Stable determination of a scattered wave from its far-field pattern: the high frequency asymptotics

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Abstract

We deal with the stability issue for the determination of outgoing time-harmonic acoustic waves from their far-field patterns. We are especially interested in keeping as explicit as possible the dependence of our stability estimates on the wavenumber of the corresponding Helmholtz equation and in understanding the high wavenumber, that is frequency, asymptotics.

Applications include stability results for the determination from far-field data of solutions of direct scattering problems with sound-soft obstacles and an instability analysis for the corresponding inverse obstacle problem.

The key tool consists of establishing precise estimates on the behavior of Hankel functions with large argument or order.

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1   Introduction

In recent years there has been an increasing attention to the study of how stability estimates for ill-posed problems involving the Helmholtz equation or the Schrödinger equation improve as the wavenumber and frequency or the energy, respectively, grows and might become extremely large.

One of the first rigorous justification of this phenomenon is due to Isakov and collaborators, [12, 33], and concerns the Cauchy problem for the Helmholtz equation.

For what concerns corresponding inverse problems, increasing stability properties has been shown in many cases by many different authors. For instance, in [2], an inverse source problem for the Helmholtz equation in the high frequencies regime was studied. The inverse problem of determining the potential in a Schrödinger equation by boundary data in the high energies regime was considered in [16, 18, 19] by geometrical optics techniques and, with a different method, in [14]. Let us notice that Isaev [13] developed a corresponding instability analysis showing the optimality of the previous estimates. In [28] similar stability estimates were proved for the inverse problem of determining by boundary data an inhomogeneous medium for the acoustic wave equation.

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We recall that typically these inverse problems are exponentially ill-posed and this is one of the main difficulties for numerical reconstruction. Such an instability character of these kinds of inverse problems was proved by Mandache, [23], for the the Schrödinger equation at zero energy and the inverse conductivity problem, see also [11] for other elliptic inverse boundary value problems and inverse scattering problems. This is the main reason for trying to obtain a more stable reconstruction by changing the frequency or energy and in particular by using high frequencies or energies. This motivated the search for stability estimates with an explicit dependence on the frequency or energy and for their high frequency or energy asymptotics. It has been shown that, asymptotically, stability estimates may be expressed as the sum of a stable Hölder or Lipschitz term with a logarithmic one which is converging to zero as the frequency or energy tends to infinity, thus making the exponential ill-posedness less severe.

Another motivation can be found in the so-called multifrequency methods (called also hopping type algorithms) to reconstruct materials or interfaces from their scattered or far-field patterns. The main advantage of using such multifrequency data is that it can help to obtain accurate reconstructions without the need for a good initial guess. Different reconstruction methods using multifrequency data have been proposed in the last two decades or so, see for instance [8, 7, 1, 4, 3, 31]. The convergence analysis of this type of algorithms was investigated in [4, 31] for the so-called recursive linearization algorithm proposed in [7]. In the analysis of these methods, the need for lower estimating the singular values of the linearized scattering problem in terms of the used frequencies arises naturally, see also [32]. In order to derive such a lower estimate, a crucial step is to estimate the scattered waves from their far-field patterns, up to the boundary of the scatterers and with an explicit dependence upon the frequency.

In this paper we are mainly interested in the direct and inverse acoustic scattering problems for impenetrable scatterers, in particular sound-soft obstacles. For the inverse problem numerical evidence that the stability improves as the frequency grows was shown in [9]. It would be extremely interesting to rigorously prove such a phenomenon by establishing suitable stability estimates in the high frequencies regime. Unfortunately this seems to be still a challenging open problem. Nevertheless, for the direct scattering problem we obtain several interesting results. Our aim is to obtain stability estimates for the values of scattered waves from their far-field patterns, in the high frequency case and on the whole exterior of the scatterer. We are able to establish this result at least for smooth convex obstacles.

A crucial step, and one of the main results of the paper, is to prove stability estimates depending on the frequency for the determination of the near-field of an outgoing acoustic time-harmonic wave by its corresponding far-field. Such a problem has been solved for a fixed frequency by Isakov, [15], see also [5]. After finishing this work, we learned that very recently and independently Isakov [17] considered the high frequency case and proved a stability estimate in the high frequencies regime. However we believe that our results are considerably stronger than his. The advantages in Isakov's result are the fact that his estimate is completely explicit and that he is able to estimate the outgoing wave on the boundary of the same ball where the a priori bound is known. Instead, we estimate the outgoing wave on the boundary of a larger ball. Nevertheless, our results have several other important advantages. First of all we require an a priori bound of L^2 type instead of an H^1 type as in Isakov's paper. Second, and more important, is the dependence of the estimate on the wavenumber k. In Isakov's result the estimate improves with respect to k in

a polynomial manner, in ours the improvement is much stronger, being of exponential type, see Theorem 4.2. Moreover, in the regime where the Isakov's estimate is more meaningful, that is when k is higher than $\log(1/\varepsilon)$, ε being an estimate of the norm of the far-field pattern, we are able to obtain a Lipschitz stability estimate completely independent of k, see Theorem 4.4. Finally, in Isakov's estimates there is a delicate issue about the a priori bound. If the a priori bound used by Isakov were of order higher than $\sqrt{k}$, then its stability estimate would not show any improvement as k grow, that is we would not have any increasing stability. Unfortunately, as we show in Theorem 5.5, in particular in (5.22), up to our knowledge even the L^2 a priori bound may be of order k and the H^1 may be even of a higher order, since u solves the Helmholtz equation with wavenumber k. In our case, instead, even if the a priori bound grows as a power of k, no matter how big, in the high frequencies regime we are still able to prove a Lipschitz stability estimate with a constant which is independent of k, see again Theorem 4.4.

Finally, by an instability analysis, we are able to evaluate from below how much the stability for the inverse scattering problem could improve as the frequency increases.

One of the main features of the paper is that in all our results we keep the dependence on the frequency as explicit as possible. We also establish estimates for the full range of frequencies, with a particular attention to the case of high frequencies. We notice that most of the results present three different regimes. The regime of bounded frequencies where the usual ill-posedness shows up. A regime of high frequencies, with a limited improvement in the stability results, and a regime of extremely high frequencies where the improvement is much more significant. Furthermore, most of the results are written for any space dimension $N \geq 2$.

Let us describe in more details the results of the paper. We consider time-harmonic acoustic waves in a homogeneous and isotropic medium in a subset of $\mathbb{R}^N$, $N \geq 2$. Such a wave is characterized by its corresponding field u which solves the *reduced wave equation* or *Helmholtz equation*

$$\Delta u + k^2 u = 0$$

where $k > 0$ is the *wavenumber*. We recall that the wavenumber k is the ratio between the corresponding frequency and the speed of sound.

We say that a time-harmonic acoustic wave in the exterior of a ball is *outgoing* if its corresponding field u satisfies the so-called *Sommerfeld radiation condition*, that is

$$\lim_{r \to \infty} r^{(N-1)/2} \left(\frac{\partial u^s}{\partial r} - \mathrm{i} k u^s \right) = 0, \quad r = \|x\|$$

where the limit is intended to hold uniformly for all directions $\hat{x} = x/\|x\| \in \mathbb{S}^{N-1}$.

The Sommerfeld radiation condition allows to characterize the asymptotic behavior of the outgoing acoustic wave, namely we have that

$$u(x) = \frac{\mathrm{e}^{\mathrm{i}k\|x\|}}{\|x\|^{(N-1)/2}} \left\{ u_\infty(\hat{x}) + O\left(\frac{1}{\|x\|} \right) \right\},$$

as $\|x\|$ goes to $+\infty$, uniformly in all directions $\hat{x} = x/\|x\| \in \mathbb{S}^{N-1}$. The function u_∞ is defined on $\mathbb{S}^{N-1}$ and is referred to as the *far-field pattern* of the field u.

Outgoing waves play a fundamental role in acoustic scattering theory. In fact, let us assume that in a homogeneous and isotropic medium in $\mathbb{R}^N$ there is a *scatterer* Σ, that is

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a compact subset of $\mathbb{R}^N$ such that $G = \mathbb{R}^N \backslash \Sigma$ is connected. We recall that a scatterer Σ is said to be an *obstacle* if Σ is the closure of an open set.

Let us assume that we send through the medium a time-harmonic acoustic wave, which is called incident wave. We call $k > 0$ its *wavenumber* and u^i its corresponding field, the *incident field*. Usually the incident wave is either a point source wave or a plane wave. We shall mainly focus on the latter case. If $\omega \in \mathbb{S}^{N-1}$ is the *direction of propagation* of the time-harmonic plane wave, then the incident field is given by $u^i(x) = e^{ik\omega \cdot x}$, $x \in \mathbb{R}^N$.

The presence of the scatterer perturbs the incident wave by producing a so-called scattered wave which is characterized by being an outgoing time-harmonic acoustic wave. Its field u^s is called the *scattered field*. The total wave is the superposition of the incident wave and the scattered wave, that is its corresponding field, the *total field u*, is simply the sum of the incident field and the scattered field. Namely the total field satisfies the following

$$(1.1) \quad \begin{cases} \Delta u + k^2 u = 0 & \text{in } \mathbb{R}^N \backslash \Sigma \\ u = u^i + u^s & \text{in } \mathbb{R}^N \backslash \Sigma \\ B.C. & \text{on } \partial \Sigma \\ \lim_{r \to \infty} r^{(N-1)/2} \left(\dfrac{\partial u^s}{\partial r} - iku^s \right) = 0 & r = \|x\|. \end{cases}$$

The boundary condition on $\partial \Sigma$ depends on the nature of the scatterer. In this paper we mainly focus on the case of impenetrable *sound-soft* scatterers to which corresponds the following homogeneous Dirichlet boundary conditions

$$u = 0 \quad \text{on } \partial \Sigma.$$

However other boundary conditions may appear in the applications, for instance the homogeneous Neumann condition for impenetrable sound-hard scatterers, the more general impedance boundary condition for impenetrable scatterers or transmission conditions for penetrable scatterers.

Since the scattered wave is outgoing the asymptotic behavior of the scattered field u^s is determined by its far-field pattern u^s_∞.

Let us recall that (1.1) is referred to as the direct scattering problem. The corresponding inverse scattering problem consists of the determination of the position and shape of a scatterer given the far-field patterns of the scattered waves corresponding to one or more incident planar waves. In this paper, precisely in Section 6, we treat the case of sound-soft scatterers and in order to perform more scattering, that is far-field, measurements, we let vary the direction of propagation of the incident fields, keeping the same wavenumber $k > 0$. Let us notice that, instead of far-field measurements, one may also use so-called near-field measurements, that is the Cauchy data of the scattered fields u^s on the boundary of a domain containing the scatterer Σ or, equivalently, the values of the scattered fields on a neighborhood of the boundary of such a domain.

One of the main results of the paper is a stability estimate for the determination of near-field data from far-field data for any outgoing solution of the Helmholtz equation, see Section 4. The technique used is the one developed by Isakov in [15], see also [5], where the result is stated just for simplicity only in dimension 3. The main novelty here, besides the fact that we state our results in any space dimension $N \geq 2$, is that we investigate how the estimate changes with respect to the wavenumber k. In Theorem 4.1 we deal with the case of k belonging to a fixed compact interval of positive numbers. This result is essentially a

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rephrasing of Isakov's result extended to any dimension $N \geq 2$ and in fact we obtain the usual logarithmic type estimate. Then we deal with the high frequency, or wavenumber, case and we notice that the stability estimate improves as k increases. Actually there are two regimes: the high frequencies regimes and the extremely high frequencies one. The first regime is treated in Theorem 4.2 and it holds for wavenumbers k which are at most of the order of $\log(1/\varepsilon)$, ε being an estimate of the norm of the far-field pattern. Here the estimate is still of logarithmic type and improves as k increases in an exponential way with respect to k. For the largest value of k for which such a regime holds, the improvement leads to a Hölder type estimate. If k is beyond such a threshold, that is in the extremely high frequencies regime, the stability estimate improves even further, see Theorem 4.3. In fact it is still at least of Hölder type and it may be actually written as the sum of a Lipschitz term plus one which is exponentially decaying with respect to k. Finally, if the a priori bound is of the order of some power of k, and k is at least of the order of $\log(1/\varepsilon)$, we obtain a Lipschitz estimate fully independent of k, see Theorem 4.4.

Let us recall the basic idea of Isakov for obtaining this kind of stability estimates. The key tool is using separation of variables for describing outgoing solutions of the Helmholtz equation. The dependence on the radial variable r is given through suitable Hankel functions of first kind evaluated at kr. If the wavenumber is below a fixed constant, the stability estimates follows by studying the asymptotic behavior of these Hankel functions with respect to their order, a classical result in the theory of special functions. However, if we let the wavenumber tend to infinity the analysis is much more involved since we need to consider the asymptotic behavior of Hankel functions in three different regimes. In the first regime, the argument, that is kr, is much larger than the order. In the second one, the argument and the order are both large but of the same magnitude. In the third one, the order is much larger than the argument. Such an asymptotic analysis is performed in Section 3 whose main result is Theorem 3.9 which contains the asymptotic behavior of Hankel functions in the previous three regimes. This is the main technical result of the paper and the essential tool for obtaining the estimates of Section 4.

In Section 5 we apply the previous estimates to direct scattering problems. We assume that Σ_1 and Σ_2 are two sound-soft obstacles and that K is the convex hull of their union. We assume that Σ_1 and Σ_2 are star-shaped and smooth enough. Let u_1 and u_2 be the solutions to the direct scattering problem with Σ replaced by Σ_1 and Σ_2, respectively. Our aim is to estimate the difference of u_1 and u_2 up to the boundary of K by the difference of the far-field patterns of the corresponding scattered waves. Since by the results of Section 4 we are able to estimate the near-field data from the far-field data, we state our stability estimates in terms of the near-field data only, Theorem 5.6. The key tool to estimate $u_1 - u_2$ up to the boundary of K from the the near-field data is given by the results of [12, 33]. In the extremely high frequencies regime we also state the stability estimate with respect to the far-field data, see Corollary 5.7. In both cases we obtain a stability estimate with an explicit dependence on the wavenumber k, however we need to note that, for the time being, we are not able to prove any increasing stability property as k grows and tend to infinity, see Corollary 5.7.

About our stability estimate, we wish to use as less as possible a priori information, namely only a priori bounds of L^2 type on the solutions. For this reason we use an integral norm to estimate the difference between u_1 and u_2. In order to use the results of [12, 33] on the whole exterior of K some technical difficulties arise which are solved by using the

L^1 norm, instead of the L^2 norm, to estimate the difference between u_1 and u_2 and by the help of a technical geometrical lemma, Lemma 5.9.

It would be desirable to proceed further with the analysis and obtain suitable stability estimates, with a precise dependence on k, up to the boundary of the unbounded connected component of $\mathbb{R}^N \backslash (\Sigma_1 \cup \Sigma_2)$ or for the corresponding inverse scattering problem. However, both seem to be rather difficult open problems.

Let us notice that a rather long preliminary part of this section contains the essential a priori estimates which are needed to implement the previously described strategy. We believe that this part may also be of independent interest. It is here that the main assumptions on Σ_1 and Σ_2, namely star-shapedness and smoothness, are needed. The key ingredient in the high frequencies regime is provided by Theorem 5.4 which follows from results due to Chandler-Wilde and Monk, [6], and to Melenk, [25]. The main result of this part concerning a priori estimates is stated as Theorem 5.5.

Finally, on Section 6 we perform an instability analysis for the corresponding inverse scattering problem with sound-soft obstacles, extending the instability result in [11] to the high frequencies regime. Let us recall that a similar result has been obtained by Isaev for the inverse problem of determining by boundary data the potential in the Schrödinger equation in the high energies regime, see [13].

The main results of this section, Theorem 6.1 and Corollary 6.2, show that the instability improves as the wavenumber k increases. The high frequencies regime holds for wavenumbers k which are at most of the order of $\log(1/\varepsilon)$, ε being an estimate of the norm of the error in the far-field pattern for all possible directions of propagation of the incident field. In this regime the improvement is not so significant, however, beyond it, that is for extremely high frequencies, the improvement is more relevant since the logarithmic instability term has a multiplicative constant converging to zero, as k goes to $+\infty$, in a polynomial way with respect to k.

Let us notice that the norm used to estimate the error in the far-field pattern is an arbitrary H^s Sobolev norm, with $s \geq 0$. For s sufficiently large this is stronger than the L^∞ norm, that is our result applies also when ε is an estimate of the superior, for all directions of propagation of the incident field, of the error in the far-field pattern measured in the L^∞ norm. Moreover, we remark that our instability results hold for star-shaped and even convex obstacles. Even if the reconstruction of star-shaped or convex obstacle is considered to be more stable, our analysis shows that star-shapedness or convexity do not provide a significant advantage.

The proof is based on the original idea by Mandache, [23], which has been generalized and applied to inverse scattering problems in [11]. However again we need to take into account the fact that the wavenumber may be arbitrarily large, therefore a careful use of the results of Section 3 is required.

The plan of the paper is the following. We begin with a preliminary section, Section 2, where we give a separation of variables description of scattered waves and their far-field patterns. Moreover we introduce suitable classes of smooth star-shaped obstacles. In Section 3 we develop the asymptotic analysis for Hankel functions. The stability estimates for the determination of near-field data from far-field data are contained in Section 4. In Section 5 we consider the application to direct scattering problems with sound-soft scatterers. In particular we apply the estimate of the previous section to the stable determination of scattered waves from their far-field patterns. Finally, in Section 6 the instability analysis of

the corresponding inverse scattering problem is considered.

2 Preliminaries

Let us fix an integer $N \geq 2$. For any $x \in \mathbb{R}^N$ and any $s > 0$, $B_s(x)$ denotes the ball contained in $\mathbb{R}^N$ with radius s and center x. Moreover, $B_s = B_s(0)$ and, finally, for any $E \subset \mathbb{R}^N$, we denote $B_s(E) = \bigcup_{x \in E} B_s(x)$. For any $E \subset \mathbb{R}^N$, $|E|$ denotes as usual the N-dimensional Lebesgue measure of E.

We fix a *scatterer* Σ in $\mathbb{R}^N$, that is a compact subset of $\mathbb{R}^N$ such that $G = \mathbb{R}^N \backslash \Sigma$ is connected. We recall that a scatterer Σ is said to be an *obstacle* if Σ is the closure of an open set. We fix the *wavenumber* $k > 0$ and a *direction of propagation* $\omega \in \mathbb{S}^{N-1}$. Then the *incident field* is the time-harmonic plane wave $u^i(x) = \mathrm{e}^{\mathrm{i}k\omega \cdot x}$, $x \in \mathbb{R}^N$. The *total field* $u = u(\omega, k, \Sigma)$ is the sum of the incident field and of the *scattered field* u^s and is the solution to the following exterior boundary value problem

$$
(2.1) \quad
\begin{cases}
\Delta u + k^2 u = 0 & \text{in } \mathbb{R}^N \backslash \Sigma \\
u = u^i + u^s & \text{in } \mathbb{R}^N \backslash \Sigma \\
u = 0 & \text{on } \partial \Sigma \\
\lim_{r \to \infty} r^{(N-1)/2} \left(\dfrac{\partial u^s}{\partial r} - \mathrm{i}k u^s \right) = 0 & r = \|x\|
\end{cases}
$$

where the last limit, the so-called *Sommerfeld radiation condition*, holds uniformly for all directions $\hat{x} = x/\|x\| \in \mathbb{S}^{N-1}$. We remark that the homogeneous Dirichlet boundary condition corresponds to a *sound-soft* scatterer Σ.

The Sommerfeld radiation condition characterizes outgoing waves and implies that the asymptotic behavior of the scattered field is given by

$$
(2.2) \quad u^s(x; \omega, k, \Sigma) = \frac{\mathrm{e}^{\mathrm{i}k\|x\|}}{\|x\|^{(N-1)/2}} \left\{ u^s_\infty(\hat{x}; \omega, k, \Sigma) + O\left(\frac{1}{\|x\|}\right) \right\},
$$

as $\|x\|$ goes to $+\infty$, uniformly in all directions $\hat{x} = x/\|x\| \in \mathbb{S}^{N-1}$. The function u^s_∞ is called the *far-field pattern* of the scattered field u^s of the solution to (2.1).

For any sound-soft scatterer Σ, we denote $\mathcal{A}(\Sigma) : \mathbb{S}^{N-1} \times \mathbb{S}^{N-1} \times (0, \infty) \mapsto \mathbb{C}$ its *far-field map*, that is, for any $\hat{x}, \omega \in \mathbb{S}^{N-1}$ and any $k > 0$,

$$
(2.3) \quad \mathcal{A}(\Sigma)(\hat{x}, \omega, k) = u^s_\infty(\hat{x}; \omega, k, \Sigma),
$$

where u^s_∞ is the far-field pattern of the scattered field u^s of the solution to (2.1).

Let us remark that the following *reciprocity relation* holds, see for instance [10, Theorem 3.13]. For any scatterer Σ and any $k \in (0, \infty)$ we have

$$
(2.4) \quad \mathcal{A}(\Sigma)(\hat{x}, \omega, k) = \mathcal{A}(\Sigma)(-\omega, -\hat{x}, k) \quad \text{for any } \hat{x}, \omega \in \mathbb{S}^{N-1}.
$$

Moreover, we have the following characterization of the L^2 norm of the far-field pattern, see for instance [29, Theorem 3.2.1] for $N = 3$,

$$(2.5) \qquad \|\mathcal{A}(\Sigma)(\cdot, \omega, k)\|^2_{L^2(\mathbb{S}^{N-1})} = 2 \left(\frac{2\pi}{k} \right)^{(N-1)/2} \Im \left(\mathrm{e}^{(N-3)\pi \mathrm{i}/4} \mathcal{A}(\Sigma)(\omega, \omega, k) \right).$$

We wish to decompose the far-field pattern in spherical harmonics. Fixed $N \geq 2$, let us consider the orthonormal basis of $L^2(\mathbb{S}^{N-1})$

$$(2.6) \qquad \{v_i\}_{i \in \mathbb{N}} = \{f_{jp} : \ j \geq 0 \text{ and } 1 \leq p \leq p_j\}$$

that consists of (real-valued) spherical harmonics, that is each f_{jp} is a *spherical harmonic* of degree j, j being a nonnegative integer, such that $\|f_{jp}\|_{L^2(\mathbb{S}^{N-1})} = 1$. The elements v_i, $i \in \mathbb{N}$, are ordered in the natural way.

