantum Langevin equation which describes the evolution of an observable rather than the states of the quantum systems.

2. We will see how the quantum Langevin equation changes when one constantly monitors the system via measurements on the environment, this step is known as *quantum filtering* [9, 20, 17, 44]. In particular, we will also show which kind of measurement we can perform in order to have a consistent quantum filter. At the end of this part, we will see how the QSME becomes an ordinary stochastic master equation (SME), in the sense that the noise terms will become classical stochastic processes rather than quantum ones. This classical reduction will be obtained by using the environment measurement outcomes to condition the state of the system.

3. In the last step we will provide instances of quantum feedback protocols [44, 49]: once we have the dynamics of a constantly monitored system, the environment measurement outcomes can be used to reinsert information in the open quantum system in order to externally drive its evolution. At the end, after some further manipulations, we will recover a Markovian master equation without any stochastic terms.

Concretely, we will firstly show the main differences between the usual Lindblad approach to open quantum dynamics and the QSME (and how to go from the latter to the former), then

we will show how to recover the QSME studying a particular setting typical of quantum optics, which is one of the field in which quantum control is mostly used.

5.1 Introduction to the quantum stochastic master equation

The irreversible evolution of an open quantum system state ρ_t without memory effects is usually described by a dynamical map Λ_t such that an initial state ρ changes as follows: $\rho \mapsto \rho_t = \Lambda_t(\rho) = \exp\{\mathcal{L}t\}\rho$. $\mathcal{L}$ is the generator of the dynamics, whose differential form is a master equation of the form

$$\dot{\rho}_t = \mathcal{L}[\rho_t] \, . \tag{5.1}$$

These maps form a semigroup, namely: $\Lambda_t \left(\Lambda_s(\rho)\right) = \Lambda_{t+s}(\rho)$ for $t, s \geq 0$. It can be proven that (under the assumptions of Markovianity, weak coupling limit and initial separability of the state of the system and of the bath) the generator $\mathcal{L}$ takes the GKSL form. To obtain this form directly from the interaction of the system with the environment containing it, some approximations must be adopted (such as the weak-coupling limit and the Markovian approximation) and the degrees of freedom of the environment must be traced out.

It is nevertheless interesting to study how the environment affects the quantum system without tracing out the former degrees of freedom. There emerges a dynamical evolution for ρ with extra stochastic terms due to quantum noise which is described by a so-called quantum stochastic master equation (QSME).

Typically, the GKSL generator arises from a QSME when the stochastic noise is averaged out. There are two main approaches to show this fact: one was investigated by Gisin and Percival [18] and involves a stochastic Schrödinger-like equation (it relies on the notion of quantum trajectories). The other one is due to Hudson and Parthasarathy [23] and is based on quantum stochastic calculus [30]. Focusing on the last method [31], it turns out that for every generator $\mathcal{L}$ in GKSL form, there exists a unitary operator U_t, acting in the composite Hilbert space of the system and the bath $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_B$ that obeys a quantum stochastic differential equation (QSDE), such that:

$$\langle \phi | \Lambda_t(X) | \psi \rangle = \langle \phi \otimes \Omega_0 | U_t^\dagger (X \otimes \mathbb{I}_B) U_t | \psi \otimes \Omega_0 \rangle \, , \tag{5.2}$$

where X belongs to $\mathcal{B}(\mathcal{H}_\mathcal{S})$, namely to the algebra of all bounded operators on the Hilbert space $\mathcal{H}_\mathcal{S}$, which is a Von Neumann algebra, as we defined in Section 1.1; ψ, $\phi \in \mathcal{H}_S$; $\mathbb{I}_B$ is the identity on $\mathcal{H}_B$ and $|\Omega_0\rangle$ is the vacuum state of the Fock space related to the bath.

As mentioned before, the unitary time evolution operator obeys a quantum stochastic differential equation. Though its derivation will be provided in the following sections; however, we anticipate its general form

$$dU_t = \left[\sum_{k=1}^{n} \left(L_k \, dB_k^\dagger - L_k^\dagger \, dB_k \right) - \left(iH + \frac{1}{2} \sum_{k=1}^{n} L_k^\dagger L_k \right) dt \right] U_t \, , \tag{5.3}$$

where L, $L^\dagger$ are operators acting on the Hilbert space related to S[1]. The Heisenberg evolution

[1] We omitted the tensor products $L \otimes \mathbb{I}_B$ and $\mathbb{I}_S \otimes dB$ to not weigh down too much the notation.

of any $X \in \mathcal{H}_S$ is therefore:

$$j_t(X) := U_t^\dagger (X \otimes \mathbb{I}_B) U_t \,, \tag{5.4}$$

$$dj_t(X) = j_t(\mathcal{L}'(X))\, dt + \sum_{k=1}^{n} \left\{ j_t\left([X, L_k]\right)\, dB_k^\dagger - j_t\left(\left[X, L_k^\dagger\right]\right)\, dB_k \right\}, \tag{5.5}$$

$$\mathcal{L}'(X) := i[H, X] + \sum_{k=1}^{n} \left(L_k^\dagger\, X\, L_k - \frac{1}{2}\{L_k^\dagger L_k,\, X\} \right) , \tag{5.6}$$

where $\mathcal{L}'$ is the usual GKSL generator in the Heinseberg picture. By taking the average over the vacuum state of the bath, we find the usual Lindblad evolution because the quantum stochastic process under exam has zero expectation value with respect to the vacuum. We can also find the evolution for the state of the total system via $\rho_t = U_t\, \rho\, U_t^\dagger$.

Here we wrapped up the general results of the quantum stochastic approach, in the next section we will see how the previous quantum stochastic equations naturally arise looking at a particular quantum optic system.

5.2 Quantum optic scenario

In this section we now show how (5.3) and (5.5) can be derived.

We start by considering the following Hamiltonian usually used to model a quantum system driven by an external laser [17]:

$$H = H_S + H_B + H_{int} \,, \tag{5.7}$$

with

$$H_B := \int_{-\infty}^{+\infty} d\omega\, \omega b^\dagger(\omega) b(\omega); \qquad H_{int} := i \int_{-\infty}^{+\infty} d\omega\, k(\omega) \left(b^\dagger(\omega) L - L^\dagger b(\omega) \right) \tag{5.8}$$

where L is an operator acting on the system S, B is the external field driving the system, namely a Bosonic operators from the environment, usually considered as a thermal bath of independent harmonic oscillators. The corresponding annihilation and creation operators of Boson states of given frequency ω satisfy the Canonical Commutation Relations:

$$\left[b(\omega), b^\dagger(\omega') \right] = \delta(\omega - \omega') \,, \tag{5.9}$$

and $k(\omega)$ describes the energy dependent coupling between the system and the bath.

Actually, the interaction H_{int} is not the most general one; indeed, only counter-rotating terms appear, while, more in general one may have:

$$H_{int} = i \int_{0}^{+\infty} d\omega\, k(\omega) \left(b^\dagger(\omega) + b(\omega) \right) (L - L^\dagger) \,. \tag{5.10}$$

If for example we consider the system S to be a dipole with resonant frequency Ω, terms such as $b^\dagger(\omega) L^\dagger$ evolve in time under the free Hamiltonians H_S and H_B as $e^{i(\omega+\Omega)t}$, which is rapidly oscillating; on the contrary, the mixed terms $b^\dagger(\omega) L \sim e^{i(\omega-\Omega)t}$ reduce to 1 in the resonant condition $\omega = \Omega$. This is called *Rotating Wave Approximation* (RWA). Assuming then that only

the resonant terms are relevant, the integration in $d\omega$ can be extended from minus infinite to infinite, since the additional terms do not contribute (this extension can be done only if we have already applied the RWA, otherwise the non-mixed terms would become resonant for $\omega = -\Omega$). The so-called *rotating wave* approximation gives rise to *quantum white noise* and will contribute to generate a Markovian evolution [17].

Moving to the interaction picture[2] [49]:

$$H_{eff} := e^{iH_B t}(H_S + H_{int})e^{-iH_B t} = H_S + i\int_{-\infty}^{+\infty} d\omega\, k(\omega)\left(e^{i\omega t}b^\dagger(\omega)L - e^{-i\omega t}L^\dagger b(\omega)\right) . \quad (5.11)$$

Now we proceed with some some simplifying assumptions and approximations [49]:

- We introduce Markovianity by fixing the coupling between the system and the bath to an energy-independent constant: $k(\omega) = \sqrt{\frac{\gamma}{2\pi}}$.
- It proves convenient to introduce the Fourier transforms of the bath operator $b(\omega)$:

$$b_{in}(t) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} d\omega\, e^{-i\omega t}b(\omega). \quad (5.12)$$

  Notice that the commutation rule becomes $\left[b_{in}(t), b_{in}^\dagger(t')\right] = \delta(t-t')$. Here the time can be interpreted as the moment in which the input laser beam meets the system S [50].

So that:

$$H_{eff} = H_S + i\sqrt{\gamma}\left(b_{in}^\dagger(t)L - L^\dagger b_{in}(t)\right) , \quad (5.13)$$

where γ is now the constant coupling with the bath. Then, the generated unitary time-evolution U_t satisfies

$$\dot{U}_t = -iH_{eff}\,U_t\,. \quad (5.14)$$

Moreover, we introduce the bath operators $B_{in}(t)$ and their differentials:

$$B_{in}(t) := \int_0^t b_{in}(\tau)\,d\tau\,, \qquad dB_{in}(t) := b_{in}(t)\,dt\,, \quad (5.15)$$

which are quantum Wiener processes, namely they respect the quantum Itô's rules (as we showed in the previous chapter).

5.2.1 Quantum Langevin equation

Going back to (5.14) and substituting in it the quantum Wiener processes (5.15), we find the following differential equation for U_t [50]:

$$dU_t = \left[-iH_S\,dt + \sqrt{\gamma}\,(dB_{in}^\dagger\,L - L^\dagger\,dB_{in})\right]U_t\,. \quad (5.16)$$

[2]There are several way to find the QSDE, for example in [17] they directly apply Heisenberg equations of motion to b and X, without going to the interaction picture.

The integrated version of the previous differential is:

$$U_t = \mathbb{I} + \int_0^t \Big[-iH_S\,dt + \sqrt{\gamma}\,(dB_{in}^\dagger\,L - L^\dagger\,dB_{in})\Big]\,U_t\,. \tag{5.17}$$

Iterating the integration, we find the formal solution:

$$U_t = \mathcal{T}\exp\Big\{\int_0^t \Big[-iH_S\,d\tau + \sqrt{\gamma}\,(dB_{in}^\dagger\,L - L^\dagger\,dB_{in})\Big]\Big\}\,, \tag{5.18}$$

where $\mathcal{T}$ is the time-ordering operator. We notice that going in the opposite direction, from (5.18) to (5.16), we use ordinary calculus. Therefore (5.18) must be understood as a quantum stochastic differential equation in Stratonovich formalism.

It is useful to find the Itô form that can be retrieved in [50] by differentiating the time evolution operator keeping terms up to the second order in the stochastic differentials and applying the quantum Itô's rules.

There however is another way to go from Stratonovich to Itô: from the definition of the stochastic integration. We saw that basically they differ for an additional term which in Itô is absent. Therefore, heuristically substituting finite differences with time-differentials and the previous expressions for the differentials of the unitary time-evolution operators U_t, one gets:

$$\begin{aligned}
\textbf{(S)}\int dB\,U(t) &= \lim\sum \frac{1}{2}\,[B(t_{i+1}) - B(t_i)]\,[U(t_{i+1}) + U(t_i)] = \\
&= \lim\sum \frac{1}{2}\,[B(t_{i+1}) - B(t_i)]\left[\underbrace{U(t_{i+1}) - U(t_i)}_{dU_t} + 2U(t_i)\right] = \\
&= \lim\sum [B(t_{i+1}) - B(t_i)]\left[\frac{1}{2}(-iH_S\,dt + \sqrt{\gamma}\,(dB^\dagger\,L - L^\dagger\,dB))U(t_i) + U(t_i)\right] = \\
&= \textbf{(I)}\int dB\,U(t) + \frac{\sqrt{\gamma}}{2}\int_0^t dt\,L\,,
\end{aligned} \tag{5.19}$$

$$\textbf{(S)}\,\sqrt{\gamma}\int dB\,L^\dagger\,U(t) = \textbf{(I)}\,\sqrt{\gamma}\int dB\,L^\dagger\,U(t) + \frac{\gamma}{2}L^\dagger L\,, \tag{5.20}$$

$$\textbf{(S)}\,\sqrt{\gamma}\int dB^\dagger\,L\,U(t) = \textbf{(I)}\,\sqrt{\gamma}\int dB^\dagger\,L\,U(t)\,, \tag{5.21}$$

where in the last step we applied Itô's rule. Then, we retrieve the time-evolution equation in Itô's form:

$$dU_t = \Big[(-iH_S - \frac{\gamma}{2}L^\dagger L)\,dt + \sqrt{\gamma}(dB_{in}^\dagger\,L - L^\dagger\,dB_{in})\Big]\,U_t\,. \tag{5.22}$$

This is precisely the expression we anticipated in (5.3) for just one Kraus operator.

From Itô's time-evolution in the Heisenberg picture, any open system operator X changes in time according to $j_t(X) = U_t^\dagger(X\otimes\mathbb{I}_B)U_t$. Such an Heisenberg dynamics is actually generated by the following differential *quantum Lagevin equation* [20]:

$$\begin{aligned}
dj_t(X) &= dU_t^\dagger(X\otimes\mathbb{I}_B)U_t + U_t^\dagger(X\otimes\mathbb{I}_B)dU_t + dU_t^\dagger(X\otimes\mathbb{I}_B)dU_t = \\
&= j_t(\mathcal{L}'(X))\,dt + \sqrt{\gamma}\Big\{dB_{in}^\dagger\,j_t\,([X,L]) + j_t\left(\Big[L^\dagger,X\Big]\right)\,dB_{in}\Big\}
\end{aligned} \tag{5.23}$$

$$\mathcal{L}'(X) = -i[X,H_S] + \gamma\,L^\dagger\,X\,L - \frac{\gamma}{2}\{L^\dagger L, X\}\,. \tag{5.24}$$

Which is of the same form of the one anticipated in (5.5) for just one mode of the bath. We remind that the quantum Langevin equation is a quantum stochastic differential equation describing the evolution of an observable $X \otimes \mathbb{I}_B$ of the composite system, driven both by noisy terms and deterministic ones.

Remark 5.1. On the basis of the algebraic structure outlined in Remark 4.2, the quantum Langevin equation (5.23) can be generalized to the case with more than one Kraus operator. This case follows from he coupling of the open quantum system with an environment with more than one discrete degrees of freedom, fopr instance an n-mode laser beam. Each mode contribute with its own annihilation and creation operators, these altogether in turn contributing with specific quantum noise. In practice, instead of just one pair L, B_{in} and $B_{in}^\dagger$ one deals with L_k, B_k, $B_k^\dagger$, $k = 1, \dots, n$ [50]. It is in this way that we recover the general quantum Langevin expression anticipated in (5.5).

Notice that, taking the expectation value of the quantum Langevin equation with respect to the vacuum state, one retrieves the usual GKSL evolution.

$$\Lambda_t(X) = \langle \Omega_0 | \, U_t^\dagger (X \otimes \mathbb{I}_B) U_t \, | \Omega_0 \rangle = e^{\mathcal{L}t} \, [X] \; . \tag{5.25}$$

In the same way as for $j_t(X)$, for a statistical operator describing system and environment together ρ_t^{S+B}, we find a *quantum stochastic master equation* in the Schrödinger picture [50]:

$$\begin{aligned} d\rho_t^{S+B} &= U_{t+dt,\,t} \, \rho_t^{S+B} \, U_{t+dt,\,t}^\dagger - \rho_t^{S+B} = \\ &= -i \Big[H_S, \rho_t^{S+B} \Big] \, dt + \gamma \, L \, \rho_t^{S+B} L^\dagger \, dt - \frac{\gamma}{2} \{ L^\dagger L \, , \rho_t^{S+B} \} + \sqrt{\gamma} dB_{in}^\dagger \, L \, \rho_t^{S+B} + \sqrt{\gamma} \rho_t^{S+B} \, L^\dagger \, dB_{in} \, , \end{aligned} \tag{5.26}$$

thus recovering the usual Lindblad form of the master equation in the Schrödinger picture by averaging with respect to the noise.

Remark 5.2. As anticipated at the beginning of this Chapter, the GKSL expression is found by averaging out the bath degrees of freedom since the very beginning, giving rise to a dissipative evolution. Instead, in what precedes, we kept all the degrees of freedom until the very final expressions, that show explicitly that those of the environment behave as quantum noise giving rise to a quantum stochastic evolution. The bridge between the two procedures is thus given by the average over the environment.

