

# On weak interaction between a ground state and a trapping potential

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## Abstract

We continue our study initiated in [4] of the interaction of a ground state with a potential considering here a class of trapping potentials. We track the precise asymptotic behavior of the solution if the interaction is weak, either because the ground state moves away from the potential or is very fast.

## 1 Introduction

Consider the nonlinear Schrödinger equation

$$iu_t = -\Delta u + \beta(|u|^2)u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^3. \quad (1.1)$$

We assume (1.1) posses a family of orbitally stable ground states  $e^{i\omega t}\phi_\omega(x)$  parametrized by  $\omega$  in some open interval  $\mathcal{O} \subset \mathbb{R}_+$ . By the translation and Galilean symmetry, (1.1) has a family of traveling wave solutions

$$e^{\frac{i}{2}\mathbf{v} \cdot x - \frac{i}{4}|\mathbf{v}|^2 t + i\omega t + i\gamma_0} \phi_\omega(x - \mathbf{v}t - y_0), \quad (1.2)$$

parametrized by  $\mathbf{v}, y_0 \in \mathbb{R}^3$  and  $\gamma_0 \in \mathbb{R}$ . As in [4], we study the the dynamics of solutions of nonlinear Schrödinger equation with a rapidly decreasing potential

$$i\mathbf{u}_t = -\Delta \mathbf{u} + V(x)\mathbf{u} + \beta(|\mathbf{u}|^2)\mathbf{u}, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^3, \quad (1.3)$$

having initial data near (1.2) with  $t = 0$  and  $|\mathbf{v}| \gg 1$  or  $|y_0| \gg 1$  (see Theorem 1.4 for the precise statement). Since  $V$  is rapidly decreasing and we are assuming that the traveling wave is very fast or far away, we can expect for the solution behavior similar to the traveling wave (1.2). Indeed in [4], under the assumption that  $-\Delta + V$  has no eigenvalue, we proved that the solution decomposes into the traveling wave and a scattering wave  $e^{i\Delta t}\eta_+$ . In this paper, we consider the case  $-\Delta + V$  has exactly one eigenvalue  $e_0 < 0$ . It is well-known that in this case (1.3) posses a family of small nonlinear bound states  $Q_w \sim w\phi_0$  where  $\phi_0$  is the eigenfunction associated to  $e_0$  and  $w \in \mathbb{C}$ ,  $|w| \ll 1$  satisfies  $\dot{w} = -iE_w w$  for some  $E_w \sim e_0 \in \mathbb{R}$  (see Proposition 1.1 below). It is also known that small solutions of (1.3) decompose into  $Q_w$  and a scattering wave [10]. Therefore, we cannot expect that the solution with the initial data near the traveling wave decomposes into the traveling wave and the scattering solution. However, as naturally expected, in this paper we show that the solution decomposes into the traveling wave, scattering wave and the small nonlinear bound states  $Q_w$ .

We now start to state our result in rigorous manner. For the linear potential  $V$  and the nonlinearity  $\beta$ , we assume the following.

(H1) We assume  $V \in \mathcal{S}(\mathbb{R}^3, \mathbb{R})$  to be a fixed Schwartz function and the set of eigenvalues  $\sigma_p(-\Delta + V)$  is formed by exactly one element:  $\sigma_p(-\Delta + V) = \{e_0\}$  with  $e_0 < 0$ . Further, we assume 0 is not a resonance (that is, if  $(-\Delta + V)u = 0$  with  $u \in C^\infty$  and  $|u(x)| \leq C|x|^{-1}$  for a fixed  $C$ , then  $u = 0$ ).

(H2)  $\beta(0) = 0$ ,  $\beta \in C^\infty(\mathbb{R}, \mathbb{R})$ .

(H3) There exists a  $p \in (1, 5)$  such that for every  $k \geq 0$  there is a fixed  $C_k$  with

$$\left| \frac{d^k}{dv^k} \beta(v^2) \right| \leq C_k |v|^{p-k-1} \quad \text{if } |v| \geq 1.$$

It is well known that under the above assumptions, (1.3) is locally wellposed.

Let  $\phi_0 \in \ker(-\Delta + V - e_0)$  be everywhere positive with  $\|\phi_0\|_{L^2} = 1$ . For  $\delta > 0$  we set  $B_{\mathbb{C}}(\delta) = \{w \in \mathbb{C} : |w| < \delta\}$ . Recall that (1.3) admits small nonlinear bound states, that is the solutions of the form  $e^{iEt}Q(x)$  with  $E \in \mathbb{R}$  and  $Q(x) > 0$ . Indeed, we have the following well known result, see [5].

**Proposition 1.1.** *There exist a constant  $a_0 > 0$  and  $Q_w \in C^\infty(B_{\mathbb{C}}(a_0), H^2)$  s.t.*

$$\begin{aligned} (-\Delta + V)Q_w + \beta(|Q_w|^2)Q_w &= E_w Q_w, \\ Q_w &= w\phi_0 + q_w, \quad \langle q_w, \phi_0 \rangle = 0, \end{aligned} \tag{1.4}$$

where  $\langle \cdot, \cdot \rangle$  is defined in (1.19) below. We have  $E_w \in C^\infty(B_{\mathbb{C}}(a_0), \mathbb{R})$  with  $|E_w - e_0| \leq C|w|^2$ , and we have  $Q_w \in C^\infty(B_{\mathbb{C}}(a_0), \Sigma_k)$  and  $\|q_w\|_{\Sigma_k} \leq C_k |w|^3$  (for  $\Sigma_k$  see (1.20) below) for any  $k$ . Furthermore, we have the identity

$$iQ_w = -w_2 \partial_{w_1} Q + w_1 \partial_{w_2} Q \quad \text{where } w_1 = \operatorname{Re} w \text{ and } w_2 = \operatorname{Im} w. \tag{1.5}$$

(1.5) is an immediate consequence of  $Q_w = e^{i\theta} Q_r$ , where  $w_1 = r \cos \theta$  and  $w_2 = r \sin \theta$ . We set the continuous modes space as follows:

$$\mathcal{H}_c[w] := \left\{ \eta \in L^2; \langle i\eta, \partial_{w_1} Q_w \rangle = \langle i\eta, \partial_{w_2} Q_w \rangle = 0 \right\}. \tag{1.6}$$

A pair  $(p, q)$  is *admissible* when

$$2/p + 3/q = 3/2, \quad 6 \geq q \geq 2, \quad p \geq 2. \tag{1.7}$$

It is shown by [10] that all small solutions decompose into nonlinear bound states given in Proposition 1.1 and scattering waves (for an analogous result with weaker hypotheses on the spectrum see [5]).

**Theorem 1.2.** *There exist  $\delta > 0$  and  $C > 0$  such that for  $\|u(0)\|_{H^1} < \delta$  then the solution  $u(t)$  of (1.3) can be written uniquely for all times as*

$$u(t) = Q_{w(t)} + \eta(t) \quad \text{with } \eta(t) \in \mathcal{H}_c[w(t)] \tag{1.8}$$

with for all admissible pairs  $(p, q)$

$$\begin{aligned} \|w\|_{L_t^\infty(\mathbb{R}_+)} + \|\eta\|_{L_t^p(\mathbb{R}_+, W_x^{1,q})} &\leq C \|u(0)\|_{H^1}, \\ \|\dot{w} + iE_w w\|_{L_t^\infty(\mathbb{R}_+) \cap L_t^1(\mathbb{R}_+)} &\leq C \|u(0)\|_{H^1}^2. \end{aligned} \tag{1.9}$$

Moreover, there exist  $w_+ \in \mathbb{C}$  with  $|w_+ - w(0)| \leq C\|u(0)\|_{H^1}^2$  and  $\eta_+ \in H^1$  with  $\|\eta_+\|_{H^1} \leq C\|u(0)\|_{H^1}$ , such that

$$\begin{aligned} \lim_{t \rightarrow +\infty} \|\eta(t, x) - e^{it\Delta} \eta_+(x)\|_{H_x^1} &= 0, \\ \lim_{t \rightarrow +\infty} w(t) e^{i \int_0^t E_{w(s)} ds} &= w_+. \end{aligned} \quad (1.10)$$

We are interested to a different class of solutions of (1.3). We think of  $V(x)u$  as a perturbation of (1.1). We assume that (1.1) has a family of orbitally stable ground states  $e^{i\omega t} \phi_\omega(x)$ . By orbital stability, we mean that for any small  $\epsilon > 0$ , there exists  $\delta > 0$  such that if  $\|\phi - u_0\|_{H^1} < \delta$ , then the solution  $u$  of (1.1) with  $u(0) = u_0$  exists globally in time and satisfies

$$\sup_{t > 0} \inf_{s \in \mathbb{R}, y \in \mathbb{R}^3} \|e^{is} \phi(\cdot - y) - u(t)\|_{H^1} < \epsilon.$$

Specifically we assume what follows, which implies by [20], the existence of orbital stability of the ground states of (1.1).

(H4) There exists an open interval  $\mathcal{O} \subset \mathbb{R}_+$  such that

$$-\Delta u + \omega u + \beta(|u|^2)u = 0 \quad \text{for } x \in \mathbb{R}^3, \quad (1.11)$$

admits a positive radial solutions  $\phi_\omega$  for all  $\omega \in \mathcal{O}$ . Furthermore the map  $\omega \mapsto \phi_\omega$  is in  $C^\infty(\mathcal{O}, \Sigma_n)$  for any  $n \in \mathbb{N}$ .

*Remark 1.3.* It suffices to assume that the map  $\omega \mapsto \phi_\omega$  is in  $C^1(\mathcal{O}, H^2)$ . Indeed this implies that  $\omega \mapsto \phi_\omega$  is in  $C^\infty(\mathcal{O}, \Sigma_n)$  for any  $n \in \mathbb{N}$ . See Appendix B.

(H5) We have  $\frac{d}{d\omega} \|\phi_\omega\|_{L^2(\mathbb{R}^3)}^2 > 0$  for  $\omega \in \mathcal{O}$ .

(H6) Let  $L_+ = -\Delta + \omega + \beta(\phi_\omega^2) + 2\beta'(\phi_\omega^2)\phi_\omega^2$  be the operator whose domain is  $H^2(\mathbb{R}^3)$ . Then we assume that  $L_+$  has exactly one negative eigenvalue and the kernel is spanned by  $\partial_{x_j} \phi_\omega$  ( $j=1,2,3$ ).

We add to the previous hypotheses few more about the linearized operator  $\mathcal{H}_\omega$  defined in (2.38).

(H7)  $\exists \mathbf{n}$  and  $0 < \mathbf{e}_1(\omega) \leq \mathbf{e}_2(\omega) \leq \dots \leq \mathbf{e}_{\mathbf{n}}(\omega)$ , s.t.  $\sigma_p(\mathcal{H}_\omega)$  consists of  $\pm \mathbf{e}_j(\omega)$  and 0 for  $j = 1, \dots, \mathbf{n}$ . We assume  $0 < N_j \mathbf{e}_j(\omega) < \omega < (N_j + 1) \mathbf{e}_j(\omega)$  with  $N_j \in \mathbb{N}$ . We set  $N = N_1$ . Here each eigenvalue is repeated a number of times equal to its multiplicity. Multiplicities and  $\mathbf{n}$  are constant in  $\omega$ .

(H8) There is no multi index  $\mu \in \mathbb{Z}^{\mathbf{n}}$  with  $|\mu| := |\mu_1| + \dots + |\mu_k| \leq 2N_1 + 3$  such that  $\mu \cdot \mathbf{e}(\omega) = \omega$ , where  $\mathbf{e}(\omega) = (\mathbf{e}_1(\omega), \dots, \mathbf{e}_{\mathbf{n}}(\omega))$ .

(H9) For  $\mathbf{e}_{j_1}(\omega) < \dots < \mathbf{e}_{j_k}(\omega)$  and  $\mu \in \mathbb{Z}^k$  s.t.  $|\mu| \leq 2N_1 + 3$ , then we have

$$\mu_1 \mathbf{e}_{j_1}(\omega) + \dots + \mu_k \mathbf{e}_{j_k}(\omega) = 0 \iff \mu = 0.$$

(H10)  $\mathcal{H}_\omega$  has no other eigenvalues except for 0 and the  $\pm \mathbf{e}_j(\omega)$ . The points  $\pm \omega$  are not resonances. For the definition of resonance, see Sect.3 [2].