The integers p_j are the dimensions of the spaces of spherical harmonics of degree j and we have that, see for instance [27, page 4],

$$p_j = \begin{cases} 1 & \text{if } j = 0, \\ \frac{(2j+N-2)(j+N-3)!}{j!(N-2)!} & \text{if } j \geq 1, \end{cases}$$

so that

$$p_j \leq 2(j+1)^{N-2}, \quad j \geq 0,$$

and

$$(2.7) \qquad \sum_{j=0}^{n} p_j \leq \sum_{j=0}^{n} 2(j+1)^{N-2} \leq 2(n+1)^{N-1}, \quad \text{for any } n \in \mathbb{N}.$$

For any spherical harmonic f, we call $\gamma(f)$ the degree of the spherical harmonic, that is $\gamma(f_{jp}) = j$. We have that $\gamma(v_i)$ is an increasing sequence, with respect to i, whose asymptotic behavior satisfies the following property. Fixed $n \in \mathbb{N}$, we have that, by (2.7), $\#\{i \in \mathbb{N} : \ \gamma(v_i) \leq n\}$ is clearly bounded from above by $2(n+1)^{N-1}$, therefore

$$(2.8) \qquad \#\{i \in \mathbb{N} : \ \gamma(v_i) \leq n\} \leq C(n+1)^p$$

with $C = 2$ and $p = N - 1$. We recall that $\#$ denotes the number of elements.

We recall that the function

$$(2.9) \qquad u_{jp}(x) = \|x\|^j f_{jp}(x/\|x\|)$$

is harmonic in $\mathbb{R}^N$ and solves the following eigenvalue problem in B_1

$$(2.10) \qquad \Delta u_{jp} = 0 \text{ in } B_1; \quad \frac{\partial u_{jp}}{\partial \nu} = j u_{jp} \text{ on } \partial B_1.$$

For any function g belonging to $L^2(\mathbb{S}^{N-1} \times \mathbb{S}^{N-1})$, we decompose it in spherical harmonics in the following way

$$g(\hat{x}, \omega) = \sum_{i,l} a_{i,l} v_i(\hat{x}) v_l(\omega) \quad (\hat{x}, \omega) \in \mathbb{S}^{N-1} \times \mathbb{S}^{N-1}$$

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where the complex-valued coefficients $a_{i,l}$ are given by

$$a_{i,l} = \iint_{\mathbb{S}^{N-1} \times \mathbb{S}^{N-1}} g(\hat{x}, \omega) v_i(\hat{x}) v_l(\omega) \mathrm{d}\hat{x} \mathrm{d}\omega.$$

For any $s \geq 0$ we define the norm of the Sobolev space $H^s(\mathbb{S}^{N-1} \times \mathbb{S}^{N-1})$ as follows

$$(2.11) \qquad \|g\|^2_{H^s(\mathbb{S}^{N-1} \times \mathbb{S}^{N-1})} = \sum_{i,l} (1 + \gamma(v_i) + \gamma(v_l))^{2s} |a_{i,l}|^2.$$

Then for any $s \geq 0$ we call $Y_s(\mathbb{S}^{N-1} \times \mathbb{S}^{N-1})$ the space

$$Y_s(\mathbb{S}^{N-1} \times \mathbb{S}^{N-1}) = \{g \in L^2(\mathbb{S}^{N-1} \times \mathbb{S}^{N-1}) : \|g\|_s < +\infty\}$$

where

$$\|g\|_s = \sup_{i,l} \left((1 + \max\{\gamma(v_i), \gamma(v_l)\})^{2s+N-1/2} |a_{i,l}| \right)$$

We notice that $Y_s(\mathbb{S}^{N-1} \times \mathbb{S}^{N-1}) \subset H^s(\mathbb{S}^{N-1} \times \mathbb{S}^{N-1})$ and the immersion is continuous, in fact we have

$$\|g\|_{H^s(\mathbb{S}^{N-1} \times \mathbb{S}^{N-1})} \leq 4\|g\|_s \quad \text{for any } g \in Y_s(\mathbb{S}^{N-1} \times \mathbb{S}^{N-1}),$$

since

$$\sum_{i,l} (1 + \gamma(v_i) + \gamma(v_l))^{2s} |a_{i,l}|^2 \leq \sum_{i,l} (1 + \max\{\gamma(v_i), \gamma(v_l)\})^{4s} |a_{i,l}|^2 \leq$$

$$16 \sup_{i,l} \left((1 + \max\{\gamma(v_i), \gamma(v_l)\})^{4s+2N-1} |a_{i,l}|^2 \right)$$

see [23, page 1439].

The decomposition of the far-field pattern in spherical harmonics is given by, for any $(\hat{x}, \omega, k) \in \mathbb{S}^{N-1} \times \mathbb{S}^{N-1} \times (0, \infty)$,

$$(2.12) \qquad \mathcal{A}(\Sigma)(\hat{x}, \omega, k) = \sum_{i,l} b_{i,l}(k) v_i(\hat{x}) v_l(\omega),$$

where the complex-valued coefficients $b_{i,l}(k)$ are given, for any $k \in (0, \infty)$, by

$$(2.13) \qquad b_{i,l}(k) = \iint_{\mathbb{S}^{N-1} \times \mathbb{S}^{N-1}} \mathcal{A}(\Sigma)(\hat{x}, \omega, k) v_i(\hat{x}) v_l(\omega) \mathrm{d}\hat{x} \mathrm{d}\omega.$$

Furthermore, we use the following characterization

$$(2.14) \qquad b_{i,l}(k) = \int_{\mathbb{S}^{N-1}} \tilde{b}_i(\omega, k) v_l(\omega) \mathrm{d}\omega,$$

where the complex-valued coefficients $\tilde{b}_i(\omega, k)$ are, for any $\omega \in \mathbb{S}^{N-1}$ and any $k \in (0, \infty)$, the Fourier coefficients, with respect to the orthonormal basis $\{v_i\}_{i \in \mathbb{N}}$, of the far-field pattern $u^s_\infty(\cdot; \omega, k, \Sigma)$ corresponding to the scattered field of the solution to (2.1), that is

$$(2.15) \qquad \tilde{b}_i(\omega, k) = \int_{\mathbb{S}^{N-1}} \mathcal{A}(\Sigma)(\hat{x}, \omega, k) v_i(\hat{x}) \mathrm{d}\hat{x}.$$

Let us assume that $\Sigma \subset \overline{B_R}$ for some positive constant R. Then for any $x \in \mathbb{R}^N \backslash \overline{B_R}$ we have

$$(2.16) \qquad u^s(x;\omega,k,\Sigma) = \sum_i \hat{b}_i(\omega,k) \frac{H^{(1)}_{\gamma(v_i)+(N-2)/2}(k\|x\|)}{(k\|x\|)^{(N-2)/2}} v_i(x/\|x\|),$$

where $\hat{b}_i = \hat{b}_i(\omega,k)$ are complex-valued coefficients given by

$$(2.17) \qquad \hat{b}_i(\omega,k) \frac{H^{(1)}_{\gamma(v_i)+(N-2)/2}(kr)}{(kr)^{(N-2)/2}} = \int_{\mathbb{S}^{N-1}} u^s(r\hat{x};\omega,k,\Sigma) v_i(\hat{x}) \mathrm{d}\hat{x}, \quad \text{for any } r > R,$$

where, for any real $\nu \geq 0$, $H^{(1)}_\nu$ denotes the *Hankel function* of first kind and order ν.

The relationship between coefficients $\tilde{b}_i$ and $\hat{b}_i$ is the following

$$(2.18) \qquad \tilde{b}_i(\omega,k) = (\pi/2)^{-1/2} k^{-(N-1)/2}(-\mathrm{i})^{\gamma(v_i)+(N-1)/2} \hat{b}_i(\omega,k).$$

Therefore, for any $r > R$,

$$(2.19) \qquad u^s(r\hat{x};\omega,k,\Sigma) = \sum_i \tilde{b}_i(\omega,k)(\pi k/2)^{1/2} \mathrm{i}^{\gamma(v_i)+(N-1)/2} \frac{H^{(1)}_{\gamma(v_i)+(N-2)/2}(kr)}{r^{(N-2)/2}} v_i(\hat{x}),$$

hence

$$(2.20) \qquad \|u^s(\cdot;\omega,k,\Sigma)\|^2_{L^2(\partial B_r)} = \frac{\pi}{2} \sum_i |\tilde{b}_i(\omega,k)|^2 kr \left| H^{(1)}_{\gamma(v_i)+(N-2)/2}(kr) \right|^2.$$

Obviously, by (2.3) and (2.15) we have

$$(2.21) \qquad \|u^s_\infty(\cdot;\omega,k,\Sigma)\|^2_{L^2(\mathbb{S}^{N-1})} = \sum_i |\tilde{b}_i(\omega,k)|^2.$$

In order to estimate the near-field from the far-field, a crucial step is to estimate the asymptotic behavior of Hankel functions $H^{(1)}_\nu(z)$ as the order ν goes to infinity, for a fixed $z = kr$. On the other hand, we are interested in the corresponding estimate as the wavenumber k goes to infinity, therefore we need to consider the behavior of $H^{(1)}_\nu(z)$ with the order ν and the argument $z = kr$ which may be both large at the same time. We deal with this issue in the following section.

We conclude this section by introducing suitable classes of obstacles. We fix integers $N \geq 2$ and $m \geq 1$ and positive constants β, R_0 and δ, $R_0 < R_0 + \delta \leq \beta$.

Let g be a strictly positive continuous function defined on $\mathbb{S}^{N-1}$. Let $\Sigma(g)$ be the compact set given by the *radial subgraph* of g, that is

$$\Sigma(g) = \{y \in \mathbb{R}^N : y = \rho\omega, \ 0 \leq \rho \leq g(\omega), \ \omega \in \mathbb{S}^{N-1}\}.$$

We denote

$$X(m,\beta,R_0,\delta) = \{\Sigma(g) : g \in C^m(\mathbb{S}^{N-1}), \ \|g\|_{C^m(\mathbb{S}^{N-1})} \leq \beta \text{ and } R_0 \leq g \leq R_0 + \delta\}.$$

We notice that $X = X(m,\beta,R_0,\delta)$ is a metric space, endowed with the Hausdorff distance d_H, that consists of obstacles in $\mathbb{R}^N$ which are star-shaped with respect to the

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origin. Moreover, for any integer $m \geq 3$, and for any positive β and R_0, there exists $\tilde{\delta} > 0$, depending on N, m, β and R_0 only, such that if $0 < \delta \leq \tilde{\delta}$, then any element of X is even convex.

Let us also notice that there exists a constant E, depending on N, m, β and R_0 only, such that

(2.22) $$\mathcal{H}^{N-1}(\Sigma) \leq E \quad \text{for any } \Sigma \in X(m, \beta, R_0, \delta),$$

where $\mathcal{H}^{N-1}$ denotes the $(N-1)$-dimensional Hausdorff measure.

Given the metric space (X, d_H) and a positive δ, $X' \subset X$ is said to be δ-*discrete* if any two distinct points x_1, $x_2 \in X'$ satisfy $d_H(x_1, x_2) \geq \delta$. We have the following result, whose proof may be easily obtained following the arguments of the proof of Lemma 2 in [23].

Proposition 2.1 *Let us fix integers $N \geq 2$ and $m \geq 1$ and positive constants β and R_0.*

Then, there exists a positive constant δ_0, depending on N, m, β and R_0 only, such that $R_0 + \delta_0 \leq \beta$ and for any δ, $0 < \delta \leq \delta_0$, we can find a δ-discrete subset of $X(m, \beta, R_0, \delta)$ with at least $\exp(2^{-N} \delta_0^{(N-1)/m} \delta^{-(N-1)/m})$ elements.

3 Estimates on the asymptotic behavior of Hankel functions

For any $\nu \geq 0$ and any $z > 0$, let $H_\nu^{(1)}(z)$ be the Hankel function of first kind of order ν and argument z. Concerning basic properties of Hankel and Bessel functions we refer to [22] and especially [34].

It is well-known that the following asymptotic behaviors of $H_\nu^{(1)}(z)$ holds true. First, fixed $\nu \geq 0$, we have that

(3.1) $$H_\nu^{(1)}(z) = \left(\frac{2}{\pi z}\right)^{1/2} e^{i(z - \nu\pi/2 - \pi/4)} \left[1 + O(z^{-1})\right] \quad \text{as } z \to +\infty.$$

On the other hand, fixed $z > 0$, we have that

(3.2) $$H_\nu^{(1)}(z) \sim -i \left(\frac{2}{\pi\nu}\right)^{1/2} \left(\frac{ez}{2\nu}\right)^{-\nu} \quad \text{as } \nu \to +\infty.$$

where $\sim$ means that the quotient between the two functions tends to 1, as $\nu \to \infty$.

We need similar estimates on the asymptotic behavior of Hankel functions which hold uniformly for suitable intervals of z and ν respectively. We obtain a full hierarchy of asymptotic behaviors depending on the relationship between z and ν as one or both of these parameters go to $+\infty$, see Theorem 3.9.

We begin by recalling that

$$J_\nu(z) = \sum_{k=0}^{+\infty} \frac{(-1)^k (z/2)^{\nu+2k}}{\Gamma(k+1)\Gamma(k+\nu+1)} \quad z > 0, \ \nu \in \mathbb{R}$$

where J_ν is the Bessel function of first kind of order ν. Here Γ denotes the Gamma function. We also recall that Y_ν, the Bessel function of second kind of order ν, is given by

$$Y_\nu(z) = \frac{J_\nu(z)\cos(\nu\pi) - J_{-\nu}(z)}{\sin(\nu\pi)} \quad z > 0, \ \nu \in \mathbb{R}\backslash\mathbb{Z},$$

whereas
$$Y_n(z) = \lim_{\nu \to n} Y_\nu(z) \quad z > 0, \; n \in \mathbb{Z}.$$

Finally,
$$H_\nu^{(1)}(z) = J_\nu(z) + \mathrm{i}Y_\nu(z) \quad z > 0, \; \nu \in \mathbb{R}.$$

Let us recall here the basic properties of the Gamma function Γ that will be repeatedly used in the sequel. We note that $\Gamma(z)$ is defined for any $z \in \mathbb{C}$ such that $z \neq 0, -1, -2, \ldots$, and we have that $\Gamma(1) = 1$. The most important property is the following

$$(3.3) \qquad \Gamma(z+1) = z\Gamma(z) \quad \text{for any } z \neq 0, -1, -2, \ldots,$$

from which we deduce that, for any $n = 0, 1, 2, \ldots$, $n! = \Gamma(n+1)$. We also use that

$$(3.4) \qquad \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \quad \text{for any } z \notin \mathbb{Z}.$$

Finally the following version of the classical Stirling formula will be used

$$(3.5) \qquad \Gamma(x) = \sqrt{2\pi}x^{x-1/2}\mathrm{e}^{-x}[1 + r(x)], \quad x > 0$$

where
$$|r(x)| \leq \mathrm{e}^{1/(12x)} - 1.$$

We begin with the following easy proposition.

Proposition 3.1 *For any a, $0 \leq a < 1$, any $z > 0$ and any $\nu > 0$, let*

$$A = A(a, z, \nu) = \sum_{k=0}^{+\infty} \frac{(-1)^k (z^2/4)^k}{\Gamma(k+a+1)\Gamma(k+\nu+1)}.$$

Let us fix $z_2 \geq 1$ and ε, $0 < \varepsilon \leq 1/\mathrm{e}$. Then there exist positive constants $\nu_0 = \nu_0(\varepsilon, z_2) \geq \mathrm{e}z_2 > 0$ and $\tilde{\nu}_0 = \tilde{\nu}_0(\varepsilon)$ such that, for any a, $0 \leq a < 1$, for any $0 < z \leq z_2$, and for any $\nu \geq \max\{\nu_0(\varepsilon, z_2), \tilde{\nu}_0(\varepsilon)\}$, we have

$$(3.6) \qquad \frac{(1-\varepsilon)}{\Gamma(a+1)}\left(\frac{1}{\sqrt{2\pi\nu}}\nu^{-\nu}\mathrm{e}^\nu\right) \leq A(a, z, \nu) \leq \frac{(1+\varepsilon)}{\Gamma(a+1)}\left(\frac{1}{\sqrt{2\pi\nu}}\nu^{-\nu}\mathrm{e}^\nu\right)$$

and, consequently, picking $a = 0$,

$$(3.7) \qquad \frac{(1-\varepsilon)}{\sqrt{2\pi\nu}}\left(\frac{\mathrm{e}z}{2\nu}\right)^\nu \leq J_\nu(z) \leq \frac{(1+\varepsilon)}{\sqrt{2\pi\nu}}\left(\frac{\mathrm{e}z}{2\nu}\right)^\nu.$$

Notice that $\nu \geq \mathrm{e}z$, therefore

$$(3.8) \qquad 0 \leq J_\nu(z) \leq \frac{(1+\varepsilon)}{\sqrt{2\pi\nu}},$$

hence there exists $\tilde{\nu}_1 = \tilde{\nu}_1(\varepsilon)$, $\tilde{\nu}_1 \geq \tilde{\nu}_0$, such that if ν is also greater than or equal to $\tilde{\nu}_1$ we have

$$(3.9) \qquad 0 \leq J_\nu(z) \leq \varepsilon.$$

PROOF. Using the basic properties of the Γ function, we can find $\nu_0 = \nu_0(\varepsilon, z_2) > 0$ such that for any $0 < z \leq z_2$ and any $\nu \geq \nu_0$ we have

$$(3.10) \qquad (1-\varepsilon)\frac{1}{\Gamma(\nu+1)\Gamma(a+1)} \leq A(a,z,\nu) \leq \frac{1}{\Gamma(\nu+1)\Gamma(a+1)}.$$

Notice that it is enough that $z_2^2/4 \leq \varepsilon(\nu_0 + 1)$, that is for instance we may set

$$\nu_0(\varepsilon, z_2) = \frac{z_2^2}{\varepsilon}.$$

Therefore, without loss of generality, we may choose $\nu_0(\varepsilon, z_2) \geq ez_2$. The rest of the proof easily follows. $\qquad \square$

We now prove some estimates on $Y_\nu(z)$ for the following particular order ν. Let us assume that $\nu = n - 1/2$ for some positive integer n. We obtain that $\cos(\nu\pi) = 0$ whereas $\sin(\nu\pi) = (-1)^{n+1}$. Therefore, for any $z > 0$, $Y_\nu(z) = (-1)^n J_{-\nu}(z)$. We notice that

$$Y_\nu(z) = (-1)^n (z/2)^{-\nu} \left[\sum_{k=0}^{\infty} \frac{(-1)^k (z^2/4)^k}{\Gamma(k+1)\Gamma(k-\nu+1)} \right] = (-1)^n (z/2)^{-\nu} [A + B] \quad z > 0,$$

where

$$B = \sum_{k=0}^{n-1} \frac{(-1)^k (z^2/4)^k}{\Gamma(k+1)\Gamma(k-\nu+1)} \quad \text{and} \quad A = \sum_{k=n}^{+\infty} \frac{(-1)^k (z^2/4)^k}{\Gamma(k+1)\Gamma(k-\nu+1)}.$$

Let us notice that by a simple change of the index, we have that

$$A = (-1)^n (z^2/4)^n A(1/2, z, n),$$

therefore

$$Y_\nu(z) = (z/2)^{2n-\nu} A(1/2, z, n) + (-1)^n (z/2)^{-\nu} B = \tilde{A} + \tilde{B}.$$

By Proposition 3.1, for any z, $0 < z \leq z_2$, and any $\nu \geq \max\{\nu_0(\varepsilon, z_2), \tilde{\nu}_0(\varepsilon)\}$, we have

$$(3.11) \qquad \frac{(1-\varepsilon)}{\Gamma(3/2)}\sqrt{\frac{z}{4\pi n}}\left(\frac{ez}{2n}\right)^n \leq \tilde{A} \leq \frac{(1+\varepsilon)}{\Gamma(3/2)}\sqrt{\frac{z}{4\pi n}}\left(\frac{ez}{2n}\right)^n.$$

We infer that

$$(3.12) \qquad 0 \leq \tilde{A} \leq \frac{(1+\varepsilon)}{\Gamma(3/2)}\sqrt{\frac{1}{2e\pi}}\left(\frac{ez}{2n}\right)^{n+1/2}.$$

hence there exists $\tilde{\nu}_2 = \tilde{\nu}_2(\varepsilon)$, $\tilde{\nu}_2 \geq \tilde{\nu}_1$, such that if ν is also greater than or equal to $\tilde{\nu}_2$ we have

$$(3.13) \qquad 0 \leq \tilde{A} \leq \varepsilon.$$

We continue by evaluating B and, correspondingly, $\tilde{B}$. A simple computation, where we use in particular (3.3) and also (3.4), leads to

$$\tilde{B} = -(z/2)^{-\nu} \sum_{k=0}^{n-1} \left(\frac{(z^2/4)^k}{\Gamma(k+1)\Gamma(1/2)} \prod_{i=1}^{n-1-k} (i - 1/2) \right) =$$

$$- (z/2)^{-\nu} \sum_{k=0}^{n-1} \left(\frac{(z^2/4)^k \Gamma(n-k-1/2)}{\Gamma(k+1)(\Gamma(1/2))^2} \right) = -\frac{(z/2)^{-\nu}}{\pi} \sum_{k=0}^{n-1} \left((z^2/4)^k \frac{\Gamma(n-k-1/2)}{\Gamma(k+1)} \right).$$

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Taking $h = n - 1 - k$, we have that

$$\tilde{B} = -\frac{(z/2)^{-\nu}}{\pi}\Gamma(\nu)\left[1 + \sum_{h=0}^{n-2}\left((z^2/4)^{n-1-h}\frac{\Gamma(h+1/2)}{\Gamma(\nu)\Gamma(n-h)}\right)\right] = -\frac{(z/2)^{-\nu}}{\pi}\Gamma(\nu)\left[1 + R\right].$$

A lengthy but straightforward computation would allow us to estimate the remainder R, by studying separately the sum with h below $(n-2)/2$ and the one with h above $(n-2)/2$. We obtain that R goes to zero as ν goes to $+\infty$. However, for our purposes, at this time it is enough to note that $R \geq 0$, therefore

$$\tilde{B} \leq -\frac{(z/2)^{-\nu}}{\pi}\Gamma(\nu) \leq -(1-\varepsilon)\left(\frac{2}{\pi\nu}\right)^{1/2}\left(\frac{ez}{2\nu}\right)^{-\nu}$$

provided $\nu \geq \tilde{\nu}_3(\varepsilon) \geq \tilde{\nu}_2(\varepsilon)$ as well. Up to changing $\tilde{\nu}_3 = \tilde{\nu}_3(\varepsilon)$ and using (3.9), we can prove the following two estimates. Let us assume that $\nu = n - 1/2$ for some positive integer n and that $\nu \geq \max\{\nu_0(\varepsilon, z_2), \tilde{\nu}_3(\varepsilon)\}$, then

(3.14)
$$Y_\nu(z) \leq -(1-\varepsilon)\left(\frac{2}{\pi\nu}\right)^{1/2}\left(\frac{ez}{2\nu}\right)^{-\nu}$$

and

(3.15)
$$|J_\nu(z)| \leq \varepsilon|Y_\nu(z)|.$$

Let us recall the following lemma, see for instance [24, Appendix B] and [21].

