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Newton's trajectories versus MOND's trajectories

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A B S T R A C T

MOND dynamics consists of a modification of the acceleration with respect to the one provided by Newtonian mechanics. In this paper, we investigate whether it can be derived from a velocity-dependent deformation of the coordinates of the systems. The conclusion is that it cannot be derived this way because of the intrinsic non-local character in time of the MOND procedure. This is a feature pointed out some time ago already by Milgrom himself.

1. Introduction

Astrophysics and cosmology have undergone great experimental advancements in the last forty years. A nice, recent review of all this can be found in ref. [1]. The theoretical explanations for several of these phenomena are still open like, for example, the anomalous rotation velocity of stars in galaxies, and of galaxies themselves in their cluster, the acceleration of the expansion of the universe, the inflationary scenario and even, perhaps, some possible anomalies in the solar system (for this last see the nice review [2]).

In this paper we will address the first of the open problems listed above for which essentially two approaches have been used by physicists so far. The first approach is the Dark Matter hypothesis which postulates the existence of non-baryonic matter which does not interact with electromagnetic radiation while the second approach is based on the MOND hypothesis; in this paper we will concentrate on this last one.

The MOND hypothesis [3–5] is a modification of Newtonian mechanics which should apply to bodies moving with a slow acceleration (less than the threshold value $a_0 \approx 1.2 \times 10^{-10}$ m s^{-2}). This modification should describe the anomalous rotation of stars around the center of their galaxies which seems to occur at the same speed whatever is the distance of the stars from the center and the same for the rotation of galaxies around the center of their cluster [6]. Actually there are counter-examples and limitations to this last statement and also to what MOND achieves, see for example [7,8]. In general MOND works if we limit its applicability to the interior of galaxies and not to galaxies in their cluster or to larger scale phenomena.

On a larger scale, for example the rotation of galaxies inside their cluster or lensing or acceleration of the growth of primordial perturbations etc., a better explanation is given by the introduction of dark matter which accounts for many other anomalous phenomena.

It is actually emerging that a complete description of all these phenomena seems to require a mixture of MOND dynamics and Dark Matter (see for example the recent paper [9]). Up to now a huge effort has been put in exploring the Dark Matter scenario while much less has been done for MOND. If at the end a mixture of them will emerge, we think it is worth to put now some more effort in studying the MOND hypothesis.

We know that many do not like to change the Newtonian laws of motion, like MOND actually does, but this is something that has been done before in physics many times. Look at the atoms: in order to describe the motion of the electron around the proton in an hydrogen atom we had to abandon the Newtonian laws of motion and build a new theory called Quantum Mechanics (QM). Not only the laws of motion changed in QM with respect to Classical Mechanics (CM) but even the basic variables to describe the system had to be modified. Imagine if, in atomic physics, we had resisted from changing the laws of CM and decided instead to introduce some Dark Matter between the electron and the nucleus. Maybe we could cook up a distribution of Dark Matter which would keep the electron in orbit. Image generations and generations of chemists forced to describe all the periodic table elements not via QM but using CM plus some Atomic Dark Matter.

There have been other changes in Newtonian mechanics (like, for example, special and general relativity whose centenary occurred this year [10]) but let us stick here to the changes brought

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in from QM. Passing from the earth–moon system which is correctly described by CM down to an atom (correctly described by QM) the jump is roughly of 20 orders of magnitude. In going upwards from the earth–moon system to the stars rotating in a galaxy the jump is also of 20 or more orders of magnitude, so we feel that also in this second jump we may have to change the Newtonian laws of motion like we did in the first jump from CM to QM. This is what MOND's theory does and we are going to describe it in section 2. MOND has been the first attempt in this direction. It may need several improvements but it is worth to be studied further and we do that in section 3. We may even have to change the basic variables with whom we describe the systems like it happens in QM and some ideas on this aspect will be presented in the conclusions.

