Richter-Peleg multi-utility representations of preorders

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Abstract

The existence of a Richter-Peleg multi-utility representation of a preorder by means of upper semicontinuous or continuous functions is discussed in connection with the existence of a Richter-Peleg utility representation. We give several applications that include the analysis of countable Richter-Peleg multi-utility representations.

Key words: Richter-Peleg multi-utility representation, Richter-Peleg representation

1 Introduction

The *multi-utility representation* of a not necessarily total preorder or quasi-ordering $\precsim$ on a decision space $\mathbf{X}$ characterizes the preorder by means of a family $\mathbf{V}$ of real-valued (isotonic) functions, in the sense that, for all elements $x, y \in \mathbf{X}$, $x \precsim y$ is required to be equivalent to $v(x) \leq v(y)$ for all functions $v \in \mathbf{V}$.
On the other hand, a function v on $\mathbf{X}$ is said to be a *Richter-Peleg utility representation* or an *order-preserving function* for a preorder $\precsim$ on $\mathbf{X}$ if it is *increasing* (i.e., $x \precsim y$ implies that $v(x) \leq v(y)$) and in addition $x \prec y$ implies that $v(x) < v(y)$, where $\prec$ stands for the *strict part* of the preorder $\precsim$. While a Richter-Peleg utility v does not characterize the preorder $\precsim$, if we are

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interested in finding a maximal element for $\precsim$, then such an element can be determined by maximizing v.

A *Richter-Peleg multi-utility representation* $\mathbf{V}$ for a preorder $\precsim$ on $\mathbf{X}$ is a multi-utility representation for $\precsim$ such that every function $v \in \mathbf{V}$ is a Richter-Peleg utility for $\precsim$. This representation notion is a synthesis of the two aforementioned approaches that preserves the advantages of both.

In this paper we prove that the existence of a single Richter-Peleg utility is necessary and sufficient for the existence of a Richter-Peleg multi-utility representation. A perfectly analogous result holds true when we require upper (or lower) semicontinuity of all the functions involved. We also show that the problem of obtaining a continuous Richter-Peleg multi-utility representation can be transformed to the problem of obtaining a continuous Richter-Peleg utility plus a continuous multi-utility representation.

These results can also be combined with the earlier findings on the existence of Richter-Peleg and multi-utility representations. For example, as a corollary of our main result, it follows that on a second countable topological space the existence of a continuous multi-utility representation implies the existence of a continuous Richter-Peleg multi-utility representation. Another notable corollary is that every preorder on a countable set has a (countable) Richter-Peleg multi-utility representation. Both of these observations follow from the fact that the existence of a countable multi-utility representation implies the existence of a Richter-Peleg utility.

As a disadvantage of our approach, we prove that it is impossible to represent a nontotal preorder on a connected topological space by means of finitely many, continuous Richter-Peleg utilities.

Seminal contributions to the literature on multi-utility representations include Levin [11] and Evren and Ok [8]. In particular, Evren and Ok develop the ordinal theory of multi-utility representations (see also the more recent paper by Bosi and Herden [6]). The case of a finite representing family was studied by Ok [13] and more recently by Kaminski [10]. The notion of a Richter-Peleg multi-utility representation was first introduced and studied by Minguzzi [12], whose focus is different than ours.

The paper is organized as follows. Section 2 contains the definitions. Section 3 presents the main results, whose applications are developed in Section 4. Section 5 focuses on the case of connected topological spaces, while Section 6 concludes.

2 Definitions and Preliminaries

Let $\mathbf{X}$ represent a *decision space* and $\precsim$ a *preorder*, also called *quasi-ordering* (reflexive, transitive binary relation) on $\mathbf{X}$. As usual, $\prec$ denotes the *strict part* of $\precsim$ and we use $x \precsim y$, resp. $x \prec y$, as a shorthand for $(x, y) \in \precsim$, resp.

