POSITIVE SOLUTIONS OF A ONE-DIMENSIONAL INDEFINITE CAPILLARITY-TYPE PROBLEM: A VARIATIONAL APPROACH

JULIÁN LÓPEZ-GÓMEZ, PIERPAOLO OMARI, AND SABRINA RIVETTI

ABSTRACT. We prove the existence and the multiplicity of positive solutions of the one-dimensional capillarity-type problem

$$-\Big(u'/\sqrt{1+(u')^2}\Big)' = a(x)f(u), \quad u'(0)=0,\ u'(1)=0,$$

where $a \in L^1(0,1)$ changes sign and $f : [0,+\infty) \to [0,+\infty)$ is continuous and has a power-like behavior at the origin and at infinity. Our approach is variational and relies on a regularization procedure that yields bounded variation solutions which are of class $W^{2,1}_{\mathrm{loc}}$, and hence satisfy the equation pointwise almost everywhere, on each open interval where the weight function a has a constant sign.

1. Introduction and statements

In this paper we are interested in the existence of positive solutions of the quasilinear Neumann problem

$$\begin{cases} -\left(\dfrac{u'}{\sqrt{1+(u')^2}}\right)' = a(x)f(u) \quad \text{in } (0,1), \\ u'(0)=0,\ u'(1)=0, \end{cases} \tag{1.1}$$

where $a \in L^1(0,1)$ changes sign and $f : [0,+\infty) \to [0,+\infty)$ is a continuous function having superlinear, or sublinear, growth at 0 and at $+\infty$.

Problem (1.1) is a particular, one-dimensional, version of the elliptic problem

$$\begin{cases} -\operatorname{div}\left(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = g(x,u) \quad \text{in } \Omega, \\ -\dfrac{\nabla u\cdot\nu}{\sqrt{1+|\nabla u|^2}} = \sigma \quad \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where Ω is a bounded regular domain in $\mathbb{R}^N$, with outward pointing normal ν, and $g : \Omega\times\mathbb{R} \to \mathbb{R}$ and $\sigma : \partial\Omega \to \mathbb{R}$ are given functions. This problem plays a relevant role in the mathematical analysis of a number of physical or geometrical issues, such as capillarity phenomena for incompressible fluids, reaction-diffusion processes where the flux features saturation at high regimes,

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or prescribed mean curvature problems for cartesian surfaces in the Euclidean space. Significant references include [41, 58, 10, 22, 31, 28, 34, 32, 30, 36, 39, 33, 40, 16].

Although there is a large amount of literature devoted to the existence of positive solutions for semilinear elliptic problems with superlinear indefinite nonlinearities, starting with [7, 1, 2, 9, 8, 3], no result is available for the problem (1.2), even in the one-dimensional case (1.1), in spite of the interest that this topic may have both mathematically and from the point of view of the applications.

As it will become clear later, according to Proposition 1.1 below, the existence of a positive solution for the homogeneous Neumann problem (1.1) forces the right hand side of the equation to change sign, thus ruling out the possibility, if f is non-negative, that the sign of the weight function a be constant. Hence, the absence of any previous result in the existing literature might be attributable to the fact that superlinear indefinite weighted problems are fraught with a number of technical difficulties which do not arise in dealing with purely sublinear or superlinear problems, even in the most classical semilinear case, not to talk about the degenerate quasilinear problem dealt with in this paper. In addition, as an effect of the spatial heterogeneities incorporated into the formulation of the problem the complexity of the structure of the solution sets might be quite intricate, even in the semilinear case [35, 48, 47, 46, 13, 14]. This an extremely challenging problem in the context of (1.1), which will be addressed elsewhere (see, e.g., [45]).

When the homogeneous Neumann boundary conditions are replaced in (1.1) by Dirichlet conditions, the existence of positive solutions is compatible with the right hand side of the equation having constant sign. As in this case technicalities are partially reduced, there are various results about existence, non-existence and multiplicity of positive solutions, even in higher dimension, assuming that both the functions a and f are non-negative (see, e.g., [51, 59, 50, 21, 20, 37, 42, 11, 53, 18]).

Our aim here is therefore to begin the analysis of the effects of spatial heterogeneities in the simplest one-dimensional prototype problem (1.1). Although part of our discussion has slightly been inspired by some available results in the context of semilinear elliptic problems, it must be stressed that the specific structure of the mean curvature operator,

$$u \mapsto \left(u'/\sqrt{1+(u')^2}\right)',$$

makes the analysis much more delicate and sophisticated, as it may determine the occurrence of discontinuous solutions [40, 11, 12, 52, 16, 54, 24, 23].

Since problem (1.1) has a variational structure, it is natural to look for its solutions as critical points of an associated action functional, such as

$$\mathcal{H}(v) = \int_0^1 (\sqrt{1+(v')^2} - 1)\, dx - \int_0^1 a\, F(v)\, dx,$$

with

$$F(s) = \int_0^s f(\xi)\, d\xi. \tag{1.3}$$

As the functional $\mathcal{H}$ grows linearly with respect to the gradient v', it is well-defined in the Sobolev space $W^{1,1}(0,1)$ of all absolutely continuous functions in $(0,1)$. Yet, this space, which might be an obvious candidate where to settle the study of $\mathcal{H}$, is not a favorable framework to deal with critical point theory. Therefore, we replace the space $W^{1,1}(0,1)$ with the space $BV(0,1)$ of all bounded variation functions in $(0,1)$, and the functional $\mathcal{H}$ with its relaxation $\mathcal{I}$ to $BV(0,1)$. Namely, we introduce the functional $\mathcal{J} : BV(0,1) \to \mathbb{R}$ defined by

$$\mathcal{J}(v) = \int_0^1 \sqrt{1+|Dv|^2} - 1, \tag{1.4}$$

where, for $v \in BV(0,1)$,

$$\int_0^1 \sqrt{1+|Dv|^2} = \sup_{\substack{w_1,w_2 \in C_0^1(0,1) \\ \|w_1^2+w_2^2\|_{L^\infty} \le 1}} \int_0^1 (vw_1 + w_2)\, dx,$$

Then, we denote by $\mathcal{I} : BV(0,1) \to \mathbb{R}$ the functional defined by

$$\mathcal{I}(v) = \mathcal{J}(v) - \mathcal{F}(v) \tag{1.5}$$

where, for $v \in BV(0,1)$,

$$\mathcal{F}(v) = \int_0^1 a\, F(v)\, dx.$$

The relaxed functional $\mathcal{I}$ is not differentiable in $BV(0,1)$, at least in the usual sense, yet it is the sum of the convex (Lipschitz) continuous functional $\mathcal{J}$ and of the continuously differentiable functional $\mathcal{F}$. Hence, following, e.g., [60], we say that a critical point of $\mathcal{I}$ is a function $u \in BV(0,1)$ such that

$$\mathcal{F}'(u) \in \partial\mathcal{J}(u),$$

where $\partial\mathcal{J}(u)$ denotes the subdifferential of $\mathcal{J}$ at the point u in the sense of convex analysis [27], or, equivalently, such that the variational inequality

$$\mathcal{J}(v) - \mathcal{J}(u) \ge \int_0^1 af(u)(v-u)\, dx \tag{1.6}$$

holds for all $v \in BV(0,1)$. Accordingly, the concept of solution used in this paper is fixed by the next definition.

Definition 1.1. *A solution of problem* (1.1) *is a function* $u \in BV(0,1)$ *such that* (1.6) *holds for all* $v \in BV(0,1)$. *In addition, a solution* u *of* (1.1) *is said to be positive if* $\operatorname{ess\,inf} u \ge 0$ *and* $\operatorname{ess\,sup} u > 0$, *and strictly positive if* $\operatorname{ess\,inf} u > 0$.

Remark 1.1. A function $u \in BV(0,1)$ satisfies the variational inequality (1.6) for all $v \in BV(0,1)$ if, and only if, u is a global minimizer in $BV(0,1)$ of the functional

$$\mathcal{K}_u(v) = \mathcal{J}(v) - \int_0^1 af(u)v\, dx.$$

Hence, we deduce from [6] that $u \in BV(0,1)$ is a solution of (1.1) if, and only if,

$$\int_0^1 \frac{(Du)^{\mathrm{a}}\,(D\phi)^{\mathrm{a}}}{\sqrt{1+|(Du)^{\mathrm{a}}|^2}}\, dx + \int_0^1 \operatorname{sgn}\left(\frac{Du}{|Du|}\right) \frac{D\phi}{|D\phi|}\, |D\phi|^{\mathrm{s}} = \int_0^1 a\, f(u)\, \phi\, dx \tag{1.7}$$

for all $\phi \in BV(0,1)$ such that $|D\phi|^{\mathrm{s}}$ is absolutely continuous with respect to $|Du|^{\mathrm{s}}$. Here, and in the sequel, for any given $v \in BV(0,1)$,

$$Dv = (Dv)^{\mathrm{a}} dx + (Dv)^{\mathrm{s}}$$

is the Lebesgue-Nikodym decomposition of the Radon measure Dv, the distributional derivative of v, in its absolutely continuous part $(Dv)^{\mathrm{a}} dx$, with density function $(Dv)^{\mathrm{a}}$, and its singular part $(Dv)^{\mathrm{s}}$, with respect to the Lebesgue measure in $\mathbb{R}$. If $|Dv|$ denotes the absolute variation of Dv,

$$|Dv| = |Dv|^{\mathrm{a}} dx + |Dv|^{\mathrm{s}}$$

is the Lebesgue-Nikodym decomposition of $|Dv|$. Moreover, $\frac{Dv}{|Dv|}$ stands for the density function of Dv with respect to its absolute variation $|Dv|$.

Note, in particular, that (1.7) implies that u is a weak solution of (1.1) if $u \in W^{1,1}(0,1)$.

The notion of solution for problem (1.1) introduced by Definition 1.1 has already been used and discussed in a series of papers, such as, e.g., [49, 42, 52, 53, 54, 56]. We just stress here its relevance because it allows to consider bounded variation solutions which arise as critical points of a different nature than minimizers of the associated action functional. However, unlike in these works, here we will go further in the investigation of the regularity properties of the bounded variation solutions we will find, by proving that they are actually $W^{2,1}_{\mathrm{loc}}$, and therefore classically satisfy the equation, on each open interval where the weight function a has a constant sign. Consequently, the discontinuities of the solutions that we construct may occur only in the nodal set of a, and we show that such discontinuity points must be 'vertical' ones. In this paper we do not address yet the issue of the existence of classical solutions: this topic will be discussed in the forthcoming paper [45], by using a different approach.

In order to better motivate the hypotheses we are going to impose on the coefficients a and f, we first observe that, if a positive solution u of (1.1) exists, then the function $a\,f(u)$ must change sign, unless it vanishes a.e. in $[0,1]$. Indeed, by choosing $v = u \pm 1$ as test functions in (1.6), or, in view of Remark 1.1, $\phi = 1$ in (1.7), we get

$$\int_0^1 a\,f(u)\,dx = 0. \tag{1.8}$$

Thus, if f has a constant sign, the function $a(x)$ must change sign in $[0,1]$. However, in the frame of (1.1) a stronger property holds if f is assumed to be increasing, as expressed by the following result. As usual, we write

$$a^+ = \max\{a, 0\} \quad \text{and} \quad a^- = -\min\{a, 0\}.$$

Proposition 1.1. *Assume that*

(a_1) $a \in L^1(0,1)$ *and* $a \neq 0$,

and

(f_1) $f \in C^1[0,+\infty)$ *is such that* $f(0) \geq 0$ *and* $f'(s) > 0$ *for all* $s > 0$.

Suppose that problem (1.1) *has a strictly positive solution. Then, the following holds*

(a_2) $a^+ \neq 0$ *and* $\displaystyle\int_0^1 a\,dx < 0.$

Remark 1.2. Even when $a \in L^1(0,1)$ satisfies (a_2), the condition (f_1) is not in general sufficient for guaranteeing the existence of a positive solution of (1.1). Indeed, suppose that there is an interval $[x_1, x_2] \subset (0,1)$ such that $a(x) > 0$ a.e. in $[x_1, x_2]$. Let ϕ_1 be a positive eigenfunction associated with the principal eigenvalue of $-d^2/dx^2$ in $H^1_0(x_1, x_2)$ and define

$$\phi(x) = \begin{cases} \phi_1(x) & \text{if } x \in [x_1, x_2], \\ 0 & \text{if } x \in [0,1] \setminus (x_1, x_2). \end{cases}$$

Suppose that (1.1) admits a positive solution u. Then, taking ϕ as a test function in (1.7) and using (f_1), we are driven to

$$\|\phi_1'\|_{L^1} \geq \int_{x_1}^{x_2} a\,f(u)\,\phi_1\,dx \geq f(\operatorname{ess\,inf} u) \int_{x_1}^{x_2} a\,\phi_1\,dx,$$

which clearly imposes a restriction on the size of f on the range of u, or on the amplitude of a in (x_1, x_2). This shows that some additional control on f, or on a, is needed.

Based on the observation that the mean curvature operator $\left(u'/\sqrt{1+(u')^2}\right)'$ behaves like the Laplace operator u'' at 0 and like the 1-Laplace operator $\left(u'/|u'|\right)'$ at infinity, and hence the functional $\mathcal{J}(u)$, defined in (1.4), behaves like $\frac{1}{2}\int_0^1 |u'|^2\,dx$ at 0 and like $\int_0^1 |u'|\,dx$ at infinity, we are led to impose on the potential F, defined in (1.3), some superquadraticity, or subquadraticity, conditions at 0 and superlinearity, or sublinearity, conditions at $+\infty$.

Our first existence result deals with the case where the potential F is superquadratic at 0 and superlinear at $+\infty$.

Theorem 1.1. *Assume that*

(a_3) $a \in L^1(0,1)$ *is such that* $\displaystyle\int_0^1 a\,dx < 0$ *and* $a(x) > 0$ *a.e. on an interval* $K \subset [0,1]$,

(f_2) $f \in C^0[0,+\infty)$ *is such that* $f(s) \geq 0$ *for* $s \geq 0$,

(f_3) *there exist* $p > 2$ *and* $L > 0$ *such that*

$$\lim_{s\to 0^+} \frac{F(s)}{s^p} = L,$$

(f_4) *there exist* $q > 1$ *and* $M > 0$ *such that*

$$\lim_{s\to +\infty} \frac{F(s)}{s^q} = M,$$

(f_5) *there exists* $\vartheta > 1$ *such that*

$$\lim_{s\to +\infty} \frac{\vartheta F(s) - f(s)\,s}{s} = 0,$$

with F defined in (1.3)*. Then, problem* (1.1) *has at least one positive solution u, with $\mathcal{I}(u) > 0$. In addition,*

$$u \in W^{2,1}_{\mathrm{loc}}(\alpha,\beta) \cap W^{1,1}(\alpha,\beta)$$

for each interval $(\alpha,\beta) \subset (0,1)$ such that $a(x) \geq 0$ a.e. in (α,β), or $a(x) \leq 0$ a.e. in (α,β). Moreover, $u \in W^{2,1}_{\mathrm{loc}}[0,\beta)$, with $u'(0) = 0$, if $\alpha = 0$, while $u \in W^{2,1}_{\mathrm{loc}}(\alpha,1]$, with $u'(1) = 0$, if $\beta = 1$. Finally, u satisfies the equation

$$-\left(\frac{u'}{\sqrt{1+(u')^2}}\right)' = a(x)f(u), \tag{1.9}$$

a.e. in each of such intervals.

Suppose further that

(f_6) *f is locally Lipschitz in $[0,+\infty)$.*

Then, for every pair of adjacent intervals, (α,β), $(\beta,\gamma) \subset (0,1)$ with $a(x) \geq 0$ a.e. in (α,β) and $a(x) \leq 0$ a.e. in (β,γ) (respectively, $a(x) \leq 0$ a.e. in (α,β) and $a(x) \geq 0$ a.e. in (β,γ)), either

$$u \in W^{2,1}_{\mathrm{loc}}(\alpha,\gamma),$$

or

$$u(\beta^-) \geq u(\beta^+) \quad \textit{and} \quad u'(\beta^-) = -\infty = u'(\beta^+)$$

$$(\textit{respectively,} \quad u(\beta^-) \leq u(\beta^+) \quad \textit{and} \quad u'(\beta^-) = +\infty = u'(\beta^+)),$$

where $u'(\beta^-), u'(\beta^+)$ are, respectively, the left and the right Dini derivatives at β.

Assume further that

(a_4) *the function a changes sign finitely many times in $(0,1)$, in the sense that there is a decomposition*

$$[0,1] = \bigcup_{i=1}^{k} [\alpha_i, \beta_i], \qquad \textit{with } \alpha_i < \beta_i = \alpha_{i+1} < \beta_{i+1}, \quad \textit{for } i = 1, \dots, k-1,$$

such that

$$(-1)^i a(x) \geq 0 \quad \textit{a.e. in } (\alpha_i, \beta_i), \quad \textit{for } i = 1, \dots, k,$$

or

$$(-1)^i a(x) \leq 0 \quad \textit{a.e. in } (\alpha_i, \beta_i), \quad \textit{for } i = 1, \dots, k.$$

Then, u is a strictly positive special function of bounded variation [5, Chapter 4].

Remark 1.3. The existence of classical positive solutions can be proved (see [45]), basically under the assumptions of Theorem 1.1, for all $\lambda > 0$ sufficiently large; whereas, for small $\lambda > 0$, it is likely that only bounded variation solutions may exist.

Remark 1.4. Suppose that (f_4) and (f_5) hold simultaneously. Condition (f_4) requires that the potential F behaves asymptotically as a power with exponent q, whereas condition (f_5), when coupled with (f_4), prescribes that the elasticity E_F of F, defined by

$$E_F(s) = \frac{sf(s)}{F(s)},$$

satisfies

$$\lim_{s \to +\infty} E_F(s) = \vartheta,$$

and hence F behaves asymptotically as a power with exponent ϑ. Thus, we conclude that $\vartheta = q$. Indeed, from

$$\lim_{s \to +\infty} \frac{\vartheta F(s) - f(s)\, s}{s} = 0,$$

we infer

$$\lim_{s \to +\infty} s^{q-1} \left(\vartheta \frac{F(s)}{s^q} - \frac{f(s)}{s^{q-1}} \right) = 0,$$

and hence, as $q > 1$,

$$\lim_{s \to +\infty} \left(\vartheta \frac{F(s)}{s^q} - \frac{f(s)}{s^{q-1}} \right) = 0.$$

Since

$$\lim_{s \to +\infty} \frac{F(s)}{s^q} = M,$$

we get

$$\lim_{s \to +\infty} \frac{f(s)}{s^{q-1}} = \vartheta M.$$

Then, L'Hospital's rule yields

$$\lim_{s \to +\infty} \frac{F(s)}{s^q} = \frac{\vartheta}{q} M$$

and hence $\vartheta = q$.