2. Review of the MOND dynamics

The basic law of MOND dynamics, like in Newton's mechanics, is the one which gives the acceleration a that a body of mass m feels under a force F (we will stick to one dimensional motion for simplicity). While in Newtonian mechanics this law is:

$$F = ma, \tag{1}$$

in MOND dynamics the equation above is deformed into the following one:

$$F = m\mu(\frac{a}{a_0})a, \tag{2}$$

where

$$\mu(\frac{a}{a_0}) \equiv (1 + \frac{a_0}{a})^{-1}. \tag{3}$$

a_0 is a fixed and very small acceleration of the order given before. The function $\mu()$ is called interpolating function because for $a_0 \to 0$ it goes to 1 and reproduces the usual Newton's law. There are other interpolating functions with this feature and they have been used in the literature [4,5]. We will stick to the one above.

Eq. (2) cannot be considered as a deformation of the mass because we will see later on that we use the same mass m in the centrifugal force. So the μ has to be considered as a deformation of the acceleration:

$$a \longrightarrow \mu(\frac{a}{a_0}).$$

The use of this deformed acceleration turns out to be very useful when we study the rotation of the stars around the center of their galaxy. This motion has a very small acceleration: $a \ll a_0$. In this limit eq. (3) gives:

$$\mu(\frac{a}{a_0}) = \frac{a}{a_0},$$

and MOND's law (2) becomes

$$F = m\frac{a^2}{a_0}. \tag{4}$$

Applying this formula when F is the gravitational force generated by a mass M and indicating with G the Newton's constant, we get:

$$m\frac{a^2}{a_0} = \frac{GmM}{r^2}$$

or equivalently:

$$a = \frac{\sqrt{GMa_0}}{r}. \tag{5}$$

The expression of the centripetal acceleration of a body like a star rotating with velocity v around the center of its galaxy at a distance r is:

$$a = \frac{v^2}{r} \tag{6}$$

if we put (5) equal to (6), we get

$$v = (GMa_0)^{\frac{1}{4}}.$$

From this expression we see that the velocity is the same at every distance from the center. This is what really happens in nature [6] and for which people have come up with the idea of dark matter.

The MOND approach has unfortunately several problems like the non-conservation of momentum [11]. This problem could be overcome if one restricts the modified acceleration to be only the gravitational one and if the associated Poisson equation is also modified [12]. One could then say that at those scales there is a modification of Newtonian gravity and nothing else. This second approach has been christened by Milgrom as Modified-Gravity MOND (MG-MOND) while the original one, in which the accelerations of all forces were modified, it was called Modified Inertia MOND (MI-MOND) [13]. In this paper we will stick to the MI-MOND. The fact that the classical momentum is not conserved is something that has happened before in passing from a theory to a new one for example when we passed from Classical Mechanics to Quantum Mechanics. In this last theory the analog of the "momentum" [14] associated to a wave-function $\psi(q)$ is given by:

$$P \equiv -i\hbar\frac{\partial\psi}{\partial q} = P_{cl} + O(\hbar). \tag{7}$$

This quantum P is conserved but it is different from the classical P_{cl} because there are $O(\hbar)$ corrections. So the same may happen in MOND where maybe the conserved momentum could be something different than the old classical one. In ref. [15] we proved that quantum mechanics could be obtained from classical mechanics via a deformation of its variables and this most probably is at the origin of the fact that the momenta in the two theories are different. Maybe we could try the same for the MI-MOND theory, that means check if we can obtain it via a deformation of the basic variables of classical mechanics. That is what we will try in the next section.

3. Configuration-space analysis

We have seen so far that MOND dynamics is a deformation of the acceleration. It seems natural to expect that this effect is induced by a deformation of the configuration-space of the system. This is not only natural but, as we said before, it already happened when we had to do the first modification of classical mechanics to pass to quantum mechanics. In fact it was proved in ref. [15] that going from CM to QM or vice versa can be understood as a change of variables once the two theories are formulated via path-integrals. In that formalism we proved that the two theories have the same functional weight but what changes are the variables over which we integrate in the path-integral. For QM we integrate over trajectories in the configuration space $q(t)$ while in CM we integrate over some different variable indicated by $Q(t)$ which are deformation of $q(t)$. This deformation is a whole multiplet and includes, besides q, the Jacobi-field variables, their symplectic duals and the response-field variables (for details see refs. [17,15]). Looking at quantization as a deformation of the basic variables is a modern version of a very old and rigorous method of quantization called *geometric quantization* (for a review see ref. [16]).

