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RESEARCH ARTICLE

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Increasing the quantum tunneling probability through a learned ancilla-assisted protocol

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Abstract
Increasing the probability of quantum tunneling between two states, while keeping constant the resources of the underlying physical system, is a task of key importance in several physical contexts and platforms, including ultracold atoms confined by double-well potentials and superconducting qubits. We propose a novel ancillary assisted protocol showing that when a quantum system—such as a qubit—is coupled to an ancilla, one can learn the optimal ancillary component and its coupling, to increase the tunneling probability. As a case study, we consider a quantum system that, due to the presence of an energy detuning between two modes, cannot transfer by tunneling the particles from one mode to the other. However, it does it through a learned coupling with an ancillary system characterized by a detuning not smaller than the one of the primary system. We provide several illustrative examples for the paradigmatic case of a two-mode system and a two-mode ancilla in the presence of interacting particles. This reduces to a qubit coupled to an ancillary qubit in the case of one particle in the system and one in the ancilla. Our proposal provides an effective method to increase the tunneling probability in all those physical situations where no direct improvement of the system parameters, such as tunneling coefficient or energy detuning, is either possible or resource efficient. Finally, we also argue that the proposed strategy is not hampered by weak coupling to noisy environments.

Keywords Quantum tunneling · Ultracold atoms · Machine learning · Double wells

1 Introduction

Quantum tunneling is a distinctive property of quantum mechanics that plays a crucial role in many physical processes (Roy 1986). It also has plenty of diverse technological applications, such as in tunnel diodes (Esaki 1958) and in scanning tunneling microscopes (Binnig and Rohrer 1986). A paradigmatic quantum tunneling scenario is provided by the Josephson effect (Barone and Paternò 1982), which finds several practical applications in high-precision measurements (Tinkham 1996) and in superconducting qubits for quantum computing (Makhlin et al. 2001; Castelvecchi 2017). Another major example is provided by the tunneling of ultracold atoms confined in double- and multi-well potentials (Morsch and Oberthaler 2006), which plays a crucial role in quantum simulations of lattice models and in ultracold Josephson junctions (Lewenstein et al. 2012).

Both in the superconducting and ultracold Josephson junctions, the underlying effective model is based on the presence of two modes: for superconducting junctions, the superconductor on the left and the one on the right, and for ultracold junctions, the condensate on the left well and the one on the right. Despite the difference in their microscopic models, in both cases, one has a tunneling between the two modes. The intensity of the tunneling—more precisely the strength of the tunneling coefficient γ (sometimes called also t or K)—depends on the details of the barrier or constriction weakly coupling the two superconductors (in superconducting Josephson junctions) or on the height of the energy barrier between the two wells of the double-well potential (in ultracold Josephson junctions). One can tune the tunneling coefficient, which is proportional to the so-called Josephson energy, but of course with some conceptual and practical limitations. For instance, in the case of ultracold atoms in a double-well potential, one can reduce the energy barrier to increase the tunneling coefficient. However, if the barrier is reduced too much, the system can no longer be accurately described by a two-mode model. In superconducting junctions and SQUIDS, one can reduce the Josephson energy by piercing the superconducting device by a magnetic field

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(Tinkham 1996; Makhlin et al. 2001), but increasing it may require another design and/or the construction of a new junction. Therefore, a general challenge in this context is the ability to increase the quantum tunneling probability in the cases one cannot operate directly on the parameters of the tunneling system at hand.

To fix the ideas, consider a typical two-mode Hamiltonian for a qubit system

$$H_S = -\gamma_S \sigma_x - \Delta_S \sigma_z, \tag{1}$$

where $\sigma_{x,z}$ are the Pauli matrices in the basis of the two modes. Then, the term proportional to σ_x with coefficient γ_s describes the tunneling between the two modes, while the term proportional to σ_z with coefficient Δ_S is the energy detuning between the two wells (e.g., in an ultracold Josephson junction, it expresses the fact that the minima of the two wells are not at the same energy). We consider the model in Eq. 1 because it is the typical two-mode Hamiltonian without interacting terms (that will be considered later) and it is routinely used to describe single qubits.

Suppose now that Δ_S is not zero: then an initial up-state along z cannot be flipped to down with certainty, at any time (see Sect. 2). To increase the tunneling probability, i.e., maximizing the probability to find the state down, one might simply reduce Δ_S. If this is not possible for the system at hand, then one wants to propose a method for increasing the tunneling probability.

The idea we propose and the novelty of the paper is the following: learning an ancillary system A, coupled to the target quantum system S, to increase the tunneling properties of the (primary) system. Since the tunneling properties have an extremely rich field of applications, let us list a few typical tasks one may want to address:

- *a)* If the detuning term Δ_S (or the presence of interaction terms in the system, see below) limits the tunneling from one mode, say up, to the other, say down, then the coupling with an ancilla can be used to increase the tunneling probability and possibly make it equal to one. It is clear that one should resort to an ancilla having a detuning Δ_A not smaller than Δ_S; otherwise, one could simply use the ancilla to perform the task.
- *b)* A related problem is decreasing the tunneling time rather than the tunneling probability, specifically, determining the first instance when the probability reaches one after starting from zero. Consider a purely tunneling Hamiltonian $H_S = -\gamma_S \sigma_x$, i.e., Eq. 1 with $\Delta_S = 0$. Then, an up-state along z can be flipped to down with certainty at a time roughly proportional to $1/\gamma_S$ (also in the presence of interaction terms). In such a case, one aims at reducing this time. The simplest strategy would be then to increase γ_S. However, for example, in superconducting qubits, this would typically require to change the Josephson current and therefore the Josephson junction with which the qubit is implemented. In SQUID geometries, one can vary the resource γ_S, but usually reduce it. Then, our approach can come to help. However, again, the sake of consistency with the given resources demands that one should use as ancilla another qubit with tunneling constant $\gamma_A \leq \gamma_S$. Otherwise, if $\gamma_A > \gamma_S$, the required resource, γ_A, would justify using the ancilla A instead of the target system S.
- *c)* A variation of the previous problem in *b)* is the following: the probability as a function of time can have a sinusoidal behavior with frequency ω, or a Fourier transform peaked at a certain ω. One may then be interested not in minimizing the tunneling time at which the probability reaches 1, but in minimizing the frequency ω. Equivalently, one could be interested in maximizing the effective tunneling coefficient or the Josephson energy without using another junction with a larger Josephson energy.
- *d)* One may have other constraints on the resources: for example, not all the couplings between the ancilla and the system can be tailored in the specific platform at hand. Then, a constraint on the form of the coupling system-ancilla has to be set, or on the strength of the coupling. Similarly, one can obtain a desired tunneling property, but at the price of preparing the ancilla in an initial state that is difficult to achieve concretely. In that case, one has to face constraints on the class of learnable initial states of the ancilla.

It is clear that one cannot reach all the goals described in *a)-d)*, or part of them, at the same time. Indeed, it could be that the coupling with the ancilla is such that the system reaches probability one starting from a zero probability, but in a longer time—or that one can decrease the tunneling time, but at the price of needing a larger tunneling coefficient or a very large detuning. In our approach, all these different problems will be addressed through coupling with an ancilla and by optimizing the desired tunneling properties within the given constraints. As a case study, we will primarily focus on problem *a)*, which involves increasing the tunneling probability in the presence of detuning, and we will discuss whether this can be achieved without resorting to an ancilla with a *smaller* detuning energy. Some illustrative results concerning point *b)* will be also presented.

We will develop our approach by considering several interacting bosons in a double-well potential. The reason is two-fold. From one side, this model can be mapped to a spin Hamiltonian with the spin of the system (ancilla) depending on the number of particles in the system (ancilla). In the case of just one particle, one gets the qubit Hamiltonian (1). Moreover, the model describes the low energy behavior of ultracold and superconducting Josephson junctions. The two-mode model we employ, beside its analytical value, permits therefore to pinpoint the main features of more complex macroscopic tunneling phenomena, which include Josephson tunneling between two superconductors (Barone and Paternò 1982) and tunneling between two Bose-Einstein condensates (Javanainen 1986; Smerzi et al. 1997). Specifically, the two-mode model involves systems with (*a*) well-separated energy levels and (*b*) a suitable coupling of two target levels, usually the lowest and the first excited one, between which tunneling occurs. Well-known concrete realizations are given by the left-right transitions in a spatial double-well trap, between up and down spin states, or between $|0\rangle$ and $|1\rangle$ states in quantum bit scenarios (Nielsen and Chuang 2010). Furthermore, the role of the interactions between the target two-level systems and external ones, as for instance large thermal baths, can also be taken into account (Kagan and Leggett 1992; Leggett et al. 1987).

In this context, a natural question arises: can we inverse-design, by learning it, the coupled ancillary systems and their interactions with the target system to amplify the quantum tunneling probability of the latter? Different methods to enhance the probability have been studied, such as chaos-assisted tunneling (Tomsovic and Ullmo 1994) and resonance tunneling (Brodier et al. 2001) or tailoring the energy barrier between the two states, (Kagan 1991; Kagan and Leggett 1992). Here, we explore a different approach consisting in *learning, the ancilla and its interaction with the assigned quantum system*, see Fig. 1.

Ancillary-assisted protocols have been employed in a variety of tasks such as quantum tomography (Altepeter et al. 2003), copying of quantum states (Anselmi and Chefles 2004), metrology (Huang et al. 2016), quantum optics (Junior et al. 2023), and quantum information theory (Lipka-Bartosik et al. 2023). To the best of the authors' knowledge, the learning of ancilla-assisted protocols involving quantum tunneling, especially in scenarios where multiple particles interact both with each other and with their environment, has yet to be explored. The novelty and purpose of our approach consist in the ability to learn and optimize: *1)* the parameters of the ancilla Hamiltonian and its initial state; *2)* the form of the ancilla-system coupling. The most noticeable outcome of our investigation is the setting up of a method that allows to increase the tunneling probability in a system with constrained resources and also in those cases when no exact or approximate analytical approaches are available.

