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Normally preordered spaces and continuous multi-utilities

Gianni Bosi [a], Alessandro Caterino [b] and Rita Ceppitelli [b]

[a] DEAMS, Università di Trieste, Piazzale Europa 1, 34127, Trieste, Italy (gianni.bosi@deams.units.it)
[b] Dipartimento di Matematica ed Informatica, Università di Perugia, Via L. Vanvitelli 1, 06123 Perugia, Italy (alessandro.caterino@unipg.it,rita.ceppitelli@unipg.it)

Abstract

We study regular, normal and perfectly normal preorders by referring to suitable assumptions concerning the preorder and the topology of the space. We also present conditions for the existence of a countable continuous multi-utility representation, hence a Richter-Peleg multi-utility representation, by assuming the existence of a countable net weight.

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1. Introduction

The present paper can be viewed, in some sense, as a continuation of the paper [4] by Bosi and Herden. In that paper, the authors proved that a topological space (X, τ) is normal iff every D-I-closed preorder $\precsim$ on X gives $(X, \tau, \precsim)$ the structure of a normally preordered space. Moreover, they discussed the existence of continuous multi-utility representations and, in addition, investigated the relation between such a representation and the concept of a *normally topological preordered space* introduced by Nachbin [18].

We recall that a (not necessarily total) preorder $\precsim$ on a topological space (X, τ) admits a *(continuous) multi-utility representation* if there exists a family $\mathcal{F}$ of (continuous) increasing real functions on the preordered topological space

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$(X, \tau, \precsim)$ such that, for all $x, y \in X$, $x \precsim y$ is equivalent to $f(x) \leq f(y)$ for all $f \in \mathcal{F}$. This kind of representation, whose main feature is to fully characterize the preorder, was first introduced by Levin [16], who called *functionally closed* a preorder admitting a multi-utility representation.

The first systematic study of multi-utility representations is due to Ok [14], who presented different conditions for the existence of continuous multi-utility representations. Minguzzi introduced the concept of a (continuous) *Richter-Peleg multi-utility representation* $\mathcal{F}$ of a preorder $\precsim$. This is a particular kind of multi-utility representation where every function $f \in \mathcal{F}$ is a *Richter-Peleg utility function* for $\precsim$ (i.e., every function $f \in \mathcal{F}$ is *order-preserving*). Richter-Peleg multi-utilities have been recently studied by Alcantud et al. [1], who in particular were concerned with the case of countable Richter-Peleg multi-utility representations.

Our attention is primarily focused on *regular*, *perfectly normal* and *strongly normal* preorders. We prove that a topological space (X, τ) is perfectly normal [normal] iff every D-I-closed preorder $\precsim$ on X gives $(X, \tau, \precsim)$ the structure of a perfectly normally [strongly normally] preordered space. A similar result does not hold for regular spaces.

Moreover, we furnish conditions for the existence of a countable continuous multi-utility representation by assuming the existence of a *countable net weight*, in the spirit of Bosi et al. [3]. The concept of a *submetrizable space* is also profitably used in this direction.

2. Notations and preliminaries

Let $\precsim$ be a *preorder*, i.e., a *reflexive* and *transitive* binary relation on some fixed given set X. The preorder $\precsim$ is said to be *total* if for any two elements $x, y \in X$ either $x \precsim y$ or $y \precsim x$.

Define, for every point $x \in X$, the sets $d(x) := \{y \in X \mid y \precsim x\}$ and $i(x) := \{z \in X \mid x \precsim z\}$. A subset D of X is said to be *decreasing* if $d(x) \subset D$ for all $x \in D$. By duality, the concept of an *increasing* subset I of X is defined. In addition, for every subset A of X we set $d(A) := \{y \in X \mid \exists\, x \in A \mid y \precsim x\}$ and $i(A) := \{z \in X \mid \exists\, x \in A \mid x \precsim z\}$, i.e., $d(A)$ is the smallest decreasing and $i(A)$ the smallest increasing subset of X that contains A.

$\Delta_X = \{(x, x) \mid x \in X\}$ is the *diagonal* of X.

If $(X, \precsim)$ is a preordered set, then a real function u on X is said to be

- (i) *isotone* (*increasing*) if, for every $x, y \in X$, $[x \precsim y \Rightarrow u(x) \leq u(y)]$,
- (ii) *order-preserving* if it is increasing and, for every $x, y \in X$, $[x \prec y \Rightarrow u(x) < u(y)]$.

