The Dirichlet problem for a prescribed anisotropic mean curvature equation: existence, uniqueness and regularity of solutions *

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Abstract

We discuss existence, uniqueness, regularity and boundary behaviour of solutions of the Dirichlet problem for the prescribed anisotropic mean curvature equation

$$-\mathrm{div}\left(\nabla u/\sqrt{1+|\nabla u|^2}\right) = -au + b/\sqrt{1+|\nabla u|^2},$$

where $a, b > 0$ are given parameters and Ω is a bounded Lipschitz domain in $\mathbb{R}^N$. This equation appears in the modeling theory of capillarity phenomena for compressible fluids and in the description of the geometry of the human cornea.

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1 Introduction

This paper is devoted to the study of existence, uniqueness, regularity and boundary behaviour of the solutions of the Dirichlet problem for the quasilinear elliptic equation

$$-\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = -au + \frac{b}{\sqrt{1+|\nabla u|^2}} \quad \text{in } \Omega. \tag{1.1}$$

We throughout suppose that $a > 0$ and $b > 0$ are given constants and Ω is a bounded domain in $\mathbb{R}^N$, with $N \geq 2$, having a Lipschitz boundary $\partial\Omega$. We remark that the case $N = 1$ has been treated in [12]. Equation (1.1) is a particular case of the prescribed anisotropic mean curvature equation

$$-\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = NH(x, u, \mathcal{N}(u)) \quad \text{in } \Omega,$$

where $H : \Omega \times \mathbb{R} \times \mathbb{R}^{N+1} \to \mathbb{R}$ is the prescribed mean curvature and $\mathcal{N}(u) = \frac{(-\nabla u, 1)}{\sqrt{1+|\nabla u|^2}}$ is the unit upper normal to the graph of u in $\mathbb{R}^{N+1}$.

Equation (1.1) has been introduced for modeling capillarity phenomena for compressible fluids, if supplemented with non-homogeneous conormal boundary conditions [18, 19, 5, 20, 4], or for describing the geometry of the human cornea, if supplemented with homogeneous Dirichlet boundary conditions [46, 47, 48, 52, 51, 50]. We refer to these papers for the derivation of the model, further discussion on the subject and an additional bibliography. Concerning the homogeneous Dirichlet problem associated with (1.1), it should be pointed out that in [46, 47, 48, 52, 51] a simplified version of (1.1) has been investigated, where the curvature operator

$$\mathrm{div}\big(\nabla u/\sqrt{1+|\nabla u|^2}\big)$$

is replaced by its linearization around 0

$$\mathrm{div}(\nabla u) = \Delta u$$

and, furthermore, Ω is supposed to be an interval in $\mathbb{R}$, or a disk in $\mathbb{R}^2$. In two previous papers [12, 13] we have instead considered the complete model (1.1) and have proved the existence of a unique classical solution for any given choice of the positive parameters a, b, but still assuming that Ω is an interval in $\mathbb{R}$, or a ball in $\mathbb{R}^N$. Some numerical experiments for approximating the solution of the 1-dimensional problem have been performed in [12, 50].

Here we wish to investigate the solvability of the homogeneous Dirichlet problem for equation (1.1), in the case of an arbitrary Lipschitz domain Ω in $\mathbb{R}^N$. Besides the interest that this study may have in view of the cited application, it appears to be challenging also from the purely mathematical point of view. Indeed, it is a well-known fact that the solvability in the classical sense of the, possibly non-homogeneous, Dirichlet problem for the prescribed mean curvature equation

$$-\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = NH(x) \quad \text{in } \Omega, \tag{1.2}$$

as well as for the capillarity equation

$$-\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = -au \quad \text{in } \Omega, \tag{1.3}$$

with $a > 0$, is intimately related to the geometric properties of $\partial\Omega$. In [53] J. Serrin established a basic criterion for the solvability of the Dirichlet problem for (1.2) and (1.3): a mean convexity assumption on $\partial\Omega$, introduced in [32, 53], was shown to be sufficient, and in a suitable sense also necessary, for the existence of a classical solution. In [53, p. 480] J. Serrin also emphasized "the delicacy of the situation when any but the simplest equations are treated".

When applying these ideas to the homogeneous Dirichlet problem for (1.1), they yield its solvability assuming a smallness condition on the coefficient b and a version of the Serrin's mean convexity condition on $\partial\Omega$: see, respectively, assumptions (2) and (3) in [39]. In [7, Remark 1] it was stated, yet without an explicit proof, that using the methods of [6] the mean convexity assumption might be suitably relaxed, allowing boundary points with negative mean curvature, at the expense however of requiring some smallness conditions both on the coefficients of the equation and on the size of the domain. We also refer to [30, 29, 31] and to the papers cited therein for further recent studies on the existence and the boundary behaviour of solutions of the Dirichlet problem for the prescribed mean curvature equation (1.2) in case the Serrin's condition is not satisfied.

In the light of this discussion our aim here is twofold. At first we provide with Theorem 1.1 a rather broad existence and uniqueness result in a suitable class of generalized solutions, without placing any additional condition either on the coefficients, or on the domain. Since in such a general setting we cannot expect to find classical solutions, we next introduce in Theorem 1.2 an explicit quantitative condition, which relates the coefficients of the equation with the geometry of the domain and guarantees that the solution previously obtained attains the homogeneous Dirichlet boundary values classically, even at points where the Serrin's mean convexity assumption fails.

To accomplish this program we must face the problem of introducing an appropriate notion of generalized solution. Following some ideas which trace back to some works of the seventies by A. Lichnewsky and R. Temam, or respectively by E. Giusti and M. Miranda, dealing with the prescribed mean curvature equation, we might define a solution as a minimizer of some related convex action functional; such solutions have been referred to as "pseudo-solutions" in [54, 15, 35, 36, 37, 38], or respectively as "generalized solutions" in [40, 26, 27, 41]. Yet, although (1.1) has a variational structure, the introduction of the associated action functional, which involves an anisotropic area term, does not appear very direct and the corresponding concepts of "pseudo-solution" and of "generalized solution" not very transparent. Therefore we prefer to adopt in our context an equivalent notion of solution, which looks more in the spirit of classical solutions and has in our opinion a more intuitive geometric interpretation. It is worthy to point out at this stage that our definition of solution is somehow implicit in the work of A. Lichnewsky [37], concerning the minimal surface equation. Indeed, in [37, Proposition 4] the author introduces a concept of lower and upper solutions that precisely yields our notion of solution for any function that is simultaneously a lower and an upper solution of the problem.

Definition 1.1. *A solution of the Dirichlet problem*

$$\begin{cases} -\operatorname{div}\left(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = -au + \dfrac{b}{\sqrt{1+|\nabla u|^2}} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.4}$$

is a function $u \in W^{1,1}(\Omega)$ such that

- $\operatorname{div}\Big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) \in L^N(\Omega)$*;*
- *u satisfies the equation in (1.4) a.e. in Ω;*
- *for $\mathcal{H}^{N-1}$-a.e. $x \in \partial\Omega$,*
  - *either $u(x) = 0$,*

– *or* $u(x) > 0$ *and* $\left[\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}, \nu\right](x) = -1$,

– *or* $u(x) < 0$ *and* $\left[\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}, \nu\right](x) = 1$,

where $\left[\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}, \nu\right] \in L^\infty(\partial\Omega)$ *denotes the weakly defined trace on* $\partial\Omega$ *of the component of* $\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}$ *with respect to the unit outer normal* ν *to* Ω.

Remark 1.1 Assuming that $u \in W^{1,1}(\Omega)$ is such that $\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) \in L^N(\Omega)$ and satisfies the equation in (1.4) a.e. in Ω is equivalent to requiring that $u \in W^{1,1}(\Omega) \cap L^N(\Omega)$ and is a distributional solution of the equation in (1.4). Note that, according to [3], the vector field $\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}$ belongs to the space $X(\Omega)_N$ and thus the weak trace $\left[\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}, \nu\right]$ on $\partial\Omega$ of the component of $\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}$ with respect to the unit outer normal ν to Ω is defined.

Remark 1.2 The concept of solution expressed by Definition 1.1 looks rather natural in this context and can heuristically be interpreted as follows: the solution u is not required to satisfy the homogeneous Dirichlet boundary condition at all points of $\partial\Omega$, but at any point of $\partial\Omega$ where the zero boundary value is not attained the unit upper normal $\mathcal{N}(u)$ to the graph of u equals the unit outer normal $(\nu, 0)$ or the unit inner normal $(-\nu, 0)$, according to the sign of u; in this case, roughly speaking, the graph of the solution might be smoothly continued by vertical segments up to the zero level. This kind of boundary behaviour for solutions of the N-dimensional prescribed mean curvature equation has already been observed and discussed in [15, 37, 26, 27, 41]; more recently, but limited to dimension $N = 1$, it has been considered in [8, 9, 44, 49, 45].

With reference to Definition 1.1 we prove the following existence, uniqueness and regularity result.

Theorem 1.1. *Let* $a, b > 0$ *be given and let* Ω *be a bounded domain in* $\mathbb{R}^N$, *with* $N \geq 2$, *having a Lipschitz boundary* $\partial\Omega$. *Then problem* (1.4) *has a unique solution* u, *which also satisfies*

(*i*) $u \in C^\infty(\Omega)$;

(*ii*) *the set of points* $x_0 \in \partial\Omega$, *where* u *is continuous and satisfies* $u(x_0) = 0$, *is non-empty;*

(*iii*) $u \in L^\infty(\Omega)$ *and* $0 < u(x) < b/a$ *for all* $x \in \Omega$;

(*iv*) u *minimizes in* $W^{1,1}(\Omega) \cap L^\infty(\Omega)$ *the functional*

$$\int_\Omega e^{-bz}\sqrt{1+|\nabla z|^2}\,dx - \frac{a}{b}\int_\Omega e^{-bz}\big(z + \tfrac{1}{b}\big)\,dx + \frac{1}{b}\int_{\partial\Omega} |e^{-bz} - 1|\,d\mathcal{H}^{N-1}.$$

The next theorem guarantees that the solution previously obtained attains the homogeneous Dirichlet boundary values provided that Ω satisfies an exterior sphere condition, in which the radius of the sphere is bounded from below by a constant depending on the coefficients a, b and the dimension N. The notion of exterior sphere condition we use is as follows.

Definition 1.2. *We say that an open set* $\Omega \subseteq \mathbb{R}^N$ *satisfies an exterior sphere condition with radius* $r > 0$ *at some point* $x_0 \in \partial\Omega$, *if there exists a point* $y \in \mathbb{R}^N$ *such that, denoting by* $B(y, r)$ *the open ball of center* y *and radius* r, *there hold*

$$B(y,r) \cap \Omega = \emptyset \quad \textit{and} \quad x_0 \in \overline{B(y,r)} \cap \partial\Omega.$$

Remark 1.3 It is fairly evident that the exterior sphere condition does not imply the above mentioned Serrin's mean convexity assumption, as it permits that all principal curvatures be negative.

Theorem 1.2. *Suppose that all assumptions of Theorem 1.1 hold and let u be the solution of* (1.4)*, whose existence is guaranteed by Theorem 1.1. Then u is continuous at x_0 and satisfies $u(x_0) = 0$ at any point $x_0 \in \partial\Omega$ where an exterior sphere condition holds with radius $r \geq (N-1)\, b/a$. Moreover, if $r > (N-1)\, b/a$, then u also satisfies a bounded slope condition at x_0, that is $\sup_{x\in\Omega} \dfrac{u(x)}{|x - x_0|} < +\infty$. In particular, if an exterior sphere condition with radius $r \geq (N-1)\, b/a$ is satisfied at every point $x_0 \in \partial\Omega$, then $u \in C^\infty(\Omega) \cap C(\overline{\Omega})$ and it is a classical solution of* (1.4).

Some further remarks follow.

Remark 1.4 As a consequence of conclusion (i) of Theorem 1.1 and of the structure of equation (1.1), classical results, such as [42, Theorem 5.8.6], guarantee that the solution u is actually analytic in Ω.

Remark 1.5 The extremality property expressed by conclusion (iv) of Theorem 1.1 is in fact crucial to infer the boundary behaviour of the solution, as required by Definition 1.1. It will actually be proved in Section 4 below that a function $u \in W^{1,1}(\Omega)$ is a solution according to Definition 1.1 if and only if it minimizes in $W^{1,1}(\Omega) \cap L^\infty(\Omega)$ the functional

$$\int_\Omega e^{-bz}\sqrt{1 + |\nabla z|^2}\, dx - \frac{a}{b}\int_\Omega e^{-bz}\big(z + \tfrac{1}{b}\big)\, dx + \frac{1}{b}\int_{\partial\Omega} |e^{-bz} - 1|\, d\mathcal{H}^{N-1}.$$

Remark 1.6 It is easily seen that the solution u is continuous and satisfies $u(x_0) = 0$ at any point $x_0 \in \partial\Omega \cap \partial\mathrm{Conv}(\overline{\Omega})$, where $\mathrm{Conv}(\overline{\Omega})$ denotes the convex hull of $\overline{\Omega}$.

Remark 1.7 If Ω exhibits some symmetry, then the solution u exhibits the same kind of symmetry. Indeed, if $\mathbb{U}(\Omega) = \Omega$ for some $\mathbb{U} \in O(N)$, $O(N)$ denoting the orthogonal group in $\mathbb{R}^N$, then $u^* = u \circ \mathbb{U}$ is still a solution and hence, by uniqueness, $u^* = u$. In particular, if Ω is rotationally invariant, then u is radially symmetric. This implies, for any annular domain, the existence of a radially symmetric solution attaining the homogeneous Dirichlet boundary conditions on the exterior sphere. This conclusion does not follow exploiting the more direct and elementary approach developed in [13] for spherical domains, due to the possible occurrence of gradient blow up phenomena on the interior sphere of the annulus.

Remark 1.8 The conclusions of Theorem 1.1 and Theorem 1.2 cannot be derived from the results in [54, 15, 7, 39]. However, the mere existence of a solution $u \in C^\infty(\Omega) \cap W^{1,1}(\Omega)$ of equation (1.1) might possibly be derived from [54, Theorem 5.1] or [15, Theorem 3.1, Chapter V], by the change of variable described below, combined with the obtention of suitable a priori estimates and appropriate truncations. Yet, this approach, relying on a vanishing viscosity method rather than the direct methods of calculus of variations as ours, would not yield the information on the boundary behaviour of u provided by Theorem 1.1 and Theorem 1.2.

The remainder of the paper is organized in several sections, which culminate with the proof of Theorem 1.1 and Theorem 1.2; throughout these sections we state and prove several auxiliary results, which we discuss in detail as they look of independent interest.

We start from the observation, already made in [18, 19, 5, 20, 6, 4], that equation (1.1) can formally be seen as the Euler equation of the functional

$$\int_\Omega e^{-bu}\sqrt{1 + |\nabla u|^2}\, dx - \frac{a}{b}\int_\Omega e^{-bu}\big(u + \tfrac{1}{b}\big)\, dx, \tag{1.5}$$

which involves the anisotropic area functional $\int_\Omega e^{-bu}\sqrt{1+|\nabla u|^2}\,dx$. The natural change of variable $v=e^{-bu}$ transforms problem (1.4) into

$$\begin{cases} -\mathrm{div}\left(\dfrac{\nabla v}{\sqrt{v^2+b^{-2}|\nabla v|^2}}\right) = -a\log(v) - \dfrac{b^2 v}{\sqrt{v^2+b^{-2}|\nabla v|^2}} & \text{in } \Omega, \\ v=1 & \text{on } \partial\Omega \end{cases} \tag{1.6}$$

and the functional in (1.5) into

$$\int_\Omega \sqrt{v^2+b^{-2}|\nabla v|^2}\,dx + \frac{a}{b^2}\int_\Omega v\,(\log(v)-1)\,dx.$$

As the first term $\int_\Omega \sqrt{v^2+b^{-2}|\nabla v|^2}\,dx$ of this functional grows linearly with respect to the gradient term, the appropriate framework where to settle its study appears to be the space of bounded variation functions. Therefore we denote by $\int_\Omega \sqrt{v^2+b^{-2}|Dv|^2}$ the relaxation of $\int_\Omega \sqrt{v^2+b^{-2}|\nabla v|^2}\,dx$ from $W^{1,1}(\Omega)$ to $BV(\Omega)$ and we define the functional

$$\mathcal{J}(v) = \int_\Omega \sqrt{v^2+b^{-2}|Dv|^2} + \frac{1}{b}\int_{\partial\Omega}|v-1|\,d\mathcal{H}^{N-1},$$

where as usual (see [28]) the term $\frac{1}{b}\int_{\partial\Omega}|v-1|\,d\mathcal{H}^{N-1}$ is introduced in order to take into account of the Dirichlet boundary conditions in (1.6).

Our aim is to find a solution of (1.6) by minimizing, on the cone $BV^+(\Omega)$ of all non-negative functions in $BV(\Omega)$, the functional

$$\mathcal{I}(v) = \mathcal{J}(v) + \int_\Omega F(v)\,dx,$$

where $F(s)$ denotes the continuous extension of the function $\frac{a}{b^2}s(\log(s)-1)$ onto $[0,+\infty[$.

Since the functional $\mathcal{J}$, and hence $\mathcal{I}$, does not seem to have been previously studied in the literature, to carry on our argument we first need to prove various facts about it, such as an alternative representation formula, its convexity, its Lipschitz continuity with respect to the norm of $BV(\Omega)$ and its lower semicontinuity with respect to the L^1-convergence in $BV(\Omega)$, as well as a lattice property, encoding a kind of maximum principle. We also prove a delicate approximation result, which plays a crucial role in the sequel of the proof.

Once this preliminary study is completed we show the existence of a global minimizer of $\mathcal{I}$ in $BV^+(\Omega)$. This positive minimizer v is, by the convexity of $\mathcal{I}$, unique, and it is bounded and bounded away from zero; moreover, v is the unique solution of an equivalent variational inequality.

Next we prove the interior regularity of v. This exploits an argument, which was introduced in [22] and used, e.g., in [23, 24, 4] for the study of capillarity problems. The procedure can be summarized as follows. We fix a point $x_0\in\Omega$ and a small open ball B centered at x_0 and compactly contained in Ω. We take a sequence $(v_n)_n$ of regular functions approximating v and satisfying $\mathcal{J}(v_n)\to\mathcal{J}(v)$, whose existence is guaranteed by the above mentioned approximation property. By a result in [39] we can solve, in the classical sense, a sequence of Dirichlet problems in B for the equation in (1.6), where the boundary values are prescribed on ∂B by the restriction of each function v_n. The gradient estimates obtained in [34] and the extremality properties enjoyed by these solutions allow us to prove their convergence, possibly within a ball of smaller radius, to a regular solution of the equation in (1.6), which by uniqueness coincides with v.

By using again the extremality of v, namely the equivalent variational inequality satisfied by v, we are eventually able to conclude that $u = -\frac{1}{b}\log(v)$ is the desired solution of (1.4) according to Definition 1.1. This solution u is unique, smooth and positive in Ω.

The final step is devoted to the study of the boundary behaviour of u, namely, we show that at any point $x_0 \in \partial\Omega$, where an exterior sphere condition of radius $r \geq (N-1)\,b/a$ is satisfied, u is continuous at x_0 and attains the value zero. This goal is achieved by first proving a comparison result valid for pairs of weak lower and upper solutions of problem (1.6) and then by constructing an appropriate upper solution of (1.4) vanishing at x_0. An elementary geometric observation guarantees that the set of points in $\partial\Omega$, where the needed exterior sphere condition holds, is always non-empty.