We also point out that (f_4) does not imply (f_5). This is shown by the function $f(s) = s^{q-1} + s^{q_0 - 1}$, with $1 \leq q_0 < q$, which obviously satisfies (f_4), but not (f_5). However, such a function satisfies both (f_4) and (f_5) whenever $0 < q_0 < 1 < q$.

Remark 1.5. Conditions (a_2) and (a_3) are equivalent if the function a is continuous. Hence, if (f_1), (f_3), (f_4), (f_5), and (a_4) are assumed, then we deduce from Proposition 1.1 and Theorem 1.1 that (a_2) is a necessary and sufficient condition for the existence of a strictly positive solution of (1.1).

A paradigmatic class of nonlinearities satisfying (f_2), (f_3), (f_4), and (f_5) is given by

$$f(s) = \min\{s^{p-1}, s^{q-1}\},$$

with $p > 2$, $q > 1$. Condition (f_1) is satisfied as well whenever $p = q$.

Our next existence result considers the case where the potential F is subquadratic at 0 and sublinear at $+\infty$.

Theorem 1.2. *Assume that*

(a_3) $a \in L^1(0,1)$ *is such that* $\displaystyle\int_0^1 a\,dx < 0$ *and* $a(x) > 0$ *a.e. on an interval* $K \subset [0,1]$,

(f_2) $f \in C^0[0,+\infty)$ *is such that* $f(s) \geq 0$ *for* $s \geq 0$,

(f_7) *there exist* $p \in (1,2)$ *and* $L > 0$ *such that*

$$\lim_{s\to 0^+} \frac{F(s)}{s^p} = L,$$

(f_8) *there exist* $q \in (0,1)$ *and* $M > 0$ *such that*

$$\lim_{s\to +\infty} \frac{F(s)}{s^q} = M,$$

with F *defined in* (1.3).

Then, problem (1.1) *has at least one positive solution* u, *with* $\mathcal{I}(u) < 0$. *In addition,* $u \in W^{2,1}_{\mathrm{loc}}(\alpha,\beta) \cap W^{1,1}(\alpha,\beta)$ *for each interval* $(\alpha,\beta) \subset (0,1)$ *such that* $a(x) \geq 0$ *a.e. in* (α,β), *or* $a(x) \leq 0$ *a.e. in* (α,β). *Moreover,* $u \in W^{2,1}_{\mathrm{loc}}[0,\beta)$, *with* $u'(0) = 0$, *if* $\alpha = 0$, *while* $u \in W^{2,1}_{\mathrm{loc}}(\alpha,1]$, *with* $u'(1) = 0$, *if* $\beta = 1$.

Remark 1.6. The existence of classical positive solutions can be proved (see [45]), under the assumptions of Theorem 1.2, for all $\lambda > 0$ sufficiently small. On the contrary, we conjecture that, for large $\lambda > 0$, only bounded variation solutions may exist.

Remark 1.7. Assume that f is locally Lipschitz in $(0,+\infty)$ and, for a pair of adjacent intervals, (α,β), $(\beta,\gamma) \subset (0,1)$, we have $a(x) \geq 0$ a.e. in (α,β) and $a(x) \leq 0$ a.e. in (β,γ) (respectively, $a(x) \leq 0$ a.e. in (α,β) and $a(x) \geq 0$ a.e. in (β,γ)).

If $u(\beta^-) > 0, u(\beta^+) > 0$, then either

$$u \in W^{2,1}_{\mathrm{loc}}(\alpha,\gamma),$$

or

$$u(\beta^-) \geq u(\beta^+) \quad \text{and} \quad u'(\beta^-) = -\infty = u'(\beta^+)$$

$$(\text{respectively,} \quad u(\beta^-) \leq u(\beta^+) \quad \text{and} \quad u'(\beta^-) = +\infty = u'(\beta^+)).$$

Whereas, if $u(\beta^-) = u(\beta^+) = 0$, then $u \in W^{1,1}(\alpha,\gamma)$ and actually $u \in W^{2,1}_{\mathrm{loc}}(\alpha,\gamma)$. Indeed, from (1.7) we infer that $\frac{u'}{\sqrt{1+(u')^2}} \in W^{1,1}_{\mathrm{loc}}(\alpha,\gamma)$ and thus the non-negativity of u yields $u'(\beta) = 0$, which in turn implies $u' \in W^{1,1}_{\mathrm{loc}}(\alpha,\gamma)$.

Finally, note that $u(\beta^-) > u(\beta^+) = 0$ (respectively, $u(\beta^+) > u(\beta^-) = 0$) cannot occur. Indeed, otherwise, by taking in (1.7) a test function $\phi \in W^{1,1}(0,1)$, having compact support in (α,γ) and such that $\phi(\beta) \neq 0$, we get, as $(D\phi)^{\mathrm{s}} = 0$,

$$\int_\alpha^\gamma a\, f(u)\,\phi\, dx = \int_\alpha^\gamma \frac{(Du)^{\mathrm{a}}\,(D\phi)^{\mathrm{a}}}{\sqrt{1+|(Du)^{\mathrm{a}}|^2}}\, dx.$$

On the other hand, since u satisfies the equation (1.9) a.e. in (α,β) and a.e. in (β,γ), multiplying by ϕ and integrating by parts on each of these two intervals, we obtain, because $u'(\beta^-) = -\infty$ and hence $\left(\frac{u'}{\sqrt{1+(u')^2}}\right)(\beta^-) = -1$,

$$\begin{aligned}
&\int_\alpha^\gamma a\, f(u)\,\phi\, dx \\
&= \int_\alpha^\gamma \frac{(Du)^{\mathrm{a}}\,(D\phi)^{\mathrm{a}}}{\sqrt{1+|(Du)^{\mathrm{a}}|^2}}\, dx - \left(\frac{u'}{\sqrt{1+(u')^2}}\right)(\beta^-)\,\phi(\beta^-) + \left(\frac{u'}{\sqrt{1+(u')^2}}\right)(\beta^+)\,\phi(\beta^+) \\
&= \int_\alpha^\gamma \frac{(Du)^{\mathrm{a}}\,(D\phi)^{\mathrm{a}}}{\sqrt{1+|(Du)^{\mathrm{a}}|^2}}\, dx + \phi(\beta)\left(1 + \left(\frac{u'}{\sqrt{1+(u')^2}}\right)(\beta^+)\right).
\end{aligned}$$

By comparison we conclude that $\left(\frac{u'}{\sqrt{1+(u')^2}}\right)(\beta^+) = -1$ and hence $u'(\beta^+) = -\infty$, which is impossible, due to the non-negativity of u.

Remark 1.8. In the framework of Theorem 1.2, one cannot guarantee the existence of a strictly positive solution. Actually, the lack of uniqueness for the Cauchy problems associated with the equation in (1.1) may even produce 'dead core' solutions. This will be illustrated through a simple example in Section 2.

A class of nonlinearities satisfying (f_7) and (f_8) is given by

$$f(s) = \min\left\{s^{p-1}, s^{q-1}\right\},$$

with $p \in (1,2)$, $q \in (0,1)$. Note also that Remark 1.7 applies to these functions.

Some variants of Theorem 1.1 and Theorem 1.2 can be proved, assuming that $f(0) = 0$ and $f'(0) \in (0,+\infty)$, and F is superlinear, or respectively sublinear, at $+\infty$. A control on the size of $f'(0)$ is however needed. These cases are, respectively, dealt with in Theorem 1.3, where we actually assume a more general quadraticity condition on F at 0, and in Theorem 1.4, where we also allow F to be asymptotically linear at $+\infty$. It is worthy to observe that now the considered assumptions do not rule out the validity of the local Lipschitz condition (f_6).

A detailed discussion of the existence and the non-existence of classical positive solutions, as well as of the development of singularities, when f has finite non-zero slope at 0 is produced in [45].

Theorem 1.3. *Assume that*

(a_3) $a \in L^1(0,1)$ *is such that* $\int_0^1 a\, dx < 0$ *and* $a(x) > 0$ *a.e. on an interval* $K \subset [0,1]$,

(f_2) $f \in C^0[0,+\infty)$ *is such that* $f(s) \geq 0$ *for* $s \geq 0$,

(f_4) *there exist* $q > 1$ *and* $M > 0$ *such that*

$$\lim_{s\to+\infty} \frac{F(s)}{s^q} = M,$$

(f_5) *there exists* $\vartheta > 1$ *such that*

$$\lim_{s\to+\infty} \frac{\vartheta F(s) - f(s)\,s}{s} = 0,$$

(f_9) *there exists* $L > 0$ *such that*

$$\lim_{s\to 0^+} \frac{F(s)}{s^2} = L,$$

with F *defined in* (1.3).

Then, there is $L^* > 0$ *such that problem* (1.1) *has at least one positive solution* u*, with* $\mathcal{I}(u) > 0$*, provided that* (f_9) *is satisfied with* $L \in (0, L^*)$*.*

In addition, $u \in W^{2,1}_{\mathrm{loc}}(\alpha,\beta) \cap W^{1,1}(\alpha,\beta)$ *for each interval* $(\alpha,\beta) \subset (0,1)$ *such that* $a(x) \geq 0$ *a.e. in* (α,β)*, or* $a(x) \leq 0$ *a.e. in* (α,β)*. Moreover,* $u \in W^{2,1}_{\mathrm{loc}}[0,\beta)$*, with* $u'(0) = 0$*, if* $\alpha = 0$*, while* $u \in W^{2,1}_{\mathrm{loc}}(\alpha,1]$*, with* $u'(1) = 0$*, if* $\beta = 1$*.*

The remaining conclusions of Theorem 1.1 *hold too, whenever* (f_6)*, or* (f_6) *and* (a_4)*, are assumed.*

The simplest prototype of functions satisfying (f_2), (f_4), (f_5) and (f_9) is obviously

$$f(s) = 2Ls,$$

for some $L > 0$.

Theorem 1.4. *Assume that*

(a_3) $a \in L^1(0,1)$ *is such that* $\displaystyle\int_0^1 a\,dx < 0$ *and* $a(x) > 0$ *a.e. on an interval* $K \subset [0,1]$*,*

(f_2) $f \in C^0[0,+\infty)$ *is such that* $f(s) \geq 0$ *for* $s \geq 0$*,*

(f_9) *there exists* $L > 0$ *such that*

$$\lim_{s\to 0^+} \frac{F(s)}{s^2} = L,$$

(f_{10}) *there exists* $M > 0$ *such that*

$$\lim_{s\to+\infty} \frac{F(s)}{s} = M,$$

with F *defined in* (1.3)*. Then, there are* $L_* > 0$ *and* $M^* > 0$ *such that problem* (1.1) *has at least one positive solution* u*, with* $\mathcal{I}(u) < 0$*, provided that* (f_9) *is satisfied with* $L > L_*$ *and* (f_{10}) *is satisfied with* $M < M^*$*.*

In addition, $u \in W^{2,1}_{\mathrm{loc}}(\alpha,\beta) \cap W^{1,1}(\alpha,\beta)$ *for each interval* $(\alpha,\beta) \subset (0,1)$ *such that* $a(x) \geq 0$ *a.e. in* (α,β)*, or* $a(x) \leq 0$ *a.e. in* (α,β)*. Moreover,* $u \in W^{2,1}_{\mathrm{loc}}[0,\beta)$*, with* $u'(0) = 0$*, if* $\alpha = 0$*, while* $u \in W^{2,1}_{\mathrm{loc}}(\alpha,1]$*, with* $u'(1) = 0$*, if* $\beta = 1$*.*

The remaining conclusions of Theorem 1.1 *hold too, whenever* (f_6)*, or* (f_6) *and* (a_4)*, are assumed.*

Remark 1.9. In Theorem 1.4 condition (f_{10}) can be replaced by condition (f_8); in this case no restriction on M is required.

A simple example of functions satisfying (f_2), (f_4), (f_5), (f_9) and (f_{10}) is

$$f(s) = \min\{2Ls, M\},$$

for some $L, M > 0$.

Our last two theorems provide the existence of multiple solutions, assuming that either F is superquadratic at 0 and F is sublinear at $+\infty$, or F is subquadratic at 0 and superlinear at $+\infty$. These multiplicity conclusions basically rely on the critical value informations we have obtained in Theorem 1.1 and Theorem 1.2. In order to state these results in a plain form, we introduce a multiplicative parameter $\lambda > 0$ into the right hand side of problem (1.1), as follows

$$\begin{cases} -\left(\dfrac{u'}{\sqrt{1+(u')^2}}\right)' = \lambda a(x) f(u) \quad \text{in } (0,1), \\ u'(0) = 0, \ u'(1) = 0. \end{cases} \tag{1.10}$$

Of course, the functional $\mathcal{I}$ associated with (1.10) now depends on the parameter λ as well, i.e., for $u \in BV(0,1)$,

$$\mathcal{I}(u) = \mathcal{I}_\lambda(u) = \mathcal{J}(u) - \lambda \int_0^1 aF(u)\,dx. \tag{1.11}$$

The corresponding variational inequality then reads

$$\mathcal{J}(v) - \mathcal{J}(u) \geq \lambda \int_0^1 af(u)(v-u)\,dx \quad \text{for all } v \in BV(0,1). \tag{1.12}$$

The left and the right pictures in Figure 1 express graphically the contents of Theorem 1.5 and Theorem 1.6, respectively. In order to keep this paper within a reasonable length, no attempt is made here to justify such bifurcation diagrams; details in this direction are given in [45].

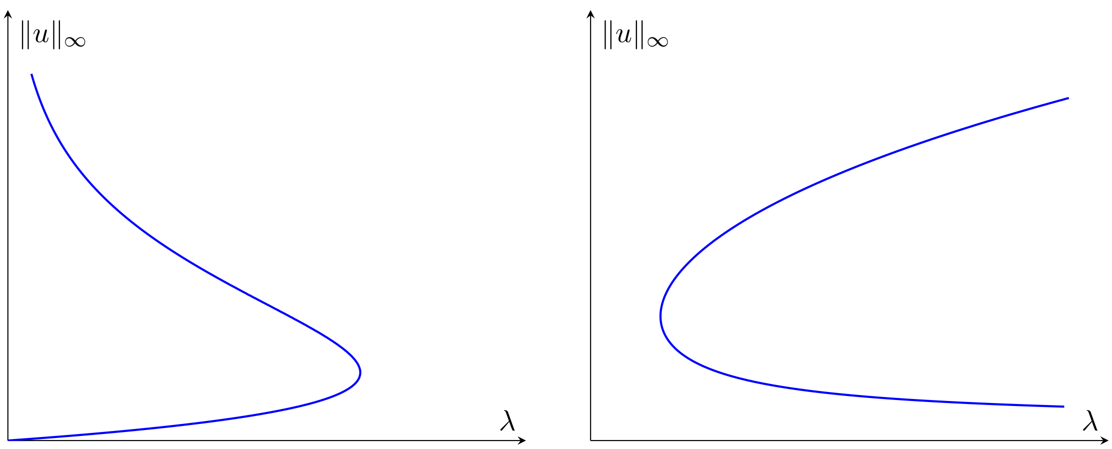

FIGURE 1. The left picture represents a possible diagram when the potential F is superquadratic at 0 and sublinear at $+\infty$, while the right one provides a diagram when F is subquadratic at 0 and superlinear at $+\infty$. In both pictures the parameter λ is plotted, in abscissas, versus $\|u\|_\infty$, in ordinates.

Theorem 1.5. *Assume that*

(a_3) $a \in L^1(0,1)$ *is such that* $\int_0^1 a\,dx < 0$ *and* $a(x) > 0$ *a.e. on an interval* $K \subset [0,1]$,

(f_2) $f \in C^0[0,+\infty)$ *is such that* $f(s) \geq 0$ *for* $s \geq 0$,

(f_3) *there exist* $p > 2$ *and* $L > 0$ *such that*

$$\lim_{s\to 0^+} \frac{F(s)}{s^p} = L,$$

(f_8) *there exist* $q \in (0,1)$ *and* $M > 0$ *such that*

$$\lim_{s\to +\infty} \frac{F(s)}{s^q} = M,$$

with F *defined in* (1.3).

Then, there is $\lambda_* > 0$ *such that problem* (1.10) *has, for all* $\lambda > \lambda_*$, *at least two positive solutions* u_1, u_2, *with* $\mathcal{I}_\lambda(u_1) < 0 < \mathcal{I}_\lambda(u_2)$.

In addition, $u_1, u_2 \in W^{2,1}_{\mathrm{loc}}(\alpha,\beta)\cap W^{1,1}(\alpha,\beta)$ *for each interval* $(\alpha,\beta) \subset (0,1)$ *such that* $a(x) \geq 0$ *a.e. in* (α,β), *or* $a(x) \leq 0$ *a.e. in* (α,β). *Moreover,* $u_1, u_2 \in W^{2,1}_{\mathrm{loc}}[0,\beta)$, *with* $u_1'(0) = u_2'(0) = 0$, *if* $\alpha = 0$, *while* $u_1, u_2 \in W^{2,1}_{\mathrm{loc}}(\alpha, 1]$, *with* $u_1'(1) = u_2'(1) = 0$, *if* $\beta = 1$.

The remaining conclusions of Theorem 1.1 *hold too, whenever* (f_6), *or* (f_6) *and* (a_4), *are assumed.*

A class of nonlinearities satisfying (f_2), (f_3) and (f_8) is given by

$$f(s) = \min\left\{s^{p-1}, s^{q-1}\right\},$$

with $p > 2$, $q \in (0,1)$.

Theorem 1.6. *Assume that*

(a_3) $a \in L^1(0,1)$ *is such that* $\int_0^1 a\,dx < 0$ *and* $a(x) > 0$ *a.e. on an interval* $K \subset [0,1]$,

(f_2) $f \in C^0[0,+\infty)$ *is such that* $f(s) \geq 0$ *for* $s \geq 0$,

(f_4) *there exist* $q > 1$ *and* $M > 0$ *such that*

$$\lim_{s\to +\infty} \frac{F(s)}{s^q} = M,$$

(f_5) *there exists* $\vartheta > 1$ *such that*

$$\lim_{s\to +\infty} \frac{\vartheta F(s) - f(s)\,s}{s} = 0,$$

(f_7) *there exist* $p \in (1,2)$ *and* $L > 0$ *such that*

$$\lim_{s\to 0^+} \frac{F(s)}{s^p} = L,$$

with F *defined in* (1.3).

Then, there is $\lambda^* > 0$ *such that problem* (1.10) *has, for all* $\lambda \in (0,\lambda^*)$, *at least two positive solutions* u_1, u_2, *with* $\mathcal{I}_\lambda(u_1) > 0 > \mathcal{I}_\lambda(u_2)$.