We denote by τ_{nat} the natural topology on the real line $\mathbb{R}$. Let τ be a topology on X. If $\precsim$ is a preorder on X, then the triplet $(X, \tau, \precsim)$ is referred to as a *preordered topological space*. For every subset A of X, we denote by $cl(A)$ its topological closure. For every subset A of X we denote, furthermore, by $D(A)$ the smallest closed decreasing subset of X that contains A. Analogously, we denote by $I(A)$ the smallest closed increasing subset of X that contains A.

A preorder $\precsim$ on a topological space (X, τ) is said to be

(i) *closed* if $\precsim$ is closed as a subset of $X \times X$ in the product topology $\tau \times \tau$;

(ii) *semi-closed* if $d(x) = d(\{x\})$ and $i(x) = i(\{x\})$ are closed subsets of X for every $x \in X$;

(iii) *D-I-closed* if for every closed subset A of X both sets $d(A)$ and $i(A)$ are closed subsets of X.

A preordered topological space $(X, \tau, \precsim)$ is said to be

(i) *regularly preordered* if for every point $x \in X$ and for every increasing closed set $F \subset X$ such that $x \notin F$ there are two disjoint open subsets U and V, decreasing and increasing respectively, such that $x \in U$ and $F \subset V$. A dual property is required if $x \notin F$ and F is decreasing.

(ii) *normally preordered* if for any two disjoint closed decreasing, respectively increasing, subsets A and B of X, there exist disjoint open decreasing, respectively increasing, subsets U and V of X such that $A \subset U$ and $B \subset V$.

(iii) *strongly normally preordered* if for any two closed subsets A and B of X such that *not* $(y \precsim x)$ for all $x \in A$ and all $y \in B$ there exist disjoint open decreasing, respectively increasing, subsets U and V of X such that $A \subset U$ and $B \subset V$.

(iv) *perfectly normally preordered* if for every $A, B \subset X$ closed disjoint subsets of X, decreasing and increasing respectively, there is a continuous isotone function $f : (X, \tau, \precsim) \to ([0, 1], \tau_{nat}, \leq)$ such that $A = f^{-1}(0)$ and $B = f^{-1}(1)$.

Finally, we remember the following

Definition 2.1. A preorder $\precsim$ on a topological space (X, τ) is said to satisfy the *continuous multi-utility representation property* if a family $\mathcal{F}$ of continuous real functions f on (X, τ) can be chosen in such a way that $x \precsim y$ if and only if $f(x) \leq f(y)$ for every $f \in \mathcal{F}$.

3. Regularly and normally preordered topological spaces

Bosi and Herden [4, Theorem 4.3] proved that a space (X, τ) is normal if and only if every D-I-closed preorder $\precsim$ gives $(X, \tau, \precsim)$ the structure of a normally preordered space. We now show that a similar result does not hold for *regular spaces*.

We note that a preordered topological space $(X, \tau, \precsim)$ is regularly preordered if and only if for every $x \in X$ and for every decreasing (increasing) open set

$A \subset X$ with $x \in A$ there are an open decreasing (increasing) set A_x and a closed decreasing (increasing) set F_x such that $x \in A_x \subset F_x \subset A$.

Proposition 3.1. *If (X, τ) is a regular non-normal space, then there is a D-I-closed preorder defined on X which does not give $(X, \tau, \preceq)$ the structure of a regularly preordered space.*

Proof. Let (X, τ) be a regular non-normal space and let $F, G \subset X$ be two disjoint closed sets which cannot be separated by disjoint open subsets of X. We consider on X the preorder $\preceq = \Delta_X \cup (F \times F)$. If $C \subset X$ then $d(C) = i(C) = C$ if $C \cap F = \varnothing$, otherwise $d(C) = i(C) = C \cup F$. Hence, $\preceq$ is D-I-closed. Now, choose an element $x \in F$ and suppose there are two open disjoint sets $U, V \subset X$, decreasing and increasing respectively, such that $x \in U$ and $G \subset V$. Then, since every decreasing set which intersects F contains F, we would have $F \subset U$ and $G \subset V$, which is a contradiction. □

We recall that a preorder $\preceq$ on a topological space (X, τ) is said to be *I-continuous* if, for every open subset A of X, $d(A)$ and $i(A)$ are both open. It is said to be *IC-continuous* if it is both I-continuous and D-I-closed (see Künzi [15]).