Lemma 3.2 *Let* $0 < z \leq \nu$. *Then*

$$J_\nu(z) > 0 \quad and \quad \frac{\partial J_\nu(z)}{\partial\nu} < 0$$

whereas

$$Y_\nu(z) < 0 \quad and \quad \frac{\partial Y_\nu(z)}{\partial\nu} < 0.$$

Moreover, as ξ *goes to* $+\infty$,

(3.16)
$$J_\xi(\xi) = \frac{\Gamma(1/3)}{3^{1/6}2^{2/3}\pi}\xi^{-1/3} + O(\xi^{-4/3})$$

and

(3.17)
$$Y_\xi(\xi) = -\frac{3^{1/3}\Gamma(1/3)}{2^{2/3}\pi}\xi^{-1/3} + O(\xi^{-4/3}).$$

Using these results we immediately infer the following corollary. The first part is a consequence of Proposition 3.1 and of the estimates (3.14) and (3.15), the second follows from Lemma 3.2.

Corollary 3.3 *Let us fix* $z_2 \geq 1$ *and* ε, $0 < \varepsilon \leq 1/e$. *Then there exists a positive constant* $\tilde{\nu}_3 = \tilde{\nu}_3(\varepsilon)$ *such that for any* $0 < z \leq z_2$, *and for any* $\nu \geq \max\{\nu_0(\varepsilon, z_2), \tilde{\nu}_3(\varepsilon)\}$, *we have*

(3.18)
$$|J_\nu(z)| \leq \varepsilon|Y_\nu(z)|$$

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hence

$$(1 - \varepsilon)|Y_\nu(z)| \leq |H_\nu^{(1)}(z)| \leq (1 + \varepsilon)|Y_\nu(z)|.$$

Furthermore, there exist absolute constants c_1, $0 < c_1 < 1$, and $\xi_0 > 0$ such that for any $\xi \geq \xi_0$ we have

$$|J_\xi(\xi)| \leq c_1|Y_\xi(\xi)|$$

therefore for any $\nu \geq z \geq \xi_0$ we have

$$|J_\nu(z)| \leq c_1|Y_\nu(z)|$$

and hence

(3.19)
$$(1 - c_1)|Y_\nu(z)| \leq |H_\nu^{(1)}(z)| \leq (1 + c_1)|Y_\nu(z)|.$$

We now consider different regimes. First, we take ν much larger than z, then we consider the case in which z is much larger than ν. Finally we shall deal with the intermediate regime when ν and z are comparable and large.

We begin with the following two results, due to Matviyenko, proved in Theorem 3.1 and Theorem 3.3 in [24], respectively.

Theorem 3.4 *Let $0 \leq \nu < z$. Then*

(3.20)
$$H_\nu^{(1)}(z) = \left(\frac{2}{\pi}\right)^{1/2} \frac{1}{(z^2 - \nu^2)^{1/4}} \exp(i\eta_1)\left[1 + R_1(z, \nu)\right]$$

where

$$\eta_1 = (z^2 - \nu^2)^{1/2} - \nu\arccos(\nu/z) - \pi/4$$

and

$$|R_1(z, \nu)| \leq \exp(\tilde{g}_1)\tilde{g}_1$$

with

$$g_1 = \frac{z - \nu}{z^{1/3}} \quad and \quad \tilde{g}_1 = \frac{2}{3g_1^{3/2}}.$$

Theorem 3.5 *Let $0 < z < \nu$. Then*

(3.21)
$$Y_\nu(z) = -\left(\frac{2}{\pi}\right)^{1/2} \frac{1}{(\nu^2 - z^2)^{1/4}} \exp(\eta_2)\left[1 + R_2(z, \nu)\right]$$

where

(3.22)
$$\eta_2 = \nu\log\left(\frac{\nu}{z} + \left(\left(\frac{\nu}{z}\right)^2 - 1\right)^{1/2}\right) - (\nu^2 - z^2)^{1/2}$$

and

$$|R_2(z, \nu)| \leq \exp(\tilde{g}_2)\tilde{g}_2$$

with

$$g_2 = \frac{\nu - z}{\nu^{1/3}} \quad and \quad \tilde{g}_2 = \frac{2}{3g_2^{3/2}}.$$

We have the following corollaries. By Theorem 3.4 we can generalize (3.1) by extending it uniformly on suitable intervals in the following way.

Corollary 3.6 *Fixed $C > 1$, for any $\nu \geq 0$ and for any positive z such that $z \geq C\nu$ we have*

$$\tilde{g}_1 \leq \frac{2}{3} \left(\frac{C}{C-1} \right)^{3/2} z^{-1} \quad and \quad 1 \leq \left(1 - \frac{\nu^2}{z^2} \right)^{-1/4} \leq \left(\frac{C^2}{C^2-1} \right)^{1/4}.$$

Therefore,

$$(3.23) \quad \left[1 - \frac{2}{3} \left(\frac{C}{C-1} \right)^{3/2} \exp\left(\frac{2}{3} \left(\frac{C}{C-1} \right)^{3/2} z^{-1} \right) z^{-1} \right] \left(\frac{2}{\pi z} \right)^{1/2} \leq |H_\nu^{(1)}(z)| \leq$$

$$\left(\frac{2}{\pi z} \right)^{1/2} \left(\frac{C^2}{C^2-1} \right)^{1/4} \left[1 + \frac{2}{3} \left(\frac{C}{C-1} \right)^{3/2} \exp\left(\frac{2}{3} \left(\frac{C}{C-1} \right)^{3/2} z^{-1} \right) z^{-1} \right].$$

Furthermore, there exists an absolute constant $C_1 > 0$ such that for any $\nu \geq 0$ and for any positive z satisfying $z - \nu \geq C_1 z^{1/3}$ then

$$(3.24) \quad \frac{1}{2} \left(\frac{2}{\pi} \right)^{1/2} \frac{1}{(z^2 - \nu^2)^{1/4}} \leq |H_\nu^{(1)}(z)| \leq \frac{3}{2} \left(\frac{2}{\pi} \right)^{1/2} \frac{1}{(z^2 - \nu^2)^{1/4}}.$$

Finally, there exists a constant C_2, depending on C only, such that for any $\nu \geq 0$ and for any positive z such that $z \geq C\nu$ we have

$$(3.25) \quad H_\nu^{(1)}(z) = \left(\frac{2}{\pi z} \right)^{1/2} e^{i(z - \nu\pi/2 - \pi/4)} \left[1 + \tilde{R}_1 \right]$$

where

$$|\tilde{R}_1| \leq C_2 \left(\frac{\nu^2}{z^2} + \min\left\{ 2, \frac{\nu^2}{z} \right\} + \exp\left(\frac{2}{3} \left(\frac{C}{C-1} \right)^{3/2} z^{-1} \right) z^{-1} \right).$$

By Theorem 3.5 we can generalize (3.2) by extending it uniformly on suitable intervals in the following way. In fact the following corollary holds and it may be coupled with Corollary 3.3.

Corollary 3.7 *Fixed $C > 1$, for any ν such that $\nu \geq Cz > 0$ we have*

$$\tilde{g}_2 \leq \frac{2}{3} \left(\frac{C}{C-1} \right)^{3/2} \nu^{-1}.$$

Thus for any ν such that $\nu \geq Cz > 0$ and $\nu \geq Cz^2$ we have

$$(3.26) \quad H_\nu^{(1)}(z) = -i \left(\frac{2}{\pi\nu} \right)^{1/2} \left(\frac{ez}{2\nu} \right)^{-\nu} \left[1 + \tilde{R}_2 \right]$$

where

$$|\tilde{R}_2| \leq \frac{|J_\nu(z)|}{|Y_\nu(z)|} + C_1 \left(1 + \frac{|J_\nu(z)|}{|Y_\nu(z)|} \right) \left(\frac{z^2}{\nu^2} + \frac{z^2}{\nu} + \exp\left(\frac{2}{3} \left(\frac{C}{C-1} \right)^{3/2} \nu^{-1} \right) \nu^{-1} \right)$$

and C_1 is an absolute constant depending on C only.

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A further important corollary is the following. Here we make use of the continuity properties of the Hankel functions with respect to both the argument and order and the fact that $|H_\nu^{(1)}(z)| > 0$ for any $\nu \geq 0$ and any $z > 0$.

Corollary 3.8 *Let us fix $0 < z_1 < z_2$. Then there exists a constant $C \geq 1$, depending on z_1 and z_2 only, such that for any z, $z_1 \leq z \leq z_2$, we have*

$$(3.27) \qquad C^{-1} \leq |H_0^{(1)}(z)| \leq C$$

and for any $\nu \geq 1/2$ we have

$$(3.28) \qquad C^{-1} \left(\frac{2}{\pi\nu} \right)^{1/2} \left(\frac{ez_2}{2\nu} \right)^{-\nu} \leq C^{-1} \left(\frac{2}{\pi\nu} \right)^{1/2} \left(\frac{ez}{2\nu} \right)^{-\nu} \leq |H_\nu^{(1)}(z)|$$

and

$$(3.29) \qquad |H_\nu^{(1)}(z)| \leq C \left(\frac{2}{\pi\nu} \right)^{1/2} \left(\frac{ez}{2\nu} \right)^{-\nu} \leq C \left(\frac{2}{\pi\nu} \right)^{1/2} \left(\frac{ez_1}{2\nu} \right)^{-\nu}.$$

This is the estimate that allows us to use Isakov's argument in [15] and prove stability estimates for the determination of the near-fields from far-field measurements, when the wavenumber k belongs to a fixed interval $[k_1, k_2]$, with $0 < k_1 < k_2$. We shall treat this case at the beginning of Section 4, in Theorem 4.1.

We now begin to investigate the more difficult case of the asymptotic behavior of Hankel functions when z and ν are both large.

We have the following theorem, which is the main result of this section.

Theorem 3.9 *There exist positive constants $z_0 \geq 1$, C_0 and $A_0 \geq 1$ such that if $z \geq z_0$ then the following asymptotic behavior of the Hankel function holds for $z > 0$ and $\nu \geq 0$.*
If

$$\nu > 0 \ and \ \frac{z - \nu}{\nu^{1/3}} \geq C_0 \quad or \quad \nu = 0$$

then

$$(3.30) \qquad A_0^{-1} \frac{1}{(z^2 - \nu^2)^{1/4}} \leq |H_\nu^{(1)}(z)| \leq A_0 \frac{1}{(z^2 - \nu^2)^{1/4}}.$$

If $\nu > 0$ and

$$\frac{|z - \nu|}{\nu^{1/3}} \leq C_0$$

then

$$(3.31) \qquad A_0^{-1} \nu^{-1/3} \leq |H_\nu^{(1)}(z)| \leq A_0 \nu^{-1/3}.$$

If

$$\frac{\nu - z}{\nu^{1/3}} \geq C_0$$

then

$$(3.32) \qquad A_0^{-1} \exp(\eta_2) \frac{1}{(\nu^2 - z^2)^{1/4}} \leq |H_\nu^{(1)}(z)| \leq A_0 \exp(\eta_2) \frac{1}{(\nu^2 - z^2)^{1/4}}$$

where η_2 is as in (3.22).

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PROOF. Let us begin with the third regime, when ν is much greater than z. Let us assume that $\nu > z \geq z_0 = \xi_0$. Then (3.32) follows immediately by Corollary 3.3 and Theorem 3.5.

For what concerns the first estimate, (3.30), let us notice that it is trivial for $\nu = 0$ and $z \geq z_0$. Therefore, without loss of generality, in what follows we shall assume $\nu > 0$. By Theorem 3.4, there exist C_0 and A_0 such that (3.30) holds provided $z - \nu \geq C_0 z^{1/3} > 0$, with no assumption that z should be greater than a constant. Clearly $z - \nu \geq C_0 z^{1/3} > 0$ implies that $z - \nu \geq C_0 \nu^{1/3} > 0$ but viceversa does not hold. The case $\nu + C_0 \nu^{1/3} \leq z < \nu + C_0 z^{1/3}$ will follow from the analysis of the intermediate regime, when z and ν have approximately the same value, which we shall now deal with.

We begin with the following remark. Fixed $C_0 > 0$, there exists $z_0 > 0$ such that if $z \geq z_0$, then $z - C_0 z^{1/3} \geq z/2$. Therefore, if $z \geq z_0$ and $0 < z - \nu < C_0 z^{1/3}$, we also have $\nu < z < \nu + C_0 2^{1/3} \nu^{1/3}$. Therefore, the open case is when $z \geq z_0$ and

$$\nu - C_1 \nu^{1/3} \leq z \leq \nu + C_1 \nu^{1/3}$$

where $C_1 = C_0 2^{1/3}$. Let us remark here that the intermediate estimate (3.31) for $\nu = z$ follows from Lemma 3.2, therefore, without loss of generality, we shall assume either $z < \nu$ or $z > \nu$, that is

$$\nu - C_1 \nu^{1/3} \leq z < \nu \quad \text{or} \quad \nu < z \leq \nu + C_1 \nu^{1/3}.$$

The analysis for these regimes is rather difficult and has been carried out by Langer in [21]. We shall here recall this asymptotic analysis. We need to introduce the following notation

$$
\begin{aligned}
&\mathrm{e}^x = \sec(\beta) \\
&\phi(x) = \tan(\beta) \\
&\xi = \nu(\tan(\beta) - \beta) \qquad \text{if } x > 0, \\
&\psi(x) = \frac{(\tan(\beta) - \beta)^{1/6}}{(\tan(\beta))^{1/2}}
\end{aligned}
$$
(3.33)

and also

$$
\begin{aligned}
&\mathrm{e}^x = \mathrm{sech}(\alpha) \\
&\phi(x) = \mathrm{i} \tanh(\beta) \\
&\xi = \mathrm{i}^3 \nu(\alpha - \tanh(\alpha)) \qquad \text{if } x < 0. \\
&\psi(x) = \frac{(\alpha - \tanh(\alpha))^{1/6}}{(\tanh(\alpha))^{1/2}}
\end{aligned}
$$
(3.34)

Then Langer in [21] showed that, fixed a positive constant M, there exists a constant E depending on M only such that if $0 < |\xi| \leq M$ then we have

(3.35) $$H_\nu^{(1)}(\nu \mathrm{e}^x) = \frac{2\psi(x)\xi^{1/3}}{3^{1/2}\nu^{1/3}} \left[\mathrm{e}^{-\pi \mathrm{i}/3} J_{-1/3}(\xi) + \mathrm{e}^{\pi \mathrm{i}/3} J_{1/3}(\xi) \right] + \frac{R(x,\nu)}{\nu^{4/3}}$$

with

$$|R(x,\nu)| \leq E.$$

First we need to investigate the term $\left[\mathrm{e}^{-\pi \mathrm{i}/3} J_{-1/3}(\xi) + \mathrm{e}^{\pi \mathrm{i}/3} J_{1/3}(\xi) \right]$. By using the classical asymptotics as $z \to 0$, see for instance [34, page 44], we have that, for any complex $z \neq 0$,

$$J_{\pm 1/3}(z) = \frac{(z/2)^{\pm 1/3}}{\Gamma(1 \pm 1/3)}(1 + R_\pm)$$

where

$$|R_\pm| < \exp\left(\frac{|z|^2/4}{1 \pm 1/3}\right) - 1.$$

Since for $z \neq 0$ we have

$$J_{1/3}(z)J'_{-1/3}(z) - J_{-1/3}(z)J'_{1/3}(z) = -2\frac{\sin(\pi/3)}{\pi z},$$

we deduce that, for any $|\xi| > 0$ defined as before, we have

$$\left|e^{-\pi i/3}J_{-1/3}(\xi) + e^{\pi i/3}J_{1/3}(\xi)\right| > 0,$$

hence there exist positive constants $B_0 < B_1$, depending on M only, such that, for any $0 < |\xi| \leq M$ defined as before, we have

(3.36) $$B_0|\xi|^{-1/3} \leq \left|e^{-\pi i/3}J_{-1/3}(\xi) + e^{\pi i/3}J_{1/3}(\xi)\right| \leq B_1|\xi|^{-1/3}.$$

Then we have that

$$\frac{2B_0}{\nu^{1/3}}\frac{|\psi(x)|}{3^{1/2}} \leq \left|\frac{2\psi(x)\xi^{1/3}}{3^{1/2}\nu^{1/3}}\left[e^{-\pi i/3}J_{-1/3}(\xi) + e^{\pi i/3}J_{1/3}(\xi)\right]\right| \leq \frac{2B_1}{\nu^{1/3}}\frac{|\psi(x)|}{3^{1/2}}.$$

It remains to analyze the term $|\psi(x)|/3^{1/2}$. We need to consider separately the case $x > 0$ and $x < 0$. For $x > 0$, we have $0 < \beta < \pi/2$ and

$$\frac{|\psi(x)|}{3^{1/2}} = \frac{(\tan(\beta) - \beta)^{1/6}}{(3\tan(\beta))^{1/2}}.$$

There exists an absolute constant β_0, $0 < \beta_0 < \pi/2$ such that for any β, $0 \leq \beta \leq \beta_0$ we have

$$\beta + \beta^3/3 \leq \tan(\beta) \leq \beta + 2\beta^3/3 \leq (3/2)\beta \quad \text{and} \quad 1 + \beta^2/4 \leq \sec(\beta) \leq 1 + 3\beta^2/4.$$

Let $\nu < z < \nu + C_1\nu^{1/3}$, then $\sec(\beta) = z/\nu = 1 + a\nu^{-2/3}$, for some a, $0 < a \leq C_1$. There exists $\tilde{z}_0 > 0$, depending on C_1 only, such that if $z \geq \tilde{z}_0$ then $\beta \leq \beta_0$. Hence

$$(2/\sqrt{3})\sqrt{a}\nu^{-1/3} \leq \beta \leq 2\sqrt{a}\nu^{-1/3}$$

that is

$$\frac{8}{9\sqrt{3}}a^{3/2} \leq \nu(\tan(\beta) - \beta) = \xi \leq \frac{16}{3}a^{3/2} \leq \frac{16}{3}C_1^{3/2} = M,$$

with M thus depending on C_1 only. An easy computation shows that there exist absolute positive constants $\tilde{B}_0 < \tilde{B}_1$ such that

$$\tilde{B}_0 \leq \frac{(\tan(\beta) - \beta)^{1/6}}{(3\tan(\beta))^{1/2}} \leq \tilde{B}_1.$$

We may conclude that there exist positive constants $\tilde{z}_0$ and $\tilde{A}_0 < \tilde{A}_1$, depending on C_1 only, such that for any $z \geq \tilde{z}_0$ and $\nu < z < \nu + C_1\nu^{1/3}$ we have

$$\frac{\tilde{A}_0}{\nu^{1/3}} \leq \left|\frac{2\psi(x)\xi^{1/3}}{3^{1/2}\nu^{1/3}}\left[e^{-\pi i/3}J_{-1/3}(\xi) + e^{\pi i/3}J_{1/3}(\xi)\right]\right| \leq \frac{\tilde{A}_1}{\nu^{1/3}}.$$

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Since the constant E depends on M, thus on C_1 only, we can find $z_0 \geq \tilde{z}_0$ such that for any $z \geq z_0$ and $\nu < z < \nu + C_1 \nu^{1/3}$ we have

$$E\nu^{-1} \leq (1/2)\tilde{A}_0,$$

thus (3.31) is proved with $A_0 = \tilde{A}_0/2$ and $A_1 = 2\tilde{A}_1$. A completely analogous argument holds for the case $x < 0$, hence (3.31) is fully proved.

It remains to consider the case when $z \geq z_0$ and $\nu + C_0 \nu^{1/3} \leq z < \nu + C_0 z^{1/3} \leq \nu + C_1 \nu^{1/3}$. We have just obtained that $A_0^{-1} \nu^{-1/3} \leq |H_\nu^{(1)}(z)| \leq A_0 \nu^{-1/3}$. On the other hand, provided $z = \nu + a\nu^{1/3}$ with $C_0 \leq a \leq C_1$, we have that

$$\frac{1}{(z^2 - \nu^2)^{1/4}} = \frac{1}{(2a\nu^{4/3} + a^2\nu^{2/3})^{1/4}} = \nu^{-1/3} \frac{1}{(2a + a^2\nu^{-2/3})^{1/4}}$$

that is

$$\frac{1}{(2C_1 + C_1^2\nu^{-2/3})^{1/4}}\nu^{-1/3} \leq \frac{1}{(z^2 - \nu^2)^{1/4}} \leq \frac{1}{(2C_0)^{1/4}}\nu^{-1/3}.$$

Provided z_0 is big enough, depending on C_0 and C_1 only, then if $z \geq z_0$ we infer that $\nu \geq 1$, thus

$$\frac{1}{(2C_1 + C_1^2)^{1/4}}\nu^{-1/3} \leq \frac{1}{(z^2 - \nu^2)^{1/4}} \leq \frac{1}{(2C_0)^{1/4}}\nu^{-1/3}$$

and the proof is concluded. $\qquad\square$

4 Stability estimates: from far-field to near-field

Throughout this section we shall assume that u^s is a radiating solution of the Helmholtz equation $\Delta u^s + k^2 u^s = 0$, for some wavenumber $k > 0$, defined in $\mathbb{R}^N \backslash \overline{B_R}$, for some fixed $R > 0$. We call u_∞^s the corresponding far-field pattern.

Throughout this section we shall also fix $R_0 > R$ and B_0 and B_1, with $1 < B_0 < B_1$. We call $\tilde{b}_0 = 1/B_0$ and we assume that for some $\varepsilon > 0$ and $M > 0$ we have

(4.1) $$\|u_\infty^s\|_{L^2(\mathbb{S}^{N-1})} = \varepsilon, \quad \|u^s\|_{L^2(\partial B_{R_0})} = M.$$

Our aim is to estimate, in terms of $\|u_\infty^s\|_{L^2(\mathbb{S}^{N-1})}$, $\|u^s\|_{L^2(\partial B_r)}$ for some $r > R$. This kind of estimate is usually referred to as the stability for the determination of the near-field from the far-field. Notice the Rellich Lemma provides the corresponding uniqueness. Such an issue has been solved by Isakov, [15], see also [5].