In addition, $u_1, u_2 \in W^{2,1}_{\mathrm{loc}}(\alpha,\beta)\cap W^{1,1}(\alpha,\beta)$ *for each interval* $(\alpha,\beta) \subset (0,1)$ *such that* $a(x) \geq 0$ *a.e. in* (α,β), *or* $a(x) \leq 0$ *a.e. in* (α,β). *Moreover,* $u_1, u_2 \in W^{2,1}_{\mathrm{loc}}[0,\beta)$, *with* $u_1'(0) = u_2'(0) = 0$, *if* $\alpha = 0$, *while* $u_1, u_2 \in W^{2,1}_{\mathrm{loc}}(\alpha, 1]$, *with* $u_1'(1) = u_2'(1) = 0$, *if* $\beta = 1$.

Remark 1.10. The regularity conclusions devised in Remark 1.7 extend to the framework of Theorem 1.6. Like in Theorem 1.2, even in the context of Theorem 1.6 one cannot guarantee the existence of a strictly positive solution (cf. Remark 1.8).

A simple prototype of functions satisfying (f_2), (f_4), (f_5) and (f_7) is given by

$$f(s) = s^{p-1},$$

with $p \in (1, 2)$.

Remark 1.11. It can be proved (see [45]) that in Theorem 1.5 and Theorem 1.6 one of the two solutions can be chosen to be classical.

The proofs of all the results stated in this paper rely on a perturbation argument, a sort of 'penalization', first introduced in [61], and further developed in [27, 44, 43, 11, 55, 54], for studying a number of boundary value problems associated with the mean curvature equation. It is worthy to point out that this approach has so far been limited to discussing cases where the discontinuities of the solutions may occur only on the boundary of the domain and not, like here, in the interior. Detecting and describing such lack of interior regularity requires developing a more refined argument, which for the moment seems to work limited to the one-dimensional case. The approximating problems are solved by using a minimax technique, in the frame of Theorem 1.1 and Theorem 1.3, and a minimization method, in the frame of Theorem 1.2 and Theorem 1.4. In all cases the obtention of $W^{1,1}$-estimates allow us to pass to the limit in the approximation scheme to get a bounded variation solution of the original problem. The multiplicity conclusions of Theorem 1.5 and Theorem 1.6 combines the preceding approaches and hinges over suitable critical value estimates. A further concavity/convexity argument, combined with ordinary differential equations techniques, finally permits to conclude the partial regularity of the obtained bounded variation solutions.

2. Some illustrating examples

First, we present a simple example showing that discontinuities of the solutions of (1.1) in the interior of the interval $[0, 1]$ may actually occur.

Example 1. Let $f : [0, +\infty) \to [0, +\infty)$ be any continuous function satisfying $f(s) > 0$ for $s > 0$. Then, the function $u \in BV(0, 1)$ given by

$$u(x) = \begin{cases} 1 + \sqrt{\frac{1}{4} - x^2} & \text{if } 0 \le x < \frac{1}{2}, \\ \frac{3}{4} - \sqrt{\frac{1}{4} - (x-1)^2} & \text{if } \frac{1}{2} \le x \le 1, \end{cases}$$

is a strictly positive solution of problem (1.1), with $a \in L^\infty(0, 1)$ defined by

$$a(x) = 2\,\mathrm{sgn}(\tfrac{1}{2} - x)\,(f(u(x)))^{-1}.$$

Indeed, for any $\phi \in BV(0, 1)$, with $|D\phi|^{\mathrm{s}}$ absolutely continuous with respect to $|Du|^{\mathrm{s}}$, we have

$$\int_0^{\frac{1}{2}} a\, f(u)\, \phi\, dx = -\int_0^{\frac{1}{2}} \left(\frac{u'}{\sqrt{1+(u')^2}}\right)' \phi\, dx = \phi(\tfrac{1}{2}^-) + \int_0^{\frac{1}{2}} \frac{u'\phi'}{\sqrt{1+(u')^2}}\, dx,$$

$$\int_{\frac{1}{2}}^1 a\, f(u)\, \phi\, dx = -\int_{\frac{1}{2}}^1 \left(\frac{u'}{\sqrt{1+(u')^2}}\right)' \phi\, dx = -\phi(\tfrac{1}{2}^+) + \int_{\frac{1}{2}}^1 \frac{u'\phi'}{\sqrt{1+(u')^2}}\, dx$$

and hence

$$\begin{aligned} \int_0^1 a\, f(u)\, \phi\, dx &= \int_0^1 \frac{(Du)^{\mathrm{a}}\,(D\phi)^{\mathrm{a}}}{\sqrt{1+|(Du)^{\mathrm{a}}|^2}}\, dx + \phi(\tfrac{1}{2}^-) - \phi(\tfrac{1}{2}^+) \\ &= \int_0^1 \frac{(Du)^{\mathrm{a}}\,(D\phi)^{\mathrm{a}}}{\sqrt{1+|(Du)^{\mathrm{a}}|^2}}\, dx + \int_0^1 \mathrm{sgn}\left(\frac{Du}{|Du|}\right) \frac{D\phi}{|D\phi|}\, |D\phi|^{\mathrm{s}}. \end{aligned}$$

Then the conclusion follows from Remark 1.1.

Next, we show with another simple example how, in the framework of Theorem 1.2, one cannot guarantee the existence of a strictly positive solution; the lack of uniqueness for the Cauchy problems associated with the equation in (1.1) may actually produce 'dead core' solutions.

Example 2. Let $f : [0,+\infty) \to [0,+\infty)$ be any continuous function with

$$f(s) = \sqrt{s} \qquad \text{for } s \in [0,1].$$

Then, the function $u \in W^{2,\infty}(0,1)$ defined by

$$u(x) = \begin{cases} 0 & \text{if } 0 \le x \le \frac{1}{3}, \\ \frac{1}{144}(x - \frac{1}{3})^4 & \text{if } \frac{1}{3} < x \le \frac{2}{3}, \\ \frac{1}{144}(\frac{2}{81} - (x-1)^4) & \text{if } \frac{2}{3} < x \le 1, \end{cases}$$

is a positive solution of (1.1), with $a \in L^\infty(0,1)$ defined by

$$a(x) = \begin{cases} -1 & \text{if } 0 \le x \le \frac{1}{3}, \\ -u''(x)\,(u(x))^{-\frac{1}{2}}\,(1 + (u'(x))^2)^{-\frac{3}{2}} & \text{if } \frac{1}{3} < x \le 1, \end{cases}$$

because (sufficiently) smooth solutions of (1.1) solve

$$\begin{cases} -u'' = a(x) f(u) \left(1 + (u')^2\right)^{\frac{3}{2}} & \text{in } (0,1), \\ u'(0) = u'(1) = 0. \end{cases}$$

3. Proof of Proposition 1.1

We point out that, although the conclusion and the proof of Proposition 1.1 is classical in the semilinear case (see, e.g., [7]), the adaptation to the present context, and in particular to the present notion of solution, requires a different argument, which relies on a rather delicate one-sided approximation argument in the space of bounded variation functions proven in [19].

Assume (a_1) and (f_1), and suppose that (1.1) admits a strictly positive solution u. We first notice that u cannot be a constant. Indeed, otherwise we get from (1.6)

$$0 = f(u) \int_0^1 a\, v\, dx \qquad \text{for all } \; v \in BV(0,1)$$

and hence $a = 0$, thus contradicting assumption (a_1).

We next observe that, since $\operatorname{ess\,inf} u > 0$ and (f_1) holds, $\frac{1}{f(u)} \in BV(0,1)$ and, due to [19, Theorem 3.3, p. 370], there exists a sequence $(v_n)_n$ in $W^{1,1}(0,1)$ such that, for all $n \ge 1$,

$$v_n \ge u \quad \text{ a.e. in } [0,1], \tag{3.1}$$

$$\lim_{n\to+\infty} v_n = u \quad \text{ and } \quad \lim_{n\to+\infty} \frac{1}{f(v_n)} = \frac{1}{f(u)} \quad \text{in } L^1(0,1) \text{ and a.e. in } [0,1], \tag{3.2}$$

and

$$\lim_{n\to+\infty} \int_0^1 |v_n'|\, dx = \int_0^1 |Du|, \qquad \lim_{n\to+\infty} \mathcal{J}(v_n) = \mathcal{J}(u). \tag{3.3}$$

Then, for any given $t > 0$, we have

$$\lim_{n\to+\infty} \Big(v_n + \frac{t}{f(v_n)}\Big) = u + \frac{t}{f(u)} \quad \text{ in } L^1(0,1)$$

and hence, by the lower semi-continuity of $\mathcal{J}$ with respect the L^1-convergence in $BV(0,1)$,

$$\mathcal{J}\Big(u+\frac{t}{f(u)}\Big)\leq \liminf_{n\to+\infty}\mathcal{J}\Big(v_n+\frac{t}{f(v_n)}\Big).$$

This entails that, along a subsequence of $(v_n)_n$, still labeled by n, we have

$$\mathcal{J}\left(u+\frac{t}{f(u)}\right)-\mathcal{J}(u)\leq \lim_{n\to+\infty}\left(\mathcal{J}\left(v_n+\frac{t}{f(v_n)}\right)-\mathcal{J}(v_n)\right). \tag{3.4}$$

Moreover, by (3.2) and (3.3), the sequence $(v_n)_n$ is bounded in $W^{1,1}(0,1)$, and therefore in $L^\infty(0,1)$, and, by (3.1), $v_n\geq u\geq \operatorname{ess\,inf} u>0$ for all $n\geq 1$. Thus, using (f_1), we infer the existence of a constant $\delta\in(0,1)$ such that

$$\delta\leq \frac{f'(v_n)}{(f(v_n))^2}\leq\frac{1}{\delta}\quad\text{for all }\ n\geq 1.$$

Hence, for any given $t\in(0,\delta)$ and all $n\geq 1$, the next chain of estimates holds

$$\begin{aligned}
&\frac{1}{t}\Big(\mathcal{J}\Big(v_n+\frac{t}{f(v_n)}\Big)-\mathcal{J}(v_n)\Big)\\
&\qquad=\frac{1}{t}\int_0^1\left(\sqrt{1+\Big(v_n'-t\frac{f'(v_n)}{(f(v_n))^2}v_n'\Big)^2}-\sqrt{1+(v_n')^2}\right)dx\\
&\qquad=\frac{1}{t}\int_0^1\frac{(v_n')^2\Big(\Big(1-t\frac{f'(v_n)}{(f(v_n))^2}\Big)^2-1\Big)}{\sqrt{1+(v_n')^2\Big(1-t\frac{f'(v_n)}{(f(v_n))^2}\Big)^2}+\sqrt{1+(v_n')^2}}\,dx\\
&\qquad\leq\frac{1}{t}\int_0^1\frac{(v_n')^2\,t\,\delta\,(t\delta-2)}{\sqrt{1+(v_n')^2\Big(1-t\frac{f'(v_n)}{(f(v_n))^2}\Big)^2}+\sqrt{1+(v_n')^2}}\,dx.
\end{aligned}$$

Consequently, since

$$t\delta-2<-1\quad\text{and}\quad\sqrt{1+(v_n')^2\Big(1-t\frac{f'(v_n)}{(f(v_n))^2}\Big)^2}\leq\sqrt{1+(v_n')^2},$$

we find that

$$\frac{1}{t}\left(\mathcal{J}\left(v_n+\frac{t}{f(v_n)}\right)-\mathcal{J}(v_n)\right)\leq-\delta\int_0^1\frac{{v_n'}^2}{2\sqrt{1+{v_n'}^2}}\,dx \tag{3.5}$$

for all $n\geq 1$.

We claim that

$$\liminf_{n\to+\infty}\int_0^1\frac{(v_n')^2}{\sqrt{1+(v_n')^2}}\,dx>0. \tag{3.6}$$

Suppose by contradiction that there exists a subsequence of $(v_n)_n$, still labeled by n, such that

$$\lim_{n\to+\infty}\int_0^1\frac{(v_n')^2}{\sqrt{1+(v_n')^2}}\,dx=0. \tag{3.7}$$

Since $x\mapsto\frac{x}{\sqrt{1+x^2}}$ is increasing, it is clear that

$$\frac{x^2}{\sqrt{1+x^2}}\geq\frac{x}{\sqrt{2}}\qquad\text{for }x\geq 1.$$

Thus we have

$$\frac{1}{\sqrt{2}}\int_{\{|v_n'|\geq 1\}} |v_n'|\,dx \leq \int_{\{|v_n'|\geq 1\}} \frac{(v_n')^2}{\sqrt{1+(v_n')^2}}\,dx \leq \int_0^1 \frac{(v_n')^2}{\sqrt{1+(v_n')^2}}\,dx,$$

and, from (3.7),

$$\lim_{n\to+\infty}\int_{\{|v_n'|\geq 1\}} |v_n'|\,dx = 0. \tag{3.8}$$

Similarly, since $x \mapsto \frac{1}{\sqrt{1+x^2}}$ is decreasing, it is evident that

$$\frac{x^2}{\sqrt{2}} \leq \frac{x^2}{\sqrt{1+x^2}} \qquad \text{for } x \in [0,1].$$

Hence we infer

$$\frac{1}{\sqrt{2}}\int_{\{|v_n'|<1\}} |v_n'|^2\,dx \leq \int_{\{|v_n'|<1\}} \frac{(v_n')^2}{\sqrt{1+(v_n')^2}}\,dx. \tag{3.9}$$

As

$$\left(\int_{\{|v_n'|<1\}} |v_n'|\,dx\right)^2 \leq \int_{\{|v_n'|<1\}} |v_n'|\,dx \leq \int_{\{|v_n'|<1\}} |v_n'|^2\,dx,$$

by (3.9), we find that

$$\frac{1}{\sqrt{2}}\left(\int_{\{|v_n'|<1\}} |v_n'|\,dx\right)^2 \leq \int_{\{|v_n'|<1\}} \frac{(v_n')^2}{\sqrt{1+(v_n')^2}}\,dx \leq \int_0^1 \frac{(v_n')^2}{\sqrt{1+(v_n')^2}}\,dx.$$

Consequently, from (3.7) and (3.8) we deduce that

$$\lim_{n\to+\infty}\int_0^1 |v_n'|\,dx = 0.$$

Hence, by (3.3), we obtain

$$\int_0^1 |Du| = 0,$$

which is impossible, because u is not a constant. Thus, (3.6) holds, as claimed above. Therefore, according to (3.4) and (3.5), we get

$$\frac{1}{t}\left(\mathcal{J}\left(u+\frac{t}{f(u)}\right)-\mathcal{J}(u)\right) \leq \lim_{n\to+\infty}\frac{1}{t}\left(\mathcal{J}\left(v_n+\frac{t}{f(v_n)}\right)-\mathcal{J}(v_n)\right) < 0.$$

Thus, by taking $v = u + \frac{t}{f(u)}$ as a test function in (1.6), we conclude that

$$\int_0^1 a\,dx \leq \frac{1}{t}\left(\mathcal{J}\left(u+\frac{t}{f(u)}\right)-\mathcal{J}(u)\right) < 0.$$

Moreover, since (1.8) holds, that is

$$\int_0^1 a\,f(u)\,dx = 0,$$

and ess inf $f(u) > 0$, the positive part a^+ of a must have a support with positive measure. The proof of Proposition 1.1 is concluded. □

4. Proof of Theorem 1.1

Since condition (f_3) implies $f(0) = 0$, we can extend f to the whole of $\mathbb{R}$ as an odd continuous function. Then, by (f_2), (f_3), (f_4) and (f_5), the following conditions hold for the odd extension of f, that we still denote by f:

(f_2^o) $f \in C^0(\mathbb{R})$ is such that $f(s)\,\mathrm{sgn}(s) \geq 0$ for all $s \in \mathbb{R}$,

(f_3^o) there exist $p > 2$ and $L > 0$ such that

$$\lim_{s \to 0} \frac{F(s)}{|s|^p} = L,$$

(f_4^o) there exist $q > 1$ and $M > 0$ such that

$$\lim_{|s| \to +\infty} \frac{F(s)}{|s|^q} = M,$$

(f_5^o) there exists $\vartheta > 1$ such that

$$\lim_{|s| \to +\infty} \frac{\vartheta F(s) - f(s)\,s}{s} = 0,$$

with F defined in (1.3).

Part 1. Solvability. The solvability of the problem will proceed after a series of steps. First we are going to describe a regularization scheme. Then, we will establish the existence of a solutions for each of the regularized problems through the mountain pass theorem of Ambrosetti and Rabinowitz [4, 57]. Since these approximating solutions possess a uniform a priori bound in $W^{1,1}$, by a compactness argument and critical value estimates one can establish the existence of a non-trivial bounded variation solution for (1.1). Finally, invoking the variational principle of Ekeland [26] we will be able to prove the existence of a positive solution for (1.1).