$(x, y) \in \prec$. The preorder $\precsim$ is *total* if for each $x, y \in \mathbf{X}$, either $x \precsim y$ or $y \precsim x$ holds true.
For every $x \in \mathbf{X}$ we define the following subsets of $\mathbf{X}$:

$$l(x) = \{y \in \mathbf{X} \mid y \prec x\}, \;\; r(x) = \{z \in \mathbf{X} \mid x \prec z\},$$

$$d(x) = \{y \in \mathbf{X} \mid y \precsim x\}, \;\; i(x) = \{z \in \mathbf{X} \mid x \precsim z\}.$$

A subset D of $\mathbf{X}$ is said to be *decreasing*, resp. *increasing*, if $d(x) \subset D$, resp. $i(x) \subset D$, for all $x \in D$.
We recall that $v : (\mathbf{X}, \precsim) \longrightarrow (\mathbb{R}, \leqslant)$ is *isotonic* or *increasing* if, for each $x, y \in \mathbf{X}$, $x \precsim y \Rightarrow v(x) \leqslant v(y)$. Furthermore, v is *strictly isotonic* or *order preserving* if it is isotonic and in addition, for each $x, y \in \mathbf{X}$, $x \prec y \Rightarrow v(x) < v(y)$. Strictly isotonic functions on $(\mathbf{X}, \precsim)$ are also called *Richter-Peleg representations* of $\precsim$ (see e.g. Peleg [14] and Richter [15]). If $\precsim$ is total, then any Richter-Peleg representation v of $\precsim$ is a standard utility representation: that is, for each $x, y \in \mathbf{X}$, $x \precsim y \Leftrightarrow v(x) \leqslant v(y)$. It is obvious that every preorder with a utility representation is total.
Following the terminology adopted by Evren and Ok [8], we say that a preorder $\precsim$ on a topological space $(\mathbf{X}, \tau)$ is *upper, resp. lower, semicontinuous* if $i(x)$, resp. $d(x)$, is a closed subset of $\mathbf{X}$ for every $x \in \mathbf{X}$. And it is *continuous* if it is both upper and lower semicontinuous.
A *multi-utility representation* of a preorder $\precsim$ on a set $\mathbf{X}$ is a family $\mathbf{V}$ of functions $v : (\mathbf{X}, \precsim) \longrightarrow (\mathbb{R}, \leqslant)$, with the property that for each $x, y \in \mathbf{X}$,

$$x \precsim y \Leftrightarrow [v(x) \leqslant v(y), \text{ for all } v \in \mathbf{V}]. \tag{1}$$

We note that each $v \in \mathbf{V}$ is an isotonic function when $\mathbf{V}$ is a multi-utility representation of $\precsim$. If $\mathbf{V}$ is a countable, resp. finite, family then we say that $\mathbf{V}$ is a *countable*, resp. *finite*, *multi-utility representation* of $\precsim$. When there is a topology τ on $\mathbf{X}$ and $\mathbf{V}$ is a family of upper semicontinuous/lower semicontinuous/continuous functions with the property that (1) holds for each $x, y \in \mathbf{X}$, then we say that $\mathbf{V}$ is an *upper semicontinuous/lower semicontinuous/continuous multi-utility representation* of $\precsim$. Combinations of these concepts (e.g., countable continuous multi-utility representation) are naturally mentioned along the paper. If $\mathbf{V}$ is a multi-utility representation of $\precsim$ then, for each $x, y \in \mathbf{X}$,

$$x \prec y \Leftrightarrow [v(x) \leqslant v(y) \text{ for all } v \in \mathbf{V}, \text{ and } v'(x) < v'(y) \text{ for some } v' \in \mathbf{V}]. \tag{2}$$

The following result is often quoted along the paper:

Proposition 2.1 *(Evren and Ok [8, Proposition 2]) Every preorder (resp., upper semicontinuous preorder) on a set (resp., on a topological space) is representable by a multi-utility (resp., an upper semicontinuous multi-utility). If the set is countable then the preorder is representable by a countable multi-utility.*

Minguzzi [12, Section 5] introduces the notion of a *Richter-Peleg multi-utility representation*, which refers to a multi-utility representation that consists of strictly isotonic functions. Therefore Richter-Peleg multi-utility representations are multi-utility representations. From the fact that there are preorders without a Richter-Peleg representation we deduce:

Corollary 2.2 *There exist preorders that do not admit a Richter-Peleg multi-utility representation.*

In particular, the existence of multi-utility representations does not secure existence of Richter-Peleg multi-utility representations. The class of preordered sets for which Richter-Peleg multi-utility representations exist has not been identified yet.