Analogously, may it be that in going now from CM to the MOND theory we have to deform $Q(t)$ to a new variable $\widetilde{Q}(t)$?. If so then we summarize the whole chain from QM to the MOND theory as follows: starting with QM and its variables $q(t)$ we made a deformation and passed to CM and its variables $Q(t)$ [15] and next we get to the MOND theory via a further deformation which brings us to the $\widetilde{Q}(t)$ like in the picture below:

$$q(t) \longrightarrow Q(t) \longrightarrow \widetilde{Q}(t). \qquad (8)$$

Somehow it is like if, going to larger and larger scales, we had to progressively deform the configurational variables we use.

For the moment let us limit ourself to the $q(t)$ component of the $Q(t)$ variables of CM (for details see ref. [15]) and let us see if for the MOND theory we can find a $\widetilde{q}(t)$ whose acceleration is equal to the MOND acceleration a_m which, according to eq. (4), is:

$$a_m \equiv (\frac{d^2q(t)}{dt^2})^2(\frac{1}{a_o}). \qquad (9)$$

So we would like to find a new configuration-space variable, $\widetilde{q}(t)$, such that:

$$\frac{d\widetilde{q}(t)}{dt^2} = (\frac{d^2q(t)}{dt^2})^2(\frac{1}{a_o}). \qquad (10)$$

Let us indicate the relation between $q(t)$ and $\widetilde{q}(t)$ as follows:

$$\widetilde{q} = \mathcal{F}(q, \dot{q}) \qquad (11)$$

where $\mathcal{F}$ is a function which should be determined using eq. (10). We have chosen the function $\mathcal{F}$ to depend not only on q but also on $\dot{q}$ otherwise, as it will be clear from the calculations which follow, there will be no chance of satisfying eq. (10). Let us now use (11) to derive the L.H.S. of (10):

$$\frac{d^2\widetilde{q}(t)}{dt^2} = \frac{\partial^2\mathcal{F}}{\partial q^2}\dot{q}^2 + 2\frac{\partial^2\mathcal{F}}{\partial\dot{q}\partial q}\dot{q}\ddot{q} +$$
$$+ \frac{\partial^2\mathcal{F}}{\partial\dot{q}\partial\dot{q}}(\ddot{q})^2 + \frac{\partial^2\mathcal{F}}{\partial\dot{q}\partial\dot{q}}\ddot{q}\dddot{q} +$$
$$+ \frac{\partial\mathcal{F}}{\partial q}\ddot{q} + \frac{\partial\mathcal{F}}{\partial\dot{q}}\dddot{q}. \qquad (12)$$

Comparing the R.H.S of eq. (12) with the R.H.S of eq. (10) we see that only the third term in the R.H.S of (12) has a form similar to the R.H.S. of eq. (10) and from this we get:

$$\frac{\partial^2\mathcal{F}}{\partial\dot{q}\partial\dot{q}} = \frac{1}{a_0}. \qquad (13)$$

Looking at this equation it is clear why we had to choose an $\mathcal{F}$ depending also on $\dot{q}$.