Notations. We conclude this introduction by setting some notations that are used throughout this paper. For each $N \geq 2$, we set $1^* = \frac{N}{N-1}$. The characteristic function of any set E is denoted by χ_E. If E is a set in $\mathbb{R}^N$ having positive finite N-dimensional Lebesgue measure and $u, v : E \to \mathbb{R}$ are given functions, we write $u \leq v$ in E (respectively, a.e. in E) whenever $u(x) \leq v(x)$ for every $x \in E$ (respectively, a.e. $x \in E$). The N-dimensional Lebesgue measure of E is denoted by $|E|$. If E is a set in $\mathbb{R}^N$ having positive finite $(N-1)$-dimensional Hausdorff measure and $u, v : E \to \mathbb{R}$ are given functions, we write $u \leq v$ on E (respectively, $\mathcal{H}^{N-1}$-a.e. on E) whenever $u(x) \leq v(x)$ for every $x \in E$ (respectively, $\mathcal{H}^{N-1}$-a.e. $x \in E$). By $\{v < w\}$ we denote the set $\{x \in E \mid v(x) < w(x) \text{ a.e. in } E\}$. We also define $u \vee v$ and $u \wedge v$ by $(u \vee v)(x) = \max\{u(x), v(x)\}$ and $(u \wedge v)(x) = \min\{u(x), v(x)\}$ for a.e. $x \in E$. The symbol δ_{ij} as usual stands for the Kronecker delta.

2 Variational setting and auxiliary results

In this section we introduce the variational setting and we prove the auxiliary results that we need for studying problem (1.6). Throughout we suppose that $b > 0$ is a given constant and $\mathcal{O}$ and $\mathcal{U}$ are two open bounded sets in $\mathbb{R}^N$, such that $\overline{\mathcal{U}} \subseteq \mathcal{O}$ and $\mathcal{U}$ has a Lipschitz boundary $\partial\mathcal{U}$.

Anisotropic area functionals

We define some functionals that are relevant for our analysis.

Definition 2.1. *For all $w \in BV(\mathcal{O})$, we set*

$$\int_{\mathcal{O}} \sqrt{w^2 + b^{-2}|Dw|^2} = \sup\Big\{ \int_{\mathcal{O}} w\big(g_{N+1} + \tfrac{1}{b}\operatorname{div}\tilde{g}\big)\,dx \mid \\ g = (\tilde{g}, g_{N+1}) = (g_1, \ldots, g_N, g_{N+1}) \in C_0^1(\mathcal{O}; \mathbb{R}^{N+1}), |g|^2 = \sum_{i=1}^{N+1} g_i^2 \leq 1 \text{ in } \mathcal{O} \Big\}.$$

Remark 2.1 We can verify that, for all $w \in C^1(\overline{\mathcal{O}})$,

$$\int_{\mathcal{O}} \sqrt{w^2 + b^{-2}|Dw|^2} = \int_{\mathcal{O}} \sqrt{w^2 + b^{-2}|\nabla w|^2}\,dx.$$

Proposition 2.1. *For all $w \in BV(\mathcal{O})$, we have*

$$\max\Big\{ \int_{\mathcal{O}} |w|\,dx, \frac{1}{b}\int_{\mathcal{O}} |Dw| \Big\} \leq \int_{\mathcal{O}} \sqrt{w^2 + b^{-2}|Dw|^2} \leq \int_{\mathcal{O}} |w|\,dx + \frac{1}{b}\int_{\mathcal{O}} |Dw|.$$

Proof. The conclusions easily follow observing that

$$\begin{aligned}
\int_{\mathcal{O}} \sqrt{w^2 + b^{-2}|Dw|^2} &= \sup\Big\{ \int_{\mathcal{O}} w\big(g_{N+1} + \tfrac{1}{b}\mathrm{div}\,\tilde{g}\big) dx \mid g \in C_0^1(\mathcal{O};\mathbb{R}^{N+1}), \sum_{j=1}^{N+1} g_j^2 \le 1 \Big\} \\
&\ge \max\Big\{ \sup\Big\{ \int_{\mathcal{O}} w\, g_{N+1}\, dx \mid g_{N+1} \in C_0^1(\mathcal{O};\mathbb{R}),\ |g_{N+1}| \le 1 \Big\}, \\
&\qquad \sup\Big\{ \frac{1}{b} \int_{\mathcal{O}} w \,\mathrm{div}\, h\, dx \mid h \in C_0^1(\mathcal{O};\mathbb{R}^N), \sum_{j=1}^{N} h_j^2 \le 1 \Big\} \Big\} \\
&= \max\Big\{ \int_{\mathcal{O}} |w|\, dx, \frac{1}{b} \int_{\mathcal{O}} |Dw| \Big\}
\end{aligned}$$

and

$$\begin{aligned}
\int_{\mathcal{O}} \sqrt{w^2 + b^{-2}|Dw|^2} &= \sup\Big\{ \int_{\mathcal{O}} w\big(g_{N+1} + \tfrac{1}{b}\mathrm{div}\,\tilde{g}\big) dx \mid g \in C_0^1(\mathcal{O};\mathbb{R}^{N+1}), \sum_{j=1}^{N+1} g_j^2 \le 1 \Big\} \\
&\le \sup\Big\{ \int_{\mathcal{O}} w\, g_{N+1}\, dx \mid g_{N+1} \in C_0^1(\mathcal{O};\mathbb{R}),\ |g_{N+1}| \le 1 \Big\} \\
&\qquad + \sup\Big\{ \frac{1}{b} \int_{\mathcal{O}} w \,\mathrm{div}\, h\, dx \mid h \in C_0^1(\mathcal{O};\mathbb{R}^N), \sum_{j=1}^{N} h_j^2 \le 1 \Big\} \\
&= \int_{\mathcal{O}} |w|\, dx + \frac{1}{b} \int_{\mathcal{O}} |Dw|.
\end{aligned}$$

□

Proposition 2.2. *For all v, $w \in BV(\mathcal{O})$, we have*

$$\left| \int_{\mathcal{O}} \sqrt{v^2 + b^{-2}|Dv|^2} - \int_{\mathcal{O}} \sqrt{w^2 + b^{-2}|Dw|^2} \right| \le \int_{\mathcal{O}} |v - w|\, dx + \frac{1}{b} \int_{\mathcal{O}} |D(v-w)|.$$

Proof. Pick any v, $w \in BV(\mathcal{O})$. By Proposition 2.1, we have

$$\begin{aligned}
&\int_{\mathcal{O}} \sqrt{v^2 + b^{-2}|Dv|^2} - \int_{\mathcal{O}} \sqrt{w^2 + b^{-2}|Dw|^2} \\
&\quad = \sup\Big\{ \int_{\mathcal{O}} v\big(g_{N+1} + \tfrac{1}{b}\mathrm{div}\,\tilde{g}\big)\, dx \mid g \in C_0^1(\mathcal{O};\mathbb{R}^{N+1}),\ |g| \le 1 \Big\} \\
&\qquad - \sup\Big\{ \int_{\mathcal{O}} w\big(g_{N+1} + \tfrac{1}{b}\mathrm{div}\,\tilde{g}\big)\, dx \mid g \in C_0^1(\mathcal{O};\mathbb{R}^{N+1}),\ |g| \le 1 \Big\} \\
&\quad \le \sup\Big\{ \int_{\mathcal{O}} (v-w)\big(g_{N+1} + \tfrac{1}{b}\mathrm{div}\,\tilde{g}\big)\, dx \mid g \in C_0^1(\mathcal{O};\mathbb{R}^{N+1}),\ |g| \le 1 \Big\} \\
&\quad = \int_{\mathcal{O}} \sqrt{(v-w)^2 + b^{-2}|D(v-w)|^2} \\
&\quad \le \int_{\mathcal{O}} |v-w|\, dx + \frac{1}{b} \int_{\mathcal{O}} |D(v-w)|.
\end{aligned}$$

Hence the conclusion follows. □

Lemma 2.3. *For any given $v \in BV(\mathcal{O})$, there exists a positive finite Radon measure μ_v on $\mathcal{O}$ such that, for every open set $A \subseteq \mathcal{O}$,*

$$\mu_v(A) = \int_A \sqrt{v^2 + b^{-2}|Dv|^2}.$$

Proof. Fix $v \in BV(\mathcal{O})$. We define $L_v : C_0(\mathcal{O};\mathbb{R}^{N+1}) \to \mathbb{R}$ by setting, for every $g = (\tilde{g}, g_{N+1})$,

$$L_v(g) = \int_{\mathcal{O}} v g_{N+1}\, dx - \frac{1}{b}\int_{\mathcal{O}} \tilde{g} Dv.$$

For any $g \in C_0(\mathcal{O};\mathbb{R}^{N+1})$, with $|g| \le 1$ in $\mathcal{O}$, we have

$$|L_v(g)| \le \max\left\{1, \frac{1}{b}\right\} \|v\|_{BV(\mathcal{O})} \|g\|_{L^\infty(\mathcal{O})}. \tag{2.1}$$

Hence, by [16, Section 1.8, Theorem 1], the map $\mu_v : \mathcal{B}(\mathcal{O}) \to \mathbb{R}$, defined, if A is an open set, by

$$\mu_v(A) = \sup\{L_v(g) \mid g \in C_0(A;\mathbb{R}^{N+1}),\ |g| \le 1 \text{ in } \mathcal{O}\},$$

and, if B is a Borel set, by

$$\mu_v(B) = \inf\{\mu_v(A) \mid A \text{ open, with } B \subseteq A\},$$

is a positive finite Radon measure. Let us prove that, for any open set $A \subseteq \mathcal{O}$,

$$\mu_v(A) = \int_A \sqrt{v^2 + b^{-2}|Dv|^2}.$$

By definition, we have

$$\int_A \sqrt{v^2 + b^{-2}|Dv|^2} = \sup\{L_v(g) \mid g \in C_0^1(A;\mathbb{R}^{N+1}),\ |g| \le 1 \text{ in } \mathcal{O}\} \le \mu_v(A).$$

It remains to show that

$$\mu_v(A) \le \int_A \sqrt{v^2 + b^{-2}|Dv|^2}. \tag{2.2}$$

Let $g \in C_0(A;\mathbb{R}^{N+1})$ such that $|g| \le 1$ in A. For any $\varepsilon > 0$, there exists $h^\varepsilon \in C_0^\infty(A;\mathbb{R}^{N+1})$ with $|h^\varepsilon| \le 1$ in A such that $\|g - h^\varepsilon\|_{L^\infty(A)} < \varepsilon$. By (2.1) and the linearity of L_v, we have

$$|L_v(g) - L_v(h^\varepsilon)| \le \max\left\{1, \frac{1}{b}\right\} \|v\|_{BV(\mathcal{O})}\, \varepsilon,$$

where g, h^ε still denote the null extensions of g, h^ε onto $\mathcal{O}$. This implies that

$$\begin{aligned} L_v(g) &\le \limsup_{\varepsilon\to 0^+} L_v(h^\varepsilon) \\ &\le \sup\{L_v(h) \mid h \in C_0^1(A;\mathbb{R}^{N+1}),\ |h| \le 1 \text{ in } \mathcal{O}\} = \int_A \sqrt{v^2 + b^{-2}|Dv|^2}. \end{aligned}$$

By generality of g, we conclude that (2.2) holds. □

In order to take into account of the Dirichlet boundary conditions, we introduce the following functional.

Definition 2.2. *Let $\varphi \in L^1(\partial\mathcal{U})$ be given. For all $v \in BV(\mathcal{U})$ we define*

$$\mathcal{J}_\varphi(v) = \int_{\mathcal{U}} \sqrt{v^2 + b^{-2}|Dv|^2} + \frac{1}{b}\int_{\partial\mathcal{U}} |v - \varphi|\, d\mathcal{H}^{N-1}.$$

In case $\varphi = 1$ we simply write $\mathcal{J}_\varphi = \mathcal{J}$, i.e.,

$$\mathcal{J}(v) = \int_{\mathcal{U}} \sqrt{v^2 + b^{-2}|Dv|^2} + \frac{1}{b}\int_{\partial\mathcal{U}} |v - 1|\, d\mathcal{H}^{N-1}.$$

We need the following result to establish some properties of $\mathcal{J}_\varphi$.

Proposition 2.4. *For any $v \in BV(\mathcal{U})$ and $w \in BV(\mathcal{O} \setminus \overline{\mathcal{U}})$, define $z : \mathcal{O} \to \mathbb{R}$, by setting*

$$z = \begin{cases} v & \text{a.e. in } \mathcal{U}, \\ w & \text{a.e. in } \mathcal{O} \setminus \overline{\mathcal{U}}. \end{cases} \tag{2.3}$$

Then $z \in BV(\mathcal{O})$ and satisfies

$$\begin{aligned} \int_{\mathcal{O}} \sqrt{z^2 + b^{-2}|Dz|^2} = \int_{\mathcal{U}} \sqrt{v^2 + b^{-2}|Dv|^2} + \int_{\mathcal{O}\setminus\overline{\mathcal{U}}} \sqrt{w^2 + b^{-2}|Dw|^2} \\ + \frac{1}{b} \int_{\partial\mathcal{U}} |v - w| \, d\mathcal{H}^{N-1}. \end{aligned} \tag{2.4}$$

Proof. By [16, Section 5.4, Theorem 1] we know that $z \in BV(\mathcal{O})$. The additivity property of the Radon measure μ_z defined in Lemma 2.3 also implies

$$\int_{\mathcal{O}} \sqrt{z^2 + b^{-2}|Dz|^2} = \int_{\mathcal{U}} \sqrt{v^2 + b^{-2}|Dv|^2} + \int_{\mathcal{O}\setminus\overline{\mathcal{U}}} \sqrt{w^2 + b^{-2}|Dw|^2} + \int_{\partial\mathcal{U}} \sqrt{z^2 + b^{-2}|Dz|^2}.$$

Therefore, we get (2.4) once we show that

$$\int_{\partial\mathcal{U}} \sqrt{z^2 + b^{-2}|Dz|^2} = \frac{1}{b} \int_{\partial\mathcal{U}} |v - w| \, d\mathcal{H}^{N-1}. \tag{2.5}$$

Let $(A_n)_n$ be a decreasing sequence of open sets such that $A_n \subseteq \mathcal{O}$, for each n, and $\bigcap_n A_n = \partial\mathcal{U}$. Then we have

$$\lim_{n\to+\infty} \int_{A_n} \sqrt{z^2 + b^{-2}|Dz|^2} = \int_{\partial\mathcal{U}} \sqrt{z^2 + b^{-2}|Dz|^2}.$$

Proposition 2.1 yields, for each n,

$$\max\left\{ \int_{A_n} |z| \, dx, \frac{1}{b} \int_{A_n} |Dz| \right\} \leq \int_{A_n} \sqrt{z^2 + b^{-2}|Dz|^2} \leq \int_{A_n} |z| \, dx + \frac{1}{b} \int_{A_n} |Dz|. \tag{2.6}$$

Moreover, the dominated convergence theorem yields

$$\lim_{n\to+\infty} \int_{A_n} |z| \, dx = \int_{\partial\mathcal{U}} |z| \, dx = 0$$

and

$$\lim_{n\to+\infty} \int_{A_n} |Dz| = \int_{\partial\mathcal{U}} |Dz|.$$

By letting n go to $+\infty$ in (2.6) and using [16, Section 5.4, Theorem 1], we infer

$$\int_{\partial\mathcal{U}} \sqrt{z^2 + b^{-2}|Dz|^2} = \frac{1}{b} \int_{\partial\mathcal{U}} |Dz| = \frac{1}{b} \int_{\partial\mathcal{U}} |v - w| \, d\mathcal{H}^{N-1}.$$

Thus (2.5) holds and the conclusion follows. □

Proposition 2.5. *Let $\varphi \in L^1(\partial\mathcal{U})$ be given. Then the following properties hold:*

(i) $\mathcal{J}_\varphi$ *is convex;*

(*ii*) *$\mathcal{J}_\varphi$ is lower semicontinuous with respect to the L^1-convergence in $BV(\mathcal{U})$, i.e., if $(v_n)_n$ is a sequence in $BV(\mathcal{U})$, which converges in $L^1(\mathcal{U})$ to $v \in BV(\mathcal{U})$, then*

$$\mathcal{J}_\varphi(v) \leq \liminf_{n\to+\infty} \mathcal{J}_\varphi(v_n).$$

Proof. The convexity of $\mathcal{J}_\varphi$ is a direct consequence of Definition 2.1 and Definition 2.2. With the aim of proving the semicontinuity property of $\mathcal{J}_\varphi$, we pick a sequence $(v_n)_n$ in $BV(\mathcal{U})$, which converges in $L^1(\mathcal{U})$ to $v \in BV(\mathcal{U})$, and an open set $\mathcal{O}$ in $\mathbb{R}^N$, with $\overline{\mathcal{U}} \subseteq \mathcal{O}$. By [21, Teorema 1.II] there exists a function $w \in W^{1,1}(\mathcal{O} \setminus \overline{\mathcal{U}})$ such that $w = \varphi$ $\mathcal{H}^{N-1}$-a.e. on $\partial\mathcal{U}$. As in Proposition 2.4, we define $z \in BV(\mathcal{O})$ by (2.3) and, for each n, $z_n \in BV(\mathcal{O})$ by

$$z_n = \begin{cases} v_n & \text{a.e. in } \mathcal{U}, \\ w & \text{a.e. in } \mathcal{O} \setminus \overline{\mathcal{U}}. \end{cases}$$

Let us take $g = (\tilde{g}, g_{N+1}) \in C^1_0(\mathcal{O}; \mathbb{R}^{N+1})$ such that $|g| \leq 1$ in $\mathcal{O}$. As $\lim\limits_{n\to+\infty} z_n = z$ in $L^1(\mathcal{O})$, we have

$$\int_{\mathcal{O}} z\big(g_{N+1} + \tfrac{1}{b}\operatorname{div}\tilde{g}\big)\,dx = \lim_{n\to+\infty} \int_{\mathcal{O}} z_n\big(g_{N+1} + \tfrac{1}{b}\operatorname{div}\tilde{g}\big)\,dx \leq \liminf_{n\to+\infty} \int_{\mathcal{O}} \sqrt{z_n^2 + b^{-2}|Dz_n|^2}$$

and hence

$$\int_{\mathcal{O}} \sqrt{z^2 + b^{-2}|Dz|^2} \leq \liminf_{n\to+\infty} \int_{\mathcal{O}} \sqrt{z_n^2 + b^{-2}|Dz_n|^2}$$

The lower semicontinuity of $\mathcal{J}_\varphi$ with respect to the L^1-convergence in $BV(\mathcal{U})$ can then be deduced from Proposition 2.4. □

An approximation property

The following approximation property plays a crucial role in the sequel; it generalizes the classical approximation property in the space of bounded variation functions with respect to the strict convergence (see, e.g., [28, Theorem 1.17]).

Proposition 2.6. *Let $\varphi \in L^1(\partial\mathcal{U})$ and $w \in BV(\mathcal{U})$ be given. Then, for each $p \in [1, 1^*[$, there exists a sequence $(w_n)_n$ in $C^\infty(\mathcal{U}) \cap W^{1,1}(\mathcal{U})$ such that*

$$\lim_{n\to+\infty} w_n = w \quad \text{in } L^p(\mathcal{U}), \tag{2.7}$$

$$\lim_{n\to+\infty} \mathcal{J}_\varphi(w_n) = \mathcal{J}_\varphi(w), \tag{2.8}$$

$$w_n = \varphi \quad \mathcal{H}^{N-1}\text{-a.e. on } \partial\mathcal{U}. \tag{2.9}$$

Moreover, if there exist $c, d \in \mathbb{R}$, with $c \leq w \leq d$ a.e. in $\mathcal{U}$ and $c \leq \varphi \leq d$ $\mathcal{H}^{N-1}$-a.e. on $\partial\mathcal{U}$, then, for each $\sigma > 0$, a sequence $(w_n)_n$, satisfying the previous conditions, can be selected such that, for all n,

$$c - \sigma \leq w_n \leq d + \sigma \quad \text{in } \mathcal{U}. \tag{2.10}$$

Proof. Let $p \in [1, 1^*[$ be fixed. Let also $(\varepsilon_n)_n$ be a sequence of positive numbers converging to 0. By [21, Teorema 1.II], there exists $u \in W^{1,1}(\mathcal{U})$ such that $u = \varphi$ $\mathcal{H}^{N-1}$-a.e. on $\partial\mathcal{U}$. Moreover, if $c \leq \varphi \leq d$ $\mathcal{H}^{N-1}$-a.e. on $\partial\mathcal{U}$, then $c \leq u \leq d$ a.e. in $\mathcal{U}$.