Step 1. A regularization scheme. Let us fix a number

$$\rho \in (1, q)\,. \tag{4.1}$$

Since in Remark 1.4 we proved that $\vartheta = q$, we have that $\rho < \vartheta$ too. Let us consider the odd C^1-diffeomorphism $\psi : \mathbb{R} \to \mathbb{R}$ given by

$$\psi(s) = \big((1 + |s|)^{\rho - 1} - 1\big)\mathrm{sgn}(s),$$

as well as the sequence of odd C^1-diffeomorphisms $\varphi_n : \mathbb{R} \to \mathbb{R}$ defined by

$$\varphi_n(s) = \frac{s}{\sqrt{1 + s^2}} + \frac{1}{n}\psi(s), \tag{4.2}$$

for all $n \in \mathbb{N}$, with $n \geq 1$. Subsequently, we set

$$\Psi(s) = \int_0^s \psi(t)\,dt = \frac{1}{\rho}\big((1 + |s|)^\rho - 1\big) - |s|, \tag{4.3}$$

$$\Phi_n(s) = \int_0^s \varphi_n(t)\,dt = \sqrt{1 + s^2} - 1 + \frac{1}{n}\Psi(s), \tag{4.4}$$

for all $s \in \mathbb{R}$, and we consider, for each $n \geq 1$, the associated functionals $\mathcal{J}_n : W^{1,\rho}(0, 1) \to \mathbb{R}$ defined by

$$\mathcal{J}_n(u) = \mathcal{J}(u) + \frac{1}{n}\int_0^1 \Psi(u')\,dx,$$

where $\mathcal{J}$ is given by (1.4), as well as $\mathcal{I}_n : W^{1,\rho}(0,1) \to \mathbb{R}$ defined by

$$\mathcal{I}_n(u) = \mathcal{J}_n(u) - \int_0^1 a\, F(u)\, dx.$$

For each $n \geq 1$, the function φ_n is increasing and therefore the functional $\mathcal{J}_n$ is convex. Moreover, as $\Psi(s) \geq 0$ for all $s \in \mathbb{R}$, it is clear that $\mathcal{J}_n(u) \geq \mathcal{J}(u)$ for all $u \in W^{1,\rho}(0,1)$ and hence

$$\mathcal{I}_n(u) \geq \mathcal{I}(u) \quad \text{for all } \; u \in W^{1,\rho}(0,1). \tag{4.5}$$

It follows from, e.g., [25, Chapter 2] or [57, Appendix B] that each $\mathcal{I}_n$ is of class C^1, with differential $\mathcal{I}_n'(u) \in \big(W^{1,\rho}(0,1)\big)^*$ given by

$$\mathcal{I}_n'(u)(v) = \int_0^1 \frac{u'\, v'}{\sqrt{1+(u')^2}}\, dx + \frac{1}{n}\int_0^1 \psi(u')\, v'\, dx - \int_0^1 a\, f(u)\, v\, dx$$

for all $u, v \in W^{1,\rho}(0,1)$. Clearly, if $u \in W^{1,\rho}(0,1)$ is a critical point of the functional $\mathcal{I}_n$, then $\varphi_n(u') \in W^{1,1}(0,1)$ and satisfies

$$\begin{cases} -\big(\varphi_n(u')\big)' = a(x) f(u) \quad \text{in} \quad (0,1), \\ u'(0) = 0, \;\; u'(1) = 0. \end{cases} \tag{4.6}$$

As $u' \in W^{1,1}(0,1)$ (see [17, Theorem 2.24]), the differential equation in (4.6) can equivalently be written as

$$-u'' = \frac{a(x)f(u)}{\varphi_n'(u')} = \frac{a(x)f(u)(1+(u')^2)^{\frac{3}{2}}}{1+\frac{1}{n}(1+(u')^2)^{\frac{3}{2}}(\rho-1)(1+|u'|)^{\rho-2}}. \tag{4.7}$$

Throughout the proof of this theorem, for every $u \in L^1(0,1)$ we set

$$r = \int_0^1 u\, dx \qquad \text{and} \qquad w = u - r,$$

so that u can be decomposed in the form

$$u = w + r, \tag{4.8}$$

with $\int_0^1 w\, dx = 0$. Then, for each $u \in BV(0,1)$, we set

$$\|u\|_{BV} = \int_0^1 |Dw| + |r|$$

and, if $u \in W^{1,\sigma}(0,1)$ for some $\sigma \geq 1$,

$$\|u\|_{W^{1,\sigma}} = \left(\int_0^1 |w'|^\sigma\, dx\right)^{\frac{1}{\sigma}} + |r| = \|w'\|_{L^\sigma} + |r|. \tag{4.9}$$

Using the representation (4.8), we can write, for all $u \in BV(0,1)$, the Poincaré-Wirtinger inequality [17, p. 102] in the form

$$\|w\|_\infty \leq \int_0^1 |Dw|,$$

and in particular, for all $u \in W^{1,1}(0,1)$,

$$\|w\|_\infty \leq \|w'\|_{L^1}. \tag{4.10}$$

Step 2. Solving the regularized problems. We will find a solution of (4.6) for any given $n \geq 1$, as a critical point of $\mathcal{I}_n$, via the mountain pass theorem.

The mountain pass geometry will follow from the next two technical results.

Lemma 4.1. *There exist constants $\delta > 0$ and $\eta > 0$ such that, setting*
$$\mathcal{S}_\eta = \{u \in W^{1,p}(0,1) : \|u\|_{W^{1,1}} = \|w'\|_{L^1} + |r| = \eta\},$$
one has, for all $n \geq 1$,
$$\inf_{u\in\mathcal{S}_\eta} \mathcal{I}_n(u) \geq \inf_{u\in\mathcal{S}_\eta} \mathcal{I}(u) \geq \delta.$$

Proof. Pick $u \in W^{1,1}(0,1)$ and use the decomposition (4.8). Due to the convexity of the function $s \mapsto \sqrt{1+s^2}$, the Jensen inequality yields
$$\sqrt{1+\|w'\|_{L^1}^2} \leq \int_0^1 \sqrt{1+|w'|^2}\,dx$$
and hence
$$\begin{aligned}
\mathcal{I}(u) &= \int_0^1 \sqrt{1+(w')^2}\,dx - 1 - \int_0^1 a\,F(w+r)\,dx \\
&\geq \sqrt{1+\|w'\|_{L^1}^2} - 1 - \int_0^1 a\,F(w+r)\,dx \\
&= \frac{\|w'\|_{L^1}^2}{1+\sqrt{1+\|w'\|_{L^1}^2}} - \int_0^1 a\,\left(F(w+r) - L|w+r|^p\right)\,dx \\
&\qquad - L\int_0^1 a\left(|w+r|^p - |r|^p\right)\,dx - L|r|^p \int_0^1 a\,dx,
\end{aligned} \tag{4.11}$$
where, according to (f_3°),
$$L = \lim_{s\to 0} \frac{F(s)}{|s|^p} > 0,$$
with $p > 2$. For any given $\varepsilon > 0$, there is $\eta_0 \in (0,1)$ such that
$$|F(s) - L|s|^p| \leq \varepsilon |s|^p, \quad \text{if } |s| \leq \eta_0. \tag{4.12}$$
Let $u \in W^{1,1}(0,1)$ satisfy $\|u\|_{W^{1,1}} \leq \eta_0$. Then, by (4.9) and (4.10), we have
$$\|w\|_\infty + |r| \leq \|w'\|_{L^1} + |r| = \|u\|_{W^{1,1}} \leq \eta_0.$$
Recall that, for all $x, y \in \mathbb{R}$ and all $p > 0$,
$$|x+y|^p \leq \max\{1, 2^{p-1}\}(|x|^p + |y|^p) \tag{4.13}$$
and hence, for all $x, y \in \mathbb{R}$ and all $p > 1$,
$$\big||x+y|^p - |x|^p\big| \leq \max\{1, 2^{p-2}\}(p\,|x|^{p-1}|y| + |y|^p). \tag{4.14}$$
Indeed, we have
$$\begin{aligned}
\big||x+y|^p - |x|^p\big| &= \left|\int_0^1 p|x+ty|^{p-2}(x+ty)y\,dt\right| \\
&\leq \int_0^1 p|x+ty|^{p-1}|y|\,dt \leq p\max\{1, 2^{p-2}\}\left(|x|^{p-1}|y| + \frac{1}{p}|y|^p\right).
\end{aligned}$$
Using (4.12), (4.13), with $p > 2$, and (4.10), we get
$$\begin{aligned}
\left|\int_0^1 a\left(F(w+r) - L|w+r|^p\right)\,dx\right| &\leq \int_0^1 |a|\,\varepsilon\,|w+r|^p\,dx \\
&\leq \varepsilon\,\|a\|_{L^1} 2^{p-1}\left(\|w\|_\infty^p + |r|^p\right) \\
&\leq \varepsilon\,\|a\|_{L^1} 2^{p-1}\left(\|w'\|_{L^1}^p + |r|^p\right).
\end{aligned}$$

Moreover, using (4.14), with $p > 2$, and (4.10), we obtain

$$\begin{aligned}\left| L\int_0^1 a\left(|w+r|^p - |r|^p\right)\,dx \right| &\leq L\|a\|_{L^1}\, \||w+r|^p - |r|^p\|_\infty \\ &\leq L\|a\|_{L^1} 2^{p-2}(p|r|^{p-1}\|w\|_\infty + \|w\|_\infty^p) \\ &\leq L\|a\|_{L^1} 2^{p-2}(p|r|^{p-1}\|w'\|_{L^1} + \|w'\|_{L^1}^p) \\ &\leq p\,2^{p-3}L\|a\|_{L^1}\left(\frac{1}{\sigma}|r|^{2(p-1)} + \sigma\|w'\|_{L^1}^2\right) \\ &\qquad\qquad + 2^{p-2}L\|a\|_{L^1}\|w'\|_{L^1}^p,\end{aligned}$$

for any given $\sigma > 0$. Then, from (4.11) we infer that, for all $u \in W^{1,1}(0,1)$ satisfying $\|w'\|_{L^1} + |r| = \|u\|_{W^{1,1}} = \eta_0$,

$$\begin{aligned}\mathcal{I}(u) \geq &\frac{1}{1+\sqrt{2}}\|w'\|_{L^1}^2 - p\,2^{p-3}L\|a\|_{L^1}\,\sigma\|w'\|_{L^1}^2 - 2^{p-2}(L+2\varepsilon)\|a\|_{L^1}\|w'\|_{L^1}^p \\ &- \left(2^{p-1}\|a\|_{L^1}\,\varepsilon + L\int_0^1 a\,dx\right)|r|^p - p\,2^{p-3}L\|a\|_{L^1}\frac{1}{\sigma}|r|^{2(p-1)}.\end{aligned}$$

Hence, taking $\varepsilon > 0$ and $\sigma > 0$ sufficiently small and using the condition $\int_0^1 a\,dx < 0$, assumed in (a_3), we can find constants $A, B, C, D > 0$ such that

$$\mathcal{I}(u) \geq A\|w'\|_{L^1}^2 - B\|w'\|_{L^1}^p + C|r|^p - D|r|^{2(p-1)}.$$

Let us set $x = \|w'\|_{L^1}$ and $y = |r|$. Taking $\eta \in (0, \eta_0)$ sufficiently small and using the condition $p > 2$, we see that

$$\left(A - Bx^{p-2}\right)x^2 + (C - Dy^{p-2})y^p \geq \frac{A}{2}x^2 + \frac{C}{2}y^p > 0,$$

for all $x \geq 0$ and $y \geq 0$, with $x + y = \eta$. Hence, setting

$$\delta = \min_{\substack{x\geq 0, y\geq 0\\ x+y=\eta}} \frac{1}{2}\left(Ax^2 + Cy^p\right) > 0,$$

we infer that, for every $n \geq 1$, the estimate

$$\inf_{u\in\mathcal{S}_\eta}\mathcal{I}_n(u) \geq \inf_{u\in\mathcal{S}_\eta}\mathcal{I}(u) \geq \delta$$

holds. This concludes the proof of Lemma 4.1. □

Lemma 4.2. *There exists $\zeta \in W^{1,\rho}(0,1)$, with $\min\zeta \geq 0$ and $\|\zeta\|_{W^{1,1}} > \eta$, such that, for all $n \geq 1$,*

$$\mathcal{I}_n(\zeta) < 0.$$

Proof. By assumption (a_3), there is an interval $K \subset (0,1)$ such that $a(x) > 0$ a.e. in K. Pick a function $z \in C^1[0,1]$, with $\operatorname{supp} z \subset K$, such that $z(x) = 1$ in an interval $K_0 \subset K$. Then, since $F(s) \geq 0$ for all $s \in \mathbb{R}$ and $F(0) = 0$, we easily get, for any given $t > 0$ and all $n \geq 1$,

$$\begin{aligned}\mathcal{I}_n(t\,z) &= \mathcal{J}_n(t\,z) - \int_K aF(t\,z)\,dx \\ &= \mathcal{J}_n(t\,z) - \int_{K_0} aF(t)\,dx - \int_{K\setminus K_0} aF(t\,z)\,dx \\ &\leq \mathcal{J}_n(t\,z) - F(t)\int_{K_0} a\,dx\end{aligned}$$

and thus, using (4.3) and (4.4),

$$\begin{aligned}
\mathcal{I}_n(t\,z) &\le \int_0^1 \Big(\big(1+(t\,z')^2\big)^{\frac{1}{2}}-1\Big)\,dx+\frac{1}{n}\int_0^1\Psi(t\,z')\,dx-F(t)\int_{K_0}a\,dx\\
&\le \int_K |t\,z'|\,dx+\frac{1}{\rho}\int_K\big((1+|t\,z'|)^\rho-1\big)\,dx-\int_K|t\,z'|\,dx-F(t)\int_{K_0}a\,dx\\
&\le t^q\left(\frac{t^{\rho-q}}{\rho}\int_K\big(t^{-1}+|z'|\big)^\rho\,dx-\frac{F(t)}{t^q}\int_{K_0}a\,dx\right).
\end{aligned}$$

Since we have, by $(f_4^{\rm o})$,

$$\lim_{t\to+\infty}\frac{F(t)}{t^q}=M>0$$

and, by (4.1), $\rho<q$, we derive

$$\lim_{t\to+\infty}\left(\frac{t^{\rho-q}}{\rho}\int_K\big(t^{-1}+|z'|\big)^\rho\,dx-\frac{F(t)}{t^q}\int_{K_0}a\,dx\right)=-M\int_{K_0}a\,dx<0.$$

Therefore, we conclude that

$$\mathcal{I}_n(t\,z)<0$$

for all sufficiently large $t>0$. By setting $\zeta=tz$, Lemma 4.2 follows. $\square$

The next result establishes the Palais–Smale condition in our framework.

Lemma 4.3. *Fix $n\in\mathbb{N}$, with $n\ge 1$. Let $(u_k)_k$ be a sequence in $W^{1,\rho}(0,1)$ satisfying*

$$\sup_{k\ge1}|\mathcal{I}_n(u_k)|<+\infty \qquad\text{and}\qquad \lim_{k\to+\infty}\mathcal{I}_n'(u_k)=0\ \text{ in }\big(W^{1,\rho}(0,1)\big)^*. \tag{4.15}$$

Then, there is a subsequence of $(u_k)_k$, still labeled by k, and $u\in W^{1,\rho}(0,1)$ such that

$$\lim_{k\to+\infty}u_k=u\ \text{ in } W^{1,\rho}(0,1). \tag{4.16}$$

Proof. We first show that $(u_k)_k$ is bounded in $W^{1,\rho}(0,1)$. By (4.15), there exist a constant $c>0$ and a sequence $(\varepsilon_k)_k$ in $(0,1)$, with $\lim_{k\to+\infty}\varepsilon_k=0$, such that

$$\mathcal{I}_n(u_k)=\int_0^1\sqrt{1+(u_k')^2}\,dx-1+\frac{1}{n}\int_0^1\Psi(u_k')\,dx-\int_0^1 aF(u_k)\,dx\le c \tag{4.17}$$

and

$$\big|\mathcal{I}_n'(u_k)(u_k)\big|=\left|\int_0^1\frac{(u_k')^2}{\sqrt{1+(u_k')^2}}\,dx+\frac{1}{n}\int_0^1\psi(u_k')u_k'\,dx-\int_0^1 af(u_k)u_k\,dx\right|\le\varepsilon_k\|u_k\|_{W^{1,\rho}}$$

for all $k\ge1$. Hence we get

$$\begin{aligned}
\vartheta\mathcal{I}_n(u_k)-\mathcal{I}_n'(u_k)(u_k)&=\int_0^1\left(\vartheta\left(\sqrt{1+(u_k')^2}-1\right)-\frac{(u_k')^2}{\sqrt{1+(u_k')^2}}\right)dx\\
&\quad+\frac{1}{n}\int_0^1\big(\vartheta\Psi(u_k')-\psi(u_k')u_k'\big)\,dx-\int_0^1 a\,(\vartheta F(u_k)-f(u_k)u_k)\,dx\\
&\le\vartheta\,c+\varepsilon_k\,\|u_k\|_{W^{1,\rho}}
\end{aligned}\tag{4.18}$$

for all $k\ge1$, where the constant $\vartheta\,(>\rho)$ comes from $(f_5^{\rm o})$. Now, observe that there exists a constant $\kappa>0$ such that the next two elementary inequalities hold

$$\vartheta\big(\sqrt{1+s^2}-1\big)-\frac{s^2}{\sqrt{1+s^2}}\ge\frac{1}{2}(\vartheta-1)\,|s|-\kappa, \tag{4.19}$$

$$\vartheta\,\Psi(s) - \psi(s)\,s \geq \frac{1}{2}\Big(\frac{\vartheta}{\rho} - 1\Big)\,|s|^{\rho} - \kappa, \tag{4.20}$$

for all $s \in \mathbb{R}$. Moreover, by (f_5^{o}), for every $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that

$$|\vartheta F(s) - f(s)s| \leq \varepsilon|s| + c_\varepsilon \qquad \text{for all } \ s \in \mathbb{R}. \tag{4.21}$$

Let us set, for all $k \geq 1$,

$$r_k = \int_0^1 u_k\,dx \quad \text{ and } \quad w_k = u_k - r_k. \tag{4.22}$$

According to (4.19), (4.20) and (4.21), we obtain from (4.9), (4.10) and (4.18) that

$$\begin{aligned}
&\vartheta\,c + \varepsilon_k\left(\|u_k'\|_{L^\rho} + |r_k|\right)\\
&\quad\geq \frac{1}{2}(\vartheta - 1)\,\|u_k'\|_{L^1} + \frac{1}{2n}\left(\frac{\vartheta}{\rho} - 1\right)\|u_k'\|_{L^\rho}^{\rho} - \varepsilon\|a\|_{L^1}\|u_k\|_\infty - c_\varepsilon\|a\|_{L^1} - 2\kappa.\\
&\quad\geq \frac{1}{2n}\left(\frac{\vartheta}{\rho} - 1\right)\|u_k'\|_{L^\rho}^{\rho} - \varepsilon\|a\|_{L^1}(\|u_k'\|_{L^1} + |r_k|) - c_\varepsilon\|a\|_{L^1} - 2\kappa.
\end{aligned}$$

Consequently, as $\|u_k\|_{L^1} \leq \|u_k\|_{L^\rho}$ for all $k \geq 1$, we obtain

$$\frac{1}{2n}\left(\frac{\vartheta}{\rho} - 1\right)\|u_k'\|_{L^\rho}^{\rho} \leq \varepsilon_k\left(\|u_k'\|_{L^\rho} + |r_k|\right) + \varepsilon\|a\|_{L^1}(\|u_k'\|_{L^\rho} + |r_k|) + c_\varepsilon\|a\|_{L^1} + 2\kappa + \vartheta c$$

for all $k \geq 1$. As in the previous estimate $\frac{\vartheta}{\rho} > 1$, $\varepsilon > 0$ is arbitrary and $\lim\limits_{k\to+\infty} \varepsilon_k = 0$, we can find a constant $d > 0$ such that

$$\|u_k'\|_{L^\rho}^{\rho} \leq |r_k| + d \qquad \text{for all } \ k \geq 1. \tag{4.23}$$

Let us suppose, by contradiction, that the sequence $(r_k)_k$ is unbounded, e.g., possibly for a subsequence still labeled by k,

$$\lim_{k\to+\infty} r_k = +\infty. \tag{4.24}$$

Then, by (4.23), for sufficiently large k, we obtain

$$\frac{\|u_k'\|_{L^\rho}}{r_k} \leq \frac{(r_k + d)^{\frac{1}{\rho}}}{r_k} = \left(r_k^{1-\rho} + d r_k^{-\rho}\right)^{\frac{1}{\rho}}$$

and hence, since $\rho > 1$,

$$\lim_{k\to+\infty} \frac{\|u_k'\|_{L^\rho}}{r_k} = 0. \tag{4.25}$$

Owing to (4.22) and (4.10), we have

$$\|w_k\|_\infty \leq \|w_k'\|_{L^1} \leq \|w_k'\|_{L^\rho} = \|u_k'\|_{L^\rho}$$

and hence, due to (4.25),

$$\lim_{k\to+\infty} \frac{\|u_k - r_k\|_\infty}{r_k} = \lim_{k\to+\infty} \frac{\|w_k\|_\infty}{r_k} = 0,$$

or, in other words,

$$\lim_{k\to+\infty} \frac{u_k}{r_k} = 1 \quad \text{ uniformly in } \quad [0,1]. \tag{4.26}$$

From (4.24) and (4.26), we infer

$$\lim_{k\to+\infty} u_k = +\infty \quad \text{ uniformly in } \quad [0,1]. \tag{4.27}$$

Thus, by (f_4°) we also have

$$\lim_{k\to+\infty} \frac{F(u_k)}{u_k^q} = M > 0 \quad \text{uniformly in} \quad [0,1]. \tag{4.28}$$

Therefore, by the dominated convergence theorem, from (4.17), (4.26) and (4.28) we find that

$$\begin{aligned} 0 &= \lim_{k\to+\infty} \frac{c}{r_k^q} \geq \lim_{k\to+\infty} \frac{\mathcal{I}_n(u_k)}{r_k^q} \\ &\geq - \lim_{k\to+\infty} \int_0^1 a(x) \frac{F(u_k(x))}{(u_k(x))^q} \left(\frac{u_k(x)}{r_k}\right)^q dx = -M \int_0^1 a\,dx, \end{aligned}$$

which is impossible, because we are assuming that $\int_0^1 a\,dx < 0$. As one can get a similar contradiction assuming that, along some subsequence relabeled by k,

$$\lim_{k\to+\infty} r_k = -\infty,$$

we conclude that the sequence $(r_k)_k$ is bounded. Consequently, by (4.23), the sequence $(u_k')_k$ is bounded in $L^\rho(0,1)$ and therefore, $(u_k)_k$ is bounded in $W^{1,\rho}(0,1)$, which was the first claim of the theorem.