Proposition 3.2. *A space (X, τ) is regular if and only if every IC-continuous preorder gives $(X, \tau, \preceq)$ the structure of a regularly preordered space.*

Proof. Let (X, τ) be regular and let $\preceq$ be a IC-continuous preorder on X. If $x \in A$, where A is an open decreasing subset of X, then, by regularity of X, there is an open subset O of X such that

$$x \in O \subset cl(O) \subset A.$$

Since A is decreasing, we have that

$$x \in O \subset cl(O) \subset d(cl(O)) \subset A$$

where $d(cl(O))$ is closed (and decresasing) because $\preceq$ is D-I-closed. Finally, we have that

$$x \in d(O) \subset d(cl(O)) \subset A,$$

where $d(O)$ is open and decreasing.

Similarly, if $x \in A$ and A is an open increasing subset of X, there is an open subset O of X such that

$$x \in i(O) \subset i(cl(O)) \subset A.$$

Conversely, since the discrete preorder $\preceq$ is IC-continuous, it easily follows the regularity of (X, τ). □

The following proposition furnishes a characterization of perfectly normally preordered spaces in terms of normally preordered spaces. The following definition is needed.

Definition 3.3. Let $(X, \tau, \precsim)$ be a preordered topological space. Then

(i) A decreasing (increasing) set $F \subset X$ is said to be $\precsim$-F_σ if it is a countable union of decreasing (increasing) closed subsets of X;

(ii) A decreasing (increasing) set $G \subset X$ is said to be $\precsim$-G_δ if it is a countable intersection of decreasing (increasing) open subsets of X.

We note that every open decreasing (increasing) subset of X is $\precsim$-F_σ if and only if every closed increasing (decreasing) subset of X is $\precsim$-G_δ.

Proposition 3.4. *Let $(X, \tau, \precsim)$ be a preordered topological space. Then $(X, \tau, \precsim)$ is perfectly normally preordered if and only if $(X, \tau, \precsim)$ is normally preordered and all closed decreasing and increasing subsets of X are $\precsim$-G_δ.*

Proof. Let $(X, \tau, \precsim)$ be perfectly normal. Then, clearly, it is normally preordered. Now, let G be a closed decreasing subset of X and let $f : X \to [0,1]$ be a continuous isotone function such that $G = f^{-1}(0)$. Then, one has

$$G = \bigcap_{n=1}^{\infty} f^{-1}\Big[0, \frac{1}{n}\Big)$$

where the sets $f^{-1}\Big[0, \frac{1}{n}\Big)$ are clearly decreasing open subsets of X. Similarly, one can prove that if G is a closed increasing subset of X then it is $\precsim$-G_δ.

Conversely, suppose that A, B are disjoint closed subsets of X, decreasing and increasing respectively. By our assumptions, we have that $X \setminus A = \bigcup_{n=1}^{\infty} A_n$, where every A_n is a closed increasing subset of X. Since X is normally preordered there is a continuous isotone function $f_n : X \to [0,1]$ such that $f_n(A) = 0$ and $f_n(A_n) = 1$. Then, $f : X \to [0,1]$ defined by

$$f = \sum_{n=1}^{\infty} \frac{f_n}{2^n} : X \to [0,1]$$

is clearly isotone, continuous because the series converges uniformly and $A = f^{-1}(0)$. Similarly, there is a continuous isotone function $g : X \to [0,1]$ such that $B = g^{-1}(1)$. Then, it is easy to construct a continuous isotone function $h : X \to [0,1]$ such that $A = h^{-1}(0)$ and $B = h^{-1}(1)$ (cf., for instance, Minguzzi [17, Proposition 5.1]). $\square$

We recall that a topological space (X, τ) is *perfectly normal* if and only if it is a normal topological space and every closed subset of X is a G_δ-set. The previous Proposition and Theorem 4.3 in Bosi and Herden [4] allow us to prove the following proposition.

Proposition 3.5. *A space (X, τ) is perfectly normal if and only if every D-I-closed preorder gives $(X, \tau, \precsim)$ the structure of perfectly normally preordered space.*

Proof. Let (X, τ) be perfectly normal and $\precsim$ be a *D-I*-closed preorder on X. From Bosi and Herden [4, Theorem 4.3], it follows that $(X, \tau, \precsim)$ is normally preordered. Now, suppose that F is a decreasing open subset of X. From the perfect normality of (X, τ), there is a countable family $\{C_n\}$ of closed subsets of X such that

$$F = \bigcup_{n=1}^{+\infty} C_n.$$

Since F is decreasing and $\preceq$ is a D-I-closed preorder, we also have that

$$F = \bigcup_{n=1}^{+\infty} D(C_n).$$

It follows that every decreasing open subsets of X is $\preceq$-F_σ. Similarly, one proves that every increasing open subsets of X is also $\preceq$-F_σ. By the above Proposition 3.5, it follows that $(X, \tau, \preceq)$ is perfectly normally preordered.