(H11) The Fermi golden rule Hypothesis (H11) in Sect. 6, see (6.18), holds.

We are interested to study how a solution  $u(t)$  of (1.3) initially close to a ground state of (1.1) which moves at a large speed is affected by the potential  $V$ . Notice that  $u(t)$  at no time has small  $H^1$  norm and so is not covered by Theorem 1.2. Unsurprisingly, in view of [4, 1, 3], we prove that the ground state survives the impact, but that as  $t \rightarrow \infty$  the solution  $u(t)$  approaches the orbit of a ground state of (1.1), up to a certain amount of radiation which satisfies Strichartz estimates, a term localized in spacetime, and a small amount of energy trapped by the Schrödinger operator  $-\Delta + V$ , which behaves like in Theorem 1.2. The difference with [4] is that in [4] we had  $\sigma_p(-\Delta + V) = \emptyset$  while here  $\sigma_p(-\Delta + V) = \{e_0\}$ .

If the initial ground state (1.2) has velocity  $\mathbf{v} \in \mathbb{R}^3$ , by setting  $u(t, x) := e^{-\frac{i}{2}\mathbf{v} \cdot x - \frac{i}{4}t|\mathbf{v}|^2} \mathbf{u}(t, x + \mathbf{v}t + y_0)$  we can equivalently assume that the ground state has initial velocity 0 and rewrite (1.3) as

$$i\dot{u} = -\Delta u + V(x + \mathbf{v}t + y_0)u + \beta(|u|^2)u, \quad u(0, x) = u_0(x). \quad (1.12)$$

Solutions of the (1.12) starting close to a positive radial solution of (1.11), for some time can be written as

$$\begin{aligned} u(t, x) &= e^{i(\frac{1}{2}v(t) \cdot x + \vartheta(t))} \phi_{\omega(t)}(x - D(t)) \\ &\quad + e^{-\frac{i}{2}\mathbf{v} \cdot x - \frac{i}{4}t|\mathbf{v}|^2} Q_{w(t)}(x + t\mathbf{v} + y_0) + r(t, x). \end{aligned} \quad (1.13)$$

**Theorem 1.4.** *Let  $\omega_1 \in \mathcal{O}$  and  $\phi_{\omega_1}(x)$  a ground state of (1.1). Assume (H1)–(H11) and assume furthermore that  $u_0 \in H^1(\mathbb{R}^3)$ . Fix  $M_0 > 1$  and  $\mathbf{v}, y_0 \in \mathbb{R}^3$  with  $|\mathbf{v}| > M_0$ . Fix a  $\varepsilon_1 > 0$ . We set*

$$\epsilon := \inf_{\theta \in \mathbb{R}} \|u_0 - e^{i\theta} \phi_{\omega_1}(\cdot)\|_{H^1} + \sup_{\text{dist}_{S^2}(\vec{e}, \frac{\mathbf{v}}{|\mathbf{v}|}) \leq \varepsilon_1} \int_0^\infty (1 + |\mathbf{v}| |\vec{e}t + y_0|^2)^{-1} dt. \quad (1.14)$$

*Then, there exist an  $\varepsilon_0 = \varepsilon_0(M_0, \omega_1, \varepsilon_1) > 0$  and a  $C > 0$  s.t. if  $u(t, x)$  is a solution of (1.12) with*

$$\epsilon < \varepsilon_0, \quad (1.15)$$

*there exist  $\omega_+ \in \mathcal{O}$ ,  $w_+ \in \mathbb{C}$ ,  $v_+ \in \mathbb{R}^3$ ,  $\theta \in C^1(\mathbb{R}_+; \mathbb{R})$ ,  $y \in C^1(\mathbb{R}_+; \mathbb{R}^3)$ ,  $w \in C^1(\mathbb{R}_+; \mathbb{C})$  and  $h_+ \in H^1$  with  $\|h_+\|_{H^1} + |\omega_+ - \omega_1| + |v_+| + |w_+| \leq C\epsilon$  such that*

$$\begin{aligned} &\lim_{t \nearrow \infty} \|u(t, x) - e^{i\theta(t) + \frac{i}{2}v_+ \cdot x} \phi_{\omega_+}(x - y(t)) \\ &\quad - e^{-\frac{i}{2}\mathbf{v} \cdot x - \frac{i}{4}t|\mathbf{v}|^2} Q_{w(t)}(x + t\mathbf{v} + y_0) - e^{it\Delta} h_+(x)\|_{H_x^1} = 0, \\ &\lim_{t \nearrow \infty} w(t) e^{i \int_0^t E_{w(s)} ds} = w_+. \end{aligned} \quad (1.16)$$

*Furthermore, there is a representation (1.13) valid for all  $t \geq 0$  such that we have  $r(t, x) = A(t, x) + \tilde{r}(t, x)$  such that  $A(t, \cdot) \in \mathcal{S}(\mathbb{R}^3, \mathbb{C})$ ,  $|A(t, x)| \leq C(t)$  with  $\lim_{t \rightarrow +\infty} C(t) = 0$  and such that for any admissible pair  $(p, q)$  we have*

$$\|\tilde{r}\|_{L_t^p(\mathbb{R}_+, W_x^{1,q})} \leq C\epsilon. \quad (1.17)$$

Theorem 1.4 extends to the case of potentials with 1 eigenvalue the result in [4]. Thanks to [5], which extends Theorem 1.2, we could have considered generic potentials with very few restrictions on the eigenvalues, but we chose to focus on this case study.

In the literature there are several results concerning the interaction between a linear potential and a fast ground state or between solitons, see the references in [4]. Here we reference [11, 12, 7], which consider in the 1-D case a fast soliton of the cubic NLS interacting with  $V(x) = q\delta_0(x)$ ,  $q \in \mathbb{R}$  and  $\delta_0$  the delta function. If  $q > 0$ ,  $-\Delta + q\delta_0$  has no eigenvalues and if  $q < 0$  it has exactly one eigenvalue. So, the situation is somewhat similar to our result in [4] and this paper. However in the

case  $V(x) = q\delta_0(x)$ , even though the interaction is fast, it is strong enough to produce a substantial modification of the soliton, which splits into two distinct solitons, one transmitted and the other reflected. In particular this means that in some obvious respect the situation is easier in our case than in [11, 12, 7], whose results, though, are less definite. In particular [11, 12] for  $q > 0$  and [7] for  $q < 0$  give some control of the solution for long but finite times. In our case, the interaction is weak, there is no splitting of ground states but we give a very detailed description of the solution for all times. It is clear that all these results, for different reasons, are very partial and that a general theory of the interaction between solitons and potentials is an interesting and largely not understood problem.

Our present paper and [4] are also related to the interaction between distinct solitons. As mentioned above, using the theory in [5] we could produce a general result on the weak interaction of a soliton with a generic potential. It is plausible that this analysis could be extended to weak interactions of pairs or of more general families of solitons. This would yield for families of weakly interacting and *generic* solitons a result more detailed than those in [14]. However, such a result would be very far from providing a sufficiently general theory of multi-solitons for non integrable systems, which remains unknown. Since currently multi-solitons are well understood in the case of integrable systems thanks to inverse scattering transform techniques, we think that the approach with the best chance to produce for some non integrable cases a setup to describe general solutions of a focusing NLS involves some combined use of inverse scattering and perturbation arguments, in the spirit of Deift and Zhou [8]. Obviously, the main issue is how to account in the non integrable case for the destruction of solitons or other patterns and for the appearance of new ones. A number of papers, like [13, 15, 16] and others quoted therein, contain insightful descriptions of specific non-integrable phenomena, but they don't provide yet a general theory for the non integrable setting.

All of this is completely beyond what we do in the present paper, in fact quite beyond of what exists in the literature, which is very fragmented and partial. Even results like Theorem 1.4 here or like in [4] require a quite sophisticated framework, which is important to perfect as a preparation for what will be a more general theory in the future. Notice that the solitons considered here are generic, while those in [17, 19] obey very restrictive hypotheses.

Our approach for Theorem 1.4 is the same of [4]. We represent solutions  $u(t)$  of (1.12) as a sum of a moving ground state of (1.1) and a small energy trapped solution of (1.12) in a way similar to the ansatz in [14, 17].

Thanks to the weakness of the interaction with the potential, we are able to show that this representation is preserved for all times and that there is a separation of moving ground state and of trapped energy. Furthermore, we prove that the stabilization processes around the energy trapped by the potential, described in Theorem 1.2, and around the ground state, described in [3, 4], continue to hold.

In [4], in the absence of trapped energy, we described  $u(t)$  in terms of the local analysis of the NLS around solitons developed in the series [1, 2, 3]. The main two novelties in [4] consisted in the fact that the coordinate changes and the effective Hamiltonian in [4] depend on the time variable and that proof of the dispersion of continuous modes require the theory of *charge transfer models* as in [18] instead of the simpler dispersive analysis of [2, 3].

These features of [4] are present here. The additional complication is that, along with a part of  $u(t)$  which has the same description as in [4],  $u(t)$  has also a term representing the energy trapped by the potential. In this paper we will describe in detail in Sect. 2 the decomposition and coordinates representation of  $u(t)$ . In the following sections we will focus mainly on the coupling terms between trapped energy and the rest of  $u(t)$ , often referring to [4]. In the proof we will assume at first that additionally

$$u_0 \in \Sigma_2, \tag{1.18}$$

see right below (1.20). Notice that in [4] it is assumed that  $u_0 \in \Sigma_n$  for sufficiently large  $n$ , but inspection of the proof shows easily that (1.18) suffices. We will then show that in fact the result extends rather easily to  $u_0 \in H^1$ .

We will make extensive use of notation and results in [1, 4]. We refer to [4] for a more extended discussion to the problem and for more references and we end the introduction with some notation.

Given two Banach spaces  $X$  and  $Y$  we denote by  $B(X, Y)$  the space of bounded linear operators from  $X$  to  $Y$ . For  $x \in X$  and  $\varepsilon > 0$ , we set

$$B_X(x, \varepsilon) := \{x' \in X \mid \|x - x'\|_X < \varepsilon\}.$$

We set  $\langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}$  and

$$\langle f, g \rangle = \operatorname{Re} \int_{\mathbb{R}^3} f(x) \bar{g}(x) dx \text{ for } f, g : \mathbb{R}^3 \rightarrow \mathbb{C}. \quad (1.19)$$

For any  $n \geq 1$  and for  $K = \mathbb{R}, \mathbb{C}$  we consider the Banach space  $\Sigma_n = \Sigma_n(\mathbb{R}^3, K^2)$  defined by

$$\|u\|_{\Sigma_n}^2 := \sum_{|\alpha| \leq n} (\|x^\alpha u\|_{L^2(\mathbb{R}^3)}^2 + \|\partial_x^\alpha u\|_{L^2(\mathbb{R}^3)}^2) < \infty. \quad (1.20)$$

We set  $\Sigma_0 = L^2(\mathbb{R}^3, K^2)$ . Equivalently we can define  $\Sigma_r$  for  $r \in \mathbb{R}$  by the norm

$$\|u\|_{\Sigma_r} := \|(1 - \Delta + |x|^2)^{\frac{r}{2}} u\|_{L^2} < \infty.$$

For  $r \in \mathbb{N}$  the two definitions are equivalent, see [3].

From now on, we identify  $\mathbb{C} = \mathbb{R}^2$  and set  $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ , so that multiplication by  $i$  in  $\mathbb{C}$  is  $J^{-1} = -J$ . Later on, we complexify  $\mathbb{R}^2$  and  $i$  will appear in such meaning. That is for  $U = {}^t(u_1, u_2)$ ,  $iU = {}^t(iu_1, iu_2)$ . So, be careful not to confuse  $-J$  with  $i$  which has the different meaning.