We recall here the stability result by Isakov, [15], which we slightly generalize to any dimension $N \geq 2$ and for the wavenumber k varying in a compact interval contained in $(0, +\infty)$. For the convenience of the reader, and since some of its arguments will be useful in the sequel, we shall prove it in detail. Before stating the result let us denote α_0 as the number such that $0 < \alpha_0 < 1$ and

$$(1/\alpha_0)^{1/\alpha_0} = \mathrm{e}.$$

Theorem 4.1 *Let $N \geq 2$. Let us fix constants k_1, k_2 with $0 < k_1 < k_2$. Under the previous notation and assumptions, the following result holds.*

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Let $z_1 = k_1 R_0$ and $z_2 = k_2 B_1 R_0$ and let $\hat{\nu}_0 = \max\{N/2, e^2 z_2/2\}$. Let $\tilde{a}_1 = \log(e/2)$. Fix $\alpha > 0$. Then there exist a constant $C_\alpha \geq 1$, depending on α only (with $C_\alpha = 1$ for any $\alpha \geq \alpha_0$), and a constant $\tilde{A}$, depending on z_1 and z_2 only, such that for any $k \in [k_1, k_2]$ and any r, $B_0 R_0 \leq r \leq B_1 R_0$, if

(4.2)
$$\log(M/\varepsilon) \geq C_\alpha \left(\frac{2}{e^{\tilde{a}_1} k_1 R_0}\right)^\alpha (\hat{\nu}_0 + 1/2)^{1+\alpha},$$

then

(4.3)
$$\|u^s\|_{L^2(\partial B_r)} \leq \sqrt{2} \tilde{A} b_0^{-1} M \exp\left(-\log(B_0)\left(\left(\frac{e^{\tilde{a}_1} k_1 R_0}{2}\right)^\alpha \frac{\log(M/\varepsilon)}{C_\alpha}\right)^{1/(1+\alpha)}\right).$$

PROOF. Let us recall there exists a sequence b_j, $j = 0, 1, \ldots$, of nonnegative numbers such that

(4.4)
$$\|u_\infty^s\|_{L^2(\mathbb{S}^{N-1})}^2 = \sum_j b_j^2$$

and for any $r > R$

(4.5)
$$\|u^s\|_{L^2(\partial B_r)}^2 = \frac{\pi}{2} \sum_j b_j^2 kr \left|H_{j+(N-2)/2}^{(1)}(kr)\right|^2.$$

Let us notice that for any $j \geq 0$,

$$b_j^2 = \sum_{i:\ \gamma(v_i)=j} |\tilde{b}_i|^2,$$

if we use the notation of Section 2 as in (2.20) and (2.21).

For any $k > 0$ and any r, $B_0 R_0 \leq r \leq B_1 R_0$, we have, for any integer $j_0 \geq 0$ to be decided later,

(4.6)
$$\|u^s\|_{L^2(\partial B_r)}^2 \leq \frac{\pi}{2} kr \sum_{j=0}^{j_0} b_j^2 \left|H_{j+(N-2)/2}^{(1)}(kr)\right|^2 + \frac{\pi}{2} kr \sum_{j=j_0+1}^{+\infty} b_j^2 \left|H_{j+(N-2)/2}^{(1)}(kr)\right|^2$$

which we estimate as follows

(4.7) $\|u^s\|_{L^2(\partial B_r)}^2 \leq \frac{\pi}{2} kr \max_{j\in\{0,1,\ldots,j_0\}} \left|H_{j+(N-2)/2}^{(1)}(kr)\right|^2 \|u_\infty^s\|_{L^2(\mathbb{S}^{N-1})}^2 +$

$$\frac{r}{R_0}\frac{\pi}{2}kR_0 \sum_{j=j_0+1}^{+\infty} b_j^2 \left|H_{j+(N-2)/2}^{(1)}(kR_0)\right|^2 \frac{\left|H_{j+(N-2)/2}^{(1)}(kr)\right|^2}{\left|H_{j+(N-2)/2}^{(1)}(kR_0)\right|^2}$$

that is

(4.8) $\|u^s\|_{L^2(\partial B_r)}^2 \leq \frac{\pi}{2} kr \max_{j\in\{0,1,\ldots,j_0\}} \left|H_{j+(N-2)/2}^{(1)}(kr)\right|^2 \|u_\infty^s\|_{L^2(\mathbb{S}^{N-1})}^2 +$

$$\frac{r}{R_0}\left(\sup_{j>j_0} \frac{\left|H_{j+(N-2)/2}^{(1)}(kr)\right|^2}{\left|H_{j+(N-2)/2}^{(1)}(kR_0)\right|^2}\right) \|u^s\|_{L^2(\partial B_{R_0})}^2.$$

By Corollary 3.8, for any ν such that $\nu = 0$ or $1/2 \leq \nu \leq \mathrm{e}z/2$, we have that there exists a constant $C \geq 1$, depending on z_1 and z_2 only, such that

$$(4.9) \qquad |H_\nu^{(1)}(z)| \leq C \frac{2}{\sqrt{\mathrm{e}\pi z}}.$$

Otherwise, if $1/2 \leq \nu$ and $\mathrm{e}z/2 \leq \nu$, we have

$$(4.10) \qquad |H_\nu^{(1)}(z)| \leq C \frac{2}{\sqrt{\mathrm{e}\pi z}} \left(\frac{2\nu}{\mathrm{e}z} \right)^{\nu-1/2}.$$

Furthermore, for any $\nu \geq 1/2$ we have

$$(4.11) \qquad \frac{\left| H_\nu^{(1)}(kr) \right|}{\left| H_\nu^{(1)}(kR_0) \right|} \leq \frac{C \frac{2}{\sqrt{\mathrm{e}\pi kr}} \left(\frac{2\nu}{\mathrm{e}kr} \right)^{\nu-1/2}}{C^{-1} \frac{2}{\sqrt{\mathrm{e}\pi kR_0}} \left(\frac{2\nu}{\mathrm{e}kR_0} \right)^{\nu-1/2}} \leq C^2 \left(\frac{R_0}{r} \right)^\nu \leq C^2 \left(1/B_0 \right)^\nu.$$

We conclude that, provided $j_0 \geq \hat{j}_0 \geq 1$, with $\hat{\nu}_0 = \hat{j}_0 + (N-2)/2 \geq \max\{N/2, \mathrm{e}^2 z_2/2\}$, and setting $\nu_0 = j_0 + (N-2)/2$,

$$(4.12) \qquad \|u^s\|^2_{L^2(\partial B_r)} \leq \frac{2C^2}{\mathrm{e}} \left(\frac{2\nu_0}{\mathrm{e}kr} \right)^{2\nu_0-1} \|u^s_\infty\|^2_{L^2(\mathbb{S}^{N-1})} + C^4 \left(\frac{R_0}{r} \right)^{2\nu_0-1} \|u^s\|^2_{L^2(\partial B_{R_0})},$$

that is, setting $\tilde{A} = \max\{2C^2/\mathrm{e}, C^4\}$ and $\tilde{a}_1$, $0 < \tilde{a}_1 < 1$, such that $\mathrm{e}^{\tilde{a}_1} = \mathrm{e}/2$, and $\tilde{b}_0 = 1/B_0$,

$$(4.13) \qquad \|u^s\|^2_{L^2(\partial B_r)} \leq \tilde{A} \left[\left(\frac{2\nu_0 - 1}{\mathrm{e}^{\tilde{a}_1} kr} \right)^{2\nu_0-1} \|u^s_\infty\|^2_{L^2(\mathbb{S}^{N-1})} + \left(\tilde{b}_0 \right)^{2\nu_0-1} \|u^s\|^2_{L^2(\partial B_{R_0})} \right].$$

Let $2n \geq 2\hat{\nu}_0 + 1$ be such that

$$(4.14) \qquad \left(\frac{2n}{\mathrm{e}^{\tilde{a}_1} kr} \right)^{2n} \varepsilon^2 = \tilde{b}_0^{2n} M^2.$$

Then there exists an integer $j_0 \in \mathbb{N}$ such that $j_0 \geq \hat{j}_0$ and $2\nu_0 - 1 \leq 2n < 2(j_0 + 1 + (N-2)/2) - 1 = 2\nu_0 + 1$. Then

$$\left(\frac{2\nu_0 - 1}{\mathrm{e}^{\tilde{a}_1} kr} \right)^{2\nu_0-1} \leq \left(\frac{2n}{\mathrm{e}^{\tilde{a}_1} kr} \right)^{2n}$$

and

$$\tilde{b}_0^{2\nu_0-1} = \tilde{b}_0^{-2} \tilde{b}_0^{2\nu_0+1} \leq \tilde{b}_0^{-2} \tilde{b}_0^{2n}.$$

Hence, if (4.14) holds we have

$$(4.15) \qquad \|u^s\|^2_{L^2(\partial B_r)} \leq 2\tilde{A} \tilde{b}_0^{-2} \left(\frac{2n}{\mathrm{e}^{\tilde{a}_1} kr} \right)^{2n} \varepsilon^2 = 2\tilde{A} \tilde{b}_0^{-2} \tilde{b}_0^{2n} M^2.$$

Let us investigate (4.14). We have $\log(\varepsilon) + n \log \left(\frac{2n}{\mathrm{e}^{\tilde{a}_1} kr} \right) = \log(M) + n \log(\tilde{b}_0)$, that is

$$\log(M/\varepsilon) = n \log \left(\frac{2n}{\tilde{b}_0 \mathrm{e}^{\tilde{a}_1} kr} \right).$$

Let us call

$$\tilde{n} = \frac{2n}{\tilde{b}_0 \mathrm{e}^{\tilde{a}_1} kr},$$

then

$$\frac{2}{\tilde{b}_0 \mathrm{e}^{\tilde{a}_1} kr} \log(M/\varepsilon) = \tilde{n} \log(\tilde{n}).$$

We notice that if $2n \geq 2\hat{\nu}_0 + 1$, then $\tilde{n} \geq \mathrm{e}$. On the other hand, for any $\alpha > 0$, there exists $C_\alpha \geq 1$ such that for any $\tilde{n} \geq \mathrm{e}$ we have

$$\tilde{n} \leq \tilde{n} \log(\tilde{n}) \leq C_\alpha \tilde{n}^{1+\alpha}.$$

Notice that we can choose $C_\alpha = 1$ for any $\alpha \geq \alpha_0$ where we recall that α_0 satisfies $0 < \alpha_0 < 1$ and

$$(1/\alpha_0)^{1/\alpha_0} = \mathrm{e}.$$

Therefore,

$$\tilde{n} \leq \frac{2}{\tilde{b}_0 \mathrm{e}^{\tilde{a}_1} kr} \log(M/\varepsilon) \leq C_\alpha \tilde{n}^{1+\alpha},$$

that is

$$\tilde{n} \geq \left(\frac{2}{\tilde{b}_0 \mathrm{e}^{\tilde{a}_1} kr} \frac{\log(M/\varepsilon)}{C_\alpha} \right)^{1/(1+\alpha)}.$$

Therefore, provided

$$\left(\frac{2}{\tilde{b}_0 \mathrm{e}^{\tilde{a}_1} kr} \frac{\log(M/\varepsilon)}{C_\alpha} \right)^{1/(1+\alpha)} \geq \frac{2\hat{\nu}_0 + 1}{\tilde{b}_0 \mathrm{e}^{\tilde{a}_1} kr},$$

that is

$$(4.16) \qquad \log(M/\varepsilon) \geq C_\alpha \left(\frac{2}{\tilde{b}_0 \mathrm{e}^{\tilde{a}_1} kr} \right)^\alpha (\hat{\nu}_0 + 1/2)^{1+\alpha},$$

we have that there exists a solution n to (4.14) such that $2n \geq 2\hat{\nu}_0 + 1$, hence

$$\|u^s\|_{L^2(\partial B_r)} \leq \sqrt{2\tilde{A}} \tilde{b}_0^{-1} M \exp(-\log(B_0) n),$$

that is

$$(4.17) \qquad \|u^s\|_{L^2(\partial B_r)} \leq \sqrt{2\tilde{A}} \tilde{b}_0^{-1} M \exp\left(-\log(B_0) \left(\left(\frac{\tilde{b}_0 \mathrm{e}^{\tilde{a}_1} kr}{2} \right)^\alpha \frac{\log(M/\varepsilon)}{C_\alpha} \right)^{1/(1+\alpha)} \right).$$

We have that (4.16) and (4.17) immediately imply (4.2) and (4.3) and the proof is concluded. $\qquad \square$

Now we investigate how the estimate changes in the high frequencies regime, starting with frequencies that are high but not extremely high. We shall use the following notation, besides the one set at the beginning of the section. Let z_0, C_0 and A_0 be as in Theorem 3.9. We observe that there exists a constant $\tilde{C}_0 \geq 2$, depending on C_0 only, such that for any

$\nu \geq 1/2$ we have $\nu + C_0\nu^{1/3} \leq \tilde{C}_0\nu$. Without loss of generality, up to taking a possibly greater z_0 but still depending only on absolute constants and on N, we assume also that

(4.18) $$z_0 \geq \tilde{C}_0 \max\{2, (N-2)/2\} \quad \text{and} \quad (1 - C_0 z_0^{-2/3}) > 0.$$

We further assume that

(4.19) $$\text{if } z_0 \leq \tilde{C}_0\nu \text{ then } C_0\nu^{1/3} \leq \nu/2.$$

Notice that if $z_0 \leq \tilde{C}_0\nu$ we also have $\nu \geq 2$.

Theorem 4.2 *Let $N \geq 2$. We keep the previous notation and assumptions. Let $k_0 > 0$ be such that $k_0 R_0 \geq z_0$, z_0 as in Theorem 3.9 and such that (4.18) and (4.19) are satisfied. Let us fix b_0, $0 < b_0 < 1$, depending on B_0 only, such that $1/B_0 < b_0 < 1$. Fix $\alpha > 0$.*

Then there exist a constant $\tilde{B}$ depending on B_0, C_0, z_0 and N only, a constant a_1, $0 < a_1 < 1$, depending on B_0, C_0 and z_0 only, a constant A depending on B_0, C_0, z_0 and A_0 only, and a constant $C_\alpha \geq 1$, depending on α only (with $C_\alpha = 1$ for any $\alpha \geq \alpha_0$), such that the following holds.

Let us assume that

$$k_0 \leq k_1(\varepsilon) = \frac{1}{B_1 R_0} \left(\frac{2}{3\tilde{B}}\right)^{1+\alpha} \left(\frac{b_0 e^{a_1}}{2}\right)^\alpha \frac{\log(M/\varepsilon)}{C_\alpha}.$$

Then for any $k \in [k_0, k_1(\varepsilon)]$ and any r, $B_0 R_0 \leq r \leq B_1 R_0$, we have

(4.20) $$\|u^s\|_{L^2(\partial B_r)} \leq \sqrt{2AB_1} b_0^{-1} M \exp\left(-\log(1/b_0)\left(\left(\frac{b_0 e^{a_1} kr}{2}\right)^\alpha \frac{\log(M/\varepsilon)}{C_\alpha}\right)^{1/(1+\alpha)}\right).$$

Furthermore, if $k = k_1(\varepsilon)$ we have

(4.21) $$\|u^s\|_{L^2(\partial B_r)} \leq \sqrt{2AB_1} b_0^{-1} M^{1-\beta_1} \varepsilon^{\beta_1},$$

where $\beta_1 = \beta_1(\alpha)$ is given by

(4.22) $$\beta_1(\alpha) = \frac{\log(1/b_0)}{C_\alpha}\left(\frac{b_0 e^{a_1}}{3\tilde{B}}\right)^\alpha \left(\frac{r}{B_1 R_0}\right)^{\alpha/(1+\alpha)} = \tilde{C}_\alpha\left(\frac{r}{B_1 R_0}\right)^{\alpha/(1+\alpha)} \leq \tilde{C}_\alpha$$

and, in the particular case $\alpha = 1$,

(4.23) $$\beta_1(1) = \log(1/b_0)\frac{b_0 e^{a_1}}{3\tilde{B}}\left(\frac{r}{B_1 R_0}\right)^{1/2} = \tilde{C}_1\left(\frac{r}{B_1 R_0}\right)^{1/2} \leq \tilde{C}_1.$$

Here $\tilde{C}_\alpha$ depends on α, B_0, C_0, z_0 and N only, and, without loss of generality by taking an eventually larger $\tilde{B}$ depending on α as well, we may assume that $0 < \beta_1(\alpha) \leq \tilde{C}_\alpha \leq 1$.

PROOF. Let us consider r, $B_0 R_0 \leq r \leq B_1 R_0$, and $k \geq k_0$.

We fix a positive constant $C \geq e^2/2$ and let $a = \sqrt{C^2 - 1}/C$, $0 < a < 1$. We assume that $C(1 - C_0 z_0^{-2/3}) \geq 1$, that $\frac{2e^{1-a}}{(1+a)B_0} \leq b_0$ and that $a \geq 2\log(4/3)$. Obviously C depends on B_0, C_0 and z_0 only. Let us notice that if $\nu \geq Ckr \geq Cz_0$ then $\nu - C_0\nu^{1/3} \geq \nu(1 - C_0\nu^{-2/3}) \geq \nu(1 - C_0 z_0^{-2/3})$. Therefore $\nu - C_0\nu^{1/3} \geq Ckr(1 - C_0 z_0^{-2/3}) \geq kr$.

Let $\hat{j}_0 = \hat{j}_0(kr) \geq 1$ be such that $\hat{\nu}_0 = \hat{\nu}_0(kr) = \hat{j}_0(kr) + (N-2)/2 \geq \max\{N/2, Ckr\} \geq 2$. Let $2n \geq 2\hat{\nu}_0(kr) + 1$, then there exists an integer $j_0 \in \mathbb{N}$ such that $j_0 \geq \hat{j}_0(kr)$ and $2\nu_0 - 1 \leq 2n < 2(j_0 + 1 + (N-2)/2) - 1 = 2\nu_0 + 1$, $\nu_0 = j_0 + (N-2)/2$ as before.

We consider an integer $j_0 \geq \hat{j}_0$, to be fixed later, and we use Theorem 3.9. If $\nu = 0$ or $\nu \geq 1/2$ and such that $kr - \nu \geq C_0 \nu^{1/3}$, we have that, by (3.30),

$$\left| H_\nu^{(1)}(kr) \right| \leq A_0 \frac{1}{((kr)^2 - \nu^2)^{1/4}} \leq A_0 \max\left\{ 1/\sqrt{z_0}, \frac{1}{(2C_0(1/2)^{4/3} + C_0^2(1/2)^{2/3})^{1/4}} \right\} = \tilde{A}_1$$

since, for $\nu = 0$, $1/\sqrt{kr} \leq 1/\sqrt{z_0}$, and for $\nu \geq 1/2$,

$$((kr)^2 - \nu^2) \geq (2C_0 \nu^{4/3} + C_0^2 \nu^{2/3}) \geq (2C_0(1/2)^{4/3} + C_0^2(1/2)^{2/3}).$$

Clearly $\tilde{A}_1$ depends on A_0, z_0 and C_0 only.

If $\nu \geq 1/2$ and $|kr - \nu| \leq C_0 \nu^{1/3}$, then, by (3.31),

$$\left| H_\nu^{(1)}(kr) \right| \leq A_0 \nu^{-1/3} \leq A_0(1/2)^{-1/3} = \tilde{A}_2,$$

with $\tilde{A}_2$ depending on A_0 only.

If $1/2 \leq \nu \leq \nu_0$ and $\nu - kr \geq C_0 \nu^{1/3}$, then, by Corollary 3.3 and Lemma 3.2 we have that

$$\left| H_\nu^{(1)}(kr) \right| \leq \frac{1 + c_1}{1 - c_1} |H_{\nu_0}^{(1)}(kr)|.$$

For any $\nu \geq \nu_0$ we can use (3.32) both for $z = kR_0$ and $z = kr$. Recalling that $a = \sqrt{C^2 - 1}/C$, $0 < a < 1$, we obtain that

$$(4.24) \qquad A_0^{-1} \frac{1}{\sqrt{\nu}} \left(\frac{(1+a)\nu}{ez} \right)^\nu \leq \left| H_\nu^{(1)}(z) \right| \leq A_0 \frac{1}{\sqrt{a\nu}} \left(\frac{2\nu}{e^a z} \right)^\nu.$$

Hence, if $1/2 \leq \nu \leq \nu_0$ and $\nu - kr \geq C_0 \nu^{1/3}$, we have

$$\left| H_\nu^{(1)}(kr) \right| \leq \frac{1 + c_1}{1 - c_1} A_0 \frac{1}{\sqrt{a\nu_0}} \left(\frac{2\nu_0}{e^a kr} \right)^{\nu_0}.$$

Since $\nu_0 \geq 2$, we have that $2\nu_0 \leq (4/3)(2\nu_0 - 1)$. Let $a_1 = a - \log(4/3)$, then

$$\left| H_\nu^{(1)}(kr) \right| \leq \tilde{A}_3 \left(\frac{e^{a_1}}{2\nu_0 - 1} \right)^{1/2} \left(\frac{2\nu_0 - 1}{e^{a_1} kr} \right)^{\nu_0},$$

with $\tilde{A}_3$ depending on A_0 and C only.

We conclude that for any $j_0 \geq \hat{j}_0(kr)$, setting $A_1 = \max\{\tilde{A}_1 \sqrt{z_0}, \tilde{A}_2 \sqrt{z_0}, \tilde{A}_3\}$, we have

$$(4.25) \qquad \max_{j \in \{0,1,\dots,j_0\}} \left| H_{j+(N-2)/2}^{(1)}(kr) \right|^2 \leq A_1^2 \left(\frac{e^{a_1}}{2\nu_0 - 1} \right) \left(\frac{2\nu_0 - 1}{e^{a_1} kr} \right)^{2\nu_0}.$$

Now we investigate the integers $j > j_0$. Again by (4.24), for any $\nu \geq \nu_0$

$$(4.26) \qquad \frac{\left| H_\nu^{(1)}(kr) \right|}{\left| H_\nu^{(1)}(kR_0) \right|} \leq \frac{A_0^2}{\sqrt{a}} \left(\frac{2e^{1-a} R_0}{(1+a)r} \right)^\nu \leq \frac{A_0^2}{\sqrt{a}} \left(\frac{2e^{1-a}}{(1+a)B_0} \right)^\nu \leq \frac{A_0^2}{\sqrt{a}} (b_0)^\nu = A_2 (b_0)^{\nu - 1/2}$$

where $A_2 = \sqrt{b_0} A_0^2 / \sqrt{a}$. We may conclude that, setting $A = \max\{(\pi/2)A_1^2, A_2^2\}$ and $\tilde{B} = (N/2)C$, if

$$\hat{\nu}_0(kr) = \tilde{B}kr \quad \text{and} \quad \nu_0 \geq \hat{\nu}_0(kr),$$

then we have

$$(4.27) \qquad \|u^s\|_{L^2(\partial B_r)}^2 \leq AB_1 \left[\left(\frac{2\nu_0 - 1}{e^{a_1} kr} \right)^{2\nu_0 - 1} \|u_\infty^s\|_{L^2(\mathbb{S}^{N-1})}^2 + (b_0)^{2\nu_0 - 1} \|u^s\|_{L^2(\partial B_{R_0})}^2 \right].$$

Notice that A depends on B_0, C_0, z_0 and A_0 only, $\tilde{B}$ depends on B_0, C_0, z_0 and N only, a_1 depends on B_0, C_0 and z_0 only, whereas $0 < b_0 < 1$ depends on B_0 only.