Concretely, to find the GKSL form, namely the evolution of ρ^S alone, we can take the expectation value over the initial state (since we are in Heinsenberg picture the state does not evolve) $\rho_0 \otimes |\Omega_0\rangle \langle \Omega_0|$, where Ω_0 is the usual vacuum state of the environment and ρ_0 is the initial state of the system S, and then apply the duality relation [31]:

$$\mathrm{tr}\{ \rho_0 \otimes |\Omega_0\rangle \langle \Omega_0| \; j_t(X) \} = \mathrm{tr}\{ \rho_0 \Lambda(X) \} = \mathrm{tr}\big\{ \rho_t^S X \big\} \, , \tag{5.27}$$

$$\rho_t^S = \mathrm{tr}_B \{ U_t \, \rho_0 \otimes |\Omega_0\rangle \langle \Omega_0| \; U_t^\dagger \} \, . \tag{5.28}$$

5.2.2 Importance of the bath state

Before continuing with the derivation of the quantum filtering equation, we make a brief digression about the role of the state of the environment, that we have so far taken to be the vacuum. If otherwise we do not specify the state of the bath, the dynamical evolution is noisy, but not necessarily stochastic in Itô or Stratonovich sense. Indeed, the quantum noise driving the dynamics depends on the bath correlation functions [17].

In particular, we are in a quantum white noise situation everytime the input state ρ_{in} of the bath satisfies:

$$\mathrm{tr}\Big\{\rho_{in} b^\dagger(t) b(t')\Big\} = \langle b^\dagger(t) b(t')\rangle = \bar{N}\delta(t-t') \tag{5.29}$$

$$\mathrm{tr}\Big\{\rho_{in} b(t) b^\dagger(t')\Big\} = \langle b(t) b^\dagger(t')\rangle = (\bar{N}+1)\delta(t-t') \tag{5.30}$$

$$\mathrm{tr}\big\{\rho_{in} b(t) b(t')\big\} = \langle b(t) b(t')\rangle = M\delta(t-t'), \tag{5.31}$$

where $\bar{N}$ is the mean number of particles (photons in the laser scenario) and M is called squeezing parameter.

Hitherto we always considered the vacuum state which is a pure state; if we consider mixed states (as long as we stay in a white noise scenario), we will find some differences in the Itô's multiplication rules [50]:

$\times$	dt	dB	$dB^\dagger$
dt	0	0	0
dB	0	$M\,dt$	$(\bar{N}+1)\,dt$
$dB^\dagger$	0	$\bar{N}\,dt$	$M^\star dt$

5.3 Quantum filtering and constantly monitored systems

We have now all the tools to introduce the notions and the techniques of quantum filtering.

As we have already remarked, quantum filtering describes what happens when the system is constantly monitored via some measurements on the bath. The most common monitoring processes are photon counting and homodyne detection. We will mostly focus on the latter. In particular we will show the conditions which the measurement process has to obey in order to build a consistent quantum filter.

After the discussion about the kinds of measurements we are allowed to perform, we will show how to exploit the string of outcomes of such measurements to modify the evolution of an observable of the system S via statistical conditioning.

5.3.1 Homodyne detection and photon counting

Homodyne detection and photon counting [12] are the most common measurements adopted in quantum optics, we now proceed with a brief description of them in our context.

Homodyne detection is a frequently used method in quantum optic to extract phase information about a laser. We remind that the laser beam interacts with the system S and it brings therefore information about the latter; hence analysing the output beam via homodyne detection

allows us to harvest information about S. The homodyne detection set up is the following: the idea is to compare the output laser beam with an additional one called Local Oscillator (LO) whose phase and amplitude are know. To achieve that the two laser beams interact via a beam splitter as shown in Figure 5.1, then we subtract the signal intensity given by the two detectors. Basically we are able to harvest information about S from the phase difference between the signal that interacted with the system and the signal which did not, namely the Local Oscillator. Clearly, the Local Oscillator can be generated by the same laser used for the interaction with S, it is sufficient to split the beam in two at the beginning and let just one of them interact with the system S while the other one takes the role of the LO.

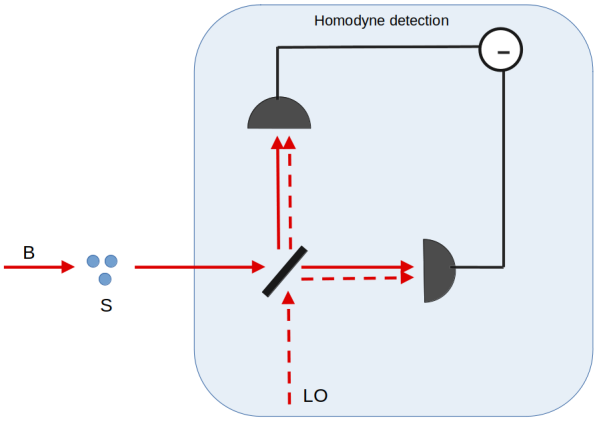

Figure 5.1: Homodyne detection scheme.

It can be proven that subtracting the signal intensity given by the two detectors corresponds to the measurement of the quadrature operator $b(t) + b^\dagger(t)$ of the bath [25]. In our context, the quadrature can better be recast as a function of the bath operators:

$$Y_{in}(t) := \mathbb{I}_S \otimes (B_{in}(t) + B_{in}^\dagger(t)) \,. \tag{5.32}$$

Here the time-dependence of the bath operators is due to the fact we took the Fourier transform of frequency-dependent creation and annihilation operators in (5.8) at the beginning of this Chapter. So the integrated quadrature above refers to the environment *at the time* of its interaction with the open quantum system, i.e. when the external laser beam has not yet hit the systems S. Since we are in the Heisenberg picture, we still have to let $Y_{in}(t)$ evolve according to the unitary dynamics of the compound system. This interaction leads to an output integrated quadrature [20]:

$$Y_{out}(t) := U_t^\dagger Y_{in}(t) U_t = B_{out}(t) + B_{out}^\dagger(t) \,, \tag{5.33}$$

$$dY_{out} = dY_{in}(t) + j_t(L + L^\dagger)\, dt \,. \tag{5.34}$$

The previous formula is called *input-output relation* and it shows how a measurement on the bath operators can bring us information about the system S. As a matter of fact Y_{out} depends on the variables of the systems S as well.

Differently from a homodyne detection setting, in a photon counting process (namely we have just one laser whose directly detected after the interaction with the system S), one measures

the following output observable:

$$Y^{\Lambda}_{out}(t) := U_t^{\dagger}(\mathbb{I}_S \otimes \Lambda_t)U_t, \tag{5.35}$$

where Λ_t is the one defined in (4.39) and corresponds to a quantum Poisson process. Then the differential from of the dynamics of such an observable amounts to

$$dY^{\Lambda}_{out}(t) = d\Lambda_t + j_t(L)\, dB^{\dagger} + j_t(L^{\dagger})\, dB + j_t(L^{\dagger}L)\, dt. \tag{5.36}$$

In the previous sections we basically showed that the quantum stochastic evolution is driven by a quantum Wiener process (under the assumption of quantum white noise). Instead, if we perform photon counting we introduce anther kind of quantum noise which follows a Possion distribution.

In the following, we will focus on homodyne detection homodyne detection as a most useful tool for quantum filtering.

5.3.2 Non-demolishing measurements

The main requirement for the measurement process on the environment in order to provide a consistent ground for quantum filtering processes is that the measured environment observables have to commute at different times [9]. Namely, they must be *compatible*, otherwise every measurement would affect the outcome of the next one. We wrap up some previous definitions and results:

$$U_t = U_{t,s}U_s \qquad t > s > 0\,, \tag{5.37}$$

$$j_t(X) := U_t^{\dagger}(X \otimes \mathbb{I})U_t\,, \tag{5.38}$$

$$Y_{out}(t) := U_t^{\dagger}\left[\mathbb{I} \otimes (B_{in}(t) + B_{in}^{\dagger}(t))\right] U_t = U_t^{\dagger} Y_{in}(t) U_t\,. \tag{5.39}$$

where, in the last expression, the observables proper to homodyne detection appear whose compatibility we now prove.

Since we are dealing with non-anticipating functions

$$[Y_{in}(s), U_{t,s}] = 0 \quad \text{for } t > s\,, \tag{5.40}$$

as a consequence, one has the following time-commutation relations

$$[Y_{out}(t), Y_{out}(s)] = U_t^{\dagger}\,[Y_{in}(t), Y_{in}(s)]\, U_t = 0 \qquad t > s\,, \tag{5.41}$$

$$[j_t(X), Y_{out}(s)] = U_t^{\dagger}\,[X \otimes \mathbb{I},\, \mathbb{I} \otimes (B_{in}(t) + B_{in}(t))]\, U_t = 0 \qquad t > s\,. \tag{5.42}$$

Therefore, we have a collection of commuting observables $\{Y_{out}(\tau), 0 \leq \tau \leq t\}$ which also commute with the evolved observable X of the system S.

We refer to (5.41) as *non-self demolition property* and to (5.42) as *Belavkin's non-demolition principle*.

While he proof of (5.42) is obvious, that of (5.41) goes as follows:

$$\begin{aligned}
[Y_{out}(t), Y_{out}(s)] &= \left[U_t^{\dagger}\, Y_{in}(t)\, U_t,\, U_s^{\dagger}\, Y_{in}(s)\, U_s\right] = \\
&= \left[U_t^{\dagger}\, Y_{in}(t)\, U_t,\, U_s^{\dagger}\, U_{t,s}^{\dagger}\, Y_{in}(s)\, U_{t,s}\, U_s\right] = \\
&= \left[U_t^{\dagger}\, Y_{in}(t)\, U_t,\, U_t^{\dagger}\, Y_{in}(s)\, U_t\right] = 0\,,
\end{aligned} \tag{5.43}$$

where we used that $\left[U_{t,s}^{\dagger},\, Y_{in}(s)\right] = 0$ due to $Y_{in}(t)$ being non-anticipating.

Now that we have physically non-interfering measurement processes, we can use their outcomes to monitor the evolution of the system.

5.3.3 Quantum filtering equations

From quantum to classical probability. The origin of the differences between classical and quantum probability theory lies in the non-commutativity of the latter: namely the set of observables acting on the Hilbert space do not necessarily commute with each other. Therefore, measuring an observable generically changes the outcomes of the measurement of another observable, this is the reason why non-self demolishing measurements are requested in order to have a consistent quantum filter.

When we monitor the bath via homodyne detection, basically for every measurement, we generate a commutative von Neumann algebra, which we defined in Section 1.1. After a certain time t, the von Neumann algebra is the one generated by all the previous measurement up to the time t [9]:

$$\mathcal{Y}_{out}(t) = vN\{Y_{out}(s),\, 0 \leq s \leq t\}\,. \tag{5.44}$$

This von Neumann algebra is still commutative because we showed in Section 5.3.2 the measurements to be compatible with each other.

Conditional expectation with respect to an algebra. We now want to infer some information about a variable X of the system S via the measurement performed on the bath. Basically, we have to evaluate the conditional probability of obtaining a certain value for X when we have a certain outcome from the homodyne detection. In particular, we will condition with respect the whole algebra of measurements we performed (see Section 1.4). We introduce the following map π_t [9] on the system observables X whose right hand side is the conditional expectation discussed in Section 1.4:

$$\pi_t(X) := \mathbb{P}(j_t(X)|\mathcal{Y}_{out}(t))\,. \tag{5.45}$$

The physical picture behind the conditioning is as follows: X is let evolve up to time t according to (5.23); then its mean value is taken by conditioning it to the commutative von Neumann algebra generated by all measurements previous to t. Basically here the von Neumann algebra associated with the homodyne measurement processes plays the role of the filtration we mentioned in Definition 4.1, it contains all the information about the system up to the time t. The commutativity of the conditioning algebra $\mathcal{Y}_{out}(t)$ is granted by the non-self demolishing property (5.41).

In order to give an explicit expression to π_t, instead to work with the observables $Y_{out}(t)$, it proves convenient to work with the quadrature operator $Z(t) = \mathbb{I}_S \otimes (B(t) + B^{\dagger}(t))$, which according to (5.33) is such that $Y_{out}(t) = U_t^{\dagger} Z(t) U_t$.

A word of caution: for the rest of the section we will omit the tensor product with the identity to lighten the notation, everytime there is a variable X related to the system S it must be intended as $X \otimes \mathbb{I}_B$ and every object B related to the bath as $\mathbb{I}_S \otimes B$. Moreover, we remind

that the unitary evolution U_t acts both on system and bath. Introducing the un-conditioned expectation

$$\mathbb{Q}_t(X) = \mathbb{P}(U_t^\dagger X\, U_t)\,, \tag{5.46}$$

the right hand side of (5.45) can be recast as

$$\mathbb{P}(j_t(X)|\mathcal{Y}_{out}(t)) = U_t^\dagger\, \mathbb{Q}_t(X|\mathcal{Z}(t))\, U_t\,, \tag{5.47}$$

where

$$\mathcal{Z}(t) = vN\{Z(s),\, 0 \le s \le t\}, \qquad \mathcal{Y}_{out}(t) = U_t^\dagger \mathcal{Z}(t) U_t\,, \tag{5.48}$$

the von Neumann algebra $\mathcal{Z}(t)$ being again commutative due to the non-self demolishing property (5.41) $U_s^\dagger Z(s) U_s = U_t^\dagger Z(s) U_t$ for $t > s$.

Proof. The relation (5.47) can be proved by the definition of conditional probability given in Section 1.4: let $\mathcal{Q}$ be a commutative sub-algebra of a von Neumann algebra $\mathcal{N}$ generated by orthogonal projectors Q_j. Then, one can define a conditional expectation from $\mathcal{N}$ onto $\mathcal{Q}$ as follows:

$$\mathbb{P}(X|\mathcal{Q}) = \sum_i \frac{\mathbb{P}(X\, Q_i)}{\mathbb{P}(Q_i)}\, Q_i = \sum_i \frac{\mathrm{tr}\{\rho\, X\, Q_i\}}{\mathrm{tr}\{\rho\, Q_i\}}\, Q_i\,, \quad X \in \mathcal{Q} \subset \mathcal{N}, \tag{5.49}$$

Assuming for sake of simplicity that the algebra $\mathcal{Y}_{out}(t)$ is generated by projectors $\widetilde{Q}_j(t) = U_t^\dagger Q_j(t) U_t$, where $Q_i(t)$ are the projectors that generate the algebra $\mathcal{Z}(t)$, the right hand side of (5.47) becomes

$$\sum_j \frac{\mathrm{tr}\Big\{\rho\, j_t(X)\, U_t^\dagger\, Q_j(t)\, U_t\Big\}}{\mathrm{tr}\Big\{\rho\, U_t^\dagger\, Q_j(t)\, U_t\Big\}} U_t^\dagger\, Q_j(t)\, U_t = U_t^\dagger \left(\sum_j \frac{\mathrm{tr}\Big\{\rho\, U_t^\dagger\, X\, Q_j(t)\, U_t\Big\}}{\mathrm{tr}\Big\{\rho\, U_t^\dagger\, Q_j(t)\, U_t\Big\}} Q_j(t) \right) U_t\,, \tag{5.50}$$

which is precisely (5.47). □

Therefore now, in order to obtain $\pi_t(X)$ we have to evaluate $\mathbb{Q}_t(X|\mathcal{Z}(t))$. We would like to use the following Bayes lemma in order to write the conditioning with respect to $\mathcal{Z}(t)$ in terms of $\mathbb{P}$ rather than $\mathbb{Q}_t$.

Lemma 5.1. (Bayes lemma [9]) Given a commutative von Neumann algebra $\mathcal{Z}$ and its commuting algebra $\mathcal{Z}'$ equipped with a state $\mathbb{P}$, we can choose an operator $V \in \mathcal{Z}'$, such that $V^\dagger V > 0$ (namely without null eigenvalues) and $\mathbb{P}(V^\dagger V) = 1$, and define a new state in $\mathcal{Z}'$ of the form $\mathbb{Q}(X) := \mathbb{P}(V^\dagger X V)$, then the conditional expectation $\mathbb{Q}(\cdot|\mathcal{Z}) : \mathcal{Z}' \mapsto \mathcal{Z}$ can be expressed as a function of $\mathbb{P}$ in the following way:

$$\mathbb{Q}(X|\mathcal{Z}) := \frac{\mathbb{P}(V^\dagger X\, V|\mathcal{Z})}{\mathbb{P}(V^\dagger V|\mathcal{Z})}\,, \quad X \in \mathcal{Z}'\,. \tag{5.51}$$

Proof. For every $K \in \mathcal{Z}$ and for $X \in \mathcal{Z}'$, (we drop the subscript ρ for $\mathbb{P}$ to lighten the notation) we have the following chain of equalities, which are all consequence of the property (1.36)

$$\begin{aligned} \mathbb{P}(\mathbb{P}(V^\dagger X V|\mathcal{Z})K) = \mathbb{P}(V^\dagger X K V) = \mathbb{Q}(XK) = \mathbb{Q}(\mathbb{Q}(X|\mathcal{Z})K) = \mathbb{P}(V^\dagger V \mathbb{Q}(X|\mathcal{Z})K)) = \\ = \mathbb{P}(\mathbb{P}(V^\dagger V \mathbb{Q}(X|\mathcal{Z})K)|\mathcal{Z})) = \mathbb{P}(\mathbb{P}(V^\dagger V|\mathcal{Z})\mathbb{Q}(X|\mathcal{Z})K), \end{aligned} \tag{5.52}$$

hence $\mathbb{P}(V^\dagger X V|\mathcal{Z}) = \mathbb{P}(V^\dagger V|\mathcal{Z})\mathbb{Q}(X|\mathcal{Z})$. □

In order to specialize the previous lemma to our case it is sufficient to substitute $\mathcal{Z}$ with $\mathcal{Z}(t)$, as a matter of fact the variable X we want to monitor commutes with $\mathcal{Z}(t)$ by construction. The expression (5.46) seems to suggest that the other substitution we have to operate in order to apply Beyes lemma is $V \mapsto U_t$. Unfortunately, this is not so because U_t does not fulfil the required commutativity with $\mathcal{Z}(t)$.