Our starting point is the analytically solvable example of a single boson interacting with N_A ancillary trapped bosons described by a two-mode Bose-Hubbard Hamiltonian, a widely used physical scenario in the study of interacting par-

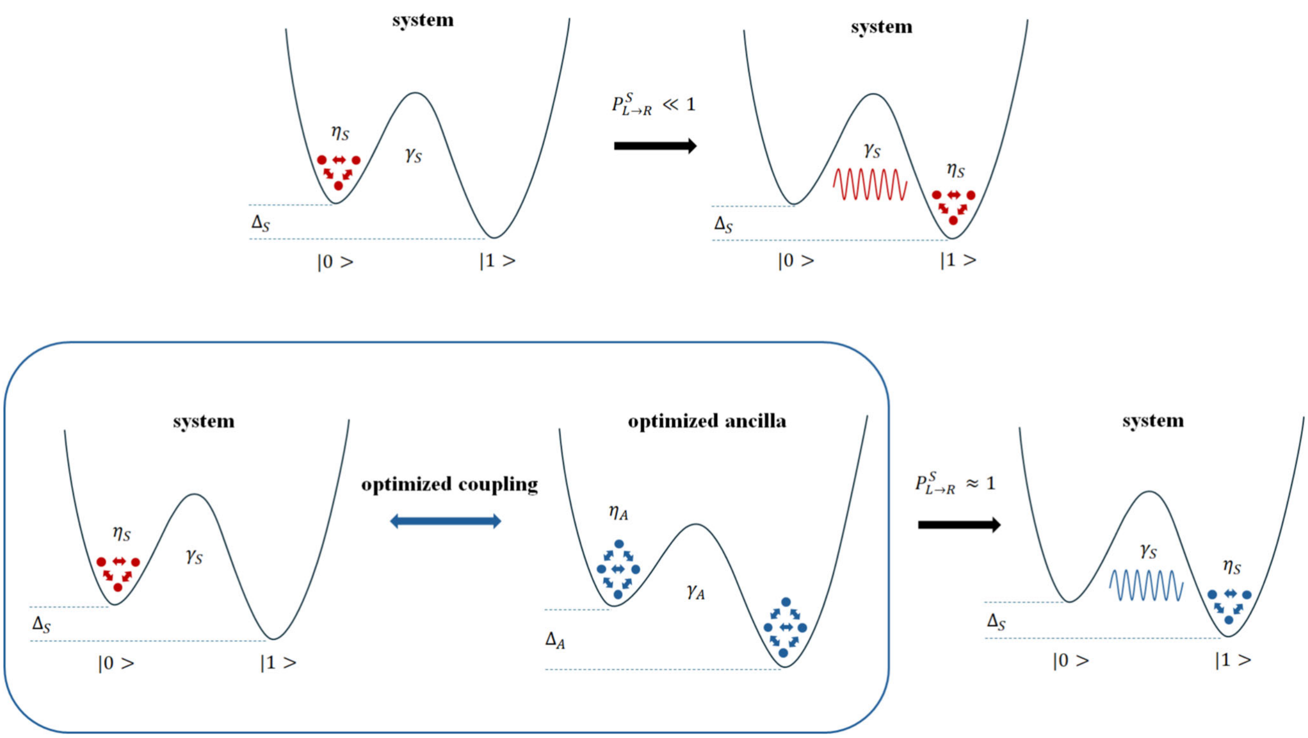


Fig. 1 (Top): In a quantum double-well asymmetric system, the transition probability of particles moving from the left to the right well can be very small, i.e., $P^S_{L\to R} \ll 1$. To implement, e.g., a quantum switch (changing from state $|0\rangle$ to $|1\rangle$), it is essential to manage and control the tunneling probability. (Bottom) Maximizing the tunneling probability can be achieved by learning an ancilla and its coupling with the system, $P^S_{L\to R} \approx 1$

ticles (Smerzi et al. 1997; Jaksch et al. 1998; Milburn et al. 1997). Specifically, we show how to enhance (or decrease) the tunneling probability or decrease the tunneling time from the left well to the right one of a system composed of N_S trapped bosons by coupling them with an automatically learned ancillary double-well system. In the context of quantum computation, such a scenario involves the transitions from a state 0 to a state 1, practically implementing a NOT-gate operation (Nielsen and Chuang 2010). Thus, enhancing the left-to-right tunneling probability or decreasing the tunneling time of a sufficiently large amount of trapped bosons becomes crucial, together with mitigating the potential noise that might reduce the quantum coherence which is necessary for tunneling.

In the following, starting from the aforementioned analytically solvable scenario, we conduct comprehensive simulations including systems and ancillas with many particles, the effects of the interactions and of the presence of a decohering noisy environment (Breuer and Petruccione 2002) affecting both the system and the coupled ancilla. The simulations performed clearly demonstrate how efficient the learning of the optimal ancilla and its coupling is to a tunneling quantum system whose tunneling probability/time one needs to maximize/minimize.

2 Two-mode bosonic systems, ancillary coupling and learning strategy

We focus on a quantum system S consisting of N_S interacting bosons trapped in a double-well potential (Smerzi et al. 1997; Milburn et al. 1997). In the absence of external couplings, the bosons evolve under a two-mode Bose-Hubbard Hamiltonian, H_S:

$$H_S = 4\eta_S(a^\dagger a^\dagger aa + b^\dagger b^\dagger bb) - 2\gamma_S(a^\dagger b + ab^\dagger) - 2\Delta_S(a^\dagger a - b^\dagger b), \quad (2)$$

where γ_S is the tunneling coefficient, Δ_S is a detuning energy, and η_S is an interaction coefficient. The operators a and b are the mode operators for the two modes, and the total number operator $N_S^{tot} = a^\dagger a + b^\dagger b$ is conserved and as well as its expectation values. For the physical system of ultracold atoms in a double-well potential represented in Fig. 1, $a^\dagger$ and $b^\dagger$ create a particles in the right and left wells, and η_S is proportional to the scattering length of the interparticle interaction in each well. For $N_S = 1$, Eq. 2 reduces simply to Eq. 1 and that for large N_S describes two weakly coupled superconductors.

As explained in Appendix A, via the Jordan-Schwinger transformation, one can pass from two-mode annihilation and creation operators to operators $J_{x,y,z}$ obeying the angular momentum algebra. With $J_z = (a^\dagger a - b^\dagger b)/2$ and $J_x = (a^\dagger b + ab^\dagger)/2$ and neglecting terms proportional to N_{tot}, one gets

$$H_S = \eta_S J_z^2 - \gamma_S J_x - \Delta_S J_z, \quad (3)$$

where γ_S controls the tunneling barrier, Δ_S the energy asymmetry, and η_S describes the boson interactions (repulsive if $\eta_S > 0$, attractive if $\eta_S < 0$) (Leggett 2001). For the physical system of ultracold atoms in a double-well potential, typical parameter values are $\gamma_S/k_B \sim 0.5\,nK$, $\Delta_S/k_B \sim 0.2\,nK$, and η_S varies in the range $k_B \cdot [0.1 - 10\,nK]$ (see Sect. 4), k_B denoting the Boltzmann constant.

The dynamics of the bosons is governed by the Liouville-von Neumann equation:

$$\partial_t \rho^{(S)}(t) = -(i/\hbar)\,[H_S, \rho^{(S)}(t)].$$

For an initial state $\rho_L^{(S)} = |N_S\rangle\langle N_S|$, where all N_S bosons are in the left well, the goal is to maximize the tunneling probability $P_{L\to R}(t_*)$ at a time t^*, representing the probability that the bosons transfer from the left to the right well. This is given by

$$P_{L\to R}(t) = \langle 0|\rho_L^{(S)}(t)|0\rangle, \quad (4)$$

with $|0\rangle$ the state with no bosons in the left well and

$$\rho_L^{(S)}(t) = \exp\left(-\frac{i}{\hbar}H_S t\right)\rho_L^{(S)}\exp\left(\frac{i}{\hbar}H_S t\right). \quad (5)$$

For $N_S = 1$, the system reduces to the two-level system (1) by $H_S = -\Delta_S\,\sigma_z - \gamma_S\,\sigma_x$. As shown in Appendix B, the tunneling probability is then

$$P_{L\to R}(t) = \frac{\sin^2(\omega t)}{1 + \frac{\Delta_S^2}{\gamma_S^2}}, \quad (6)$$

with $\hbar\omega = \sqrt{\Delta_S^2 + \gamma_S^2}$. Thus, for asymmetric traps ($\Delta_S \neq 0$), the tunneling probability cannot reach $P_{L\to R}(t) = 1$. Notice that the maximum probability $P_{L\to R}^{max} = \dfrac{\gamma_S^2}{\Delta_S^2 + \gamma_S^2}$ is reached at time $t^* = \dfrac{\pi}{2\hbar\sqrt{\Delta_S^2 + \gamma_S^2}}$. Probability 1 can thus only be achieved by setting $\Delta_S = 0$ at time $t^* = \pi/2\hbar\gamma_S$. The latter can then be decreased by increasing γ_S.

In what follow, we exactly consider the case when the energy asymmetry Δ_S cannot be made too low, nor possibly the tunneling strength too large, as discussed in the Introduction. To solve these limitations and nevertheless increase the tunneling probability, we propose to consider two double-well potentials interacting via a density-density interaction among bosons. The first system, S, with N_S bosons ($\rho^{(S)}$), and governed by a fixed Bose-Hubbard Hamiltonian, is the target system. The second, A, serves as a learnable ancillary system with N_A bosons ($\rho^{(A)}$) and adjustable parameters $\eta_A, \gamma_A, \Delta_A$ that fix the ancilla's Hamiltonian H_A; moreover, a learnable interaction strength α couples the two systems through $H_{int} = \alpha J_z \otimes J_z$:

$$H_A = \eta_A J_z^2 - \gamma_A J_x - \Delta_A J_z, \quad H_{int} = \alpha\, J_z \otimes J_z.$$

Such interaction could be experimentally implemented, e.g., using dipolar or Rydberg atoms (Defenu et al. 2021).