Conversely, since the discrete preorder $\preceq$ is a D-I-closed preorder, it easily follows the perfect normality of X. □

Minguzzi [17, Proposition 5.2] proved that every regularly preordered Lindelöf space $(X, \tau, \preceq)$, where $\preceq$ is semi-closed, is normally preordered. It turns out that this is true even if the preorder is not semi-closed. Formally :

Proposition 3.6. *Every regularly preordered Lindelöf space is normally preordered.*

We recall that a topological space (X, τ) is said to be *hereditarily Lindelöf* if for every collection $\{O_i\}_{i \in I}$ of open subsets of X there exists a countable subset J of I such that $\bigcup_{i \in I} O_i = \bigcup_{j \in J} O_j$.

The following result about perfectly normally preordered spaces improves Theorem 5.3 in Minguzzi [17].

Proposition 3.7. *Every regularly preordered hereditarily Lindelöf space is perfectly normally preordered.*

Proof. Let $(X, \tau, \preceq)$ be a regularly preordered hereditarily Lindelöf space. By the previous proposition such a space is normally preordered. It remains to prove that every open decreasing (or increasing) subset of X is $\preceq$-F_σ. Let A be an open decreasing (increasing) subset of X and let $x \in A$. Since the space is regularly preordered, for every $x \in A$ there are an open decreasing (increasing) set U_x and a closed decreasing (increasing) set V_x such that $x \in U_x \subset V_x \subset A$. Since the space is hereditarily Lindelöf, the open cover $\{U_x : x \in A\}$ of A has a countable subcover $\{U_{x_n} : n \in \mathbb{N}\}$. Clearly

$$A = \bigcup_{n \in \mathbb{N}} U_{x_n} = \bigcup_{n \in \mathbb{N}} V_{x_n}$$

and so A is $\preceq$- F_σ. □

The next proposition follows from Theorem 4.3 and from the proof of Theorem 4.8 in Bosi and Herden [4].

Proposition 3.8. *A topological space (X, τ) is normal if and only if every D-I-closed preorder $\precsim$ on (X, τ) gives $(X, \tau, \precsim)$ the structure of a strongly normally preordered space.*

We recall that a topological space (X, τ) is said to be *limit point compact* if every infinite subset L of X has an accumulation point.

Proposition 3.9. *Let (X, τ) be a T_2 limit point compact topological space. Then the following conditions are equivalent*
(i) (X, τ) is normal
(ii) every closed preorder $\precsim$ on X gives $(X, \tau, \precsim)$ the structure of a strongly normally preordered space.

Proof. Without supposing any separation axiom on X, as in Bosi and Herden [4, Theorem 4.8], it is possible to prove the implication $(i) \Longrightarrow (ii)$. Conversely, if we assume (ii), then the discrete order $\precsim$ on X, which is closed since (X, τ) is T_2, gives $(X, \tau, \precsim)$ the structure of a strongly normally preordered space, that is to say, (X, τ) is normal. □

We observe that there exist non-normal T_1 compact spaces (hence limit point compact) that satisfies property (ii) in the above proposition. To this purpose, consider the following example.

Example 3.10. Let X be an infinite set endowed with the cofinite topology τ_1, that is $\tau_1 = \{A \subset X \ : \ |X \setminus A| < \aleph_0\} \cup \{\varnothing\}$. It is easy to see that $cl(\Delta_X) = X \times X$. Hence the only closed preorder of X is $\precsim = X \times X$. It follows that there is no pair (A, B) of non-empty closed disjoint subsets of X such that $\neg(x \precsim y)$ for all $x \in A$ and $y \in B$. So $(X, \tau_1, \precsim)$ is strongly normally preordered.

The compact space (X, τ_1) is also an example of a non-normal T_1 space, every closed preorder of which has a continuous multi-utility representation. Therefore, separation hypotheses stronger than T_1 are needed in order to guarantee the validity of the necessary conditions of Theorem 4.8 and Theorem 4.9 in Bosi and Herden [4].

Another example of a T_0 but not T_1 space is (Y, τ_2), where $Y = \{a, b_1, b_2\}$ and $\tau_2 = \{X, \varnothing, \{a\}, \{a, b_1\}, \{a, b_2\}\}$.