Let $2n \geq 3\hat{\nu}_0(kr) \geq 2\hat{\nu}_0(kr) + 1$ be such that

$$(4.28) \qquad\qquad \left(\frac{2n}{e^{a_1} kr} \right)^{2n} \varepsilon^2 = b_0^{2n} M^2.$$

Let $j_0 \in \mathbb{N}$ be such that $2\nu_0 - 1 \leq 2n < 2\nu_0 + 1$. Then $j_0 \geq \hat{j}_0(kr)$ and

$$\left(\frac{2\nu_0 - 1}{e^{a_1} kr} \right)^{2\nu_0 - 1} \leq \left(\frac{2n}{e^{a_1} kr} \right)^{2n}$$

and

$$b_0^{2\nu_0 - 1} = b_0^{-2} b_0^{2\nu_0 + 1} \leq b_0^{-2} b_0^{2n}.$$

Hence, if (4.28) holds we have

$$(4.29) \qquad\qquad \|u^s\|_{L^2(\partial B_r)}^2 \leq 2AB_1 b_0^{-2} \left(\frac{2n}{e^{a_1} kr} \right)^{2n} \varepsilon^2 = 2AB_1 b_0^{-2} b_0^{2n} M^2.$$

We argue exactly as in the previous case, with $\tilde{A}$, $\tilde{b}_0$ and $\tilde{a}_1$ replaced by AB_1, b_0 and a_1, respectively. Fixed $\alpha > 0$ and C_α as before, provided

$$(4.30) \qquad\qquad \log(M/\varepsilon) \geq C_\alpha \left(\frac{2}{b_0 e^{a_1} kr} \right)^\alpha ((3/2)\hat{\nu}_0(kr))^{1+\alpha},$$

we have that there exists a solution n to (4.28) such that $2n \geq 3\hat{\nu}_0(kr) \geq 2\hat{\nu}_0(kr) + 1$, hence

$$(4.31) \quad \|u^s\|_{L^2(\partial B_r)} \leq \sqrt{2AB_1} b_0^{-1} M \exp\left(-\log(1/b_0) \left(\left(\frac{b_0 e^{a_1} kr}{2} \right)^\alpha \frac{\log(M/\varepsilon)}{C_\alpha} \right)^{1/(1+\alpha)} \right).$$

Clearly such an estimate improves as k becomes larger. However, if we call

$$k_1(\varepsilon, r) = \frac{1}{r} \left(\frac{2}{3\tilde{B}} \right)^{1+\alpha} \left(\frac{b_0 e^{a_1}}{2} \right)^\alpha \frac{\log(M/\varepsilon)}{C_\alpha},$$

and we assume that $k_1(\varepsilon, r) \geq k_0$, then we obtain that the estimate remains valid in the following regime

$$(4.32) \qquad\qquad k_0 \leq k \leq k_1(\varepsilon, r) = \frac{1}{r} \left(\frac{2}{3\tilde{B}} \right)^{1+\alpha} \left(\frac{b_0 e^{a_1}}{2} \right)^\alpha \frac{\log(M/\varepsilon)}{C_\alpha}.$$

If we pick the optimal choice of k, that is $k = k_1(\varepsilon, r)$, then (4.31) reduces to

$$\|u^s\|_{L^2(\partial B_r)} \le \sqrt{2AB_1} b_0^{-1} M \exp\left(-\log(1/b_0)(3\tilde{B}/2)k_1(\varepsilon, r)r\right) =$$
$$\sqrt{2AB_1} b_0^{-1} M \exp\left(-\log(1/b_0)\left(\frac{b_0 e^{a_1}}{3\tilde{B}}\right)^\alpha \frac{\log(M/\varepsilon)}{C_\alpha}\right)$$

that is

$$(4.33) \qquad \|u^s\|_{L^2(\partial B_r)} \le \sqrt{2AB_1} b_0^{-1} M^{1-\beta} \varepsilon^\beta$$

where $\beta = \beta(\alpha)$ is given by

$$(4.34) \qquad \beta(\alpha) = \frac{\log(1/b_0)}{C_\alpha}\left(\frac{b_0 e^{a_1}}{3\tilde{B}}\right)^\alpha$$

and, in the particular case $\alpha = 1$,

$$(4.35) \qquad \beta(1) = \log(1/b_0)\frac{b_0 e^{a_1}}{3\tilde{B}}.$$

Let us conclude the proof by noticing that, without loss of generality by taking an eventually larger $\tilde{B}$ depending on α as well, we may assume that $0 < \beta(\alpha) \le 1$. □

In the final part of this section, we are interested in understanding what happens if $k > k_1(\varepsilon, r)$, that is in the extremely high frequencies regime. Besides having the assumptions of Theorem 4.2, we need to make an additional assumption, namely that

$$(4.36) \qquad B_0 \ge \tilde{B}_0 = 3e\tilde{C}_0/2 \ge 3e.$$

The result is the following.

Theorem 4.3 *Let $N \ge 2$. We keep the previous notation and assumptions. Let $k_0 > 0$ be such that $k_0 R_0 \ge z_0$, z_0 as in Theorem 3.9 and such that (4.18) and (4.19) are satisfied.*

Let us assume that B_0 satisfies (4.36). Then there exists a constant $\tilde{A}$, depending on N only, such that for any $k \ge k_0$ and any r, $B_0 R_0 \le r \le B_1 R_0$, we have

$$(4.37) \qquad \|u^s\|_{L^2(\partial B_r)} \le \sqrt{\tilde{A}B_1}\left[\varepsilon^2 + M^2(kr/\tilde{C}_0)\left(\frac{2}{3}\right)^{2kr/\tilde{C}_0}\right]^{1/2}.$$

In particular, if $k \ge k_1(\varepsilon)$, where $k_1(\varepsilon)$ is as in Theorem 4.2 and satisfies $k_0 \le k_1(\varepsilon)$, we have that

$$(4.38) \qquad \|u^s\|_{L^2(\partial B_r)} \le \sqrt{\tilde{A}B_1}\left[\varepsilon^2 + M^2(k_1(\varepsilon)B_0 R_0/\tilde{C}_0)\left(\frac{2}{3}\right)^{2k_1(\varepsilon)B_0 R_0/\tilde{C}_0}\right]^{1/2}.$$

We postpone the proof of the theorem to the end of the section. Here we make a few comments on this result and discuss an interesting conclusion, one of the main of this section, Theorem 4.4, where a Lipschitz stability estimate is proved.

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We notice that there exists an absolute constant $C_1 \geq 1$ such that if $k_0 \leq k_1(\varepsilon)$ we have

$$(k_1(\varepsilon)B_0R_0/\tilde{C}_0)(2/3)^{2k_1(\varepsilon)B_0R_0/\tilde{C}_0} \leq C_1(3/4)^{2k_1(\varepsilon)B_0R_0/\tilde{C}_0}.$$

Hence (4.38) may be written in the following simpler form

$$\|u^s\|_{L^2(\partial B_r)} \leq \sqrt{\tilde{A}B_1}\left[\varepsilon^2 + C_1 M^{2(1-\tilde{\beta})}\varepsilon^{2\tilde{\beta}}\right]^{1/2} \leq \sqrt{2C_1\tilde{A}B_1}\,M^{(1-\tilde{\beta})}\varepsilon^{\tilde{\beta}}$$

where we assume that $\varepsilon \leq M$ and

$$\tilde{\beta} = \frac{\log(4/3)}{\tilde{C}_0 C_\alpha}\frac{B_0}{B_1}\left(\frac{2}{3\tilde{B}}\right)^{1+\alpha}\left(\frac{b_0 e^{a_1}}{2}\right)^\alpha.$$

Finally, we state the following crucial remark. Let us assume that for some positive exponent τ and a constant C_2 we have, for any $k \geq k_0$,

(4.39) $$M \leq C_2 k^\tau.$$

Hence, for any $r \geq B_0R_0$,

$$M \leq C_2 k^\tau \leq C_2(\tilde{C}_0/r)^\tau (kr/\tilde{C}_0)^\tau \leq C_2(\tilde{C}_0/B_0R_0)^\tau(kr/\tilde{C}_0)^\tau.$$

Then there exists an absolute constant $C(\tau)$, depending on τ only, such that

$$M^2(kr/\tilde{C}_0)\left(\frac{2}{3}\right)^{2kr/\tilde{C}_0} \leq C_2^2(\tilde{C}_0/B_0R_0)^{2\tau}C_3(\tau)\left(\frac{3}{4}\right)^{2kr/\tilde{C}_0} \leq$$

$$C_2^2(\tilde{C}_0/B_0R_0)^{2\tau}C_3(\tau)\left(\frac{3}{4}\right)^{2kB_0R_0/\tilde{C}_0}.$$

Therefore, the following corollary with a Lipschitz stability estimate holds.

Theorem 4.4 *Under the assumptions of Theorem 4.3, let us further assume that, for some positive exponent τ and a constant C_2, (4.39) holds for any $k \geq k_0$.*
If $\varepsilon \leq 1/\mathrm{e}$ and

$$k \geq \frac{\tilde{C}_0}{\log(4/3)B_0R_0}\log(1/\varepsilon)$$

we have

(4.40) $$\|u^s\|_{L^2(\partial B_r)} \leq \left(\tilde{A}B_1(1 + C_2^2(\tilde{C}_0/B_0R_0)^{2\tau}C_3(\tau))\right)^{1/2}\varepsilon.$$

PROOF OF THEOREM 4.3. We recall that $k_0 > 0$ is such that $k_0R_0 \geq z_0$. We fix r, $B_0R_0 \leq r \leq B_1R_0$, and $k \geq k_0$.

We consider all nonnegative integers j such that $\nu = j + (N-2)/2$ satisfies $kr \geq \tilde{C}_0\nu$ and we call $\tilde{j}_0 = \tilde{j}_0(kr)$ the largest of these integers and $\tilde{\nu}_0 = \tilde{\nu}_0(kr) = \tilde{j}_0(kr) + (N-2)/2$. Notice that $\tilde{j}_0(kr) \geq 0$ by our assumption (4.18) on z_0. Then, by (3.30),

$$kr\left|H_\nu^{(1)}(kr)\right|^2 \leq kr A_0^2\frac{1}{((kr)^2-\nu^2)^{1/2}} \leq A_0^2\frac{1}{(1-(\nu/(kr))^2)^{1/2}} \leq A_0^2\frac{\tilde{C}_0}{(\tilde{C}_0^2-1)^{1/2}}.$$

Let us now assume that $j > \tilde{j}_0(kr)$, that is $kr < \tilde{C}_0\nu$. First of all, by our more restrictive assumption on B_0 and by (4.19), we have that

$$kR_0 \leq \tilde{C}_0\nu/B_0 \leq 2\nu/(3\mathrm{e}) \leq \nu/2 \leq \nu - C_0\nu^{1/3}.$$

We have three different cases. In the first, we have that $kr \geq \nu + C_0\nu^{1/3}$. In the second, we have that $\nu + C_0\nu^{1/3} > kr > \nu - C_0\nu^{1/3}$. In the third case, kr is less than or equal to $\nu - C_0\nu^{1/3}$.

In all cases we can use (3.32) for $z = kR_0$ and obtain that

$$A_0^{-1}\frac{1}{\sqrt{\nu}}\left(\frac{\nu}{\mathrm{e}kR_0}\right)^\nu \leq \left|H_\nu^{(1)}(kR_0)\right|.$$

In the first case, using (3.30), we have that

$$\left|H_\nu^{(1)}(kr)\right| \leq A_0\frac{1}{((kr)^2 - \nu^2)^{1/4}} \leq A_0\frac{1}{(2C_0\nu^{4/3} + C_0^2\nu^{2/3})^{1/4}} \leq \frac{A_0}{(2C_0)^{1/4}}\nu^{-1/3}.$$

In the second case, by (3.31), we have that

$$\left|H_\nu^{(1)}(kr)\right| \leq A_0\nu^{-1/3},$$

whereas in the third, using again (3.32), we obtain

$$\left|H_\nu^{(1)}(kr)\right| \leq A_0\left(\frac{2\nu}{kr}\right)^\nu\frac{1}{(\nu^2 - (kr)^2)^{1/4}}\exp\left(-(\nu^2 - (kr)^2)^{1/2}\right).$$

Since $(\nu^2 - (kr)^2) \geq (\nu^2 - (\nu - C_0\nu^{1/3})^2) \geq 2C_0\nu^{4/3} - C_0^2\nu^{2/3}$, then by (4.18) we have that $(\nu^2 - (kr)^2) \geq C_0\nu^{4/3} \geq C_0z_0^{4/3}$. Hence there exists a constant $\hat{C}_0 \geq 1$, depending on N only, such that

$$\left|H_\nu^{(1)}(kr)\right| \leq \hat{C}_0A_0\left(\frac{2\nu}{kr}\right)^\nu.$$

We conclude that in the first and second cases we have

(4.41)
$$\frac{\left|H_\nu^{(1)}(kr)\right|}{\left|H_\nu^{(1)}(kR_0)\right|} \leq \max\{1, 1/(2C_0)^{1/4}\}A_0^2\nu^{1/6}\left(\frac{\mathrm{e}kR_0}{\nu}\right)^\nu,$$

whereas in the third we have

(4.42)
$$\frac{\left|H_\nu^{(1)}(kr)\right|}{\left|H_\nu^{(1)}(kR_0)\right|} \leq \hat{C}_0A_0^2\sqrt{\nu}\left(\frac{2\mathrm{e}R_0}{r}\right)^\nu \leq \hat{C}_0A_0^2\sqrt{\nu}\left(\frac{2\mathrm{e}}{B_0}\right)^\nu.$$

Using the fact that $\nu \geq 2$ and our assumption on B_0, we conclude that for any $j > \tilde{j}_0(kr)$ we have

(4.43)
$$\frac{\left|H_\nu^{(1)}(kr)\right|}{\left|H_\nu^{(1)}(kR_0)\right|} \leq \hat{A}_0\sqrt{\nu}\left(\frac{2}{3}\right)^\nu \leq \hat{A}_0\sqrt{\tilde{\nu}_0(kr) + 1}\left(\frac{2}{3}\right)^{\tilde{\nu}_0(kr)+1},$$

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where $\hat{A}_0 = \max\{1, 1/(2C_0)^{1/4}\}\hat{C}_0 A_0^2$ is a constant depending on N only.

Therefore, taking

$$\tilde{A} = \max\left\{(\pi/2)A_0^2\frac{\tilde{C}_0}{(\tilde{C}_0^2-1)^{1/2}}, \hat{A}_0^2\right\},$$

and using (4.8) with $j_0 = \tilde{j}_0(kr)$ we have

$$(4.44) \quad \|u^s\|_{L^2(\partial B_r)}^2 \leq \tilde{A}B_1\left[\|u_\infty^s\|_{L^2(\mathbb{S}^{N-1})}^2 + (\tilde{\nu}_0(kr)+1)\left(\frac{2}{3}\right)^{2(\tilde{\nu}_0(kr)+1)}\|u^s\|_{L^2(\partial B_{R_0})}^2\right].$$

Let us note that $\tilde{A}$ depends on N only. Moreover, $2(\tilde{\nu}_0(kr)+1) \geq 2kr/\tilde{C}_0 \geq 2z_0/\tilde{C}_0 \geq 4$. Hence

$$(\tilde{\nu}_0(kr)+1)\left(\frac{2}{3}\right)^{2(\tilde{\nu}_0(kr)+1)} \leq (kr/\tilde{C}_0)\left(\frac{2}{3}\right)^{2kr/\tilde{C}_0}.$$

Finally, assuming that $k_1(\varepsilon, r) \geq k_0$, we obtain that if $k \geq k_1(\varepsilon, r)$ then at least we have

$$(4.45) \quad \|u^s\|_{L^2(\partial B_r)} \leq$$

$$\sqrt{\tilde{A}B_1}\left[\|u_\infty^s\|_{L^2(\mathbb{S}^{N-1})}^2 + (k_1(\varepsilon,r)r/\tilde{C}_0)\left(\frac{2}{3}\right)^{2k_1(\varepsilon,r)r/\tilde{C}_0}\|u^s\|_{L^2(\partial B_{R_0})}^2\right]^{1/2}.$$

The proof is concluded. $\qquad\square$

5 Stability estimates: from far-field up to the obstacle

We begin this section by establishing suitable a priori estimates for the solution to the direct scattering problem (2.1). In particular we are interested in the high frequencies regime and these a priori estimates will be stated in Theorem 5.5

We recall that, for any $k > 0$, by Φ_k we denote the fundamental solution to the Helmholtz equation $\Delta u + k^2 u = 0$ which is given by

$$\Phi_k(x,y) = \frac{i}{4}\left(\frac{k}{2\pi\|x-y\|}\right)^{(N-2)/2} H_{(N-2)/2}^{(1)}(k\|x-y\|) \quad \text{for any } x, y \in \mathbb{R}^N, \ x \neq y.$$

We remark that for $N = 2, 3$ this reduces to the well known formulas

$$\Phi_k(x,y) = \frac{i}{4}H_0^{(1)}(k\|x-y\|) \quad \text{for any } x, y \in \mathbb{R}^2, \ x \neq y.$$

and

$$\Phi_k(x,y) = \frac{e^{ik\|x-y\|}}{4\pi\|x-y\|} \quad \text{for any } x, y \in \mathbb{R}^3, \ x \neq y.$$

Let now Σ be a scatterer which may be characterized as the closure of a bounded Lipschitz open set. We recall that a bounded open set D is said to be Lipschitz if for any $x \in \partial D$ there exist a positive r and a Lipschitz function $\varphi : \mathbb{R}^{N-1} \to \mathbb{R}$, such that $\varphi(0) = 0$ and, up to a rigid change of coordinates, we have $x = 0$ and

$$B_r(x) \cap D = \{y = (y', y_N) \in B_r(x) : \ y_N < \varphi(y')\}.$$

For any density $\psi \in L^2(\partial\Sigma)$, let us define w and v as the corresponding *single-layer* and *double-layer potentials* with density ψ, namely

$$(5.1) \qquad w(x) = \int_{\partial\Sigma} \psi(y)\Phi_k(x,y)\mathrm{d}\mathcal{H}^{N-1}(y) \quad x \in \mathbb{R}^N\backslash\Sigma,$$

and

$$(5.2) \qquad v(x) = \int_{\partial\Sigma} \psi(y)\frac{\partial\Phi_k(x,y)}{\partial\nu(y)}\mathrm{d}\mathcal{H}^{N-1}(y) \quad x \in \mathbb{R}^N\backslash\Sigma,$$

where ν is the exterior normal to Σ. We notice that w and v satisfy the Helmholtz equation as well as the Sommerfeld radiation condition. We call w_∞ and v_∞ their far-field patterns, respectively. Then the following result holds.

Proposition 5.1 *Under the previous notation and assumptions we have the following estimates. For any $x \in G = \mathbb{R}^N\backslash\Sigma$ let $d = \mathrm{dist}(x,\Sigma)$. Then there exists a constant C, depending on N only, such that for any $k > 0$ and any $x \in G$ we have*

$$(5.3) \qquad |w(x)| \le C(\mathcal{H}^{N-1}(\partial\Sigma))^{1/2}\|\psi\|_{L^2(\partial\Sigma)}\frac{1}{d^{N-2}}\max\{1,(kd)^{(N-3)/2}\} \quad \text{for } N \ge 3$$

and

$$(5.4) \qquad |w(x)| \le (1/4)(\mathcal{H}^{N-1}(\partial\Sigma))^{1/2}\|\psi\|_{L^2(\partial\Sigma)}|H_0^{(1)}(kd)| \quad \text{for } N = 2,$$

and

$$(5.5) \qquad |v(x)| \le C(\mathcal{H}^{N-1}(\partial\Sigma))^{1/2}\|\psi\|_{L^2(\partial\Sigma)}\frac{1}{d^{N-1}}\max\{1,(kd)^{(N-1)/2}\} \quad \text{for } N \ge 2.$$

Furthermore, for any $N \ge 2$, any $k > 0$ and any $\hat{x} \in \mathbb{S}^{N-1}$, we have

$$(5.6) \qquad w_\infty(\hat{x}) = \frac{\mathrm{i}}{2}\frac{\mathrm{e}^{-(N-1)\pi\mathrm{i}/4}}{(2\pi)^{(N-1)/2}}k^{(N-3)/2}\int_{\partial\Sigma}\psi(y)\mathrm{e}^{-\mathrm{i}k\hat{x}\cdot y}\mathrm{d}\mathcal{H}^{N-1}(y)$$

and

$$(5.7) \qquad v_\infty(\hat{x}) = \frac{\mathrm{i}}{2}\frac{\mathrm{e}^{-(N-1)\pi\mathrm{i}/4}}{(2\pi)^{(N-1)/2}}k^{(N-3)/2}\int_{\partial\Sigma}\psi(y)\frac{\partial\mathrm{e}^{-\mathrm{i}k\hat{x}\cdot y}}{\partial\nu(y)}\mathrm{d}\mathcal{H}^{N-1}(y).$$

Therefore, for any $N \ge 2$, any $k > 0$ and any $\hat{x} \in \mathbb{S}^{N-1}$, we have

$$(5.8) \qquad |w_\infty(\hat{x})| \le \frac{k^{(N-3)/2}}{2(2\pi)^{(N-1)/2}}(\mathcal{H}^{N-1}(\partial\Sigma))^{1/2}\|\psi\|_{L^2(\partial\Sigma)}$$

and

$$(5.9) \qquad |v_\infty(\hat{x})| \le \frac{k^{(N-1)/2}}{2(2\pi)^{(N-1)/2}}(\mathcal{H}^{N-1}(\partial\Sigma))^{1/2}\|\psi\|_{L^2(\partial\Sigma)}.$$

PROOF. Let us notice that for any x, $y \in \mathbb{R}^N$, $x \neq y$, we have

$$\nabla_y \Phi_k(x,y) =$$
$$\frac{\mathrm{i}}{4} \frac{k^{N-2} k}{(2\pi)^{(N-2)/2}} \left[\frac{H_{(N-4)/2}^{(1)}(k\|x-y\|)}{(k\|x-y\|)^{(N-2)/2}} - (N-2) \frac{H_{(N-2)/2}^{(1)}(k\|x-y\|)}{(k\|x-y\|)^{N/2}} \right] \frac{y-x}{\|y-x\|}.$$

We also remark that for $N = 2, 3$ this reduces to

$$\nabla_y \Phi_k(x,y) = -\frac{\mathrm{i}}{4} k \left[H_1^{(1)}(k\|x-y\|) \right] \frac{y-x}{\|y-x\|} \quad N = 2$$

and

$$\nabla_y \Phi_k(x,y) = \frac{\mathrm{i}}{4\sqrt{2\pi}} k^2 \left[\frac{\mathrm{i} H_{1/2}^{(1)}(k\|x-y\|)}{(k\|x-y\|)^{1/2}} - \frac{H_{1/2}^{(1)}(k\|x-y\|)}{(k\|x-y\|)^{3/2}} \right] \frac{y-x}{\|y-x\|} \quad N = 3.$$

Then the estimates (5.3) and (5.5) follow by straightforward, although lengthy, computations. The main ingredient is the asymptotic behavior of Hankel functions $H_\nu(z)$, with $\nu \geq 0$, as $z \to 0^+$ and $z \to +\infty$. The latter is given in (3.1), while the former is the following

(5.10) $$H_\nu^{(1)}(z) \sim \begin{cases} -\mathrm{i}\dfrac{2}{\pi} \log(2/z) & \text{for } \nu = 0 \\[2mm] -\mathrm{i}\dfrac{\Gamma(\nu)}{\pi}(2/z)^\nu & \text{for } \nu > 0 \end{cases} \quad \text{as } z \to 0^+.$$

For what concerns (5.4), the estimate follows trivially from this remark. We have that $|H_\nu^{(1)}(z)|^2$ is a decreasing function of $z > 0$ for any fixed $\nu \geq 0$, see for instance [34, page 446]. This remark and the fact that

$$|H_0^{(1)}(z)| \sim \frac{2}{\pi} \log(2/z) \text{ as } z \to 0^+ \quad \text{and} \quad |H_0^{(1)}(z)| \sim \left(\frac{2}{\pi z} \right)^{1/2} \text{ as } z \to +\infty$$

provide also the correct way to interpret (5.4).