## 2 The Ansatz

We consider the energy

$$\begin{aligned} \mathbf{E}(u) &= \mathbf{E}_0(u) + \mathbf{E}_V(u) \\ \mathbf{E}_0(u) &:= \frac{1}{2} \|\nabla u\|_{L^2}^2 + \mathbf{E}_P(u), \quad \mathbf{E}_P(u) := \frac{1}{2} \int_{\mathbb{R}^3} B(|u|^2) dx \\ \mathbf{E}_V(u) &:= \frac{1}{2} \langle V(\cdot + \mathbf{v}t + y_0)u, u \rangle, \end{aligned} \quad (2.1)$$

with  $B(0) = 0$  and  $B'(t) = \beta(t)$ . It is well known that  $\mathbf{E}_0$  is conserved by the flow of (1.1). For  $u \in H^1(\mathbb{R}^3, \mathbb{C})$ , its charge and momenta, invariants of motion of (1.1), are defined as follows:

$$\begin{aligned} \Pi_4(u) &= \frac{1}{2} \|u\|_{L^2}^2 = \frac{1}{2} \langle \Diamond_4 u, u \rangle, \quad \Diamond_4 := 1; \\ \Pi_a(u) &= \frac{1}{2} \operatorname{Im} \langle u_{x_a}, u \rangle = \frac{1}{2} \langle \Diamond_a u, u \rangle, \quad \Diamond_a := J \partial_{x_a} \text{ for } a = 1, 2, 3. \end{aligned} \quad (2.2)$$

The charge  $\Pi_4$  is conserved by the flow of both (1.3) and (1.1). However,  $\Pi_a$ ,  $a = 1, 2, 3$  are conserved only by (1.1) but not by the perturbed equation (1.3) which is not translation invariant. We set

$\Pi(u) = (\Pi_1(u), \dots, \Pi_4(u))$ . We have  $\mathbf{E} \in C^2(H^1(\mathbb{R}^3, \mathbb{C}), \mathbb{C})$  and  $\Pi_j \in C^\infty(H^1(\mathbb{R}^3, \mathbb{C}), \mathbb{C})$ . Recall the following formulas

$$\begin{aligned}\Pi_4(e^{-\frac{1}{2}Jv \cdot x}u) &= \Pi_4(u); \\ \Pi_a(e^{-\frac{1}{2}Jv \cdot x}u) &= \Pi_a(u) + \frac{1}{2}v_a \Pi_4(u) \text{ for } a = 1, 2, 3; \\ \mathbf{E}_0(e^{-\frac{1}{2}Jv \cdot x}u) &= \mathbf{E}_0(u) + v \cdot \Pi(u) + \frac{v^2}{4} \Pi_4(u), \quad v \cdot \Pi(u) = \sum_{a=1}^3 v_a \Pi_a(u).\end{aligned}\tag{2.3}$$

By (H5) and (2.2), setting  $p = \Pi(e^{-\frac{1}{2}Jv \cdot x}\phi_\omega)$ , we have

$$\frac{\partial p}{\partial(\omega, v)} = \begin{pmatrix} \frac{1}{2}\Pi_4(\phi_\omega)I_3 & * \\ 0 & 2\frac{d}{d\omega}\|\phi_\omega\|_{L^2}^2 \end{pmatrix},$$

where  $I_3$  is  $3 \times 3$  identity matrix. Therefore, we have that  $(\omega, v) \rightarrow p = \Pi(e^{-\frac{1}{2}Jv \cdot x}\phi_\omega)$  is a diffeomorphism into an open subset of  $\mathcal{P} \subset \mathbb{R}^4$ . For  $p = p(\omega, v) \in \mathcal{P}$  set  $\Phi_p = e^{-\frac{1}{2}Jv \cdot x}\phi_\omega$  for  $p = \Pi(e^{-\frac{1}{2}Jv \cdot x}\phi_\omega)$ .

## 2.1 Linearized operator and its generalized null space

We will consider the group  $\tau = (D, -\vartheta) \rightarrow e^{J\tau \cdot \diamond}u(x) := e^{i\vartheta}u(x - D)$ . The  $\Phi_p$  are constrained critical points of  $\mathbf{E}_0$  with associated Lagrange multipliers  $\lambda(p) \in \mathbb{R}^4$  so that  $\nabla \mathbf{E}_0(\Phi_p) = \lambda(p) \cdot \diamond \Phi_p$ , where we have

$$\lambda_4(p) = -\omega(p) - \frac{v^2(p)}{4}, \quad \lambda_a(p) := v_a(p) \text{ for } a = 1, 2, 3.\tag{2.4}$$

We set also

$$d(p) := \mathbf{E}_0(\Phi_p) - \lambda(p) \cdot \Pi(\Phi_p).\tag{2.5}$$

For any fixed vector  $\tau_0$  a function  $u(t) := e^{J(t\lambda(p) + \tau_0) \cdot \diamond} \Phi_p$  is a solitary wave solution of (1.1). We now introduce the linearized operator

$$\mathcal{L}_p := J(\nabla^2 \mathbf{E}_0(\Phi_p) - \lambda(p) \cdot \diamond)\tag{2.6}$$

where  $\nabla^2 \mathbf{E}_0 \in C^0(H^1, B(H^1, H^{-1}))$  is the differential of  $\nabla \mathbf{E}_0 \in C^0(H^1, H^{-1})$ . By an abuse of notation, we set

$$\mathcal{L}_\omega := \mathcal{L}_p \text{ when } v(p) = 0 \text{ and } \omega(p) = \omega.\tag{2.7}$$

We have the following identity, see [1] Sect.7, which implies  $\sigma(\mathcal{L}_p) = \sigma(\mathcal{L}_{\omega(p)})$ ,

$$\mathcal{L}_p = e^{-\frac{1}{2}Jv(p) \cdot x} \mathcal{L}_{\omega(p)} e^{\frac{1}{2}Jv(p) \cdot x},\tag{2.8}$$

and which follows by

$$e^{-\frac{1}{2}Jv \cdot x}(-\Delta)e^{\frac{1}{2}Jv \cdot x} = -\Delta - v \cdot \diamond + \frac{|v|^2}{4}.$$

Hypothesis (H5) implies that  $\text{rank} \left[ \frac{\partial \lambda_i}{\partial p_j} \right]_{i \downarrow, j \rightarrow} = 4$ . This and (H6) imply

$$\begin{aligned}\ker \mathcal{L}_p &= \text{Span}\{J \diamond_j \Phi_p : j = 1, \dots, 4\} \text{ and} \\ N_g(\mathcal{L}_p) &= \text{Span}\{J \diamond_j \Phi_p, \partial_{p_j} \Phi_p : j = 1, \dots, 4\},\end{aligned}\tag{2.9}$$

where  $N_g(L) := \cup_{j=1}^{\infty} \ker(L^j)$ . Recall that we have a well known decomposition

$$L^2 = N_g(\mathcal{L}_p) \oplus N_g^\perp(\mathcal{L}_p^*), \quad (2.10)$$

$$N_g(\mathcal{L}_p^*) = \text{Span}\{\diamond_j \Phi_p, J^{-1} \partial_{\lambda_j} \Phi_p : j = 1, \dots, 4\}. \quad (2.11)$$

We denote by  $P_{N_g}(p)$  the projection on  $N_g(\mathcal{L}_p)$  and by  $P(p)$  the projection on  $N_g^\perp(\mathcal{L}_p^*)$  associated to (2.10).

$$P_{N_g}(p) = -J \diamond_j \Phi_p \langle \cdot, J^{-1} \partial_{p_j} \Phi_p \rangle + \partial_{p_j} \Phi_p \langle \cdot, \diamond_j \Phi_p \rangle, \quad P(p) = 1 - P_{N_g}(p). \quad (2.12)$$

We now decompose the solution of (1.12) into the large solitary wave given in (H4), small bound state given in Prop. 1.1 and the remainder part which will belong in both the  $N_g^\perp(\mathcal{L}_p^*)$  and the galilean transform of  $\mathcal{H}_c[w]$ .

**Proposition 2.1.** *Fix  $\varepsilon_1 > 0$  and  $\omega_1 \in \mathcal{O}$ . Let  $\varkappa \in \mathcal{P}$  be s.t.  $v(\varkappa) = 0$  and  $\omega(\varkappa) = \omega_1$ . Then there exists  $\varepsilon_2 > 0$  s.t. if*

$$\sup_{\text{dist}_{S^2}(\vec{e}, \frac{\mathbf{v}}{|\mathbf{v}|}) \leq \varepsilon_1} \int_0^\infty (1 + |\mathbf{v}| |\vec{e}t + y_0|^2)^{-1} dt < \varepsilon_2 \quad (2.13)$$

and for all  $t \geq 0$ ,  $\tau_0 \in B_{\mathbb{R}^3}(0, \varepsilon_2 \langle t \rangle) \times \mathbb{R}$  and  $u \in e^{J\tau_0 \diamond} B_{H^1}(\Phi_\varkappa, \varepsilon_2)$ , there exists

$$(\tau, p, w) \in C^\infty(B(\varepsilon_2); \mathbb{R}^4 \times \mathbb{R}^4 \times \mathbb{R}^2),$$

where

$$B(\varepsilon_2) := \{(t, u) \in [0, \infty) \times H^1 \mid \exists \tau \in B_{\mathbb{R}^3}(0, \varepsilon_2 \langle t \rangle) \times \mathbb{R} \text{ s.t. } u \in e^{J\tau \diamond} B_{H^1}(\Phi_\varkappa, \varepsilon_2)\}, \quad (2.14)$$

s.t.

$$p(t, e^{J\tau_0 \diamond} \phi_{\omega_1}) = \varkappa, \quad \tau(t, e^{J\tau_0 \diamond} \phi_{\omega_1}) = \tau_0 \text{ and } w(t, e^{J\tau_0 \diamond} \phi_{\omega_1}) = 0, \quad (2.15)$$

$$\mathcal{F}_j(t, u, \tau(t, u), p(t, u), w(t, u)) = \mathcal{G}_j(t, u, \tau(t, u), p(t, u), w(t, u)) = 0 \text{ for } j = 1, 2, 3, 4 \text{ and}$$

$$\mathcal{L}_j(t, u, \tau(t, u), p(t, u), w(t, u)) = 0 \text{ for } j = 1, 2$$

with

$$\mathcal{F}_j(t, u, \tau, p, w) := \left\langle \tilde{R}(t, u, \tau, p, w), e^{J\tau \diamond} J^{-1} \partial_{p_j} \Phi_p \right\rangle = 0, \quad j = 1, 2, 3, 4, \quad (2.16)$$

$$\mathcal{G}_j(t, u, \tau, p, w) := \left\langle \tilde{R}(t, u, \tau, p, w), e^{J\tau \diamond} \diamond_j \Phi_p \right\rangle = 0, \quad j = 1, 2, 3, 4, \quad (2.17)$$

$$\mathcal{L}_j(t, u, \tau, p, w) := \left\langle \tilde{R}(t, u, \tau, p, w), e^{J(\frac{1}{2}v \cdot x + \frac{t}{4}|v|^2)} \partial_{w_j} Q_w(\cdot + tv + y_0) \right\rangle = 0, \quad j = 1, 2 \quad (2.18)$$

where

$$\tilde{R}(t, u, \tau, p, w) := u - e^{J\tau \diamond} \Phi_p - e^{J(\frac{1}{2}v \cdot x + \frac{t}{4}|v|^2)} Q_w(\cdot + tv + y_0). \quad (2.19)$$

**Remark 2.2.** The solution  $u$  which we consider in Theorem 1.4 will always belong to  $(t, u(t)) \in B(\varepsilon_2)$  provided  $\varepsilon_0$  sufficiently small. Therefore, we can always decompose the solution as

$$u = e^{J\tau \diamond} \Phi_p + e^{J(\frac{1}{2}v \cdot x + \frac{t}{4}|v|^2)} Q_w(\cdot + tv + y_0) + e^{J\tau \diamond} R,$$

where  $\tilde{R} = e^{J\tau \diamond} R$ .



Proposition 2.1 is a direct consequence of the following two lemmas.