Now we will prove that Theorem 4.9 in [4] holds under the hypothesis T_3.

Proposition 3.11. *Let (X, τ) be a T_3-space. If every closed preorder $\precsim$ on (X, τ) has a continuous multi-utility representation then (X, τ) is normal.*

Proof. First we prove that if $\precsim$ is a D-I closed preorder then $\precsim$ is also closed. Suppose that $\neg\,[x \succsim y]$. Then, $x \notin i(y)$ and $i(y)$ is a closed subset of X since (X, τ) is T_1 and $\precsim$ is D-I closed. So, by regularity of X, there is an open set O in X such that $x \in O \subset Cl(O) \subset X \setminus i(y)$. Moreover, since $\precsim$ is D-I-closed, we also get $x \in O \subset Cl(O) = d(Cl(O)) \subset X \setminus i(y)$. It follows that $U = Cl(O)$ and $V = X \setminus Cl(O)$ are disjoint neighborhoods of x and y, decreasing and increasing respectively. As known, this is equivalent to say that $\precsim$ is closed.

By hypothesis, we obtain that every D-I closed preorder has a continuous multi-utility representation. Finally, from Theorem 4.5 [4], that is true for T_1 spaces, the thesis follows. □

We note that in Proposition 3.11 the space X is not assumed to be limit point compact. Instead, this hypothesis allows to prove the sufficient condition of Theorem 4.9 in [4]:

Proposition 3.12. *Let (X, τ) be a limit point compact space. If (X, τ) is a normal space then every closed preorder on (X, τ) has a continuous multi-utility representation.*

In fact, if (X, τ) is limit point compact then, as showed in the proof of Theorem 4.8 in [4], every closed preorder $\preceq$ on X is D-I closed. Thus, if $\preceq$ is a closed preorder, $(X, \tau, \preceq)$ is normally preordered and, as known, $\preceq$ has a continuous multi-utility representation.

Propositions 3.11 and 3.12 investigate the relation between the existence of continuous multi-utility representation for closed preorders and the normality of the topological space.

4. Countable continuous multi-utilities

Minguzzi [17] proved that, if $(X, \tau, \preceq)$ is a second-countable regularly preordered space, where $\preceq$ is a closed preorder, then there is a countable continuous multi-utility representation for $\preceq$. Actually, using the same proof, the following result holds.

Proposition 4.1. *Let $(X, \tau, \preceq)$ be a regularly preordered space with a closed preorder. If the product topology $\tau \times \tau$ on $X \times X$ is hereditarily Lindelöf, then $\preceq$ has a countable continuous multi-utility representation.*

A family $\mathcal{N}$ of subsets of a topological space (X, τ) is called a *network* for X if every open subset of X is a union of elements of $\mathcal{N}$.
The *network weight* (or *net weight*) of (X, τ) is defined by

$$nw(X, \tau) = \min\{|\mathcal{N}| : \mathcal{N} \text{ is a network for } (X, \tau)\} + \aleph_0.$$

We just mention the fact that the existence of a countable net weight generalizes the concept of *second countability* of a topology (i.e., the existence of a countable *basis*). Moreover, we remember that if a topological space X has a countable netweight then $X \times X$ is hereditarily Lindelöf.

Corollary 4.2. *Let $(X, \tau, \preceq)$ be a regularly preordered space with a countable net weight and assume that $\preceq$ is a closed preorder. Then $\preceq$ has a countable continuous multi-utility representation.*

The following example shows that, even if $(X, \tau, \preceq)$ is a regularly preordered hereditarily Lindelöf space and $\preceq$ is a total closed preorder (hence $(X, \tau, \preceq)$ is perfectly normally preorded), then, in general, it doesn't exist a countable continuous multi-utility representation for $\preceq$.

Example 4.3. Let $J = (-1, 1] \subset \mathbb{R}$ endowed with the Sorgenfrey topology τ and with the preorder $\precsim$ defined by

$$x \precsim y \Leftrightarrow \mid x \mid > \mid y \mid \text{ or } (\mid x \mid = \mid y \mid \text{ and } x < 0) \text{ or } x = y \ \forall x, y \in (-1, 1].$$

It is well-known that the space (J, τ) is hereditarily Lindelöf. Moreover, it is easy to verify that $\precsim$ is semiclosed. In fact, if $a > 0$, then one has

$$d(-a) = (-1, -a] \cup (a, 1] \ , \ i(-a) = [-a, a]$$

and

$$d(a) = (-1, -a] \cup [a, 1] \ , \ i(a) = (-a, a].$$

Since $\precsim$ is total, then it is also closed. Now, we observe that the preorder $\precsim$ defined on J is neither D-I-closed nor I-continuous, and so Proposition 3.2 cannot apply to prove that $(J, \tau, \precsim)$ is regularly preordered. In fact, if $-1 \leq -a < -b \leq 0$, then $i((-a, -b]) = (-a, a)$ which is open but not closed. Moreover, if $0 \leq a < b \leq 1$, then $i((a, b]) = (-1, -a] \cup [a, 1]$ which is closed but not open.