The relationships (5.6) and (5.7) may be proved, for $N = 3$, by the argument used to prove Theorem 2.5 in [10]. For $N \neq 3$, a standard modification is needed.

Finally, (5.8) and (5.9) follow immediately from (5.6) and (5.7), respectively. $\square$

We also need the following easy lemma.

Lemma 5.2 *Let $\Sigma \subset \overline{B}_R$, $R > 0$, be any scatterer, with no regularity assumption. Let us fix $k > 0$ and $\omega \in \mathbb{S}^{N-1}$. Let $R < r_1 < r$. We let $u = u(\omega, k, \Sigma)$ be the solution to (2.1). Then there exists a constant C, depending on R, r_1 and r only, such that*

$$\|u\|_{H^1(B_{r_1} \setminus \Sigma)} \leq C \max\{1, k\} \|u\|_{L^2(B_r \setminus \Sigma)}.$$

PROOF. We fix $\tilde{r}$ such that $R < r_1 < \tilde{r} < r$, $\tilde{r}$ depending on R, r_1 and r only. Then by a standard Caccioppoli's inequality, we infer that

$$\|u\|_{H^1(B_{\tilde{r}}\setminus\overline{B_{r_1}})} \leq C_1 \max\{1,k\}\|u\|_{L^2(B_r\setminus\Sigma)},$$

for some constant C_1 depending on R, r_1 and r only.

By integrating over $B_{\tilde{r}}\setminus\overline{B_{r_1}}$ in spherical coordinates, we infer that there exists ρ, $r_1 < \rho < \tilde{r}$, such that

$$\int_{\partial B_\rho} |u|^2 \leq \frac{3}{\tilde{r}-r_1}\|u\|^2_{L^2(B_r\setminus\Sigma)} \quad \text{and} \quad \int_{\partial B_\rho} \left|\frac{\partial u}{\partial\nu}\right|^2 \leq \frac{3}{\tilde{r}-r_1}C_1^2\max\{1,k^2\}\|u\|^2_{L^2(B_r\setminus\Sigma)}.$$

Since

$$\int_{B_\rho\setminus\Sigma} \|\nabla u\|^2 = k^2 \int_{B_\rho\setminus\Sigma} |u|^2 + \int_{\partial B_\rho} \frac{\partial u}{\partial\nu}\overline{u},$$

the thesis immediately follows. $\qquad\square$

Let us remark that, for the same ρ as in the previous proof, if $u^s = u^s(\omega,k,\Sigma)$ we have

$$\|u^s\|_{L^2(\partial B_\rho)} \leq \left(\frac{3}{\tilde{r}-r_1}\right)^{1/2}\|u\|_{L^2(B_r\setminus\Sigma)} + (\mathcal{H}^{N-1}(\partial B_\rho))^{1/2}$$

and

$$\left\|\frac{\partial u^s}{\partial\nu}\right\|_{L^2(\partial B_\rho)} \leq \left(\frac{3}{\tilde{r}-r_1}\right)^{1/2}C_1\max\{1,k\}\|u\|_{L^2(B_r\setminus\Sigma)} + k(\mathcal{H}^{N-1}(\partial B_\rho))^{1/2}.$$

Therefore, since u^s has the following Helmholtz representation

$$(5.11) \qquad u^s(x) = \int_{\partial B_\rho} \frac{\partial u^s(y)}{\partial\nu}\Phi_k(x,y) - u^s(y)\frac{\partial\Phi_k(x,y)}{\partial\nu(y)}\mathrm{d}\mathcal{H}^{N-1}(y) \quad \|x\| > \rho,$$

we can use all the results given in Proposition 5.1. In particular we have for any $k > 0$

$$(5.12) \quad \|\mathcal{A}(\Sigma)(\cdot,\omega,k)\|_{L^2(\mathbb{S}^{N-1})} \leq$$

$$\mathcal{H}^{N-1}(\mathbb{S}^{N-1})\frac{k^{(N-3)/2}}{2(2\pi)^{(N-1)/2}}\tilde{r}^{(N-1)/2}\left[k\|u^s\|_{L^2(\partial B_\rho)} + \left\|\frac{\partial u^s}{\partial\nu}\right\|_{L^2(\partial B_\rho)}\right] \leq$$

$$C(N)\left(\frac{k\tilde{r}}{2\pi}\right)^{(N-1)/2}\left[\left(\frac{3}{\tilde{r}-r_1}\right)^{1/2}(1+C_1\max\{k^{-1},1\})\|u\|_{L^2(B_r\setminus\Sigma)} + 2(\mathcal{H}^{N-1}(\partial B_{\tilde{r}}))^{1/2}\right]$$

where $C(N) = \mathcal{H}^{N-1}(\mathbb{S}^{N-1})/2$. Furthermore, by (2.5), we also have for any $k > 0$

$$(5.13) \quad \|\mathcal{A}(\Sigma)(\cdot,\omega,k)\|^2_{L^2(\mathbb{S}^{N-1})} = 2\left(\frac{2\pi}{k}\right)^{(N-1)/2}\Im\left(e^{(N-3)\pi i/4}\mathcal{A}(\Sigma)(\omega,\omega,k)\right) \leq$$

$$k^{-1}(\mathcal{H}^{N-1}(\partial B_\rho))^{1/2}\left[k\|u^s\|_{L^2(\partial B_\rho)} + \left\|\frac{\partial u^s}{\partial\nu}\right\|_{L^2(\partial B_\rho)}\right] \leq$$

$$(\mathcal{H}^{N-1}(\partial B_{\tilde{r}}))^{1/2}\left[\left(\frac{3}{\tilde{r}-r_1}\right)^{1/2}(1+C_1\max\{k^{-1},1)\}\|u\|_{L^2(B_r\setminus\Sigma)} + 2(\mathcal{H}^{N-1}(\partial B_{\tilde{r}}))^{1/2}\right].$$

We can conclude that there exists a constant C, depending on N, R, r and $\|u\|_{L^2(B_r\setminus\Sigma)}$ only, such that

(5.14) $$\|\mathcal{A}(\Sigma)(\cdot,\omega,k)\|_{L^2(\mathbb{S}^{N-1})} \leq C \quad \text{for any } N \geq 3 \text{ and any } k > 0$$

and

(5.15) $$\|\mathcal{A}(\Sigma)(\cdot,\omega,k)\|_{L^2(\mathbb{S}^{N-1})} \leq C \max\{k^{-1/2}, 1\} \quad \text{for } N = 2 \text{ and any } k > 0.$$

We continue by establishing suitable a priori estimates on the solutions to direct scattering problems with sound-soft scatterers. In particular we are interested in the high frequency asymptotics. The main tool will be provided by the results established by Chandler-Wilde and Monk, [6].

Let us fix an integer $m \geq 1$ and positive constants β, R_0, and δ_0 such that $R_0 < R_0 + \delta_0 = R \leq \beta$. Let us also fix $r > R$.

We begin with the case in which k lies in a bounded interval, that is we fix constants $0 < k_0 < k_1$ and we denote, for any $N \geq 2$,

(5.16) $$I_N = \begin{cases} [k_0, k_1] & \text{if } N = 2, \\ (0, k_1] & \text{if } N \geq 3. \end{cases}$$

We have the following a priori estimate which follows from arguments used in [15]. We notice that a much more general class of scatterers may be used, see for instance [30] for sound-soft scatterers and [26] for the corresponding sound-hard case.

Proposition 5.3 *Under the previous notation, there exists a constant C, depending on N, m, β, R_0, R, r and I_N only, such that for any $k \in I_N$, any $\omega \in \mathbb{S}^{N-1}$ and any $\Sigma \in X(m,\beta,R_0,\delta_0)$ we have*

(5.17) $$\|u(\omega,k,\Sigma)\|_{L^2(B_r\setminus\Sigma)} \leq C.$$

We now consider the high frequency asymptotics. We limit ourselves to the case $m \geq 2$ and $N = 2, 3$. Fix $\Sigma \in X(m,\beta,R_0,\delta_0)$, $k > 0$ and $\omega \in \mathbb{S}^{N-1}$. We have that $u^s = u^s(\omega,k,\Sigma)$, the scattered field of the solution to the direct scattering problem (2.1), may be described as the sum of a double- and a single-layer potential in the following way

(5.18) $$u^s(x) = \int_{\partial\Sigma} \psi(y) \left[\frac{\partial\Phi_k(x,y)}{\partial\nu(y)} - ik\Phi_k(x,y) \right] d\mathcal{H}^{N-1}(y) \quad x \in \mathbb{R}^N\setminus\Sigma,$$

where $\psi \in C^0(\partial\Sigma)$, see for instance [10]. Here ψ solves the following integral equation

$$(I + K_k - ikS_k)\psi = A\psi = -2u^i \quad \text{on } \partial\Sigma$$

where K_k and S_k are the double- and single-layer operators, respectively, defined by, see Chapter 3 of [10],

$$K_k(\psi)(x) = \int_{\partial\Sigma} \frac{\partial\Phi_k(x,y)}{\partial\nu(y)} \psi(y) d\mathcal{H}^{N-1}(y), \quad S_k(\psi)(x) = \int_{\partial\Sigma} \Phi_k(x,y)\psi(y) d\mathcal{H}^{N-1}(y),$$

for almost every $x \in \partial\Sigma$. We also denote for any $x \in \mathbb{R}^N\setminus\partial\Sigma$

$$\tilde{K}_k(\psi)(x) = \int_{\partial\Sigma} \frac{\partial\Phi_k(x,y)}{\partial\nu(y)} \psi(y) d\mathcal{H}^{N-1}(y), \quad \tilde{S}_k(\psi)(x) = \int_{\partial\Sigma} \Phi_k(x,y)\psi(y) d\mathcal{H}^{N-1}(y).$$

We have that $A = A(k, \Sigma)$ is bounded and bijective from $C^0(\partial\Sigma)$ onto itself and also from $L^2(\partial\Sigma)$ onto itself.

In the following theorem, we state some useful estimates on $A(k, \Sigma)^{-1}$, $\tilde{K}_k$ and $\tilde{S}_k$.

Theorem 5.4 *Under the previous notation, let $m \geq 2$ and $N = 2, 3$. There exist constants $\tilde{C}$, depending on N, m, β and R_0 only, and $\tilde{C}_1$, depending on N, m, β, R_0 and r only, such that for any k satisfying $kR_0 \geq 1$ and any $\Sigma \in X(m, \beta, R_0, \delta_0)$ we have*

$$(5.19) \qquad \|A(k, \Sigma)^{-1}\|_{\mathcal{L}(L^2(\partial\Sigma), L^2(\partial\Sigma))} \leq \tilde{C}.$$

Furthermore we also have

$$(5.20) \qquad \|\tilde{K}_k\|_{\mathcal{L}(L^2(\partial\Sigma), L^2(B_r \setminus \Sigma))} \leq \tilde{C}_1 k$$

and

$$(5.21) \qquad \|\tilde{S}_k\|_{\mathcal{L}(L^2(\partial\Sigma), L^2(B_r \setminus \Sigma))} \leq \tilde{C}_1.$$

PROOF. The estimate (5.19) is an immediate corollary of Theorem 4.3 and Corollary 4.4 in [6], whereas (5.20) follows from Theorem 5.2 in [25].

For what concerns the estimate on $\tilde{S}_k$, the argument depends on the dimension N. For $N = 3$ we have that $|\Phi_k(x, y)| \leq \Phi_0(x, y) = 1/(4\pi\|x - y\|)$, therefore

$$\|\tilde{S}_k\|_{\mathcal{L}(L^2(\partial\Sigma), L^2(B_r \setminus \Sigma))} \leq \|\tilde{S}_0\|_{\mathcal{L}(L^2(\partial\Sigma), L^2(B_r \setminus \Sigma))}$$

where $\tilde{S}_0$ is the corresponding operator with Φ_k replaced by Φ_0, the fundamental solution of the Laplacian for $N = 3$. For $N = 2$ the argument is slightly more involved. We recall that $|H_0^{(1)}(z)|$ is a decreasing function of $z > 0$, therefore, if we set for the time being $k_0 = 1/R_0$, for any $k \geq k_0$ we have $|\Phi_k(x, y)| \leq |\Phi_{k_0}(x, y)|$. Furthermore, there exist positive constants C_1 and C_2, depending on R_0 and r only, such that for any $x, y \in B_r \subset \mathbb{R}^2$ we have

$$|\Phi_{k_0}(x, y)| \leq C_1 \Phi_0(x, y) + C_2$$

where $\Phi_0(x, y) = -(2\pi)^{-1} \log(\|x - y\|)$ is the fundamental solution of the Laplacian for $N = 2$. Here we have made use of (5.10). Therefore, for any $x \in B_r \setminus \Sigma$ and any $k \geq k_0$, we have

$$|\tilde{S}_k(\psi)(x)| \leq \int_{\partial\Sigma} |\Phi_{k_0}(x, y)||\psi(y)| d\mathcal{H}^{N-1}(y) \leq C_1 \tilde{S}_0(|\psi|)(x) + C_2 E^{1/2}\|\psi\|_{L^2(\partial\Sigma)},$$

E as in (2.22), and the conclusion immediately follows. $\qquad\square$

We conclude this part devoted to a priori estimates stating the following result.

Theorem 5.5 *Under the previous notation, let $m \geq 2$ and $N = 2, 3$.*

There exists a constant C_1, depending on N, m, β, R_0, δ_0 and r only, such that for any k satisfying $kR_0 \geq 1$, any $\omega \in \mathbb{S}^{N-1}$ and any $\Sigma \in X(m, \beta, R_0, \delta_0)$ we have the following estimates. We let $u = u(\omega, k, \Sigma)$, $u^s = u^s(\omega, k, \Sigma)$ and $u_\infty^s = u_\infty^s(\omega, k, \Sigma)$. Then

$$(5.22) \qquad\qquad \|u(\omega, k, \Sigma)\|_{L^2(B_r \setminus \Sigma)} \le C_1 k$$

and, for any $x \in G = \mathbb{R}^N \setminus \Sigma$, if $d = \operatorname{dist}(x, \Sigma)$ we have

$$(5.23) \qquad |u^s(x)| \le 4C\tilde{C}E \left[\max \left\{ \frac{1}{d^{N-1}}, \left(\frac{k}{d} \right)^{(N-1)/2} \right\} \right] \quad \text{for } N = 3,$$

and

$$(5.24) \quad |u^s(x)| \le 2\tilde{C}E \left[(1/4)k|H_0^{(1)}(kd)| + C \max \left\{ \frac{1}{d^{N-1}}, \left(\frac{k}{d} \right)^{(N-1)/2} \right\} \right] \quad \text{for } N = 2,$$

where C is as in Proposition 5.1. Notice that, if we fix $d_0 > 0$, then

$$k|H_0^{(1)}(kd)| \le \hat{C} \sqrt{\frac{k}{d}} \le \hat{C} \max \left\{ \frac{1}{d}, \sqrt{\frac{k}{d}} \right\} \quad \text{for any } d \ge d_0$$

where $\hat{C}$ depends on d_0/R_0 only. Finally, we have for any $\hat{x} \in \mathbb{S}^{N-1}$

$$(5.25) \qquad\qquad |u_\infty^s(\hat{x})| \le 2 \frac{k^{(N-1)/2}}{(2\pi)^{(N-1)/2}} \tilde{C}E$$

hence by (2.5)

$$(5.26) \qquad\qquad \|u_\infty^s\|_{L^2(\mathbb{S}^{N-1})}^2 \le 4\tilde{C}E.$$

PROOF. We sketch the proof only of estimate (5.22), all the others follow in a standard way from the previously stated results.

We notice that if $\psi = A(k, \Sigma)^{-1}(-2u^i)$, then

$$(5.27) \qquad\qquad \|\psi\|_{L^2(\partial\Sigma)} \le 2\tilde{C}(\mathcal{H}^{N-1}(\partial\Sigma))^{1/2} \le 2\tilde{C}E^{1/2},$$

where $\tilde{C}$ is as in (5.19) and E is as in (2.22).

From (5.18) we have

$$\|u^s(\omega, k, \Sigma)\|_{L^2(B_r \setminus \Sigma)} \le \left[\|\tilde{K}_k\|_{\mathcal{L}(L^2(\partial\Sigma), L^2(B_r \setminus \Sigma))} + k\|\tilde{S}_k\|_{\mathcal{L}(L^2(\partial\Sigma), L^2(B_r \setminus \Sigma))} \right] \|\psi\|_{L^2(\partial\Sigma)}.$$

Recalling that $\|u^i\|_{L^2(B_r \setminus \Sigma)} \le |B_1|^{1/2} r^{N/2}$, the estimate (5.22) follows then from (5.20), (5.21) and (5.27). $\qquad\qquad\square$

We remark that with the use of Theorem 5.5 and of the results obtained by coupling Proposition 5.1, Lemma 5.2 and the following discussion, in particular (5.14) and (5.15), with Proposition 5.3, we may obtain a priori estimates for solutions of the direct scattering problem (2.1) for any $k > 0$ if $N = 3$ and for any $k \ge k_0$ for $N = 2$, k_0 being a fixed positive constant. The interested reader will be easily able to deduce the corresponding estimates from our previous statements. We also notice that an estimate related to (5.26) in the high frequencies regime was obtained in [20].

Now we pass to the main topic of this section. Let us introduce the geometrical setting. Under the previous notation, let $m \geq 1$ and $N \geq 2$. We consider Σ_1, $\Sigma_2 \in X(m, \beta, R_0, \delta_0)$ and we call K the convex hull of $\Sigma_1 \cup \Sigma_2$. We begin by observing that K is closed, convex with not empty interior. Moreover, $B_{R_0} \subset K \subset \overline{B_R}$.

We fix $k > 0$ and $\omega \in \mathbb{S}^{N-1}$. Let $u_1 = u(\omega, k, \Sigma_1)$ and $u_2 = u(\omega, k, \Sigma_2)$ be the solutions to (2.1) with Σ replaced by Σ_1 and Σ_2 respectively. Let the corresponding scattered fields be denoted by $u_1^s = u^s(\omega, k, \Sigma_1)$ and $u_2^s = u^s(\omega, k, \Sigma_2)$ and their far-field patterns by $\mathcal{A}(\Sigma_1)(\cdot, \omega, k)$ and $\mathcal{A}(\Sigma_2)(\cdot, \omega, k)$ respectively. We call $v = u_1 - u_2 = u_1^s - u_2^s$. Obviously, $v = v(\omega, k)$.

We wish to estimate v outside K in terms of the difference between the far-field patterns $\mathcal{A}(\Sigma_1)(\cdot, \omega, k)$ and $\mathcal{A}(\Sigma_2)(\cdot, \omega, k)$. We are clearly interested only in the high frequencies regime. By the results of the previous section we know how to estimate v in the near-field from its far-field pattern. Therefore, we shall mainly focus on estimating v outside K from the values of v in the near-field, Theorem 5.6. We need to introduce some notation and assumptions.

Let $z_0 \geq 4$, C_0 and A_0 be as in Theorem 3.9 and such that (4.18) and (4.19) are satisfied. We fix a constant $k_0 \geq 1$ such that $k_0 R_0 \geq z_0$. Given $\tilde{B}_0$ as in (4.36) and $\tilde{B}_1 \geq \tilde{B}_0$, to be fixed later, we denote $\rho_1 = \tilde{B}_0(R+1)$ and $\rho_2 = \tilde{B}_1(R+1)$.

We notice that there exists ρ_0, $R < \rho_0 < R+1$, such that

$$(5.28) \qquad \int_{\partial B_{\rho_0}} |v|^2 \leq \int_{B_{R+1} \setminus \overline{B_R}} |v|^2.$$

Then $\rho_1 = B_0 \rho_0$ where $\tilde{B}_0 \leq B_0 \leq \tilde{B}_0(R+1)/R$ and $\rho_2 = B_1 \rho_0$ where $\tilde{B}_1 \leq B_1 \leq \tilde{B}_1(R+1)/R$.