Part 1. Construction of a sequence $(\hat{w}_h)_h$ in $BV(\mathcal{U})$ satisfying (2.7), (2.8), (2.9) *and* (2.10).

Consider the sequence of functions $(\psi_h)_h$ in $C^{0,1}(\overline{\mathcal{U}})$ defined by

$$\psi_h(x) = \min\left\{\max\left\{\tfrac{h+1}{h}[1 - h\, d(x, \partial\mathcal{U})], 0\right\}, 1\right\}.$$

By [11, Lemma 7.3], we know that

(*i*) $\lim\limits_{h\to+\infty} \psi_h = 0$ a.e. in $\mathcal{U}$ and in $L^1(\mathcal{U})$;

(*ii*) for each h, $\mathrm{supp}(1-\psi_h)$ is a compact subset of $\mathcal{U}$;

(*iii*) for each h, $0 \le \psi_h \le 1$ in $\mathcal{U}$;

(*iv*) for each h, $\psi_h = 0$ in $\mathcal{U}\setminus S_h$, with $S_h = \{x \in \mathcal{U} \mid d(x, \partial\mathcal{U}) \le 1/h\}$;

(*v*) for each $z \in BV(\mathcal{U})$, $\limsup\limits_{h\to+\infty} \int_{\mathcal{U}} |z|\,|D\psi_h| \le \int_{\partial\mathcal{U}} |z|\, d\mathcal{H}^{N-1}$.

Consider the sequence of functions $(\hat{w}_h)_h$ in $BV(\mathcal{U})$ defined by $\hat{w}_h = \psi_h\, u + (1-\psi_h)\, w$. We easily have

(*a*) $\lim\limits_{h\to+\infty} \hat{w}_h = w$ a.e. in $\mathcal{U}$ and in $L^p(\mathcal{U})$;

(*b*) for each h, $\hat{w}_h = u = \varphi$ $\mathcal{H}^{N-1}$-a.e. on $\partial\mathcal{U}$;

(*c*) If $c \le w \le d$ a.e. in $\mathcal{U}$ and $c \le \varphi \le d$ $\mathcal{H}^{N-1}$-a.e. on $\partial\mathcal{U}$, then, for each h, $c \le \hat{w}_h \le d$ a.e. in $\mathcal{U}$.

It remains to prove that

$$\lim_{h\to+\infty} \mathcal{J}_\varphi(\hat{w}_h) = \mathcal{J}_\varphi(w). \tag{2.11}$$

By Proposition 2.2, we have

$$\int_{\mathcal{U}} \sqrt{\hat{w}_h^2 + b^{-2}|D\hat{w}_h|^2} \le \int_{\mathcal{U}} \sqrt{w^2 + b^{-2}|Dw|^2} + \int_{\mathcal{U}} |\hat{w}_h - w|\, dx + \frac{1}{b}\int_{\mathcal{U}} |D(\hat{w}_h - w)|$$

where, by [16, Proposition 3.2],

$$\int_{\mathcal{U}} |D(\hat{w}_h - w)| = \int_{\mathcal{U}} |D(\psi_h(u-w))| \le \int_{\mathcal{U}} \psi_h |D(u-w)| + \int_{\mathcal{U}} |u-w|\,|D\psi_h|.$$

From (*i*), (*iii*) and the dominated convergence theorem, we deduce

$$\lim_{h\to+\infty} \int_{\mathcal{U}} \psi_h |D(u-w)| = 0;$$

whereas, from (*v*) and $u = \varphi$ $\mathcal{H}^{N-1}$-a.e. on $\partial\mathcal{U}$, we infer

$$\limsup_{h\to+\infty} \int_{\mathcal{U}} |u-w|\,|D\psi_h| \le \int_{\partial\mathcal{U}} |w-\varphi|\, d\mathcal{H}^{N-1}.$$

Therefore, we have

$$\limsup_{h\to+\infty} \mathcal{J}_\varphi(\hat{w}_h) \le \mathcal{J}_\varphi(w).$$

By Proposition 2.5, we also get

$$\mathcal{J}_\varphi(w) \le \liminf_{h\to+\infty} \mathcal{J}_\varphi(\hat{w}_h).$$

Hence we conclude that (2.11) holds.

Part 2. Regularization of a given $\hat{w} \in BV(\mathcal{U})$.
Step 1. Construction of an approximating sequence $(w_n)_n$ in $L^p(\mathcal{U}) \cap C^\infty(\mathcal{U})$ such that

$$\lim_{n\to+\infty} w_n = \hat{w} \quad \text{in } L^p(\mathcal{U}).$$

Fix $n \in \mathbb{N}$ and, for each $m \in \mathbb{N}_0$, define the set

$$\mathcal{U}(m) = \Big\{x \in \mathcal{U} \mid \operatorname{dist}(x, \partial\mathcal{U}) > \tfrac{1}{m}\Big\}.$$

Let $\chi_{\mathcal{U}(m)}$ be the characteristic function of $\mathcal{U}(m)$. The dominated convergence theorem yields

$$\begin{aligned}
\lim_{m\to+\infty} \int_{\mathcal{U}\setminus\mathcal{U}(m)} |D\hat{w}| &= \lim_{m\to+\infty} \int_{\mathcal{U}} (1-\chi_{\mathcal{U}(m)})|D\hat{w}| = 0,\\
\lim_{m\to+\infty} \int_{\mathcal{U}\setminus\mathcal{U}(m)} |\hat{w}|\,dx &= \lim_{m\to+\infty} \int_{\mathcal{U}} (1-\chi_{\mathcal{U}(m)})|\hat{w}|\,dx = 0.
\end{aligned}$$

Hence we can choose $M_n \in \mathbb{N}_0$ such that

$$\int_{\mathcal{U}\setminus\mathcal{U}(M_n)} |D\hat{w}| < \varepsilon_n \quad \text{and} \quad \int_{\mathcal{U}\setminus\mathcal{U}(M_n)} |\hat{w}|\,dx < \varepsilon_n. \tag{2.12}$$

For simplicity, we relabel the sequence $(\mathcal{U}(M_n+i))_{i\in\mathbb{N}}$ as $(\mathcal{U}_i)_{i\in\mathbb{N}}$. Let us introduce the family $\{A_i\}_{i\in\mathbb{N}_0}$ of open subsets of $\mathcal{U}$ given by

$$A_1 = \mathcal{U}_2 = \Big\{x \in \mathcal{U} \mid \operatorname{dist}(x, \partial\mathcal{U}) > \tfrac{1}{M_n+2}\Big\},$$

and, for $i \geq 2$,

$$A_i = \mathcal{U}_{i+1} \setminus \overline{\mathcal{U}_{i-1}} = \Big\{x \in \mathcal{U} \mid \tfrac{1}{M_n+i+1} < \operatorname{dist}(x, \partial\mathcal{U}) < \tfrac{1}{M_n+i-1}\Big\}.$$

The family $\{A_i\}_{i\in\mathbb{N}_0}$ is an open covering of $\mathcal{U}$. Moreover, for all $i \geq 2$, we have

$$\begin{gathered}
\overline{A_1} \subseteq \mathcal{U}_3, \qquad \overline{A_i} \subseteq \mathcal{U}_{i+2} \setminus \overline{\mathcal{U}_{i-2}},\\
\overline{A_i} \subseteq \mathcal{U} \setminus \mathcal{U}_0 = \Big\{x \in \mathcal{U} \mid \operatorname{dist}(x, \partial\mathcal{U}) \leq \tfrac{1}{M_n}\Big\},\\
A_i \cap A_k = \emptyset \quad \text{for all } k \geq i+2.
\end{gathered} \tag{2.13}$$

Let now $(\phi_i)_{i\in\mathbb{N}_0}$ be a partition of unity on $\mathcal{U}$ subordinate to the open covering $\{A_i\}_{i\in\mathbb{N}_0}$, i.e.,

$$\begin{gathered}
\phi_i|_{\mathcal{U}\setminus A_i} = 0, \quad \phi_i|_{A_i} \in C_0^\infty(A_i),\\
0 \leq \phi_i \leq 1 \quad \text{in } \mathcal{U},\\
\sum_{i=1}^{+\infty} \phi_i = 1 \quad \text{in } \mathcal{U}.
\end{gathered} \tag{2.14}$$

Let η be a positive radial mollifier centered at 0 and consider the sequence

$$(\eta_{\delta_i} * \hat{w}\phi_i)_{i\in\mathbb{N}_0},$$

where, using (2.13), the sequence $(\delta_i)_{i\in\mathbb{N}_0} = (\delta_i(n))_{i\in\mathbb{N}_0}$ has been chosen such that

$$\begin{aligned}
&\operatorname{supp}(\eta_{\delta_1} * (\hat{w}\phi_1)) \subseteq \mathcal{U}_3,\\
&\operatorname{supp}(\eta_{\delta_i} * (\hat{w}\phi_i)) \subseteq \mathcal{U}_{i+2} \setminus \overline{\mathcal{U}_{i-2}} \quad \text{for all } i \geq 2,
\end{aligned} \tag{2.15}$$

and, by [10, Theorem 4.22], for all $i \in \mathbb{N}_0$,

$$\begin{aligned} \|(\eta_{\delta_i} * (\hat{w}\phi_i)) - \hat{w}\phi_i\|_{L^p(\mathcal{U})} &< \frac{\varepsilon_n}{2^i}, \\ \|(\eta_{\delta_i} * (\hat{w}\nabla\phi_i)) - \hat{w}\nabla\phi_i\|_{L^p(\mathcal{U})} &< \frac{\varepsilon_n}{2^i}. \end{aligned} \tag{2.16}$$

To conclude we define

$$w_n = \sum_{i=1}^{+\infty} \eta_{\delta_i} * (\hat{w}\phi_i).$$

It is easy to see that $w_n \in L^p(\mathcal{U}) \cap C^\infty(\mathcal{U})$. Using conditions (2.14) and (2.16), we compute

$$\|\hat{w} - w_n\|_{L^p} = \Big\|\sum_{i=1}^{+\infty} \big(\hat{w}\phi_i - \eta_{\delta_i} * (\hat{w}\phi_i)\big)\Big\|_{L^p} \le \sum_{i=1}^{+\infty} \Big\|\hat{w}\phi_i - \eta_{\delta_i} * (\hat{w}\phi_i)\Big\|_{L^p} < \sum_{i=1}^{+\infty} \frac{\varepsilon_n}{2^i} = \varepsilon_n,$$

and hence

$$\lim_{n\to+\infty} w_n = \hat{w} \quad \text{in } L^p(\mathcal{U}).$$

Step 2. For each n, $w_n = \hat{w}$ $\mathcal{H}^{N-1}$-a.e. in $\partial\mathcal{U}$. By definition of trace [28, Theorem 2.10], we prove that, for all n and for all $x \in \partial\mathcal{U}$,

$$\lim_{\rho\to 0^+} \frac{1}{\rho^N} \int_{B(x,\rho)\cap\mathcal{U}} |w_n - \hat{w}|\, dy = 0. \tag{2.17}$$

Fix $x \in \partial\mathcal{U}$ and $\rho > 0$. For all $i < i_0(\rho) = \left\lceil \frac{1}{\rho} \right\rceil - M - 2$, we have $B(x,\rho) \cap \mathcal{U}_{i+2} = \emptyset$ and hence $\eta_{\delta_i} * (\hat{w}\phi_i) - \hat{w}\phi_i = 0$ in $B(x,\rho) \cap \mathcal{U}$. Then, for all n and for a.e. $y \in B(x,\rho) \cap \mathcal{U}$, there holds

$$\begin{aligned} w_n(y) - \hat{w}(y) &= \sum_{i=1}^{+\infty} \eta_{\delta_i} * (\hat{w}\phi_i)(y) - \hat{w}(y) \\ &= \sum_{i=1}^{+\infty} \Big(\eta_{\delta_i} * (\hat{w}\phi_i)(y) - \hat{w}(y)\phi_i(y)\Big) = \sum_{i=i_0}^{+\infty} \Big(\eta_{\delta_i} * (\hat{w}\phi_i)(y) - \hat{w}(y)\phi_i(y)\Big). \end{aligned}$$

By construction of ϕ_i, we know that there exists $C > 0$ such that, for all $i \in \mathbb{N}_0$,

$$\int_{\mathcal{U}} |\eta_{\delta_i} * (\hat{w}\phi_i) - \hat{w}\phi_i|\, dy \le \frac{C}{2^i}.$$

This implies that

$$\int_{B(x,\rho)\cap\mathcal{U}} |w_n - \hat{w}|\, dy \le \sum_{i=i_0}^{+\infty} \int_{B(x,\rho)\cap\mathcal{U}} |\eta_{\delta_i} * (\hat{w}\phi_i) - \hat{w}\phi_i|\, dy \le \sum_{i=i_0}^{+\infty} \frac{C}{2^i} = \frac{2C}{2^{i_0}}.$$

Notice that, by definition of i_0, we have

$$\lim_{\rho\to 0^+} 2^{i_0}\rho^N = \infty.$$

Then we conclude that (2.17) is satisfied, for all n and for all $x \in \partial\mathcal{U}$.

Step 3. For each n, $w_n \in W^{1,1}(\mathcal{U})$. As $w_n \in L^1(\mathcal{U}) \cap C^\infty(\mathcal{U})$, for proving that $w_n \in W^{1,1}(\mathcal{U})$ it is enough to verify that

$$\int_{\mathcal{U}} |Dw_n| = \sup\Big\{ \int_{\mathcal{U}} w_n \operatorname{div} h \, dx \mid h = (h_1, \dots, h_N) \in C_0^1(\mathcal{U}; \mathbb{R}^N), \ \sum_{j=1}^N h_j^2 \le 1 \Big\} < +\infty.$$

Take $h \in C_0^1(\mathcal{U}; \mathbb{R}^N)$ satisfying $\displaystyle\sum_{j=1}^N h_j^2 \le 1$.

Claim 1. For each n,

$$\int_{\mathcal{U}} w_n \operatorname{div} h \, dx = \sum_{i=1}^\infty \int_{\mathcal{U}} \hat{w} \operatorname{div}\left(\phi_i(\eta_{\delta_i} * h)\right) dx - \sum_{i=1}^\infty \int_{\mathcal{U}} h \left(\eta_{\delta_i} * (\hat{w} \nabla \phi_i) - \hat{w} \nabla \phi_i\right) dx.$$

By definition of w_n, using [10, Proposition 4.16, Proposition 4.20] and recalling that η_{δ_i} is an even function, we have

$$\begin{aligned} \int_{\mathcal{U}} w_n \operatorname{div} h \, dx &= \sum_{i=1}^\infty \int_{\mathcal{U}} \left(\eta_{\delta_i} * (\hat{w}\phi_i)\right) \operatorname{div} h \, dx \\ &= \sum_{i=1}^\infty \int_{\mathcal{U}} \hat{w}\phi_i (\eta_{\delta_i} * \operatorname{div} h) \, dx = \sum_{i=1}^\infty \int_{\mathcal{U}} \hat{w}\phi_i \operatorname{div}(\eta_{\delta_i} * h) \, dx \end{aligned}$$

Since

$$\operatorname{div}\big(\phi_i(\eta_{\delta_i} * h)\big) = \phi_i \operatorname{div}(\eta_{\delta_i} * h) + \nabla\phi_i \, (\eta_{\delta_i} * h),$$

Claim 1 is proved if we show that, for all $i \in \mathbb{N}_0$,

$$\int_{\mathcal{U}} \hat{w} \, \nabla \phi_i \, (\eta_{\delta_i} * h) \, dx = \int_{\mathcal{U}} h \, (\eta_{\delta_i} * (\hat{w}\nabla\phi_i) - \hat{w}\nabla\phi_i) \, dx.$$

The even character of η_{δ_i} and [10, Proposition 4.16] imply

$$\int_{\mathcal{U}} h \, (\eta_{\delta_i} * (\hat{w}\nabla\phi_i) - \hat{w}\nabla\phi_i) \, dx = \int_{\mathcal{U}} \hat{w} \, \nabla\phi_i \, (\eta_{\delta_i} * h) \, dx - \int_{\mathcal{U}} \hat{w} \, h \, \nabla\phi_i \, dx.$$

Using $\displaystyle\sum_{i=1}^\infty \phi_i = 1$, we deduce that $\displaystyle\sum_{i=1}^\infty \frac{\partial \phi_i}{\partial x_j} = 0$, for all $j \in \{1, \dots, N\}$. Hence the result follows.

Claim 2. There exists $C > 0$, such that, for all n, we have

$$\int_{\mathcal{U}} w_n \operatorname{div} h \, dx \le \int_{\mathcal{U}} \hat{w} \operatorname{div}\left(\phi_1(\eta_{\delta_1} * h)\right) dx + C\varepsilon_n.$$

For any $i \ge 2$, the function $\phi_i(\eta_{\delta_i} * h) \in C^\infty(\mathcal{U}; \mathbb{R}^N)$ is such that $\operatorname{supp}(\phi_i(\eta_{\delta_i} * h)) \subseteq A_i$ and, for $x \in \mathcal{U}$,

$$\sum_{j=1}^N \phi_i^2(x)(\eta_{\delta_i} * h_j)^2(x) \le 1.$$

Hence, by the characterization of the total variation of $\hat{w}$ and (2.12), we have

$$\begin{aligned} \sum_{i=2}^{+\infty} \int_{\mathcal{U}} \hat{w} \operatorname{div}\big(\phi_i(\eta_{\delta_i} * h)\big) \, dx &= \sum_{i=2}^{+\infty} \int_{A_i} \hat{w} \operatorname{div}\big(\phi_i(\eta_{\delta_i} * h)\big) \, dx \\ &\le \sum_{i=2}^{+\infty} \int_{A_i} |D\hat{w}| \le 2 \int_{\mathcal{U} \setminus \mathcal{U}_0} |D\hat{w}| \ < 2\varepsilon_n. \end{aligned}$$

On the other hand, by (2.16), we obtain

$$\begin{aligned}\Big|\sum_{i=1}^{+\infty}\int_{\mathcal{U}} h\left(\eta_{\delta_i} * (\hat{w}\nabla\phi_i) - \hat{w}\nabla\phi_i\right)dx\Big| &\leq \sum_{i=1}^{+\infty}\|h\|_{L^q(\mathcal{U})}\|\eta_{\delta_i} * (\hat{w}\nabla\phi_i) - \hat{w}\nabla\phi_i\|_{L^p(\mathcal{U})}\\ &\leq \sum_{i=1}^{+\infty}\frac{\|h\|_{L^q(\mathcal{U})}\varepsilon_n}{2^i} \leq |\mathcal{U}|^{1/q}\varepsilon_n.\end{aligned}$$

The conclusion follows taking $C = |\mathcal{U}|^{1/q} + 2$.

Step 4. We have

$$\limsup_{n\to+\infty}\int_{\mathcal{U}}\sqrt{w_n^2 + b^{-2}|\nabla w_n|^2}\,dx \leq \int_{\mathcal{U}}\sqrt{\hat{w}^2 + b^{-2}|D\hat{w}|^2}.$$

Take $g = (g_1, \dots, g_N, g_{N+1}) = (\tilde{g}, g_{N+1}) \in C_0^1(\mathcal{U};\mathbb{R}^{N+1})$ satisfying $\sum_{j=1}^{N+1} g_j^2 \leq 1$. Recall that

$$\int_{\mathcal{U}}\sqrt{w_n^2 + b^{-2}|\nabla w_n|^2}dx = \sup\Big\{\int_{\mathcal{U}} w_n\big(g_{N+1} + \tfrac{1}{b}\mathrm{div}\,\tilde{g}\big)dx \mid g \in C_0^1(\mathcal{U};\mathbb{R}^{N+1}), \sum_{j=1}^{N+1} g_j^2 \leq 1\Big\}.$$

Hence, the conclusion will follow if we prove that, for all $n \in \mathbb{N}$,

$$\int_{\mathcal{U}} w_n\big(g_{N+1} + \tfrac{1}{b}\mathrm{div}\,\tilde{g}\big)\,dx \leq \int_{\mathcal{U}}\hat{w}\big(G_{N+1} + \tfrac{1}{b}\mathrm{div}\,\tilde{G}\big)\,dx + C\varepsilon_n, \tag{2.18}$$

with $G = (G_1, \dots, G_N, G_{N+1}) = (\tilde{G}, G_{N+1}) = \phi_1(\eta_{\delta_1} * g)$.