Happily, there is a way out. Indeed, we may construct an operator $V_t \in \mathcal{Z}'(t)$ such that

$$\mathbb{P}(V_t^\dagger X V_t) = \mathbb{P}(U_t^\dagger X U_t)$$

and use it in Bayes lemma. Letting U_t in (5.22) act on an initial state which is the tensor product of a state of the open quantum system with the environment vacuum Ω_0 one gets:

$$\begin{aligned} U_t\,\psi\otimes\Omega_0 &= \left[\mathbb{I} + \int_0^t L\,U_s\,dB^\dagger - \int_0^t L^\dagger\,U_s\,dB - \int_0^t \left(\frac{1}{2}L^\dagger L + iH\right) U_s\,ds\right]\psi\otimes\Omega_0 = \\ &= \left[\mathbb{I} + \int_0^t L\,U_s\,dB^\dagger - \int_0^t \left(\frac{1}{2}L^\dagger L + iH\right) U_s\,ds\right]\psi\otimes\Omega_0\,, \end{aligned} \tag{5.53}$$

because any stochastic differential dB annihilates the vacuum. Hence, we can arbitrary modify the dB integral in U_t without modifying its action on the vacuum state. In particular, we can introduce an operator V_t satisfying the following QSME:

$$dV_t = \left[L(dB^\dagger + dB) - \left(\frac{1}{2}L^\dagger L + iH\right)dt\right] V_t \tag{5.54}$$

which is identical to the one satisfied by U_t, see (5.22), except for the dB contribute. Now, in the V_t expression we have $dB^\dagger + dB$ rather than $L dB^\dagger - L^\dagger dB$, clearly the first one commutes with $\mathcal{Z}_{out}$, for it is the algebra of the quadrature operators, while the latter does not.

Moreover, the action of V_t on the state of the system $\psi\otimes\Omega_0$ is identical to the U_t action, again because $dB|\Omega_0\rangle = 0$:

$$V_t\,\psi\otimes\Omega_0 = \left[\mathbb{I} + \int_0^t L\,V_s\,dB^\dagger - \int_0^t \left(\frac{1}{2}L^\dagger L + iH\right) V_s\,ds\right]\psi\otimes\Omega_0\,. \tag{5.55}$$

We can now use V_t in Bayes lemma to re-write $\mathbb{Q}_t(X|\mathcal{Z}(t))$ and obtain $\pi_t(X)$ from (5.46)-(5.45):

$$\pi_t(X) = U_t^\dagger \frac{\mathbb{P}(V_t^\dagger X\,V_t|\mathcal{Z}(t))}{\mathbb{P}(V_t^\dagger V_t|\mathcal{Z}(t))}\,U_t\ . \tag{5.56}$$

It is useful to introduce the non-normalized conditional expectation

$$\sigma_t(X) := U_t^\dagger \mathbb{P}(V_t^\dagger X\,V_t|\mathcal{Z}(t))U_t\ \in \mathcal{Y}_{out}(t). \tag{5.57}$$

Notice that $\sigma_t(\mathbb{I}) = U_t^\dagger\mathbb{P}(V_t^\dagger\,V_t|\mathcal{Z}(t))U_t$ belongs to the same commutative algebra $\mathcal{Y}_{out}(t)$ and $\sigma_t(\mathbb{I}) > 0$ because we assumed $V_t^\dagger V_t > 0$. It thus makes sense to define the following ratio of commutative non-vanishing expressions:

$$\pi_t(X) = \frac{\sigma_t(X)}{\sigma_t(\mathbb{I})}\ \in \mathcal{Y}_{out}(t). \tag{5.58}$$

In filtering theory, the latter relation is known as Kallianpur-Striebel formula [24]. Applying Itô's differentiation rule to the previous equation, we firstly find the quantum counterpart of the so-called Zakai equation [48], which basically is the stochastic differential equation satisfied by the non-normalized conditional expectation:

$$d\sigma_t(X) = \sigma_t(\mathcal{L}(X))\,dt + \sigma_t(L^\dagger X + XL)\,dY_t\,. \tag{5.59}$$

In order to derive such a differential expression, we observe that for any adapted process K_s:

$$\mathbb{P}\left(\int_0^t K_s\,ds\Big|\mathcal{Z}_t\right) = \int_0^t \mathbb{P}\left(K_s|\mathcal{Z}_s\right)ds, \qquad \mathbb{P}\left(\int_0^t K_s\,dB\Big|\mathcal{Z}_t\right) = \int_0^t \mathbb{P}\left(K_s|\mathcal{Z}_s\right)dB. \tag{5.60}$$

By means of the quantum Zakai equation and using again Itô's differentiation rules, we now recover the stochastic differential equation satisfied by the normalized conditional expectation: $\pi_t(X)$ [3]:

$$d\pi_t(X) = \pi_t(\mathcal{L}(X))\,dt + \Big(\pi_t(L^\dagger X + XL) - \pi_t(L^\dagger + L)\pi_t(X)\Big)\,\left(dY_t^{out} - \pi_t(L^\dagger + L)\,dt\right). \tag{5.61}$$

The latter is known as the *quantum filtering equation*, where dY_t^{out} is the so-called innovation process. Notice that this is now a *classical* stochastic differential equation because it is a recursive equation driven by the observations Y_t^{out}, which commute at different times.

The previous result can be formulated for a statistical operator on S, switching from the Heisenberg to the Schrödinger picture via the duality relation $\mathrm{tr}\{\pi_t(X)\rho\} = \mathrm{tr}\{X\pi_t^*(\rho)\}$, where π_t^* is the dual map of π_t such that $\rho_c(t) := \pi^*(\rho(t)) = \mathbb{P}(\rho(t)|\mathcal{Y}_{out}(t))$ [49]:

$$\begin{aligned} d\rho_c(t) = &-i[\rho_c(t), H_S]dt + \frac{\gamma}{2}\Big(2L\rho_c(t)L^\dagger - L^\dagger L\rho_c(t) - \rho_c L^\dagger L\Big)\,dt \\ &+ \sqrt{\gamma}\left[L\rho_c(t) - \rho_c(t)L^\dagger - \mathrm{tr}\Big\{(L+L^\dagger)\rho_c(t)\Big\}\rho_c\right]dW_t \end{aligned} \tag{5.62}$$

$$dW_t = dY_t^{out} - \mathrm{tr}\Big\{(L+L^\dagger)\rho_c(t)\Big\}\,dt \tag{5.63}$$

In the previous formula, $\rho_c(t)$ is a random density matrix and dW_t is a classical Wiener process (in the sense that it can be proven $dW^2 \sim dt$). We stress that the Wiener process expresses the difference between what we do actually observe, $Y_{out}(t)$, and what we would have expected to observe at that time according to our previous knowledge of the system [20]. This is a peculiarity of homodyne detection: it generates a Gaussian stochastic process (if we had used photon counting rather than homodyne detection, we would have found a Poisson process) [9].

To summarize, the quantum filtering procedure amounts to [36]:

1. Perform a measurement Y_t
2. Evaluate the innovation process dW_t
3. Use the previous result to update the conditional density matrix $\rho_c(t)$

By means of the description of how information via homodyne detection can be gathered from a continuously monitored open quantum system we can proceed to discuss how, on account of that, one can control the dynamics of the system.

[3] Since we are dealing with commuting operators we can use the following Itô's relations $d\left(\frac{X}{Y}\right) = \frac{X}{Y}\left(\frac{dX}{X} - \frac{dY}{Y} - \frac{dX}{X}\frac{dY}{Y} + \left(\frac{dY}{Y}\right)^2\right)$ since $d\left(\frac{1}{Y}\right) = -\frac{dY}{Y^2} + \frac{(dY)^2}{Y^3}$

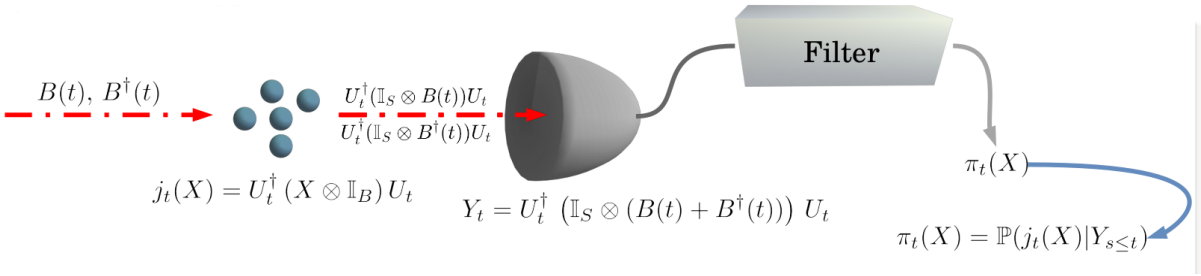

Figure 5.2: Through an external laser interacting with the environment, we measure environment quadratures after the later has interacted with the open quantum system. Then, by conditioning on the measurement outcomes, we recover the dynamics of the observables X of the quantum system [9].

5.4 Quantum feedback

The idea is to perform a quantum feedback control before the state collapses into a completely classical state after the measurement. Important points are: what observable is the optimal one to measure and how to choose the feedback forces as a function of the stream of measurement outcomes [49].

The necessity of introducing a feedback in the Hamiltonian is due to the fact that solving the stochastic master equation in (5.63) is not computationally easy in a real time experiment. So we can undertake the opposite path: without processing the measurements, an extra term proportional to the measurement outcome is added to the Hamiltonian at each time. We now show how to mathematically express this process, to do so we firstly wrap up the quantum filtering results.

A general filtering equation obtained via homodyne detection reads as follows:

$$d\rho_c = -\,i[H,\rho_c]dt + \sum_\mu \left(\mathcal{D}\left[L_\mu\right]\rho_c\,dt + \sqrt{\eta}\,\mathcal{S}\left[L_\mu\right]\rho_c\,dW_\mu\right)\,, \tag{5.64}$$

$$\mathcal{D}\left[L\right]\rho = L\rho L^\dagger - \frac{1}{2}L^\dagger L\,\rho - \frac{1}{2}\rho\,L^\dagger L\,, \tag{5.65}$$

$$\mathcal{S}\left[L\right]\rho = L\rho + \rho L^\dagger - \langle L + L^\dagger\rangle\,\rho\,; \tag{5.66}$$

where L is a bath operator, ρ_c is the normalized conditional state after a measurement and $\eta \in [0,1]$ takes into account the efficiency of the measurement process (in the ideal case $\eta = 1$, note that for $\eta = 0$ we recover an ordinary master equation because there is no detection).

If we consider homodyne detection with a single Kraus operator L (as done in the previous sections), then the increment dW is in the following relation with the measurement Y (we are dropping some subscripts):

$$dY = \langle L + L^\dagger\rangle\,dt + \frac{1}{\sqrt{\eta}}\,dW\,. \tag{5.67}$$

Introducing the white noise:

$$\xi(t) = dW/dt\,, \tag{5.68}$$

characterized by the following properties, as stated in Section 4.1.2:

$$\langle \xi(t) \rangle = 0, \qquad \langle \xi(t)\xi(t') \rangle = \delta(t - t'), \tag{5.69}$$

we can write:

$$\dot{\rho}_c = -i[H, \rho_c] + \mathcal{D}[L]\,\rho_c + \mathcal{S}[L]\,\rho_c\,\xi(t) \tag{5.70}$$

$$I(t) = \langle L + L^\dagger \rangle_t + \frac{1}{\sqrt{\eta}}\xi(t) \tag{5.71}$$

Where $I(t) = dY/dt$ is called *homodyne photocurrent* (the output signal). Quantum feedback consists in exploiting this quantity to control the system dynamics, as exemplified in Figure 5.3. Concretely, we add to the conditional evolution in (5.70) an arbitrary super-operator $\mathcal{F}$ depending on the previous homodyne outcomes [44]:

$$[\dot{\rho}_c(t)]_f = \mathcal{F}\left[t, I_{[0,t[}\right]\rho_c(t)\,, \tag{5.72}$$

where $I_{[0,t[} = \{I(\tau)\,|\,\tau \in [0,t[\}$ is the collection of all the previous measurements, the lowercase f stands for feedback. The previous expression is the most general way to write the quantum

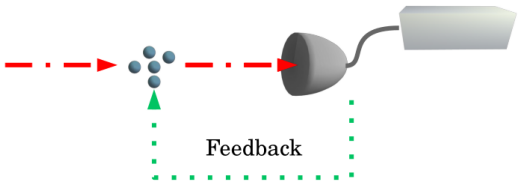

Figure 5.3: Feedback scheme, the information is re-inserted back in the system after the measurement.

feedback action, in the following section we will focus to a particular way to express the super-operator $\mathcal{F}$, in particular we will introduce the conditions for a Markovian quantum feedback in order to recover a simpler Markovian master equation.

5.4.1 Markovian feedback

A feedback protocol is called Markovian when the resulting master equation is Markovian as well. An example of feedback protocol which is not Markovian is the Bayesian feedback [42, 44]. The Bayesian Feedback is more complex to deal with, it exploits all the string of measurements to externally drive the system and it takes into account the delay between the measurement process and the consequent feedback action. It is called Bayesian because the state of the subsystem is constantly updated as soon as more information are harvested due to the filtering process (in analogy with the Bayes theorem illustrated in Section 1.2.1). For the interested reader other kinds of feedback protocols can be found in [49].

Instead, the main idea of Markovian feedback is to use only the last measurement outcome rather than all the stream of measurements as in (5.72).

We want to modify the master equation in (5.63) with something like [44]:

$$[\dot{\rho}_c(t)]_f = I(t-\tau)\mathcal{K}[\rho_c(t)]\,, \qquad \mathcal{K}[\rho] = -i[F, \rho]\,. \tag{5.73}$$

Namely, we modify the master equation at time t by adding a new Hamiltonian term depending on the homodyne measurement at the previous instant and on an arbitrary Hermitean operator F of the system S. In this way, one re-introduces to the system the information gained through the measurement.

Since we are interested in finding a Markovian feedback, we assume no time interval passes between the measurement and the feedback drive: $\tau \mapsto 0$.

In order to find an explicit feedback master equation, we have to put (5.73) and (5.70) together. The feedback depends on the current $I(t)$ which contains a stochastic term, therefore we have to understand if (5.73) has to be interpreted in Itô or Stratonovich sense. The filtering equation (5.70) has been derived using Itô's calculus; on the other hand, in (5.73) the feedback drive has been added by hand, therefore its interpretation is not so straightforward [43]. It has been shown [43, 44] that (5.73) has to be intended in Stratonovich sense, otherwise the resulting master equation is not physically consistent.

Therefore, we ought to write (5.70) as a Stratonovich equation (see (4.20)), namely:

$$\text{(S)}\ \dot{\rho_{\mathrm{c}}} = \left[\mathcal{L} + \sqrt{\eta}\,\xi(\mathrm{t})\mathcal{S} - \frac{1}{2}\eta\mathcal{S}^2\right]\rho_{\mathrm{c}}\,. \tag{5.74}$$

Then, we add the feedback contribution (5.73) and we substitute the current (5.71) (we drop the subscript f for simplicity):

$$\text{(S)}\ \dot{\rho_{\mathrm{c}}} = \left[\mathcal{L} - \frac{1}{2}\eta\mathcal{S}^2 + \langle \mathrm{L} + \mathrm{L}^\dagger\rangle\mathcal{K} + \sqrt{\eta}\xi(\mathrm{t})(\mathcal{S} + \eta^{-1}\mathcal{K})\right]\rho_{\mathrm{c}}\,. \tag{5.75}$$

Finally, we can go back to Itô's formalism:

$$\dot{\rho_c} = \mathcal{L}\rho_c - \mathcal{K}(L\rho_c + \rho_c L^\dagger) + \frac{1}{2\eta}\mathcal{K}^2\rho_c + \sqrt{\eta}\xi(t)(\mathcal{S} + \eta^{-1}\mathcal{K})\rho_c\,. \tag{5.76}$$

It can be proven [43] that this master equation is trace preserving. The great advantage of Itô's calculus is that the noise $\xi(t)$ is independent from $\rho_c(t)$, thus it is easy to take the ensamble average and to recover the reduced system density matrix $\rho = \langle\rho_c\rangle$, where the expectation is taken with respect to bath vacuum state:

$$d\rho(t) = \left(-i[H,\rho(t)] + \mathcal{D}(L)\rho_t - i\Big[F, L\rho(t) + \rho(t)L^\dagger\Big] + \frac{1}{\eta}\mathcal{D}(F)\rho(t)\right)dt + \mathcal{S}\left[\sqrt{\eta}L - \frac{i}{\sqrt{\eta}}F\right]\rho(t)\,dW\,, \tag{5.77}$$

where we substituted $\mathcal{S}$ and $\mathcal{K}$ from (5.66) and (5.73) respectively.