**Lemma 2.3.** Fix  $\delta > 0$ . Set

$$X(\tau, t) = \max_{j,l=1,2,3,4,k,l=1,2,a+b=1} \left| \left\langle e^{J(\frac{1}{2}\mathbf{v} \cdot \mathbf{x} + \frac{t}{4}|\mathbf{v}|^2)} J^{k-1} \phi_0(\cdot + t\mathbf{v} + y_0), e^{J\tau \diamond} J^{l-1} \partial_{p_j}^a \diamond_l^b \Phi_{\mathbf{x}} \right\rangle \right|$$

and

$$\mathcal{T}(t, \delta) = \{\tau \in \mathbb{R}^4 \mid X(\tau, t) < \delta\}.$$

Then, there exists  $\varepsilon = \varepsilon(\delta) > 0$  s.t. if (2.13) is satisfied with  $\varepsilon_2$  replaced to  $\varepsilon$ , then

$$B_{\mathbb{R}^3}(0, \varepsilon \langle t \rangle) \times \mathbb{R} \subset \mathcal{T}(t, \delta), \quad \forall t \geq 0.$$

**Lemma 2.4.** There exists  $\delta > 0$  s.t. for any  $t_0 \geq 0$  and any  $\tau_0 \in \mathcal{T}(t_0, \delta)$ , there exists  $(\tau, p, w) \in C^1(X; \mathbb{R}^4 \times \mathbb{R}^4 \times \mathbb{R}^2)$ , with  $X := (t_0 - \delta, t_0 + \delta) \times e^{J\tau_0 \diamond} B_{H^1}(\Phi_{\mathbf{x}}, \delta)$ , which satisfies (2.15)–(2.19). Furthermore, in any open subset of  $X$  there is only one such function  $(\tau, p, w)$ .

*Proof of Lemma 2.3.* First, notice that if  $|\mathbf{v}| \geq C\delta^{-1}$  for some constant  $C > 0$ , then we have  $\mathcal{T}(t, \delta) = \mathbb{R}^4$ . This can be easily shown by integration by parts. Therefore, we can assume  $|\mathbf{v}| \leq C\delta^{-1}$ . Notice that there is an  $M = M(\delta)$  such that, if

$$\inf_{\text{dist}_{S^2}(\vec{\mathcal{C}}, \frac{\mathbf{v}}{|\mathbf{v}|}) \leq \varepsilon_1} |\mathbf{v}| \vec{\mathcal{C}} \tilde{t} + y_0| \geq M, \quad \text{for all } \tilde{t} > 0, \quad (2.20)$$

then for sufficiently small  $\varepsilon > 0$ , we have  $B_{\mathbb{R}^3}(0, \varepsilon \langle t \rangle) \times \mathbb{R} \subset \mathcal{T}(t, \delta)$  for all  $t \geq 0$ . Indeed, for any  $\tau = (D, -\vartheta) \in B_{\mathbb{R}^3}(0, \varepsilon \langle t \rangle) \times \mathbb{R}$ , there exists  $\mathbf{v}_\varepsilon \in \mathbb{R}^3$  with  $|\mathbf{v}_\varepsilon| < \varepsilon$  and  $y_\varepsilon \in \mathbb{R}^3$  with  $|y_\varepsilon| < \varepsilon$  s.t.  $D = \mathbf{v}_\varepsilon t + y_\varepsilon$ . Therefore,

$$|\mathbf{v}t + y_0 - D| = |(\mathbf{v} - \mathbf{v}_\varepsilon)t + y_0 - y_\varepsilon| \geq \left| |\mathbf{v}| \left( \frac{\mathbf{v} - \mathbf{v}_\varepsilon}{|\mathbf{v} - \mathbf{v}_\varepsilon|} \right) \left( \frac{|\mathbf{v} - \mathbf{v}_\varepsilon|}{|\mathbf{v}|} t \right) - y_0 \right| - |y_\varepsilon| \geq M - \varepsilon,$$

where we have used (2.20) with  $\vec{\mathcal{C}} = \frac{\mathbf{v} - \mathbf{v}_\varepsilon}{|\mathbf{v} - \mathbf{v}_\varepsilon|}$  and  $\tilde{t} = \frac{|\mathbf{v} - \mathbf{v}_\varepsilon|}{|\mathbf{v}|} t$ . This in turn implies  $X(\tau, t) < \delta$  for all  $t \geq 0$  if  $M$  is large enough, and so  $\tau \in \mathcal{T}(t, \delta)$ .

We fix such an  $M$  and suppose now that for some  $t > 0$  and some  $\tilde{\mathbf{v}} = |\mathbf{v}| \vec{\mathcal{C}}$  with  $\text{dist}_{S^2}(\vec{\mathcal{C}}, \frac{\mathbf{v}}{|\mathbf{v}|}) < \varepsilon_1$ , we have  $|\tilde{\mathbf{v}}t + y_0| < M$ . We will show that for  $\varepsilon$  small this is incompatible with  $|\mathbf{v}| < C\delta^{-1}$ . We have

$$t^2 |\mathbf{v}|^2 + 2t \tilde{\mathbf{v}} \cdot y_0 + |y_0|^2 - M^2 < 0. \quad (2.21)$$

Next we claim that for  $\varepsilon$  sufficiently small we have  $|y_0| \geq A := \max\left(16 \frac{M^2}{\varepsilon_1^2}, 2M + C\delta^{-1}\right)$  with  $\varepsilon_1 > 0$  the fixed constant used in (1.14). Indeed, if this is not the case, then

$$\int_0^\infty \langle |\mathbf{v}|t + |y_0| \rangle^{-2} dt \leq \int_0^\infty \langle \tilde{\mathbf{v}}t + y_0 \rangle^{-2} dt \leq \varepsilon \Rightarrow |\mathbf{v}| \geq \left(\frac{\pi}{2} - \arctan A\right) \varepsilon^{-1}. \quad (2.22)$$

But for  $\varepsilon \in (0, \varepsilon_0)$ , with  $\varepsilon_0 > 0$  small enough this contradicts with  $|\mathbf{v}| < C\delta^{-1}$ . So we can assume  $|y_0| \geq A$ . Further, we can assume  $t \geq 1$  since if  $0 < t < 1$ , then

$$|\tilde{\mathbf{v}}t + y_0| \geq |y_0| - |\mathbf{v}| \geq A - C\delta^{-1} \geq M.$$

For  $\hat{y} := \frac{y}{|y|}$  and  $\hat{\mathbf{v}} := \frac{\tilde{\mathbf{v}}}{|\tilde{\mathbf{v}}|}$  the discriminant of the quadratic in  $t$  polynomial in (2.21) is positive:

$$\cos^2 \alpha > 1 - M^2 |y_0|^{-2} > 1 - \frac{\varepsilon_1^2}{16} \text{ where } -\hat{y}_0 \cdot \hat{\mathbf{v}} = \cos(\alpha) \quad (2.23)$$

with  $\alpha = \text{dist}_{S^2}(-\hat{y}_0, \hat{\mathbf{v}})$  the angle between  $-\hat{y}_0$  and  $\hat{\mathbf{v}}$ . (2.21) requires also  $\cos(\alpha) > 0$ , so

$$\cos(\alpha) > \sqrt{1 - \varepsilon_1^2/16}. \quad (2.24)$$

Since  $\varepsilon_1$  has been chosen sufficiently small, from (2.24) we obtain  $\alpha < \varepsilon_1/3$ . This implies

$$\varepsilon \geq \int_0^\infty \langle -|\mathbf{v}| \hat{y}_0 t + y_0 \rangle^{-2} dt = |\mathbf{v}|^{-1} \int_0^\infty \langle t - |y_0| \rangle^{-2} dt \geq \frac{\pi}{2} |\mathbf{v}|^{-1}. \quad (2.25)$$

But this again contrasts with  $|\mathbf{v}| < C\delta^{-1}$ . Hence we conclude that  $|\mathbf{v}| < C\delta^{-1}$  and  $\varepsilon$  sufficiently small imply  $|\tilde{\mathbf{v}}t + y_0| \geq M$  for all  $t > 0$  for any preassigned  $M$ .  $\square$

*Proof of Lemma 2.4.* We apply the implicit function theorem (Theorem A.1) to  $X = \mathbb{R} \times H^1(\mathbb{R}^3)$ ,  $Y = \mathbb{R}^{10}$  and  $\mathbf{F} \in C^\infty([0, \infty) \times H^1 \times \mathbb{R}^4 \times \mathcal{P} \times B_{\mathbb{R}^2}(a_0), \mathbb{R}^{10})$  for

$$\mathbf{F} = (\mathcal{F}_1, \dots, \mathcal{F}_4, -\mathcal{G}_1, \dots, -\mathcal{G}_4, -\mathcal{L}_1, \mathcal{L}_2).$$

We first compute the Jacobian matrix of  $\mathbf{F}$ . We compute the derivatives of  $\tilde{R}$ .

$$\begin{aligned} \partial_{\tau_k} \tilde{R} &= -e^{J\tau\Diamond} J\Diamond_k \Phi_p, \quad k = 1, 2, 3, 4, \\ \partial_{p_k} \tilde{R} &= -e^{J\tau\Diamond} \partial_{p_k} \Phi_p, \quad k = 1, 2, 3, 4, \\ \partial_{w_k} \tilde{R} &= -e^{J(\frac{1}{2}vx + \frac{t}{4}|v|^2)} \partial_{w_k} Q_w(\cdot + tv + y_0), \quad k = 1, 2. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \partial_{\tau_k} \mathcal{F}_j &= -\langle e^{J\tau\Diamond} J\Diamond_k \Phi_p, e^{J\tau\Diamond} J^{-1} \partial_{p_j} \Phi_p \rangle + \langle \tilde{R}, e^{J\tau\Diamond} \Diamond_k \partial_{p_j} \Phi_p \rangle \\ &= \delta_{jk} + \langle \tilde{R}, e^{J\tau\Diamond} \Diamond_k \partial_{p_j} \Phi_p \rangle \\ \partial_{p_k} \mathcal{F}_j &= -\langle e^{J\tau\Diamond} \partial_{p_k} \Phi_p, e^{J\tau\Diamond} J^{-1} \partial_{p_j} \Phi_p \rangle + \langle \tilde{R}, e^{J\tau\Diamond} J^{-1} \partial_{p_k} \partial_{p_j} \Phi_p \rangle \\ &= \langle \tilde{R}, e^{J\tau\Diamond} J^{-1} \partial_{p_k} \partial_{p_j} \Phi_p \rangle \\ \partial_{w_k} \mathcal{F}_j &= -\langle e^{J(\frac{1}{2}vx + \frac{t}{4}|v|^2)} \partial_{w_k} Q_w(\cdot + tv + y_0), e^{J\tau\Diamond} J^{-1} \partial_{p_j} \Phi_p \rangle \\ &= -\langle e^{J(\frac{1}{2}vx + \frac{t}{4}|v|^2)} J^{k-1} \phi_0(\cdot + tv + y_0), e^{J\tau\Diamond} J^{-1} \partial_{p_j} \Phi_p \rangle \\ &\quad - \langle e^{J(\frac{1}{2}vx + \frac{t}{4}|v|^2)} \partial_{w_k} q_w(\cdot + tv + y_0), e^{J\tau\Diamond} J^{-1} \partial_{p_j} \Phi_p \rangle, \end{aligned}$$

where we have used

$$-\langle e^{J\tau\Diamond} J\Diamond_k \Phi_p, e^{J\tau\Diamond} J^{-1} \partial_{p_j} \Phi_p \rangle = \frac{1}{2} \partial_{p_j} \langle \Diamond_k \Phi_p, \Phi_p \rangle = \partial_{p_j} \Pi_k(\Phi_p) = \partial_{p_j} p_k = \delta_{jk}.$$