If we start "integrating" eq. (13) we get:

$$\frac{\partial\mathcal{F}}{\partial\dot{q}} = \frac{1}{a_0}\dot{q} + \mathcal{G}(q) \qquad (14)$$

where $\mathcal{G}$ is a function to be determined. The remaining terms on the R.H.S of eq. (12), besides the third one, must sum up to zero and using in them the relation (14), we get:

$$\frac{\partial^2\mathcal{G}}{\partial q^2}\dot{q}^2 + \frac{\partial\mathcal{G}}{\partial q}\ddot{q} + [(\frac{1}{a_0})\dot{q} + \mathcal{G}]\dddot{q} = 0. \qquad (15)$$

If we consider this as a differential equation for $\mathcal{G}$, the fact that as "coefficients" in this equation we have terms depending on $\dot{q}$, $\ddot{q}$, $\dddot{q}$, implies that as a general solution for $\mathcal{G}$ we will get a function that will depend, besides q, also on $\dot{q}$, $\ddot{q}$, $\dddot{q}$. But this is contradictory because in (14) the $\mathcal{G}$ was dependent only on q. So we conclude

that there is no solution to our equation (11), i.e. there is no manner to build a $\widetilde{q}(t)$ from a $q(t)$ which is the thing we wanted to do in (11).

The reader may think that by choosing a more general $\mathcal{F}$ in (11), depending also on $\ddot{q}$ and $\dddot{q}$, things could be fixed up, but this is not true. In fact it turns out that the analog of eq. (15) would then depend also on $\ddddot{q}$ implying that $\mathcal{F}$ would have to depend also on this variable leading in this way to a contradiction.

The reader may think that another way out could be to have no $\mathcal{G}$ at all in equation (14). In this way many terms in (12) would disappear but then, besides the third term, we would be left with the last one that is $\frac{\partial\mathcal{F}}{\partial\dot{q}}\dddot{q}$ which is not zero and so the problem is not solved.

Another attempt could be based on giving more freedom to our equations by letting even the time t change and not just $q(t)$. We would then have trajectories indicated by $\widetilde{q}(t')$ where t' is:

$$t' = \mathcal{E}(t) \qquad (16)$$

with $\mathcal{E}$ a free function to be determined together with the function $\mathcal{F}$ from the analog of eq. (10). It is not difficult to prove that even in this case we would end up in some contradiction like with eq. (15). Let us start by simplifying the calculations via an $\mathcal{F}$ depending only on $\dot{q}$. The equation we get as analog of (12) is:

$$\frac{d^2\widetilde{q}(t')}{dt'^2} = \frac{\partial^2\mathcal{F}}{\partial\dot{q}\partial\dot{q}}\left[\frac{1}{\mathcal{E}'}\frac{-\mathcal{E}''}{(\mathcal{E}')^2}\frac{dq}{dt} + \right.$$
$$\left. + \frac{1}{(\mathcal{E}')^2}\frac{d^2q}{dt^2}\right]^2 + \frac{\partial\mathcal{F}}{\partial\dot{q}}\left[\frac{d^3q(t')}{dt'^3}\right]. \qquad (17)$$

Remember: we wanted that the expression above be equal to:

$$\frac{1}{a_0}\left(\frac{d^2q}{dt^2}\right)^2 \qquad (18)$$

and this implies from (17) that:

$$\frac{\partial^2\mathcal{F}}{\partial\dot{q}\partial\dot{q}}\frac{1}{[\mathcal{E}']^4} = \frac{1}{a_0} \qquad (19)$$

while all the other terms on the R.H.D. of (17) must sum-up to zero, i.e.:

$$\frac{\partial^2\mathcal{F}}{\partial\dot{q}\partial\dot{q}}\frac{1}{\mathcal{E}'^6}(\mathcal{E}'')^2(\frac{dq}{dt})^2 - 2\frac{\partial^2\mathcal{F}}{\partial\dot{q}\partial\dot{q}}\frac{\mathcal{E}''}{\mathcal{E}'^5}\dot{q}\ddot{q} + \frac{\partial\mathcal{F}}{\partial\dot{q}}\frac{d^3q(t')}{dt'^3} = 0. \qquad (20)$$

The coefficients of this equation depend not only on $\dot{q}$ but also on $\ddot{q}$ and $\dddot{q}$ and as a consequence also its solution $\mathcal{F}$ will depend on these higher derivatives of q while we had made the choice at the beginning of having an $\mathcal{F}$ depending only on $\dot{q}$. We could think of playing with the $\mathcal{E}$ in order to cancel the first two terms of the equation (20), but there is no way to eliminate the third term. So this is the contradiction which cannot be removed even with the presence of the re-parametrization function $\mathcal{E}$.