Starting from an uncorrelated state $\rho_L^{(S)} \otimes \rho^{(A)}$, where all N_S bosons are initially in the left well, i.e., $\rho_L^{(S)} = |N_S\rangle\langle N_S|$, the time evolution of the system S is given by

$$\rho_L^{(S)}(t) = \mathrm{Tr}_A\Big(e^{-iH_{SA}t}\rho_L^{(S)} \otimes \rho^{(A)} e^{iH_{SA}t}\Big), \tag{7}$$

with total Hamiltonian:

$$H_{SA} = H_S \otimes I + I \otimes H_A + H_{int}\ ,$$

where I denotes the identity operator. Automatic differentiation is used to optimize α, η_A, Δ_A, γ_A, the initial state $\rho^{(A)}$ of the ancilla, and the time t, to maximize the tunneling probability $P_{L\to R}(t)$ of the N_S bosons in S. A learnable t (within a fixed time window $t \in [0, T_{max}]$) in this context would mean that the algorithm is trying to find the optimal time to reach the maximal probability, within the fixed time window, gradually increasing the evolution time. For the initial uncorrelated state $\rho_L^{(S)} \otimes \rho^{(A)}$, the probability at time t that k bosons in the system are found in the right well, and thus $N_S - k$ in the left one, is given by

$$P_{N_S \to N_S - k}(t) = \langle N_S - k|\rho_L^{(S)}(t)|N_S - k\rangle,$$

where the reduced state of the system S at time t is given by Eq. 7.

Our goal is to maximize the tunneling probability of the N_S bosons in the system S, first in the absence of noise and then in its presence. By means of learning techniques, one aims at increasing the tunneling probability

$$P(\rho_A, \eta_A, \gamma_A, \Delta_A, \alpha, t) = \mathrm{Tr}\Big(\rho_R^{(S)}\rho_L^{(S)}(t)\Big)\ ,$$

where $\rho_R^{(S)} = |0\rangle\langle 0|$ is the state of the system with all particles in the right well and $\rho_L^{(S)}(t)$, in the noiseless case, is given by Eq. 7.

In the presence of noise instead, the system evolves accordingly to an open, irreversible dynamics generated by the following master equation of the so-called Gorini-Kossakowski-Sudarshan-Lindblad form Breuer and Petruccione (2002):

$$\partial_t \rho(t) = -i[H, \rho(t)] + \lambda\left(J_z\rho(t)J_z - \frac{1}{2}\left\{J_z^2, \rho(t)\right\}\right), \tag{8}$$

where $\{X, Y\}$ denotes the anti-commutator and λ is a constant, much smaller than the typical system energy, that measures the strength of the coupling of the system to the environment. This dynamics asymptotically lead to equally distributed mixtures of $0 \le k \le N$ bosons localized in the left well and $N-k$ in the right one (see Appendix D). Also in the noisy case, our strategy consists in coupling the (noisy) system with a (noisy) ancilla. The resulting dynamics is governed by

$$\begin{aligned}\partial_t \rho^{(SA)}(t) &= -i[H_{SA}, \rho^{(SA)}(t)] \\ &+\lambda_A\Big((I\otimes J_z)\,\rho^{(SA)}(t)\,(I\otimes J_z) - \frac{1}{2}\{I\otimes J_z^2,\ \rho^{(SA)}(t)\}\Big) \\ &+\lambda_S\Big((J_z\otimes I)\,\rho^{(SA)}(t)\,(J_z\otimes I) - \frac{1}{2}\{J_z^2\otimes I,\ \rho^{(SA)}(t)\}\Big)\ ,\end{aligned} \tag{9}$$

that has the same form of Eq. 8 but now describes the system S and the ancilla A both independently coupled to two independent environments responsible for the same kind of decoherence effects.

In this case, to calculate the evolved system, we used a fourth-order standard Runge–Kutta method (see Devries and Hasburn 2011, pg. 215). Concretely, the maximization of the tunneling probability P was performed by defining an equivalent minimization problem introducing the loss function $\mathcal{L} : \mathbb{C}^{N_A\times N_A} \times \mathbb{R}^5 \to \mathbb{R}_+$ with $\mathcal{L} = 1 - P$. In the case we want instead to minimize the tunneling probability (see Fig. 5), the loss is set to $\mathcal{L} = P$.

Optimization

We employed a very effective and widely used optimizer in machine learning, ADAM (Kingma and Ba 2015), with a learning rate $lr = 0.01$ and automatic differentiation in PyTorch (Paszke et al. 2017), a machine learning library of the Python programming language (Paszke et al. 2019). Although the use of automatic differentiation is not novel in physical problems and has been used for more then a decade (e.g., Abdelhafez et al. 2019, Leung et al. (2017) 2017, Tamayo-Mendoza et al. (2018) 2018, Di Matteo and Woloshyn (2022) 2022 to mention few among many), it is important to note that it has never been applied in the specific context analyzed in this work, i.e., learning an ancilla and its interaction with the target system with the specific goal of increasing the tunneling probability or decreasing the tunneling time of the target system. In particular, among the many optimization methods we could have chosen, we used the ADAM optimizer since it offers several nice features including adaptive learning rates for each learned parameter, stabilization, and fast convergence.

The learnable parameters η_A, γ_A, Δ_A, α, and t were initialized to 1 without sign constraints during the optimization. The time t was also initialized to 1 and remained positive during learning. Changing the initialization randomly in the range [0, 1] did not change the results. The ancilla $\rho^{(A)}$ was initialized to a random complex density matrix. The number of iterations was chosen to guarantee the convergence of all learned parameters.

At each iteration of the algorithm, the following steps were performed: (1) calculation of the evolved state together with the ancilla, (2) calculation of the tunneling probability, and (3) update of the learnable parameters to better match the target probability.

After learning, the optimized parameters were utilized to construct H_A and H_{int} and consequently H_{SA}. The Hamiltonians H_{SA}, H_S were then employed to produce the plots of the tunneling probability evolution over time using the same evolution functions as during the learning process. The evolution for the noisy system alone (no ancilla) was obtained with the same Runge–Kutta algorithm with $\lambda_A = 0$ and $J_z \otimes I \rightarrow J_z$. This approach allowed us to observe and analyze the behavior of the system's tunneling probability over time, based on the optimized parameters, providing valuable insights into the effectiveness of the learning process and the achieved probability.

3 Numerical results

3.1 1-boson system

In order to test the learning procedure, we started to study the simplest analytically solvable case with $N_S = N_A = 1$, namely, the case of one non-self-interacting ($\eta_S = 0$), tunneling ($\gamma_S \neq 0$) boson with a fixed energy asymmetry Δ_S, coupled to an ancillary double-well potential also trapping a single non-self-interacting and non-tunneling boson ($\eta_A = \gamma_A = 0$) with same energy asymmetry, $\Delta_A = \Delta_S$. We fixed $\Delta_S = \Delta_A = 1$; see more details in Appendix C.

The plots shown in Fig. 2 (and Fig. 8 in the Appendix) provide clear evidence that the learning method rapidly converges to the optimal parameter values that maximize the tunneling probability. In Fig. 2a, one has $\Delta_S = 1$, and therefore, the state cannot tunnel with certainty from state-up to state-down; indeed, the system without the ancilla reaches a maximum probability of 0.2 (with $\gamma_S = 0.5$). Then, we couple the system to a learnable ancilla, reaching unit tunneling probability. In this particular case, we fix also $\gamma_A = 0$, and what is learned is the coupling coefficient α, the initial state of the ancilla, and t. The finding is a main result of the paper showing that if we cannot reduce the energy detuning of the primary system, then through the coupling with an ancilla—crucially not having a smaller energy detuning—one can reach probability one. Notice though that the time of the period of the oscillation is larger. This confirms that one cannot reach objectives *a)* and *b)* of the Introduction simultaneously, and then one has to have clear what is the goal and what resource one can use. We also considered (not shown in figure) the case when γ_A and/or Δ_A are learnable: as an example with $\gamma_S = 0.5$, $\Delta_S = 1$ (as before, the maximum probability without ancilla is 0.2), the learned ancilla parameters are $\gamma_A \approx 0$, $\Delta_A \approx 1.6$ and $\alpha \approx 1$, with maximum tunneling probability being one. We noticed that the fact that ancilla wants to have a small tunneling is peculiar of the non-interacting limit $N_S = N_A = 1$ with $\eta_S = \eta_A = 0$.

To clarify the point that one cannot reach all objectives at the same time, let consider the results presented in Fig. 2b. There, $\Delta_S = 0$, so tunneling probability can reach the value one. Suppose now that one wants to decrease the tunneling time, in this case maintaining time periodic oscillations of the probability. When $\Delta_S = 0$ and the system already reaches unit probability, minimizing the tunneling time can be achieved by incorporating a time-penalizing term into the learning protocol, expressed as $\lambda \|t\|^2$ (we chose $\lambda = 0.01$). In this case, we have fixed $\Delta_A = 1$ and $\gamma_A = 0$. After optimization, we see that the learned interaction strength α works as a resource to decrease the tunneling time: one finds that the learned $\alpha \approx 1$. In this particular cases, also note that we used an ancilla having a smaller tunneling coefficient compared to the system ($\gamma_S = 0.5$, $\gamma_A = 0$). Whether these choices are convenient or possible in the specific context (be qubits, or ultracold atoms or superconducting junctions/SQUIDS) remains again to be seen. It is clear that these results raise the need of a systematic investigation of the possibility to reach with ancillas the desired objective (it could be *a*, *b*, *c*, or *d* or possibly even another) with the available resources and requirements on the ancillary system(s) and the coupling system-ancilla.