By Claim 2, there exists $C > 0$, independent of $g \in C_0^1(\mathcal{U};\mathbb{R}^{N+1})$ with $\sum_{j=1}^{N+1} g_j^2 \leq 1$, such that, for all n,

$$\int_{\mathcal{U}} w_n\,\mathrm{div}\,\tilde{g}\,dx \leq \int_{\mathcal{U}}\hat{w}\,\mathrm{div}\,\tilde{G}\,dx + C\varepsilon_n.$$

Hence it remains to prove that

$$\int_{\mathcal{U}} w_n g_{N+1}\,dx \leq \int_{\mathcal{U}}\hat{w}G_{N+1}\,dx + C\varepsilon_n = \int_{\mathcal{U}}\hat{w}\phi_1(\eta_{\delta_1} * g_{N+1})\,dx + C\varepsilon_n.$$

By definition of w_n and [10, Proposition 4.16], we have

$$\int_{\mathcal{U}} w_n g_{N+1}\,dx = \sum_{i=1}^{+\infty}\int_{\mathcal{U}}\hat{w}\phi_i(\eta_{\delta_i} * g_{N+1})\,dx.$$

Observe that, for $i \geq 2$, $\phi_i(\eta_{\delta_i} * g_{N+1}) \in C^\infty(\mathcal{U})$, vanishes outside of A_i and satisfies

$$|\phi_i(x)(\eta_{\delta_i} * g_{N+1})(x)| \leq \|\phi_i\|_{L^\infty(\mathcal{U})}\|g_{N+1}\|_{L^\infty(\mathcal{U})}\int_{\mathbb{R}^N}\eta_{\varepsilon_i}(y)\,dy \leq 1$$

in A_i. Hence, by applying (2.13) and (2.12), we have

$$\begin{aligned}\int_{\mathcal{U}}\hat{w}_n g_{N+1}\,dx - \int_{\mathcal{U}}\hat{w}\phi_1(\eta_{\delta_1} * g_{N+1})\,dx &= \sum_{i=2}^{+\infty}\int_{\mathcal{U}}\hat{w}\phi_i(\eta_{\delta_i} * g_{N+1})\,dx\\ &= \sum_{i=2}^{+\infty}\int_{A_i}\hat{w}\phi_i(\eta_{\delta_i} * g_{N+1})\,dx\\ &\leq 2\int_{\mathcal{U}\setminus\mathcal{U}_0}|\hat{w}| < 2\varepsilon_n.\end{aligned}$$

Finally, as (2.18) holds for every $g \in C_0^1(\mathcal{U};\mathbb{R}^{N+1})$ with $\sum_{j=1}^{N+1} g_j^2 \le 1$, we have by definition

$$\int_{\mathcal{U}} \sqrt{w_n^2 + b^{-2}|Dw_n|^2} \le \int_{\mathcal{U}} \sqrt{\hat{w}^2 + b^{-2}|D\hat{w}|^2} + C\varepsilon_n,$$

and hence

$$\limsup_{n\to+\infty} \int_{\mathcal{U}} \sqrt{w_n^2 + b^{-2}|Dw_n|^2} \le \int_{\mathcal{U}} \sqrt{\hat{w}^2 + b^{-2}|D\hat{w}|^2}.$$

Step 5. We have

$$\lim_{n\to+\infty} \int_{\partial\mathcal{U}} |w_n - \varphi|\, d\mathcal{H}^{N-1} = \int_{\partial\mathcal{U}} |\hat{w} - \varphi|\, d\mathcal{H}^{N-1}.$$

This conclusion follows from [28, Theorem 2.11] if we prove that

$$\lim_{n\to+\infty} \int_{\mathcal{U}} |Dw_n| = \int_{\mathcal{U}} |D\hat{w}|. \tag{2.19}$$

By Claim 2, we have

$$\limsup_{n\to+\infty} \int_{\mathcal{U}} |Dw_n| \le \int_{\mathcal{U}} |D\hat{w}|$$

The lower semicontinuity of the total variation with respect to the L^1-convergence in $BV(\mathcal{U})$ (see [28, Theorem 1.9]) yields (2.19).

Step 6. We have

$$\lim_{n\to+\infty} \mathcal{J}_\varphi(w_n) = \mathcal{J}_\varphi(\hat{w}).$$

By Step 1, Step 3 and Proposition 2.5 we know that

$$\mathcal{J}_\varphi(\hat{w}) \le \liminf_{n\to+\infty} \mathcal{J}_\varphi(w_n).$$

On the other hand, by Step 4 and Step 5, we obtain

$$\limsup_{n\to+\infty} \mathcal{J}_\varphi(w_n) \le \mathcal{J}_\varphi(\hat{w}).$$

Thus the conclusion follows.

Step 7. If $c \le \hat{w} \le d$ a.e. in $\mathcal{U}$, then, for each $\sigma > 0$ the sequence $(w_n)_n$ can be selected such that, for all n, (2.10) *is satisfied.* Let fix $\sigma > 0$. Since, by construction, for each $i \in \mathbb{N}_0$, the function ϕ_i is uniformly continuous in $\overline{\mathcal{U}}$, there exists $\eta_i = \eta_i(\sigma) > 0$ such that, for all x, $z \in \overline{\mathcal{U}}$, with $|x - z| < \eta_i$, there holds

$$|\phi_i(x) - \phi_i(z)| < \tfrac{1}{\max\{|c|,|d|\}}\, \tfrac{\sigma}{2^i}.$$

Up to now, for any n, for any $i \ge 1$, the constants $\delta_i = \delta_i(n)$ have been chosen small enough in order that (2.15) and (2.16) are satisfied. Hence, reducing δ_i if necessary, we can assume that $\delta_i \le \eta_i$. In this way, for all n, we obtain

$$\begin{aligned} w_n(x) &= \sum_{i=1}^{+\infty} (\eta_{\delta_i} * (\hat{w}\phi_i))(x) = \sum_{i=1}^{+\infty} \int_{B(0,\delta_i)} \eta_{\delta_i}(y)\phi_i(x-y)\hat{w}(x-y)\,dy \\ &\le \sum_{i=1}^{+\infty} \int_{B(0,\delta_i)} \eta_{\delta_i}(y)\big[\phi_i(x)\hat{w}(x-y) + |\phi_i(x-y)-\phi_i(x)|\,|\hat{w}(x-y)|\big]\,dy \\ &\le \sum_{i=1}^{+\infty} \Big(d\,\phi_i(x) + \tfrac{1}{\max\{|c|,|d|\}}\,\tfrac{\sigma}{2^i}\max\{|c|,|d|\}\Big) \int_{B(0,\delta_i)} \eta_{\delta_i}(y)\,dy \\ &\le \left(d\sum_{i=1}^{+\infty}\phi_i(x) + \sum_{i=1}^{+\infty}\frac{\sigma}{2^i}\right) \le d + \sum_{i=1}^{+\infty}\frac{\sigma}{2^i} = d+\sigma, \end{aligned}$$

for all $x \in \mathcal{U}$. The proof that $w_n \ge c - \sigma$ in $\mathcal{U}$ is similar.

Part 3. Conclusion of the proof: construction of the sequence $(w_n)_n$. By Part 1, for each n we find $\hat{w}_{h_n} \in BV(\mathcal{U})$ such that

$$\|\hat{w}_{h_n} - w\|_{L^p(\mathcal{U})} \le \tfrac{1}{2n},$$
$$|\mathcal{J}_\varphi(\hat{w}_{h_n}) - \mathcal{J}_\varphi(w)| \le \tfrac{1}{2n},$$
$$\hat{w}_{h_n} = \varphi \quad \mathcal{H}^{N-1}\text{-a.e. in } \partial\mathcal{U}.$$

Moreover, if $c \le w \le d$ a.e. in $\mathcal{U}$, and $c \le \varphi \le d$ $\mathcal{H}^{N-1}$-a.e. on $\partial\mathcal{U}$, then

$$c \le \hat{w}_{h_n} \le d \quad \text{a.e. in } \mathcal{U}.$$

On the other hand, by Part 2 letting $\hat{w} = \hat{w}_{h_n}$, we find $w_n \in W^{1,1}(\mathcal{U}) \cap C^\infty(\mathcal{U})$ such that

$$\|w_n - \hat{w}_{h_n}\|_{L^p(\mathcal{U})} \le \tfrac{1}{2n},$$
$$|\mathcal{J}_\varphi(w_n) - \mathcal{J}_\varphi(\hat{w}_{h_n})| \le \tfrac{1}{2n},$$
$$w_n = \hat{w}_{h_n} = \varphi, \quad \mathcal{H}^{N-1}\text{-a.e. on } \partial\mathcal{U}.$$

Moreover, if $c \le w \le d$ a.e. in $\mathcal{U}$, and $c \le \varphi \le d$ $\mathcal{H}^{N-1}$-a.e. on $\partial\mathcal{U}$, then, for each $\sigma > 0$, (2.10) is satisfied. Thus the conclusion follows. $\square$

In the particular case where $\varphi = 1$, i.e., $\mathcal{J}_\varphi = \mathcal{J}$, we can restate Proposition 2.6 as follows.

Corollary 2.7. *Let $w \in BV(\mathcal{U})$ be given. Then, for each $p \in [1, 1^*[$, there exists a sequence $(w_n)_n$ in $C^\infty(\mathcal{U}) \cap W^{1,1}(\mathcal{U})$ such that*

$$\lim_{n\to+\infty} w_n = w \quad \textit{in } L^p(\mathcal{U}),$$
$$\lim_{n\to+\infty} \mathcal{J}(w_n) = \mathcal{J}(w),$$
$$w_n = 1 \quad \mathcal{H}^{N-1}\textit{-a.e.on } \partial\mathcal{U}.$$

Moreover, if there exist $c, d \in \mathbb{R}$, with $c \le 1 \le d$ and $c \le w \le d$ a.e. in $\mathcal{U}$, then, for each $\sigma > 0$, a sequence $(w_n)_n$, satisfying the previous condition, can be selected such that, for all n,

$$c - \sigma \le w_n \le d + \sigma \quad \textit{a.e. in } \mathcal{U}.$$

Remark 2.2 In the same way, we can prove that for any given $\varphi \in L^1(\partial\mathcal{U})$ and $w \in BV(\mathcal{U})$, there exists a sequence $(w_n)_n$ in $C^\infty(\mathcal{U}) \cap W^{1,1}(\mathcal{U})$ such that

$$\lim_{n\to+\infty} w_n = w \quad \text{in } L^p(\mathcal{U}),$$
$$\lim_{n\to+\infty} \mathcal{J}_\varphi(w_n) = \mathcal{J}_\varphi(w),$$
$$w_n = w \quad \mathcal{H}^{N-1}\text{-a.e. on } \partial\mathcal{U}.$$

To this aim we just have to apply directly Part 2 of the above proof to w instead of applying it to $\hat{w}_h$.

A lattice property

We finally show that $\mathcal{J}_\varphi$ satisfies the following lattice property, which encodes a kind of maximum principle.

Proposition 2.8. *Let $\varphi \in L^1(\partial\mathcal{U})$ be given. For any v, $w \in BV(\mathcal{U})$, we have*

$$\mathcal{J}_\varphi(v \wedge w) + \mathcal{J}_\varphi(v \vee w) \leq \mathcal{J}_\varphi(v) + \mathcal{J}_\varphi(w). \tag{2.20}$$

Proof. We first observe that, for any $v, w \in W^{1,1}(\mathcal{U})$,

$$\mathcal{J}_\varphi(v \wedge w) + \mathcal{J}_\varphi(v \vee w) = \mathcal{J}_\varphi(v) + \mathcal{J}_\varphi(w). \tag{2.21}$$

This easily follows by using, e.g., [55, Theorem 1.56], which guarantees that $v \wedge w, v \vee w \in W^{1,1}(\mathcal{U})$ and

$$\begin{aligned} |\nabla(v \wedge w)| &= \chi_{\{v<w\}}|\nabla v| + \chi_{\{v \geq w\}}|\nabla w|, \\ |\nabla(v \vee w)| &= \chi_{\{v<w\}}|\nabla w| + \chi_{\{v \geq w\}}|\nabla v|. \end{aligned}$$

Next, we pick any v, $w \in BV(\mathcal{U})$. By [2] we know that $v \wedge w$, $v \vee w \in BV(\mathcal{U})$. Corollary 2.7 assures the existence of two sequences $(v_n)_n$, $(w_n)_n$ in $W^{1,1}(\mathcal{U})$ such that

$$\begin{aligned} &\lim_{n\to+\infty} v_n = v, \qquad \lim_{n\to+\infty} w_n = w \quad \text{in } L^1(\mathcal{U}), \\ &\lim_{n\to+\infty} \mathcal{J}_\varphi(v_n) = \mathcal{J}_\varphi(v), \qquad \lim_{n\to+\infty} \mathcal{J}_\varphi(w_n) = \mathcal{J}_\varphi(w). \end{aligned}$$

Moreover, we have

$$\begin{aligned} &\lim_{n\to+\infty} v_n \wedge w_n = v \wedge w \quad \text{in } L^1(\mathcal{U}), \\ &\lim_{n\to+\infty} v_n \vee w_n = v \vee w \quad \text{in } L^1(\mathcal{U}). \end{aligned}$$

Hence, by the lower semicontinuity of $\mathcal{J}_\varphi$ with respect to the L^1-convergence, we deduce from (2.21)

$$\begin{aligned} \mathcal{J}_\varphi(v \wedge w) + \mathcal{J}_\varphi(v \vee w) &\leq \liminf_{n\to+\infty} \mathcal{J}_\varphi(v_n \wedge w_n) + \liminf_{n\to+\infty} \mathcal{J}_\varphi(v_n \vee w_n) \\ &\leq \liminf_{n\to+\infty} \big(\mathcal{J}_\varphi(v_n) + \mathcal{J}_\varphi(w_n)\big) \\ &= \mathcal{J}_\varphi(v) + \mathcal{J}_\varphi(w), \end{aligned}$$

that is (2.20) holds. $\square$

3 Global minimization

In this section we prove that the action functional, naturally associated with problem (1.6), has a unique global minimizer in the cone of non-negative functions of $BV(\Omega)$, which is bounded, strictly positive and regular in Ω; in addition, it satisfies a suitable variational inequality.

Definition 3.1. *Let us set*

$$BV^+(\Omega) = \{w \in BV(\Omega) \mid w \geq 0 \text{ a.e. in } \Omega\}$$

and define the functional $\mathcal{I} : BV^+(\Omega) \to \mathbb{R}$ by setting

$$\mathcal{I}(v) = \mathcal{J}(v) + \mathcal{F}(v),$$

where $\mathcal{J}$ has been introduced in Definition 2.2, with $\mathcal{U} = \Omega$, and $\mathcal{F} : BV^+(\Omega) \to \mathbb{R}$ is the potential functional

$$\mathcal{F}(v) = \int_\Omega F(v)\, dx,$$

with $F : [0, +\infty[\to \mathbb{R}$ the continuous extension of the function $\frac{a}{b^2} s(\log(s) - 1)$.

Existence, uniqueness and localization of the global minimizer

Proposition 3.1. *The functional $\mathcal{I}$ has a unique global minimizer $v \in BV^+(\Omega)$, which satisfies*

$$\exp\big(-\tfrac{b^2}{a}\big) \le v \le 1 \quad a.e. \ in \ \Omega.$$

Proof. Let us fix $p \in]1, 1^*[$.

Step 1. $\mathcal{I}$ is lower semicontinuous with respect to the L^p-convergence. By Proposition 2.5, $\mathcal{J}$ is lower semicontinuous with respect to the L^1-convergence and therefore with respect to the L^p-convergence in $BV(\Omega)$. On the other hand, there exists $c > 0$ such that F satisfies

$$|F(s)| \le c(|s|^p + 1), \qquad \text{for all } s \ge 0,$$

and then, by, e.g., [17, Theorem 2.3], $\mathcal{F}$ is continuous with respect to the L^p-convergence in $BV^+(\Omega)$. Hence the conclusion follows.

Step 2. Existence of a global minimizer. Let $(v_n)_n$ be a minimizing sequence of $\mathcal{I}$ in $BV^+(\Omega)$. By Proposition 2.1, we have

$$\begin{aligned}\max\Big\{\int_\Omega |v_n|\,dx, \frac{1}{b}\int_\Omega |Dv_n|\Big\} &\le \int_\Omega \sqrt{v_n^2 + b^{-2}|Dv_n|^2} \\ &\le \mathcal{J}(v_n) + \int_\Omega \big(F(v_n) - \min_{[0,+\infty[} F\big)\,dx = \mathcal{I}(v_n) + \frac{a}{b^2}|\Omega|.\end{aligned}$$

Hence $(v_n)_n$ is bounded in $BV(\Omega)$. By [1, Corollary 3.49, Proposition 3.6], there exists a subsequence of $(v_n)_n$, still denoted by $(v_n)_n$, and $v \in BV^+(\Omega)$ such that $\lim_{n\to+\infty} v_n = v$ in $L^p(\Omega)$. The lower semicontinuity of $\mathcal{I}$ with respect to the L^p-convergence yields

$$\mathcal{I}(v) \le \lim_{n\to+\infty} \mathcal{I}(v_n) = \inf_{BV^+(\Omega)} \mathcal{I},$$

that is v is a global minimizer of $\mathcal{I}$ in $BV^+(\Omega)$.

Step 3. Uniqueness of the global minimizer. Since $\mathcal{J}$ is convex $BV(\Omega)$ and $\mathcal{F}$ is strictly convex in $BV^+(\Omega)$, because F is strictly convex in $[0, +\infty[$, the functional $\mathcal{I}$ is strictly convex in $BV^+(\Omega)$. This implies the uniqueness of the global minimizer.

Step 4. We have $v \ge \exp\big(-\frac{b^2}{a}\big)$ a.e. in Ω. Let us set, for convenience, $\varepsilon = \exp\big(-\frac{b^2}{a}\big)$. As v is a global minimizer, by Proposition 2.8, we have

$$0 \le \mathcal{I}(v \vee \varepsilon) - \mathcal{I}(v) \le \mathcal{J}(\varepsilon) - \mathcal{J}(v \wedge \varepsilon) + \mathcal{F}(v \vee \varepsilon) - \mathcal{F}(v).$$

Using Proposition 2.1 and $\varepsilon \in]0, 1]$, we obtain

$$\begin{aligned}\mathcal{J}(\varepsilon) - \mathcal{J}(v \wedge \varepsilon) &= \int_\Omega \varepsilon - \int_\Omega \sqrt{(v\wedge\varepsilon)^2 + b^{-2}|D(v\wedge\varepsilon)|^2} \\ &\qquad + \frac{1}{b}\int_{\partial\Omega} (|\varepsilon - 1| - |(v\wedge\varepsilon) - 1|)\,d\mathcal{H}^{N-1} \\ &\le \int_\Omega (\varepsilon - |v \wedge \varepsilon|)\,dx + \frac{1}{b}\int_{\partial\Omega} ((v\wedge\varepsilon) - \varepsilon)\,d\mathcal{H}^{N-1} \\ &\le \int_\Omega (\varepsilon - (v\wedge\varepsilon))\,dx = \int_{\{v<\varepsilon\}} (\varepsilon - v)\,dx.\end{aligned}$$

Thus we have

$$0 \le \mathcal{I}(v \vee \varepsilon) - \mathcal{I}(v) \le \int_{\{v<\varepsilon\}} (\varepsilon - v + F(\varepsilon) - F(v))\,dx.$$

Since the function $G : [0,\infty[\to \mathbb{R}$, defined by $G(s) = s + F(s)$, is strictly decreasing in $[0,\varepsilon]$, we conclude that

$$0 \le \mathcal{I}(v \vee \varepsilon) - \mathcal{I}(v) \le \int_{\{v<\varepsilon\}} (G(\varepsilon) - G(v))\, dx \le 0,$$

where the last inequality is strict if $|\{v < \varepsilon\}| > 0$. This implies that $|\{v < \varepsilon\}| = 0$, i.e., $v \ge \varepsilon = \exp\big(-\frac{b^2}{a}\big)$ a.e. in Ω.