Further, we have

$$\begin{aligned}
\partial_{\tau_k} \mathcal{G}_j &= -\langle e^{J\tau\Diamond} J\Diamond_k \Phi_p, e^{J\tau\Diamond} \Diamond_j \Phi_p \rangle + \langle \tilde{R}, J e^{J\tau\Diamond} \Diamond_k \Diamond_j \Phi_p \rangle \\
&= \langle \tilde{R}, J e^{J\tau\Diamond} \Diamond_k \Diamond_j \Phi_p \rangle \\
\partial_{p_k} \mathcal{G}_j &= -\langle e^{J\tau\Diamond} \partial_{p_k} \Phi_p, e^{J\tau\Diamond} \Diamond_j \Phi_p \rangle + \langle \tilde{R}, e^{J\tau\Diamond} \Diamond_j \partial_{p_k} \Phi_p \rangle \\
&= -\delta_{jk} + \langle \tilde{R}, e^{J\tau\Diamond} \Diamond_j \partial_{p_k} \Phi_p \rangle \\
\partial_{w_k} \mathcal{G}_j &= -\langle e^{J(\frac{1}{2}vx + \frac{t}{4}|v|^2)} \partial_{w_k} Q_w(\cdot + tv + y_0), e^{J\tau\Diamond} \Diamond_j \Phi_p \rangle \\
&= -\langle e^{J(\frac{1}{2}vx + \frac{t}{4}|v|^2)} J^{k-1} \phi_0(\cdot + tv + y_0), e^{J\tau\Diamond} \Diamond_j \Phi_p \rangle \\
&\quad - \langle e^{J(\frac{1}{2}vx + \frac{t}{4}|v|^2)} \partial_{w_k} q_w(\cdot + tv + y_0), e^{J\tau\Diamond} \Diamond_j \Phi_p \rangle,
\end{aligned}$$

and

$$\begin{aligned}
\partial_{\tau_k} \mathcal{L}_j &= -\langle e^{J\tau\Diamond} J\Diamond_k \Phi_p, e^{J(\frac{1}{2}vx + \frac{t}{4}|v|^2)} \partial_{w_j} Q_w(\cdot + tv + y_0) \rangle \\
&= -\langle e^{J\tau\Diamond} J\Diamond_k \Phi_p, e^{J(\frac{1}{2}vx + \frac{t}{4}|v|^2)} J^{j-1} \phi_0(\cdot + tv + y_0) \rangle \\
&\quad - \langle e^{J\tau\Diamond} J\Diamond_k \Phi_p, e^{J(\frac{1}{2}vx + \frac{t}{4}|v|^2)} \partial_{w_j} q_w(\cdot + tv + y_0) \rangle \\
\partial_{p_k} \mathcal{L}_j &= -\langle e^{J\tau\Diamond} \partial_{p_k} \Phi_p, e^{J(\frac{1}{2}vx + \frac{t}{4}|v|^2)} \partial_{w_j} Q_w(\cdot + tv + y_0) \rangle \\
&= -\langle e^{J\tau\Diamond} \partial_{p_k} \Phi_p, e^{J(\frac{1}{2}vx + \frac{t}{4}|v|^2)} J^{j-1} \phi_0(\cdot + tv + y_0) \rangle \\
&\quad - \langle e^{J\tau\Diamond} \partial_{p_k} \Phi_p, e^{J(\frac{1}{2}vx + \frac{t}{4}|v|^2)} \partial_{w_j} q_w(\cdot + tv + y_0) \rangle \\
\partial_{w_k} \mathcal{L}_j &= -\langle e^{J(\frac{1}{2}vx + \frac{t}{4}|v|^2)} \partial_{w_k} Q_w(\cdot + tv + y_0), e^{J(\frac{1}{2}vx + \frac{t}{4}|v|^2)} \partial_{w_j} Q_w(\cdot + tv + y_0) \rangle \\
&\quad + \langle \tilde{R}, e^{J(\frac{1}{2}vx + \frac{t}{4}|v|^2)} \partial_{w_k} \partial_{w_j} Q_w(\cdot + tv + y_0) \rangle, \\
&= -\langle \partial_{w_k} Q_w, \partial_{w_j} Q_w \rangle + \langle \tilde{R}, e^{J(\frac{1}{2}vx + \frac{t}{4}|v|^2)} \partial_{w_k} \partial_{w_j} q_w(\cdot + tv + y_0) \rangle,
\end{aligned}$$

Now, since  $Q_w = (w_1 - Jw_2)\phi_0 + q_w$ , and  $\partial_{w_j} q_w = O(|w|^2)$ , we have

$$-\langle \partial_{w_k} Q_w, \partial_{w_j} Q_w \rangle = \langle (-J)^{k-1} \phi_0, (-J)^{j-1} \phi_0 \rangle + O(|w|^2) = (-1)^j \langle \phi_0, J^{k+j-2} \phi_0 \rangle + O(|w|^2). \tag{2.26}$$

Therefore,

$$\begin{aligned}
\begin{pmatrix} -\partial_{w_1} \mathcal{L}_1 & -\partial_{w_2} \mathcal{L}_1 \\ \partial_{w_1} \mathcal{L}_2 & \partial_{w_2} \mathcal{L}_2 \end{pmatrix} &= \begin{pmatrix} \langle \partial_{w_1} Q_w, \partial_{w_2} Q_w \rangle & \langle \partial_{w_2} Q_w, \partial_{w_2} Q_w \rangle \\ -\langle \partial_{w_1} Q_w, \partial_{w_1} Q_w \rangle & -\langle \partial_{w_2} Q_w, \partial_{w_1} Q_w \rangle \end{pmatrix} + O(|w|^2) \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + O(|w|^2)
\end{aligned}$$

Therefore, we have

$$\frac{\partial \mathbf{F}}{\partial(\tau, p, w)} = I_{10} + A.$$

where  $I_{10}$  is the unit matrix and each component in  $A$  can be bounded by

$$C\|u - e^{J\tau \cdot \diamond} \Phi_p\|_{L^2} + C|w| + X(\tau, t), \quad (2.27)$$

where  $C$  is independent of  $(p, \tau, w) \in \mathcal{P} \times \mathbb{R}^4 \times B_{\mathbb{R}^2}(0; \delta_0)$ .

Now, there exists a universal constant  $\tilde{\delta}$  s.t. if the absolute value of each component of  $A$  is less than  $\tilde{\delta}$ , then  $(I_{10} + A)^{-1}$  exists and its operator norm is bounded by 2. Now we claim there exists  $\tilde{\delta}_1 > 0$  s.t. if  $(\tau, p, w) \in B_{\mathbb{R}^{10}}((\tau_0, \varkappa, 0), \tilde{\delta}_1)$  and  $(t, u) \in (t_0 - \tilde{\delta}_1, t_0 + \tilde{\delta}_1) \times B_{H^1}(\Phi_\varkappa, \tilde{\delta}_1)$ , we have  $\|(I_{10} + A)^{-1}\| \leq 2$ . The bounds for  $C\|u - e^{J\tau \cdot \diamond} \Phi_p\|_{L^2} + C|w|$  is obvious so we only consider the bound of  $X(\tau, t)$ . Notice that if  $|\mathbf{v}| \geq C\tilde{\delta}^{-1}$ , then since  $\mathcal{T}(t, \tilde{\delta}) = \mathbb{R}^4$ , we only have to consider the case  $|\mathbf{v}| \leq C\tilde{\delta}^{-1}$ . In this case, since

$$|\langle e^{J(\frac{1}{2}\mathbf{v} \cdot x + \frac{t}{4}|\mathbf{v}|^2)} J^{k-1} \phi_0(\cdot + t\mathbf{v} + y_0), e^{J\tau \cdot \diamond} J^{l-1} \partial_{p_j}^a \diamond_l^b \Phi_\varkappa \rangle| \quad (2.28)$$

$$= |\langle e^{J\frac{t-t_0}{4}|\mathbf{v}|} e^{J(\frac{1}{2}\mathbf{v} \cdot x + \frac{t}{4}|\mathbf{v}|^2)} J^{k-1} \phi_0(\cdot + t_0\mathbf{v} + y_0 + (t - t_0)\mathbf{v}), e^{J\tau \cdot \diamond} J^{l-1} \partial_{p_j}^a \diamond_l^b \Phi_\varkappa \rangle| \quad (2.29)$$

$$\leq C\tilde{\delta} + C|e^{J\frac{t-t_0}{4}|\mathbf{v}|} - 1| + \|\phi_0(\cdot + (t - t_0)\mathbf{v}) - \phi_0\|_{L^2}. \quad (2.30)$$

Therefore, we see there exists  $\tilde{\delta}_1$  which satisfies the claim.

Finally, setting  $\delta_1 = \delta_2 = \tilde{\delta}_1$ , by Theorem , there exists  $\delta_3, \delta_4 > 0$  independent to the choice of  $t_0, \tau_0$  s.t. the desired  $(\tau, p, w) \in C^1((t_0 - \delta, t_0 + \delta) \times e^{J\tau \cdot \diamond} B_{H^1}(\Phi_\varkappa, \delta); B_{\mathbb{R}^{10}}((\tau_0, \varkappa, 0), \delta_4))$  exists.  $\square$

We choose  $p_0, v_0, \omega_0$  such that if  $u_0$  is the initial value in (1.12), then

$$\Pi(\Phi_{p_0}) = \Pi(u_0), \quad v_0 = v(p_0) \text{ and } \omega_0 = \omega(p_0). \quad (2.31)$$

We fix  $\pi \in \mathcal{P}$ . Now, Proposition 2.1 can be reframed as follows.

**Lemma 2.5.** *For  $|\pi - p_0| < \delta_0$  and  $|\varkappa - p_0| < \delta_0$  for sufficiently small  $\delta_0$  and for  $(t, u) \in B(\varepsilon_2)$  as in Proposition 2.1, there exists  $r \in N_g^\perp(\mathcal{L}_{p_0}^*)$  s.t. for the  $(\tau, w)$  of Propostion 2.1 we have*

$$u = U[t, u] + Q[t, u] \text{ where } U[t, u] := e^{J\tau \cdot \diamond} (\Phi_p + P(p)P(\pi)r) \text{ and} \quad (2.32)$$

$$Q[t, u] := e^{J\Theta \cdot \diamond} Q_w \text{ for } \Theta := (-\mathbf{v}t - y_0, 2^{-1}\mathbf{v} \cdot x + 4^{-1}t|\mathbf{v}|^2)$$

with

$$\langle e^{J\tau \cdot \diamond} P(p)P(\pi)r, J e^{J\Theta \cdot \diamond} \partial_{w_i} Q_w \rangle = 0 \text{ for } i = 1, 2. \quad (2.33)$$

$\square$

Notice that  $e^{J\Theta \cdot \diamond} Q_w(x) = e^{J(\frac{1}{2}\mathbf{v} \cdot x + \frac{t}{4}|\mathbf{v}|^2)} Q_w(\cdot + \mathbf{v}t + y_0)$ .

Eventually we will set  $\pi = \Pi(U[u(t)])$ , but for the moment we will take  $\pi$  as a parameter.

We will consider the following notation:

$$\tilde{Q} := Q[t, u], \quad U := U[t, u], \quad \tilde{H} := -\Delta + V(\cdot + \mathbf{v}t + y_0). \quad (2.34)$$

Since  $w \in C^\infty(B(\varepsilon_2), \mathbb{R}^2)$ ,  $w(0, e^{J\tau \cdot \diamond} \phi_{\omega_1}) = 0$  for all  $\tau \in \mathcal{T}(0, \delta_1)$ ,  $\tau_j \in C^\infty(B(\varepsilon_2), \mathbb{R})$ ,  $\tau_j(0, e^{i\vartheta} \phi_{\omega_1}) = 0$  for  $j \leq 3$  and by the definition of  $\epsilon$  in Theor. 1.4 we have  $|w(0, u_0)| \leq c\epsilon$  and  $|\tau_j(0, u_0)| \leq c\epsilon$  for  $j \leq 3$  for a fixed  $c$ . For another fixed  $c$  we have

$$\inf_{\theta \in \mathbb{R}} \|U[0, u_0] - e^{i\theta} \phi_{\omega_1}(\cdot)\|_{H^1} \leq c\epsilon. \quad (2.35)$$

## 2.2 Spectral coordinates associated to $\mathcal{L}_p$

We will summarize in this section a number of facts about equation (1.3) when  $V \equiv 0$  which have been proved in [1, 4] or which can be easily proved following the ideas therein.