Summarizing, in the simple case considered in Fig. 2, the learning algorithm optimizes the coupling α between the two double-well traps, which, in its turn, determines the tunneling probability (see Eq. C13 with $N = 1$ in Appendix C). Specifically, the model consistently learns the optimal value $\alpha = -2$ (expected for maximizing the probability when $N = 1$ and the ancilla is initialized to $|\Psi_A\rangle = |L\rangle$, see Appendix C, Eq. C13). Similarly, a value of $\alpha = +2$ is learned when the ancilla is initialized to $|\Psi_A\rangle = |R\rangle$ (not shown). We also report the time t needed to reach the maximal probability: as expected from the theory $t^* \approx \pi$. Interestingly, in the case when the max probability reached by the system is already one, adding a time minimization constraint in the optimization (see Sect. 2) decreases the time to reach the maximal probability (b) (as expected from Eq. C21 in Appendix C). In the next section, where a more general (not analytically

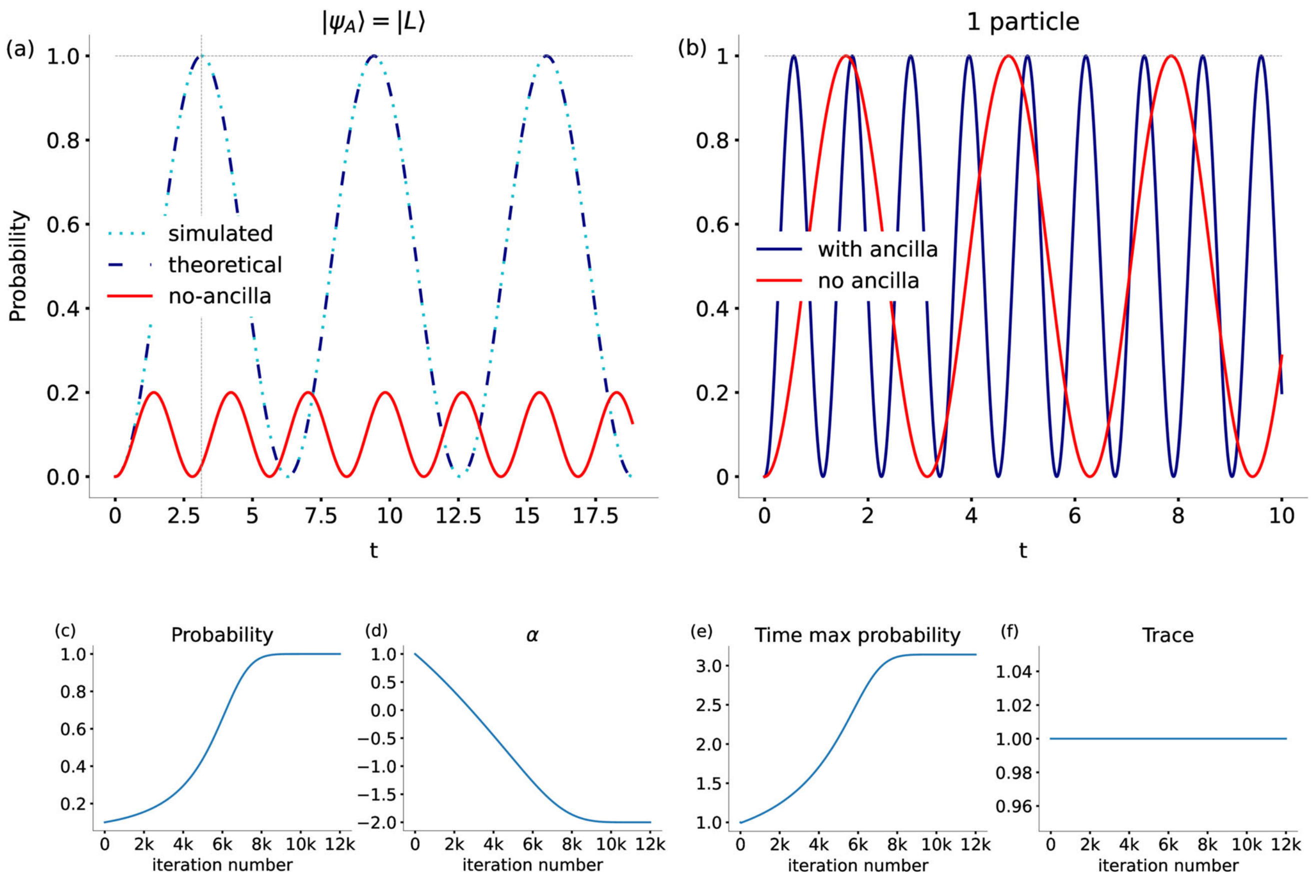


Fig. 2 **a** (Case: $\eta_S = \eta_A = 0, \gamma_S = 0.5, \gamma_A = 0, \Delta_S = \Delta_A = 1$), after learning α, t. Time evolution of the tunneling probability for the system coupled with the ancilla compared to the system without the ancilla, reporting both the theoretical predictions and the optimized results. **b** (Case: $\eta_S = \eta_A = 0, \gamma_S = 1, \gamma_A = 0, \Delta_S = 0, \Delta_A = 1$), after learning α, t. Case of a system that already reaches unit probability: the maximal tunneling probability is reached earlier when the ancilla is coupled to the system. In both cases **a** and **b**, time is reported in $\hbar/\Delta_S$ units, each corresponding to $\sim 10ms$. Evolution, during optimization for the case in **a**, of the tunneling probability (**c**), the learnable parameters α (**d**) and of the time to reach the maximum probability (**e**). Also, we report the trace of the learned ancilla (**f**) as a sanity check

solvable) setting will be considered, the state of the ancilla will not be fixed; rather, it will be part of the optimization process.

3.2 Multi-particle noiseless tunneling and physical resources

The noiseless case, where the number of bosons grows in both the double-well system and ancilla, represents the next natural step in increasing model complexity. In this case, no analytical solution is available. It also functions as a reference to explore the relationships among the learnable parameters governing the phenomenon.

Figure 3 confirms that, despite the tunneling probability of N_S bosons from the left to the right well being vanishingly small in the absence of coupling to an ancilla, it is nevertheless possible to achieve $P^{max}_{L\to R} = 1$ for various configurations involving different sizes of the system and ancilla by learning the interaction between them and the initial state of the ancilla (the method was tested up to 10 system particles with similar results). Notably, we tested the results (in this and the following sections) for robustness: randomly perturbing the learned parameters in the range [1, 5]% of their magnitude did not change significantly the tunneling probability.

Interpreting the learnable parameters $\eta_A, \gamma_A, \Delta_A, \alpha$ as learning resources a natural question is how they behave with respect to varying the number of bosons in the system and ancilla (we refer to Sect. 4 for a discussion of a realistic set-up in which such parameters can be tuned).

Qualitatively, the plots of Fig. 4 indicate that, for larger systems, there is a rapid initial decrease of $\eta_A, \gamma_A, \Delta_A, \alpha$ with increasing ancilla size followed by a stabilization. Note that not directly constraining the learned tunneling coefficient of the ancilla γ_A always results in a larger one compared to that of the system ($\gamma_A \in [0.5, 28]$, $\gamma_S = 0.5$). From one side, this can be a limitation and confirms the qualitative conclusion that it is necessary to define the specific quantity to be optimized (e.g., the tunneling probability) while simultaneously considering the available resources. However, notice that, also in this case, the learned ancillary detuning is always

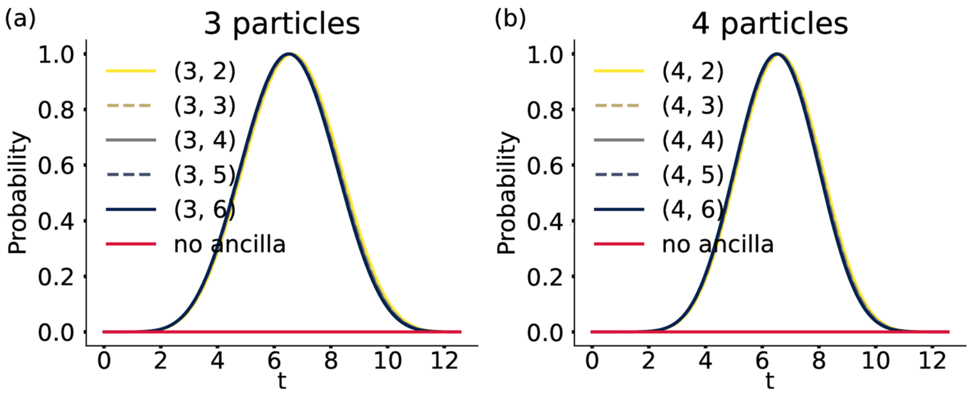


Fig. 3 **a**, **b** Time evolution of tunneling probability for (N_S, N_A) particles in the system ($\eta_S = 1$, $\gamma_S = 0.5$, $\Delta_S = 1$) and ancilla with $N_S = 3, 4$ and $N_A = 2, \cdots, 6$ compared with that of the system with no coupling (red line)