Now that every term involved is statistically independent, we can take the average over the quantum white noise and all the terms with dW vanish.

$$d\rho(t) = \left(-i[H,\rho(t)] + \mathcal{D}(L)\rho_t - i\Big[F, L\rho(t) + \rho(t)L^\dagger\Big] + \frac{1}{\eta}\mathcal{D}(F)\rho(t)\right)dt\,. \tag{5.78}$$

We can also write the feedback master equation in Lindblad form [44]:

$$\dot{\rho}(t) = -i\left[H + \frac{1}{2}(L^\dagger F + FL),\,\rho\right] + \mathcal{D}\left[L - iF\right]\rho + \frac{1-\eta}{\eta}\mathcal{D}\left[F\right]\rho\,, \tag{5.79}$$

and can generalize it to multiple Kraus operators:

$$\dot{\rho} = -i\left[H + \frac{1}{2}(L^{(1)\dagger}F + FL^{(1)}),\, \rho\right] + \mathcal{D}\left[L^{(1)} - iF\right]\rho + \frac{1-\eta}{\eta}\mathcal{D}\left[F\right]\rho + \sum_{\mu=2}^{6}\mathcal{D}\left[L^{(\mu)}\right]\rho\,. \quad (5.80)$$

The previous one is the Markovian master equation of a system subject to quantum feedback and homodyne detection. We notice that we have an extra Hamiltonian term, which we will call H_{FB} later; moreover the action of the feedback adds a dissipation term $\mathcal{D}\left[F\right]$ which goes to zero when we are in the ideal condition $\eta = 1$; $\mathcal{D}\left[L^{(1)} - iF\right]$ shows the degree of freedom on which the feedback is acting (it is related to the observable we decide to detect during the filtering).

Example: Forcing a pure state via Markovian feedback in a two-level atom

Example taken from [42]: we consider an atom with two levels $|e\rangle$, $|g\rangle$. We define $\sigma = |g\rangle\langle e|$ is the lowering operator.

The atom is merged in a classical resonant driving field with Rabi frequency proportional to 2α.

The master equation is the following:

$$\dot{\rho} = \mathcal{D}(\sigma)\rho - i\alpha\left[\sigma_y, \rho\right] + \Gamma\mathcal{D}(\sigma_z)\rho\,, \quad (5.81)$$

$$\mathcal{D}(\sigma)\rho = \sigma\rho\sigma^\dagger = \{\sigma^\dagger\sigma, \rho\}\,, \quad (5.82)$$

$$\sigma_x = \sigma + \sigma^\dagger \qquad \sigma_y = i(\sigma - \sigma^\dagger)\,, \quad (5.83)$$

where Γ is the dephasing rate.

We adopt Bloch representation:

$$\rho = \frac{1}{2}(\mathbb{I} + x\sigma_x + y\sigma_y + z\sigma_z)\,. \quad (5.84)$$

Substituting in the master equation, we find:

$$\dot{\rho} = \frac{1}{2}\begin{pmatrix} -1 - 2\alpha x - z & \frac{1+4\Gamma}{2}x + \alpha z + i\frac{1+4\Gamma}{2}y \\ \frac{1+4\Gamma}{2}x + \alpha z - i\frac{1+4\Gamma}{2}y & 1 + 2\alpha x + z \end{pmatrix}. \quad (5.85)$$

Therefore, the stationary solution is:

$$x_{ss} = -\frac{4\alpha}{(1+2\Gamma) + 8\alpha^2}\,, \qquad y_{ss} = 0\,, \qquad z_{ss} = -\frac{1+2\Gamma}{(1+2\Gamma) + 8\alpha^2}\,. \quad (5.86)$$

The effect of the driving field is to rotate the Bloch sphere around the y-axes.

The family of solutions lies on the xz plane, we can then parametrize: $x = r\sin\theta$, $z = r\cos\theta$ with $\theta \in [-\pi, \pi]$

The purity is given by:

$$p = 2\,\mathrm{tr}\left(\rho^2\right) - 1 = x^2 + y^2 + z^2\,, \quad (5.87)$$

hence in Bloch representation purity corresponds to the ray.

$$r = \sqrt{x^2 + z^2} \qquad \begin{cases} r = 1 & \text{pure state} \\ r = 0 & \text{maximally mixed state} \end{cases}. \quad (5.88)$$

The stationary states we are able to achieve by driving the atom are limited and generally far from being pure.

Now we add Markovian feedback via homodyne detection. Calling $I(t)$ the homodyne current and $\mathcal{M}$ the measurement super operator:

$$I(t)dt = \sqrt{\eta}\,\mathrm{tr}\Big(\rho_I(\sigma+\sigma^\dagger)\Big)dt + dW(t)\,, \tag{5.89}$$

$$\mathcal{M}\rho = \sqrt{\eta}(\sigma\rho + \rho\sigma^\dagger)\,. \tag{5.90}$$

We can now implement the previous in the quantum filtering equation (5.70), after some algebra we find:

$$\dot{\rho}_I = \sigma^\dagger\rho_I\sigma - \frac{1}{2}\{\sigma^\dagger\sigma,\rho_I\} - i\alpha\,[\sigma_y,\rho_I] + \Gamma(\sigma_z\rho_I\sigma_z - \rho_I) + \sqrt{\eta}(I-x)(\sigma\rho_I + \rho_I\sigma^\dagger - x\rho_I)\,, \tag{5.91}$$

$$x = \mathrm{tr}\Big\{\rho\,(\sigma+\sigma^\dagger)\Big\}\,. \tag{5.92}$$

The last expression is the full SME for a system under homodyne detection, it is equal to (5.81), with the extra noisy term given by the measurement. The equation without the last term is a deterministic master equation (DME), due to the very last term, the evolution is stochastic.

The aim is to make the stationary state of the atom as close as possible to a pure state $|\theta_0\rangle$:

$$|\theta_0\rangle = \cos\frac{\theta_0}{2}|e\rangle + \sin\frac{\theta_0}{2}|g\rangle\,, \tag{5.93}$$

where basically $r=1$ and $\theta=\theta_0$.

The desired state lies in $y=0$, then we can add an H_{FB} depending on σ_y.

$$H_{FB} = \frac{1}{\sqrt{\eta}}I(t)\lambda\sigma_y\,; \tag{5.94}$$

the parameter λ is the one we can externally control in order to maintain a pure state during the dissipative evolution. We notice that the feedback Hamiltonian is analogous to the field driving the system $\alpha\sigma_y$, hence the feedback protocol consists in modulating the driving field.

Now, following the results of the Markovian feedback (5.79) we find the quantum feedback master equation:

$$\dot{\rho} = -i[\alpha\sigma_y,\rho] + \mathcal{D}\left[\sigma - i\lambda\sigma_y\right]\rho + \frac{\lambda^2}{\eta}\mathcal{D}\left[\sigma_y\right]\rho + \Gamma\mathcal{D}\left[\sigma_z\right]\rho\,. \tag{5.95}$$

We use again the Block representation to find the stationary state of the previous ME as a function of λ:

$$\begin{aligned} x_{ss} &= \frac{-4\alpha(1+2\lambda)}{D}\,,\\ y_{ss} &= 0\,, \qquad (5.96)\\ z_{ss} &= -\frac{(1+2\lambda)(1+4\lambda+2\Gamma+4\lambda^2/\eta)}{D}\,,\\ D &= 8\alpha^2 + (1+4\lambda+2\Gamma+4\lambda^2/\eta))(1+2\lambda+2\lambda^2/\eta)\,. \qquad (5.97) \end{aligned}$$

In order to have a pure state we have to maximize the Bloch ray $r = \sqrt{x^2 + y^2}$, using the polar parametrization we have:

$$\tan\theta = \frac{x_{ss}}{z_{ss}}\,. \tag{5.98}$$

Letting $\theta \mapsto \theta_0$, we find:

$$\alpha = \left(\frac{1}{4} + \lambda + \frac{\Gamma}{2} + \frac{\lambda^2}{\eta}\right)\tan\theta_0 \tag{5.99}$$

and for the Block ray:

$$r_{ss} = \sqrt{x_{ss}^2 + y_{ss}^2} = \frac{(1+2\lambda)\cos\theta_0}{1 + 2\lambda + 2\lambda^2/\eta + (\Gamma + 1/2)\sin^2\theta_0}\,. \tag{5.100}$$

We have to find the parameter λ which maximizes this ray for every θ_0 we wish to achieve; in particular the maximum is given by:

$$\lambda = -\frac{\eta}{2}\left(1 + \frac{\cos\theta_0}{r_0}\right)\,. \tag{5.101}$$

We now proceed in the next Chapter in applying the quantum feedback protocol to the case analysed in 3.2 with the scope of entanglement enhancement.

Chapter 6

Quantum Feedback for entanglement enhancement

The present chapter is the focal point of the thesis namely, the study of whether and how feedback protocols can be devised to enhance the entanglement generation and persistence during a dissipative evolution.

The scope of this chapter is to present the major results of the thesis and the techniques employed to achieve them, relegating most of the mathematics to Appendix C. All the results presented in the following are original and novel outcomes of the thesis.

In Section 3.2 we showed under which circumstances, two initially separable qubits can be entangled by a dissipative dynamics and remain entangled asymptotically in time. The aim of this Chapter is to exploit the quantum feedback protocol via homodyne detection illustrated in Chapter 5 to enhance the entanglement generation via dissipation in a two qubits scenario similar to the one adopted in Section 3.2. Thus, we will show how the dynamics generated by a dissipator as in (3.32) changes when we apply the feedback drive and how the latter can be modulated in order to facilitate the entanglement generation, both in the short and long-time regimes.

As in Section 3.2, we will focus on a 2-qubit sub-system interacting with the same bath; as usual, we will refer to the sub-system with the letter S and to the environment with B.

6.1 Master equation with Markovian feedback for open 2-qubit systems

Firstly, we proceed to write the master equation for a system subjected to Markovian feedback via homodyne detection specializing the expression (5.80) for a case of a 2-qubit system immersed in a dissipative environment.

We write the master equation related to Markovian feedback via homodyne detection in the following compact form:

$$\frac{\partial \rho(t)}{\partial t} = \mathbb{H}\left[\rho(t)\right] + \mathbb{D}'\left[\rho(t)\right], \tag{6.1}$$

where the first term in the right hand side is the generator of the Hamiltonian evolution, given by the commutator of the state with the free Hamiltonian of the sub-system S plus corrections due to the interaction with the environment and to the feedback action; the second one is a dissipative term in which, again, there will be a feedback contribution.

We now focus on the structure of the right hand side of the previous compact expression when $\rho(t)$ describes a two-qubit system.

The dissipator The master equation related to a Markovian feedback process contains the following dissipative term, as derived in Section 5.4.1 (we omit the temporal dependence of the quantum state for sake of simplicity):

$$\mathbb{D}'[\rho] = \mathcal{D}[L - iF]\rho + \frac{1-\eta}{\eta}\mathcal{D}[F]\rho, \tag{6.2}$$

$$\text{with} \quad \mathcal{D}[L]\rho = L\rho L^\dagger - \frac{1}{2}\{L^\dagger L, \rho\}, \tag{6.3}$$

where L is a Kraus operator of the system without feedback and F is the action of the feedback we externally add to drive the system.

We now consider the case in which ρ describes two qubits, therefore we can expand L and F with respect to the matrix basis given by the tensor product of the Pauli matrices. Since we are interested in operations that act on the two qubits separately, (we want to see how the interaction with the bath generates entanglement, not our possibly non-local actions on them), we expand as follows:

$$L = \sum_{i=1}^{3} \ell_i \, \sigma_i \otimes \mathbb{I} + \sum_{i=1}^{3} r_i \, \mathbb{I} \otimes \sigma_i, \tag{6.4}$$

$$F = \sum_{i=1}^{3} f_i \, \sigma_i \otimes \mathbb{I} + \sum_{i=1}^{3} g_i \, \mathbb{I} \otimes \sigma_i \quad \text{with } f_i = f_i^*, \; g_i = g_i^*. \tag{6.5}$$

Substituting in the dissipator, we find:

$$\mathbb{D}'[\rho] = \tag{6.6}$$

$$\begin{aligned} = \sum_{i,j=1}^{3} \Bigg\{ & \left(\ell_i \ell_j^* + f_i f_j \, \frac{2\eta - 1}{\eta} + i\ell_i f_j - i f_i \ell_j^* \right) \left[(\sigma_i \otimes \mathbb{I}) \, \rho \, (\sigma_j \otimes \mathbb{I}) - \frac{1}{2}\{(\sigma_j \sigma_i \otimes \mathbb{I}), \, \rho\} \right] + \\ & + \left(r_i r_j^* + g_i g_j \, \frac{2\eta - 1}{\eta} + i r_i g_j - i g_i r_j^* \right) \left[(\mathbb{I} \otimes \sigma_i) \, \rho \, (\mathbb{I} \otimes \sigma_j) - \frac{1}{2}\{(\mathbb{I} \otimes \sigma_j \sigma_i), \, \rho\} \right] + \\ & + \left(\ell_i r_j^* + f_i g_j \, \frac{2\eta - 1}{\eta} + i \ell_i g_j - i f_i r_j^* \right) \left[(\sigma_i \otimes \mathbb{I}) \, \rho \, (\mathbb{I} \otimes \sigma_j) - \frac{1}{2}\{(\sigma_i \otimes \sigma_j), \, \rho\} \right] + \\ & + \left(r_i \ell_j^* + g_i f_j \, \frac{2\eta - 1}{\eta} + i r_i f_j - i g_i \ell_j^* \right) \left[(\mathbb{I} \otimes \sigma_i) \, \rho \, (\sigma_j \otimes \mathbb{I}) - \frac{1}{2}\{(\sigma_j \otimes \sigma_i), \, \rho\} \right] \Bigg\}, \end{aligned} \tag{6.7}$$

where now the feedback drive is controlled by the choice of the coefficients f_i and g_i.

For general dissipative dynamics whose dissipator consists of more Kraus operators as in (3.36), the feed-back driven dissipator can be generalized to:

$$\mathbb{D}'[\rho] = \mathcal{D}\left[L^{(1)} - iF\right]\rho + \frac{1-\eta}{\eta}\mathcal{D}[F]\rho + \sum_{\mu=2}^{6} \mathcal{D}\left[L^{(\mu)}\right]\rho \, . \tag{6.8}$$

The fact that the feedback action has been chosen such to modify just one Kraus operator corresponds to the physical setting where the filtering process is based on the following measurement of the environment. Only one homodyne detection of the form $B + B^\dagger$ as illustrated in (5.33) is performed at different times: as shown in Section 5.3.2 these homodyne measurements at different times commute. This is not true for different degrees of freedom that would be mapped into non-commuting quadratures at different times because the corresponding dynamics is determined by non-commuting Kraus operators L_k. This is important since as we remarked in Section 5.3.2: the measurements must commute among each other, otherwise it is not possible to monitor the system in a single realization of the latter. The choice of which one is completely arbitrary, it makes us manipulate different degrees of freedom of the qubit system.

By using again the expansion with respect to the Pauli matrices,

$$L^{(\mu)} = \sum_{i=1}^{3} \ell_i^{(\mu)}\, \sigma_i \otimes \mathbb{I} + \sum_{i=1}^{3} r_i^{(\mu)}\, \mathbb{I} \otimes \sigma_i \ , \tag{6.9}$$

the dissipator takes the following form:

$$\begin{aligned}
\mathbb{D}'[\rho] = & \\
\sum_{i,j=1}^{3} \Bigg\{ & \left(\sum_{\mu=1}^{6} \ell_i^{(\mu)} \ell_j^{(\mu)*} + f_i f_j \, \frac{2\eta - 1}{\eta} + i\ell_i^{(1)} f_j - i f_i \ell_j^{(1)*} \right) \left[(\sigma_i \otimes \mathbb{I})\, \rho\, (\sigma_j \otimes \mathbb{I}) - \frac{1}{2} \{ (\sigma_j \sigma_i \otimes \mathbb{I}),\, \rho \} \right] + \\
& + \left(\sum_{\mu=1}^{6} r_i^{(\mu)} r_j^{(\mu)*} + g_i g_j \, \frac{2\eta - 1}{\eta} + i r_i^{(1)} g_j - i g_i r_j^{(1)*} \right) \left[(\mathbb{I} \otimes \sigma_i)\, \rho\, (\mathbb{I} \otimes \sigma_j) - \frac{1}{2} \{ (\mathbb{I} \otimes \sigma_j \sigma_i),\, \rho \} \right] + \\
& + \left(\sum_{\mu=1}^{6} \ell_i^{(\mu)} r_j^{(\mu)*} + f_i g_j \, \frac{2\eta - 1}{\eta} + i \ell_i^{(1)} g_j - i f_i r_j^{(1)*} \right) \left[(\sigma_i \otimes \mathbb{I})\, \rho\, (\mathbb{I} \otimes \sigma_j) - \frac{1}{2} \{ (\sigma_i \otimes \sigma_j),\, \rho \} \right] + \\
& + \left(\sum_{\mu=1}^{6} r_i^{(\mu)} \ell_j^{(\mu)*} + g_i f_j \, \frac{2\eta - 1}{\eta} + i r_i^{(1)} f_j - i g_i \ell_j^{(1)*} \right) \left[(\mathbb{I} \otimes \sigma_i)\, \rho\, (\sigma_j \otimes \mathbb{I}) - \frac{1}{2} \{ (\sigma_j \otimes \sigma_i),\, \rho \} \right] .
\end{aligned} \tag{6.10}$$

In order to compare the feed-back driven dissipator $\mathbb{D}'[\rho]$ with $\mathbb{D}[\rho]$ in (3.36), we need a connection between the $\mathcal{F}$ operators in (3.32) and the L operators in (6.8). In order to find the connection we can diagonalize the Kossakowski matrix $K = [K_{\alpha\beta}] \geq 0$ in (3.32):

$$K = UDU^\dagger, \qquad \text{where } D \text{ is a diagonal matrix with elements } \lambda_j, \tag{6.11}$$

$$K_{\alpha\beta} = \sum_{\mu=1}^{6} \lambda_\mu U_{\alpha\mu} U^*_{\beta\mu}, \tag{6.12}$$

hence

$$\sum_{\alpha,\beta=1}^{6} K_{\alpha\beta} \mathcal{F}_\alpha \rho \mathcal{F}_\beta = \sum_{\alpha,\beta,\mu=1}^{6} \lambda_\mu\, U_{\alpha\mu}\, \mathcal{F}_\alpha\, \rho\, \mathcal{F}_\beta\, U^*_{\beta\mu} = \sum_{\mu=1}^{6} L^{(\mu)} \rho\, L^{(\mu)\dagger}, \tag{6.13}$$

where

$$L^{(\mu)} = \sqrt{\lambda_\mu} \sum_{\alpha=1}^{6} U_{\alpha\mu}\, \mathcal{F}_\alpha. \tag{6.14}$$

Notice that the Kraus operator in (6.4) read

$$L^{(\mu)} = \sum_{\alpha=1}^{6} \ell_\alpha^{(\mu)}\, \mathcal{F}_\alpha \qquad \text{with } \ell_\alpha = (\ell_i, r_i),\ \ i = 1,2,3\ . \tag{6.15}$$

Therefore we find the relation

$$\ell_\alpha^{(\mu)} = \sqrt{\lambda_\mu}\, U_{\alpha\mu}, \tag{6.16}$$

which allows us to identify the terms in (3.36) with the ones in (6.10), in particular:

$$\mathcal{A}_{ij} := \sum_{\mu=1}^{6} \ell_i^{(\mu)} \ell_j^{(\mu)*}, \tag{6.17}$$

$$\widetilde{\mathcal{A}}_{ij} := f_i f_j\, \frac{2\eta - 1}{\eta} + i\ell_i^{(1)} f_j - i f_i \ell_j^{(1)*}. \tag{6.18}$$

The tilde quantities contain feedback contributions which are absent in (3.36), the other correspondences for $\mathcal{B}$, $\mathcal{C}$ and $\widetilde{\mathcal{B}}$, $\widetilde{\mathcal{C}}$ being then straightforwardly computed.