However, it is not difficult to prove directly that $(J, \tau, \precsim)$ is regularly preordered.

Let $x \in J$ and suppose $x = -a \leq 0$. If $A \subset J$ is an open increasing set containing x, then $x \in i(x) \subset A$, hence $x \in [-a, a] \subset A$. Since A is open and increasing, then there is $b \in J$ with $a < b < 1$ such that $x \in (-b, b) \subset A$. If $c \in J$, with $a < c < b$, then $x \in U = (-c, c] \subset A$, where U is closed, open and increasing. Now, suppose that $A \subset J$ is an open decreasing set containing x. Then one has $x \in d(x) \subset A$ and so $x \in V = (-1, -a] \cup (a, 1] \subset A$. Clearly, V is closed, open and increasing.

Now, let $x \in J$ and suppose $x = a > 0$. If $A \subset J$ is an open increasing set containing x, then $x \in i(x) \subset A$, hence $x \in U = (-a, a] \subset A$. As above U is closed, open and increasing. If $A \subset J$ is an open decreasing set containing x then $x \in d(x) \subset A$ and it follows that $x \in V = (-1, -a] \cup [a, 1] \subset A$. Since A is open and decreasing, then there is $b \in J$, with $0 < b < a < 1$, such that $x \in (-1, -b) \cup (b, 1] \subset A$. Finally, if $c \in J$, with $b < c < a$, then $x \in U = (-1, -c] \cup (c, 1] \subset A$, where U is closed, open and increasing.

In conclusion, $\precsim$ has a continuous multi-utility representation, since $(J, \tau, \precsim)$ is (perfectly) normally preordered and $\precsim$ is closed. But it does not have a countable one, otherwise (see Minguzzi [17, Lemma 5.4]) it should also have a countable continuous multi-utility representation consisting of continuous utility functions. But this is not true (see Bosi et al. [3]).

We recall that a topological space (X, τ) is said to be *submetrizable* if there is a metric topology τ' on X which is coarser than τ. Moreover, (X, τ) is *hemicompact* if there is a countable family $\{K_n\}$ of compact subsets of X such that every compact subset of X is contained in some K_n. Finally, X is a k-*space* if a subset $A \subset X$ is open if and only if $A \cap K$ is open in K, for every compact subset K of X.

In Minguzzi [17] it is proved that a k_ω-space, that is a hemicompact k-space, is normally preordered with respect to every closed preorder. Caterino et al.

[12, Theorem 3.5] showed that every closed preorder defined on a submetrizable k_ω-space has a continuous utility representation. An important example of a submetrizable k_ω-space is the space of the tempered distributions [9, 10, 11].

A submetrizable k_ω-space X is a union of a countable family of compact metrizable subspaces K_n. Since every K_n has a countable basis $\mathcal{B}_n$, then $\mathcal{B} = \cup_{n=1}^{\infty}\mathcal{B}_n$ is a countable network for X. Therefore, we have the following result.

Corollary 4.4. *Let $(X, \tau, \preceq)$ be a submetrizable k_ω-space, where $\preceq$ is a closed preorder. Then $\preceq$ has a countable continuous multi-utility representation.*

5. Conclusions

In this paper we have presented several results on regularly and normally preordered spaces and we have investigated the existence of continuous multi-utility representations for not necessarily total preorders defined on these spaces. Different topological conditions are considered to this aim, which can be thought of as interesting in mathematical economics, such as submetrizability or the existence of a countable net weight, as well as suitable "continuity conditions" of the preorder, such as the assumption according to which the preorder is closed.

In the last section of this paper, we have proved some sufficient conditions that guarantee the existence of a countable continuous multi-utility representation for every closed preorder. We hope that in the future we shall be able to characterize the preordered spaces in which every closed preorder has a countable continuous multi-utility representation. We note that Theorem 3.4 in [5] represents a result of this type with respect to the existence of continuous multi-utility representations of every closed preorder.

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