Step 4. We have $v \le 1$ a.e. in Ω. By Proposition 2.8, we have

$$\mathcal{I}(v \wedge 1) - \mathcal{I}(v) \le \mathcal{J}(1) - \mathcal{J}(v \vee 1) + \mathcal{F}(v \wedge 1) - \mathcal{F}(v).$$

On the one hand, by Proposition 2.1, we get

$$\begin{aligned}\mathcal{J}(1) \; - \mathcal{J}(v \vee 1) &= \int_\Omega 1 - \int_\Omega \sqrt{(v \vee 1)^2 + b^{-2}|D(v \vee 1)|^2} - \frac{1}{b}\int_{\partial\Omega} |(v \vee 1) - 1|\, d\mathcal{H}^{N-1} \\ &\le \int_\Omega (1 - |v \vee 1|)\, dx \le 0.\end{aligned}$$

On the other hand, since F is increasing in $[1, +\infty[$, we infer

$$\mathcal{F}(v \wedge 1) - \mathcal{F}(v) = \int_{\{v \ge 1\}} (F(1) - F(v))\, dx \le 0.$$

We then obtain

$$\mathcal{I}(v \wedge 1) \le \mathcal{I}(v).$$

As v is the unique global minimizer of $\mathcal{I}$, this implies that $v \wedge 1 = v$, i.e., $v \le 1$ a.e. in Ω. $\square$

Interior $C^{1,\alpha}$-regularity of the global minimizer

In order to prove the local $C^{1,\alpha}$-regularity in Ω of the global minimizer v of $\mathcal{I}$, we use an argument which requires a preliminary study of the problem

$$\begin{cases} -\mathrm{div}\Big(\dfrac{\nabla z}{\sqrt{z^2 + b^{-2}|\nabla z|^2}}\Big) = -a\log(z) - \dfrac{b^2 z}{\sqrt{z^2 + b^{-2}|\nabla z|^2}} & \text{in } B_r, \\ z = \psi & \text{on } \partial B_r, \end{cases} \tag{3.1}$$

where $B_r = B(x_0, r)$ is a ball of center $x_0 \in \Omega$ and radius $r > 0$, with $\overline{B}_r \subseteq \Omega$, and $\psi \in C^{2,\alpha}(\overline{B_r})$, for some $\alpha \in]0,1[$, is a given function, with

$$\tfrac{1}{2}\exp\big(-\tfrac{b^2}{a}\big) \le \psi \le \tfrac{3}{2} \quad \text{in } \overline{B_r}. \tag{3.2}$$

We associate with problem (3.1) the functional $\mathcal{I}_r : BV^+(B_r) \to \mathbb{R}$, defined by

$$\mathcal{I}_r(w) = \int_{B_r} \sqrt{w^2 + b^{-2}|Dw|^2} + \frac{1}{b}\int_{\partial B_r} |w - \psi|\, d\mathcal{H}^{N-1} + \int_{B_r} F(w)\, dx,$$

where $BV^+(B_r) = \{w \in BV(B_r) \mid w \ge 0 \text{ a.e. in } B_r\}$ and F has been introduced in Definition 3.1.

Lemma 3.2. *Fix any $x_0 \in \Omega$. Then there exists $r_0 > 0$ such that, for any given $r \in]0, r_0[$ and every $\psi \in C^{2,\alpha}(\overline{B_r})$ satisfying (3.2), problem (3.1) has a unique solution $z \in C^{2,\alpha}(\overline{B_r})$ such that*

(i) $\frac{1}{2}\exp\big(-\frac{b^2}{a}\big) \le z \le \frac{3}{2}$ *in B_r;*

(*ii*) *there exist* $\beta = \beta(a, b, N, r) > 0$ *and* $C = C(a, b, N, r) > 0$, *independent of* ψ, *such that* $\|z\|_{C^{1,\beta}(\overline{B_{r/4}})} \leq C$;

(*iii*) z *is a global minimizer of* $\mathcal{I}_r$ *in* $BV^+(B_r)$.

Proof. Set $\phi = -\frac{1}{b}\log(\psi)$. By (3.2), we have

$$-\tfrac{1}{b}\log\bigl(\tfrac{3}{2}\bigr) \leq \phi \leq \tfrac{b}{a} + \tfrac{1}{b}\log 2 \quad \text{in } B_r.$$

It is clear that $z \in C^{2,\alpha}(\overline{B_r})$ is a solution of (3.1), satisfying (*i*) and (*ii*), if and only if $u = -\frac{1}{b}\log(z) \in C^{2,\alpha}(\overline{B_r})$ is a solution of

$$\begin{cases} -\mathrm{div}\Bigl(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Bigr) = -au + \dfrac{b}{\sqrt{1+|\nabla u|^2}} & \text{in } B_r, \\ u = \phi & \text{on } \partial B_r, \end{cases} \tag{3.3}$$

satisfying

(*i'*) $-\frac{1}{b}\log\bigl(\frac{3}{2}\bigr) \leq u \leq \frac{b}{a} + \frac{1}{b}\log 2 \quad$ in B_r;

(*ii'*) there exist $\beta = \beta(a, b, N, r) > 0$ and $D = D(a, b, N, r) > 0$, independent of ϕ, such that $\|u\|_{C^{1,\beta}(\overline{B_{r/4}})} \leq D$.

Step 1. There exists $r_0 > 0$ *such that, if* $r \in]0, r_0[$, *then problem* (3.3) *has a unique solution* $u \in C^{2,\alpha}(\overline{B_r})$. The conclusion will follow by [39, Corollary 1]; here we use the notations there introduced. Let us define $H : \mathbb{R} \times \mathbb{R}^{N+1} \to \mathbb{R}$ by $H(z, \xi_1, \ldots, \xi_{N+1}) = \frac{1}{N}(az - b\xi_{N+1})$. Condition (*sc*) in [39] is satisfied, with $H_1(z, \xi_1, \ldots, \xi_{N+1}) = \frac{a}{N}z$ and $H_2(z, \xi_1, \ldots, \xi_{N+1}) = -\frac{b}{N}$. Moreover, we have $\partial_z H = \partial_z H_1 \geq 0$ in $\mathbb{R}$. Let us therefore verify condition (2) therein. For all $\eta \in C^1_0(\Omega)$, we have

$$\left|\int_{B_r} H(0, \nabla\eta)\,\eta\,dx\right| \leq \frac{b}{N}\int_{B_r}|\eta|\,dx \leq \frac{b}{N}C(B_r)\int_{B_r}|\nabla\eta|\,dx,$$

with $C(B_r) = \frac{r}{N}$ (see [14, Example 3.2]). Hence condition (2) in [39] holds if $r < \frac{N}{b}$. Finally, as the mean curvature of the sphere ∂B_r is $H_{\partial B_r} = \frac{1}{r}$, condition (3) in [39] is satisfied provided $r \in \bigl]0, \bigl(\frac{a}{N-1}\bigl(\frac{b}{a} + \frac{1}{b}\log 2\bigr)\bigr)^{-1}\bigr]$. Set $r_0 = \min\{\mathrm{dist}(x_0, \partial\Omega), \bigl(\frac{a}{N-1}\bigl(\frac{b}{a} + \frac{1}{b}\log 2\bigr)\bigr)^{-1}\}$. Observing that $r_0 < \frac{N}{b}$, Corollary 1 in [39] applies and yields the conclusion.

Step 2. Condition (*i'*) *holds.* The equation in (3.3) can be written as

$$\frac{\Delta u}{\sqrt{1+|\nabla u(x)|^2}} - \sum_{i,j=1}^{N}\frac{\partial_{x_i}u\,\partial_{x_j}u\,\partial^2_{x_jx_i}u}{(1+|\nabla u(x)|^2)^{3/2}} = au - \frac{b}{\sqrt{1+|\nabla u(x)|^2}} \quad \text{in } B_r. \tag{3.4}$$

If we assume that $\max_{\overline{B_r}} u > \frac{b}{a} + \frac{1}{b}\log 2$, then the boundary conditions imply that $\max_{\overline{B_r}} u$ is attained at some point $\bar{x} \in B_r$. Then evaluating (3.4) at x_0 yields the contradiction

$$0 \geq \Delta u(\bar{x}) = au(\bar{x}) - b > 0.$$

Hence we conclude that $\max_{\overline{B_r}} u \leq \frac{b}{a} + \frac{1}{b}\log 2$. Similarly one proves that $\min_{\overline{B_r}} u \geq -\frac{1}{b}\log\bigl(\frac{3}{2}\bigr)$.

Step 3. There exists $d = d(a, b, N, r) > 0$ *such that* $\|\nabla u\|_{L^\infty(B_{r/2})} \leq d$. Let us show that all the conditions required by [34, Theorem 4] are fulfilled; here we keep the notations there introduced. Let us set $M = \frac{b}{a} + \frac{1}{b}\log 2$,

$$a_j(\xi) = \frac{\xi_j}{\sqrt{1+|\xi|^2}}, \qquad \text{for } j \in \{1, \ldots, N\}, \qquad \text{and} \qquad a(s, \xi) = as - \frac{b}{\sqrt{1+|\xi|^2}},$$

for all $s \in [-\frac{1}{b}\log\frac{3}{2}, M]$ and $\xi \in \mathbb{R}^N$. It is easy to check that conditions (2.2), (2.3), (2.4) of [34, Theorem 4] are satisfied, with $\mu_0 = 1$, $\mu_1 = \mu_2 = 1$ and $\mu_3 = a(\frac{b}{a} + \frac{1}{b}\log 2) + b$. To verify condition (2.5) of [34, Theorem 4], we observe that

$$\begin{aligned}\sum_{i,j=1}^N a_{ij}(\nabla u)\xi_i\xi_j &= \sum_{i,j=1}^N \frac{\delta_{ij}(1+|\nabla u|^2) - \partial_{x_i}u\,\partial_{x_j}u}{(1+|\nabla u|^2)^{3/2}}\xi_i\xi_j \\ &= \frac{|\xi|^2}{\sqrt{1+|\nabla u|^2}} - \frac{(\nabla u\,\xi)^2}{(1+|\nabla u|^2)^{3/2}} = \frac{|\xi'|^2}{\sqrt{1+|\nabla u|^2}}.\end{aligned}$$

Hence condition (2.5) of [34, Theorem 4] is satisfied with $\mu_4 = \mu_5 = 1$. On the other hand, we have

$$A(u,\nabla u) = -b\sum_{i,l=1}^N \frac{\partial_{x_i}u\,\partial_{x_l}u}{(1+|\nabla u|^2)^{3/2}}\delta_i\partial_{x_l}u - b\sum_{i,l=1}^N \frac{(\partial_{x_i}u)^2\partial_{x_l}u}{(1+|\nabla u|^2)^{3/2}}\delta_{N+1}\partial_{x_l}u - a|\nabla u|^2.$$

Since

$$\frac{|\partial_{x_i}u\,\partial_{x_l}u|}{(1+|\nabla u|^2)^{3/2}} \le 1 \quad \text{and} \quad \frac{|(\partial_{x_i}u)^2\partial_{x_l}u|}{(1+|\nabla u|^2)^{3/2}} \le 1,$$

for all $i, l \in \{1,\dots,N\}$, we obtain

$$A(u,\nabla u) \le bN\sum_{i=1}^{N+1}\sum_{l=1}^N |\delta_i\partial_{x_l}u| \le bN\sqrt{N(N+1)}|\delta\nabla u|.$$

Hence condition (2.22) of [34, Theorem 4] is satisfied with $\mu_6 = bN\sqrt{N(N+1)}$ and $\mu_7 = 0$. This shows that [34, Theorem 4] applies and yields the existence of $d = d(a,b,N,r) > 0$ such that

$$\|\nabla u\|_{L^\infty(B_{r/2})} \le d.$$

Step 4. Condition (ii′) holds. By Step 2, it is enough to prove that $\|\nabla u\|_{C^{0,\beta}(\overline{B_{r/4}})} \le D$ for some $\beta = \beta(a,b,N,r) > 0$ and $D = D(a,b,N,r) > 0$. This can be easily deduced from [33, Chapter 6, Theorem 1.1], by using Step 2 and Step 3 above and noticing that the equation in (3.3) can be written as

$$\sum_{i,j=1}^N a_{ij}(\nabla u)\partial_{x_ix_j}u + a(u,\nabla u) = 0 \quad \text{in } B_r,$$

with

$$a_{ij}(\xi) = \frac{\delta_{ij}(1+|\xi|^2) - \xi_i\,\xi_j}{(1+|\xi|^2)^{3/2}}, \quad \text{for } i,j \in \{1,\dots,N\}, \quad \text{and} \quad a(s,\xi) = -as + \frac{b}{\sqrt{1+|\xi|^2}},$$

for all $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$.

Step 5. Conditions (i) and (ii) hold. This follows from Step 2 and Step 4 by the change of variable $u = -\frac{1}{b}\log(z)$.

Step 6. For all $w \in W^{1,1}(B_r)$, we have

$$\begin{aligned}\int_{B_r}\sqrt{w^2+b^{-2}|\nabla w|^2}\,dx + \frac{1}{b}\int_{\partial B_r}|w-\psi|\,d\mathcal{H}^{N-1} - \int_{B_r}\sqrt{z^2+b^{-2}|\nabla z|^2}\,dx \\ - \frac{1}{b}\int_{\partial B_r}|z-\psi|\,d\mathcal{H}^{N-1} \ge -\frac{a}{b^2}\int_{B_r}\log(z)\,(w-z)\,dx. \qquad (3.5)\end{aligned}$$

Pick any $w \in W^{1,1}(B_r)$. We first multiply the equation in (3.1) by $w - z$ and integrate by parts using [43, Section 3.1.2, Theorem 1.1]. Then the convexity in $[0, +\infty[\times \mathbb{R}^N$ and the differentiability in $]0, +\infty[\times \mathbb{R}^N$ of the map $(s, \xi) \mapsto \sqrt{s^2 + b^{-2}|\xi|^2}$, together with condition (i), yield

$$\begin{aligned}\int_{B_r} \sqrt{w^2 + b^{-2}|\nabla w|^2}\,dx \,-\, &\int_{B_r} \sqrt{z^2 + b^{-2}|\nabla z|^2}\,dx \\ &\geq \frac{1}{b^2}\int_{B_r} \frac{\nabla z\,\nabla(w-z)}{\sqrt{z^2 + b^{-2}|\nabla z|^2}}\,dx + \int_{B_r} \frac{z(w-z)}{\sqrt{z^2 + b^{-2}|\nabla z|^2}}\,dx \\ &= \frac{1}{b^2}\int_{\partial B_r} \frac{\nabla z\,\nu}{\sqrt{z^2 + b^{-2}|\nabla z|^2}}\,(w-z)\,d\mathcal{H}^{N-1} - \frac{a}{b^2}\int_{B_r} \log(z)\,(w-z)\,dx \\ &\geq -\frac{1}{b}\int_{\partial B_r} |w-z|\,d\mathcal{H}^{N-1} - \frac{a}{b^2}\int_{B_r} \log(z)\,(w-z)\,dx,\end{aligned}$$

from which (3.5) follows.

Step 7. Condition (iii) holds. Pick any $w \in BV^+(B_r)$. Proposition 2.6 guarantees the existence of a sequence $(w_n)_n$ in $W^{1,1}(B_r)$ such that $\lim\limits_{n\to+\infty} w_n = w$ in $L^1(B_r)$ and

$$\begin{aligned}\lim_{n\to+\infty}\left(\int_{B_r} \sqrt{w_n^2 + b^{-2}|\nabla w_n|^2}\,dx + \frac{1}{b}\int_{\partial B_r} |w_n - \psi|\,d\mathcal{H}^{N-1}\right)& \\ = \int_{B_r} \sqrt{w^2 + b^{-2}|Dw|^2} + \frac{1}{b}\int_{\partial B_r} |w-\psi|\,d\mathcal{H}^{N-1}&.\end{aligned} \tag{3.6}$$

By (3.5) we have, for all n,

$$\begin{aligned}\int_{B_r} \sqrt{w_n^2 + b^{-2}|\nabla w_n|^2}\,dx + \frac{1}{b}\int_{\partial B_r} |w_n - \psi|\,d\mathcal{H}^{N-1} - \int_{B_r} \sqrt{z^2 + b^{-2}|\nabla z|^2}\,dx& \\ - \frac{1}{b}\int_{\partial B_r} |z-\psi|\,d\mathcal{H}^{N-1} \geq -\frac{a}{b^2}\int_{B_r} \log(z)\,(w_n - z)\,dx&.\end{aligned} \tag{3.7}$$

Passing to the limit in (3.7) and using (3.6), we obtain

$$\begin{aligned}\int_{B_r} \sqrt{w^2 + b^{-2}|Dw|^2} + \frac{1}{b}\int_{\partial B_r} |w-\psi|\,d\mathcal{H}^{N-1} - \int_{B_r} \sqrt{z^2 + b^{-2}|\nabla z|^2}\,dx& \\ - \frac{1}{b}\int_{\partial B_r} |z-\psi|\,d\mathcal{H}^{N-1} \geq -\frac{a}{b^2}\int_{B_r} \log(z)\,(w-z)\,dx&.\end{aligned}$$

Conclusion (iii) then follows using condition (i) and the convexity in $[0, +\infty[$ and the differentiability in $]0, +\infty[$ of the function F. □

Proposition 3.3. *The global minimizer $v \in BV^+(\Omega)$ of $\mathcal{I}$ belongs to $W^{1,1}(\Omega)$, and, for every open set Ω_1, with $\overline{\Omega_1} \subseteq \Omega$, there exists $\alpha > 0$ such that $v \in C^{1,\alpha}(\overline{\Omega_1})$.*

Proof. Let $p \in\,]1, 1^*[$ be fixed. By Corollary 2.7, there exists a sequence $(v_n)_n$ in $C^\infty(\Omega) \cap W^{1,1}(\Omega)$ such that

$$\lim_{n\to+\infty} v_n = v \quad \text{in } L^p(\Omega),$$

$$\lim_{n\to+\infty} \mathcal{J}(v_n) = \mathcal{J}(v),$$

$$\tfrac{1}{2}\exp\big(-\tfrac{b^2}{a}\big) \leq v_n \leq \tfrac{3}{2}, \quad \text{in } \Omega.$$

The continuity of the potential operator $\mathcal{F}$ in $L^p(\Omega)$ also implies

$$\lim_{n\to+\infty} \mathcal{I}(v_n) = \mathcal{I}(v). \tag{3.8}$$

Let now fix $x_0 \in \Omega$ and take $r \in]0, r_0[$, with $r_0 > 0$ given by Lemma 3.2. For each n, consider the problem

$$\begin{cases} -\mathrm{div}\left(\dfrac{\nabla z}{\sqrt{z^2 + b^{-2}|\nabla z|^2}}\right) = -a\log(z) - \dfrac{b^2 z}{\sqrt{z^2 + b^{-2}|\nabla z|^2}} & \text{in } B_r, \\ z = v_n & \text{on } \partial B_r \end{cases}$$

and denote by $z_n \in C^2(\overline{B_r})$ its unique solution provided by Lemma 3.2. Define the sequence $(w_n)_n$ in $BV(\Omega)$ by setting

$$w_n = \begin{cases} z_n & \text{a.e. in } \overline{B_r} \\ v_n & \text{a.e. in } \Omega \setminus \overline{B_r}. \end{cases}$$