We feel that what we have derived has a lot to do with the "non-locality" in the time "t" which Milgrom [18] discovered as a peculiar feature of the MI-MOND. This non-locality implies the dependence of the transformations on all the higher order derivatives of q. This is the same we would be forced to do in our case in order to have a consistent solution to our equations. To discover that non-locality Milgrom instead had to impose that the MI-MOND equation should be derivable from a Galilei invariant action. For us instead the request was that the MOND configuration variables should be derivable from deformation of the Newtonian ones. Both approaches point in the same direction: the non-locality in "t" intrinsic in the MI-MOND theory.

4. Conclusions and outlook

The conclusions we can draw from the calculations of the previous sections on the MI-MOND theory seem rather depressing but there may be a way out. As we said in the Introduction, in ref. [15] we showed that the true important coordinates of CM seem to be a set of variables whose configurational part we indicated with $Q(t)$ in order to distinguish it from $q(t)$. The $Q(t)$ is a full multiplet which contains not only the $q(t)$ but also the Jacobi fields, which we indicated with the symbol $c(t)$, its symplectic dual, $\bar{c}(t)$, and what is known in statistical mechanics as the response field $\lambda(t)$. For more details on these variables see ref. [17].

We indicated in formula (8) that it is this $Q(t)$ which should be deformed into a $\widetilde{Q}(t)$ in order to go into the MI-MOND theory. This would imply that the transformation of the first component of $\widetilde{Q}(t)$ which is $\widetilde{q}(t)$ would not be anymore of the form (11) but would have in $\mathcal{F}$ also a dependence on the other components of Q that we indicated above with c, $\bar{c}$ and λ. This more general $\mathcal{F}$ may lead to a solution of our problem.

The reader may wonder why we insist so much in deforming the Q. These variables were studied in ref. [15] from a mathematical point of view but we feel they have also a very important physical meaning which is now under investigation. What seems to emerge is that Q, together with its momenta P, do not represent points in phase-space like q, p but blocks of phase-space of dimension $\gg \hbar$. These blocks are the true degrees of freedom of CM because CM cannot handle objects in phase-space which have a volume less than $\hbar$. Applications of this idea have been done in ref. [19]. Being these the true degrees of freedom of CM they are the correct objects which had to be deformed to get to the MI-MOND theory. Most probably we were not getting the correct MI-MOND variables because we did not start from the true-degrees of freedom of CM.

In ref. [15] we described a mathematical procedure to go from QM (whose path-integral variables were indicated with $q(t)$, $p(t)$) to CM whose path-integral variables were indicated $Q(t)$, $P(t)$. As these last seem to be blocks of phase space [19], the original mathematical steps [15] to go from QM to CM can most probably be rewritten in physical terms as a "renormalization-group-like" procedure where the Q(t) are the analog of the block-spin variables in the renormalization process. This procedure requires the construction of what are known as the β and γ functions (for a review see for example [20]). In ref. [15] we basically proved (in a different language) that the β-function is identically zero. This is related to the fact that the masses and the couplings of the theory remain invariant, i.e. they do not change in going from QM to CM. In fact in ref. [15] we proved that the path-integrals in CM and QM have the same weight but with different fields. This actually implies that what is different from zero is the γ-function. In fact the γ-function tells us how the field changes under this renormalization group. Our field changes from $q(t)$ in QM to $Q(t)$ in CM so our γ-function must be different from zero.

We feel that QM must be an ultraviolet fixed point of this γ-function because up to now no violations to the laws of QM have been found so we should consider it the correct theory at the smallest possible scale (in phase-space) and this is equivalent to saying that QM is an ultraviolet stable fixed point. For CM instead there are many indications (like for example the rotation curves of stars) that it may not be the correct theory at the largest possible scale, so most probably it is not the infrared stable fixed point of the γ function that we mentioned before. This infrared stable fixed point will be given by the MOND theory (or some modifications of it). We say "some modifications of it" because we know that the MOND theory does not work well at scales larger than a galaxy.