We assume that $k \geq k_0$ and that for some $0 < \varepsilon \leq M \leq \tilde{M}$

$$(5.29) \qquad \|\mathcal{A}(\Sigma_1)(\cdot, \omega, k) - \mathcal{A}(\Sigma_2)(\cdot, \omega, k)\|_{L^2(\mathbb{S}^{N-1})} \leq \varepsilon, \quad \|v\|_{L^2(B_{R+1}) \setminus \overline{B_R}} \leq M$$

and

$$(5.30) \qquad \|u_1\|_{L^2(B_{\rho_2} \setminus \Sigma_1)}, \ \|u_2\|_{L^2(B_{\rho_2} \setminus \Sigma_2)} \leq \tilde{M}/2.$$

We define the *error in the near-field* by η_1 as follows

$$(5.31) \qquad \|v\|_{L^2(B_{\rho_2} \setminus \overline{B_{\rho_1}})} \leq \eta_1$$

We note that the results of the previous section allow us to estimate precisely η_1 in terms of ε, k, M and $\tilde{B}_1$. In fact

$$\|v\|_{L^2(\partial B_{\rho_0})} \leq M$$

and

$$\|v\|_{L^2(B_{\rho_2} \setminus \overline{B_{\rho_1}})} \leq ((B_1 - B_0)(R+1))^{1/2} \max_{r \in [B_0 \rho_0, B_1 \rho_0]} \|v\|_{L^2(\partial B_r)}.$$

Finally, by Theorem 5.5 we clearly have

$$(5.32) \qquad M \leq \tilde{M} \leq C_1 k$$

where C_1 depends on N, m, β, R_0, δ_0 and $\tilde{B}_1$ only. Therefore, provided $\varepsilon \leq M \leq \tilde{M} \leq C_1 k$ we have that

$$(5.33) \qquad \eta_1 \leq \eta_1(\varepsilon, k, M, \tilde{B}_1) \leq \eta_1(\varepsilon, k, \tilde{M}, \tilde{B}_1) \leq \eta_1(\varepsilon, k, C_1 k, \tilde{B}_1)$$

where the explicit expression of $\eta_1(\varepsilon, k, C_1 k, \tilde{B}_1)$ may be deduced in the different regimes by the results of Section 4. However we shall explicit the value of η_1 only in a particular case in the extremely high frequencies regime, see Corollary 5.7, leaving the other cases to the interested reader.

We finally set, for suitable constants $\tilde{C}_1$ and $\tilde{C}_2$ to be fixed later,

$$(5.34) \qquad \eta_2(\eta_1, k, \tilde{M}) = \left(\tilde{C}_1^2 k^2 \eta_1^2 + \frac{\tilde{C}_2^2 k^2 \tilde{M}^2}{\left(- \log \left((\tilde{C}_1/\tilde{C}_2)\eta_1/\tilde{M} \right) + k \right)^{1/8}} \right)^{1/2}.$$

The result is the following.

Theorem 5.6 *Under the previous notation and assumptions, let us assume that $N = 2, 3$, $m \geq 2$ and $k_0 \geq 1$ such that $k_0 R_0 \geq z_0 \geq 4$. We let $k \geq k_0$, $\omega \in \mathbb{S}^{N-1}$ and Σ_1 and $\Sigma_2 \in X(m, \beta, R_0, \delta_0)$.*

Let us also fix μ, $0 < \mu \leq 1$, and

$$(5.35) \qquad \tilde{B}_1 = 2\tilde{B}_0 + 5, \quad \rho_1 = \tilde{B}_0(R + 1), \quad \rho_2 = \tilde{B}_1(R + 1).$$

Then there exist positive constants $\tilde{C}_1, \ldots, \tilde{C}_3$, depending on N, m, β, R_0 and δ_0 only, and a constant $\tilde{F}(\mu)$, depending on μ, N, m, β, R_0 and δ_0 only, such that the following holds.

Let us assume that (5.30) holds and that

$$\|u_1 - u_2\|_{L^2(B_{\rho_2} \setminus \overline{B_{\rho_1}})} \leq \eta_1 \leq (\tilde{C}_2/\tilde{C}_1)\tilde{M}.$$

Then we have the stability estimate

$$(5.36) \qquad \|u_1 - u_2\|_{L^1(B_{\rho_1} \setminus K)} \leq \eta(\eta_1)$$

where $\eta(\eta_1)$ satisfies the following. Let $\eta_2 = \eta_2(\eta_1, k, \tilde{M})$ be given as in (5.34).
For $N = 2$, we pick $\mu = 1$ and we have

$$(5.37) \qquad \eta(\eta_1) \leq \tilde{F}(1)\tilde{C}_3 \eta_2(\eta_1, k, \tilde{M}).$$

For $N = 3$, we pick μ, $0 < \mu < 1$, and we have

$$(5.38) \qquad \eta(\eta_1) \leq \tilde{F}(\mu)\tilde{C}_3^{a_1} \tilde{M}^{a_2} (\eta_2(\eta_1, k, \tilde{M}))^{a_1},$$

where a_1 and a_2 are given by the following formulas

$$(5.39) \qquad a_1 = \frac{\mu}{2 - \mu}, \quad a_2 = \frac{2(1 - \mu)}{2 - \mu}.$$

The proof of this result will be divided in several steps. Before attempting it, in the next corollary we show how to use the previous theorem to estimate the difference between u_1 and u_2 up to the boundary of K from the difference between their respective far-field patterns, at least in the extremely high frequencies regime.

Corollary 5.7 *Let us assume that $N = 2, 3$, $m \geq 2$ and $k_0 \geq 1$ such that $k_0 R_0 \geq z_0 \geq 4$. We let $k \geq k_0$, $\omega \in \mathbb{S}^{N-1}$ and Σ_1 and $\Sigma_2 \in X(m, \beta, R_0, \delta_0)$.*
Let

$$\|\mathcal{A}(\Sigma_1)(\cdot, \omega, k) - \mathcal{A}(\Sigma_2)(\cdot, \omega, k)\|_{L^2(\mathbb{S}^{N-1})} \leq \varepsilon \leq C_1 k$$

with C_1 as in (5.32)
Then there exist constants $\hat{C}_0$ and $\hat{C}_1$, depending on N, m, β, R_0 and δ_0 only, such that if $\varepsilon \leq 1/e$ and

$$k \geq \frac{\hat{C}_0}{\log(4/3)\tilde{B}_0 R_0} \log(1/\varepsilon),$$

we have that

(5.40) $\|u_1 - u_2\|_{L^1(B_{\rho_1}\setminus K)} \leq$

$$F(\mu)\tilde{C}_3^{a_1} C_1^{a_2} k^{a_2} \left((\tilde{C}_1 \hat{C}_1)^2 k^2 \varepsilon^2 + \frac{(\tilde{C}_2 C_1)^2 k^4}{\left(-\log\left(\tilde{C}_1 \hat{C}_1 \varepsilon / (\tilde{C}_2 C_1 k) \right) + k \right)^{1/8}} \right)^{a_1/2}$$

where $0 < \mu < 1$ for $N = 3$ and $\mu = 1$ for $N = 2$, and $\tilde{C}_1$, $\tilde{C}_2$, $\tilde{C}_3$ and $F(\mu)$ are as in the previous theorem.

PROOF. We begin by noticing that (4.39) holds for any $k \geq k_0$, with $\tau = 1$ and a constant C_1 depending on N, m, β, R_0 and δ_0 only, as in (5.32). Hence, if $\varepsilon \leq 1/e$ and

$$k \geq \frac{\tilde{C}_0}{\log(4/3)\tilde{B}_0 R_0} \log(1/\varepsilon),$$

we can apply Theorem 4.4 and obtain that

$$\eta_1(\varepsilon, k, C_1 k, \tilde{B}_1) \leq \left((\rho_2 - \rho_1)(\tilde{A}\tilde{B}_1(R+1)(1 + C(1)C_1^2(\tilde{C}_0/\tilde{B}_0 R_0)^2)/R) \right)^{1/2} \varepsilon = \hat{C}_1 \varepsilon,$$

where $C(1)$ is an absolute constant and $\hat{C}_1$ clearly depends on N, m, β, R_0 and δ_0 only. Therefore, by taking $\hat{C}_0 \geq \tilde{C}_0$ such that $\varepsilon \leq \tilde{C}_2 C_1 k/(\tilde{C}_1 \hat{C}_1)$, we can easily conclude the proof. $\qquad\square$

It is very important to notice that we have an estimate with an explicit dependence on k, in the extremely high frequencies regime. However we have to point out that we do not have any increasing stability phenomenon as k grows.

We conclude the section by proving Theorem 5.6. The main tool to go from the near-field up to the boundary of K is provided by the results in [33], a consequence of those in [12].

We begin with the following intermediate case. Let us take $P \in \partial K$. Let π be a supporting hyperplane for K passing through P and let S^+ be the open half-space with boundary π not intersecting K. We wish to estimate the L^2 norm of v on $B_{\rho_1} \cap S^+$. We argue in the following way. Let ν be the normal to π pointing inside S^+ and let P_1 be the point of π intersecting the half-line $l = \{s\nu : \ s \geq 0\}$. We notice that $0 < R_0 \leq \|P_1\| \leq R$. For any fixed r, $\rho_1 + 1 \leq r \leq \rho_1 + 2$, we consider the open cylinder T_r, contained in S^+, whose lower

base is contained in π, is centered in P_1 and has radius r, and whose height is r. We call Γ_r its upper base, that is the one contained in S^+. We recall that

(5.41) $$\tilde{B}_1 = 2\tilde{B}_0 + 5 \text{ and } \rho_2 = \tilde{B}_1(R+1).$$

We notice that for any r, $\rho_1 + 1 \le r \le \rho_1 + 2$, we have

$$B_{\rho_1} \cap S^+ \subset T_r \subset B_{\rho_2-1} \cap S^+ \quad \text{and} \quad \Gamma_r \subset B_{\rho_2-1}\backslash\overline{B_{\rho_1+1}}.$$

Then, by the same argument used in Lemma 5.2, we infer that there exists r, $\rho_1 + 1 < r < \rho_1 + 2$, such that

(5.42) $$\int_{\Gamma_r} |v|^2 \le 3\|v\|^2_{L^2(B_{\rho_2}\backslash\overline{B_{\rho_1}})} \quad \text{and} \quad \int_{\Gamma_r} |\nabla v|^2 \le 3C^2k^2\|v\|^2_{L^2(B_{\rho_2}\backslash\overline{B_{\rho_1}})}$$

for some constant C depending on N, R_0 and δ_0 only. We conclude that

(5.43) $$\|v\|_{L^2(\Gamma_r)} + \|\nabla v\|_{L^2(\Gamma_r)} \le \sqrt{3}(1 + Ck)\|v\|_{L^2(B_{\rho_2}\backslash\overline{B_{\rho_1}})} \le C_2 k\eta_1$$

where $C_2 = \sqrt{3}(1 + C)$.

Again by Lemma 5.2, there exists a constant C_3, depending on N, R_0 and δ_0 only, such that

(5.44) $$\|v\|_{H^1(T_r)} \le C_3 k\tilde{M}.$$

We obtain the following lemma, an immediate consequence of Theorem 1.1 in [33].

Lemma 5.8 *Under the previous notation and assumptions, setting $\tilde{C}_1 = C_2$ and $\tilde{C}_2 = C_3$ and recalling that*

$$\eta_1 \le (\tilde{C}_2/\tilde{C}_1)\tilde{M},$$

we have that for any $k \ge k_0$

(5.45) $$\|v\|_{L^2(B_{\rho_1}\cap S^+)} \le C_4\eta_2(\eta_1, k, \tilde{M})$$

where C_4 depends on N, R_0 and δ_0 only and $\eta_2(\eta_1, k, \tilde{M})$ is defined in (5.34).

Up to now all the results hold for $m = 1$ and any $N \ge 2$ as well. To proceed further we need an additional assumption, namely that $m \ge 2$. Let us begin by studying some geometrical properties of K, the convex hull of $\Sigma_1 \cup \Sigma_2$. Let P be any point belonging to ∂K. Without loss of generality, let us assume that $P = s_0 e_N$, where $s_0 > 0$ and $e_1, \ldots, e_N$ denote the canonical base in $\mathbb{R}^N$. Then there exist constants $r_0 > 0$ and θ_0, $0 < \theta_0 < \pi/2$, depending on m, β, R_0 and δ_0 only, such that the following holds. There exists a point Q, depending on P, such that $\|P - Q\| = r_0$, $B_{r_0}(Q) \subset K$ and finally the angle between $-e_N$ and the vector $Q - P$ is at most θ_0. We immediately infer a few interesting properties. First of all, there exists a unique supporting hyperplane for K passing through P, the hyperplane whose normal is given by the vector $Q - P$. For any direction $\hat{x} \in \mathbb{S}^{N-1}$ there exists a unique $s_0(\hat{x}) > 0$ such that $s\hat{x} \in K$ for any $0 \le s \le s_0(\hat{x})$ and $s\hat{x} \notin K$ for any $s > s_0(\hat{x})$. Clearly $P(\hat{x}) = s_0(\hat{x})\hat{x}$ is the only one of these points belonging to ∂K and $R_0 \le s_0(\hat{x}) \le R$ for any $\hat{x} \in \mathbb{S}^{N-1}$. For any $\hat{x} \in \mathbb{S}^{N-1}$ we denote by $\pi(\hat{x})$ the unique supporting hyperplane for K passing through $P(\hat{x})$ and by $S^+(\hat{x})$ the open half-space with boundary $\pi(\hat{x})$ not intersecting K. A further crucial geometrical property of K is given in the following lemma.

Lemma 5.9 *Let $m \geq 2$ and $N \geq 2$. Let us consider Σ_1, $\Sigma_2 \in X(m, \beta, R_0, \delta_0)$ and let K be the convex hull of $\Sigma_1 \cup \Sigma_2$.*

Let $\hat{x} \in \mathbb{S}^{N-1}$ and let $P = P(\hat{x}) = s_0(\hat{x})\hat{x}$ belong to ∂K. For any $d > 0$ we denote $P_d = (s_0(\hat{x}) + d)\hat{x}$.

Then there exists a positive constant E_0, depending on N, m, β, R_0 and δ_0 only, such that

$$(5.46) \qquad \mathcal{H}^{N-1}\left(\{\hat{x}_1 \in \mathbb{S}^{N-1} : \ P_d \in S^+(\hat{x}_1)\}\right) \geq E_0 \min\{d^{(N-1)/2}, 1\}.$$

PROOF. We sketch the proof of the lemma. First of all we notice that the distance of P_d from Q is bounded from below by $r_0 + \cos(\theta_0)d$ and from above by $r_0 + d$. Let us call T^+ the portion of $\partial B_{\rho_0}(Q)$ that is formed by points $x \in \partial B_{\rho_0}(Q)$ such that the segment connecting x to P_d intersects $\partial B_{\rho_0}(Q)$ only at x. A simple computation shows that

$$\mathcal{H}^{N-1}(T^+) \geq E_1 \min\{d^{(N-1)/2}, 1\}$$

where E_1 is a positive constant depending on N, m, β, R_0 and δ_0 only. In fact, T^+ is the intersection of $\partial B_{\rho_0}(Q)$ with a symmetric cone with vertex in Q and bisecting line passing through P_d and whose amplitude is given by an angle α, α being of the order of $\sqrt{d}$

Let $\pi(\hat{x})$ be the supporting hyperplane at P. Let us call D the open region which is enclosed by $\partial B_{\rho_0}(Q)$ and all tangent lines to $\partial B_{\rho_0}(Q)$ passing through P_d. Notice that this is a portion of a symmetric cone with vertex in P_d and bisecting line l containing Q. We wish to prove that

$$(5.47) \qquad \mathcal{H}^{N-1}(\pi(\hat{x}) \cap D) \geq E_2 \min\{d^{(N-1)/2}, 1\}$$

where E_2 is a positive constant depending on N, m, β, R_0 and δ_0 only. In order to prove this property, let us begin with the following intermediate step. We call Π the plane of $\mathbb{R}^N$ containing P and Q, and consequently P_d. Let us take the two points x_1 and x_2 which are the intersections of $\pi(\hat{x})$ with the two lines in Π passing through P_d and tangent to $B_{\rho_0}(Q) \cap \Pi$. It is convenient to perform a rigid change of variables such that, in this new coordinate system, $Q = 0$ and $e_N = P_d - Q/\|P_d - Q\|$. We show that

$$\|x_1 - x_2\| \geq E_3 \min\{\sqrt{d}, 1\}$$

where E_3 is a positive constant depending on m, β, R_0 and δ_0 only. In order to prove this, we begin with the case in which P belongs to the segment connecting Q to P_d. Then a simple geometric construction, using the properties of the angle α defined before, implies that $\|x_1 - x_2\| = 2\|x_1 - P\|$ and $a_1 = \|x_1 - P\|$ is of order $\sqrt{d}$. In the general case, we always have that $\|x_1 - x_2\| \geq a_1$. Let us notice that this concludes the proof of (5.47) at least for $N = 2$.

For $N > 2$, the key step, which follows from elementary calculations, is to prove that the distance of P from the line l passing through Q and P_d is bounded by a constant times d. Then we take the point x_3 which is the intersection of the segment connecting x_1 to x_2 with l. Then we construct the point x_4 which is one of the intersections of $\partial D \cap \Pi$ with the hyperplane passing through x_3 with normal $P_d - Q$. Another computation leads to show that $\|x_3 - x_4\|$ is of order $\sqrt{d}$. From this last property (5.47) easily follows.

For any $\hat{x}_1 \in \mathbb{S}^{N-1}$, let us call $l(\hat{x}_1) = \{x = s\hat{x}_1 : \ s \geq 0\}$. We have that if $l(\hat{x}_1)$ intersects $\pi(\hat{x}) \cap D$ then $P_d \in S^+(\hat{x}_1)$. Then the thesis immediately follows from (5.47). $\qquad \square$

PROOF OF THEOREM 5.6. We now finally restrict ourselves to $N = 2, 3$, $m \geq 2$ and $k_0 \geq 1$ such that $k_0 R_0 \geq z_0 \geq 4$. We let $k \geq k_0$ and, first of all, by (5.22), we recall that we can estimate $\tilde{M}$ as follows

$$(5.48) \qquad\qquad \tilde{M} \leq C_1 k$$

where C_1 depends on N, m, β, R_0 and δ_0 only.

Then we proceed in the following manner. For any direction $\hat{x} \in \mathbb{S}^{N-1}$, the previous Lemma 5.8 allows us to estimate

$$\int_{B_{\rho_1} \cap S^+(\hat{x})} |v|^2 = \int_{S^+(\hat{x})} |v|^2(x) \chi_{B_{\rho_1}}(x) \mathrm{d}x \leq C_4^2 \eta_2^2(\eta_1, k, \tilde{M})$$

where χ denotes characteristic functions. Therefore

$$\int_{\mathbb{S}^{N-1}} \left(\int_{S^+(\hat{x})} |v|^2(x) \chi_{B_{\rho_1}}(x) \mathrm{d}x \right) \mathrm{d}\mathcal{H}^{N-1}(\hat{x}) \leq \mathcal{H}^{N-1}(\mathbb{S}^{N-1}) C_4^2 \eta_2^2(\eta_1, k, \tilde{M}).$$

But, by Fubini's Theorem,

$$\int_{\mathbb{S}^{N-1}} \left(\int_{S^+(\hat{x})} |v|^2(x) \chi_{B_{\rho_1}}(x) \mathrm{d}x \right) \mathrm{d}\mathcal{H}^{N-1}(\hat{x}) =$$

$$\int_{\mathbb{S}^{N-1}} \left(\int_{\mathbb{R}^N} |v|^2(x) \chi_{B_{\rho_1}}(x) \chi_{S^+(\hat{x})}(x) \mathrm{d}x \right) \mathrm{d}\mathcal{H}^{N-1}(\hat{x}) =$$

$$\int_{\mathbb{R}^N} |v|^2(x) \chi_{B_{\rho_1}}(x) \left(\int_{\mathbb{S}^{N-1}} \chi_{S^+(\hat{x})}(x) \mathrm{d}\mathcal{H}^{N-1}(\hat{x}) \right) \mathrm{d}x =$$

$$\int_{B_{\rho_1}} |v|^2(x) f(x) \mathrm{d}x = \int_{B_{\rho_1} \setminus K} |v|^2(x) f(x) \mathrm{d}x$$

where for any $x \in \mathbb{R}^N$

$$f(x) = \int_{\mathbb{S}^{N-1}} \chi_{S^+(\hat{x})}(x) \mathrm{d}\mathcal{H}^{N-1}(\hat{x})$$

and we used the fact that $f(x) = 0$ for any $x \in K$.

Then let us fix a constant μ, $0 < \mu \leq 1$. By a repeated use of Hölder inequality we have

$$(5.49) \qquad \int_{B_{\rho_1} \setminus K} |v| \leq \left(\int_{B_{\rho_1} \setminus K} |v|^2 f \right)^{a_1/2} \left(\int_{B_{\rho_1} \setminus K} |v|^2 \right)^{a_2/2} \left(\int_{B_{\rho_1} \setminus K} f^{-\gamma} \right)^{1/2}$$

where $0 < \gamma \leq 1$, $a_1 + a_2 = 1$ and their are given by the following formulas

$$(5.50) \qquad\qquad \gamma = \frac{\mu}{2 - \mu}, \quad a_1 = \frac{\mu}{2 - \mu}, \quad a_2 = \frac{2(1 - \mu)}{2 - \mu}.$$

The crucial remark is the following. For any $0 < \mu < 1$ we have that $\gamma = \mu/(2 - \mu) < 1$ and there exists a constant $F(\mu)$, depending on μ, N, m, β, R_0 and δ_0 only, such that

$$(5.51) \qquad\qquad \left(\int_{B_{\rho_1} \setminus K} f^{-\gamma} \right)^{1/2} \leq F(\mu).$$

We have that (5.51) follows from Lemma 5.9 and this construction. We integrate in spherical coordinates

$$(5.52) \quad \int_{B_{\rho_1} \backslash K} f^{-\gamma} = \int_{\mathbb{S}^{N-1}} \left(\int_{s_0(\hat{x})}^{\rho_1} f^{-\gamma}(s\hat{x}) s^{N-1} ds \right) d\mathcal{H}^{N-1}(\hat{x}) \leq$$

$$\int_{\mathbb{S}^{N-1}} \left(\int_0^{\rho_1 - s_0(\hat{x})} (E_0 \min\{s^{(N-1)/2}, 1\})^{-\gamma} (s_0(\hat{x}) + s)^{N-1} ds \right) d\mathcal{H}^{N-1}(\hat{x}) \leq F(\mu)^2.$$

Let us note that, for $N = 2$, $F(1)$ is also bounded therefore we may allow $\mu = 1$, hence $\gamma = 1$, $a_1 = 1$, and $a_2 = 0$.