Remark 2.3 *It is immediate to check that a Richter-Peleg multi-utility representation* $\boldsymbol{V}$ *of a preordered set* $(\boldsymbol{X}, \precsim)$ *also characterizes the strict part* $\prec$ *of* $\precsim$*, in the sense that for each* $x, y \in \boldsymbol{X}$*,*

$$x \prec y \Leftrightarrow [v(x) < v(y), \text{ for all } v \in \boldsymbol{V}]. \tag{3}$$

It is also worth noting that a Richter-Peleg multi-utility representation $\mathbf{V}$ admits a multi-self interpretation in the sense of Evren [7, Section 5]. Each $v \in \mathbf{V}$ gives what Evren calls "a description of a possible self of the agent defined by $\precsim$" because on a given choice set, any maximal element according to v is also maximal according to $\precsim$. This follows from the fact that $\mathbf{V}$ consists of strictly isotonic functions. By contrast, given a multi-utility representation $\mathbf{U}$ of the preorder, maximization of an individual $u \in \mathbf{U}$ does not necessarily produce elements that are maximal according to the preorder.

3 Main results

The existence of a Richter-Peleg representation implies the existence of a Richter-Peleg multi-utility representation. Indeed, the following theorem holds.

Theorem 3.1 *Let* $\precsim$ *be a preorder on a set* $\boldsymbol{X}$*. The following conditions are equivalent:*

1. $\precsim$ *can be represented by a Richter-Peleg multi-utility.*

2. *There is a Richter-Peleg representation of* $\precsim$*.*

The equivalence remains true if there is a topology on $\boldsymbol{X}$ *and we insert the term 'upper/lower semicontinuous' in each of the clauses of the statement.*

Proof. Since the implication $1 \Rightarrow 2$ is obvious we only need to prove that $2 \Rightarrow 1$. Let $\mathbf{V}$ be a multi-utility representation of $\precsim$, and let f be a Richter-Peleg representation of $\precsim$. Then it is easily checked that $\mathbf{U} = \{v + \alpha f : v \in \mathbf{V}, \alpha \in \mathbb{Q}, \alpha > 0\}$ is a Richter-Peleg multi-utility representation of $\precsim$.

This argument serves for the corresponding equivalence under upper/lower semicontinuity too. □

Proposition 3.2 *Let $\precsim$ be a preorder on a topological space $\boldsymbol{X}$. The following conditions are equivalent:*

1. $\precsim$ can be represented by a continuous Richter-Peleg multi-utility.

2. $\precsim$ can be represented by a continuous multi-utility, and there are continuous Richter-Peleg representations of $\precsim$.

Proof. The implication $1 \Rightarrow 2$ is trivial. The implication $2 \Rightarrow 1$ can be proven by mimicking the proof of Theorem 3.1. □

These results show that the notion of (resp., upper, lower semicontinuous) Richter-Peleg multi-utility is more demanding than the notion of (resp., upper, lower semicontinuous) multi-utility, and it is also more demanding than Richter-Peleg utility representation in the continuous case.

4 Applications

In this section we demonstrate how our main findings can be utilized to obtain further representation results.
In contrast to the general observations above, the existence of a countable multi-utility representation implies the existence of a countable Richter-Peleg multi-utility representation. Indeed, if $\mathbf{V} = \{v_1, v_2, ...\}$ is a multi-utility representation of $\precsim$, then the function $f := \sum_{n\in\mathbb{N}^+} 2^{-n} v_n$ is a Richter-Peleg representation of $\precsim$, where without loss of generality we assume that $\mathbf{V}$ consists of uniformly bounded functions. We can then invoke Theorem 3.1 to obtain a Richter-Peleg multi-utility representation of $\precsim$. It is also clear that this Richter-Peleg multi-utility representation will involve countably many functions, and that a continuous analogue of this observation follows from Proposition 3.2. We thus have the following result.

Proposition 4.1 *Let $\precsim$ be a preorder on a topological space $\boldsymbol{X}$. The following conditions are equivalent:*

1. $\precsim$ can be represented by a countable continuous Richter-Peleg multi-utility.

2. $\precsim$ can be represented by a countable continuous multi-utility.

The equivalence remains true if the term 'continuous' is deleted from each clause, or replaced with 'upper/lower semicontinuous'.