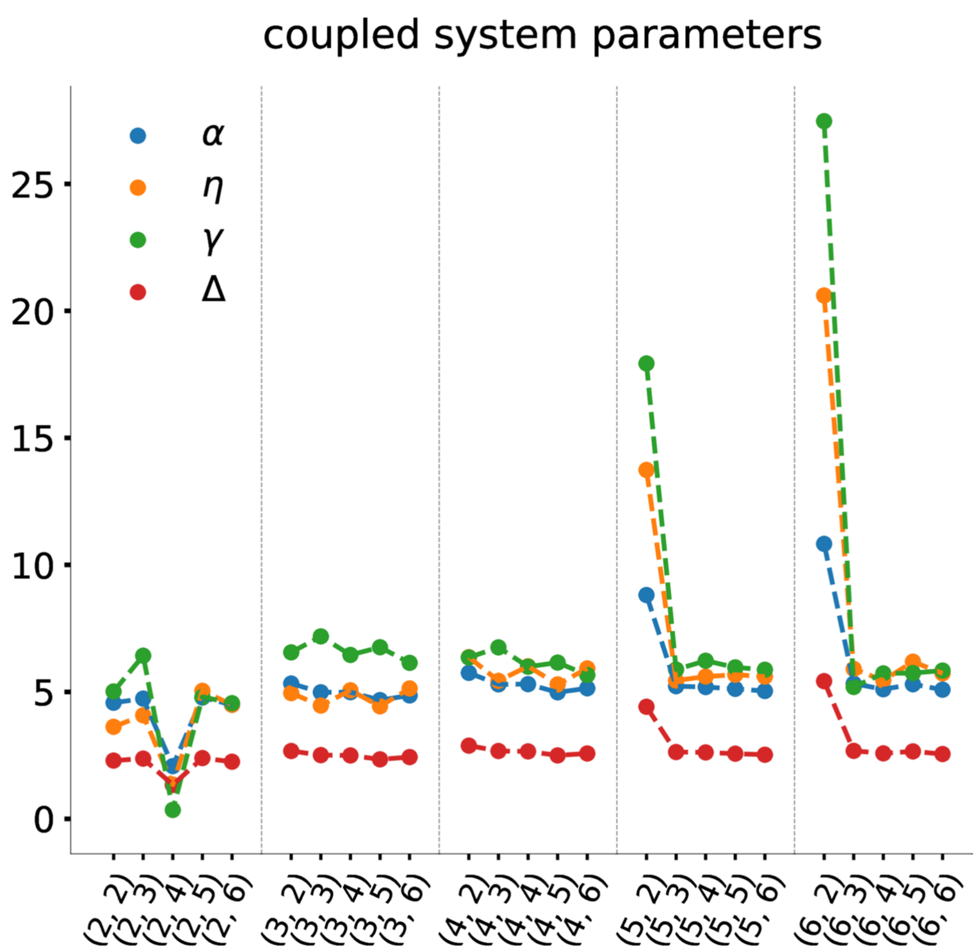
larger than the system one ($\Delta_A \in [2, 6]$, $\Delta_S = 1$). As a consequence, one cannot directly use the learned ancillary system to optimize the tunneling probability and boost it to unit. Thus, our method remains particularly useful in this context.

We conclude the noiseless case observing that, interestingly, as shown in Fig. 5, the same strategy can be applied to decrease the tunneling probability.

3.3 Decoherence: a proof of concept

We now show that the method developed in the previous sections is, as a matter of principle, applicable even in the presence of decoherence without severely affecting the ancilla parameters and initial state so to make them physically unattainable. As already mentioned, the dynamics asymptotically lead to equally distributed mixtures of $0 \le k \le N$

Fig. 4 Variability of the learned parameters α, η, γ and Δ in function of the couples (N_S, N_A) reporting system and ancilla particle numbers (x axis). The system parameters are $\eta_S = 1$, $\gamma_S = 0.5$, and $\Delta_S = 1$. The learned parameters belong to the following ranges: $\alpha \in [2, 11]$, $\eta_A \in [1.5, 21]$, $\gamma_A \in [0.5, 28]$, $\Delta_A \in [2, 6]$. Their units of measure are reported in the text at the beginning of Sect. 2



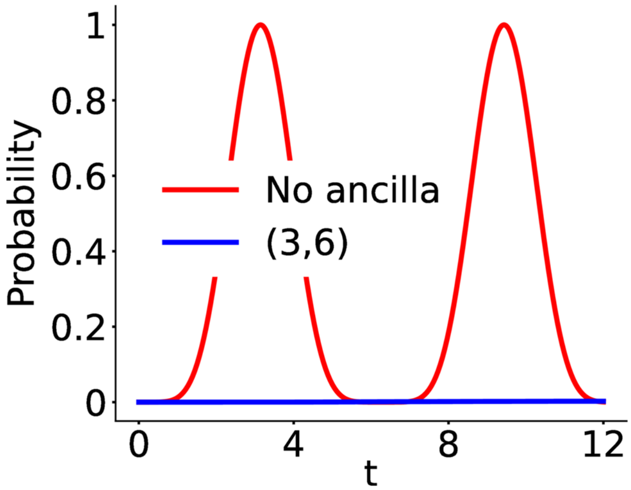


Fig. 5 Time evolution of tunneling probability after optimization for a system of $N_S = 3$ particles with $\eta_S = 0$, $\gamma_S = 1$, and $\Delta_S = 0$ coupled with an ancilla of $N_A = 6$ particles. Red line: system with no coupling

bosons localized in the left well and $N-k$ in the right one (see Appendix D). As a consequence of the coupling to an ancilla independently affected by decoherence, the time evolution of the joint initial state of the system and ancilla is given by the master Eq. 10, where λ_S, λ_A are positive constants determining the noise strength which, for our simulations, have been fixed, in the spirit of the so-called weak-coupling limit, to a value of 0.01.

Figure 6a reports the evolution of the tunneling probability where the learnable parameters are in this case randomly chosen. We note that, in agreement with Eq. D23 (see Appendix D), the asymptotic state reaches a maximal tunneling probability of 0.2. This is the case both for a system of 4 particles without ancilla and a system plus ancilla both with 4 particles. Figure 6b reports the evolution of the tunneling probability in time for $N_S = 4$ with $N_A = 4, 5, 6$ particles. Interestingly, we note that, although the same asymptotic probability is attained, when we optimize the parameters, (1) the asymptotic state probability is reached within orders of magnitude faster, and (2) for a brief period, a significantly higher probability can be achieved compared to the non-optimized scenarios. To increase the experimental implementability of the learned ancilla states, we also constrained them, during optimization, to be diagonal in the well-occupation number states. Results are qualitatively similar to those reported in Fig. 6 (see Appendix, Section E).

4 Experimental considerations

A possible experimental platform in which one has the possibility to couple the system with an ancilla and tune the parameters of both the ancilla and the system-ancilla coupling could be provided by dipolar ultracold atoms in optical potentials (Bloch et al. 2008; Lahaye et al. 2009; Kawaguchi and Ueda 2012). Notice that here, our goal is not to make a systematic study of the proposal of a quantum simulation of coupling ancilla system (that would anyway be interesting *per se*), but just to exhibit a system in which the coupling with a tunable ancilla is possible and to do simple estimates of the involved parameters.

The setup would envisage as system of an ultracold gas in a double-well potential, realized, for instance, by combining a harmonic potential and an optical lattice potential (Albiez et al. 2005) (see Fig. 7).

One could then consider an ancilla as a spatially separated ultracold gas, also in a double-well potential. The geometric configuration needed for such a setup could be provided by a ladder configuration, such as that introduced in Hofferberth et al. (2007). Tuning the parameters of the external potentials for the system and the ancilla and the distance between the two traps, one could control the parameters described

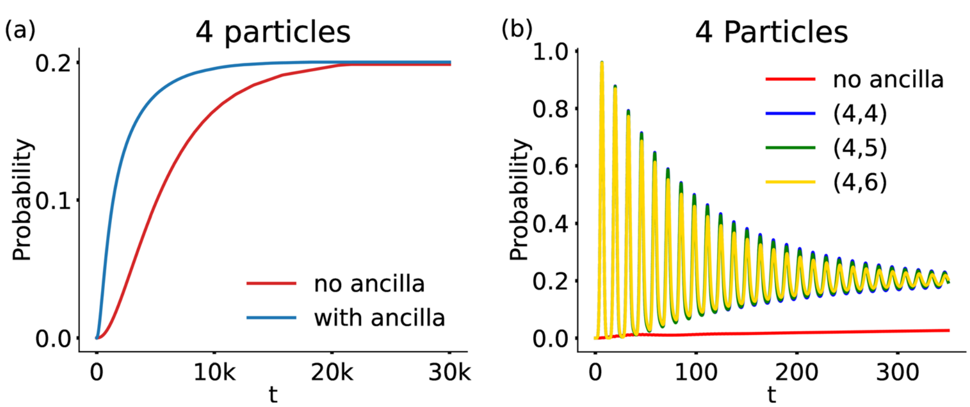


Fig. 6 **a** The asymptotic behavior of the tunneling probability for a 4 particles system (red) and system coupled with ancilla (blue) with the same number of particles (random fixed parameters, no learning). **b** Tunneling probability evolution after optimization for a system of 4 particles coupled with an ancilla of 4, 5, 6 particles. Red line: system with no coupling. Note the difference in the time scales between the right and left plots. The system parameters for **a** and **b** are $\eta_S = 1$, $\gamma_S = 0.5$, and $\Delta_S = 1$

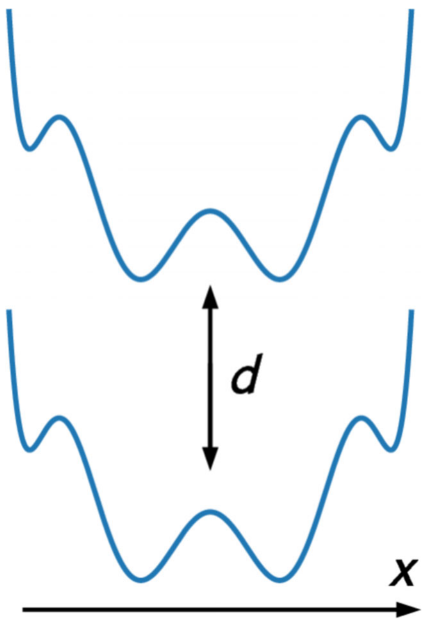


Fig. 7 Schematic of system-ancilla tuning in a ultracold system of, e.g., dipolar and Rydberg atoms

in the text. To have an estimate of the parameters γ, Δ, η and the coupling α, consider, e.g., the following form for the potential $V(x, y, z)$ for the ultracold system as the sum of a contribution of a (magnetic) harmonic potential V_h and of an optical lattice potential V_l along the x-direction:

$$V(x, y, z) = V_h(x, y, z) + V_l(x),$$
$$V_h(x, y, z) = \frac{m}{2}\left[\omega_x^2 x^2 + \omega_r^2(y^2 + z^2)\right),$$
$$V_l(x) = V_0 \cos^2(kx)$$

where V_h is the harmonic potential and V_l is the periodic potential. Typically (see Morsch and Oberthaler 2006), one has $k = 2\pi/\lambda$, where $\lambda = \lambda_{laser} \sin(\theta/2)$, with λ_{laser} being the wavelength of the lasers giving raise to the optical potential and θ the angle between the counter-propagating laser beams (we consider simply $\theta = \pi$ in the following). Energies are commonly expressed in units of the recoil energy $E_R = \hbar^2 k^2/2m$, m begin the mass of the atom. To have a one-dimensional geometry, one needs to have $\omega_r \gg \omega_x$. By varying the parameters ω_x and V_0, one can have double- or multi-well configurations (Albiez et al. 2005; Macrì and Trombettoni 2013). Moreover, with $V_l(x) = V_0 \cos^2(kx + \phi)$, by varying the phase ϕ (or equivalently by shifting the center of the x-part of the harmonic potential V_h), one can induce an energy imbalance ΔE between the adjacent wells.