We can then rewrite (6.10) in a more compact form:

$$\begin{aligned}
\mathbb{D}'\left[\rho\right] = \sum_{i,j=1}^{3} \Bigg\{ & \left(\mathcal{A}_{ij} + \widetilde{\mathcal{A}}_{ij}\right) \left[(\sigma_i \otimes \mathbb{I})\, \rho\, (\sigma_j \otimes \mathbb{I}) - \frac{1}{2}\{(\sigma_j\sigma_i \otimes \mathbb{I}),\, \rho\} \right] + \\
& + \left(\mathcal{C}_{ij} + \widetilde{\mathcal{C}}_{ij}\right) \left[(\mathbb{I} \otimes \sigma_i)\, \rho\, (\mathbb{I} \otimes \sigma_j) - \frac{1}{2}\{(\mathbb{I} \otimes \sigma_j\sigma_i),\, \rho\} \right] + \\
& + \left(\mathcal{B}_{ij} + \widetilde{\mathcal{B}}_{ij}\right) \left[(\sigma_i \otimes \mathbb{I})\, \rho\, (\mathbb{I} \otimes \sigma_j) - \frac{1}{2}\{(\sigma_i \otimes \sigma_j),\, \rho\} \right] + \\
& + \left(\mathcal{B}_{ij}^\dagger + \widetilde{\mathcal{B}}_{ij}^\dagger\right) \left[(\mathbb{I} \otimes \sigma_i)\, \rho\, (\sigma_j \otimes \mathbb{I}) - \frac{1}{2}\{(\sigma_j \otimes \sigma_i),\, \rho\} \right], \qquad (6.19)
\end{aligned}$$

Putting the feedback to zero means setting $f_i = g_i = 0$ in (6.10); one thus recovers the Kossakowski matrix in (3.32) with no feedback effects.

When $\ell_i^{(\mu)} = r_i^{(\mu)}$ for all μ and i one gets $\mathcal{A} = \mathcal{B} = \mathcal{C}$ and then the Kossakowski matrix $K = \mathcal{A} \otimes \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. By further choosing $f_i = g_i$ one remains with a new Kossakowski matrix of the same form. This allows one to introduce the operators defined in (3.54) and (3.55) and to write the dissipator as:

$$\mathbb{D}'\left[\rho(t)\right] = \sum_{i,j=1}^{3} \left(\mathcal{A}_{ij} + \widetilde{\mathcal{A}}_{ij}\right) \left[\Sigma_j\, \rho(t)\, \Sigma_i - \frac{1}{2}\{\Sigma_i\Sigma_j,\, \rho(t)\} \right], \tag{6.20}$$

which is the form we will use in the rest of this work.

Remark 6.1. We emphasize that the correspondence given in (6.17) leaves us a certain freedom in the choice of the ℓ_i coefficients. Indeed, the summation (6.17) must give back the Kossakowski matrix K of the dissipative dynamics without the feedback action: this latter K is insensitive to the multiplication of the coefficients $\ell_i^{(\mu)}$ by a common phase $e^{i\varphi_\mu}$. It is important to observe that the coefficients $\ell_i^{(\mu)}$ also appear in the feedback contribution $\widetilde{\mathcal{A}}_{ij}$ but in a way that instead make the new Kossakowski matrix sensitive to changes of phase. Thus, we can control the feedback action either via the f_i free parameters, or via the phases of $\ell_i^{(\mu)}$.

Given the short time entanglement generation condition (3.49), we that the worst scenario is when the coefficients $\ell_i^{(\mu)}$ are all real.

The Hamiltonian evolution We can now focus on the first term of the right hand side of (6.1). As shown in Section 5.4.1,

$$\mathbb{H}\left[\rho(t)\right] = -i\left[H + \frac{1}{2}(L^{(1)\dagger}F + FL^{(1)}),\, \rho(t)\right], \tag{6.21}$$

where H is the Hamiltonian with Lamb-shift terms, but without feedback effects.

Taking the decomposition (6.9) with $\ell_i^{(1)} = r_i^{(1)}$ and $g_i = f_i$, as done for the dissipative term, and using again the operators (3.54) and (3.55), we have:

$$\frac{1}{2}(L^{(1)\dagger}F + FL^{(1)}) = \frac{1}{2}\sum_{i,j=1}^{3}\left(\ell_i^{(1)^*} f_j \Sigma_i \Sigma_j + \ell_i^{(1)} f_j \Sigma_j \Sigma_i\right) = \tag{6.22}$$

$$= \sum_{i,j=1}^{3}\left(2\,\mathrm{Re}\{\ell_i^{(1)}\} f_j \mathbb{I}\otimes\mathbb{I} + \epsilon_{ijk}\mathrm{Im}\{\ell_i^{(1)}\} f_j \Sigma_k + \mathrm{Re}\{\ell_i^{(1)}\} f_j S_{ij}\right), \tag{6.23}$$

where the matrices S_{ij} are as in (3.55). Given the master equation specified by the above requests on Hamiltonian and dissipative contributions to the generator, in order to proceed analytically as far as possible, we now restrict the possible matrices $\mathcal{A}$ and $\widetilde{\mathcal{A}}$.

6.2 The physical setting

We discuss the dissipative dynamics of a two-qubit system immersed in a bosonic bath when a quantum feedback is performed.

The general form of the master equation is the one discussed in the previous section,

$$\frac{\partial\rho(t)}{\partial t} = \mathbb{H}\left[\rho(t)\right] + \mathbb{D}'\left[\rho(t)\right]. \tag{6.24}$$

but with the following additional constraints:

- The 6×6 Kossakowski matrix is made by four 3×3 identical blocks $\mathcal{A}_{ij}$;
- The initial $\mathcal{A}_{ij}$, without feedback, is diagonal with real and positive eigenvalues $\mathcal{A} = diag(a_{11},\, a_{22},\, a_{33})$;

- We fix the phase degree of freedom choosing $\ell_1^{(1)} = \sqrt{a_{11}}$ and we further choose the feedback parameters $f_1 = f_3 = 0$, while f_2 is free.
- We ideally set condition $\eta = 1$ for the environment monitoring process.

These conditions yield the following **dissipator**:

$$\mathbb{D}'[\rho(t)] = \sum_{i,j=1}^{3} (\mathcal{A}_{ij} + \widetilde{\mathcal{A}}_{ij}) \left[\Sigma_j \rho(t) \Sigma_i - \frac{1}{2}\{\Sigma_i\Sigma_j, \rho(t)\}\right] \quad \text{with}$$

$$\mathcal{A}_{ij} = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}, \qquad \widetilde{\mathcal{A}}_{ij} = \begin{pmatrix} 0 & i\sqrt{a_{11}}\, f_2 & 0 \\ -i\sqrt{a_{11}}\, f_2 & f_2^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{6.25}$$

Under the same conditions, the **Hamiltonian** contribution to the generator is:

$$\mathbb{H}[\rho] = -i\,[H_{free} + H_{FB},\, \rho]\,, \tag{6.26}$$

where H_{free} is the Hamiltonian of the system without feedback and H_{FB} is the **feedback correction** given by

$$H_{FB} = \frac{1}{2}(L^{(1)\dagger}F + FL^{(1)}) = \sum_{i,j=1}^{3} \left(2\,\mathrm{Re}\{\ell_i^{(1)}\} f_j \mathbb{I}\otimes\mathbb{I} + \epsilon_{ijk}\mathrm{Im}\{\ell_i^{(1)}\} f_j \Sigma_k + \mathrm{Re}\{\ell_i^{(1)}\} f_j S_{ij}\right)$$
$$= 2\sqrt{a_{11}}\, f_2\, \mathbb{I}\otimes\mathbb{I} + \sqrt{a_{11}}\, f_2\, S_{12}. \tag{6.27}$$

It is important to stress that the Hamiltonians H_{free} arising from the weak coupling limit are such that the action of the unitary evolution related to the free Hamiltonian and the purely dissipative ones (in absence of feedback) must commute.

From this requirement, we find (see Appendix C.1) that all the elements in $\mathcal{A}_{ij}$ must be identical to each other, $a = a_{11} = a_{22} = a_{33}$ so that the free Hamiltonian must then be of the following form:

$$H_{free} = h\,\Sigma_1 + b\,\Sigma_2 + c\,\Sigma_3 + d\,(S_{11} + S_{22} + S_{33}). \tag{6.28}$$

Since we want to inspect the dissipative entanglement capabilities of the environment, we neglect Hamiltonian entangling terms as the two-qubit matrices S_{ii} and choose for simplicity

$$H_{free} = \Sigma_3. \tag{6.29}$$

Therefore, our working scenario becomes:

$$\frac{\partial\rho(t)}{\partial t} = -i\left[\Sigma_3 + \sqrt{a} f_2 S_{12},\, \rho(t)\right] + \sum_{i,j=1}^{3} (\mathcal{A}_{ij} + \widetilde{\mathcal{A}}_{ij}) \left[\Sigma_j \rho(t) \Sigma_i - \frac{1}{2}\{\Sigma_i\Sigma_j, \rho(t)\}\right] \tag{6.30}$$

with

$$\mathcal{A}_{ij} = a\mathbb{I}_3\,, \tag{6.31}$$

$$\widetilde{\mathcal{A}}_{ij} = \begin{pmatrix} 0 & i\sqrt{a}\, f_2 & 0 \\ -i\sqrt{a}\, f_2 & f_2^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{6.32}$$

We now proceed to study how the feedback parameter f_2 contributes to the entanglement generation in the short and long-time regimes.

6.2.1 Feedback and short time entanglement

Let us now investigate if there exists at least one initial state which becomes entangled as soon as the system starts evolving in time accordingly to the above master equation (6.30).

For this purpose we consider the following initial states (which are the subsystem of the X-states which are both pure and separable):

$$\begin{aligned}\rho_a(0) &= |0\rangle\langle 0| \otimes |0\rangle\langle 0|,\\ \rho_b(0) &= |1\rangle\langle 1| \otimes |0\rangle\langle 0|,\\ \rho_c(0) &= |0\rangle\langle 0| \otimes |1\rangle\langle 1|,\\ \rho_d(0) &= |1\rangle\langle 1| \otimes |1\rangle\langle 1|.\end{aligned} \tag{6.33}$$

The general condition for entanglement creation at short times is given by

$$\langle u|(\mathcal{A}+\widetilde{\mathcal{A}})|u\rangle\langle v|(\mathcal{C}^T+\widetilde{\mathcal{C}}^T)|v\rangle < \left|\langle u|\mathrm{Re}\{\mathcal{B}+\widetilde{\mathcal{B}}\}+iH^{(12)}|v\rangle\right|^2. \tag{6.34}$$

In the previous formula $H^{(12)}$ is the coefficients matrix of the interacting Hamiltonian $H_{int}=\sum_{i,j=1}^{3} H_{ij}^{(12)}\,(\sigma_i\otimes\sigma_j)=\sqrt{a}\,f_2\,S_{12}$. In our case:

$$H^{(12)} = \begin{pmatrix} 0 & \sqrt{a}\,f_2 & 0\\ \sqrt{a}\,f_2 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}. \tag{6.35}$$

In the particular case in which the block matrices in the Kossakowski matrix are all equal to each other, the condition (6.34) reads

$$\langle u|(\mathcal{A}+\widetilde{\mathcal{A}})|u\rangle\langle v|\,(\mathcal{A}^T+\widetilde{\mathcal{A}}^T)\,|v\rangle < \left|\langle u|\mathrm{Re}\{\mathcal{A}+\widetilde{\mathcal{A}}\}+iH^{(12)}|v\rangle\right|^2. \tag{6.36}$$

Explicit calculations of the previous quantity (see Appendix (C.3)) leads to

1. For $\rho_a(0)$ and $\rho_d(0)$: $(a+\sqrt{a}f_2)^2<0$ which is never satisfied.
2. For $\rho_b(0)$ and $\rho_c(0)$: $-4\,a\,f_2^2<0$ which is always satisfied for $f_2\neq 0$.

Notice that (a must be greater than zero for the Kossakowski matrix to be positive definite). Moreover, the Hamiltonian feed-back term $H^{(12)}$ does not contribute since $H^{(12)}|v\rangle=0$ for all $|v\rangle$ associated to the chosen separable X-states. Therefore, the entanglement generation at short times is only due to dissipative processes.

Hence, there is a class of initial states, those that are convex combinations of $\rho_a(0)$ and $\rho_d(0)$ that are never entangled by the dissipative dynamics even when controlled by feedback, if chosen the way we did. On the other hand, there there are states which become entangled only thanks to the chosen feedback. General initial states are of course possible giving rise to a rich phenomenology whose features are essentially expounded by the above two classes and their mixing.

Rather then investigating the plethora of different scenarios attached to generic initial states, we will focus upon knowing if the entanglement generated by the dissipation can be preserved in the asymptotic limit, which is the aim of the next Section.

6.2.2 Feedback and entanglement in the asymptotic state

In order to study the entanglement behaviour in the long-time regime, we need to find the asymptotic state of the system. The shape of the differential equations of the evolution is very similar to the one already studied in Section 3.2, however, there we focused only on the dissipator of the evolution, here we also have an Hamiltonian term. Unfortunately the Hamiltonian term in (6.30) does not commute with the asymptotic state (3.69) found for the dynamics in Section 3.2, we have therefore to find a faithful state of the dynamics we are currently studying in order to build the asymptotic manifold. Interestingly enough all the analysis done in Section 3.2, a part for what just stated about the asymptotic state, is still valid in this new context. Based on Section 3.2, in the following we mostly focus on the action of the feedback on the entanglement of the asymptotic state.