Following the proof of Proposition 2 by Evren and Ok [8], it can easily be shown that every preorder on a countable set admits a countable multi-utility representation. Thus the next result is a corollary of Proposition 4.1.

Corollary 4.2 *Let $\precsim$ be a preorder on a countable set $\boldsymbol{X}$. Then there are countable Richter-Peleg multi-utility representations of $\precsim$.*

Our next result shows that on second countable spaces, the existence of a continuous Richter-Peleg multi-utility representation is equivalent to that of a continuous multi-utility representation.

Proposition 4.3 *Suppose that a preorder $\precsim$ on a second countable topological space (X, τ) has a continuous multi-utility representation $\boldsymbol{V}$. Then $\precsim$ has a countable continuous Richter-Peleg multi-utility representation $\boldsymbol{V}$.*

Proof. We benefit from a technique in Minguzzi [12, Theorem 5.5]. Define $G(\precsim) = \{(x, y) \in X \times X : x \precsim y\}$ and $G_v = \{(x, y) \in X \times X : v(x) \leq v(y)\}$ for each $v \in \mathbf{V}$. Then $G(\precsim) = \bigcap_{v \in \mathbf{V}} G_v$ and each G_v is closed by continuity of v. The product space $X \times X$ is second countable (Willard [17, 16E]) hence hereditary Lindelöff (Hocking and Young [9, Exercise 2-17]), which ensures the existence of a countable family $\mathbf{V}' \subseteq \mathbf{V}$ such that $G(\precsim) = \bigcap_{v \in \mathbf{V}'} G_v$. This means that $\mathbf{V}'$ is a countable continuous multi-utility representation of $\precsim$. In order to conclude we invoke Proposition 4.1. □

We recall that a preorder $\precsim$ on a topological space $(\mathbf{X}, \tau)$ is said to be *weakly continuous* if for every pair $(x, y) \in \prec$ there exists a continuous increasing real-valued function f_{xy} on $(\mathbf{X}, \tau)$ such that $f_{xy}(x) < f_{xy}(y)$.
A preorder $\precsim$ on $(\mathbf{X}, \tau)$ is said to satisfy the *continuous analogue of the Dushnik and Miller theorem* (see Bosi and Herden [4,5]) if it is the intersection of all continuous total preorders $\leq$ extending it (i.e., all continuous total preorders $\leq$ such that $\precsim \subset \leq$ and $\prec \subset <$).
The next result shows that these two properties jointly imply the existence of a continuous Richter-Peleg multi-utility on second countable spaces.

Proposition 4.4 *Let $\precsim$ be a weakly continuous preorder on a second countable topological space $(\boldsymbol{X}, \tau)$. If $\precsim$ satisfies the continuous analogue of the Dushnik-Miller theorem, then $\precsim$ has a countable continuous Richter-Peleg multi-utility representation $\boldsymbol{V}$.*

Proof. By Bosi and Herden [6, Proposition 3.4], there is a continuous multi-utility representation of $\precsim$. In addition, there is a continuous Richter-Peleg representation of $\precsim$ by Bosi et al. [3, Theorem 3.1]. Therefore the conclusion follows from Proposition 3.2 and Proposition 4.3. □

We recall that a preorder $\precsim$ on a topological space $(\mathbf{X}, \tau)$ is said to be *closed* if $\precsim$ is a closed subset of $\mathbf{X} \times \mathbf{X}$ with respect to the product topology on $\mathbf{X} \times \mathbf{X}$ that is induced by τ. The following corollary of Proposition 4.3 easily follows from Evren and Ok [8, Corollary 1], who proved that every closed preorder $\precsim$ on a locally compact metrizable topological space $(\mathbf{X}, \tau)$ has a continuous Richter-Peleg multi-utility representation $\mathbf{V}$

Corollary 4.5 *Let $(\mathbf{X}, \tau)$ be a locally compact metrizable topological space. If τ is a second countable topology, then every closed preorder $\precsim$ on $(\mathbf{X}, \tau)$ has a countable continuous Richter-Peleg multi-utility representation $\boldsymbol{V}$.*

Banerjee and Dubey [2, Proposition 1] show that an *ethical social welfare relation*[1] (SWR) does not admit a Richter-Peleg representation. Then in their Theorem 2 they prove that no ethical SWR admits a countable multi-utility representation. An appeal to the above arguments permits to derive the latter result from the former immediately. On the other hand Alcantud and Dubey [1] show that there are SWRs that have both multi-utility representations continuous with respect to the product topology (with the set of utilities being countable infinite) and Richter-Peleg representations. Theorem 3.1 ensures that such SWRs admit countable Richter-Peleg multi-utility representations continuous in the product topology.