Let us summarize: we start from the variables $q(t)$ of QM on the ultraviolet fixed point and then proceed with the block-spin procedure to the variables $Q(t)$ of CM. By continuing with the block-spin procedure we should arrive at the infrared stable fixed point which is MI-MOND (or some modification of it) with its degrees of freedom $\widetilde{Q}(t)$. From this renormalization-group procedure it is clear why we say that, to get to the MI-MOND theory, we have to deform the whole $Q(t)$ and not just its component $q(t)$ as we did in this paper. They are in fact the $Q(t)$ the variables that we get in CM via the renormalization-group and from these we have to start to continue with the same procedure in order to get to the infrared stable fixed point, that is the MI-MOND (or some modification of it). As the Q of CM are blocks of phase-space $\gg \hbar$ we envision that the MOND-$\widetilde{Q}$ will be huge blocks of phase space like those of a star or a galaxy. To understand how to deform the Q we need to know the γ function and at the moment we have no idea on how to get it, but it must be a derivation as simple as the derivation of the β function that we implicitly performed (without realizing) in [15] and proved to be identically zero.

In this picture CM is somehow in between QM and MI-MOND theory and it is not the infrared stable fixed point of the γ function. If so then CM should be unstable under the renormalization group flow. Actually It seems to be unstable under the change of some extra parameters [15] entering $Q(t)$, parameters which are formally two partners of time but whose product has the dimension of the inverse of an action. These parameters were never before introduced in CM and that is the reason nobody realized that CM was unstable. This whole analysis is also bringing to light the special role that the action plays in physics. We know that when the action gets very small, of the order of $\hbar$, we have to change the laws of motion from those of CM to the one of QM. The study we are performing on the γ function indicates that also for very large value of the action we may have to change the laws of motion and pass from CM to the MI-MOND (or some modification of it). The very large action we talk about are those of a galaxy or a cluster of galaxies which have some of the largest value of the action among the objects present in the universe.

Another project worth pursuing is the following one. The MG-MOND [12,13] theory does not present the non-locality problems of the MI-MOND besides being a theory where all conservation laws are respected. So it would be nice to see if it can be obtained via a deformation of the standard Poisson equation. Basically the MG-MOND [12] postulates a different Poisson equation with respect to the Newtonian one relating the gravitational potential $U(x)$ to the matter density $\rho(x)$. It has the form:

$$\nabla \times \left\{ \mu\left(\frac{\|\nabla U\|}{a_0}\right) \nabla U(x) \right\} = 4\pi G \rho(x). \tag{21}$$

G is Newton gravitational constant, $\mu(\cdot)$ is an interpolating function like in the MI-MOND theory and when $\mu = 1$ we get the Poisson equation of the Newton Theory. Eq. (21) is a non-linear equation which gives rise to very peculiar phenomena in physics. The most interesting one is the so called "External Field Effect" or (EFE) [12]. It basically says that, differently than in Newtonian or Einstein gravity, an object at a distance r from a center O does not only feel the gravitational force created by the matter inside the sphere of radius r with center in O but also the external gravitational field created by other objects lying outside the sphere. For example for a planet of the solar system it would feel also the external field created by the galaxy. Note that this effect would take place on objects, like the solar planets, which have an acceleration which is not small with respect to a_0 of the MOND laws. There is an extensive literature on this topic [21]. The EFE effect is in principle also present in the MI-MOND but, due to the non-local

features of this formulation, it is more difficult to calculate it and in some model it could even be brought to zero [13].

Using MG-MOND an effect of EFE on the internal planets of the solar system would manifest itself for example with a retrograde motion of the perihelion of Saturn and extensive numerical calculations of the modified Poisson equation have been performed [22]. From the latest data on Saturn the effect is unfortunately very very small. Where instead the effect should not be small is on the Oort cloud and some very interesting work has been done [23]. In particular it turns out that the orbits of objects in the Oort cloud would be quite deformed.