We adopt the Fano decomposition of the two-qubit density matrix:

$$\rho(t) = \frac{1}{4}\left(\mathbb{I}\otimes\mathbb{I} + \sum_{i=1}^{3}\rho_{0i}(t)\,\mathbb{I}\otimes\sigma_i + \sum_{i=1}^{0}\rho_{i0}(t)\,\sigma_i\otimes\mathbb{I} + \sum_{i,j=1}^{0}\rho_{ij}(t)\,\sigma_i\otimes\sigma_j\right). \tag{6.37}$$

Considering the full dynamics in (6.30), we find a faithful state of the following form, (see Appendix C.4 for the explicit derivation and Section 3.2 for the generator properties):

$$\hat{\rho}_0 = \frac{1}{4}\left(\mathbb{I}\otimes\mathbb{I} + M\,\Sigma_3 - N\,(S_{11}-S_{22}) + L\,S_{12} + R\,S_{33}\right). \tag{6.38}$$

The explicit expressions of the coefficients L, M and N are derived in Appendix C.4: we report them here only to highlight their non-trivial dependence on the parameter a of $\mathcal{A}$ of the free dynamics and on the feedback parameter f_2:

$$M = -\frac{2\sqrt{a}f_2\left(18a^3+15a^2f_2^2+2a\left(f_2^4+1\right)+f_2^2\right)}{36a^4+60a^3f_2^2+a^2\left(21f_2^4+4\right)+2af_2^2\left(f_2^4+2\right)+f_2^4} \tag{6.39}$$

$$N = \frac{a^2f_2^2\left(6a+f_2^2\right)}{36a^4+60a^3f_2^2+a^2\left(21f_2^4+4\right)+2af_2^2\left(f_2^4+2\right)+f_2^4} \tag{6.40}$$

$$R = \frac{af_2^2\left(18a^2+3af_2^2+2\right)}{36a^4+60a^3f_2^2+a^2\left(21f_2^4+4\right)+2af_2^2\left(f_2^4+2\right)+f_2^4} \tag{6.41}$$

$$L = \frac{4a^2f_2^2}{36a^4+60a^3f_2^2+a^2\left(21f_2^4+4\right)+2af_2^2\left(f_2^4+2\right)+f_2^4}\,. \tag{6.42}$$

As for the dynamics in Section 3.2, the commutant associated with the generator of (6.30) is still given by $\{\mathbb{I}, S\}$ (where $S=\sum_{i=1}^{3}S_{ii}$), hence generated by the orthogonal projectors $P=\frac{1}{4}\left(1-\frac{S}{2}\right)$ and $Q=1-P$. Then, elements of the asymptotic convex manifold, namely the generic stationary states are obtained as the convex combinations

$$\rho_{asy} = \lambda\,\frac{P\hat{\rho}_0P}{\mathrm{tr}(P\hat{\rho}_0)} + (1-\lambda)\frac{Q\hat{\rho}_0Q}{\mathrm{tr}(Q\hat{\rho}_0)}, \tag{6.43}$$

where $0\le\lambda\le1$. In Appendix C.4 it is shown that the parameter λ parameter in (6.43) can be expressed as a function of the constant of motion $\tau=\sum_{i=1}^{3}\rho_{ii}$ (where ρ_{ii} are the coefficients in

the last term of the Fano decomposition in (6.37)):

$$\rho_{asy} = \frac{P\hat{\rho}_0 P}{\mathrm{tr}(P\hat{\rho}_0)}\frac{1-\tau}{4} + \frac{Q\hat{\rho}_0 Q}{\mathrm{tr}(Q\hat{\rho}_0)}\frac{3+\tau}{4}. \tag{6.44}$$

As seen in Section 2.3, any initial state is mapped into an element of the asymptotic manifold; indeed, the commutant algebra of the Kraus operators in (6.30) coincides with the commutant of the set containing the Hamiltonian beside the Kraus operators. Thus, all initial states with the same τ will be mapped into the same asymptotic state (6.44).

Furthermore, the asymptotic states are all X-states (see Appendix C.2 for a brief mathematical motivation of this property). One has then an analytic expression for the asymptotic concurrence of any initial state $C_\infty(\rho)$:

$$C_\infty(\rho) = 2\ \max\{0, D_1, D_2\}, \tag{6.45}$$

$$D_1 = \sqrt{\frac{(\tau-2R)^2}{(2R+3)^2}} - \frac{1}{2}\sqrt{\frac{(\tau+3)^2\left((2R+1)^2-4M^2\right)}{(2R+3)^2}}, \tag{6.46}$$

$$D_2 = \sqrt{\frac{(\tau+3)^2\left(4N^2+L^2\right)}{(2R+3)^2}} - \frac{1}{2}\sqrt{\frac{(2R(\tau+1)+\tau-3)^2}{(2R+3)^2}}. \tag{6.47}$$

Plotting D_2 we find it to be always negative, while for D_1 we find the graph in Figure 6.1 (putting for simplicity $a = 10$):

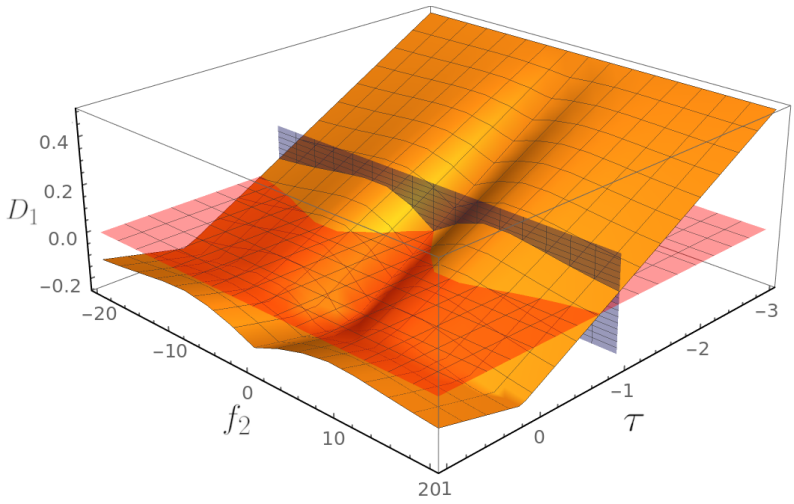

Figure 6.1: D_1 with respect to the feedback f_2 and the constant of motion τ. The red plane highlights when D_1 becomes bigger than zero (namely there is entanglement in the asymptotic state); while the blue plane locates the states with $\tau = -1$.
The curve corresponding to $f_2 = 0$ gives the behaviour of the concurrence in absence of the feedback action, we can see then that some state entanglement is preserved thanks to the feedback; in particular it is possible to evaluate a specific value for f_2 for which the concurrence has a local maximum.

To better show the action of the feedback, we draw a 2D graph, considering the case with no feedback ($f_2 = 0$) with the case $f_2 = 5$ (which is a value close to the local maximum of the concurrence):

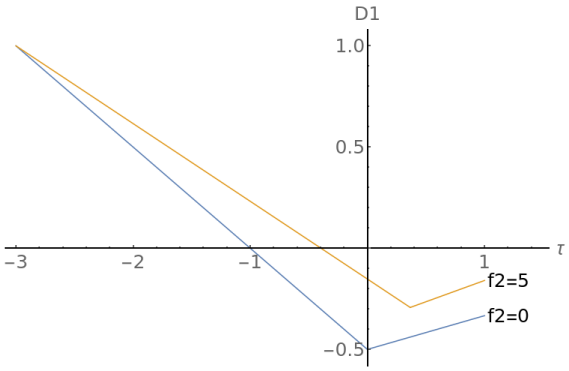

Figure 6.2: D_1 with respect to τ for two fixed values of the feedback. We remind again that when $D_1 > 0$ the state characterized by τ is entangled.

From Figure 6.2, it is possible to see that there are some states whose entanglement is preserved in the asymptotic regime thanks to the feedback protocol. In particular, considering again the initial states (6.33), which we remind to be pure and separable, we find that $\tau_{\rho_b} = \tau_{\rho_c} = -1$, hence from Figure 6.1, we see that the feedback action not only keeps the entanglement in the asymptotic regimes, it is also able to generate it!

Hitherto we kept the discussion on an abstract and as general as possible level without specifying any experimental set up where the theoretical ideas discussed above might be concretely implemented: there indeed exists a variety of scenarios, such as atom optics and cavity QED, opto-mechanical systems, superconducting circuits and quantum dots. Possible realizations of the theoretical results just presented are therefore many and may differ a lot according to the kind of set up one decides to settle. The discussion of this matter opens a wide topic, involving efficiency tests as well, which goes beyond the scopes of this thesis. We recommend [49] for a review of quantum control experiments actually performed in laboratory, where efficiency comparison between different protocols and set ups are illustrated. However it is worth noticing that, while some studies about entanglement generation via feedback already exist (see [49] and its references), the main novelty of our approach is to generate entanglement via a dissipative action: namely to engineer the environment in a way that by itself it becomes able to produce entanglement and make it persist asymptotically. Concretely, we act via measurement-based feedback in order to modify conveniently the Kossakowski matrix of the master equation, rather than trying to stabilize the state of the qubits.

Conclusions

In this thesis we addressed the issue of entanglement generation and persistence in open quantum systems.

It was already known that by suitably engineering the environment one can generate entanglement and make it persist asymptotically. In this thesis we have instead tried to understand how to use the monitoring and feedback (namely quantum control) in order to somehow engineer the entanglement production by gathering information about the system dynamics through the monitoring of the environment in which the system evolves.

In the thesis, we showed how the feedback protocol can be exploited to enhance, preserve and even generate entanglement between two qubits immersed in a dissipative environment. This issue is particularly relevant because the ability of preserving entanglement is extremely important in view of the amount of applications entanglement has in modern quantum technologies and because of its fragility with respect to dissipation.

In particular our original idea was to adapt the feedback protocol in order to modify conveniently the Kossakowski matrix of the master equation, rather than trying to stabilize the qubit state (as it is usually done). In other words, we act in order to create entanglement through dissipative processes by manipulating via feedback the correlation functions of the environment, which are embodied in the Kossakowski matrix of the system.

The principal advantage of this approach is to create environments capable of generating entanglement rather than directly acting on the qubit state.

To do so, we firstly addressed the conditions for entanglement generation in a dissipative environment. In particular, we studied the dynamics of two qubits interacting with an environment, resulting in a master equation with a definite structure given in (3.32). We illustrated the conditions on the bath correlations (namely on the Kossakowski matrix), according to the initial state of the 2-qubit system, in order to have entanglement generation even in the long time regime. A first major achievement of this thesis, that can be found in the last part of Chapter 3, is the extension of the preliminary results given in [5, 6, 7]: as a matter of fact, in the articles cited, the dynamics under exam was purely dissipative without considering any Hamiltonian contribution to the GKSL master equation. In this thesis we were able to extend the validity of the results related to entanglement generation and preservation to wider scenarios with more general GKSL master equations, including unitary terms as well.

The second part of the thesis is instead devoted to the quantum control protocol. After illustrating the main known features of quantum control and the mathematical tools of stochastic calculus, we added and specialized the feedback protocol to the previously mentioned dynamics in order to enhance the generation of entanglement. We managed to explicit find the asymptotic

states for a particular class of two-qubit open dynamics: namely, two qubits interacting with an environment subject to quantum filtering via homodyne detection and Markovian quantum feedback.

In the last Chapter are contained our major results relative to the feedback protocol. There we show how to adapt the known feedback master equation to our specific dynamics and we show how to enhance the generation of entanglement by externally tuning the feedback parameter. As we mentioned at the beginning of the Conclusions, the main original scope was to engineer entanglement generating environment by modifying the Kossakowski matrix of the system. Choosing a particularly simple feedback procedure with only one control parameter in order to simplify the dynamics and find an analytical result, we showed that it is possible to generate entanglement through dissipation in cases in which it would not happen without the feedback action. The major drawback of this protocol is the introduction of a new noise source in the dynamics, which is negligible only in an ideal case.

This thesis thus opens the way to the study of how to optimize more structured and complex feedback drives, i.e. a more complex markovian feedback protocol with several tunable feedback parameters. Or other kind of feedback protocols may be implemented, for instance the cited Bayesian feedback, where the resulting master equation is non-Markovian. It is supposed that Bayesian feedback may perform better than the markovian one, since it constantly updates the state of the system as soon as a measurement is performed via filtering, but the former increases dramatically the computational complexity.

Appendix A

Operator Algebras

A.1 Definitions

Algebras are collections of objects, most commonly measurable sets, commuting functions defined over a set Ω, or non-commuting operators on a Hilbert space, which enjoy certain closure properties. According on the particular kind of closure properties they satisfy, they take different names. The algebras encountered in this work are the following.

- **A *measure algebra*** $\mathcal{F}$ is characterized by the following properties:

  1. $\mathcal{F}$ includes the empty subset.
  2. $\mathcal{F}$ is closed under complementarity: if $A \subset \mathcal{F}$ also the complementary A_c of A is a subset of $\mathcal{F}$.
  3. $\mathcal{F}$ is closed under *pairwise* unions: $A \cup B \subseteq \mathcal{F}$ if $A, B \subseteq \mathcal{F}$.

- **A measure σ-*algebra*** $\mathcal{F}$ is characterized by the following properties:

  1. $\mathcal{F}$ includes the empty subset.
  2. $\mathcal{F}$ is closed under complementarity: if $A \subset \mathcal{F}$ also the complementary A_c of A is a subset of $\mathcal{F}$.
  3. $\mathcal{F}$ is closed under *countable* unions: $\bigcup_{n=1}^{\infty} A_n \subseteq \mathcal{F}$ if $A_n \subseteq \mathcal{F}$.

- **A C^*-*algebra*** $\mathcal{F}$ is a complex algebra whose elements are bounded and continuous linear operators defined on an Hilbert space. It is characterized by the following properties, given two elements $A, B \in \mathcal{F}$ and two complex numbers α, β

  1. $\left(A^\dagger\right)^\dagger = A$.
  2. $(\alpha\, A + \beta\, B)^\dagger = \alpha^* A^\dagger + \beta^* B^\dagger$.

3. $(AB)^\dagger = B^\dagger A^\dagger$.

4. The algebra is endowed with a norm satisfying $\|A^\dagger\| = \|A\|\ \|A^\dagger A\| = \|A\|^2$, with respect to which all Cauchy sequences $\hat{X}_n \in \mathcal{F}$ such that $\|\hat{X}_n - \hat{X}_m\| \mapsto 0$ when $n, m \to \infty$, converge to $\hat{X} \in \mathcal{F}$.

The norm $\|\cdot\|$ is the standard one for operators on a Hilbert space $\mathcal{H}$:

$$\|\hat{X}\| = \sup_{\varphi\in\mathcal{H}} \frac{\|\hat{X}\varphi\|}{\|\varphi\|} \ .$$

Beside $\|A^\dagger\| = \|A\|\ \|A^\dagger A\| = \|A\|^2$, it also satisfies the standard relations:

1. $||A|| \geq 0$ with the identity satisfied if and only if $A = 0$.

2. $||\alpha\, A|| = ||\alpha||\ ||A||$.

3. $||A + B|| \leq ||A|| + ||B||$.

4. $||AB|| \leq ||A||\ ||B||$.

• **A *von Neumann-algebra*** $\mathcal{F}$ is a special case of C^*-algebra, it fulfils weaker closure requirements than being closed with respect to the norm. Indeed, von Neumann algebra are closed with respect to both the strong and weak operator topologies defined by the following convergence requirements: given a sequence $\{\hat{X}_n\}_{n\in\mathbb{N}}$ of bounded operators acting on the Hilbert space $\mathcal{H}$

1. weak-convergence: $\hat{X}_n$ converges weakly to $\hat{X} \in \mathcal{F}$ if for all $|\varphi_{1,2}\rangle \in \mathcal{H}$

$$\lim_{n\to+\infty} \langle\varphi_1|(\hat{X}_n - \hat{X})\varphi_2\rangle = 0 \ ;$$

2. strong convergence: $\hat{X}_n$ converges strongly to $\hat{X} \in \mathcal{F}$ if for all $|\varphi\rangle \in \mathcal{H}$

$$\lim_{n\to+\infty} \|(\hat{X}_n - \hat{X})|\varphi\rangle\| = 0 \ .$$

The algebra $\mathcal{B}(\mathcal{H})$ of all bounded operators on on $\mathcal{H}$ is a C^*-algebra, the simplest example being given by finite dimensional matrix algebras. Given a sub-algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$,

1. the *commutant* $\mathcal{A}' \subseteq \mathcal{B}(\mathcal{H})$ of $\mathcal{A}$ is the set of all linear operators commuting with all elements of $\mathcal{A}$;

2. the *bi-commutant* $\mathcal{A}'' \subseteq \mathcal{B}(\mathcal{H})$ of $\mathcal{A}$ is the set of all linear operators commuting with all elements of $\mathcal{A}'$.

3. In general $\mathcal{A} \subseteq \mathcal{A}$"; $\mathcal{A}$ is *commutative* if $\mathcal{A} \subseteq \mathcal{A}'$.

Notice that norm-convergence implies strong convergence which in its turn implies weak-convergence, therefore, the norm closure of a sub-algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is contained in the strong-closure which is in its turn contained in the weak-closure. However, weak and strong closures coincide and equal the bi-commutant $\mathcal{A}$" (von Neumann theorem).

For finite level systems, matrix algebras are norm, weakly and strongly closed, so that the notion of C^* and von Neumann algebras coincide. It is not so on an infinite dimensional Hilbert space $\mathcal{H}$; for instance, the norm closure of the algebra of matrices of any dimension yields the C^* algebra of all compact operators $\hat{X}$, those bounded operators such that $\sqrt{\hat{X}\hat{X}}$ has discrete spectrum with only finite multiplicities apart, possibly, for the zero eigenvalue. Instead, the weak and strong-closures are the whole C^* algebra $\mathcal{B}(\mathcal{H})$ of bounded operators whose commutant is trivial, $\mathcal{B}(\mathcal{H})' = \{\lambda\mathbb{I}\}$ and whose bi-commutant is then $\mathcal{B}(\mathcal{H})$ itself.

For a comprehensive study of von Neumann and C^*-algebras, we recommend [10].