The coefficient γ_S in Eq. 1 is proportional to the tunneling coefficient, typically denoted by K, of the two-mode model (Smerzi et al. 1997). One can perform a very simple estimate of K/E_R by considering a variational ansatz of the (Wannier) spatial wavefunctions localized in the well (Trombettoni et al. 2005). Putting $V_0 = sE_R$ and using $\lambda \sim 800nm$, one finds for (bosonic) Yb atoms $\gamma_S \sim k_B 0.5nK$ with $s = 10$. The coefficient Δ_S in Eq. 1 main text depends on the energy asymmetry ΔE: with $\omega_x \sim 2\pi 100Hz$ and again $s = 10$, one gets $\Delta_S \sim 0.2nK$ for $\phi = 0.5$. The coefficient η_S is proportional to the interaction coefficient, usually denoted by U, of the two-mode model. It depends on the transverse size, i.e., in the $-z$ directions of the system and of course on the number of particle (Smerzi and Trombettoni 2003; Ananikian and Bergeman 2006): one finds with $\sigma_\perp$ of few microns values from fraction to some nK with typical values (Cataliotti et al. 2001; Morsch and Oberthaler 2006).

Similar estimates can be done for the coefficients of the ancilla system. Finally, the coefficient α of the coupling is the one in front the of the $z - z$ interaction and clearly depends on the distance between the system and the ancilla. A simple estimate can be obtained by writing the localized wavefunction of the system, say $\Phi_S(\vec{r})$, and of the ancilla, say $\Phi_A(\vec{r})$, of course respectively centered in the wells of the system of the ancilla. Then, one has

$$\alpha \propto \int d\vec{r} d\vec{r}' \Phi_S(\vec{r}) \mathcal{V}(\vec{r} - \vec{r}') \Phi_A(\vec{r}'),$$

where $\mathcal{V}$ is the non-local potential. A discussion of it for dipolar and Rydberg atoms can be found in Lahaye et al. (2009); Defenu et al. (2023). When the angle between the dipoles is $\pi/2$, one can write the 2-body potential in the form $\mathcal{V}(\vec{r} - \vec{r}') = \mathcal{C}/|\vec{r} - \vec{r}'|^3$. One can tune the distance between the system and the ancilla and crucially as well the coefficient $\mathcal{C}$, therefore making possible the tuning of the coefficient α.

To conclude we observe that, when the energies are expressed in units of γ_S and time in units of $\hbar/\gamma_S$, with $\gamma_S \sim 0.5nK$, one has that the unit of time is $\sim 10ms$, so hundreds of oscillations correspond to a total time of the experiment of order up to few seconds. Similar estimates hold when the energies are measured in units of Δ_S.

5 Conclusions and outlook

In this work, we addressed the issue of whether it is possible to learn a double-well bosonic trap A and its interaction with a target double-well trap S so to increase the latter tunneling properties given a fixed physical scenario in terms of a system S energy asymmetry and a tunneling strength *on which we cannot act directly*. We have shown that the learned ancilla physical setting and its coupling with the target system maximize the system tunneling probability without requiring an ancilla that, by itself, would outperform the system S; in which case, the ancilla A should evidently better be used in the place of S.

The idea of learning an ancillary system and its coupling to a specific quantum system to enhance particular quantum features of the latter extends beyond merely improving tunneling properties. Rather, we believe our analysis introduces a comprehensive framework with diverse applications. For instance, another significant area of research involves enhancing the quantum efficiency of energy harvesting systems. A recent study by Paternostro et al. (2022) has demonstrated how optimizing the local energies of a Fenna-Matthews-Olson complex can achieve highly efficient excitation transfer, even under varying environmental conditions. A closely related interesting line of research regards the ability of preserving the coherence properties of the quantum system throughout time (see also Ullah et al. 2023, Anselmi et al. 2024).

Furthermore, the implementation of interacting ancillary systems holds great promise in increasing the efficiency and reliability of quantum algorithms. For instance, as already mentioned, quantum tunneling plays a crucial role in the successful realization of quantum gates. Additionally, it has the potential to contribute to the development of more effective error correction methods, thereby ensuring a more robust preservation of quantum information.

In summary, by unlocking the potential for changing tunneling probabilities through learning, this work opens the doors for advancements in many fields such as classical and quantum computing, materials science, and energy harvesting devices.

Appendix A. Jordan-Schwinger representation

In the following, we shall be concerned with a typical ultracold atom experimental setup consisting in a double-well potential confining N particles of bosonic type described by creation and annihilation operators $a, a^\dagger, b, b^\dagger$ satisfying the commutation relations $[a, a^\dagger] = [b, b^\dagger] = 1$, while all other commutators vanish. If $|vac\rangle$ denotes the vacuum state such that $a|vac\rangle = b|vac\rangle = 0$, then $a^\dagger|vac\rangle$ creates a particle in the left well and $b^\dagger|vac\rangle$ a particle in the right one. It follows that states with $0 \le k \le N$ particles in the left well together with $N-k$ in the other one are represented by

$$|k\rangle = \frac{(a^\dagger)^k (b^\dagger)^{N-k}}{\sqrt{k!\,(N-k)!}}|vac\rangle, \quad k = 1, \cdots, N\,. \tag{A1}$$

Notice indeed that these vectors fulfill

$$a^\dagger b|k\rangle = \sqrt{(k+1)(N-k)}\,|k+1\rangle$$
$$ab^\dagger|k\rangle = \sqrt{k(N-k+1)}\,|k-1\rangle$$

and are thus eigenstates of the number operators $a^\dagger a$ and $b^\dagger b$:

$$a^\dagger\, a\,|k\rangle \,=\, k\,|k\rangle\,, \quad b^\dagger\, b\,|k\rangle \,=\, (N-k)\,|k\rangle$$

As such, they constitute an orthonormal basis for the Hilbert space $\mathbb{C}^{N+1}$ associated with the system S. Moreover, in the Jordan-Schwinger representation of the $su(2)$ algebra, the operators

$$J_x \equiv \frac{a^\dagger b + ab^\dagger}{2}\,, \quad J_y \equiv \frac{ab^\dagger - a^\dagger b}{2i}\,, \quad J_z \equiv \frac{b^\dagger b - a^\dagger a}{2}$$

satisfy the algebraic relations proper to the generators of the rotation group:

$$[J_x, J_y] = i\,J_z\,,$$

their cyclic permutations, together with

$$[N, J_x] = [N, J_y] = [N, J_z] = 0$$

where N denotes the total number operator

$$N \equiv a^\dagger a \,+\, b^\dagger b\,.$$

Their matrix elements with respect to the ONB (A1) are

$$\langle j|J_x|k\rangle = \frac{\sqrt{(k+1)(N-k)}\,\delta_{j,k+1} + \sqrt{k\,(N-k+1)}\,\delta_{j,k-1}}{2}$$
$$\langle j|J_y|k\rangle = \frac{\sqrt{k\,(N-k+1)}\,\delta_{j,k-1} - \sqrt{(k+1)(N-k)}\,\delta_{j,k+1}}{2i}$$
$$\langle j|J_z|k\rangle = \frac{N-2k}{2}\delta_{j,k}.$$

Appendix B. Trapped two-mode bosonic systems

In the following, we focus on a quantum system S consisting of N_S interacting bosons trapped in a double-well potential (Smerzi et al. 1997; Milburn et al. 1997).

In the absence of couplings to external systems or to an environment, the trapped bosons evolve in time according to a two-mode Bose-Hubbard Hamiltonian, which, using the Jordan-Schwinger representation, reads:

$$H_S = \eta_S\,J_z^2 \,-\, \gamma_S\,J_x \,-\, \Delta_S\,J_z \tag{B2}$$

The real coefficient γ_S governs the energy barrier's height that controls the tunneling between the left and right wells, Δ_S measures the energy asymmetry, while η_S quantifies the repulsion ($\eta_S > 0$) or the attraction ($\eta_S < 0$) among bosons and is proportional to the s-wave scattering length (Leggett 2001). An estimate of the parameters γ_S, Δ_S gives $\gamma_S/k_B \sim 0.5nK$ and $\Delta_S/k_B \sim 0.2nK$, while η_S can be varied in the interval $\sim k_B \cdot [0.1 - 10nK]$ (see Sect. 4 for more details), where k_B is the Boltzmann constant. The number-preserving, reversible dynamics for the states (density matrices) $\rho^{(S)}$ of the N_S trapped bosons is thus generated by the master equation

$$\partial_t \rho^{(S)}(t) = -i[H_S\,,\ \rho^{(S)}(t)]\,.$$

When all bosons are initially confined in the left well at time $t = 0$, their state is $\rho_L^{(S)} = |N_S\rangle\langle N_S|$, where $|k\rangle$ describes the state with k boson in the left well and $N_S - k$ in the right one and satisfies $J_z|k\rangle = (N_S - 2k)|k\rangle$. The states $\rho_L^{(S)}$ might be used to encode the logical qubit state $|0\rangle$, which then evolve into the logical qubit state $|1\rangle$ when all N_S bosons have tunneled to the right well turning the initial boson state into $\rho_R^{(S)} = |0\rangle\langle 0|$. Within this setting, one is interested in maximizing the tunneling probability $P_{L\to R}(t_*)$ of the N_S bosons from the left to the right well at a certain time t^*. Such a probability is given by

$$P_{L\to R}(t) = \mathrm{Tr}(\rho_L^{(S)}(t)\rho_R^{(S)}) = \langle 0|\rho_L^{(S)}(t)|0\rangle\ , \tag{B3}$$

where

$$\rho_L^{(S)}(t) = \exp\left(-\frac{i}{\hbar}\,H_S\,t\right)\,\rho_L^{(S)}\,\exp\left(+\frac{i}{\hbar}\,H_S\,t\right)\ . \tag{B4}$$

Notice that, for $N_S = 1$, a two-mode N_S-body system trapped by a double-well potential reduces to a two-level system described by the Hamiltonian

$$H_S = -\Delta_S\,\sigma_z - \gamma_S\,\sigma_x\ ,$$

where $\sigma_{x,z}$ are the Pauli matrices, while $J_z^2 = \sigma_z^2/4 = I/4$ in Eq. B2 can be neglected. The states with one particle in the left, respectively right well are the eigenstates $|L\rangle$, respectively $|R\rangle$ of σ_z, corresponding to eigenvalues -1, respectively $+1$.