We should remind the reader that there are also other MOND-like effects [24]. In general many of these MOND-like effects could be similar to those generated by a distant planet [25] and a lot of interesting work is being performed at the moment on this issue.

As most of the work on these MG-MOND effects is numerical, it would be nice to find the analytic deformation transformation that bring the modified Poisson equation into the standard Newtonian one. All the physical effects that we mentioned have to be buried into this transformation.

This project and the previous one presented in these conclusions are under investigations and we hope to come back soon with more details.

Acknowledgements

References

[1] I. Debono, G.F. Smoot, Universe 2 (4) (2016) 23.
[2] L. Iorio, Int. J. Mod. Phys. D 24 (6) (2015) 1530015.
[3] M. Milgrom, Astron. J. 270 (1983) 365;
    M. Milgrom, Astron. J. 270 (1983) 371;
    M. Milgrom, Astron. J. 270 (1983) 384.
[4] M. Milgrom, Scholarpedia 9 (6) (2014) 31410.
[5] B. Famaey, et al., Living Rev. Relativ. 15 (1) (2012) 10.
[6] F. Zwicky, Helv. Phys. Acta 6 (1993) 110;
    V.C. Rubin, et al., Astron. J. 159 (1970) 379;
    V.C. Rubin, et al., Astron. J. 238 (1980) 471.
[7] N. Frusciante, et al., Mon. Not. R. Astron. Soc. 426 (2012) 751.
[8] P. Salucci, et al., Mon. Not. R. Astron. Soc. 378 (41) (2007).
[9] J. Khoury, Phys. Rev. D 91 (2015) 024022, arXiv:1409.0012.
[10] L. Iorio, Universe 1 (1) (2015) 38–81.
[11] J.E. Felten, Astron. J. 286 (1984) 3.
[12] J. Bekenstein, M. Milgrom, Astron. J. 286 (1984) 7.
[13] M. Milgrom, Acta Phys. Pol. B 43 (2011) 2175.
[14] R.A. Leacock, M. Padgett, Phys. Rev. Lett. 50 (1993) 3.
[15] A.A. Abrikosov Jr., E. Gozzi, D. Mauro, Ann. Phys. 317 (2005) 24.
[16] N. Woodhouse, Geometric Quantization, Claredon Press, Oxford, 1980.
[17] E. Gozzi, M. Reuter, W.D. Thacker, Phys. Rev. D 40 (10) (1989) 3363;
    E. Gozzi, M. Reuter, W.D. Thacker, Phys. Rev. D 46 (2) (1992) 757.
[18] M. Milgrom, Ann. Phys. 229 (1994) 384.
[19] E. Gozzi, C. Pagani, Phys. Rev. Lett. 105 (2010) 150604;
    E. Cattaruzza, E. Gozzi, Phys. Lett. A 376 (2012) 3017, arXiv:1207.5706.
[20] M. Le Bellac, Quantum and Statistical Field Theory, Claredon Press, Oxford, 1992.
[21] M. Milgrom, Astron. J. 302 (1986) 617;
    M. Milgrom, Mon. Not. R. Astron. Soc. 399 (1) (2009) 474;
    L. Iorio, AIP Conf. Proc. 1241 (2010) 935.
[22] L. Iorio, Astron. J. 3 (2010) 1;
    L. Blanchet, J. Novak, Mon. Not. R. Astron. Soc. 412 (2011) 2503;
    A. Hees, et al., Phys. Rev. D 89 (2014) 102002.
[23] L. Iorio, Astron. J. 3 (2010) 156.
[24] M. Milgrom, Mon. Not. R. Astron. Soc. 426 (2012) 673;
    L. Iorio, Class. Quantum Gravity 30 (2013) 165018.
[25] L. Iorio, Celest. Mech. Dyn. Astron. 112 (2012) 117;
    Astrophys. Space Sci. 362 (2017) 11, arXiv:1512.05288.