A.2 The algebra of symmetric Kraus operators Σ_k and S_{ij}

The multiplication rules of the operator defined in (3.54) and (3.55) are as follows:

$$\begin{aligned}
\Sigma_i\Sigma_j &= 2\delta_{ij}\mathbb{I}_2\otimes\mathbb{I}_2 + i\sum_{k=1}^{3}\epsilon_{ijk}\Sigma_k + S_{ij},\\
S_{ij}\Sigma_k &= \delta_{ik}\Sigma_j + \delta_{jk}\Sigma_i + i\Sigma_{l=1}^{3}\epsilon_{ikl}S_{lj} + i\Sigma_{l=1}^{3}\epsilon_{jkl}S_{il},\\
\Sigma_k S_{ij} &= \delta_{ik}\Sigma_j + \delta_{jk}\Sigma_i - i\Sigma_{l=1}^{3}\epsilon_{ikl}S_{lj} - i\Sigma_{l=1}^{3}\epsilon_{jkl}S_{il},\\
S_{ij}S_{kl} &= 2\left(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}\right)\left(\mathbb{I}_2\otimes\mathbb{I}_2\right) + i\sum_{r=1}^{3}\left(\delta_{ik}\epsilon_{jlr}+\delta_{jk}\epsilon_{ilr}+\delta_{il}\epsilon_{jkr}\delta_{jl}\epsilon_{ikr}\right)\Sigma_r\\
&\quad - \left(2\delta_{ij}\delta_{kl}-\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk}\right)S + 2\left(\delta_{ij}S_{kl}+\delta_{kl}S_{ij}\right)\\
&\quad - \delta_{ik}S_{jl} - \delta_{il}S_{jk} - \delta_{jk}S_{il} - \delta_{jl}S_{ik}.
\end{aligned} \tag{A.1}$$

where $S := \sum_{i=1}^{3} S_{ii}$.

One thus sees that by multiplying linear combinations of the the operators Σ_k and S_{ij} one obtains again linear combinations of them; they then constitute an algebra.

It is also useful to provide the explicit forms of the commutators:

$$[S_{ij}, S_{kl}] = 2i\left[\delta_{ik}\epsilon_{jlr}+\delta_{jk}\epsilon_{ilr}+\delta_{il}\epsilon_{jkr}+\delta_{jl}\epsilon_{ikr}\right]\Sigma_r, \tag{A.2}$$

$$[\Sigma_i, \Sigma_j] = 2i\epsilon_{ijk}\Sigma_k, \tag{A.3}$$

$$[\mathcal{S}_{ij}, \Sigma_k] = 2i\left(\epsilon_{ikl}S_{lj}+\epsilon_{jkl}S_{li}\right). \tag{A.4}$$

Explicitly they are

$$
\begin{array}{lll}
[\Sigma_1, \Sigma_2] = 2i\Sigma_3 & [\Sigma_1, \Sigma_3] = -2i\Sigma_2 & [\Sigma_2, \Sigma_3] = 2i\Sigma_1 \\
[S_{22}, \Sigma_1] = -4iS_{32} & [S_{11}, \Sigma_2] = 4iS_{31} & [S_{11}, \Sigma_3] = -4iS_{21} \\
[S_{33}, \Sigma_1] = 4iS_{32} & [S_{33}, \Sigma_2] = -4iS_{31} & [S_{22}, \Sigma_3] = 4iS_{21} \\
[S_{12}, \Sigma_1] = -2iS_{31} & [S_{12}, \Sigma_2] = 2iS_{32} & [S_{12}, \Sigma_3] = 2i\left(S_{11} - S_{22}\right) \\
[S_{13}, \Sigma_1] = 2iS_{21} & [S_{13}, \Sigma_2] = 2i\left(S_{33} - S_{11}\right) & [S_{13}, \Sigma_3] = -2iS_{23} \\
[S_{23}, \Sigma_1] = 2i\left(S_{22} - S_{33}\right) & [S_{23}, \Sigma_2] = -2iS_{21} & [S_{23}, \Sigma_3] = 2iS_{13} \\
[S_{12}, S_{13}] = 2i\Sigma_1 & [S_{12}, S_{23}] = -2i\Sigma_2 & [S_{13}, S_{23}] = 2i\Sigma_3 \\
[S_{11}, S_{12}] = 4i\Sigma_3 & [S_{11}, S_{13}] = -4i\Sigma_2 & [S_{11}, S_{23}] = 0 \\
[S_{22}, S_{12}] = -4i\Sigma_3 & [S_{22}, S_{23}] = 4i\Sigma_1 & [S_{22}, S_{13}] = 0 \\
[S_{33}, S_{13}] = 4i\Sigma_2 & [S_{33}, S_{23}] = -4i\Sigma_1 & [S_{33}, S_{12}] = 0.
\end{array}
\tag{A.5}
$$

From the previous expressions it is immediate to notice that the commutant of the dynamics in (3.56) and (6.30) is given by the identity and $S = \sum_{i=1}^{3} S_{ii}$.

Appendix B

Completely positive semigroups and constants of the motion

Complete positivity is the key issue underlying the techniques regarding the stationary states of dynamical semigroups briefly introduced in Section 2.3.

Indeed, if a map Λ^* on a C^* algebra $\mathcal{B}(\mathcal{H})$ of bounded operators on a Hilbert space $\mathcal{H}$ is completely positive and unital, $\Lambda^*[\mathbb{I}] = \mathbb{I}$, then it is in particular 2-positive and thus Schwartz-positive; namely,

$$\Lambda^*[\hat{X}^\dagger\,\hat{X}] \geq \Lambda^*[\hat{X}^\dagger]\,\Lambda[\hat{X}] = (\Lambda^*[\hat{X}])^\dagger\,\Lambda^*[\hat{X}] \geq 0\ . \tag{B.1}$$

This can be proved as follows: the 2×2 operator matrix

$$P_X = \begin{pmatrix} \hat{X}^\dagger X & \hat{X}^\dagger \\ \hat{X} & \mathbb{I} \end{pmatrix} = \begin{pmatrix} \hat{X}^\dagger \\ \mathbb{I} \end{pmatrix} \begin{pmatrix} \hat{X} & \mathbb{I} \end{pmatrix}$$

is positive semi-definite. Thus, such is also

$$\Lambda^* \otimes \mathrm{id}_2[P_X] = \begin{pmatrix} \Lambda^*[\hat{X}^\dagger\,\hat{X}] & \Lambda^*[\hat{X}^\dagger] \\ \Lambda^*[\hat{X}] & \mathbb{I} \end{pmatrix}\ .$$

Therefore, in particular,

$$\begin{aligned} \begin{pmatrix} \langle\psi|, & -\langle\psi|\Lambda[\hat{X}^\dagger] \end{pmatrix} \Lambda^* \otimes \mathrm{id}_2[P_X] \begin{pmatrix} |\psi\rangle \\ -\Lambda[\hat{X}^\dagger]|\psi\rangle \end{pmatrix} &= \langle\psi|\Lambda^*[\hat{X}^\dagger\,\hat{X}]|\psi\rangle\ -\ \langle\psi|\Lambda^*[\hat{X}^\dagger]\,\Lambda^*[\hat{X}]|\psi\rangle \\ &= \langle\psi|D_t[\hat{X},\hat{X})|\psi\rangle \geq 0\ , \end{aligned}$$

for al $|\psi\rangle \in \mathcal{H}$, where

$$D_t(\hat{X},\hat{Y}) := \Lambda_t^*[\hat{X}^\dagger\hat{Y}]\ -\ \Lambda_t^*[\hat{X}^\dagger]\,\Lambda_t^*[\hat{Y}]$$

is the quantity introduced in (2.110). Given a faithful state, that is a density matrix $\hat{\rho}$ with non zero eigenvalues and thus invertible, over $\mathcal{B}(\mathcal{H})$, the following expression defines a positive semi-definite bilinear form over $\mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H})$,

$$\rho(\hat{X},\hat{Y}) := \mathrm{Tr}\left(\hat{\rho}\,D_t(\hat{X},\hat{Y})\right)\ . \tag{B.2}$$

Namely, this form is linear in the second argument, anti-linear in the first one and such that $\rho(\hat{X}, \hat{X}) \geq 0$, the equality holding if and only f $\hat{X}$ is Λ_t^*-invariant. It then satisfies the Cauchy Schwartz inequality

$$\Big||\rho(\hat{X}, \hat{Y})\Big|^2 \leq \rho(\hat{X}, \hat{X})\, \rho(\hat{Y}, \hat{Y}) \ . \tag{B.3}$$

Thus, if $\Lambda_t^*[\hat{X}] = \hat{X}$, then $D_t(\hat{X}, \hat{X}) = 0$ and $\rho\Big(D_t(\hat{X}, \hat{Y})\Big) = 0$, so that the invertibility of $\hat{\rho}$ yields

$$\Lambda_t^*[\hat{X}^\dagger\, \hat{Y}] = \hat{X}^\dagger\, \Lambda_t^*[\hat{Y}] \ , \tag{B.4}$$

for all $\hat{Y} \in \mathcal{B}(\mathcal{H})$.

Appendix C

Dissipative dynamics and entanglement generation

In this Appendix we aim to give all the mathematical procedures adopted to find the asymptotic state of the dynamics in (6.30).

C.1 Hamiltonian evolution and Dissipator commutation

As shown in Section 2.2.3, the Hamiltonian terms in the master equations arising from applying the weak-coupling limit techniques must commute with the dissipative ones; namely, for all open system operators $\hat{X}$ it must hold that

$$\mathbb{H}_{free}[\mathbb{D}[\hat{X}]] = \mathbb{D}[\mathbb{H}_{free}[\hat{X}]] \ . \tag{C.1}$$

. In the following we derive these Hamiltonian terms for the evolution under exam in Chapter 6, namely for

$$\mathbb{H}_{free}[\rho] = -i \left[H_{\text{free}} \ , \rho \right], \tag{C.2}$$

$$\mathbb{D}[\rho] = \sum_{i,j=1}^{3} \mathcal{A}_{ij} \left[\Sigma_j \rho \Sigma_i - \frac{1}{2} \left\{ \Sigma_i \Sigma_j, \rho \right\} \right], \tag{C.3}$$

where $\mathcal{A}_{ij} = \text{diag}\left\{a_{11}, a_{22}, a_{33}\right\}$.

Notice that the constraint (C.1) does involve the Hamiltonian without feedback, H_{free}: this is because the Markovian approximation is performed before the introduction of the feedback protocol.

Now we want to impose that the action of (C.3) and (C.2) commute over any state ρ

$$[\mathbb{H}, \mathbb{D}] = 0. \tag{C.4}$$

To do so, it is convenient to restrict the calculation in the 3×3 subspace given by the first three Bell states $\left\{|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle\right\}$.

Basically we rewrite every operator X as $PXP + QXQ$, where $P = |\psi_4\rangle\langle\psi_4|$ and $Q = 1 - P$ are orthogonal projectors, and then we work only with the 3×3 sub-matrix of QXQ (a row and a column will be full of zeros, so it is immediate to find the 3×3 sub-matrix).

In this subspace we label the action of Σ_k and S_{ij} as Σ_k^Q and S_{ij}^Q, the explicit expression of these operators and of the Bell states is given in (C.28) and (C.33).

The most general density matrix in this subspace is

$$\rho^Q = \begin{pmatrix} \rho_{11} & \rho_{r12} + i\rho_{i12} & \rho_{r13} + i\rho_{i13} \\ \rho_{r12} - i\rho_{i12} & \rho_{22} & \rho_{r23} + i\rho_{i23} \\ \rho_{r13} - i\rho_{i13} & \rho_{r23} - i\rho_{i23} & \rho_{33} \end{pmatrix}, \tag{C.5}$$

where the subscript r and i stand for real and imaginary part.

The action of the Hamiltonian evolution and of the dissipator can be written as a 9×9 matrix acting on a 9-vector whose entries are the one of the previous general density matrix

$$\mathbb{D}_{9\times9}\left[\rho_9^Q\right] = \begin{pmatrix} -4\left(a_{11}+a_{33}\right) & 4a_{11} & 4a_{33} & 0 \\ 4a_{11} & -4\left(a_{11}+a_{22}\right) & 4a_{22} & 0 \\ 4a_{33} & 4a_{22} & -4\left(a_{22}+a_{33}\right) & 0 \\ 0 & 0 & 0 & R \end{pmatrix} \begin{pmatrix} \rho_{11} \\ \rho_{22} \\ \rho_{33} \\ \rho_{r12} \\ \rho_{i12} \\ \rho_{r13} \\ \rho_{i13} \\ \rho_{r23} \\ \rho_{i23} \end{pmatrix}, \tag{C.6}$$

where R is a 6×6 matrix

$$R = -2\, diag\{a_{22}+a_{33},\, 4a_{11}+a_{22}+a_{33},\, a_{11}+a_{22},\, a_{11}+a_{22}+4a_{33},\, a_{11}+4a_{22}+a_{33},\, a_{11}+a_{33}\}. \tag{C.7}$$

Now we do the same for the Hamiltonian part, we consider the most general Hamiltonian in the 3×3 subspace

$$H^Q_{\text{free}} = \begin{pmatrix} h_{11} & h_{r12} + ih_{i12} & h_{r13} + ih_{i13} \\ h_{r12} - ih_{i12} & h_{22} & h_{r23} + ih_{i23} \\ h_{r13} - ih_{i13} & h_{r23} - ih_{i23} & h_{33} \end{pmatrix}, \tag{C.8}$$

then we write its action as a 9×9 matrix

$$\mathbb{H}[\rho_9^Q] = \begin{pmatrix} 0 & 0 & 0 & 2h_{i12} & -2h_{r12} & 2h_{i13} & -2h_{r13} & 0 & 0 \\ 0 & 0 & 0 & -2h_{i12} & 2h_{r12} & 0 & 0 & 2h_{i23} & -2h_{r23} \\ 0 & 0 & 0 & 0 & 0 & -2h_{i13} & 2h_{r13} & -2h_{i23} & 2h_{r23} \\ -h_{i12} & h_{i12} & 0 & 0 & h_{11}-h_{22} & h_{i23} & -h_{r23} & h_{i13} & -h_{r13} \\ h_{r12} & -h_{r12} & 0 & h_{22}-h_{11} & 0 & h_{r23} & h_{i23} & -h_{r13} & -h_{i13} \\ -h_{i13} & 0 & h_{i13} & -h_{i23} & -h_{r23} & 0 & h_{11}-h_{33} & h_{i12} & h_{r12} \\ h_{r13} & 0 & -h_{r13} & h_{r23} & -h_{i23} & h_{33}-h_{11} & 0 & -h_{r12} & h_{i12} \\ 0 & -h_{i23} & h_{i23} & -h_{i13} & h_{r13} & -h_{i12} & h_{r12} & 0 & h_{22}-h_{33} \\ 0 & h_{r23} & -h_{r23} & h_{r13} & h_{i13} & -h_{r12} & -hi12 & h_{33}-h_{22} & 0 \end{pmatrix} \rho_9^Q. \tag{C.9}$$

We can now impose the condition (C.4), finding that the most general free Hamiltonian compatible with the weak coupling limit takes the form

$$H^Q_{free} = \alpha\Sigma_1^Q + \beta\Sigma_2^Q + \gamma\Sigma_3^Q + \delta\left(S_{11}^Q + S_{22}^Q + S_{33}^Q\right), \tag{C.10}$$

where the coefficients $\alpha, \beta, \gamma, \delta$ are free parameters. Moreover. the commutativity condition puts constrains on the element of the Kossakowski matrix as well: in particular we found that all the diagonal elements of $\mathcal{A}$ must be identical to each other.

These results hold in the 3×3 sub-space characterized by the first three Bell states. The extension to the whole space is easily done: it is sufficient to add back the PXP contribution to every operator X we previously split. In the Bell basis, this corresponds to simply add a row and a column full of zeros but for the $(4,4)$. Since we are adding a diagonal element both to the Hamiltonian and the dissipator, they still commute. Hence we can make use of the Hamiltonian form in (C.10) getting rid of the Qs.

C.2 Mapping X-states into X-states

From the relation gave in Appendix A.2, it is possible to infer that the commutant of $\{\Sigma_i\}_{i=1}^3$ is given by $\{\mathbb{I}_4, S\}$. This is an interesting results because it implies that X-states are always mapped into X-states by the dynamics in (6.30) [35]. As a matter of fact all the X-states can always be written in the following Fano decomposition

$$\rho_X = \frac{1}{4}\left(\mathbb{I}\otimes\mathbb{I} + A\,\sigma_3\otimes\mathbb{I} + B\,\mathbb{I}\otimes\sigma_3 + \sum_{i=1}^{3} C_i\sigma_i\otimes\sigma_i + D_1\,\sigma_1\otimes\sigma_2 + D_2\,\sigma_2\otimes\sigma_1\right), \tag{C.11}$$

namely using only the following set of operators:

$$\mathcal{S} := \{\mathbb{I}_2\otimes\mathbb{I}_2, \sigma_3\otimes\mathbb{I}_2, \mathbb{I}_2\otimes\sigma_3, \sigma_1\otimes\sigma_1, \sigma_2\otimes\sigma_2, \sigma_3\otimes\sigma_3, \sigma_1\otimes\sigma_2, \sigma_2\otimes\sigma_1\}, \tag{C.12}$$

which are closed under multiplication (namely an element of $\mathcal{S}$ multiplied by an element of $\mathcal{S}$ is still an element of $\mathcal{S}$). The remaining 8 operators needed to complete the 16 elements of the basis in the 2-qubit Hilbert space are

$$\mathcal{S}' := \{\sigma_1\otimes\mathbb{I}_2, \mathbb{I}_2\otimes\sigma_1, \sigma_2\otimes\mathbb{I}_2, \mathbb{I}_2\otimes\sigma_2, \sigma_1\otimes\sigma_3, \sigma_2\otimes\sigma_3, \sigma_3\otimes\sigma_1, \sigma_3\otimes\sigma_2\}. \tag{C.13}$$

Given the operators $A, B \in \mathcal{S}$ and $C, D \in \mathcal{S}'$, their compositions satisfy the following rules: $AB \in \mathcal{S}$, $CD \in \mathcal{S}$, $AC \in \mathcal{S}'$.