First of all we observe that we have coordinates  $(\tau, p, r)$  for the quantity  $U$  defined by

$$\begin{aligned} \mathbb{R}^4 \times \{p : |p - p_0| < a\} \times (N_g^\perp(\mathcal{L}_{p_0}^*) \cap \Sigma_k) &\rightarrow \Sigma_k(\mathbb{R}^3, \mathbb{R}^2), \\ (\tau, p, r) &\rightarrow U = e^{J\tau \cdot \diamond}(\Phi_p + P(p)P(\pi)r). \end{aligned} \quad (2.36)$$

$(\tau, p, r)$  are coordinates for  $U$  in an open set

$$\mathbb{N} = \cup_{\tau \in \mathbb{R}^4} e^{J\tau \cdot \diamond} B_{H^1}(\phi_{\omega_1}, \delta) \quad (2.37)$$

with  $\delta > 0$  sufficiently small. For any  $U \in H^1(\mathbb{R}^3, \mathbb{R}^2)$  we have also  $\Pi_j = \Pi_j(U)$ . Then  $(\tau, \Pi, r)$  is also a system of coordinates in  $\mathbb{N}$ . The functions  $(\tau, \Pi)$  depend smoothly in  $U$  while we have  $r \in C^l(\mathbb{N} \cap \Sigma_k, \Sigma_{k-l})$ . Obviously, if we set  $(t, u) \rightarrow U = U[t, u]$ , which is a smooth function, functions  $(t, u) \rightarrow (\tau, \Pi, r)$  remain defined.

The next task is to further decompose the variable  $r$ . This is done in terms of the spectral decomposition of the operator  $\mathcal{L}_{p_0}$  as we explain now.

We now consider the complexification of  $L^2(\mathbb{R}^3, \mathbb{R}^2)$  into  $L^2(\mathbb{R}^3, \mathbb{C}^2)$  and think of  $\mathcal{L}_p$  and  $J$  as operators in  $L^2(\mathbb{R}^3, \mathbb{C}^2)$ . Then we set

$$\mathcal{H}_p := i\mathcal{L}_p \text{ with } \mathcal{H}_\omega := \mathcal{H}_p \text{ when } v(p) = 0 \text{ and } \omega(p) = \omega. \quad (2.38)$$

We have

$$\mathcal{H}_\omega = iJ(-\Delta + \omega) + iJ \begin{pmatrix} \beta(\phi_\omega^2) + 2\beta'(\phi_\omega^2)\phi_\omega^2 & 0 \\ 0 & \beta(\phi_\omega^2) \end{pmatrix}. \quad (2.39)$$

and

$$\begin{aligned} M^{-1}\mathcal{H}_\omega M &= \mathcal{K}_\omega, \\ \mathcal{K}_\omega &:= \sigma_3(-\Delta + \omega) + \begin{pmatrix} \beta(\phi_\omega^2) + \beta'(\phi_\omega^2)\phi_\omega^2 & \beta'(\phi_\omega^2)\phi_\omega^2 \\ -\beta'(\phi_\omega^2)\phi_\omega^2 & -\beta(\phi_\omega^2) - \beta'(\phi_\omega^2)\phi_\omega^2 \end{pmatrix} \\ M &:= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}, \quad M^{-1} = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned} \quad (2.40)$$

*Remark 2.6.* Notice that  $M \begin{pmatrix} u \\ \bar{u} \end{pmatrix} = \begin{pmatrix} \text{Re } u \\ \text{Im } u \end{pmatrix}$ .

We extend the bilinear map  $\langle \cdot, \cdot \rangle$  and  $\Omega(\cdot, \cdot) = \langle J^{-1} \cdot, \cdot \rangle$  as bilinear maps in  $L^2(\mathbb{R}^3, \mathbb{C}^2)$ . That is, for  $u = (u_1, u_2)$ ,  $v = (v_1, v_2) \in L^2(\mathbb{R}^3, \mathbb{C}^2)$ , we have  $\langle u, v \rangle = \int_{\mathbb{R}^3} u_1 v_1 + u_2 v_2$ . In particular,  $\langle \cdot, \cdot \rangle$  extends into a bilinear form in

$$\mathcal{S}'(\mathbb{R}^3, \mathbb{C}^2) \times L_d^2(\mathcal{H}_p^*), L_d^2(\mathcal{H}_p^*) := N_g(\mathcal{H}_p^*) \oplus \left( \oplus_{\mu \in \sigma_p(\mathcal{H}_p^*) \setminus \{0\}} \ker(\mathcal{H}_p^* - \mu) \right).$$

Set now  $(L_d^2(\mathcal{H}_p^*))^\perp$  the subspace of  $\mathcal{S}'$  orthogonal to  $L_d^2(\mathcal{H}_p^*)$ .

**Lemma 2.7.** *Let  $\lambda$  be the non-zero eigenvalue of  $\mathcal{H}_p$ . Then algebraic and geometric multiplicity of  $\lambda$  coincide. Furthermore, for  $\lambda > 0$  and  $\xi \in \ker(H_p - \lambda)$ , we have  $-i \langle J^{-1} \xi, \bar{\xi} \rangle > 0$ .*

*Proof.* By (2.8), it suffices to consider  $p$  with  $\omega(p) = \omega$  and  $v(p) = 0$ . First, we show there are no  $\xi \in \ker(H_p - \lambda)$  s.t.  $\langle (\nabla^2 \mathbf{E}_0(\Phi_p) + \omega)\xi, \bar{\xi} \rangle = 0$ . Suppose, there exists such  $\xi$ . Then, by [9] Corollary 3.3.1 p.171, we have  $\xi = aJ^{-1}\Phi_p$ . However, since  $\xi \in N_g(\mathcal{H}_p^*)^\perp$  and  $N_g(\mathcal{H}_p) \cap N_g(\mathcal{H}_p^*)^\perp = \{0\}$ , we have  $\xi = 0$ . So, we see there are no  $\xi \in \ker(H_p - \lambda)$  s.t.  $\langle (\nabla^2 \mathbf{E}_0(\Phi_p) + \omega)\xi, \bar{\xi} \rangle = 0$ . Therefore, Assumption 2.8 of [6] is satisfied and by [6] Corollary 2.12, we see that  $-\mathrm{i} \langle J^{-1}\xi, \bar{\xi} \rangle > 0$  for  $\lambda > 0$ .  $\square$

**Lemma 2.8.** *There is a neighborhood  $\mathcal{P}_{p_0}$  of  $p_0$  in  $\mathcal{P}$  and a  $C^\infty(\mathcal{P}_{p_0}, \Sigma_m^n)$  map (for any preassigned  $m$ )  $\pi \rightarrow (\xi_1(\pi), \dots, \xi_n(\pi))$  such that the following facts hold.*

- (1)  $\xi_j(\pi) \in \ker(\mathcal{H}_\pi - \mathbf{e}_j)$  for all  $j$ .
- (2)  $-\mathrm{i} \langle J^{-1}\xi_j(\pi), \xi_k(\pi) \rangle = 0$  for all  $j$  and  $k$  and  $-\mathrm{i} \langle J^{-1}\xi_j(\pi), \bar{\xi}_k(\pi) \rangle = \delta_{jk}$ .

*Proof.* For the proof of the existence of a such a frame for any fixed  $\pi$  we refer to Lemma 5.2 [1]. Here we discuss the fact that the dependence in  $\pi$  is smooth. Let us pick  $l_1 = 1 < l_2 < \dots < l_k \leq \mathbf{n}$  and set  $l_{k+1} = \mathbf{n} + 1$ , with  $\mathbf{e}_j(\omega) = \mathbf{e}_i(\omega)$  if and only if  $j, i \in [l_a, l_{a+1})$  for some  $a$ . The numbers  $l_1, \dots, l_k$  do not depend on  $\omega$  by the constancy of multiplicity in Hypothesis (H7).

By (2.8) we can set  $\xi_j(\pi) = e^{-\frac{1}{2}Jv(\pi) \cdot x} \tilde{\xi}_j(\omega(\pi))$ , with  $\tilde{\xi}_j(\omega) \in \ker(\mathcal{H}_\omega - \mathbf{e}_j(\omega(\pi)))$  appropriate vectors dependent now only on  $\omega$ . It is easy to conclude that it is enough to focus on the case  $v(\pi) \equiv 0$ .

For  $\omega_0 = \omega(p_0)$  we can suppose we have a frame  $\{\hat{\xi}_j(\omega)\}$  satisfying the equalities in claim (2), that is for  $\omega = \omega_0$ ,  $L = \mathbf{n} + 1$  and  $\ell = 1$  we have

$$-\mathrm{i} \langle J^{-1}\hat{\xi}_j(\omega), \bar{\xi}_k(\omega) \rangle = \delta_{jk} \text{ for } j, k \in [\ell, L). \quad (2.41)$$

For  $\delta V_\omega := \mathcal{H}_\omega - \mathcal{H}_{\omega_0}$  we have that  $\omega \rightarrow \delta V_\omega \in C^\infty(I_{\omega_0}, B(\Sigma_m, \Sigma_m))$  for any  $m$  for a small interval  $I_{\omega_0}$  with center  $\omega_0$ . Fix now an index  $l_a$  and let  $\gamma_a$  be a small circle with counter clock orientation and centered in  $\mathbf{e}_{l_a}(\omega_0)$ . By taking  $I_{\omega_0}$  small we can assume that  $\mathbf{e}_{l_a}(\omega)$  is for all  $\omega \in I_{\omega_0}$  contained in a compact subset of the interior of the disk encircled by  $\gamma_a$ . Then the following is a projection on  $\ker(\mathcal{H}_\omega - \mathbf{e}_{l_a}(\omega))$ :

$$P_a(\omega) = \frac{\mathrm{i}}{2\pi} \oint_{\gamma} \frac{1}{\mathcal{H}_\omega - z} dz. \quad (2.42)$$

We have  $\omega \rightarrow P_a(\omega) \in C^\infty(I_{\omega_0}, B(\Sigma_{-m}, \Sigma_m))$ . Now focus on the frame  $\{\hat{\xi}_j(\omega_0)\}$  for  $j \in [l_a, l_{a+1})$  s.t. (2.41) is true for  $\omega = \omega_0$ ,  $L = l_{a+1}$  and  $\ell = l_a$ . We first set  $\tilde{\xi}_1(\omega) = P_a(\omega)\hat{\xi}_1(\omega_0)$  which we can normalize into a  $\hat{\xi}_1(\omega)$  s.t.  $-\mathrm{i} \langle J^{-1}\hat{\xi}_1(\omega), \bar{\xi}_1(\omega) \rangle = 1$ . Suppose now that we have for some  $l < l_{a+1}$  a frame  $\{\hat{\xi}_j(\omega) : j \in [l_a, l)\}$  which is  $C^\infty$  in  $\omega \in I_{\omega_0}$  and s.t. (2.41) is true for all  $\omega \in I_{\omega_0}$ , for  $L = l$  and  $\ell = l_a$ . Set now

$$\tilde{\xi}_l(\omega) = P_a(\omega)\hat{\xi}_l(\omega_0) + \mathrm{i} \sum_{j \in [l_a, l)} \hat{\xi}_j(\omega) \langle J^{-1}P_a(\omega)\hat{\xi}_l(\omega_0), \bar{\xi}_j(\omega) \rangle.$$

Then  $\langle J^{-1}\tilde{\xi}_l(\omega), \bar{\xi}_j(\omega) \rangle = 0$  for all  $j \in [l_a, l)$ . Notice that  $\tilde{\xi}_l(\omega)$  depends smoothly on  $\omega$  and that  $\tilde{\xi}_l(\omega_0) = \hat{\xi}_l(\omega_0)$ . Then by continuity  $-\mathrm{i} \langle J^{-1}\tilde{\xi}_l(\omega), \bar{\xi}_l(\omega) \rangle =: a^2(\omega) > 0$ . Setting  $\hat{\xi}_l(\omega) = a^{-1}(\omega)\tilde{\xi}_l(\omega)$  we obtain a frame  $\{\hat{\xi}_j(\omega) : j \in [l_a, l]\}$  which is  $C^\infty$  in  $\omega \in I_{\omega_0}$  and s.t. (2.41) is true for all  $\omega \in I_{\omega_0}$ , for  $L = l + 1$  and  $\ell = l_a$ .