The previous two-level Hamiltonian can always be written in the form (Sakurai 1994)

$$H_S = -\Delta\,\sigma_z\, -\, \gamma\,\sigma_x = -\hbar\,\omega\,\frac{1+\vec{n}\cdot\vec{\sigma}}{2}\, +\, \hbar\,\omega\,\frac{1-\vec{n}\cdot\vec{\sigma}}{2}$$

with $\hbar\,\omega = \sqrt{\Delta^2+\gamma^2}$, $\sigma_{x,y,z}$ the Pauli matrices and $\dfrac{1\pm\vec{n}\cdot\vec{\sigma}}{2}$ its orthogonal eigenprojections, where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ and $\vec{n} = (\gamma, 0, \Delta)/(\hbar\,\omega)$. The states corresponding to one particle in the left, respectively right well satisfy $\sigma_z|L\rangle = -|L\rangle$ and $\sigma_z|R\rangle = |R\rangle$. Starting at $t = 0$ in the left well, at time t, $|L\rangle$ will evolve into

$$|L\rangle_t = \Big(\cos(\omega t)\, +\, i\,\vec{n}\cdot\vec{\sigma}\,\sin(\omega t)\Big)|L\rangle\ .$$

Then, the left-to-right tunneling probability at time t is

$$P_{L\to R}(t) = |\langle R|L\rangle_t|^2 = \frac{\gamma_S^2}{\hbar^2\omega^2}\,\sin^2(\omega t) = \frac{\sin^2(\omega t)}{1+\frac{\Delta_S^2}{\gamma_S^2}} \tag{B5}$$

with $\hbar\,\omega = \sqrt{\Delta_S^2+\gamma_S^2}$. Thus, the tunneling probability can never reach its maximum $P_{L\to R}(t) = 1$ for asymmetric traps, that is, when $\Delta_S \neq 0$.

Appendix C. Noiseless coupling to ancilla systems

We now consider two double-well potentials and a density-density interaction between them of dipolar origin The first double-well potential S, containing N_S bosons and characterized by a fixed Bose-Hubbard Hamiltonian, is the target system. The second double-well potential, denoted by A, serves as a learnable ancillary system and accommodates N_A trapped bosons. The ancillary system is also governed by the Hamiltonian in Eq. B2, but with learnable energy parameters $\eta_A, \gamma_A, \Delta_A$. We consider a density-density interaction $H_{int} = \alpha\, J_z \otimes J_z$ between the bosons of the two double-wells, with an interaction strength α, which is also learnable. This interaction could be experimentally implemented using, e.g., dipolar or Rydberg atoms (Defenu et al. 2021).

In the following, we distinguish two different physical contexts:

- The first one refers to a system S consisting of an asymmetric trap ($\Delta_S \neq 0$), for which reaching certainty of left-to-right tunneling is impossible. One is then interested to *achieve unit tunneling probability* by suitable coupling S to an ancilla A. In the numerical simulations, it will be convenient to measure energies in units of the system energy asymmetry Δ_S and time in units of $\hbar/\Delta_S$, equivalently setting $\hbar = \Delta_S = 1$.
- The second physical context regards a single boson in an energy-symmetric trap ($\Delta_S = 0$), whereby one is interested to devise a suitable coupling to ancillas, so to reach perfect tunneling of all system bosons from left to right in a *shorter time* with respect to the case without ancillas. Unlike in the previous one, in such a case, it is convenient to express energies in unit of γ_S and time in units of $\hbar/\gamma_S$. In both cases, the time unit for a system of coupled ultracold dipolar gases (see Sect. 4 for details) is of the order of $10ms$.

We consider an initial uncorrelated $S+A$ state $\rho_L^{(S)}\otimes\rho^{(A)}$, where $\rho_L^{(S)} = |N_S\rangle\langle N_S|$ denotes the initial state with all N_S bosons of the system of interest trapped in the left well, while

$\rho^{(A)}$ denotes any initial state of the ancillary double-well potential. At $t > 0$, the system S state is

$$\rho_L^{(S)}(t) = \mathrm{Tr}_A\Big(e^{-i\,H_{SA}\,t}\,\rho_L^{(S)} \otimes \rho^{(A)} e^{i\,H_{SA}\,t}\Big)\,, \tag{C6}$$

where $H_{SA} = H_S + H_A + H_{int}$ is the total Hamiltonian of the compound system $S+A$ and Tr_A denotes the partial trace with respect to the ancillary system.

We learn the optimal coupling strength α, the energy parameters $\eta_A, \Delta_A, \gamma_A$ together with the full initial state $\rho^{(A)}$, and the time t. All these parameters will be learned in such a way to increase the transfer probability $P_{L\to R}(t) = \langle 0|\rho_L^{(S)}(t)|0\rangle$ of the N_S bosons of the target system S from the left to the right well.

System *S*: single qubit, $N_S = 1$

As anticipated above, in order to inspect how the coupling to an ancilla affects the tunneling probability, we disregard, for the moment, the presence of an environment and consider a double-well containing one boson, $N_S = 1$, coupled to an ancillary double-well with $N_A \equiv N$ non-interacting and non-tunneling bosons.

- Case 1: Energy asymmetric trap: $\Delta_S = \Delta \neq 0$.

Consider an ancilla A with the same energy asymmetry as S, single system Hamiltonians

$$H_S = -\Delta\,\sigma_z - \gamma\,\sigma_x\,, \quad H_A = -\Delta\,J_z\,, \tag{C7}$$

and interaction Hamiltonian

$$H_{int} = \alpha\,\sigma_z \otimes J_z\,. \tag{C8}$$

By means of the eigenprojections $P_k = |k\rangle\langle k|$ of J_z,

$$P_k P_\ell = \delta_{k\ell}\,P_\ell \qquad \sum_{k=0}^{N} P_k = I \qquad J_z = \sum_{k=0}^{N} \frac{N-2k}{2}\,P_k$$

one finds $H_{SA} = \sum_{k=0}^{N} H_k \otimes P_k$, where neglecting terms proportional to the identity that do not affect the dynamics of the system S observables,

$$H_k = -\left(\Delta - \alpha\frac{N-2k}{2}\right)\sigma_z - \gamma\sigma_x\,. \tag{C9}$$

Then, the unitary time evolution generated by H_{SA} reads

$$\mathrm{e}^{-i\,t\,H_{SA}} = \sum_{k=0}^{N} \mathrm{e}^{-i\,t\,H_k} \otimes P_k\,, \tag{C10}$$

where $\hbar$ has been absorbed into the energy parameters. Consider a joint factorized initial state $\rho^{(SA)} = \rho_L^{(S)} \otimes \rho^{(A)}$ corresponding to the system with its single boson localized in the left well, $\rho_L^{(S)} = P_1$, and to the ancilla being in a generic pure state $\rho^{(A)} = |\psi_A\rangle\langle\psi_A|$ of its N bosons. According to Eq. C6, the system's initial state projector evolves into

$$\rho_L^{(S)}(t) = \sum_{k=0}^{N} \Big|\langle k|\psi_A\rangle\Big|^2\,\mathrm{e}^{-i\,t\,H_k}\,|L\rangle\langle L|\,\mathrm{e}^{i\,t\,H_k}\,. \tag{C11}$$

Therefore, the left-to-right transition probability at time $t \geq 0$ amounts to

$$P_{L\to R}(t) = \langle R|\rho_L^{(S)}(t)|R\rangle = \sum_{k=0}^{N} \Big|\langle k|\psi_A\rangle\Big|^2\,\Big|\langle R|\mathrm{e}^{-i\,t\,H_k}|L\rangle\Big|^2\,, \tag{C12}$$

where $|R\rangle$ is the state with all system bosons to the right. Using Eq. B5 with H_k as in Eq. C9, one finally finds

$$P_{L\to R}(t) = \sum_{k=0}^{N} \Big|\langle k|\psi_A\rangle\Big|^2\,\frac{\gamma^2}{\omega_k^2}\,\sin^2(\omega_k t), \tag{C13}$$

$$\omega_k := \sqrt{\left(\Delta - \alpha\frac{N-2k}{2}\right)^2 + \gamma^2}\,. \tag{C14}$$

Unit probability can be reached only if $|\psi_A\rangle$ is an eigenstate, say $|k\rangle$, of J_z and the coupling α together with the system energy asymmetry satisfy

$$\Delta_k := \Delta - \alpha\frac{N-2k}{2} = 0\,. \tag{C15}$$

Then, $P_{L\to R}(t^*) = 1$ at $t^* = \frac{\pi}{2\gamma}$. Equation C15 allows for a physical interpretation behind the increase of the tunneling probability in the system: each ancilla eigenstate induces in the system an effective energy asymmetry Δ_k that can be made vanish by choosing $\alpha = \dfrac{2\Delta}{N-2k}$.