5 Continuous Richter-Peleg multi-utilities on connected spaces

For the purposes of optimization, it would be useful to obtain a finite, continuous Richter- Peleg multi-utility representation. Unfortunately, however, such a representation does not exist on connected spaces unless the preorder in question is total. To prove this negative result, let us first recall that a preorder $\precsim$ on a set $\mathbf{X}$ is said to be *nontrivial* if there exist two elements $x, y \in \mathbf{X}$ such that $x \prec y$. The following lemma is well known and widely cited in the literature.

Lemma 5.1 (Schmeidler [16]) *Let $\precsim$ be a nontrivial preorder on a connected topological space (X, τ). If for every $x \in X$ the sets $d(x)$ and $i(x)$ are closed and the sets $l(x)$ and $r(x)$ are open, then the preorder $\precsim$ is total.*

[1] A social welfare relation (i.e. a preorder on $[0,1]^{\mathbb{N}}$) is said to be *ethical* if it is *anonymous* and *strong Pareto.*

Proposition 5.2 *If a nontrivial preorder $\precsim$ on a connected topological space $(\mathbf{X}, \tau)$ has a continuous Richter-Peleg multi-utility representation $\mathbf{V} = \{v_1, ..., v_n\}$ then $\precsim$ is total and every v_i is a continuous utility representation of $\precsim$.*

Proof. It suffices to check that $\precsim$ is total, because in that case any Richter-Peleg representation of $\precsim$ is a utility representation and each v_i is Richter-Peleg representation of $\precsim$ by assumption. It is immediate to check that if a preorder $\precsim$ on a topological space $(\mathbf{X}, \tau)$ has a continuous multi-utility representation then both $d(x)$ and $i(x)$ are closed subsets of $\mathbf{X}$ for all $x \in \mathbf{X}$ (see e.g. Proposition 5 in Bosi and Herden [6] or Theorem 3.1 in Kaminski [10] for a restricted version). Therefore, by using Lemma 5.1, it suffices to show that under our assumptions, both $l(x)$ and $r(x)$ are open subsets of $\mathbf{X}$ for all $x \in \mathbf{X}$. To prove this fact we observe that, from Remark 2.3,

$$\begin{aligned} l(x) &= \{y \in \mathbf{X} \mid y \prec x\} = \{y \in \mathbf{X} \mid v_i(y) < v_i(x), \text{ for all } i \in \{1, ..., n\}\} = \\ &= \bigcap_{i=1}^{n} v_i^{-1}(]-\infty, v_i(x)[), \text{ and} \\ r(x) &= \{y \in \mathbf{X} \mid x \prec y\} = \{y \in \mathbf{X} \mid v_i(x) < v_i(y), \text{ for all } i \in \{1, ..., n\}\} = \\ &= \bigcap_{i=1}^{n} v_i^{-1}(]v_i(x), +\infty[) \end{aligned}$$

for each $x \in \mathbf{X}$. From these equalities and continuity of the functions v_i, the conclusion follows immediately. □

6 Concluding Remarks

In this paper, we have studied multi-utility representations that consist of Richter-Peleg utility functions. Our results show that, in general, this representation notion is more demanding that the notion of a multi-utility representation. Yet, the two representation notions turn out to be equivalent in many cases of interest. The advantage of the former representation notion is that any alternative that maximizes any one of the representing functions on a given choice set is guaranteed to be a maximal element of that set. On the other hand, when the space of alternatives is connected, this approach necessitates infinitely many utility functions to characterize a nontotal preorder. An interesting venue for further research can be the study of an alternative notion of a multi-utility representation that was recently proposed by Evren [7]. The distinctive feature of Evren's approach is that it does not necessitate the preorder to be closed even when the representing utility functions are continuous. Consequently, this approach is compatible with finitely many continuous Richter-Peleg utility functions even if the domain is a connected space.

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