Then, setting $\tilde{C}_1 = C_2$, $\tilde{C}_2 = C_3$ and $\tilde{C}_3 = C_4(\mathcal{H}^{N-1}(\mathbb{S}^{N-1}))^{1/2}$, and by Lemma 5.8, we may conclude that

$$(5.53) \quad \int_{B_{\rho_1} \backslash K} |v| \leq \begin{cases} F(1)\tilde{C}_3 \eta_2(\eta_1, k, \tilde{M}) & N = 2 \\ \\ F(\mu)\tilde{C}_3^{a_1} \tilde{M}^{a_2} (\eta_2(\eta_1, k, \tilde{M}))^{a_1} & N = 3, \ 0 < \mu < 1. \end{cases}$$

Thus we have finally proved our result. □

6 Instability for the inverse scattering problem

In this section we fix $N = 2, 3$. We also fix an integer $m \geq 2$ and positive constants β and R_0. We fix $\delta_0 > 0$ as defined in Proposition 2.1 and we set $R = R_0 + \delta_0$. We fix a positive constant k_0 and we denote

$$(6.1) \qquad\qquad \tilde{I}_N = \begin{cases} [k_0, +\infty) & \text{if } N = 2, \\ (0, +\infty) & \text{if } N = 3. \end{cases}$$

We fix $\Sigma \in X(m, \beta, R_0, \delta_0)$ and we consider its far-field pattern $\mathcal{A}(\Sigma)$ and its decomposition in spherical harmonics. Then, by (2.17) and (2.18) we have, for any $\omega \in \mathbb{S}^{N-1}$, any $k > 0$, any index i, and any $r > R$

$$|\tilde{b}_i(\omega, k)| \leq \sqrt{\frac{2}{\pi}} k^{-(N-1)/2} \frac{(kr)^{(N-2)/2}}{|H_{\gamma(v_i)+(N-2)/2}^{(1)}(kr)|} \left| \int_{\mathbb{S}^{N-1}} u^s(r\hat{x}; \omega, k, \Sigma) v_i(\hat{x}) d\hat{x} \right|.$$

Hence, by collecting the estimates of the previous section, we have for any $\omega \in \mathbb{S}^{N-1}$, any $k \in \tilde{I}_N$, and any index i

$$|\tilde{b}_i(\omega, k)| \leq C_1 (kr)^{-1/2} \frac{1}{|H_{\gamma(v_i)+(N-2)/2}^{(1)}(kr)|} \max\{1, k^{(N-1)/2}\}, \quad \text{for any } r \geq R + 1,$$

where C_1 depends on N, m, β, R_0, R and, only if $N = 2$, k_0. We call $\tilde{R} = R + 1$. If $k \leq 1/\tilde{R}$ we choose $r = 1/k$ and we obtain

$$|\tilde{b}_i(\omega, k)| \leq \frac{C_1}{|H_{\gamma(v_i)+(N-2)/2}^{(1)}(1)|}, \quad \text{for } k \in \tilde{I}_N, \ k \leq 1/\tilde{R}.$$

If $k \geq 1/\tilde{R}$ we choose $r = \tilde{R}$ and we obtain

$$|\tilde{b}_i(\omega, k)| \leq C_1 \frac{(k\tilde{R})^{(N-2)/2}}{|H^{(1)}_{\gamma(v_i)+(N-2)/2}(k\tilde{R})|}, \quad \text{for } k \in \tilde{I}_N, \ k \geq 1/\tilde{R}.$$

Finally, if we set $z(k) = \max\{1, k\tilde{R}\}$, we have for any $\omega \in \mathbb{S}^{N-1}$ and any index i

$$(6.2) \qquad |\tilde{b}_i(\omega, k)| \leq C_1 \frac{(z(k))^{(N-2)/2}}{|H^{(1)}_{\gamma(v_i)+(N-2)/2}(z(k))|}, \quad \text{for } k \in \tilde{I}_N.$$

On the other hand, we also have for any $\omega \in \mathbb{S}^{N-1}$ and any index i

$$(6.3) \qquad |\tilde{b}_i(\omega, k)| \leq C_2, \quad \text{for } k \in \tilde{I}_N$$

where C_2 depends on N, m, β, R_0, R and, only if $N = 2$, k_0.

Then we use Corollary 3.8 and Theorem 3.9 to obtain the following result. For any $k \in \tilde{I}_N$, and any arbitrary $\omega \in \mathbb{S}^{N-1}$, we have two different cases. For any index i such that $\gamma(v_i) \geq \mathrm{e}z(k)$ we have

$$(6.4) \qquad |\tilde{b}_i(\omega, k)| \leq C_3(z(k))^{(N-1)/2} \left(\frac{a\nu(i)}{\mathrm{e}z(k)}\right)^{-(\nu(i)-1/2)}$$

where $\nu(i) = \gamma(v_i) + (N-2)/2$, $a = 1 + \sqrt{\mathrm{e}^2 - 1}/\mathrm{e}$ and C_3 depends on N, m, β, R_0, R and, only if $N = 2$, k_0. Let us also notice that $\nu(i) \geq \mathrm{e}$, therefore $\nu(i) - 1/2 \geq 2$ and, obviously, $a\nu(i)/(\mathrm{e}z(k)) \geq a > 1$.

We note that there exists a constant $\tilde{c} \geq 1$, depending on N only, such that for any $z \geq 1$ and any t such that $t \geq \tilde{c}\mathrm{e}z$ we have

$$z^{(N-1)/2} \left(a(t + (N-2)/2)/(\mathrm{e}z)\right)^{-(t+(N-2)/2-1/2)} \leq z^{(N-1)/2} \left(at/(\mathrm{e}z)\right)^{-(t+(N-2)/2-1/2)} \leq 1.$$

We notice that, since $\mathrm{e}\log(a) > 1$, for $N = 2, 3$ we may actually choose $\tilde{c} = 1$. On the other hand, we recall that for any index i we have (6.3).

Let $\tilde{C} = (\mathcal{H}^{N-1}(\mathbb{S}^{N-1})^{1/2} \max\{C_2, C_3\}$. Obviously, $\tilde{C}$ depends on N, m, β, R_0, R and, only if $N = 2$, k_0. Without loss of generality we may assume that $\tilde{C} \geq 2$.

By the reciprocity relation (2.4) we conclude that for any $k \in \tilde{I}_N$, and for any indexes i, l, we have

$$(6.5) \qquad |b_{i,l}(k)| \leq \tilde{C}$$

and for any $k \in \tilde{I}_N$, and for any indexes i, l such that $\max\{\gamma(v_i), \gamma(v_l)\} \geq \tilde{c}\mathrm{e}z(k)$, we have

$$(6.6) \qquad |b_{i,l}(k)| \leq \tilde{C}(z(k))^{(N-1)/2} \left(\frac{a\max\{\gamma(v_i), \gamma(v_l)\}}{\mathrm{e}z(k)}\right)^{-(\max\{\gamma(v_i), \gamma(v_l)\}+(N-3)/2)} \leq \tilde{C}.$$

For any $\Sigma \in X(m, \beta, R_0, \delta_0)$ and any $k \in \tilde{I}_N$, we have $\mathcal{A}(\Sigma)(\cdot, \cdot, k) \in Y_s(\mathbb{S}^{N-1} \times \mathbb{S}^{N-1})$ for any $s \geq 0$. We also recall that, again for any $s \geq 0$,

$$\|\mathcal{A}(\Sigma)(\cdot, \cdot, k)\|_{H^s} = \|\mathcal{A}(\Sigma)(\cdot, \cdot, k)\|_{H^s(\mathbb{S}^{N-1} \times \mathbb{S}^{N-1})} \leq C_4 \|\mathcal{A}(\Sigma)(\cdot, \cdot, k)\|_s$$

where $C_4 = 4$.

For any fixed $k \in \tilde{I}_N$, we denote the set

$$\tilde{Y}(k) = \{\mathcal{A}(\Sigma)(\cdot, \cdot, k) : \ \Sigma \in X(m, \beta, R_0, \delta_0)\}.$$

We notice that for any fixed $s \geq 0$, $\tilde{Y}(k) \subset H^s(\mathbb{S}^{N-1} \times \mathbb{S}^{N-1})$ and it may be considered as a metric space endowed with the distance induced by the H^s norm. We recall that, for a given positive ε, a subset $\tilde{Y}' \subset \tilde{Y}(k)$ is an ε-net for $\tilde{Y}(k)$, with respect to the H^s norm, if for every $y \in \tilde{Y}(k)$ there exists $y' \in \tilde{Y}'$ whose H^s distance from y is less than or equal to ε.

Before stating the main instability theorem, let us introduce the following notation. We begin by noticing that for any t such that $t \geq \max\{\tilde{c}ez(k), 2s + N\}$ we have that

$$f(t) = (1 + t)^{2s+N-1/2} \left(\frac{at}{ez(k)} \right)^{-(t+(N-3)/2)}$$

is a decreasing function of t. We call

(6.7) $$\tilde{\varepsilon}(k) = 2C_4 \tilde{C}(z(k))^{(N-1)/2} f(\max\{\tilde{c}e^2, 4s + (3N/2) + 1\}z(k)).$$

Let us notice that $\tilde{\varepsilon}(k)$ is a decreasing function of k which goes to 0 as $k \to \infty$. Finally, we denote

(6.8) $$\tilde{B}(s) = \max\{\tilde{c}e^2, 4s + (3N/2) + 1\} \quad Z(k) = \tilde{B}(s)\max\{1, (R+1)k\}$$
$$C_5 = (2C_4)(2\tilde{C} + 1).$$

We observe that $\tilde{B}(s)$ depends on s and N only, whereas C_5 depends on N, m, β, R_0, R and, only if $N = 2$, k_0.

Theorem 6.1 *Fixed $s \geq 0$ and $k \in \tilde{I}_N$, for every ε, $0 < \varepsilon < 1/\mathrm{e}$, there exists $\delta = \delta(\varepsilon, k)$, $0 < \delta \leq \delta_0$, and two obstacles Σ_1 and Σ_2 belonging to $X(m, \beta, R_0, \delta)$ such that*

(6.9) $$d_H(\Sigma_1, \Sigma_2) \geq \delta \quad and \quad \|\mathcal{A}(\Sigma_1)(\cdot, \cdot, k) - \mathcal{A}(\Sigma_2)(\cdot, \cdot, k)\|_{H^s(\mathbb{S}^{N-1} \times \mathbb{S}^{N-1})} \leq 2\varepsilon.$$

If $\varepsilon \geq \tilde{\varepsilon}(k)$, then

(6.10) $$\delta(\varepsilon, k) = \delta_0 \frac{2^{-m(N+3)/(N-1)}}{(1 + Z(k))^{2m}} \left[\log \left(C_5 (1 + Z(k))^{(2s+N-1/2)} / \varepsilon \right) \right]^{-m/(N-1)},$$

hence

(6.11) $$d_H(\Sigma_1, \Sigma_2) \geq$$
$$\delta_0 \frac{2^{-m(N+3)/(N-1)}}{(1 + Z(k))^{2m}} \left[\log \left(\frac{2C_5(1 + Z(k))^{(2s+N-1/2)}}{\|\mathcal{A}(\Sigma_1)(\cdot, \cdot, k) - \mathcal{A}(\Sigma_2)(\cdot, \cdot, k)\|_{H^s}} \right) \right]^{-m/(N-1)}.$$

If $0 < \varepsilon < \tilde{\varepsilon}(k)$, then

(6.12) $$\delta(\varepsilon, k) = \delta_0 \frac{2^{-m(N+3)/(N-1)}}{(1 + \tilde{t})^{2m}} \left[\log \left(C_5 (1 + \tilde{t})^{(2s+N-1/2)} / \varepsilon \right) \right]^{-m/(N-1)}$$

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hence

(6.13) $\quad d_H(\Sigma_1, \Sigma_2) \geq$

$$\delta_0 \frac{2^{-m(N+3)/(N-1)}}{(1+\tilde{t})^{2m}} \left[\log \left(\frac{2C_5(1+\tilde{t})^{(2s+N-1/2)}}{\|\mathcal{A}(\Sigma_1)(\cdot, \cdot, k) - \mathcal{A}(\Sigma_2)(\cdot, \cdot, k)\|_{H^s}} \right) \right]^{-m/(N-1)},$$

where $\tilde{t} > Z(k)$ satisfies

(6.14) $$2C_4\tilde{C}(z(k))^{(N-1)/2}f(\tilde{t}) = \varepsilon.$$

PROOF. Let us then fix $s \geq 0$, $k \in \tilde{I}_N$, and ε, $0 < \varepsilon < 1/e$. The crucial step is constructing an ε-net for $\tilde{Y}(k)$, with respect to the H^s norm, and estimating its number of elements. We distinguish two regimes. First we treat the case when $\varepsilon \geq \tilde{\varepsilon}(k)$, then we shall deal with the case $0 < \varepsilon < \tilde{\varepsilon}(k)$.

If $\varepsilon \geq \tilde{\varepsilon}(k)$, for any integer n such that $n \geq Z(k)$ we have

$$2C_4\tilde{C}(z(k))^{(N-1)/2}f(n) \leq \tilde{\varepsilon}(k) \leq \varepsilon.$$

Let $\tilde{n}$ be the integer part of $Z(k)$. Let $\varepsilon' = (1+\tilde{n})^{-(2s+N-1/2)}\varepsilon/(2C_4)$ and $\Psi_\varepsilon = [-\tilde{C}, \tilde{C}] \cap \varepsilon'\mathbb{Z}$. We remark that Ψ_ε is a finite subset of $\mathbb{R}$ and we have that $\#\Psi_\varepsilon \leq (2\tilde{C}+1)/\varepsilon'$.

Let us define the following subset of $L^2(\mathbb{S}^{N-1} \times \mathbb{S}^{N-1})$

$$\hat{Y}(\varepsilon) = \{g \in L^2(\mathbb{S}^{N-1} \times \mathbb{S}^{N-1}) : a_{i,l} \in \Psi_\varepsilon \text{ if } \max\{\gamma(v_i), \gamma(v_l)\} \leq \tilde{n} \text{ and } a_{i,l} = 0 \text{ otherwise}\}.$$

We may count the number of elements of $\hat{Y}(\varepsilon)$ as follows. If we set

$$s = \#\{(i,l) : \max\{\gamma(v_i), \gamma(v_l)\} \leq \tilde{n}\}$$

we obtain that

$$s \leq 4(1+\tilde{n})^{2N-2}.$$

Then we have that $\#\hat{Y}(\varepsilon) = (\#\Psi_\varepsilon)^s$ and hence

$$\#\hat{Y}(\varepsilon) \leq ((2\tilde{C}+1)/\varepsilon')^s \leq \left((2C_4)(2\tilde{C}+1)(1+\tilde{n})^{(2s+N-1/2)}/\varepsilon \right)^s \leq$$

$$\left(C_5(1+\tilde{n})^{(2s+N-1/2)}/\varepsilon \right)^{4(1+\tilde{n})^{2N-2}}.$$

It is now easy to construct an ε-net for $\tilde{Y}(k)$ with respect to the H^s norm with at most

$$\exp \left(4(1+Z(k))^{2N-2} \log \left(C_5(1+Z(k))^{(2s+N-1/2)}/\varepsilon \right) \right)$$

elements, see for instance the proof of Lemma 2.3 in [11].

We conclude the proof of the theorem in the first regime. Let us assume that there exists δ, $0 < \delta \leq \delta_0$, such that

$$\exp(2^{-N}(\delta_0/\delta)^{(N-1)/m}) > \exp \left(4(1+Z(k))^{2N-2} \log \left(C_5(1+Z(k))^{(2s+N-1/2)}/\varepsilon \right) \right).$$

Then, there exist two elements Σ_1 and Σ_2 of $X(m, \beta, R_0, \delta)$ satisfying (6.9). This is true if

$$2^{-N}(\delta_0/\delta)^{(N-1)/m} > 4(1 + Z(k))^{2N-2} \log\left(C_5(1 + Z(k))^{(2s+N-1/2)}/\varepsilon\right)$$

that is, for instance, when δ is given by (6.10). Therefore, also (6.11) immediately follows and the theorem is fully proved in the first regime.

Let us now consider the second regime, that is when $0 < \varepsilon < \tilde{\varepsilon}(k)$. Let $\tilde{t} > Z(k)$ be such that $2C_4\tilde{C}(z(k))^{(N-1)/2}f(\tilde{t}) = \varepsilon$. Then we repeat exactly the same procedure just by replacing $\tilde{n}$ with the integer part of $\tilde{t}$ and the proof is concluded. $\qquad\square$

We conclude this section with a few comments on the results contained in the previous instability theorem. First of all we make estimates (6.12) and (6.13) more readable by estimating in a suitable way $\tilde{t}$. In the theorem, we are assuming the wavenumber k fixed and we are establishing how the instability changes with respect to the error ε. However, in order to understand the high frequency asymptotics, we then consider ε to be fixed and discuss the changes in the instability as k increases.

We being with the first remark. We notice that, since $\varepsilon < \tilde{\varepsilon}(k)$, we have $\tilde{t} \geq Z(k)$ and consequently

$$f(\tilde{t}) \leq f_1(\tilde{t}) = \left(\frac{3ez(k)}{2a}\right)^{2s+N-1/2}\left(\frac{a\tilde{t}}{ez(k)}\right)^{-\tilde{t}/2}.$$

It is enough to find $\hat{t}$ such that

$$2C_4\tilde{C}(z(k))^{(N-1)/2}f_1(\hat{t}) = \varepsilon$$

to deduce that $Z(k) \leq \tilde{t} \leq \hat{t}$. Hence our result holds true if we replace $\tilde{t}$ with $\hat{t}$ in (6.12) and (6.13). Finally, a straightforward computation shows that

$$\hat{t} \leq 2\log\left(\frac{\tilde{b}(k, s)}{\varepsilon}\right)$$

where

(6.15) $$\tilde{b}(k, s) = 2C_4\tilde{C}(3e/(2a))^{2s+N-1/2}(z(k))^{2s+(3N/2)-1}.$$

Let us now notice that

(6.16) $$C_5(1 + \hat{t})^{2s+N-1/2}/\varepsilon = \frac{2\tilde{C} + 1}{\tilde{C}(z(k))^{(N-1)/2}}\left(\frac{a\tilde{t}}{ez(k)}\right)^{\hat{t}+(N-3)/2}.$$

We deduce that

$$C_5(1 + \hat{t})^{2s+N-1/2}/\varepsilon \leq \frac{2\tilde{C} + 1}{\tilde{C}(z(k))^{(N-1)/2}}\left(\frac{a\hat{t}}{ez(k)}\right)^{(8/3)(\hat{t}/2)} = \frac{2\tilde{C} + 1}{\tilde{C}(z(k))^{(N-1)/2}}\left(\frac{\tilde{b}(k, s)}{\varepsilon}\right)^{8/3}.$$

Since $\tilde{C} \geq 2$ and $z(k) \geq 1$ we conclude that we can replace $\delta(\varepsilon, k)$ in (6.12) by

$$\delta = \frac{\delta_0}{2^{m(N+3)/(N-1)}}\left(1 + (8/3)\log\left(\frac{\tilde{b}(k, s)}{\varepsilon}\right)\right)^{-2m-m/(N-1)}$$

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and (6.13) may be replaced by

$$d_H(\Sigma_1, \Sigma_2) \geq$$

$$\frac{\delta_0}{2^{m(N+3)/(N-1)}} \left(1 + (8/3) \log \left(\frac{2\tilde{b}(k,s)}{\|\mathcal{A}(\Sigma_1)(\cdot,\cdot,k) - \mathcal{A}(\Sigma_2)(\cdot,\cdot,k)\|_{H^s}} \right) \right)^{-2m-m/(N-1)}$$

where $\tilde{b}(k,s)$ is as in (6.15).

Let us now consider a fixed ε, $0 < \varepsilon < 1/e$. Let us consider that for some $k \in \tilde{I}_N$ we have $\tilde{\varepsilon}(k) > \varepsilon$, otherwise we have that $\varepsilon \geq \tilde{\varepsilon}(k)$ for any $k \in \tilde{I}_N$ and it is easy to understand the instability behavior since we are always in the first regime and (6.10) and (6.11) apply.

Then let $k(\varepsilon) \in \tilde{I}_N$ be the first $k \in \tilde{I}_N$ such that $\tilde{\varepsilon}(k(\varepsilon)) = \varepsilon$. An easy computation shows that $k(\varepsilon)$ grows essentially like a constant times $\log(1/\varepsilon)$. As long as $k < k(\varepsilon)$ we have that the improvement in the instability as k increases is not very big since we need to use (6.12) and (6.13). Let us notice that as $k < k(\varepsilon)$ we have $Z(k) < \tilde{t} \leq Z(k(\varepsilon))$, therefore as $k < k(\varepsilon)$ increases and converges to $k(\varepsilon)$ we have that $\tilde{t}$ increases and converges to $Z(k(\varepsilon))$. From $k(\varepsilon)$ onwards, that is in the very high frequencies regime, (6.10) and (6.11) apply and the improvement in the instability is more evident. We state these observations in this final corollary.

Corollary 6.2 *Let us fix ε, $0 < \varepsilon < 1/e$. Let us assume that for some $k \in \tilde{I}_N$ we have $\tilde{\varepsilon}(k) > \varepsilon$ and let $k(\varepsilon) \in \tilde{I}_N$ be the first $k \in \tilde{I}_N$ such that $\tilde{\varepsilon}(k(\varepsilon)) = \varepsilon$.*

If $k < k(\varepsilon)$, then

$$\delta = \frac{\delta_0}{2^{m(N+3)/(N-1)}} \left(1 + (8/3) \log \left(\frac{\tilde{b}(k,s)}{\varepsilon} \right) \right)^{-2m-m/(N-1)}$$

hence

$$d_H(\Sigma_1, \Sigma_2) \geq$$

$$\frac{\delta_0}{2^{m(N+3)/(N-1)}} \left(1 + (8/3) \log \left(\frac{2\tilde{b}(k,s)}{\|\mathcal{A}(\Sigma_1)(\cdot,\cdot,k) - \mathcal{A}(\Sigma_2)(\cdot,\cdot,k)\|_{H^s}} \right) \right)^{-2m-m/(N-1)}$$

where $\tilde{b}(k,s)$ is as in (6.15).

If $k \geq k(\varepsilon)$, then

$$\delta = \delta_0 \frac{2^{-m(N+3)/(N-1)}}{(1 + Z(k))^{2m}} \left[\log \left(C_5 (1 + Z(k))^{(2s+N-1/2)} / \varepsilon \right) \right]^{-m/(N-1)},$$

hence

$$d_H(\Sigma_1, \Sigma_2) \geq$$

$$\delta_0 \frac{2^{-m(N+3)/(N-1)}}{(1 + Z(k))^{2m}} \left[\log \left(\frac{2 C_5 (1 + Z(k))^{(2s+N-1/2)}}{\|\mathcal{A}(\Sigma_1)(\cdot,\cdot,k) - \mathcal{A}(\Sigma_2)(\cdot,\cdot,k)\|_{H^s}} \right) \right]^{-m/(N-1)} .$$

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