Hence only $\mathcal{S}$ is closed under multiplication, while combinations of two elements of $\mathcal{S}'$ are mapped in $\mathcal{S}$, and cross-combinations of the two subsets lie in $\mathcal{S}'$.

Now, considering the dynamics in (6.30), the Hamiltonian part contains operators belonging to $\mathcal{S}$, so when they act on an X-state ρ_X they preserve its X shape. The only contributions of the dissipator containing operators of $\mathcal{S}'$ are the diagonal terms $\Sigma_i\rho_X\Sigma_i$ with $i = 1, 2$ (the other $\mathcal{S}'$ elements are absent because of the null entries of the Kossakowski matrix). However, the previous contributions are a composition of two $\mathcal{S}'$ elements and they therefore belong to $\mathcal{S}$, preserving once again the X-shape of the state.

In our case the absence of Kraus elements which destroy the X-structure of the state is due to the peculiar form of the Kossakowski matrix, which has null elements in the entries $(i = 1, j = 3)$ and $(i = 2, j = 3)$.

C.3 Short time entanglement for pure X-states

We want to study the short time entanglement generation given by the dynamics is (6.30).

We consider a generic X-state of two qubits

$$\rho_X = \begin{pmatrix} a & 0 & 0 & w \\ 0 & b & z & 0 \\ 0 & z* & c & 0 \\ w* & 0 & 0 & d \end{pmatrix}, \tag{C.14}$$

where $d = 1 - a - b - c$ is fixed by the normalization. It can also be written as

$$\begin{aligned} \rho_X = & a|0\rangle\langle 0| \otimes |0\rangle\langle 0| + b|0\rangle\langle 0| \otimes |1\rangle\langle 1| + \\ & + c|1\rangle\langle 1| \otimes |0\rangle\langle 0| + d|1\rangle\langle 1| \otimes |1\rangle\langle 1| + \\ & + z|0\rangle\langle 1| \otimes |1\rangle \langle 0| + z^* |1\rangle \langle 0| \otimes |0\rangle\langle 1| + \\ & + w|0\rangle\langle 1| \otimes |0\rangle \langle 1| + w^* |1\rangle \langle 0| \otimes |1\rangle\langle 0|, \end{aligned} \tag{C.15}$$

The partial transposition of the X-state corresponds to

$$\rho_X^{\mathbb{I}\otimes T} = \begin{pmatrix} a & 0 & 0 & z \\ 0 & b & w & 0 \\ 0 & w* & c & 0 \\ z* & 0 & 0 & d \end{pmatrix}. \tag{C.16}$$

We are interested in studying pure and separable states, in order to see if they become entangled by the evolution (we have already stated that considering only pure states do not reduce the generality of the result).

The condition for separability are given by the PPT criterion (see Section 3.1.1)

$$\begin{cases} bc - w^2 \geq 0 \\ ad - z^2 \geq 0 \end{cases}, \tag{C.17}$$

while the ones for the purity of the states are

$$\begin{cases} a = d = w = 0 \\ bc = z^2 \end{cases} \quad \text{or} \quad \begin{cases} b = c = z = 0 \\ ad = w^2 \end{cases}. \tag{C.18}$$

Combining these constrains together, we find that the only pure and separable X-state are the ones with only one of the four real coefficients a, b, c, d different from zero (and in particular equal to one to keep the normalization). Adopting the decomposition in (C.16), they are

$$\begin{aligned} \rho_a &= |0\rangle\langle 0| \otimes |0\rangle\langle 0|, \\ \rho_b &= |1\rangle\langle 1| \otimes |0\rangle\langle 0|, \\ \rho_c &= |0\rangle\langle 0| \otimes |1\rangle\langle 1|, \\ \rho_d &= |1\rangle\langle 1| \otimes |1\rangle\langle 1|. \end{aligned} \tag{C.19}$$

Now we can check the entanglement formation for this class of state at short time via (3.49),

$$\langle u|\mathcal{A} + \widetilde{\mathcal{A}}|u\rangle \left\langle v \left| \mathcal{A}^T + \widetilde{\mathcal{A}}^T \right| v \right\rangle < \left| \left\langle u \left| \mathrm{Re}\{\mathcal{A} + \widetilde{\mathcal{A}}\} + iH^{(12)} \right| v \right\rangle \right|^2, \tag{C.20}$$

where $\mathcal{A}$ and $\widetilde{\mathcal{A}}$ are given by (6.31)-(6.32), while $H^{(12)}$ corresponds to (6.35).

Hence, to evaluate the entanglement formation at short times, we need to evaluate the components of $|u\rangle$ and $|v\rangle$, to do so we need the vectors $|\psi\rangle, |\tilde{\psi}\rangle, |\phi\rangle, |\tilde{\psi}\rangle$ in (3.46)-(3.47) and the rotation matrices U and V in the same reference, from which we recover the elements of $|u\rangle$ and $|v\rangle$ according to (3.50).

In the following we illustrate just one case of the whole calculation just for completeness, the other ones are straightforward.

Case $a = 1$. The vectors $|\psi\rangle, |\tilde{\psi}\rangle, |\phi\rangle, |\tilde{\psi}\rangle$ in this case are

$$\begin{aligned} |\phi\rangle &= |0\rangle \quad \text{and} \quad |\tilde{\phi}\rangle = |1\rangle \\ |\psi\rangle &= |0\rangle \quad \text{and} \quad |\tilde{\psi}\rangle = |1\rangle, \end{aligned} \tag{C.21}$$

hence $U = V = \sigma_1$ and

$$\mathcal{U} = \mathcal{V} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{C.22}$$

Therefore, the vectors u and v are

$$u = \frac{1}{2}\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \quad \text{and} \quad v = \frac{1}{2}\begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}. \tag{C.23}$$

Plugging them in (C.20), we find

$$\left(a - \sqrt{a} f_2\right)^2 < 0 \tag{C.24}$$

which is never satisfied, hence ρ_a is never entangled by the dynamics under exam for every choice of the feedback parameter f_2.

C.4 Finding the asymptotic state

We devote this appendix to the procedure adopted to find the asymptotic state of the dynamics in (6.30). The commutant algebra of both Hamiltonian and Kraus operators appearing in the master equation in (6.30) is given by:

$$\mathcal{D}' = \{\mathbb{I}, S\}, \qquad S = \sum_{i=1}^{3} S_{ii} \tag{C.25}$$

where S_{ij} are the operators defined in (3.55). The commutant algebra is clearly commutative and generated by the orthogonal projection operators

$$P := \frac{1}{4}\left(\mathbb{I} - \frac{S}{2}\right), \quad Q := \mathbb{I} - P\,. \tag{C.26}$$

From the general theory [15], one knows how to use them to build the family of asymptotic states: given an initial state $\rho(0)$ which asymptotically tends to $\hat{\rho}$ and a faithful asymptotic state $\hat{\rho}_0$, one retrieves $\hat{\rho}$ from.

$$\hat{\rho} = \frac{P\hat{\rho}_0 P}{\mathrm{tr}(P\hat{\rho}_0)}\,\mathrm{tr}(P\rho(0)) + \frac{Q\hat{\rho}_0 Q}{\mathrm{tr}(Q\hat{\rho}_0)}\,\mathrm{tr}(Q\rho(0)). \tag{C.27}$$

Let's now consider the Bell states:

$$|\psi_1\rangle := \frac{|00\rangle + |11\rangle}{\sqrt{2}}, \quad |\psi_2\rangle := \frac{|01\rangle + |10\rangle}{\sqrt{2}}, \quad |\psi_3\rangle := \frac{|00\rangle - |11\rangle}{\sqrt{2}}, \quad |\psi_4\rangle := \frac{|01\rangle - |10\rangle}{\sqrt{2}}. \tag{C.28}$$

We notice the following equivalence

$$P = \frac{1}{4}\left(\mathbb{I} - \frac{S}{2}\right) = |\psi_4\rangle\langle\psi_4|. \tag{C.29}$$

Now, if we substitute in (C.27) $\rho(0) = \hat{\rho}_0$ (namely we use the faithful state as initial state), we find:

$$\hat{\rho} = P\hat{\rho}_0 P + Q\hat{\rho}_0 Q = \hat{\rho}_0. \tag{C.30}$$

On the Bell basis, we have $P = diag\{0,0,0,1\}$ and $Q = diag\{1,1,1,0\}$, hence:

$$Q\hat{\rho}_0 Q = \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} & 0 \\ \rho_{12}^* & \rho_{22} & \rho_{23} & 0 \\ \rho_{13}^* & \rho_{23}^* & \rho_{33} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad P\hat{\rho}_0 P = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho_{44} \end{pmatrix}. \tag{C.31}$$

As a consequence, the faithful state has to be of the following shape:

$$\hat{\rho}_0 = \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} & 0 \\ \rho_{12}^* & \rho_{22} & \rho_{23} & 0 \\ \rho_{13}^* & \rho_{23}^* & \rho_{33} & 0 \\ 0 & 0 & 0 & \rho_{44} \end{pmatrix} = \begin{pmatrix} & & & 0 \\ & \rho^Q & & 0 \\ & & & 0 \\ 0 & 0 & 0 & \rho_{44} \end{pmatrix}. \tag{C.32}$$

We have seen that the 4×4 density matrix can be decomposed onto two orthogonal subspaces using the two projectors obtained from the set of the invariant operators.

As a consequence, we can reduce the investigation of the dynamics just to the 3×3 subspace characterized by the projection Q, because the matrix element ρ_{44} will be then uniquely determined by the normalization of the state.

Moreover, in this 3×3 subspace, we can restrict the action of the operators $\{\Sigma_i,\, S_{ij}\}$ onto the subspace spanned by the Bell states $\{\psi_1, \psi_2, \psi_3\}$ which is indeed mapped into itself by $\{\Sigma_i,\, S_{ij}\}$.

We shall denote by $\{\Sigma_i^Q,\, S_{ij}^Q\}$ the restrictions: they read

$$\Sigma_1^Q = 2\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix},\quad \Sigma_2^Q = 2\begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix},\quad \Sigma_3^Q = 2\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix},$$
$$S_{11}^Q = 2\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\quad S_{22}^Q = 2\begin{pmatrix}-1&0&0\\0&1&0\\0&0&1\end{pmatrix},\quad S_{33}^Q = 2\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix},$$
$$S_{12}^Q = 2\begin{pmatrix}0&0&i\\0&0&0\\-i&0&0\end{pmatrix},\quad S_{13}^Q = 2\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix},\quad S_{23}^Q = 2\begin{pmatrix}0&i&0\\-i&0&0\\0&0&0\end{pmatrix}. \tag{C.33}$$

We can thus also restrict the master equation (6.30) to states supported by the same subspace, where it becomes:

$$\frac{\partial\rho_Q(t)}{\partial t} = -i\left[\Sigma_3^Q + \sqrt{a}f_2 S_{12}^Q,\, \rho_Q\right] + \sum_{i,j=1}^{3}(\mathcal{A}_{ij} + \widetilde{\mathcal{A}}_{ij})\left[\Sigma_j^Q\,\rho_Q(t)\,\Sigma_i^Q - \frac{1}{2}\{\Sigma_i^Q\Sigma_j^Q,\,\rho_Q(t)\}\right] \tag{C.34}$$

Where ρ_Q is the 3×3 sub-matrix in (C.32).

We found the faithful asymptotic state of the dynamics in the 3×3 space and on the Bell basis, then we found the solution in the 4×4 space by adding a row and a column at the end of the 3×3 density matrix, in order to have an asymptotic state as in (C.32). In particular, after imposing the normalization, we found:

$$\hat{\rho}_0 = \begin{pmatrix}A&0&W&0\\0&B&0&0\\W^*&0&C&0\\0&0&0&B\end{pmatrix} \tag{C.35}$$

Then we build the family of asymptotic state via the following convex combination:

$$\hat{\rho} = \frac{P\hat{\rho}_0 P}{tr(P\hat{\rho}_0)}\lambda + \frac{Q\hat{\rho}_0 Q}{tr(Q\hat{\rho}_0)}(1-\lambda) = \tag{C.36}$$
$$= \begin{pmatrix}A'(\lambda-1)&0&W'(1-\lambda)&0\\0&C'(\lambda-1)&0&0\\W'(1-\lambda)^*&0&D'(\lambda-1)&0\\0&0&0&\lambda\end{pmatrix} \tag{C.37}$$

Hence, we notice that the parameter λ of the convex combination corresponds to the last diagonal term of the asymptotic state. Moreover, it can be proven that is is a constant of motion and it is related to the τ parameter:

$$\lambda = \mathrm{tr}(P\rho(0)) = \frac{1-\tau}{4} \tag{C.38}$$

Where $\rho(0)$ is the initial state and τ is the sum of the $i = j$ terms of the Fano decomposition of the initial state.

For completeness, the coefficient in the (C.37) are

$$A' = \frac{\left(36a^4 + 72a^3 f_2^2 + a^2\left(23f_2^4 + 4\right) + 2af_2^2\left(f_2^4 + 4\right) + f_2^4\right)}{108a^4 + 216a^3 f_2^2 + 3a^2\left(23f_2^4 + 4\right) + 2af_2^2\left(3f_2^4 + 8\right) + 3f_2^4} \tag{C.39}$$

$$B' = \frac{\left(36a^4 + 24a^3 f_2^2 + a^2\left(15f_2^4 + 4\right) + 2af_2^6 + f_2^4\right)}{108a^4 + 216a^3 f_2^2 + 3a^2\left(23f_2^4 + 4\right) + 2af_2^2\left(3f_2^4 + 8\right) + 3f_2^4} \tag{C.40}$$

$$C' = \frac{\left(36a^4 + 120a^3 f_2^2 + a^2\left(31f_2^4 + 4\right) + 2af_2^2\left(f_2^4 + 4\right) + f_2^4\right)}{108a^4 + 216a^3 f_2^2 + 3a^2\left(23f_2^4 + 4\right) + 2af_2^2\left(3f_2^4 + 8\right) + 3f_2^4} \tag{C.41}$$

$$W' = -\frac{4\sqrt{a}f2\left(2ia^{3/2}f_2 + 18a^3 + 15a^2 f_2^2 + 2a\left(f_2^4 + 1\right) + f_2^2\right)}{108a^4 + 216a^3 f_2^2 + 3a^2\left(23f_2^4 + 4\right) + 2af_2^2\left(3f_2^4 + 8\right) + 3f_2^4} \tag{C.42}$$

Moving to the canonic base, we find the following correspondences:

$$Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad P = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{C.43}$$

$$\hat{\rho}_0 = \begin{pmatrix} a' & 0 & 0 & w' \\ 0 & b' & 0 & 0 \\ 0 & 0 & c' & 0 \\ w'^* & 0 & 0 & d' \end{pmatrix} \qquad \hat{\rho}_= \begin{pmatrix} a & 0 & 0 & w \\ 0 & b & z & 0 \\ 0 & z & b & 0 \\ w'^* & 0 & 0 & d \end{pmatrix}. \tag{C.44}$$

The asymptotic state is then

$$\hat{\rho}_0 = \frac{1}{4}\left[\mathbb{I}_2 \otimes \mathbb{I}_2 + M\Sigma_3 - N\left(S_{11} + S_{22}\right) + RS_{33} - LS_{12}\right], \tag{C.45}$$

with coefficients

$$M = -\frac{2\sqrt{a}f_2\left(18a^3 + 15a^2 f_2^2 + 2a\left(f_2^4 + 1\right) + f_2^2\right)}{36a^4 + 60a^3 f_2^2 + a^2\left(21f_2^4 + 4\right) + 2af_2^2\left(f_2^4 + 2\right) + f_2^4} \tag{C.46}$$

$$N = \frac{a^2 f_2^2\left(6a + f_2^2\right)}{36a^4 + 60a^3 f_2^2 + a^2\left(21f_2^4 + 4\right) + 2af_2^2\left(f_2^4 + 2\right) + f_2^4} \tag{C.47}$$

$$R = \frac{af_2^2\left(18a^2 + 3af_2^2 + 2\right)}{36a^4 + 60a^3 f_2^2 + a^2\left(21f_2^4 + 4\right) + 2af_2^2\left(f_2^4 + 2\right) + f_2^4} \tag{C.48}$$

$$L = \frac{4a^2 f_2^2}{36a^4 + 60a^3 f_2^2 + a^2\left(21f_2^4 + 4\right) + 2af_2^2\left(f_2^4 + 2\right) + f_2^4} \tag{C.49}$$

Remarkably enough the initial state is always mapped into the asymptotic manifold because from (A.5) it is possible to see that the commutant of the algebra formed by the dissipator operators coincides with the commutant of the algebra given by the Hamiltonian operators and the dissipator together, in accordance with the theory of asymptotic states outlined in Section 2.3.

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