Step 1. The sequence $(w_n)_n$ is bounded in $BV(\Omega)$ and satisfies $\lim_{n\to+\infty} w_n = v$ in $L^p(\Omega)$. Using Proposition 2.4 and conclusion (iii) in Lemma 3.2 with $\psi = v_n$, we obtain

$$\begin{aligned} \mathcal{I}(w_n) &= \int_\Omega \sqrt{w_n^2 + b^{-2}|Dw_n|^2} + \frac{1}{b}\int_{\partial\Omega} |w_n - 1|\, d\mathcal{H}^{N-1} + \int_\Omega F(w_n)\, dx \\ &= \int_{B_r} \sqrt{z_n^2 + b^{-2}|Dz_n|^2} + \int_{\Omega\setminus\overline{B_r}} \sqrt{v_n^2 + b^{-2}|Dv_n|^2} \; + \frac{1}{b}\int_{\partial B_r} |z_n - v_n|\, d\mathcal{H}^{N-1} \\ &\quad + \frac{1}{b}\int_{\partial\Omega} |v_n - 1|\, d\mathcal{H}^{N-1} + \int_{B_r} F(z_n)\, dx + \int_{\Omega\setminus B_r} F(v_n)\, dx \\ &= \mathcal{I}_r(z_n) + \int_{\Omega\setminus\overline{B_r}} \sqrt{v_n^2 + b^{-2}|Dv_n|^2} + \frac{1}{b}\int_{\partial\Omega} |v_n - 1|\, d\mathcal{H}^{N-1} + \int_{\Omega\setminus B_r} F(v_n)\, dx \\ &\le \mathcal{I}_r(v_n) + \int_{\Omega\setminus\overline{B_r}} \sqrt{v_n^2 + b^{-2}|Dv_n|^2} + \frac{1}{b}\int_{\partial\Omega} |v_n - 1|\, d\mathcal{H}^{N-1} + \int_{\Omega\setminus B_r} F(v_n)\, dx \\ &= \mathcal{I}(v_n). \end{aligned}$$

As a consequence of (3.8), we may assume that $\mathcal{I}(v_n) \le \mathcal{I}(v) + 1$, for all n. Hence, by Proposition 2.1, we obtain, as in Proposition 3.1,

$$\begin{aligned} \max\left\{\frac{1}{b}\int_\Omega |Dw_n|, \int_\Omega |w_n|\, dx\right\} &\le \int_\Omega \sqrt{w_n^2 + b^{-2}|Dw_n|^2} \le \mathcal{I}(w_n) + \frac{a}{b^2}|\Omega| \\ &\le \mathcal{I}(v_n) + \frac{a}{b^2}|\Omega| \le \mathcal{I}(v) + 1 + \frac{a}{b^2}|\Omega|. \end{aligned}$$

This implies that $(w_n)_n$ is bounded in $BV(\Omega)$ and hence, by [1, Corollary 3.49, Proposition 3.6], there exists a subsequence of $(w_n)_n$, still denoted by $(w_n)_n$, which converges in $L^p(\Omega)$ and a.e. in Ω to some $w \in BV(\Omega)$. As $w_n \ge \frac{1}{2}\exp\left(-\frac{b^2}{a}\right)$ in Ω for all n, we have that $w \in BV^+(\Omega)$. The lower semicontinuity of $\mathcal{I}$ with respect to the L^p-norm then yields

$$\mathcal{I}(w) \le \liminf_{n\to+\infty} \mathcal{I}(w_n) \le \liminf_{n\to+\infty} \mathcal{I}(v_n) = \mathcal{I}(v).$$

We finally conclude that $v = w$ by uniqueness of the minimizer of $\mathcal{I}$ in $BV^+(\Omega)$.

Step 2. For every open set Ω_1, with $\overline{\Omega_1} \subseteq \Omega$, there exists $\alpha > 0$ such that $v \in C^{1,\alpha}(\overline{\Omega_1})$. By Lemma 3.2, the sequence $(z_n)_n$ is bounded in $C^{1,\beta}(\overline{B_{r/4}})$. Therefore, for any given $\alpha \in]0, \beta[$, possibly passing to a subsequence, $(z_n)_n$ converges in $C^{1,\alpha}(\overline{B_{r/4}})$ to some $z \in C^{1,\alpha}(\overline{B_{r/4}})$. By uniqueness of the limit, we have $v = z \in C^{1,\alpha}(\overline{B_{r/4}})$. As this holds for every $x_0 \in \overline{\Omega}_1$, by compactness, we conclude that $v \in C^{1,\alpha}(\overline{\Omega}_1)$.

Step 3. v belongs to $W^{1,1}(\Omega)$. As $v \in C^1(\Omega) \cap BV(\Omega)$, we have $Dv = \nabla v\, dx$ and $\int_\Omega |\nabla v|\, dx = \int_\Omega |Dv|$ and then $v \in W^{1,1}(\Omega)$. □

A variational inequality

We prove now a characterization of the global minimizer v of $\mathcal{I}$ as a solution of an associated variational inequality.

Proposition 3.4. *Let $v \in BV(\Omega)$ be such that $\operatorname*{ess\,inf}_{\Omega} v > 0$. Then v is the global minimizer of $\mathcal{I}$ in $BV^+(\Omega)$ if and only if v satisfies the variational inequality*

$$\mathcal{J}(w) - \mathcal{J}(v) \geq -\frac{a}{b^2} \int_\Omega \log(v)\,(w-v)\,dx \tag{3.9}$$

for all $w \in BV(\Omega)$.

Proof. Step 1. If $v \in BV^+(\Omega)$ is the global minimizer of $\mathcal{I}$ in $BV^+(\Omega)$, then v satisfies (3.9) *for all $w \in BV(\Omega) \cap L^\infty(\Omega)$.* Let $w \in BV(\Omega) \cap L^\infty(\Omega)$ be fixed. By Proposition 3.1 we know that $v \in L^\infty(\Omega)$ and $\operatorname*{ess\,inf}_{\Omega} v > 0$. Hence there exists $\bar{t} > 0$ such that, for all $t \in [0, \bar{t}[$,

$$\tfrac{1}{2} \operatorname*{ess\,inf}_{\Omega} v \leq v + t(w-v) \leq 2 \operatorname*{ess\,sup}_{\Omega} v \quad \text{a.e. in } \Omega. \tag{3.10}$$

The convexity of $\mathcal{J}$ implies that, for all $t > 0$,

$$\mathcal{J}(w) - \mathcal{J}(v) = \frac{(1-t)\mathcal{J}(v) + t\mathcal{J}(w) - \mathcal{J}(v)}{t} \geq \frac{\mathcal{J}(v + t(w-v)) - \mathcal{J}(v)}{t}. \tag{3.11}$$

Moreover, as v is a global minimizer of $\mathcal{I} = \mathcal{J} + \mathcal{F}$ in $BV^+(\Omega)$, we have, for all $t \in]0, \bar{t}[$,

$$\frac{\mathcal{J}(v + t(w-v)) - \mathcal{J}(v)}{t} \geq -\int_\Omega \frac{F(v + t(w-v)) - F(v)}{t}\,dx. \tag{3.12}$$

On the other hand, as $F :]0, +\infty[\to \mathbb{R}$ is continuously differentiable, with $F'(s) = \frac{a}{b^2}\log(s)$, and (3.10) holds, we get, by the dominated convergence theorem,

$$\lim_{t \to 0^+} \int_\Omega \frac{F(v + t(w-v)) - F(v)}{t}\,dx = \frac{a}{b^2} \int_\Omega \log(v)\,(w-v)\,dx. \tag{3.13}$$

The conclusion then follows from (3.11), (3.12) and (3.13).

Step 2. If $v \in BV^+(\Omega)$ is the global minimizer of $\mathcal{I}$ in $BV^+(\Omega)$, then v satisfies (3.9) *for all $w \in BV(\Omega)$.* Let $w \in BV(\Omega)$ be fixed. By Corollary 2.7, there exists a sequence $(w_n)_n$ in $C^\infty(\Omega) \cap W^{1,1}(\Omega)$ such that

$$\lim_{n \to +\infty} w_n = w \quad \text{in } L^1(\Omega) \qquad \text{and} \qquad \lim_{n \to +\infty} \mathcal{J}(w_n) = \mathcal{J}(w).$$

For each n, let us define $\tilde{w}_n = (w_n \wedge n) \vee -n$. We have $\tilde{w}_n \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ and $\mathcal{J}(\tilde{w}_n) \leq \mathcal{J}(w_n)$. Therefore, from Step 1, we infer

$$\mathcal{J}(w_n) - \mathcal{J}(v) \geq \mathcal{J}(\tilde{w}_n) - \mathcal{J}(v) \geq -\frac{a}{b^2} \int_\Omega \log(v)\,(\tilde{w}_n - v)\,dx. \tag{3.14}$$

Since $\lim_{n \to +\infty} \tilde{w}_n = w$ in $L^1(\Omega)$, we get

$$\lim_{n \to +\infty} \int_\Omega \log(v)\,(\tilde{w}_n - v)\,dx = \int_\Omega \log(v)\,(w-v)\,dx.$$

By passing to the limit in (3.14), we conclude that (3.9) holds.

Step 3. If v satisfies (3.9) *for all $w \in BV(\Omega)$, then v is the global minimizer of $\mathcal{I}$ in $BV^+(\Omega)$.* Since F is convex and continuously differentiable in $]0,+\infty[$, with $F'(s) = \frac{a}{b^2}\log(s)$, and $\operatorname*{ess\,inf}_{\Omega} v > 0$, from (3.9) we get

$$\begin{aligned}\mathcal{I}(w) = \mathcal{J}(w) + \int_\Omega F(w)\,dx &\geq \mathcal{J}(w) + \int_\Omega F(v)\,dx + \int_\Omega F'(v)(w-v)\,dx \\ &\geq \mathcal{J}(v) + \int_\Omega F(v)\,dx = \mathcal{I}(v),\end{aligned}$$

for all $w \in BV^+(\Omega)$. Hence v is the global minimizer of $\mathcal{I}$ in $BV^+(\Omega)$. □

As a consequence of Proposition 3.4 we can show that v satisfies the equation in (1.6) in the weak sense.

Corollary 3.5. *The global minimizer $v \in W^{1,1}(\Omega)$ of $\mathcal{I}$ in $BV^+(\Omega)$ satisfies*

$$\int_\Omega \frac{\nabla v \nabla \phi}{\sqrt{v^2 + b^{-2}|\nabla v|^2}}\,dx + \int_\Omega \frac{b^2 v\,\phi}{\sqrt{v^2 + b^{-2}|\nabla v|^2}}\,dx + a\int_\Omega \log(v)\,\phi\,dx = 0 \tag{3.15}$$

for all $\phi \in C_0^1(\Omega)$.

Proof. Pick $\phi \in C_0^1(\Omega)$. As $v \in W^{1,1}(\Omega)$ satisfies (3.9), we have, for all $t > 0$,

$$\int_\Omega \frac{1}{t}\Big(\sqrt{(v+t\phi)^2 + b^{-2}|\nabla(v+t\phi)|^2} - \sqrt{v^2 + b^{-2}|\nabla v|^2}\Big)\,dx + \frac{a}{b^2}\int_\Omega \log(v)\,\phi\,dx \geq 0.$$

Using $\operatorname*{ess\,inf}_{\Omega} v > 0$, we can pass to the limit as $t \to 0^+$ and get

$$\frac{1}{b^2}\int_\Omega \frac{\nabla v\,\nabla\phi}{\sqrt{v^2 + b^{-2}|\nabla v|^2}}\,dx + \int_\Omega \frac{v\,\phi}{\sqrt{v^2 + b^{-2}|\nabla v|^2}}\,dx + \frac{a}{b^2}\int_\Omega \log(v)\,\phi\,dx \geq 0.$$

By replacing ϕ with $-\phi$, we then conclude that (3.15) holds. □

Interior smoothness of the global minimizer

We are finally in position of proving the smoothness in Ω of the global minimizer v of $\mathcal{I}$.

Proposition 3.6. *The global minimizer $v \in BV^+(\Omega)$ of $\mathcal{I}$ belongs to $C^\infty(\Omega) \cap W^{1,1}(\Omega)$.*

Proof. As $\operatorname*{ess\,inf}_{\Omega} v > 0$, using Corollary 3.5 and Proposition 3.3, we have that, for any smooth subdomain Ω_1, with $\overline{\Omega_1} \subseteq \Omega$, v is a weak solution of the linear Dirichlet problem

$$\begin{cases} \displaystyle\sum_{i,j=1}^N a^{ij}(x)\partial_{x_i x_j} z = g(x) & \text{in } \Omega_1, \\ z = v & \text{on } \partial\Omega_1, \end{cases}$$

with coefficients

$$a^{ij} = \frac{\delta_{ij}}{\sqrt{v^2 + b^{-2}|\nabla v|^2}} - \frac{\partial_{x_i} v\,\partial_{x_j} v}{b^2(v^2 + b^{-2}|\nabla v|^2)^{3/2}},$$

for $i, j \in \{1, \dots, N\}$, and

$$g = \frac{v\,|\nabla v|^2}{(v^2 + b^{-2}|\nabla v|^2)^{3/2}} + a\,\log(v) + \frac{b^2 v}{\sqrt{v^2 + b^{-2}|\nabla v|^2}}$$

belonging to $C^{0,\alpha}(\overline{\Omega_1})$. The result can then be deduced from [25, Theorem 6.13] and iterated applications of [25, Theorem 6.17]. $\square$

4 Minimizers towards solutions

We show here the equivalence between problem (1.4) and the variational inequality (3.9), which by Proposition 3.4 is in turn equivalent to the minimization of $\mathcal{I}$ in $BV^+(\Omega)$.

We start proving a localization result for any solution of (1.4).

Proposition 4.1. *Let u be a solution of (1.4). Then $u \in L^\infty(\Omega)$ and $0 \le u \le b/a$ a.e. in Ω.*

Proof. From the equation in (1.4), we see that $u \in L^N(\Omega)$. Then multiplying the equation by u^-, using the integration by parts formula in [3, Proposition 1.3], which holds according to Remark 1.1, and the boundary conditions satisfied by u, we get

$$\begin{aligned}
0 &\ge -\int_\Omega \frac{|\nabla u^-|^2}{\sqrt{1+|\nabla u^-|^2}}\,dx - \int_{\partial\Omega} u^-\,d\mathcal{H}^{N-1} \\
&= \int_\Omega \frac{\nabla u\,\nabla u^-}{\sqrt{1+|\nabla u|^2}}\,dx - \int_{\partial\Omega}\left[\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}, \nu\right] u^-\,d\mathcal{H}^{N-1} \\
&= -\int_\Omega \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) u^-\,dx \\
&= -\int_\Omega a u u^-\,dx + \int_\Omega \frac{b\,u^-}{\sqrt{1+|\nabla u|^2}}\,dx \\
&= \int_\Omega a(u^-)^2\,dx + \int_\Omega \frac{b\,u^-}{\sqrt{1+|\nabla u^-|^2}}\,dx \ge 0
\end{aligned}$$

and hence $u(x) \ge 0$ for a.e. $x \in \Omega$. In a completely similar way, multiplying now by $(u - \frac{b}{a})^+$, we prove that $u(x) \le \frac{b}{a}$ for a.e. $x \in \Omega$. $\square$

Proposition 4.2. *Let $v \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$, with $0 < \operatorname*{ess\,inf}_\Omega v \le \operatorname*{ess\,sup}_\Omega v \le 1$, satisfy (3.9) for all $w \in BV(\Omega)$. Then $u = -\frac{1}{b}\log(v) \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ is a solution of (1.4).*

Proof. Step 1. The function u is such that $u \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$, $\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) \in L^\infty(\Omega)$ and u satisfies the equation in (1.4) a.e. in Ω. As $v \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ and $0 < \operatorname*{ess\,inf}_\Omega v \le \operatorname*{ess\,sup}_\Omega v \le 1$, we have $u = -\frac{1}{b}\log(v) \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ and $u \ge 0$ a.e. in Ω. By Proposition 3.4 and Corollary 3.5 we know that, for any $\phi \in C_0^\infty(\Omega)$, v satisfies (3.15) and hence u satisfies

$$-\frac{1}{b}\int_\Omega \frac{\nabla u\,\nabla\phi}{\sqrt{1+|\nabla u|^2}}\,dx + \int_\Omega \frac{\phi}{\sqrt{1+|\nabla u|^2}}\,dx - \frac{a}{b}\int_\Omega u\,\phi\,dx = 0.$$

We then conclude that $\operatorname{div}\Big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) \in L^\infty(\Omega)$ and

$$-\operatorname{div}\Big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) = -au + \frac{b}{\sqrt{1+|\nabla u|^2}}, \tag{4.1}$$

a.e. in Ω.

Step 2. For $\mathcal{H}^{N-1}$-a.e. $x \in \partial\Omega$, either $u(x) = 0$, or both $u(x) > 0$ and $\Big[\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}, \nu\Big](x) = -1$ hold.
Let us fix $\phi \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ such that, for $\mathcal{H}^{N-1}$-a.e. $x \in \partial\Omega$, $\phi(x) = 0$ whenever $v(x) = 1$. Pick $t > 0$. By assumption v satisfies (3.9) and hence we have

$$\begin{aligned}\int_\Omega \frac{1}{t}\Big(\sqrt{(v+t\phi)^2 + b^{-2}|\nabla(v+t\phi)|^2} - \int_\Omega \sqrt{v^2+b^{-2}|\nabla v|^2}\Big)\,dx& \\ +\frac{1}{b}\int_{\partial\Omega} \frac{1}{t}\Big(|v+t\phi-1| - |v-1|\Big)\,d\mathcal{H}^{N-1} + \frac{a}{b^2}\int_\Omega \log(v)\,\phi\,dx \geq 0.&\end{aligned} \tag{4.2}$$

Since $v \in W^{1,1}(\Omega)$ satisfies $v \leq 1$ a.e. in Ω, there also holds $v \leq 1$ $\mathcal{H}^{N-1}$-a.e. on $\partial\Omega$ (see [16, Theorem 5.3.2]). Moreover, according to the choice of ϕ, we have, for $\mathcal{H}^{N-1}$-a.e. $x \in \partial\Omega$,

$$\lim_{t\to 0^+} \frac{1}{t}\Big(|v(x)+t\phi(x)-1| - |v(x)-1|\Big) = -\phi(x).$$

On the other hand, we can easily verify that, for all $t > 0$ and $\mathcal{H}^{N-1}$-a.e. $x \in \partial\Omega$,

$$\Big|\frac{1}{t}\Big(|v(x)+t\phi(x)-1| - |v(x)-1|\Big)\Big| \leq |\phi(x)|.$$

Hence, the dominated convergence theorem yields

$$\lim_{t\to 0^+} \frac{1}{b}\int_{\partial\Omega} \frac{1}{t}\Big(|v+t\phi-1| - |v-1|\Big)\,d\mathcal{H}^{N-1} = -\frac{1}{b}\int_{\partial\Omega} \phi\,d\mathcal{H}^{N-1}.$$

Accordingly, passing to the limit as $t \to 0^+$ in (4.2), we get

$$\frac{1}{b^2}\int_\Omega \frac{\nabla v\,\nabla\phi}{\sqrt{v^2+b^{-2}|\nabla v|^2}}\,dx + \int_\Omega \frac{v\,\phi}{\sqrt{v^2+b^{-2}|\nabla v|^2}}\,dx + \frac{a}{b^2}\int_\Omega \log(v)\,\phi\,dx - \frac{1}{b}\int_{\partial\Omega}\phi\,d\mathcal{H}^{N-1} \geq 0.$$

Replacing ϕ with $-\phi$, we obtain

$$\frac{1}{b^2}\int_\Omega \frac{\nabla v\,\nabla\phi}{\sqrt{v^2+b^{-2}|\nabla v|^2}}\,dx + \int_\Omega \frac{v\,\phi}{\sqrt{v^2+b^{-2}|\nabla v|^2}}\,dx + \frac{a}{b^2}\int_\Omega \log(v)\,\phi\,dx - \frac{1}{b}\int_{\partial\Omega}\phi\,d\mathcal{H}^{N-1} = 0.$$

The change of variable $u = -\frac{1}{b}\log(v)$ gives

$$\int_\Omega \frac{\nabla u\,\nabla\phi}{\sqrt{1+|\nabla u|^2}}\,dx = \int_\Omega \Big(-au + \frac{b}{\sqrt{1+|\nabla u|^2}}\Big)\phi\,dx - \int_{\partial\Omega}\phi\,d\mathcal{H}^{N-1}.$$

By the integration by parts formula in [3, Proposition 1.3], we infer

$$\int_\Omega \Big(\operatorname{div}\Big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) - au + \frac{b}{\sqrt{1+|\nabla u|^2}}\Big)\phi\,dx = \int_{\partial\Omega}\Big(\Big[\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}, \nu\Big] + 1\Big)\phi\,d\mathcal{H}^{N-1}.$$

Hence, using (4.1), we have

$$\int_{\partial\Omega}\Big(\Big[\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}, \nu\Big] + 1\Big)\phi\,d\mathcal{H}^{N-1} = 0. \tag{4.3}$$

Since (4.3) holds for all $\phi \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ such that, for $\mathcal{H}^{N-1}$-a.e. $x \in \partial\Omega$, $\phi(x) = 0$ whenever $u(x) = 0$, we conclude that

$$\left[\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}, \nu\right](x) = -1,$$

for $\mathcal{H}^{N-1}$-a.e. $x \in \partial\Omega$ such that $u(x) > 0$. □

Proposition 4.3. *Let u be a solution of problem* (1.4). *Then $v = e^{-bu} \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ satisfies* (3.9) *for all $w \in BV(\Omega)$.*

Proof. From Proposition 4.1 it follows that $v = e^{-bu} \in W^{1,1}(\Omega)$ and $\exp(-\frac{b^2}{a}) \le \operatorname*{ess\,inf}_\Omega v \le \operatorname*{ess\,sup}_\Omega v \le 1$.