More in general, by means of the coupling just investigated, it is possible to beat the bound $\dfrac{\gamma^2}{\omega^2}$ to the left-to-right tunneling probability that follows from Eq. B5. Indeed, the tunneling probability $P_{L\to R}(t)$ in Eq. C13 consists of $N+1$ positive terms each bounded by $\dfrac{\gamma^2}{\omega_k^2}$.

- Case 2: Energy symmetric trap: $\Delta_S = \Delta = 0$.

Consider now a single trapped boson with $\Delta = 0$ coupled via the interaction Hamiltonian

$$H_{int} = \alpha\,\sigma_x \otimes J_z\,. \tag{C16}$$

to an asymmetric, non-interacting, and non-tunneling ancilla with single system Hamiltonians:

$$H_S = -\gamma\,\sigma_x\,, \quad H_A = -\Delta\,J_z\,. \tag{C17}$$

Then, (see Eq. B5) without any coupling to an ancilla, namely with $\alpha = 0$, $\Delta_S = 0$, the system S achieves unit tunneling probability at $t^* = \pi/(2\gamma)$. Switching on the coupling (C16), the S reduced dynamics is given by Eq. C10 with

$$H_k = -\Big(\gamma - \alpha\frac{N-2k}{2}\Big)\,\sigma_x\,. \tag{C18}$$

Therefore, the coupling keeps the energy symmetry, but induces an effective tunneling strength

$$\gamma_k := \gamma - \alpha\frac{N-2k}{2}\,, \tag{C19}$$

and an associated tunneling probability

$$P_{L\to R}(t) = \sum_{k=0}^{N}\Big|\langle k|\psi_A\rangle\Big|^2\,\sin^2(\omega_k t)\,, \qquad \omega_k = \gamma_k\,. \tag{C20}$$

Therefore, by choosing $|\psi_A\rangle = |k\rangle$, $P_{L\to R}(t_k) = 1$, at time t_k shorter than the corresponding time with $\alpha = 0$:

$$t_k = \frac{\pi}{2\gamma_k} = \frac{\pi}{2\gamma\Big(1 - \alpha\,\frac{N-2k}{2\gamma}\Big)} \le \frac{\pi}{2\gamma} \tag{C21}$$

for $\alpha \ge 0$ and $N \le 2k$ or $\alpha \le 0$ and $N \ge 2k$.

Appendix D. Decoherence

In many-body systems, the presence of noise and dissipation makes decoherence hardly negligible. We thus consider the optimization of the tunneling probability when both system S and ancilla A cannot be considered isolated from their environment even if they interact very weakly with it. In such a case, one expects tunneling to be affected by decoherence. Such a possibility can be investigated by considering both system S and ancilla A independently undergoing reduced irreversible dynamics generated by a master equation of the so-called Gorini-Kossakowski-Sudarshan-Lindblad form (Breuer and Petruccione 2002):

$$\partial_t\rho(t) = -i[H,\rho(t)] + \lambda\left(J_z\rho(t)J_z - \frac{1}{2}\left\{J_z^2,\rho(t)\right\}\right), \tag{D22}$$

where $\{X,\,Y\}$ denotes the anti-commutator and λ is a constant, smaller than the typical system energy, that measures the strength of the coupling of the system to the environment. The right-hand side of Eq. D22 generates the irreversible dynamics of S as an open quantum system in a particular weak interaction with its environment. The latter contributes by adding to the commutator with the Hamiltonian two terms: the first one amounting to a particular kind of quantum noise, embodied in the operators J_z, and the second one to a damping term that restores probability preservation. The chosen modification of the otherwise reversible dynamics is based upon the fact that it tends to eliminate the off-diagonal terms $\langle k|\rho(t)|\ell\rangle$ of the time-evolving density matrices, which are those sustaining the left-to-right tunneling probability. Indeed, the operator J_z has the vectors $|k\rangle$ as eigenvectors so that the corresponding eigenprojectors are left invariant by the dissipative term:

$$J_z\,|k\rangle\langle k|\,J_z - \frac{1}{2}\left\{J_z^2\,,\,|k\rangle\langle k|\right\} = 0\,.$$

Then, the dissipative open dynamics (8) is expected to favor states that are mixtures of left/right localized states $|k\rangle\langle k|$ with weights that depend on the initial state. Notice however that the projectors $|k\rangle\langle k|$ are not left invariant by the Hamiltonian when it contains a non-vanishing tunneling term ($\gamma \neq 0$). Indeed, the only state of the two-well open system that is left invariant by the master Eq. 8 is the completely mixed state

$$\rho_{mix} := \frac{1}{N_S+1}\sum_{k=0}^{N_S}|k\rangle\langle k|$$

to which all eigenprojectors of J_z equally contribute. Such a state is the only one that commutes with both a generic Hamiltonian H_S and J_z. Therefore, it represents a stationary state of the master Eq. 8, and all initial states ρ_S of the N_S trapped bosons, which evolve into $\rho^{(S)}(t)$ by means of Eq. 8, actually tend to ρ_{mix} asymptotically in time (Frigerio 1978):

$$\lim_{t\to+\infty}\rho^{(S)}(t) = \rho_{mix} \qquad \forall\rho^{(S)}\,. \tag{D23}$$

Appendix E. Supplementary figures

E.1. Ancilla different initialization

Figure 8 shows the same plots as in Fig. 2 in the main text but when the ancilla is fixed to a superposed state of right and left one particle during the optimization. The results show full agreement with the theoretical predictions. Interestingly, the reached max probability is not one, highlighting the importance of the choice of the ancilla.

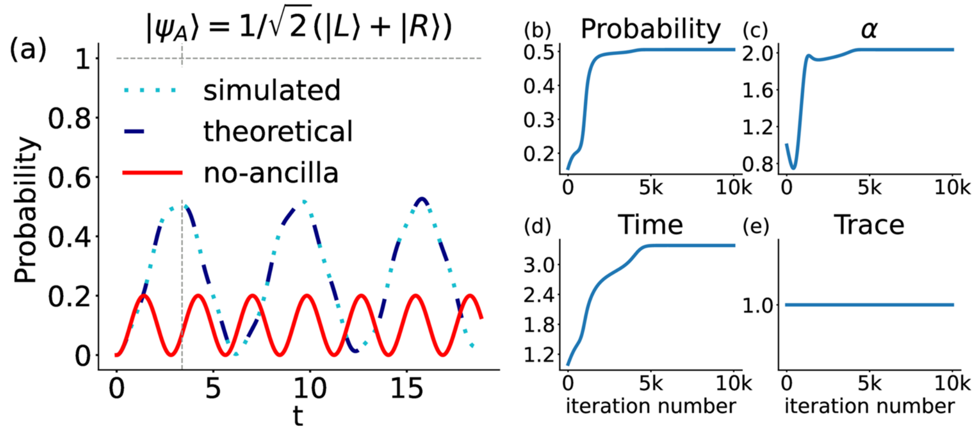


Fig. 8 **a** After learning. Time evolution of the tunneling probability for a system of $N_S = 1$ particle coupled with an ancilla of $N_A = 1$ particle with $|\psi_A\rangle = 1/\sqrt{2}(|L\rangle + |R\rangle)$), compared to the system without the ancilla, reporting both the theoretical predictions and the optimized results ($\eta_S = 0$, $\gamma_S = 0.5$, $\Delta_S = 1$). Evolution, during optimization, of the tunneling probability (**b**), interaction strength α (**c**), time to reach the maximum probability (**d**), and trace of the learned ancilla (**e**) as sanity check

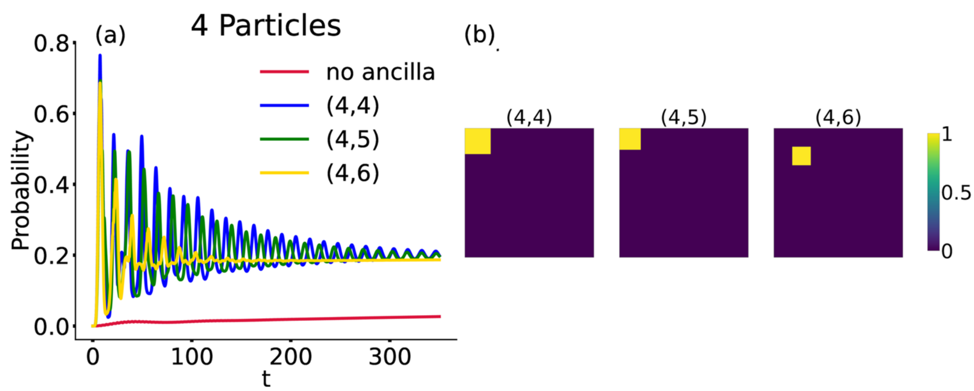


Fig. 9 **a** Tunneling probability evolution after optimization for a system of $N_S = 4$ particles coupled with an ancilla of $N_A = [4, 5, 6]$ particles constrained to be diagonal. Red line: system with no coupling. **b** The learned ancillas. The system parameters are $\eta_S = 1$, $\gamma_S = 0.5$, and $\Delta_S = 1$

E.2. Further results on the noisy case

Figure 9a reports the evolution of the tunneling probabilities for $N_S = 4$ (as in Fig. 5 in the main text) but with the constraint of diagonal ancillas (**b**). Interestingly, although not explicitly imposed in the learning, the ancilla state converges not to a superposition state of particles but to a state with a precise number of particles in the right and left well. The constraint on the learned ancilla has been imposed as follows: at each step of the optimization, we extracted the diagonal elements from ρ_A, generated a new ancilla density matrix with those diagonal elements, and then normalized the result. In specific at each iteration step, we perform

$$\rho^{(A)} \leftarrow \mathrm{diag}(\rho^{(A)})$$

$$\rho^{(A)} \leftarrow \frac{\rho^{(A)}}{\mathrm{Tr}(\rho^{(A)})}.$$

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Declarations

Conflict of interest The authors declare no competing interests.

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