Finally, notice that if  $\mathbf{e}_j(\omega) \neq \mathbf{e}_k(\omega)$ , then  $\langle J^{-1}\widehat{\xi}_j(\omega), \bar{\widehat{\xi}}_k(\omega) \rangle = 0$ . So we have built a frame smooth in  $\omega$  which satisfies (2.41) for  $L = \mathbf{n} + 1$  and  $\ell = 1$  and for all  $\omega \in I_{\omega_0}$ . The identities  $\langle J^{-1}\widehat{\xi}_j(\omega), \widehat{\xi}_k(\omega) \rangle = 0$  hold for all  $j, k$ , see Lemma 5.2 [1]. So Lemma 2.8 is proved.  $\square$

The following spectral decomposition remains determined

$$\begin{aligned} N_g^\perp(\mathcal{L}_p^*) &= N_g^\perp(\mathcal{H}_p^*) = \left( \oplus_{\mu \in \sigma_p(\mathcal{H}_p^*) \setminus \{0\}} \ker(\mathcal{H}_p - \mu) \right) \oplus L_c^2(p) \\ L_c^2(p) &:= L^2(\mathbb{R}^3, \mathbb{C}^2) \cap (L_d^2(\mathcal{H}_p^*))^\perp. \end{aligned} \quad (2.43)$$

Correspondingly for any  $r \in N_g^\perp(\mathcal{H}_{p_0}^*)$  with  $r = \bar{r}$  we have, for a  $z \in \mathbb{C}^{\mathbf{n}}$  and an  $f \in L_c^2(p_0)$ ,

$$P(\pi)r = \sum_{j=1}^{\mathbf{n}} z_j \xi_j(\pi) + \sum_{j=1}^{\mathbf{n}} \bar{z}_j \bar{\xi}_j(\pi) + P_c(\pi)f, \quad (2.44)$$

with a frame  $\{\xi_j(\pi) : j \in 1, \dots, \mathbf{n}\}$  as in Lemma 2.8. Notice that  $\langle J^{-1}\xi_j(\pi), P_c(\pi)f' \rangle = 0$ . We also have

$$P_c(p) = 1 - P_{N_g}(p) + \sum_{j=1}^{\mathbf{n}} i \langle J^{-1} \cdot, \bar{\xi}_j(p) \rangle \xi_j(p) + i \langle J^{-1} \cdot, \xi_j(p) \rangle \bar{\xi}_j(p). \quad (2.45)$$

The representation (2.44) is possible because of the following fact.

**Lemma 2.9.** *Under (H4)–(H7) and (H10), given  $p_0$  and for any fixed  $n \in \mathbb{N}$ , there exists  $a > 0$  such that for  $\pi \in \mathcal{P}$  with  $|\pi - p_0| < a$  the maps*

$$P_c(\pi)P_c(p_0) : L_c^2(p_0) \cap \Sigma_k(\mathbb{R}^3, \mathbb{R}^2) \rightarrow L_c^2(\pi) \cap \Sigma_k(\mathbb{R}^3, \mathbb{R}^2) \quad (2.46)$$

for all  $k \geq -n$  are isomorphisms.

*Proof.* Consider the composition  $P_c(p_0)P_c(\pi)P_c(p_0)$ . Then in  $L_c^2(p_0) \cap \Sigma_k$  its restriction equals

$$\begin{aligned} P_c(p_0)P_c(\pi)P_c(p_0) &= 1 + P_c(p_0)(P_{N_g}(\pi) - P_{N_g}(p_0))P_c(p_0) \\ &+ \sum_{j=1}^{\mathbf{n}} P_c(p_0) \left\{ (\xi_j(\pi) \langle \cdot, iJ^{-1}\bar{\xi}_j(\pi) \rangle - \xi_j(p_0) \langle \cdot, iJ^{-1}\bar{\xi}_j(p_0) \rangle) P_c(p_0) \right. \\ &\left. - (\bar{\xi}_j(\pi) \langle \cdot, iJ^{-1}\xi_j(\pi) \rangle - \bar{\xi}_j(p_0) \langle \cdot, iJ^{-1}\xi_j(p_0) \rangle) \right\} P_c(p_0). \end{aligned} \quad (2.47)$$

Using now the fact that  $\xi_j(\pi) \in C^\infty(\mathcal{P}, \Sigma_k)$ , we conclude that if  $|\pi - p_0| < a_k$  with  $a_k > 0$  sufficiently small, the operator in (2.47) is an isomorphism in  $L_c^2(p_0) \cap \Sigma_k$ . Similarly,  $P_c(\pi)P_c(p_0)P_c(\pi)$  is an isomorphism in  $L_c^2(\pi) \cap \Sigma_k$ . Finally, by the argument in Lemma 2.3 [1], we can pick a fixed  $a_k$  for all  $k \geq -n$ .  $\square$

### 3 Change of coordinate

To distinguish between an initial system of coordinates obtained from Lemma 2.5 and the further decomposition of  $r$  due to (2.44) and a "final" system of coordinates in Theorem 3.5 below, we will

add a "prime" to the initial coordinates, except for the pair  $(\Pi, w)$ . In particular we have functions  $(t, u) \rightarrow (\tau', \Pi, z', f')$ . In particular, with  $\aleph$  defined in (2.37), we have

$$\begin{aligned} (t, \pi, U) &\rightarrow f' \in C^l(\mathbb{R} \times \{|\pi - p_0| < a\} \times (\aleph \cap \Sigma_k), \Sigma_{k-l}) \text{ and} \\ (t, \pi, U) &\rightarrow z' \text{ smooth.} \end{aligned} \quad (3.1)$$

We introduce now appropriate symbols.

**Definition 3.1.** Let  $\mathcal{A}$  be a neighborhood of  $(p_0, p_0, 0, 0, 0)$  in the  $(\pi, \Pi, \varrho, z, f)$  space with  $(\pi, \Pi, \varrho) \in \mathbb{R}^{12}$ ,  $z \in \mathbb{C}^n$  and  $f \in L_c^2(p_0) \cap \Sigma_{-n}(\mathbb{R}^3, \mathbb{R}^2)$ . Let  $I \subset \mathbb{R}$  be an interval. Then we say that  $F \in C^m(I \times \mathcal{A}, \mathbb{R})$  is  $\mathcal{R}_{n,m}^{i,j}$  if there exists a  $C > 0$  and a smaller neighborhood  $\mathcal{A}'$  of  $(p_0, p_0, 0, 0, 0)$ , s.t. in  $I \times \mathcal{A}'$

$$|F(t, \pi, \Pi, \varrho, z, f)| \leq C(\|f\|_{\Sigma_{-n}} + |z|)^j(\|f\|_{\Sigma_{-n}}|z| + |\varrho| + |\Pi - \pi|)^i. \quad (3.2)$$

We will write also  $F = \mathcal{R}_{n,m}^{i,j}$  or  $F = \mathcal{R}_{n,m}^{i,j}(t, \pi, \Pi, \varrho, z, f)$ .

**Definition 3.2.** A  $T \in C^m(I \times \mathcal{A}, \Sigma_n(\mathbb{R}^3, \mathbb{R}^2))$ , with  $I$  and  $\mathcal{A}$  like above, is  $\mathbf{S}_{n,m}^{i,j}$  and we write as above  $T = \mathbf{S}_{n,m}^{i,j}$  or  $T = \mathbf{S}_{n,m}^{i,j}(t, \pi, \Pi, \varrho, z, f)$ , if there exists a  $C > 0$  and a smaller neighborhood  $\mathcal{A}'$  of  $(p_0, p_0, 0, 0)$ , s.t. in  $\mathcal{A}'$

$$\|T(t, \pi, \Pi, \varrho, z, f)\|_{\Sigma_n} \leq C(\|f\|_{\Sigma_{-n}} + |z|)^j(\|f\|_{\Sigma_{-n}}|z| + |\varrho| + |\Pi - \pi|)^i. \quad (3.3)$$

Notice that in the coordinates  $u \rightarrow (\tau, \Pi, z, f)$  introduced using (2.36) and (2.44) (and omitting the "primes"), we have we have  $p_j = \Pi_j - \varrho_j + \mathcal{R}_{n,m}^{0,2}(\pi, \Pi, \varrho, z, f)$  with  $\varrho = \Pi(f)$ . Then we have

$$\begin{aligned} U &= e^{J\tau \cdot \diamond} \Phi_p + \sum_{j=1}^n z_j e^{J\tau \cdot \diamond} P(p) \xi_j(\pi) + \sum_{j=1}^n \bar{z}_j e^{J\tau \cdot \diamond} P(p) \bar{\xi}_j(\pi) + e^{J\tau \cdot \diamond} P(p) P_c(\pi) f \\ &= \mathbf{S}_{n,m}^{0,0}(\pi, \Pi, \varrho, z, f) + \mathbf{S}_{n,m}^{0,1}(\pi, \Pi, \varrho, z, f) + e^{J\tau \cdot \diamond} P(p) P_c(\pi) f \end{aligned} \quad (3.4)$$

for arbitrary  $(n, m)$  and for  $\varrho = \Pi(f)$ .

We introduce now

$$K_0(\pi, U) := \mathbf{E}_0(U) - \mathbf{E}_0(\Phi_\pi) + \lambda(p(U)) \cdot (\Pi(U) - \pi). \quad (3.5)$$

**Definition 3.3** (Normal Forms). A function  $Z(z, f, \varrho, \pi, \Pi)$  is in normal form if  $Z = Z_0 + Z_1$  where  $Z_0$  and  $Z_1$  are finite sums of the following type:

$$Z_1 = i \sum_{\mathbf{e}(\omega(\pi)) \cdot (\mu - \nu) \in \sigma_e(\mathcal{H}_\pi)} z^\mu \bar{z}^\nu \langle JG_{\mu\nu}(\pi, \Pi, \varrho), f \rangle \quad (3.6)$$

where the vector  $\mathbf{e}(\omega)$  is introduced in (H8) and where  $G_{\mu\nu}(\cdot, \pi, \Pi, \varrho) \in C^m(\tilde{U}, \Sigma_k(\mathbb{R}^3, \mathbb{C}^2))$  for fixed  $k, m \in \mathbb{N}$ , with  $\tilde{U} = \{p : |p - p_0| < a\}^2 \times U$  and  $U \subseteq \mathbb{R}^4$  a neighborhood of 0;

$$Z_0 = \sum_{\mathbf{e}(\omega(\pi)) \cdot (\mu - \nu) = 0} g_{\mu\nu}(\pi, \Pi, \varrho) z^\mu \bar{z}^\nu \quad (3.7)$$

and  $g_{\mu\nu} \in C^m(\tilde{U}, \mathbb{C})$ . We assume furthermore that  $Z_0$  and  $Z_1$  are real valued for  $f = \bar{f}$ , and hence their coefficients satisfy the following symmetries:  $\bar{g}_{\mu\nu} = g_{\nu\mu}$  and  $\bar{G}_{\mu\nu} = -G_{\nu\mu}$ .

We have the following elementary fact, proved in Remark 5.6 [5], which tells us that the pairs  $(\mu, \nu)$  in Def. 3.3 in the case of the polynomials which interest us, do not depend on  $\pi$ .



**Lemma 3.4.** Consider the  $N$  in (H7). Then there exists an  $\delta_0 > 0$  such that for  $|\pi - p_0| < \delta_0$  the following are independent of  $\pi$ :

- (1) the formula  $\omega(\pi) \cdot (\mu - \nu) \in \sigma_e(\mathcal{H}_\pi)$  for  $|\mu + \nu| \leq N + 1$ ;
- (2) the equality  $\mathbf{e}(\omega(\pi)) \cdot (\mu - \nu) = 0$  for  $|\mu + \nu| \leq 2N + 2$ .