Accordingly, thanks to the theorem of Eberlein-Shmulyan, there exist a subsequence of $(u_k)_k$, labeled again by k, and a function $u \in W^{1,\rho}(0,1)$ such that

$$\lim_{k\to+\infty} u_k = u \ \text{ weakly in } \ W^{1,\rho}(0,1) \ \text{ and strongly in } \ L^\infty(0,1), \tag{4.29}$$

because, due to the theorem of Rellich-Kondrachov, the imbedding of $W^{1,\rho}(0,1)$ into $C^0[0,1]$ is compact.

Next, we consider the generalized Dirichlet form

$$\mathcal{A}_n(u,v) := \int_0^1 \frac{u'\,v'}{\sqrt{1+(u')^2}}\,dx + \frac{1}{n}\int_0^1 \psi(u')\,v'\,dx - \int_0^1 |u|^{\rho-2}u\,v\,dx,$$

for all $u,v \in W^{1,\rho}(0,1)$. Since all assumptions of [15, Lemma 3] hold, the condition (S) therein will guarantee that $(u_k)_k$ is convergent to u strongly in $W^{1,\rho}(0,1)$ provided that

$$\lim_{k\to+\infty} \big(\mathcal{A}_n(u_k, u_k-u) - \mathcal{A}_n(u, u_k-u)\big) = 0. \tag{4.30}$$

Therefore, to complete the proof of the lemma, it suffices to establish (4.30).

As, due to (4.15), $\lim_{k\to+\infty} \mathcal{I}_n'(u_k) = 0$ in $\big(W^{1,\rho}(0,1)\big)^*$ and $(u_k)_k$ is bounded in $W^{1,\rho}(0,1)$, we find that

$$\lim_{k\to+\infty} \mathcal{I}_n'(u_k)(u_k-u) = 0. \tag{4.31}$$

Moreover, since from (4.29) $\lim_{k\to+\infty} u_k = u$ in $L^\infty(0,1)$, it is clear that

$$\lim_{k\to+\infty} \int_0^1 |u_k|^{\rho-2}u_k(u_k-u)\,dx = 0, \qquad \lim_{k\to+\infty} \int_0^1 af(u_k)(u_k-u)\,dx = 0. \tag{4.32}$$

Therefore, since

$$\mathcal{A}_n(u_k, u_k-u) = \mathcal{I}_n'(u_k)(u_k-u) + \int_0^1 |u_k|^{\rho-2}u_k(u_k-u)\,dx + \int_0^1 a\,f(u_k)\,(u_k-u)\,dx$$

for all $k \geq 1$, we find from (4.31) and (4.32) that

$$\lim_{k\to+\infty} \mathcal{A}_n(u_k, u_k-u) = 0. \tag{4.33}$$

Similarly, for every $k \geq 1$, we have

$$\mathcal{A}_n(u, u_k - u) = \mathcal{I}_n'(u)(u_k - u) + \int_0^1 |u|^{\rho-2} u(u_k - u)\, dx + \int_0^1 a\, f(u)\, (u_k - u)\, dx.$$

Since the linear functional $\mathcal{I}_n'(u)$ is weakly continuous in $W^{1,\rho}(0,1)$, condition (4.29) implies that

$$\lim_{k\to+\infty} \mathcal{I}_n'(u)(u_k - u) = 0.$$

Moreover, since $\lim_{k\to+\infty} u_k = u$ in $L^\infty(0,1)$, we infer

$$\lim_{k\to+\infty} \int_0^1 |u|^{\rho-2} u(u_k - u)\, dx = 0, \qquad \lim_{k\to+\infty} \int_0^1 a\, f(u)\, (u_k - u)\, dx = 0.$$

Consequently, we get

$$\lim_{k\to+\infty} \mathcal{A}_n(u, u_k - u) = 0$$

and combining this with (4.33), (4.30) holds. This ends the proof of Lemma 4.3. □

Subsequently, we consider the set of continuous curves linking the origin to the point ζ constructed in Lemma 4.2, i.e.,

$$\Gamma = \{\gamma \in C^0([0,1], W^{1,\rho}(0,1)) \; : \; \gamma(0) = 0, \; \gamma(1) = \zeta\}$$

and, for every $n \geq 1$, we set

$$c_n = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \mathcal{I}_n(\gamma(t)).$$

The mountain pass theorem then guarantees the existence, for any given $n \geq 1$, of a critical point $u_n \in W^{1,\rho}(0,1)$ of the functional $\mathcal{I}_n$, satisfying

$$\mathcal{I}_n(u_n) = c_n = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \mathcal{I}_n(\gamma(t)). \tag{4.34}$$

As $\varphi_n(u') \in W^{1,1}(0,1)$, u_n is a solution of (4.6) with $u_n \in W^{2,1}(0,1)$. Morerover, since $\mathcal{I}_{n+1}(v) \leq \mathcal{I}_n(v)$ for each $n \geq 1$ and every $v \in W^{1,\rho}(0,1)$, we find, using Lemma 4.1 too, that

$$0 < \delta \leq c_{n+1} \leq c_n \leq \cdots \leq c_1 \tag{4.35}$$

for all $n \geq 1$.

Step 3. Uniform estimates in $W^{1,1}(0,1)$ for the mountain pass solutions. The main conclusion of this section is the next one.

Lemma 4.4. *Let, for each $n \geq 1$, $u_n \in W^{1,\rho}(0,1)$ be the solution of* (4.6) *satisfying* (4.34). *Then, we have*

$$\sup_{n\geq 1} \|u_n\|_{W^{1,1}} < +\infty. \tag{4.36}$$

Proof. We can easily adapt the first part of the proof of Lemma 4.3 to the present setting. Indeed, according to (4.35), we have

$$\begin{aligned}\delta \leq \mathcal{I}_n(u_n) = \int_0^1 &\left(\sqrt{1 + (u_n')^2} - 1\right) dx \\ &+ \frac{1}{n} \int_0^1 \Psi(u_n')\, dx - \int_0^1 aF(u_n)\, dx = c_n \leq c_1\end{aligned} \tag{4.37}$$

and

$$\mathcal{I}_n'(u_n)(u_n) = \int_0^1 \frac{(u_n')^2}{\sqrt{1 + (u_n')^2}}\, dx + \frac{1}{n} \int_0^1 \psi(u_n') u_n'\, dx - \int_0^1 a f(u_n) u_n\, dx = 0$$

for all $n \geq 1$. Thus, we get

$$\vartheta \mathcal{I}_n(u_n) - \mathcal{I}_n'(u_n)(u_n) = \int_0^1 \left(\vartheta \left(\sqrt{1+(u_n')^2} - 1 \right) - \frac{(u_n')^2}{\sqrt{1+(u_n')^2}} \right) dx$$

$$+ \frac{1}{n} \int_0^1 \left(\vartheta \Psi(u_n') - \psi(u_n')u_n' \right) dx - \int_0^1 a \left(\vartheta F(u_n) - f(u_n)u_n \right) dx \leq \vartheta\, c_1$$

and hence, using (4.19), (4.20) and (4.21),

$$\vartheta c_1 \geq \frac{1}{2}(\vartheta-1)\|u_n'\|_{L^1} + \frac{1}{2n}\left(\frac{\vartheta}{\rho} - 1\right)\|u_n'\|_{L^\rho}^\rho - \varepsilon\|a\|_{L^1}(\|u_n'\|_{L^1} + |r_n|) - c_\varepsilon\|a\|_{L^1} - 2\kappa$$

$$\geq \frac{1}{2}(\vartheta-1)\|u_n'\|_{L^1} - \varepsilon\|a\|_{L^1}(\|u_n'\|_{L^1} + |r_n|) - c_\varepsilon\|a\|_{L^1} - 2\kappa.$$

Therefore, for every $\varepsilon > 0$ there exists $d_\varepsilon > 0$ such that

$$\|u_n'\|_{L^1} \leq \varepsilon|r_n| + d_\varepsilon \quad \text{for all } \; n \geq 1. \tag{4.38}$$

Let us suppose by contradiction that the sequence $(r_n)_n$ is unbounded, e.g., possibly for a subsequence, relabeled by n,

$$\lim_{n\to+\infty} r_n = +\infty.$$

Then, by (4.38), for sufficiently large k, we obtain

$$\frac{\|u_n'\|_{L^1}}{r_n} \leq \varepsilon + \frac{d_\varepsilon}{r_n}$$

and hence, letting $n \to +\infty$,

$$\lim_{n\to+\infty} \frac{\|u_n'\|_{L^1}}{r_n} \leq \varepsilon.$$

As $\varepsilon > 0$ is arbitrary, we infer

$$\lim_{n\to+\infty} \frac{\|u_n'\|_{L^1}}{r_n} = 0.$$

Then, arguing as in the proof of Lemma 4.3, we find

$$\lim_{n\to+\infty} \frac{u_n}{r_n} = 1 \quad \text{uniformly in} \quad [0,1],$$
$$\lim_{n\to+\infty} u_n = +\infty \quad \text{uniformly in} \quad [0,1],$$
$$\lim_{n\to+\infty} \frac{F(u_n)}{u_n^q} = M > 0 \quad \text{uniformly in} \quad [0,1].$$

Therefore, it readily follows from (4.37) that

$$0 = \lim_{n\to+\infty} \frac{c_1}{r_n^q} \geq \lim_{n\to+\infty} \frac{\mathcal{I}_n(u_n)}{r_n^q}$$
$$\geq - \lim_{n\to+\infty} \int_0^1 a(x) \frac{F(u_n(x))}{(u_n(x))^q} \left(\frac{u_n(x)}{r_n}\right)^q dx = -M \int_0^1 a\, dx.$$

which is impossible, because $\int_0^1 a\, dx < 0$. Consequently, $(r_n)_n$ is bounded. Condition (4.38) implies that $(u_n')_n$ is bounded in $L^1(0,1)$ and hence $(u_n)_n$ is bounded in $W^{1,1}(0,1)$. This ends the proof of Lemma 4.4. $\square$

Step 4. Existence of a solution of (1.1)**.** Condition (4.36) implies, by [5, Theorem 3.23, Proposition 3.13], that there exist a subsequence of $(u_n)_n$, labeled again by n, and a function $u \in BV(0,1)$ such that

$$\sup_{n\geq 1} \|u_n\|_\infty < +\infty \quad \text{and} \quad \lim_{n\to+\infty} u_n = u \ \text{ in } \ L^1(0,1) \ \text{ and a.e. in } \ [0,1]. \tag{4.39}$$

Let us prove that u is a solution of (1.1), i.e., u satisfies (1.6) for all $v \in BV(0,1)$. Since the functional $\mathcal{J}_n : W^{1,\rho}(0,1) \to \mathbb{R}$ is convex and of class C^1 and $\mathcal{I}_n'(u_n) = 0$ for all $n \geq 1$, we have that

$$\int_0^1 af(u_n)(z-u_n)\,dx = \mathcal{J}_n'(u_n)(z-u_n) \leq \mathcal{J}_n(z) - \mathcal{J}_n(u_n) \tag{4.40}$$

for every $z \in W^{1,\rho}(0,1)$. Since by (4.39)

$$\sup_{n\geq 1} \|f(u_n)(z-u_n)\|_\infty < +\infty$$

and

$$\lim_{n\to+\infty} af(u_n)(z-u_n) = af(u)(z-u) \quad \text{ a.e. in } \quad [0,1],$$

the dominated convergence theorem implies that

$$\int_0^1 af(u)(z-u)\,dx = \lim_{n\to+\infty} \int_0^1 af(u_n)(z-u_n)\,dx.$$

Thus, by (4.40) and the lower semicontinuity of $\mathcal{J}$ with respect to the L^1-convergence in $BV(0,1)$, we infer

$$\begin{aligned}\int_0^1 af(u)(z-u)\,dx &\leq \limsup_{n\to+\infty} (\mathcal{J}_n(z) - \mathcal{J}_n(u_n)) \\ &\leq \lim_{n\to+\infty} \left(\mathcal{J}(z) + \frac{1}{n}\int_0^1 \Psi(z')\,dx\right) - \liminf_{n\to+\infty} \left(\mathcal{J}(u_n) + \frac{1}{n}\int_0^1 \Psi(u_n')\,dx\right) \\ &\leq \mathcal{J}(z) - \mathcal{J}(u).\end{aligned}$$

Since $\mathcal{J}$ is a continuous functional on $W^{1,1}(0,1)$ and $W^{1,\rho}(0,1)$ is dense in $W^{1,1}(0,1)$, it is easily seen that

$$\mathcal{J}(z) - \mathcal{J}(u) \geq \int_0^1 af(u)(z-u)\,dx \quad \text{for all } \ z \in W^{1,1}(0,1). \tag{4.41}$$

Fix now $v \in BV(0,1)$. The approximation property in $BV(0,1)$ stated in [6, Fact 3.3] guarantees the existence of a sequence $(z_n)_n$ in $W^{1,1}(0,1)$ such that

$$\lim_{n\to+\infty} z_n = v \quad \text{in} \ \ L^1(0,1) \ \text{ and a.e. in } \ [0,1]$$

and

$$\lim_{n\to+\infty} \mathcal{J}(z_n) = \mathcal{J}(v).$$

The last condition also implies that

$$\sup_{n\geq 1} \|z_n\|_\infty \leq \sup_{n\geq 1} \|z_n\|_{W^{1,1}} < +\infty$$

and hence, thanks to the dominated convergence theorem,

$$\lim_{n\to+\infty} \int_0^1 af(u)z_n = \int_0^1 af(u)v\,dx.$$

Therefore, since

$$\mathcal{J}(z_n) - \mathcal{J}(u) \geq \int_0^1 af(u)(z_n - u)\,dx \quad \text{for all } \; n \geq 1,$$

letting $n \to +\infty$ in this inequality yields (1.6) for all $v \in BV(0,1)$.

Step 5. A critical value estimate. There is a subsequence of $(u_n)_n$, still labeled by n, such that

$$\lim_{n\to+\infty} \mathcal{I}(u_n) = \mathcal{I}(u). \tag{4.42}$$

Since condition (4.39) and the dominated convergence theorem imply that

$$\lim_{n\to+\infty} \int_0^1 aF(u_n)\,dx = \int_0^1 aF(u)\,dx,$$

it is evident that condition (4.42) will follow, if we prove that

$$\lim_{n\to+\infty} \mathcal{J}_n(u_n) = \mathcal{J}(u). \tag{4.43}$$

We first observe that

$$\liminf_{n\to+\infty} \mathcal{J}_n(u_n) \geq \mathcal{J}(u). \tag{4.44}$$

Indeed, we have

$$\mathcal{J}_n(u_n) = \mathcal{J}(u_n) + \frac{1}{n}\int_0^1 \Psi(u_n')\,dx \geq \mathcal{J}(u_n)$$

and hence, by the lower semicontinuity of $\mathcal{J}$ with respect to the L^1-convergence in $BV(0,1)$, we get (4.44). Next, we show that

$$\limsup_{n\to+\infty} \mathcal{J}_n(u_n) \leq \mathcal{J}(u). \tag{4.45}$$

Recall that each u_n satisfies the variational inequality (4.40) for all $z \in W^{1,\rho}(0,1)$. Hence, using again (4.39) and the dominated convergence theorem, we infer

$$\begin{aligned}
\limsup_{n\to+\infty} \mathcal{J}_n(u_n) &\leq \limsup_{n\to+\infty} \left(\mathcal{J}_n(z) - \int_0^1 af(u_n)(z-u_n)\,dx \right) \\
&= \limsup_{n\to+\infty} \left(\mathcal{J}(z) + \frac{1}{n}\int_0^1 \Psi(z')\,dx - \int_0^1 af(u_n)(z-u_n)\,dx \right) \\
&= \mathcal{J}(z) - \int_0^1 af(u)(z-u)\,dx.
\end{aligned}$$

Pick any $w \in W^{1,1}(0,1)$. By the density of $W^{1,\rho}(0,1)$ in $W^{1,1}(0,1)$, we find a sequence $(z_k)_k$ in $W^{1,\rho}(0,1)$ such that $\lim\limits_{k\to+\infty} z_k = w$ in $W^{1,1}(0,1)$. Therefore we have, for all $k \geq 1$,

$$\limsup_{n\to+\infty} \mathcal{J}_n(u_n) \leq \mathcal{J}(z_k) - \int_0^1 af(u)(z_k - u)\,dx$$

and hence, letting $k \to +\infty$,

$$\limsup_{n\to+\infty} \mathcal{J}_n(u_n) \leq \mathcal{J}(w) - \int_0^1 af(u)(w-u)\,dx.$$

By the approximation property in $BV(0,1$ [6, Fact 3.3], we find a sequence $(w_k)_k$ in $W^{1,1}(0,1)$ such that

$$\lim_{n\to+\infty} w_k = u \quad \text{in } \ L^1(0,1) \ \text{ and a.e. in } \ [0,1],$$
$$\lim_{n\to+\infty} \mathcal{J}(w_k) = \mathcal{J}(u) \ \text{ and } \ \sup_{k\geq 1} \|w_k\|_\infty < +\infty.$$

This yields, for all $k \geq 1$,

$$\limsup_{n\to+\infty} \mathcal{J}_n(u_n) \leq \mathcal{J}(w_k) - \int_0^1 af(u)(w_k - u)\,dx$$

and hence, letting $k \to +\infty$ and using the dominated convergence theorem,

$$\limsup_{n\to+\infty} \mathcal{J}_n(u_n) \leq \mathcal{J}(u),$$

which is precisely (4.45).