Step 1. Inequality (3.9) *holds for all $w \in W^{1,1}(\Omega)$ such that $w = 1$ $\mathcal{H}^{N-1}$-a.e. on Ω.* Let $w \in W^{1,1}(\Omega)$ satisfy $w = 1$ $\mathcal{H}^{N-1}$-a.e. on Ω and set $\phi = w - v$. Observe that $\phi \in W^{1,1}(\Omega)$ is such that, for $\mathcal{H}^{N-1}$-a.e. $x \in \partial\Omega$, $\phi(x) = 0$ whenever $u(x) = 0$, or equivalently $v(x) = 1$. According to the boundary behaviour of u, we have

$$\int_{\partial\Omega}\left(\left[\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}, \nu\right] + \operatorname{sgn}(u)\right)\phi \, d\mathcal{H}^{N-1} = 0.$$

On the other hand, multiplying by ϕ the equation in (1.4), integrating over Ω and applying the integration by parts formula in [3, Proposition 1.3], we obtain

$$\begin{aligned}\int_\Omega \frac{\nabla u\, \nabla\phi}{\sqrt{1+|\nabla u|^2}}\, dx &= \int_\Omega\left(-au + \frac{b}{\sqrt{1+|\nabla u|^2}}\right)\phi\, dx + \int_{\partial\Omega}\left[\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}, \nu\right]\phi\, d\mathcal{H}^{N-1} \\ &= \int_\Omega\left(-au + \frac{b}{\sqrt{1+|\nabla u|^2}}\right)\phi\, dx - \int_{\partial\Omega}\operatorname{sgn}(u)\,\phi\, d\mathcal{H}^{N-1}.\end{aligned}$$

The change of variable $v = e^{-bu}$ yields

$$\begin{aligned}&\frac{1}{b^2}\int_\Omega \frac{\nabla v\, \nabla\phi}{\sqrt{v^2 + b^{-2}|\nabla v|^2}}\, dx + \int_\Omega \frac{v\,\phi}{\sqrt{v^2+b^{-2}|\nabla v|^2}}\, dx \\ &\qquad = -\frac{a}{b^2}\int_\Omega \log(v)\,\phi\, dx + \frac{1}{b}\int_{\partial\Omega}\operatorname{sgn}(1-v)\,\phi\, d\mathcal{H}^{N-1}.\end{aligned}$$

Then the convexity in $[0,+\infty[\, \times \mathbb{R}^N$ and the differentiability in $]0,+\infty[\, \times \mathbb{R}^N$ of the map $(s,\xi) \mapsto \sqrt{s^2 + b^{-2}|\xi|^2}$, together with the condition $\operatorname*{ess\,inf}_\Omega v > 0$, yield

$$\begin{aligned}&\int_\Omega \sqrt{(v+\phi)^2 + b^{-2}|\nabla(v+\phi)|^2}\, dx - \int_\Omega \sqrt{v^2+b^{-2}|\nabla v|^2}\, dx \\ &\qquad \ge -\frac{a}{b^2}\int_\Omega \log(v)\,\phi\, dx + \frac{1}{b}\int_{\partial\Omega}\operatorname{sgn}(1-v)\,\phi\, d\mathcal{H}^{N-1}.\end{aligned}$$

Since

$$\operatorname{sgn}(1-v)\,\phi + |v+\phi-1| - |v-1| \ge 0 \quad \mathcal{H}^{N-1}\text{-a.e. in } \partial\Omega,$$

we infer that

$$\mathcal{J}(v+\phi) - \mathcal{J}(v) \ge -\frac{a}{b^2}\int_\Omega \log(v)\,\phi\, dx,$$

which is (3.9) as $v + \phi = w$.

Step 2. Inequality (3.9) *holds for all* $w \in BV(\Omega)$. Pick $w \in BV(\Omega)$. According to Proposition 2.6, there exists a sequence $(w_n)_n$ in $C^\infty(\Omega) \cap W^{1,1}(\Omega)$ such that

$$\lim_{n\to+\infty} w_n = w \quad \text{in } L^1(\Omega), \qquad \lim_{n\to+\infty} \mathcal{J}(w_n) = \mathcal{J}(w),$$
$$w_n = 1 \quad \mathcal{H}^{N-1}\text{-a.e. in } \partial\Omega.$$

By Step 1, for all n we have

$$\mathcal{J}(w_n) - \mathcal{J}(v) \geq -\frac{a}{b^2}\int_\Omega \log(v)\,(w_n - v)\,dx.$$

Then, passing to the limit as $n \to +\infty$, we obtain (3.9). □

5 Comparison results

We present here a comparison principle and we state some of its consequences.

Proposition 5.1. *Let* γ, $\delta \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ *satisfy* $\operatorname*{ess\,inf}_\Omega \gamma > 0$, $\operatorname*{ess\,inf}_\Omega \delta > 0$,

$$\mathcal{J}(\gamma + z) - \mathcal{J}(\gamma) \geq -\frac{a}{b^2}\int_\Omega \log(\gamma)\,z\,dx, \tag{5.1}$$

for all $z \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ *with* $z \leq 0$ *a.e. in* Ω, *and*

$$\mathcal{J}(\delta + z) - \mathcal{J}(\delta) \geq -\frac{a}{b^2}\int_\Omega \log(\delta)\,z\,dx, \tag{5.2}$$

for all $z \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$, *with* $z \geq 0$ *a.e. in* Ω. *Then* $\gamma \leq \delta$ *a.e. in* Ω.

Proof. Taking $z = -(\delta - \gamma)^-$ in (5.1) and $z = (\delta - \gamma)^-$ in (5.2), we obtain

$$\mathcal{J}(\gamma \wedge \delta) - \mathcal{J}(\gamma) = \mathcal{J}(\gamma - (\delta - \gamma)^-) - \mathcal{J}(\gamma) \geq \frac{a}{b^2}\int_\Omega \log(\gamma)\,(\delta - \gamma)^-\,dx \tag{5.3}$$

and

$$\mathcal{J}(\gamma \vee \delta) - \mathcal{J}(\delta) = \mathcal{J}(\delta + (\delta - \gamma)^-) - \mathcal{J}(\delta) \geq -\frac{a}{b^2}\int_\Omega \log(\delta)\,(\delta - \gamma)^-\,dx. \tag{5.4}$$

Summing up (5.3) and (5.4) and using Proposition 2.8, we conclude that

$$0 \geq \frac{a}{b^2}\int_\Omega (\log(\gamma) - \log(\delta))\,(\delta - \gamma)^-\,dx \geq 0,$$

which implies, by the strictly increasing character of the logarithm, that $\gamma \leq \delta$ a.e. in Ω. □

We introduce a notion of upper and lower solutions for problem (1.4). It has already been considered in [37, Proposition 4] for studying the minimal surface equation.

Definition 5.1. *Let* $\beta \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ *be such that* $\operatorname{div}\Big(\frac{\nabla\beta}{\sqrt{1+|\nabla\beta|^2}}\Big) \in L^N(\Omega)$. *We say that* β *is an upper solution of problem* (1.4) *if*

$$-\operatorname{div}\left(\frac{\nabla\beta}{\sqrt{1+|\nabla\beta|^2}}\right) \geq -a\beta + \frac{b}{\sqrt{1+|\nabla\beta|^2}} \quad \textit{a.e. in } \Omega \tag{5.5}$$

and, for $\mathcal{H}^{N-1}$-a.e. $x \in \partial\Omega$, either $\beta(x) \geq 0$ or both $\beta(x) < 0$ and $\left[\frac{\nabla\beta}{\sqrt{1+|\nabla\beta|^2}}, \nu\right](x) = 1$.

A lower solution α is defined similarly by reversing the inequality in (5.5) and assuming that, for $\mathcal{H}^{N-1}$-a.e. $x \in \partial\Omega$, either $\alpha(x) \leq 0$ or both $\alpha(x) > 0$ and $\left[\frac{\nabla\alpha}{\sqrt{1+|\nabla\alpha|^2}}, \nu\right](x) = -1$.

Remark 5.1 It is clear that a function u is a solution of problem (1.4) if and only if u is simultaneously a lower solution and an upper solution of the problem.

Lemma 5.2. *Let β be an upper solution of (1.4) and set $\gamma = e^{-b\beta}$. Then $\gamma \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$, $\operatorname*{ess\,inf}_\Omega \gamma > 0$, $\operatorname{div}\Big(\frac{\nabla\gamma}{\sqrt{\gamma^2+b^{-2}|\nabla\gamma|^2}}\Big) \in L^N(\Omega)$ and*

$$\mathcal{J}(\gamma + z) - \mathcal{J}(\gamma) \geq -\frac{a}{b^2}\int_\Omega \log(\gamma)\, z\, dx, \tag{5.6}$$

for all $z \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$, with $z \leq 0$ a.e. in Ω.

Proof. From the assumptions on β it is easy to deduce that $\gamma = e^{-b\beta} \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$, $\operatorname*{ess\,inf}_\Omega \gamma > 0$, $\operatorname{div}\Big(\frac{\nabla\gamma}{\sqrt{\gamma^2+b^{-2}|\nabla\gamma|^2}}\Big) \in L^N(\Omega)$,

$$-\operatorname{div}\left(\frac{\nabla\gamma}{\sqrt{\gamma^2 + b^{-2}|\nabla\gamma|^2}}\right) \leq -a\log(\gamma) - \frac{b^2\gamma}{\sqrt{\gamma^2 + b^{-2}|\nabla\gamma|^2}} \quad \text{a.e. in } \Omega \tag{5.7}$$

and, for $\mathcal{H}^{N-1}$-a.e. $x \in \partial\Omega$, either $\gamma(x) \leq 1$, or both $\gamma(x) > 1$ and

$$\left[\frac{\nabla\gamma}{\sqrt{\gamma^2 + b^{-2}|\nabla\gamma|^2}}, \nu\right](x) = -b. \tag{5.8}$$

Indeed, to verify that (5.8) holds whenever $\gamma(x) > 1$, let us show that

$$\left[\frac{\nabla\gamma}{\sqrt{\gamma^2 + b^{-2}|\nabla\gamma|^2}}, \nu\right] = -b\left[\frac{\nabla\beta}{\sqrt{1 + |\nabla\beta|^2}}, \nu\right] \tag{5.9}$$

in $L^\infty(\partial\Omega)$. A direct calculation yields

$$\int_\Omega \frac{\nabla\gamma\,\nabla z}{\sqrt{\gamma^2 + b^{-2}|\nabla\gamma|^2}}\, dx = -b\int_\Omega \frac{\nabla\beta\,\nabla z}{\sqrt{1 + |\nabla\beta|^2}}\, dx,$$

for all $z \in W^{1,1}(\Omega) \cap L^N(\Omega)$ and, in particular, for all $z \in C_0^\infty(\Omega)$. Then, by definition of distributional divergence, we obtain

$$\operatorname{div}\left(\frac{\nabla\gamma}{\sqrt{\gamma^2 + b^{-2}|\nabla\gamma|^2}}\right) = -b\operatorname{div}\left(\frac{\nabla\beta}{\sqrt{1 + |\nabla\beta|^2}}\right).$$

According to formula (1.9) of [3, Proposition 1.3], we deduce

$$\begin{aligned}\int_{\partial\Omega}\left[\frac{\nabla\gamma}{\sqrt{\gamma^2 + b^{-2}|\nabla\gamma|^2}}, \nu\right] z\, d\mathcal{H}^{N-1} &= -b\left(\int_\Omega \operatorname{div}\left(\frac{\nabla\beta}{\sqrt{1 + |\nabla\beta|^2}}\right) z\, dx + \int_\Omega \frac{\nabla\beta\,\nabla z}{\sqrt{1 + |\nabla\beta|^2}}\, dx\right)\\ &= -b\int_{\partial\Omega}\left[\frac{\nabla\beta}{\sqrt{1 + |\nabla\beta|^2}}, \nu\right] z\, d\mathcal{H}^{N-1},\end{aligned}$$

for all $z \in W^{1,1}(\Omega) \cap L^N(\Omega)$. Therefore (5.9) holds. This allows to conclude that (5.8) holds, if $\gamma(x) > 1$ or, equivalently, $\beta(x) < 0$, by using the condition

$$\Big[\frac{\nabla\beta}{\sqrt{1+|\nabla\beta|^2}},\nu\Big](x) = 1.$$

In order to prove (5.6), let us fix $z \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ with $z \le 0$ a.e. in Ω. Multiplying (5.7) by z, integrating over Ω and using again formula (1.9) in [3, Proposition 1.3], we obtain

$$\int_\Omega \frac{\nabla\gamma\,\nabla z}{\sqrt{\gamma^2+b^{-2}|\nabla\gamma|^2}}\,dx - \int_{\partial\Omega}\Big[\frac{\nabla\gamma}{\sqrt{\gamma^2+b^{-2}|\nabla\gamma|^2}},\nu\Big] z\,d\mathcal{H}^{N-1} + \int_\Omega \frac{b^2\gamma z}{\sqrt{\gamma^2+b^{-2}|\nabla\gamma|^2}}\,dx \ge -a\int_\Omega \log(\gamma)\,z\,dx.$$

Hence, using the convexity in $[0,+\infty[\,\times\mathbb{R}^N$ and the differentiability in $]0,+\infty[\,\times\mathbb{R}^N$ of the map $(s,\xi)\mapsto\sqrt{s^2+b^{-2}|\xi|^2}$ in $]0,+\infty[\,\times\mathbb{R}^N$, together with the condition $\operatorname*{ess\,inf}_\Omega \gamma > 0$, we get

$$\begin{aligned}\int_\Omega \sqrt{(\gamma+z)^2+b^{-2}|\nabla(\gamma+z)|^2}\,dx &- \int_\Omega\sqrt{\gamma^2+b^{-2}|\nabla\gamma|^2}\,dx\\ &\ge \frac{1}{b^2}\int_\Omega \frac{\nabla\gamma\,\nabla z}{\sqrt{\gamma^2+b^{-2}|\nabla\gamma|^2}}\,dx + \int_\Omega \frac{\gamma z}{\sqrt{\gamma^2+b^{-2}|\nabla\gamma|^2}}\,dx\\ &\ge -\frac{a}{b^2}\int_\Omega \log(\gamma)\,z\,dx + \frac{1}{b^2}\int_{\partial\Omega}\Big[\frac{\nabla\gamma}{\sqrt{\gamma^2+b^{-2}|\nabla\gamma|^2}},\nu\Big] z\,d\mathcal{H}^{N-1}\end{aligned}$$

and therefore

$$\begin{aligned}\mathcal{J}(\gamma+z)-\mathcal{J}(\gamma) \ge -\frac{a}{b^2}\int_\Omega \log(\gamma) z\,dx &+ \frac{1}{b^2}\int_{\partial\Omega}\Big[\frac{\nabla\gamma}{\sqrt{\gamma^2+b^{-2}|\nabla\gamma|^2}},\nu\Big] z\,d\mathcal{H}^{N-1}\\ &+ \frac{1}{b}\Big(\int_{\partial\Omega}|\gamma+z-1|\,d\mathcal{H}^{N-1} - \int_{\partial\Omega}|\gamma-1|\,d\mathcal{H}^{N-1}\Big).\end{aligned}$$

We write

$$\begin{aligned}&\frac{1}{b^2}\int_{\partial\Omega}\Big[\frac{\nabla\gamma}{\sqrt{\gamma^2+b^{-2}|\nabla\gamma|^2}},\nu\Big] z\,d\mathcal{H}^{N-1} + \frac{1}{b}\Big(\int_{\partial\Omega}|\gamma+z-1|\,d\mathcal{H}^{N-1} - \int_{\partial\Omega}|\gamma-1|\,d\mathcal{H}^{N-1}\Big)\\ &\quad= \frac{1}{b^2}\int_{\partial\Omega\cap\{\gamma\le1\}}\Big[\frac{\nabla\gamma}{\sqrt{\gamma^2+b^{-2}|\nabla\gamma|^2}},\nu\Big] z\,d\mathcal{H}^{N-1} + \frac{1}{b}\int_{\partial\Omega\cap\{\gamma\le1\}}\Big(|\gamma+z-1|-|\gamma-1|\Big)\,d\mathcal{H}^{N-1}\\ &\qquad+ \frac{1}{b^2}\int_{\partial\Omega\cap\{\gamma>1\}}\Big[\frac{\nabla\gamma}{\sqrt{\gamma^2+b^{-2}|\nabla\gamma|^2}},\nu\Big] z\,d\mathcal{H}^{N-1} + \frac{1}{b}\int_{\partial\Omega\cap\{\gamma>1\}}\Big(|\gamma+z-1|-|\gamma-1|\Big)\,d\mathcal{H}^{N-1}.\end{aligned}$$

Let us consider the set $\partial\Omega\cap\{\gamma\le1\}$. On the one hand, as $z \le 0$ a.e. in Ω, we have

$$|\gamma+z-1|-|\gamma-1| = 1-\gamma-z+\gamma-1 = |z|$$

and, on the other hand, by [3, Theorem 1.1]

$$\begin{aligned}\frac{1}{b^2}\Big|\int_{\partial\Omega\cap\{\gamma\le1\}}&\Big[\frac{\nabla\gamma}{\sqrt{\gamma^2+b^{-2}|\nabla\gamma|^2}},\nu\Big] z\,d\mathcal{H}^{N-1}\Big|\\ &\le \frac{1}{b^2}\Big\|\frac{\nabla\gamma}{\sqrt{\gamma^2+b^{-2}|\nabla\gamma|^2}}\Big\|_{L^\infty(\Omega)}\int_{\partial\Omega\cap\{\gamma\le1\}}|z|\,d\mathcal{H}^{N-1} \le \frac{1}{b}\int_{\partial\Omega\cap\{\gamma\le1\}}|z|\,d\mathcal{H}^{N-1}.\end{aligned}$$

Let us now consider $\partial\Omega \cap \{\gamma > 1\}$. Using again the condition $z \le 0$ a.e. in Ω, we see that

$$\begin{aligned}\frac{1}{b^2}\int_{\partial\Omega\cap\{\gamma>1\}}\left[\frac{\nabla\gamma}{\sqrt{\gamma^2+b^{-2}|\nabla\gamma|^2}},\nu\right]z\,d\mathcal{H}^{N-1}+\frac{1}{b}\int_{\partial\Omega\cap\{\gamma>1\}}\Big(|\gamma+z-1|-|\gamma-1|\Big)\,d\mathcal{H}^{N-1}\\ =\frac{1}{b}\int_{\partial\Omega\cap\{\gamma>1\}}\Big(-z+|\gamma+z-1|-|\gamma-1|\Big)\,d\mathcal{H}^{N-1}\\ =\frac{1}{b}\int_{\partial\Omega\cap\{\gamma>1\}}\Big(|z|+|\gamma+z-1|-|\gamma-1|\Big)\,d\mathcal{H}^{N-1}\ge 0.\end{aligned}$$

This implies that, for all $z \in W^{1,1}(\Omega)\cap L^\infty(\Omega)$ with $z \le 0$ a.e. in Ω,

$$\mathcal{J}(\gamma+z)-\mathcal{J}(\gamma)\ge -\frac{a}{b^2}\int_\Omega \log(\gamma)\,z\,dx,$$

which is the conclusion. □

Proposition 5.3. *Let β be an upper solution of* (1.4) *and u be a solution of* (1.4)*. Then $u \le \beta$ in Ω.*

Proof. The conclusion follows from Lemma 5.2, Proposition 4.3 and Proposition 5.1. □

Remark 5.2 Statements similar to Lemma 5.2 and Proposition 5.3 hold for lower solutions too.

Lemma 5.4. *Assume that Ω is a bounded domain in $\mathbb{R}^N$, which satisfies an exterior sphere condition with radius $r \ge (N-1)\frac{b}{a}$ at some $x_0 \in \partial\Omega$, Then there exists an upper solution β of problem* (1.4) *such that $\beta(x_0)=0$. In case $r > (N-1)\frac{b}{a}$, the upper solution can be chosen in such a way to satisfy a bounded slope condition at x_0, that is $\sup\limits_{x\in\Omega}\dfrac{\beta(x)}{|x-x_0|}<+\infty$.*

Proof. According to Definition 1.2, there exist $r \ge (N-1)\frac{b}{a}$ and $y \in \mathbb{R}^N$ such that $B(y,r)\cap\Omega=\emptyset$ and $x_0 \in \overline{B(y,r)}\cap\partial\Omega$. Pick a constant $R \ge r+\frac{b}{a}$ such that

$$\Omega \subseteq A_{r,R}=\{x\in\mathbb{R}^N : r<|x-y|<R\}.$$

Next define a function $\eta : [r,R]\to\mathbb{R}$, by

$$\eta(t)=\begin{cases}\sqrt{\left(\frac{b}{a}\right)^2-\left(t-\left(r+\frac{b}{a}\right)\right)^2} & \text{if } r\le t<r+\frac{b}{a},\\ \frac{b}{a} & \text{if } r+\frac{b}{a}\le t\le R,\end{cases}$$

and a function $\beta : \overline{A_{r,R}}\to\mathbb{R}$ by

$$\beta(x)=\eta(|x-y|). \tag{5.10}$$

Note that $\beta \in W^{1,1}(\Omega)\cap C(\overline{\Omega})$ satisfies $\beta(x_0)=0$ and $\beta \ge 0$ on $\partial\Omega$. In addition, we can easily verify that $\operatorname{div}\Big(\frac{\nabla\beta}{\sqrt{1+|\nabla\beta|^2}}\Big)\in L^\infty(\Omega)$. To check that β is an upper solution of (1.4), as β is radially symmetric, we show that, for a.e. $t \in]r,R[$,

$$-\left(\frac{t^{N-1}\eta'(t)}{\sqrt{1+\eta'(t)^2}}\right)'\ge t^{N-1}\left(-a\eta(t)+\frac{b}{\sqrt{1+\eta'(t)^2}}\right).$$

On the one hand, for $t \in]r, r + \frac{b}{a}[$, we have

$$
\begin{aligned}
-\Big(\frac{t^{N-1}\eta'(t)}{\sqrt{1+\eta'(t)^2}} \Big)' &= t^{N-2} \frac{a}{b} \Big(t - (N-1)\Big(r + \frac{b}{a} - t \Big) \Big) \\
&\geq t^{N-2} \frac{a}{b} \Big(r - (N-1)\frac{b}{a} \Big) \geq 0 = t^{N-1} \Big(-a\eta(t) + \frac{b}{\sqrt{1+\eta'(t)^2}} \Big).
\end{aligned}
$$

On the other hand, for $t \in]r + \frac{b}{a}, R[$, we have

$$
-\Big(\frac{t^{N-1}\eta'(t)}{\sqrt{1+\eta'(t)^2}} \Big)' = 0 = t^{N-1} \Big(-a\eta(t) + \frac{b}{\sqrt{1+\eta'(t)^2}} \Big).
$$

This yields the conclusion if $r = (N-1)\frac{b}{a}$.

In case $r > (N-1)\frac{b}{a}$, we modify the definition of η as follows

$$
\eta(x) = \begin{cases} c\sqrt{\left(\frac{b}{a}\right)^2 + \varepsilon^2 - \left(t - \left(r + \frac{b}{a}\right)\right)^2} - \varepsilon c & \text{if } r \leq t < r + \frac{b}{a}, \\ \frac{b}{a} & \text{if } r + \frac{b}{a} \leq t \leq R, \end{cases}
$$

where $c = \frac{a}{b}\Big(\varepsilon + \sqrt{\left(\frac{b}{a}\right)^2 + \varepsilon^2}\Big)$, for some $\varepsilon > 0$ suitably chosen. It is then easy to see that the function β defined by (5.10) is an upper solution of (1.4), which satisfies $\beta(x_0) = 0$ as well as a bounded slope condition at x_0. $\square$

6 Conclusions

We are now in position of proving the main result of this paper, of which Theorem 1.1 and Theorem 1.2 are direct consequences.

Theorem 6.1. *Let $a, b > 0$ be given and let Ω be a bounded domain in $\mathbb{R}^N$, with $N \geq 2$, having a Lipschitz boundary $\partial\Omega$. Then there exists a unique solution u of* (1.4)*, which also satisfies*

- *(j) $u \in C^\infty(\Omega)$;*
- *(jj) at each point $x_0 \in \partial\Omega$ where an exterior sphere condition with radius $R \geq (N-1)\,b/a$ holds, u is continuous and satisfies $u(x_0) = 0$; moreover, if $R > (N-1)\,b/a$, then u also satisfies a bounded slope condition at x_0, that is $\sup_{x\in\Omega} \dfrac{u(x)}{|x - x_0|} < +\infty$;*
- *(jjj) $u \in L^\infty(\Omega)$ and $0 < u(x) < b/a$ for all $x \in \Omega$;*
- *(jv) u minimizes in $W^{1,1}(\Omega) \cap L^\infty(\Omega)$ the functional*

$$
\int_\Omega e^{-bz}\sqrt{1+|\nabla z|^2}\,dx - \frac{a}{b}\int_\Omega e^{-bz}\big(z + \tfrac{1}{b}\big)\,dx + \frac{1}{b}\int_{\partial\Omega} |e^{-bz} - 1|\,d\mathcal{H}^{N-1}.
$$

Proof. The proof is divided in some steps.

Step 1. Problem (1.4) *has a solution u satisfying (j) and $0 \leq u(x) \leq b/a$ for all $x \in \Omega$.* By Proposition 3.1 and Proposition 3.6, we know that the functional $\mathcal{I}$ admits a unique global minimizer v in $BV^+(\Omega)$, which satisfies $v \in C^\infty(\Omega) \cap W^{1,1}(\Omega)$ and $\exp\big(-\frac{b^2}{a}\big) \leq v \leq 1$ in Ω. Hence Proposition 3.4 , Proposition 4.1 and Proposition 4.2 yield the conclusion.

Step 2. Uniqueness of the solution. This conclusion follows from Proposition 5.3, using Step 1 too.

Step 3. The function u minimizes in $W^{1,1}(\Omega) \cap L^\infty(\Omega)$ the functional

$$\int_\Omega e^{-bz}\sqrt{1+|\nabla z|^2}\,dx - \frac{a}{b}\int_\Omega e^{-bz}\big(z+\tfrac{1}{b}\big)\,dx + \frac{1}{b}\int_{\partial\Omega}|e^{-bz}-1|\,d\mathcal{H}^{N-1}.$$

This can be easily deduced from the fact that

$$\int_\Omega e^{-bz}\sqrt{1+|\nabla z|^2}\,dx - \frac{a}{b}\int_\Omega e^{-bz}\big(z+\tfrac{1}{b}\big)\,dx + \frac{1}{b}\int_{\partial\Omega}|e^{-bz}-1|\,d\mathcal{H}^{N-1} = \mathcal{I}(e^{-bz})$$

and $v = e^{-bu}$ minimizes $\mathcal{I}$ in $BV^+(\Omega)$.

Step 4. The function u is such that $u(x) > 0$ for all $x \in \Omega$. We already know that $u(x) \geq 0$ for all $x \in \Omega$. Assume by contradiction that there exists $x_0 \in \Omega$ such that $u(x_0) = 0$. Note that the equation in (1.4) can be written as

$$\frac{\Delta u}{\sqrt{1+|\nabla u(x)|^2}} - \sum_{i,j=1}^{N}\frac{\partial_{x_i}u\,\partial_{x_j}u\,\partial^2_{x_jx_i}u}{(1+|\nabla u(x)|^2)^{3/2}} = au - \frac{b}{\sqrt{1+|\nabla u(x)|^2}} \quad \text{in } \Omega. \tag{6.1}$$

By evaluating (6.1) at x_0, we obtain $\Delta u(x_0) = -b < 0$, thus contradicting the fact that x_0 is a minimum point of u in Ω.

Step 5. The function u is such that $u(x) < b/a$ for all $x \in \Omega$. Let B be an open ball in $\mathbb{R}^N$ such that $\overline{\Omega} \subseteq B$. According to [13, Theorem 1.1], there exists a unique solution $\beta \in C^2(\overline{B})$ of

$$\begin{cases} -\mathrm{div}\left(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = -au + \dfrac{b}{\sqrt{1+|\nabla u|^2}} & \text{in } B,\\ u = 0 & \text{on } \partial B, \end{cases}$$

which in addition satisfies $\beta(x) < b/a$ for all $x \in \overline{B}$. In particular, β is an upper solution of (1.4). The conclusion then follows from Proposition 5.3.

Step 6. At each point $x_0 \in \partial\Omega$ where an exterior sphere condition with radius $R \geq (N-1)\,b/a$ holds, u is continuous and satisfies $u(x_0) = 0$. By the first part of Lemma 5.4, problem (1.4) has an upper solution β, with $\beta(x_0) = 0$. The conclusion then follows from Proposition 5.3.

Step 7. At each point $x_0 \in \partial\Omega$, where an exterior sphere condition with radius $R > (N-1)\,b/a$ holds, u also satisfies a bounded slope condition. By the second part of Lemma 5.4, problem (1.4) has an upper solution β, with $\beta(x_0) = 0$, having bounded slope at x_0. The conclusion then follows from Proposition 5.3. □

Proof of Theorem 1.1. From Theorem 6.1 we immediately infer the validity of conclusions (*i*), (*iii*), (*iv*) of Theorem 1.1. In order to verify (*ii*) it is enough to observe that, for every point $x_0 \in \partial\Omega \cap \mathrm{Conv}(\overline{\Omega})$, an exterior sphere condition holds for any given radius $R > 0$. □

Proof of Theorem 1.2. Theorem 1.2 follows from a direct application of Theorem 6.1. □

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References

[1] L. Ambrosio, N. Fusco and D. Pallara, Functions of Bounded Variation and Free Discontinuity Problems, Clarendon Press, Oxford, 2000.

[2] L. Ambrosio, S. Mortola and V.M. Tortorelli, Functionals with linear growth defined on vector valued BV functions, J. Math. Pures Appl. 70 (1991), 269–323.

[3] G. Anzellotti, Pairings between measures and bounded functions and compensated compactness, Ann. Mat. Pura Appl. 135 (1983), 293–318.

[4] M. Athanassenas and J. Clutterbuck, A capillarity problem for compressible liquids, Pacific J. Math. 243 (2009), 213–232.

[5] M. Athanassenas and R. Finn, Compressible fluids in a capillary tube, Pacific J. Math. 224 (2006), 201–229.

[6] M. Bergner, The Dirichlet problem for graphs of prescribed anisotropic mean curvature in $\mathbb{R}^{n+1}$, Analysis (Munich) 28 (2008), 149–166.

[7] M. Bergner, On the Dirichlet problem for the prescribed mean curvature equation over general domains, Differential Geom. Appl. 27 (2009), 335–343.

[8] D. Bonheure, P. Habets, F. Obersnel and P. Omari, Classical and non-classical solutions of a prescribed curvature equation, J. Differential Equations 243 (2007), 208–237.

[9] D. Bonheure, P. Habets, F. Obersnel and P. Omari, Classical and non-classical positive solutions of a prescribed curvature equation with singularities, Rend. Istit. Mat. Univ. Trieste 39 (2007), 63–85.

[10] H. Brezis, Functional Analysis, Sobolev Spaces and Partial Differential Equations, Springer, New York, 2010.

[11] M. Carriero, G. Dal Maso, A. Leaci and E. Pascali, Relaxation of the non-parametric Plateau problem with an obstacle, J. Math. Pures Appl. 67 (1988), 359-396.

[12] I. Coelho, C. Corsato and P. Omari, A one-dimensional prescribed curvature equation modeling the corneal shape, Bound. Value Probl. 2014, 2014:127.

[13] C. Corsato, C. De Coster and P. Omari, Radially symmetric solutions of an anisotropic mean curvature equation modeling the corneal shape, Discr. Cont. Dyn. Systems (2015), in press. Available at http://www.dmi.units.it/pubblicazioni/Quaderni_Matematici/642_2014.pdf

[14] F. Demengel, Functions locally almost 1-harmonic, Appl. Anal. 83 (2004), 865–896.

[15] I. Ekeland and R. Temam, Convex Analysis and Variational Problems, Society for Industrial and Applied Mathematics, Philadelphia, 1999.

[16] L. Evans and R.F. Gariepy, Measure Theory and Fine Properties of Functions, CRC Press, Boca Raton, 1992.

[17] D.G. de Figueiredo, Lectures on the Ekeland Variational Principle with Applications and Detours, Tata Institute of Fundamental Research Lectures on Mathematics and Physics, 81, Springer, Berlin, 1989.

[18] R. Finn, On the equations of capillarity, J. Math. Fluid Mech. 3 (2001), 139–151.

[19] R. Finn, Capillarity problems for compressible fluids, Mem. Differential Equations Math. Phys. 33 (2004), 47–55.

[20] R. Finn and G. Luli, On the capillary problem for compressible fluids, J. Math. Fluid Mech. 9 (2007), 87–103.

[21] E. Gagliardo, Caratterizzazioni delle tracce sulla frontiera relative ad alcune classi di funzioni in n variabili, Rend. Sem. Mat. Univ. Padova 27 (1957), 284–305.

[22] C. Gerhardt, Existence and regularity of capillary surfaces, Boll. Un. Mat. Ital. (4) 10 (1974), 317–335.

[23] C. Gerhardt, Existence, regularity, and boundary behavior of generalized surfaces of prescribed mean curvature, Math. Z. 139 (1974), 173–198.

[24] C. Gerhardt, On the regularity of solutions to variational problems in $BV(\Omega)$, Math. Z. 149 (1976), 281–286.

[25] D. Gilbarg and N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer, New York, 2001.

[26] E. Giusti, On the equation of surfaces of prescribed mean curvature. Existence and uniqueness without boundary conditions, Invent. Math. 46 (1978), 111–137.

[27] E. Giusti, Generalized solutions for the mean curvature equation, Pacific J. Math. 88 (1980), 297–321.

[28] E. Giusti, Minimal Surfaces and Functions of Bounded Variations, Birkhäuser, Basel, 1984.

[29] K. Hayasida and Y. Ikeda, Prescribed mean curvature equations under the transformation with non-orthogonal curvilinear coordinates, Nonlinear Anal. 67 (2007), 1–25.

[30] K. Hayasida and M. Nakatani, On the Dirichlet problem of prescribed mean curvature equations without H-convexity condition, Nagoya Math. J. 157 (2000), 177–209.

[31] R. Huff and J. McCuan, Minimal graphs with discontinuous boundary values, J. Aust. Math. Soc. 86 (2009), 75–95.

[32] H. Jenkins and J. Serrin, The Dirichlet problem for the minimal surface equation in higher dimensions, J. Reine Angew. Math. 229 (1968), 170–187.

[33] G.A. Ladyzhenskaya and N.N. Ural'tseva, Linear and Quasilinear Elliptic Equations, Academic Press, New York, 1968.

[34] G.A. Ladyzhenskaya and N.N. Ural'tseva, Local estimates for gradients of solutions of non-uniformly elliptic and parabolic equations, Comm. Pur. Appl. Math. 23 (1970), 677–703.

[35] A. Lichnewsky, Principe du maximum local et solutions généralisées de problèmes du type hypersurfaces minimales, Bull. Soc. Math. France 102 (1974), 417–433.

[36] A. Lichnewsky, Sur le comportement au bord des solutions généralisées du problème non paramétrique des surfaces minimales, J. Math. Pures Appl. 53 (1974), 397–425.

[37] A. Lichnewsky, Solutions généralisées du problème des surfaces minimales pour des données au bord non bornées, J. Math. Pures Appl. 57 (1978), 231–253.

[38] A. Lichnewsky and R. Temam, Pseudosolutions of the time-dependent minimal surface problem, J. Differential Equation 30 (1978), 340–364.

[39] T. Marquardt, Remark on the anisotropic prescribed mean curvature equation on arbitrary domains, Math. Z. 264 (2010), 507–511.

[40] M. Miranda, Superfici minime illimitate, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 4 (1977), 313–322.

[41] M. Miranda, Maximum principles and minimal surfaces, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 25 (1997), 667–681.

[42] C.B. Morrey Jr., Multiple Integrals in the Calculus of Variations, Springer, New York, 1966.

[43] J. Nečas, Direct Methods in the Theory of Elliptic Equations, Springer, New York, 2012.

[44] F. Obersnel, Classical and non-classical sign-changing solutions of a one-dimensional autonomous prescribed curvature equation, Adv. Nonlinear Stud. 7 (2007), 671–682.

[45] F. Obersnel and P. Omari, Existence, regularity and boundary behaviour of bounded variation solutions of a one-dimensional capillarity equation, Discrete Contin. Dyn. Syst. 33 (2013), 305–320.

[46] W. Okrasiński and Ł. Płociniczak, A nonlinear mathematical model of the corneal shape, Nonlinear Anal. Real World Appl. 13 (2012), 1498–1505.

[47] W. Okrasiński and Ł. Płociniczak, Bessel function model of corneal topography, Appl. Math. Comput. 223 (2013), 436–443.

[48] W. Okrasiński and Ł. Płociniczak, Regularization of an ill-posed problem in corneal topography, Inverse Probl. Sci. Eng. 21 (2013), 1090–1097.

[49] H. Pan and R. Xing, Time maps and exact multiplicity results for one-dimensional prescribed mean curvature equations. II, Nonlinear Anal. 74 (2011), 3751–3768.

[50] Ł. Płociniczak, G.W. Griffiths and W.E. Schiesser, ODE/PDE analysis of corneal curvature, Computers in Biology and Medicine 53 (2014), 30–41.

[51] Ł. Płociniczak and W. Okrasiński, Nonlinear parameter identification in a corneal geometry model, Inverse Probl. Sci. Eng. 23 (2015), 443–456.

[52] Ł. Płociniczak, W. Okrasiński, J.J. Nieto and O. Domínguez, On a nonlinear boundary value problem modeling corneal shape, J. Math. Anal. Appl. 414 (2014), 461–471.

[53] J. Serrin, The problem of Dirichlet for quasilinear elliptic differential equations with many independent variables, Phil. Trans. R. Soc. Lond. A 264 (1969), 413–496.

[54] R. Temam, Solutions généralisées de certaines équations du type hypersurfaces minima, Arch. Rational Mech. Anal. 44 (1971/72), 121–156.

[55] G.M. Troianiello, Elliptic Differential Equations and Obstacle Problems, Plenum Press, New York, 1987.