Step 6. Existence of a non-trivial solution of (1.1)**.** Relation (4.42), together with (4.35) and (4.34), imply that the solution u of (1.1) we have found satisfies

$$\mathcal{I}(u) \geq \delta > 0$$

and hence, in particular, $u \neq 0$.

Step 7. Existence of a positive solution of (1.1)**.** In order to prove that actually u provides us with a positive solution of (1.1) it suffices to show that $u_n \geq 0$ for all $n \geq 1$ a.e. in $[0,1]$. We recall that, for each $n \geq 1$, u_n satisfies

$$\mathcal{I}_n(u_n) = c_n = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \mathcal{I}_n(\gamma(t))$$

where

$$\Gamma = \{\gamma \in C^0([0,1], W^{1,\rho}(0,1)) : \gamma(0) = 0, \ \gamma(1) = \zeta\}.$$

According to Lemma 4.2, $\min\zeta \geq 0$ and hence $|\gamma| \in \Gamma$, whenever $\gamma \in \Gamma$. Moreover, since $\mathcal{I}_n$ is even, we find that, for every $n \geq 1$,

$$\mathcal{I}_n(|\gamma(t)|) = \mathcal{I}_n(\gamma(t)) \quad \text{for all } \ t \in [0,1].$$

Therefore, for every $k \geq 1$ there exists $\gamma_{n,k} \in \Gamma$ such that, for each $t \in [0,1]$,

$$\gamma_{n,k}(t) \geq 0 \qquad \text{a.e. in } \quad [0,1] \tag{4.46}$$

and

$$c_n \leq \max_{t\in[0,1]} \mathcal{I}_n(\gamma_{n,k}(t)) < c_n + \frac{1}{k}.$$

Hence, by the Ekeland variational principle [26], for every $k \geq 1$, there exist $\gamma^*_{n,k} \in \Gamma$ and $t_{n,k} \in [0,1]$ such that

$$c_n \leq \max_{t\in[0,1]} \mathcal{I}_n(\gamma^*_{n,k}(t)) \leq \max_{t\in[0,1]} \mathcal{I}_n(\gamma_{n,k}(t)) < c_n + \frac{1}{k}, \tag{4.47}$$

$$\max_{t\in[0,1]} \|\gamma_{n,k}(t) - \gamma^*_{n,k}(t)\|_{W^{1,\rho}} < \frac{1}{\sqrt{k}}, \tag{4.48}$$

$$c_n - \frac{1}{k} < \mathcal{I}_n(u^*_{n,k}) < c_n + \frac{1}{k} \quad \text{with} \quad u^*_{n,k} = \gamma^*_{n,k}(t_{n,k}), \tag{4.49}$$

and

$$|\mathcal{I}'_n(u^*_{n,k})(v)| \leq \frac{1}{\sqrt{k}}\|v\|_{W^{1,\rho}} \quad \text{for all } \ v \in W^{1,\rho}(0,1).$$

By the Palais-Smale condition satisfied by $\mathcal{I}_n$ (cf. Lemma 4.3), there exist a subsequence of $(u^*_{n,k})_k$, still labeled by k, and a function $u^*_n \in W^{1,\rho}(0,1)$ such that

$$\lim_{k\to+\infty} u^*_{n,k} = u^*_n \quad \text{in} \quad W^{1,\rho}(0,1).$$

Thus, from (4.48) and (4.49) we may infer that

$$\lim_{k\to+\infty} \gamma_{n,k}(t_{n,k}) = \lim_{k\to+\infty} \gamma^*_{n,k}(t_{n,k}) = \lim_{k\to+\infty} u^*_{n,k} = u^*_n \quad \text{in } W^{1,\rho}(0,1).$$

Therefore, thanks to (4.46), we find that $u^*_n \geq 0$ a.e. in $[0,1]$. Moreover, by (4.47) and (4.49), we get

$$\mathcal{I}_n(u^*_n) = c_n.$$

Finally, reasoning as in Steps 3 and 4, it is plain that we can extract a subsequence of $(u^*_n)_n$, relabeled by n, such that

$$\lim_{n\to+\infty} u^*_n = u^* \quad \text{a.e. in } [0,1],$$

with $u^* \geq 0$ a.e. in $[0,1]$ and $u^* \neq 0$. This is the positive solution of (1.1) whose existence was claimed in Theorem 1.1.

Part II. Regularity. Let u be a positive solution of (1.1) and let $(u_n)_n$ be a sequence in $W^{2,1}(0,1)$ of positive solutions of (4.6) such that, for some constant $C > 0$,

$$\sup_{n\geq 1} \|u_n\|_\infty < C \ \text{ and } \ \lim_{n\to+\infty} u_n = u \ \text{ in } \ L^1(0,1) \ \text{ and a.e. in } \ [0,1]. \tag{4.50}$$

By Step 4 of Part I, these conditions hold for the solution constructed in Part I. The proof of the regularity will be divided into three steps, accordingly to the statement of Theorem 1.1.

Step 1. Behavior on a single interval. Let $[\alpha,\beta] \subset [0,1]$ be an interval such that $a(x) \geq 0$ a.e. in $[\alpha,\beta]$. Since, by (f_2), we have $f(s) \geq 0$ for $s \geq 0$, it follows from (4.7) that u_n is concave in $[\alpha,\beta]$ and hence u'_n is non-increasing in $[\alpha,\beta]$. Consequently, the next simple result holds.

Lemma 4.5. *For every $n \geq 1$ and $\delta \in \left(0, \frac{\beta-\alpha}{2}\right)$,*

$$\max_{[\alpha+\delta,\beta-\delta]} |u'_n| \leq \frac{C}{\delta},$$

where $C > 0$ is the constant introduced in (4.50)*.*

Proof: As u'_n is non-increasing in $[\alpha,\beta]$, we have

$$\max_{[\alpha+\delta,\beta-\delta]} u'_n = u'_n(\alpha+\delta) \quad \text{and} \quad \min_{[\alpha+\delta,\beta-\delta]} u'_n = u'_n(\beta-\delta).$$

Thus, it remains to prove that

$$u'_n(\alpha+\delta) \leq \frac{C}{\delta} \quad \text{and} \quad u'_n(\beta-\delta) \geq -\frac{C}{\delta}. \tag{4.51}$$

By the concavity of u_n, we have

$$u_n(x) \leq u_n(\alpha+\delta) + u'_n(\alpha+\delta)(x-\alpha-\delta) \quad \text{for all } x \in [\alpha,\alpha+\delta].$$

In particular, letting $x = \alpha$ yields

$$0 \leq u_n(\alpha) \leq u_n(\alpha+\delta) - \delta u'_n(\alpha+\delta) \leq C - \delta u'_n(\alpha+\delta)$$

whence the first estimate of (4.51) follows. Similarly, we have

$$u_n(x) \leq u_n(\beta-\delta) + u'_n(\beta-\delta)(x-\beta+\delta) \quad \text{for all } x \in [\beta-\delta,\beta]$$

and hence, letting $x = \beta$, we get

$$0 \le u_n(\beta) \le C + u_n'(\beta - \delta)\delta.$$

Therefore, the second estimate of (4.51) also holds. □

According to Lemma 4.5, from (4.7) we infer the existence of a function $K_\delta \in L^1(\alpha+\delta, \beta-\delta)$ such that

$$|u_n''(x)| \le K_\delta(x) \ \text{ for a.e. } x \in [\alpha+\delta, \beta-\delta].$$

This implies that

$$|u_n'(x) - u_n'(y)| \le \left| \int_x^y K_\delta(t)\, dt \right| \quad \text{for all } \ x, y \in [\alpha+\delta, \beta-\delta].$$

Therefore, the sequence $(u_n')_n$ is bounded in $C^0[\alpha+\delta, \beta-\delta]$ and uniformly equicontinuous. Thus, the theorem of Arzelà-Ascoli yields the existence of a subsequence of $(u_n)_n$, labeled again by n, which converges to u in $C^1[\alpha+\delta, \beta-\delta]$. Since $\delta \in (0, \frac{\beta-\alpha}{2})$ is arbitrary, we may conclude from (4.7) that $(u_n)_n$ converges to u in $W^{2,1}_{\mathrm{loc}}(\alpha, \beta)$. In particular, u satisfies a.e. in (α, β) the differential equation

$$-u'' = a(x)\, f(u)\, (1 + (u')^2)^{\frac{3}{2}}. \tag{4.52}$$

As a byproduct, u is concave in (α, β) and the following limits are well defined

$$u(\alpha^+) = \lim_{x\to\alpha^+} u(x) \in [0, C], \qquad u(\beta^-) = \lim_{x\to\beta^-} u(x) \in [0, C],$$

$$u'(\alpha^+) = \lim_{x\to\alpha^+} u'(x) \in \mathbb{R} \cup \{+\infty\}, \qquad u'(\beta^-) = \lim_{x\to\beta^-} u'(x) \in \mathbb{R} \cup \{-\infty\}.$$

Moreover, since u_n solves (4.6) for all $n \ge 1$, we have that $u_n'(0) = u_n'(1) = 0$ and hence, if, e.g., $\alpha = 0$, for each $\delta \in (0, \frac{\beta}{2})$,

$$\max_{[0,\beta-\delta]} |u_n'| \le \frac{C}{\delta}.$$

Therefore we conclude that $u \in W^{2,1}_{\mathrm{loc}}[0, \beta)$ satisfies (4.52) a.e. in $[0, \beta)$. Similar conclusions hold if $\beta = 1$.

In a completely similar fashion we can discuss the case where $a(x) \le 0$ a.e. in $[\alpha, \beta]$.

Step 2. Behavior on adjacent intervals. Now we further assume (f_6), that is, f is locally Lipschitz in $[0, +\infty)$; this condition guarantees the uniqueness of solutions of the Cauchy problems associated with the differential equations in (1.1) and in (4.6).

Let (α, β) and (β, γ) be two adjacent intervals such that $a(x) \ge 0$ a.e. in (α, β) and $a(x) \le 0$ a.e. in (β, γ). The case $a(x) \le 0$ a.e. in (α, β) and $a(x) \ge 0$ a.e. in (β, γ) can be discussed similarly. The convexity properties of u guarantee that there exist

$$u'(\beta^-) \in \mathbb{R} \cup \{-\infty\} \quad \text{and} \quad u'(\beta^+) \in \mathbb{R} \cup \{-\infty\}.$$

Assume $u'(\beta^-) \in \mathbb{R}$. Then, u can be continued to the right of β as a solution of (4.52). The continuous dependence on initial conditions and parameters [29, Theorem 6, Chapter 1.1] implies that there exists a neighborhood of β where the approximating sequence $(u_n)_n$ is C^1-bounded and therefore $W^{2,1}$-bounded. This entails that $u \in C^1_{\mathrm{loc}}(\alpha, \gamma)$ and therefore, $u \in W^{2,1}_{\mathrm{loc}}(\alpha, \gamma)$. Similarly, we see that $u \in W^{2,1}_{\mathrm{loc}}(\alpha, \gamma)$ if $u'(\beta^+) \in \mathbb{R}$.

The previous argument also shows that $u'(\beta^-) = -\infty$ if, and only if, $u'(\beta^+) = -\infty$. Suppose this occurs and let us show that

$$u(\beta^-) \ge u(\beta^+). \tag{4.53}$$

Indeed, if, on the contrary, $u(\beta^-) < u(\beta^+)$, then there exist $\delta > 0$ and $n \geq 1$ such that

$$u_n(\beta - \delta) < \frac{u(\beta^+) + u(\beta^-)}{2} < u_n(\beta + \delta), \quad u_n'(\beta - \delta) < 0, \quad u_n'(\beta + \delta) < 0.$$

Thus, the concavity and the convexity of u_n on $[\alpha, \beta]$ and $[\beta, \gamma]$, respectively, yield

$$u_n(\beta) \leq u_n(\beta - \delta) < \frac{u(\beta^+) + u(\beta^-)}{2} < u_n(\beta + \delta) \leq u_n(\beta),$$

which is impossible. Therefore, (4.53) holds.

Step 3. Strict positivity. In order to prove that

$$\operatorname{ess\,inf} u > 0,$$

here we assume, in addition to (f_6), condition (f_7), that is, $a(x)$ changes of sign in $[0, 1]$ finitely many times, as specified in the statement of Theorem 1.1. Assume that, e.g., $a(x) \leq 0$ a.e. in $[\alpha_1, \beta_1] = [0, \beta_1]$, and pick $i \in \{2, \ldots, k-1\}$ such that

$$\begin{aligned} a(x) &\leq 0 \text{ a.e. in } [\alpha_{i-1}, \beta_{i-i}], \\ a(x) &\geq 0 \text{ a.e. in } [\alpha_i, \beta_i], \\ a(x) &\leq 0 \text{ a.e. in } [\alpha_{i+1}, \beta_{i+1}]. \end{aligned}$$

Let us prove that

$$\inf_{(\alpha_i, \beta_i)} u > 0. \tag{4.54}$$

Suppose, by contradiction, that there exists $x_0 \in (\alpha_i, \beta_i)$ such that $u(x_0) = 0$. Then, we have $u'(x_0) = 0$ and, as $f(0) = 0$, by uniqueness of solutions of the Cauchy problems associated with the equation in (1.1), $u = 0$ in (α_i, β_i). Thus, we get

$$\lim_{n \to +\infty} u_n = 0 \text{ in } C^1_{\text{loc}}(\alpha_i, \beta_i)$$

and, due to [38, Lemma 3.1, p. 24], we find that

$$\lim_{n \to +\infty} u_n = 0 \text{ uniformly in } [0, 1],$$

which implies $u = 0$, a contradiction because u is positive. Therefore, $u(x) > 0$ for all $x \in (\alpha_i, \beta_i)$.

Suppose, by contradiction, that $u(\beta_i^-) = 0$. By concavity, we have $u'(\beta_i^-) \leq 0$ and, actually, $u'(\beta_i^-) = -\infty$. Indeed, if $u'(\beta_i^-) \in (-\infty, 0)$, then u can be extended to the right of β_i as a solution of (4.52). By continuous dependence, taking a sufficiently large $n \geq 1$, we obtain that $u_n(x) < 0$ for some $x > \beta_i$, which is impossible, because u_n is positive. Similarly, if $u'(\beta_i^-) = 0$, then, arguing as above, we find that $u(x) = 0$ in $(\alpha_i, \beta_i]$, which is again a contradiction. Thus, we have $u'(\beta_i^-) = -\infty$. As $(u_n)_n$ converges to u in $C^1_{\text{loc}}(\alpha_i, \beta_i)$, the concavity implies that $u_n'(\beta_i) < 0$ for sufficiently large n and, along some subsequence, we have that

$$\lim_{n \to +\infty} u_n'(\beta_i) = -\infty,$$

as otherwise $u'(\beta_i^-)$ should be finite. But this also implies $u'(\beta_i^+) = -\infty$, because otherwise the continuous dependence would yield the boundedness of the sequence $(u_n'(\beta_i))_n$. Consequently, we have that

$$u(\beta_i^-) = 0 \quad \text{and} \quad u'(\beta_i^+) = -\infty.$$

Note that, according to (4.53), we also have that

$$u(\beta_i^+) = 0 \quad \text{and} \quad u'(\beta_i^+) = -\infty,$$

which entails the negativity of u in $[\beta_i, \beta_i + \eta]$ for some $\eta > 0$. As this is impossible, because u is positive, we get $u(\beta_i^-) > 0$. Similarly, we can prove that $u(\alpha_i^+) > 0$. Therefore, (4.54) holds true.

The previous argument can be easily adapted to cover the case when

$$\begin{aligned}&a(x) \geq 0 \ \text{ a.e. in } [\alpha_{i-1}, \beta_{i-i}],\\&a(x) \leq 0 \ \text{ a.e. in } [\alpha_i, \beta_i],\\&a(x) \geq 0 \ \text{ a.e. in } [\alpha_{i+1}, \beta_{i+1}],\end{aligned}$$

for some $i \in \{2, \dots, k-1\}$.

Finally, the cases $i = 1$ and $i = k$ can be treated in an analogous way, taking also into account that, since $f(0) = 0$ and $u'(0) = 0 = u'(1)$, the uniqueness of the solutions of the associated Cauchy problems imply $u(0) > 0$ and $u(1) > 0$. This ends the proof of Theorem 1.1. □

5. Proof of Theorem 1.2

We begin as in the proof of Theorem 1.1. Since condition (f_7) implies $f(0) = 0$, we can extend f to the whole of $\mathbb{R}$ as an odd continuous function. Then, by (f_2), (f_7) and (f_8), the following conditions hold for the odd extension of f, that we still denote by f:

(f_2^o) $f \in C^0(\mathbb{R})$ is such that $f(s)\,\mathrm{sgn}(s) \geq 0$ for all $s \in \mathbb{R}$,

(f_7^o) there exist $p \in (1, 2)$ and $L > 0$ such that

$$\lim_{s \to 0} \frac{F(s)}{|s|^p} = L,$$

(f_8^o) there exist $q \in (0, 1)$ and $M > 0$ such that

$$\lim_{|s| \to +\infty} \frac{F(s)}{|s|^q} = M.$$

with F defined in (1.3).

Step 1. A regularization scheme. Like in the proof of Theorem 1.1, we define a sequence of approximating problems, taking here $\rho = 2$; that is, we consider, for each $n \in \mathbb{N}$, with $n \geq 1$, the problem

$$\begin{cases} -\big(\varphi_n(u')\big)' = a(x) f(u) \quad \text{in } (0,1), \\ u'(0) = 0, \ u'(1) = 0, \end{cases} \tag{5.1}$$

where $\varphi_n : \mathbb{R} \to \mathbb{R}$ is given by

$$\varphi_n(s) = \frac{s}{\sqrt{1+s^2}} + \frac{1}{n} s \quad \text{for all } \ s \in \mathbb{R}. \tag{5.2}$$

The differential equation in (5.1) can be equivalently written as

$$-u'' = \frac{a(x)\, f(u)}{\varphi_n(u')} = a(x) f(u) \frac{(1 + (u')^2)^{\frac{3}{2}}}{1 + \frac{1}{n}(1 + (u')^2)^{\frac{3}{2}}}.$$

Subsequently, all the notations introduced in the proof of Theorem 1.1 will be kept. For any fixed $n \geq 1$, we will find a positive solution u_n of (5.1) as a global minimizer of the functional $\mathcal{I}_n : H^1(0,1) \to \mathbb{R}$, defined by

$$\mathcal{I}_n(v) = \mathcal{J}_n(v) - \int_0^1 a\, F(v)\, dx,$$

with

$$\mathcal{J}_n(v) = \mathcal{J}(v) + \frac{1}{2n}\int_0^1 (v')^2\,dx.$$

Step 2. Solving the regularized problems. Let $n \geq 1$ be given. We first prove that $\mathcal{I}_n$ is coercive and bounded from below in $H^1(0,1)$. Indeed, according to (f_8^o), for every $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that

$$|F(s) - M|s|^q| \leq \varepsilon\,|s|^q + c_\varepsilon \quad \text{for all } \ s \in \mathbb{R}. \tag{5.3}$$

Hence, setting, as in (4.8),

$$r = \int_0^1 u\,dx \quad \text{and} \quad w = u - r \quad \text{for every } \ u \in H^1(0,1),$$

it follows from the Jensen inequality that

$$\begin{aligned} \mathcal{I}_n(u) &= \int_0^1 \sqrt{1+(w')^2}\,dx - 1 + \frac{1}{2n}\int_0^1 (w')^2\,dx - \int_0^1 a\,F(w+r)\,dx \\ &\geq \sqrt{1+\|w'\|_{L^1}^2} - 1 + \frac{1}{2n}\|w'\|_{L^2}^2 - \int_0^1 a\,F(u)\,dx. \end{aligned} \tag{5.4}$$

On the other hand, since $q \in (0,1)$, we find from (4.10) that, for all $x \in [0,1]$,

$$|u(x)|^q - |r|^q \leq \big||w(x)+r|^q - |r|^q\big| \leq |w(x)|^q \leq \|w\|_\infty^q \leq \|w'\|_{L^1}^q.$$

Hence, thanks to (5.3), we get

$$\begin{aligned} \int_0^1 aF(u)\,dx &= \int_0^1 a\left(F(u) - M|u|^q\right)\,dx \\ &\quad + M\int_0^1 a\left(|u|^q - |r|^q\right)\,dx + M|r|^q\int_0^1 a\,dx \\ &\leq \int_0^1 |a|\left(\varepsilon|u|^q + c_\varepsilon\right)\,dx + M\|a\|_{L^1}\|w'\|_{L^1}^q + M|r|^q\int_0^1 a\,dx \\ &\leq \|a\|_{L^1}\left((\varepsilon+M)\|w'\|_{L^1}^q + \varepsilon|r|^q + c_\varepsilon\right) + M|r|^q\int_0^1 a\,dx. \end{aligned}$$

Consequently, applying this estimate to (5.4) easily yields

$$\begin{aligned} \mathcal{I}_n(u) \geq \frac{1}{2n}&\|w'\|_{L^2}^2 + \|w'\|_{L^1} - \|a\|_{L^1}(\varepsilon+M)\|w'\|_{L^1}^q \\ &- M\left(\int_0^1 a\,dx + \varepsilon\|a\|_{L^1}\right)|r|^q - c_\varepsilon\|a\|_{L^1} - 1. \end{aligned}$$

Thus, taking $\varepsilon > 0$ so small that

$$\int_0^1 a\,dx + \varepsilon\|a\|_{L^1} < 0,$$

which is possible because we are assuming that $\int_0^1 a\,dx < 0$, it is plain that we can find two positive constants $A, B > 0$, independent of n, such that

$$\mathcal{I}_n(u) \geq \frac{1}{2n}\|w'\|_{L^2}^2 + A\big(\|w'\|_{L^1} + |r|^q\big) - B \tag{5.5}$$

and, in particular,

$$\mathcal{I}_n(u) \geq \ A\big(\|w'\|_{L^1} + |r|^q\big) - B. \tag{5.6}$$

Condition (5.5) implies that

$$\lim_{\|u\|_{H^1}\to+\infty} \mathcal{I}_n(u) = +\infty \quad \text{and} \quad \inf_{u\in H^1(0,1)} \mathcal{I}_n(u) > -\infty.$$

Since $\mathcal{I}_n$ is weakly lower semicontinuous in $H^1(0,1)$, it is a classical fact [17] that it possesses a global minimizer $u_n \in H^1(0,1)$. As $\mathcal{I}_n$ is of class C^1, u_n must be a critical point of $\mathcal{I}_n$ and hence a solution of (5.1).

Now, we will prove that u_n is non-trivial. It suffices to show that $\mathcal{I}_n(u_n) < 0$. We will follow the argument used to prove Lemma 4.2. By (a_3) there is an interval $K \subset (0,1)$ such that $a(x) > 0$ a.e. in K. Then we pick a function $z \in C^1[0,1]$, with $\operatorname{supp} z \subset K$, such that $z(x) = 1$ in an interval $K_0 \subset K$. Since $F(s) \geq 0$ for all $s \in \mathbb{R}$ and $F(0) = 0$, we infer from (f_7^{o}) that, for a sufficiently small $t > 0$,

$$\begin{aligned}
\mathcal{I}_n(t\,z) &= \mathcal{J}_n(t\,z) - \int_{K_0} a\,F(t)\,dx - \int_{K\setminus K_0} a\,F(t\,z)\,dx \\
&\leq \int_0^1 \frac{t^2\,(z')^2}{1+\sqrt{1+t^2\,(z')^2}}\,dx + \frac{1}{2n}\int_0^1 t^2 (z')^2\,dx - F(t)\int_{K_0} a\,dx \qquad (5.7)\\
&\leq t^p \left(t^{2-p}\int_0^1 (z')^2\,dx \;-\; \frac{F(t)}{t^p}\int_{K_0} a\,dx \right) < 0,
\end{aligned}$$

because $2 - p > 0$. This implies that there exists a constant $\eta > 0$, independent of n, such that

$$\inf_{u\in H^1(0,1)} \mathcal{I}_n(u) \leq -\eta. \tag{5.8}$$

Finally, we show that u_n can be chosen positive. Indeed, since

$$\mathcal{I}_n(|u|) = \mathcal{I}_n(u) \quad \text{for all } \; u \in H^1(0,1),$$

we see that if u_n is a global minimizer of $\mathcal{I}_n$, then $|u_n|$ is a global minimizer too.

Step 3. Existence of a positive solution of (1.1)**.** By (5.6) and (5.8), we have

$$0 \geq \mathcal{I}_n(u_n) \geq A\big(\|w_n'\|_{L^1} + |r_n|^q\big) - B \quad \text{for all } \; n \geq 1. \tag{5.9}$$

Hence, the sequence $(u_n)_n$ is bounded in $W^{1,1}(0,1)$ and, therefore, there exist a subsequence of $(u_n)_n$, still labeled by n, and a function $u \in BV(0,1)$ such that

$$\sup_n \|u_n\|_\infty < +\infty$$

and

$$\lim_{n\to+\infty} u_n = u \; \text{ in } L^1(0,1) \text{ and a.e. in } [0,1].$$

This implies in particular that $u(x) \geq 0$ a.e. in $[0,1]$. As the functional $\mathcal{J}_n$ is lower semicontinuous with respect to the L^1-convergence in $BV(0,1)$ and

$$\lim_{n\to+\infty} \int_0^1 a\,F(u_n)\,dx = \int_0^1 a\,F(u)\,dx,$$

we conclude by (5.8) that

$$\begin{aligned}
\mathcal{I}(u) = \mathcal{J}(u) - \int_0^1 a\,F(u)\,dx &\leq \liminf_{n\to+\infty} \left(\mathcal{J}(u_n) - \int_0^1 a\,F(u_n)\,dx \right) \\
&\leq \liminf_{n\to+\infty} \mathcal{I}_n(u_n) \leq -\eta.
\end{aligned}$$

Therefore, u is non-trivial and hence positive. The fact that u satisfies (1.6) for all $v \in BV(0,1)$ can be verified as in the proof of Theorem 1.1. Also the $W^{2,1}_{\mathrm{loc}}$-regularity of u on every interval where a has a constant sign follows as in the proof of Theorem 1.1. This ends the proof of Theorem 1.2. $\square$

6. Proof of Theorem 1.3

We follow the same steps and patterns of the proof of Theorem 1.1, with the exception of Lemma 4.1, which now requires a few changes, that are reported in Lemma 6.1 below. Since condition (f_9) implies $f(0)=0$, we extend, as above, f to the whole of $\mathbb{R}$ as an odd continuous function that we still denote by f. Then, the odd extension of f satisfies conditions (f_2°), (f_4°), (f_5°) and

(f_9°) there exists $L>0$ such that

$$\lim_{s\to 0}\frac{F(s)}{s^2}=L.$$

Lemma 6.1. *There exist constants $L^*>0$, $\delta>0$ and $\eta>0$ such that, if (f_9°) holds with $L\in(0,L^*)$, then, setting*

$$\mathcal{S}_\eta=\{u\in H^1(0,1):\|u\|_{W^{1,1}}=\|w'\|_{L^1}+|r|=\eta\},$$

one has, for all $n\geq 1$,

$$\inf_{u\in\mathcal{S}_\eta}\mathcal{I}_n(u)\geq\inf_{u\in\mathcal{S}_\eta}\mathcal{I}(u)\geq\delta.$$

Proof. Pick $u\in W^{1,1}(0,1)$ and use the decomposition (4.8). As in Lemma 4.1, using the Jensen inequality we get

$$\begin{aligned}\mathcal{I}(u)&=\int_0^1\sqrt{1+(w')^2}\,dx-1-\int_0^1 a\,F(w+r)\,dx\\&\geq\sqrt{1+\|w'\|_{L^1}^2}-1-\int_0^1 a\,F(w+r)\,dx\\&=\frac{\|w'\|_{L^1}^2}{1+\sqrt{1+\|w'\|_{L^1}^2}}-\int_0^1 a\,\left(F(w+r)-L(w+r)^2\right)\,dx\\&\qquad\qquad-L\int_0^1 a\left((w+r)^2-r^2\right)\,dx-Lr^2\int_0^1 a\,dx,\end{aligned}$$

where

$$L=\lim_{s\to 0}\frac{F(s)}{s^2}>0.$$

For any given $\varepsilon>0$, there is $\eta\in(0,1)$ such that

$$|F(s)-Ls^2|\leq\varepsilon s^2,\quad\text{if }|s|\leq\eta.$$

Let $u\in W^{1,1}(0,1)$ satisfy $\|u\|_{W^{1,1}}\leq\eta$ and hence

$$\|w\|_\infty+|r|\leq\|w'\|_{L^1}+|r|=\|u\|_{W^{1,1}}\leq\eta.$$

Using elementary inequalities, we obtain

$$\begin{aligned}\left|\int_0^1 a\left(F(w+r)-L(w+r)^2\right)\,dx\right|&\leq\int_0^1|a|\,\varepsilon\,(w+r)^2\,dx\\&\leq 2\varepsilon\,\|a\|_{L^1}\left(\|w\|_\infty^2+r^2\right)\\&\leq 2\varepsilon\,\|a\|_{L^1}\left(\|w'\|_{L^1}^2+r^2\right)\end{aligned}$$

and, for any $\sigma > 0$,

$$\begin{aligned}\left|L\int_0^1 a\left((w+r)^2 - r^2\right)\,dx\right| &\le L\|a\|_{L^1}\left\|(w+r)^2 - r^2\right\|_\infty \\ &\le L\|a\|_{L^1}(2|r|\|w\|_\infty + \|w\|_\infty^2) \\ &\le L\|a\|_{L^1}(2|r|\|w'\|_{L^1} + \|w'\|_{L^1}^2) \\ &\le L\|a\|_{L^1}\left(\sigma r^2 + \frac{1}{\sigma}\|w'\|_{L^1}^2\right) + L\|a\|_{L^1}\|w'\|_{L^1}^2.\end{aligned}$$

Therefore, for all $u \in W^{1,1}(0,1)$ satisfying $\|u\|_{W^{1,1}} = \eta$, we have

$$\begin{aligned}\mathcal{I}(u) \ge &\left(\frac{1}{1+\sqrt{2}} - 2\varepsilon\|a\|_{L^1} - \left(1+\frac{1}{\sigma}\right)L\|a\|_{L^1}\right)\|w'\|_{L^1}^2 \\ &+ L\left(-\frac{2}{L}\varepsilon\|a\|_{L^1} - \sigma\|a\|_{L^1} - \int_0^1 a\,dx\right)r^2.\end{aligned}$$

Since $\int_0^1 a\,dx < 0$, we can fix

$$\sigma \in \left(0, -\|a\|_{L^1}^{-1}\int_0^1 a\,dx\right)$$

and

$$L^* \in \left(0, \frac{1}{1+\sqrt{2}}\frac{\sigma}{1+\sigma}\|a\|_{L^1}^{-1}\right).$$

Then, for any given $L \in (0, L^*)$ we can find $\varepsilon > 0$ and $\eta > 0$ so small that, for all $u \in W^{1,1}(0,1)$ satisfying $\|u\|_{W^{1,1}} = \eta$,

$$\mathcal{I}(u) \ge A\|w'\|_{L^1}^2 + Br^2,$$

for some constants $A, B > 0$. Therefore, setting

$$\delta = \min_{\substack{x\ge 0, y\ge 0\\ x+y=\eta}}\left(Ax^2 + By^2\right) > 0$$

and using (4.5), we infer that, for all $n \ge 1$, the estimate

$$\inf_{u\in\mathcal{S}_\eta}\mathcal{I}_n(u) \ge \inf_{u\in\mathcal{S}_\eta}\mathcal{I}(u) \ge \delta$$

holds. This ends the proof of Lemma 6.1. □

7. Proof of Theorem 1.4

We basically repeat the argument of the proof Theorem 1.2 with only few minor changes. Like there we extend f to the whole of $\mathbb{R}$ as an odd continuous function that we still denote by f. Then, the odd extension of f satisfies conditions (f_2^{o}), (f_9^{o}) and

(f_{10}^{o}) there exists $M > 0$ such that

$$\lim_{|s|\to+\infty}\frac{F(s)}{|s|} = M.$$

A first change occurs for proving that, for any fixed n, the functional $\mathcal{I}_n$ is coercive and bounded from below in $H^1(0,1)$. Indeed, according to (f_{10}^{o}), for every $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that

$$|F(s) - M|s|| \le \varepsilon\,|s| + c_\varepsilon \quad \text{for all } \; s \in \mathbb{R}.$$

Hence, for every $u = w + r \in H^1(0,1)$, we get

$$\begin{aligned}\int_0^1 aF(u)\,dx &= \int_0^1 a\left(F(u) - M|u|\right)\,dx \\ &\quad + M\int_0^1 a\left(|u| - |r|\right)\,dx + M|r|\int_0^1 a\,dx \\ &\le \int_0^1 |a|\left(\varepsilon|u| + c_\varepsilon\right)\,dx + M\|a\|_{L^1}\|w'\|_{L^1} + M|r|\int_0^1 a\,dx \\ &\le \|a\|_{L^1}\,\big((\varepsilon + M)\|w'\|_{L^1} + \varepsilon|r| + c_\varepsilon\big) + M|r|\int_0^1 a\,dx.\end{aligned}$$

Consequently, applying this estimate we easily obtain

$$\begin{aligned}\mathcal{I}_n(u) &= \int_0^1 \sqrt{1+(u')^2}\,dx - 1 + \frac{1}{2n}\int_0^1 (u')^2\,dx - \int_0^1 a\,F(u)\,dx \\ &\ge \sqrt{1+\|w'\|_{L^1}^2} - 1 + \frac{1}{2n}\|w'\|_{L^2}^2 - \int_0^1 a\,F(u)\,dx \\ &\ge \frac{1}{2n}\|w'\|_{L^2}^2 + \left(1 - \|a\|_{L^1}(\varepsilon + M)\right)\|w'\|_{L^1} \\ &\quad - M\left(\int_0^1 a\,dx + \varepsilon\|a\|_{L^1}\right)|r| - c_\varepsilon\|a\|_{L^1} - 1.\end{aligned}$$

Fix $M^* \in (0, \|a\|_{L^1}^{-1})$ and pick $M \in (0, M^*)$. Thus, taking $\varepsilon > 0$ so small that

$$(\varepsilon + M)\|a\|_{L^1} < 1 \quad \text{and} \quad \int_0^1 a\,dx + \varepsilon\|a\|_{L^1} < 0,$$

which is possible because we are assuming that $\int_0^1 a\,dx < 0$, we can find two positive constants $A, B > 0$, independent of n, such that

$$\mathcal{I}_n(u) \ge \frac{1}{2n}\|w'\|_{L^2}^2 + A\big(\|w'\|_{L^1} + |r|\big) - B \tag{7.1}$$

and therefore

$$\lim_{\|u\|_{H^1}\to+\infty} \mathcal{I}_n(u) = +\infty \quad \text{and} \quad \inf_{u\in H^1(0,1)} \mathcal{I}_n(u) > -\infty.$$

A second change is needed for proving the existence of $\eta > 0$ such that, for all $n \ge 1$,

$$\inf_{u\in H^1(0,1)} \mathcal{I}_n(u) \le -\eta.$$

Indeed, by (a_3) there is an interval $K \subset (0,1)$ such that $a(x) > 0$ a.e. in K. Then, we pick a function $z \in C^1[0,1]$, with $\operatorname{supp} z \subset K$, such that $z(x) = 1$ in an interval $K_0 \subset K$. Since $F(s) \ge 0$ for all $s \in \mathbb{R}$ and $F(0) = 0$, we have, for all $n \ge 1$ and $t > 0$,

$$\begin{aligned}\mathcal{I}_n(t\,z) &= \mathcal{J}_n(t\,z) - \int_{K_0} a\,F(t)\,dx - \int_{K\setminus K_0} a\,F(t\,z)\,dx \\ &\le \int_0^1 \frac{t^2\,(z')^2}{1+\sqrt{1+t^2\,(z')^2}}\,dx + \frac{1}{2n}\int_0^1 t^2(z')^2\,dx - F(t)\int_{K_0} a\,dx \\ &\le t^2\left(\int_0^1 (z')^2\,dx - \frac{F(t)}{t^2}\int_{K_0} a\,dx\right)\end{aligned}$$

and hence, using $(f_9^{\rm o})$,

$$\limsup_{t\to 0^+} \frac{\mathcal{I}_n(t\,z)}{t^2} \leq \int_0^1 (z')^2\,dx \,-\, L\int_{K_0} a\,dx.$$

Fix $L_* > \left(\int_{K_0} a\,dx\right)^{-1} \int_0^1 (z')^2\,dx$ and set $\eta = \int_0^1 (z')^2\,dx - L_* \int_{K_0} a\,dx > 0$. Then, the conclusion follows for all $L > L_*$.

The last change occurs in (5.9) which, due to (7.1), now reads

$$0 \geq \mathcal{I}_n(u_n) \geq A\big(\|w_n'\|_{L^1} + |r_n|\big) - B \quad \text{for all } \ n \geq 1,$$

but this modification, clearly, does not affect the remainder of the argument. $\square$

8. Proof of Theorem 1.5

Similarly as in the previous proofs, we extend f to the whole of $\mathbb{R}$ as an odd function still denoted by f. The corresponding assumptions for such an extension are indicated by $(f_2^{\rm o})$, $(f_3^{\rm o})$ and $(f_8^{\rm o})$. Along this proof, which is divided into two steps, we make use, like in the proof of Theorem 1.2, of the regularized problems

$$\begin{cases} -\big(\varphi_n(u')\big)' = \lambda a(x) f(u) & \text{in } (0,1), \\ u'(0) = 0, \ u'(1) = 0, \end{cases} \tag{8.1}$$

with φ_n given by (5.2) for each $n \geq 1$. The functional associated with (8.1) is defined in $H^1(0,1)$. As it depends on the parameter λ, it is convenient to denote it by $\mathcal{I}_{n,\lambda}$, i.e., we set

$$\mathcal{I}_{n,\lambda}(u) = \mathcal{J}_n(u) - \lambda \int_0^1 aF(u)\,dx, \tag{8.2}$$

for all $u \in H^1(0,1)$.

Step 1. Existence of a positive solution u_1 with $\mathcal{I}_\lambda(u_1) < 0$. The existence of a first positive solution u_1, with $\mathcal{I}_\lambda(u_1) < 0$, is obtained following the patterns of the proof of Theorem 1.2. However, since (f_7) is not anymore assumed, we have to slightly modify the argument used therein to obtain condition (5.8). Indeed, our aim here is to prove that there are constants $\lambda_* > 0$ and $\eta > 0$ such that, for all $n \geq 1$ and all $\lambda > \lambda_*$,

$$\inf_{u\in H^1(0,1)} \mathcal{I}_{n,\lambda}(u) \leq -\eta. \tag{8.3}$$

Pick a point $t_0 > 0$ such that $F(t_0) > 0$ and let $z \in C^1[0,1]$ and $K_0 \subset K$ be like in the proof of Theorem 1.4; as there, we have

$$\begin{aligned} \mathcal{I}_{n,\lambda}(t_0\,z) &= \mathcal{J}_n(t_0\,z) - \lambda \int_{K_0} aF(t_0)\,dx - \lambda \int_{K\setminus K_0} aF(t_0\,z)\,dx \\ &\leq \int_0^1 \frac{t_0^2\,(z')^2}{1+\sqrt{1+t_0^2\,(z')^2}}\,dx + \frac{1}{2n}\int_0^1 t_0^2\,(z')^2\,dx \,-\, \lambda F(t_0)\int_{K_0} a\,dx \\ &\leq t_0^2 \int_0^1 (z')^2\,dx \,-\, \lambda F(t_0)\int_{K_0} a\,dx. \end{aligned}$$

Hence we immediately derive the existence of $\lambda_* > 0$ and $\eta > 0$ such that (8.3) holds, for all $n \geq 1$ and all $\lambda > \lambda_*$.

Step 2. Existence of a positive solution u_2 with $\mathcal{I}_\lambda(u_2) > 0$. The existence of a second positive solution u_2, with $\mathcal{I}_\lambda(u_2) > 0$, is obtained following the patterns of the proof of Theorem 1.1. However, since (f_4) and (f_5) are not anymore assumed, the argument used there requires a few changes. As already noticed, we are now working in $H^1(0,1)$ with the regularized problems (5.1) and the associated functionals $\mathcal{I}_{n,\lambda}$. It is plain that Lemma 4.1 still holds with $\eta > 0$ and $\delta > 0$ now depending on λ, whereas the conclusion of Lemma 4.2 follows from (8.3), provided that $\lambda > \lambda_*$. In order to prove Lemma 4.3, we observe that, as assumptions (f_8) and (a_3) imply the coercivity condition in $H^1(0,1)$ expressed by (5.5), any sequence $(u_k)_k$ in $H^1(0,1)$ satisfying (4.15) must be bounded in $H^1(0,1)$. Therefore we can find a subsequence of $(u_k)_k$, labeled again by k, and $u \in H^1(0,1)$ such that (4.29) holds with $\rho = 2$. Once this conclusion is achieved, the proof of Lemma 4.3 carries on unchanged. As in the proof of Theorem 1.1, the mountain pass theorem then applies and yields the existence, for any given $\lambda > \lambda_*$ and for all $n \geq 1$, of a critical point $u_n \in H^1(0,1)$ of the functional $\mathcal{I}_{n,\lambda}$, which satisfies

$$c_1 \geq \mathcal{I}_{n,\lambda}(u_n) \geq \delta, \tag{8.4}$$

with c_1 and $\delta > 0$ possibly depending on λ, but independent of n. The positivity of each u_n is proved without changes. Finally, the coercivity condition in $W^{1,1}(0,1)$, expressed by (5.5), implies that a subsequence of $(u_n)_n$, still labeled by n, converges to a positive solution $u \in BV(0,1)$ of (1.1). This solution satisfies $\mathcal{I}_\lambda(u) > 0$, by the conditions (8.4) and

$$\mathcal{I}_\lambda(u) = \lim_{n\to+\infty} \mathcal{I}_{n,\lambda}(u_n),$$

which can be proved exactly as (4.42).

From the two previous steps we infer that problem (1.1) has at least of two positive solutions; the remaining conclusions of Theorem 1.5 then follow as in the proof of Theorem 1.1. $\square$

9. Proof of Theorem 1.6

Similarly as in the proofs of Theorem 1.1 and Theorem 1.5 we extend f to the whole of $\mathbb{R}$ as an odd function still denoted by f. The corresponding assumptions for such an extension are indicated by (f_2^o), (f_4^o), (f_5^o) and (f_7^o). We also make use, for each $n \geq 1$, of the regularized problem (8.1), with φ_n given by (4.2). The associated functional, denoted by $\mathcal{I}_{n,\lambda}$, is defined by (8.2) for all $u \in W^{1,\rho}(0,1)$, with ρ satisfying (4.1).

Step 1. Existence of a positive solution u_1 with $\mathcal{I}_\lambda(u_1) > 0$. The existence of a first positive solution u_1, with $\mathcal{I}_\lambda(u_1) > 0$, is obtained following the patterns of the proof of Theorem 1.1. However, since (f_3) is replaced by (f_7), we have to slightly modify the argument used there to get Lemma 4.1, whose counterpart now reads as follows.

Lemma 9.1. *There exist constants $\lambda^* > 0$ and $\eta > 0$ such that, for every $\lambda \in (0, \lambda^*)$, there is $\delta_\lambda > 0$ for which, setting*

$$\mathcal{S}_\eta = \{u \in W^{1,\rho}(0,1) : \|u\|_{W^{1,1}} = \|w'\|_{L^1} + |r| = \eta\},$$

one has, for all $n \geq 1$,

$$\inf_{u\in\mathcal{S}_\eta} \mathcal{I}_{n,\lambda}(u) \geq \inf_{u\in\mathcal{S}_\eta} \mathcal{I}_\lambda(u) \geq \delta_\lambda.$$

Proof. Pick $u \in W^{1,1}(0,1)$ and consider the decomposition (4.8). As in Lemma 4.1, we get

$$\begin{aligned}\mathcal{I}_\lambda(u) \geq \frac{\|w'\|_{L^1}^2}{1+\sqrt{1+\|w'\|_{L^1}^2}} &- \lambda \int_0^1 a\ (F(w+r) - L|w+r|^p)\ dx \\ &- \lambda L \int_0^1 a\,(|w+r|^p - |r|^p)\ dx - \lambda L |r|^p \int_0^1 a\, dx,\end{aligned}$$

where

$$L = \lim_{s\to 0} \frac{F(s)}{|s|^p} > 0, \tag{9.1}$$

with $p \in (1,2)$. For any given $\varepsilon > 0$, there is $\eta \in (0,1)$ such that

$$|F(s) - L|s|^p| \leq \varepsilon |s|^p, \quad \text{if } |s| \leq \eta.$$

Therefore, still arguing as in Lemma 4.1, we obtain, for every $u \in W^{1,1}(0,1)$ satisfying $\|w'\|_{L^1} + |r| = \|u\|_{W^{1,1}} = \eta$, we get

$$\left| \int_0^1 a\,(F(w+r) - L|w+r|^p)\ dx \right| \leq \varepsilon\, \|a\|_{L^1} 2^{p-1} \left(\|w'\|_{L^1}^p + |r|^p \right)$$

and, for any $\sigma > 0$,

$$\begin{aligned}\left| L \int_0^1 a\,(|w+r|^p - |r|^p)\ dx \right| &\leq L\|a\|_{L^1} (p|r|^{p-1}\|w'\|_{L^1} + \|w'\|_{L^1}^p) \\ &\leq \frac{p}{2} L\|a\|_{L^1} \left(\sigma |r|^{2(p-1)} + \frac{1}{\sigma}\|w'\|_{L^1}^2 \right) + L\|a\|_{L^1}\|w'\|_{L^1}^p.\end{aligned}$$

Accordingly, we obtain

$$\begin{aligned}\mathcal{I}(u) \geq &\left(\frac{1}{1+\sqrt{2}} - \lambda \frac{p}{2} L \|a\|_{L^1} \frac{1}{\sigma} \right) \|w'\|_{L^1}^2 - \lambda (L + 2^{p-1}\varepsilon) \|a\|_{L^1} \|w'\|_{L^1}^p \\ &- \lambda \left(2^{p-1} \|a\|_{L^1}\, \varepsilon + L \int_0^1 a\, dx \right) |r|^p - \lambda \frac{p}{2} L \|a\|_{L^1}\, \sigma |r|^{2(p-1)},\end{aligned}$$

for any $\sigma > 0$. Hence, using the condition $\int_0^1 a\,dx < 0$ and taking $\varepsilon > 0$ and $\eta > 0$ sufficiently small, we can find constants $A, B, C, D, E > 0$ such that

$$\mathcal{I}_\lambda(u) \geq \left(A - \lambda B \frac{1}{\sigma} \right) \|w'\|_{L^1}^2 - \lambda C \|w'\|_{L^1}^p + \lambda D |r|^p - \lambda \sigma E |r|^{2(p-1)},$$

for every $u \in W^{1,1}(0,1)$ with $\|u\|_{W^{1,1}} = \eta$. Let us set $x = \|w'\|_{L^1}$ and $y = |r|$. We want to show that, for a suitable choice of $\sigma > 0$, there is $\lambda^* > 0$ such that, for every $\lambda \in (0, \lambda^*)$,

$$\min_{\substack{x\geq 0, y\geq 0\\ x+y=\eta}} \left(\left(A - \lambda B \frac{1}{\sigma} \right) x^2 - \lambda C x^p + \lambda D y^p - \lambda \sigma E y^{2(p-1)} \right) > 0. \tag{9.2}$$

Indeed, if $\sigma > 0$ is taken sufficiently small, there exists $x^* \in (0, \eta)$ such that

$$-Cx^p + D(\eta - x)^p - \sigma E (\eta - x)^{2(p-1)} > 0, \quad \text{for all } x \in [0, x^*],$$

and then, for every $\lambda > 0$ small enough,

$$\left(A - \lambda B \frac{1}{\sigma} \right) x^2 + \lambda \left(-Cx^p + D(\eta - x)^p - \sigma E(\eta - x)^{2(p-1)} \right) > 0, \quad \text{for all } x \in [0, x^*].$$

On the other hand, possibly further reducing $\lambda > 0$, we have

$$Ax^2 + \lambda\left(-B\frac{1}{\sigma}x^2 - Cx^p + D(\eta - x)^p - \sigma E(\eta - x)^{2(p-1)}\right) > 0, \quad \text{for all } x \in [x^*, \eta].$$

Hence, we conclude that there exists $\lambda^* > 0$ such that (9.2) holds, for all $\lambda \in (0, \lambda^*)$. Then, setting, for each $\lambda \in (0, \lambda^*)$,

$$\delta_\lambda = \min_{\substack{x\geq 0, y\geq 0\\ x+y=\eta}} \left(\left(A - \lambda B\frac{1}{\sigma}\right)x^2 - \lambda C x^p + \lambda D y^p - \lambda\sigma E y^{2(p-1)}\right) > 0,$$

we infer from (4.5) that, for every $n \geq 1$, the estimate

$$\inf_{u\in\mathcal{S}_\eta} \mathcal{I}_{n,\lambda}(u) \geq \inf_{u\in\mathcal{S}_\eta} \mathcal{I}_\lambda(u) \geq \delta_\lambda$$

holds. □

The remainder of the proof of the existence of a positive solution u, with $\mathcal{I}_\lambda(u) > 0$, of problem (1.1) then proceeds, without changes, as that of Theorem 1.1.

Step 2. Existence of a positive solution u_2 with $\mathcal{I}_\lambda(u_2) < 0$. The existence of a second positive solution u_2, with $\mathcal{I}_\lambda(u_2) < 0$, is obtained following the patterns of the proof of Theorem 1.2. However, since (f_8) is not presently assumed, the argument used there requires a few changes. Indeed, the existence, for each $n \geq 1$, of a second solution of the regularized problem (8.1), is now proved by minimizing the functional $\mathcal{I}_{n,\lambda}$ in the open set

$$\Omega_\eta = \{u \in W^{1,\rho}(0,1) : \|u\|_{W^{1,1}} = \|w'\|_{L^1} + |r| < \eta\},$$

where ρ satisfies (4.1) and η comes from Lemma 9.1.

We first observe that, for each $n \geq 1$ and every $\lambda \in (0, \lambda^*)$, $\mathcal{I}_{n,\lambda}$ is bounded from below in Ω_η. Indeed, we have

$$\inf_{u\in\Omega_\eta} \mathcal{I}_{n,\lambda}(u) \geq \inf_{u\in\Omega_\eta} \mathcal{I}_\lambda(u) \geq \inf_{u\in\Omega_\eta}\left(-\lambda\int_0^1 aF(u)\,dx\right) \geq -\lambda^*\|a\|_{L^1}\max_{|s|\leq\eta}|F(s)|.$$

Moreover, assumptions (a_3) and (f_7) imply, for each $\lambda > 0$, the existence of a constant $\kappa_\lambda > 0$, independent of n, such that

$$\inf_{u\in\Omega_\eta} \mathcal{I}_{n,\lambda}(u) < -\kappa_\lambda. \tag{9.3}$$

Indeed, let $z \in C^1[0,1]$ and $K_0 \subset K$ be chosen as in the proof of Theorem 1.2. We have, for all $t > 0$,

$$\begin{aligned}
\mathcal{I}_{n,\lambda}(t\,z) &= \mathcal{J}_n(t\,z) - \lambda\int_{K_0} aF(t)\,dx - \lambda\int_{K\setminus K_0} aF(t\,z)\,dx\\
&\leq \int_0^1 \left(\left(1 + (t\,z')^2\right)^{\frac{1}{2}} - 1\right)dx + \frac{1}{n}\int_0^1 \Psi(t\,z')\,dx - F(t)\int_{K_0} a\,dx\\
&\leq t^p\left(\int_0^1 \frac{\left(1+(t\,z')^2\right)^{\frac{1}{2}} - 1}{t^p}\,dx + \int_0^1 \frac{\Psi(t\,z')}{t^p}\,dx \;-\; \lambda\frac{F(t)}{t^p}\int_{K_0} a\,dx\right).
\end{aligned}$$

where, as $p \in (1,2)$,

$$\lim_{t\to 0^+}\int_0^1 \frac{\left(1+(t\,z')^2\right)^{\frac{1}{2}} - 1}{t^p}\,dx = 0 \quad \text{and} \quad \lim_{t\to 0^+}\int_0^1 \frac{\Psi(t\,z')}{t^p}\,dx = 0.$$

Hence we infer

$$\lim_{t\to 0^+} \frac{\mathcal{I}_{n,\lambda}(t\,z)}{t^p} = -\lambda \lim_{t\to 0^+} \frac{F(t)}{t^p} \left(\int_{K_0} a\,dx \right) < 0,$$

which implies (9.3).

Next, fix $n \geq 1$ and let $(u_{n,k})_k$ be a minimizing sequence of $\mathcal{I}_{n,\lambda}$ in Ω_η. Condition (9.3) implies in particular that

$$\mathcal{I}_{n,\lambda}(u_{n,k}) < 0,$$

for all sufficiently large k. Hence, the elementary inequality,

$$\Psi(s) \geq \frac{1}{\rho}(|s|^\rho - 1), \text{ for all } s,$$

and the structure of the functional $\mathcal{I}_{n,\lambda}$ yield

$$\frac{1}{n\rho}(\|u'_{n,k}\|_{L^\rho} - 1) \leq \mathcal{J}_n(u_{n,k}) \leq \lambda^* \|a\|_{L^1} \max_{|s|\leq \eta} |F(s)|,$$

thus implying that $(u_{n,k})_k$ is bounded in $W^{1,\rho}(0,1)$. Therefore, there exist a subsequence of $(u_{n,k})_k$, still labeled by $(u_{n,k})_k$, and a function $u_n \in W^{1,\rho}(0,1)$ such that

$$\lim_{k\to+\infty} u_{n,k} = u_n \quad \text{weakly in } W^{1,\rho}(0,1).$$

Since $\mathcal{I}_{n,\lambda}$ is weakly lower semicontinuous in $W^{1,\rho}(0,1)$, we conclude that

$$\mathcal{I}_{n,\lambda}(u_n) \leq \inf_{u\in\Omega_\eta} \mathcal{I}_{n,\lambda}(u) < 0. \tag{9.4}$$

Notice that the set

$$\{u \in W^{1,\rho}(0,1) : \|u\|_{W^{1,1}} = \|w'\|_{L^1} + |r| \leq \eta\}$$

is closed and convex in $W^{1,\rho}(0,1)$ and hence it is weakly closed. Accordingly, we have $\|u_n\|_{W^{1,1}} \leq \eta$ and actually, from (9.4) and Lemma 9.1,

$$u_n \in \Omega_\eta.$$

As u_n is a local minimizer of $\mathcal{I}_{n,\lambda}$ and $\mathcal{I}_{n,\lambda}$ is differentiable in $W^{1,\rho}(0,1)$, we conclude that u_n is a solution of (8.1).

The proof of the existence of a positive solution u of problem (1.1), with $\mathcal{I}_\lambda(u) < 0$, then proceeds as that of Theorem 1.2 without any change.

From the two previous steps we infer that problem (1.1) has at least of two positive solutions; the remaining conclusions of Theorem 1.5 then follow as in the proof of Theorem 1.1. □

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Universidad Complutense de Madrid, Departamento de Matemática Aplicada, Plaza de Ciencias 3, 28040 Madrid, Spain
E-mail address: julian@mat.ucm.es

Università degli Studi di Trieste, Dipartimento di Matematica e Geoscienze, Sezione di Matematica e Informatica, Via A. Valerio 12/1, 34127 Trieste, Italy
E-mail address: omari@units.it
E-mail address: sabrina.rivetti@gmail.com