□

The main result of [1], see also [4], is the following.

**Theorem 3.5.** There is an  $\varepsilon_3 > 0$  and a map

$$\begin{aligned}\tau' &= \tau + \mathcal{T}(\pi, \Pi, \Pi(f), z, f), & \Pi' &= \Pi, \\ z' &= z + \mathcal{Z}(\pi, \Pi, \Pi(f), z, f), \\ f' &= e^{Jq(\pi, \Pi, \Pi(f), z, f) \cdot \diamond} (f + \mathbf{S}(\pi, \Pi, \Pi(f), z, f))\end{aligned}\tag{3.8}$$

which is in

$$C^1(\mathbb{R}^4 \times B_{\mathbb{C}^n}(\varepsilon_3) \times (\Sigma_2 \cap B_{H^1}(\varepsilon_3) \cap L_c^2(p_0)), \mathbb{R}^4 \times \mathbb{C}^n \times (H^1 \cap L_c^2(p_0)))\tag{3.9}$$

$$C^0(\mathbb{R}^4 \times B_{\mathbb{C}^n}(\varepsilon_3) \times (B_{H^1}(\varepsilon_3) \cap L_c^2(p_0)), \mathbb{R}^4 \times \mathbb{C}^n \times (H^1 \cap L_c^2(p_0)))\tag{3.10}$$

$$C^0(\mathbb{R}^4 \times B_{\mathbb{C}^n}(\varepsilon_3) \times (\Sigma_2 \cap B_{H^1}(\varepsilon_3) \cap L_c^2(p_0)), \mathbb{R}^4 \times \mathbb{C}^n \times (\Sigma_2 \cap L_c^2(p_0))),\tag{3.11}$$

in the sense of (3.10)–(3.11) is a homeomorphism in its image with the image containing  $\mathbb{R}^4 \times B_{\mathbb{C}^n}(\frac{\varepsilon_3}{2}) \times (B_{H^1}(\frac{\varepsilon_3}{2}) \cap L_c^2(p_0))$  in the case of (3.10) (resp.  $\mathbb{R}^4 \times B_{\mathbb{C}^n}(\frac{\varepsilon_3}{2}) \times (\Sigma_2 \cap B_{H^1}(\frac{\varepsilon_3}{2}) \cap L_c^2(p_0))$  in the case of (3.11)) and such that in the new variables  $(\tau, \Pi, z, f)$  we have

$$K_0(\pi, U) = \psi(\pi, \Pi, \Pi(f)) + H'_2 + Z_0 + Z_1 + \mathcal{R} + \mathbf{E}_P(f)\tag{3.12}$$

where we have for  $k, m \in \mathbb{N}$  preassigned and arbitrarily large:

- (1)  $\psi$  is smooth and with  $\psi(\Pi, \Pi, \Pi(f)) = O(|\Pi(f)|^2)$  near 0.
- (2)  $H'_2 = \sum_{j=1}^n a_j(\pi, \Pi, \Pi(f)) |z_j|^2 - \frac{i}{2} \langle J^{-1} \mathcal{H}_\pi P_c(\pi) f, P_c(\pi) f \rangle$  where we have  $a_j(\pi, \Pi, \Pi(f)) = \mathbf{e}_j + O(|\Pi - \pi| + |\Pi(f)|)$ .
- (3)  $Z_0$  is in normal form as in (3.7) with  $|\mu + \nu| \leq 2N + 2$ .
- (4)  $Z_1$  is in normal form as in (3.6) with  $|\mu + \nu| \leq N + 1$ .
- (5) We have  $\mathcal{R} \in C^1$  with  $\|\nabla_f \mathcal{R}\|_{\Sigma_k} \leq C(|z|^{N+2} + \|f\|_{L^{2,-k}} \|f\|_{H^1})$  near the origin and similarly with  $|\nabla_z \mathcal{R}| \leq C(|z|^{2N+2} + \|f\|_{L^{2,-k}} \|f\|_{H^1})$ .
- (6) The functions  $q, \mathcal{T}_j, \mathcal{Z}_j$  in (3.8) are of type  $\mathcal{R}_{k,m}^{1,2}$ , see Def. 3.2 above.
- (7) The function  $\mathbf{S}$  in (3.8) is of type  $\mathbf{S}_{k,m}^{1,1}$ , see Def. 3.1 above.
- (8) For each fixed  $\pi$ , the pullback of  $\Omega = \langle J^{-1}, \cdot \rangle$  by means of the map (3.9) equals

$$\Omega^{(\pi)} = \sum_{j=1}^4 d\tau_j \wedge d\Pi_j + i \sum_{j=1}^n dz_j \wedge d\bar{z}_j + \Omega(P_c(\pi)df, P_c(\pi)df).\tag{3.13}$$

□

Here we skip the proof of Theorem 3.5 which is a minor modification of the arguments in [1]. It is important to observe that here and in [4] the role of the fixed  $p_0$  in the normal forms argument is taken by the time varying  $\pi(t)$ , with  $\pi(t) = \Pi(u(t))$  in [4] and by  $\pi(t) = \Pi(U[t, u(t)])$  in here.

It is important to check the dependence of various coordinates on the variables  $(\pi, u)$  and  $(\pi, U)$ .

**Lemma 3.6.** *Consider the variables  $(\tau, z, f)$  in Theorem 3.5. Set  $\varrho = \Pi(f)$ . Then, for any preassigned pair  $(k, m)$ , they have the following dependence on  $(\pi, U)$ . Then there exists an  $a > 0$  such that for  $B_{\mathbb{R}^4}(\kappa, a)$  (resp.  $B_{H^1}(\phi_{\omega_1}, a)$ ), and for  $\mathcal{V} = \cup_{\tau \in \mathbb{R}^4} e^{J\tau \cdot \diamond} B_{H^1}(\phi_{\omega_1}, a) = \cup_{\tau \in \mathbb{R}^4} B_{H^1}(e^{J\tau \cdot \diamond} \phi_{\omega_1}, a)$ , we have:*

- (1)  $\tau(\pi, U), \varrho(\pi, U) \in C^1(B_{\mathbb{R}^4}(\kappa, a) \times \mathcal{V}, \mathbb{R}^4)$ ;
- (2)  $z(\pi, U) \in C^1(B_{\mathbb{R}^4}(\kappa, a) \times \mathcal{V}, \mathbb{R}^4)$ ;
- (3)  $f(\pi, U) \in C^i(B_{\mathbb{R}^4}(\kappa, a) \times \mathcal{V}, H^{1-i})$  for  $i = 0, 1$ .

□

For this and more see Lemmas 6.1–6.2 [4].

Notice that since we initially are assuming (1.18), that is  $u_0 \in \Sigma_2$ , we have  $u(t) \in \Sigma_2$  and so also  $U(t) := U[t, u(t)] \in \Sigma_2$  and that for  $t \in [0, T]$  for some  $T > 0$  we have that the coordinates  $(\tau(t, U(t)), p(t, U(t)), z(t, U(t)), f(t, U(t)))$  belong to the image of the maps in (3.9)–(3.11). Notice that later we will drop (1.18) and assume only  $u_0 \in H^1$ .

## 4 Equations

Equation (1.12) can be written as  $u_t = J\nabla \mathbf{E}(u) = X_{\mathbf{E}}(u) = \{u, \mathbf{E}\}$  where we have the following notions:

- the exterior differential  $dF(u)$  of a Frechét differentiable function  $F$  defined in an open subset of  $H^1$ ;
- the gradient  $\nabla F(u)$  defined by  $\langle \nabla F(u), X \rangle = dF(u)X$ ;
- the symplectic form  $\Omega(X, Y) := \langle J^{-1}X, Y \rangle$ ;
- the Hamiltonian vectorfield  $X_F$  of  $F$  with respect to a  $\Omega$  defined by  $\Omega(X_F, Y) = dFY$ , that is  $X_F = J\nabla F$ ;
- the Poisson bracket of two scalar functions  $\{F, G\} := dFX_G$ ,
- if  $\mathcal{G}$  has values in a given Banach space  $\mathbb{E}$  and is Frechét differentiable with Frechét derivative  $d\mathcal{G}$ , and if  $G$  is a scalar valued function, then we set  $\{\mathcal{G}, G\} := d\mathcal{G}X_G$ .

We have introduced in Lemma 2.5 the functional  $B(\varepsilon_2) \ni u \rightarrow U[t, u]$  for the set  $B(\varepsilon_2)$  defined in (2.14). The following elementary lemma relates Poisson brackets associated to  $\Omega$  in the  $u$  and the  $U$  space.

**Lemma 4.1.** Consider the map  $B(\varepsilon_2) \ni (t, u) \rightarrow U = U[t, u]$  and fix  $t$ . Then, given a differentiable function  $u \rightarrow \mathcal{E}(u)$  and a differentiable function  $U \rightarrow F(U)$ , we have, for  $\tilde{Q} := Q[t, u]$ , see (2.32) and (3.5),

$$\{F(U[t, u]), \mathcal{E}\} = d_U F(U[t, u]) J \nabla_u \mathcal{E}(u) - \sum_{k=1}^2 \{w_k, \mathcal{E}\} d_U F(U[t, u]) \partial_{w_k} \tilde{Q}. \quad (4.1)$$

For  $\mathcal{E}(u) = G(U[t, u])$ , summing on repeated indices we have

$$\begin{aligned} \{F(U[t, u]), G(U[t, u])\} &= dF(U[t, u]) J \nabla_U G(U[t, u]) - dw_j (J \nabla_U G(U[t, u])) dF(U[t, u]) \partial_{w_j} \tilde{Q} \\ &\quad - \langle \nabla G(U[t, u]), \partial_{w_k} \tilde{Q} \rangle dF(U[t, u]) \partial_{w_k} \tilde{Q} + \langle \nabla_U G(U[t, u]), \partial_{w_k} \tilde{Q} \rangle \{w_j, w_k\} dF(U[t, u]) \partial_{w_j} \tilde{Q}. \end{aligned} \quad (4.2)$$

*Proof.* We have, summing on repeated indices

$$\begin{aligned} \{F(U[t, u]), \mathcal{E}\} &= d_u F(U[t, u]) J \nabla_u \mathcal{E} \text{ with} \\ d_u F(U[t, u]) &= d_U F(U[t, u]) d_u U[t, u] = d_U F(U[t, u]) - d_U F(U[t, u]) \left( \partial_{w_k} \tilde{Q} \right) d_u w_k. \end{aligned}$$

This yields (4.1). (4.2) follows for  $\mathcal{E}(u) = G(U[t, u])$  if we use also  $\nabla_u G(U[t, u]) = \nabla_U G(U[t, u]) - \langle \nabla_U G(U[t, u]), \partial_{w_j} \tilde{Q} \rangle \nabla_u w_j$ .  $\square$

The following lemma will play an important role later.

**Lemma 4.2.** Set  $\mathcal{E} = \mathbf{E}$  in (4.1), with  $\mathbf{E}$  the energy in (2.1). Consider a solution  $u = u(t)$  of  $u_t = J \nabla \mathbf{E}(u)$  with  $(t, u(t)) \in B(\varepsilon_2)$  over an interval of time. Then we have

$$\begin{aligned} \frac{d}{dt} F(U[t, u]) &= d_U F(U) J \nabla_U \mathbf{E}(U) + d_U F(U) \mathbf{A} \\ \mathbf{A} &:= J \mathbf{f}(U, \tilde{Q}) - (\dot{w}_1 - E_w w_2) \partial_{w_1} \tilde{Q} - (\dot{w}_2 + E_w w_1) \partial_{w_2} \tilde{Q} \end{aligned} \quad (4.3)$$

where for

$$\tilde{\beta}(u) := \beta(|u|^2)u \quad (4.4)$$

we have

$$\mathbf{f}(U, \tilde{Q}) := \int_{[0,1]^2} \partial_t \partial_s [\tilde{\beta}(tU + s\tilde{Q})] dtds. \quad (4.5)$$

*Proof.* It is elementary that, summing on repeated indices,

$$\partial_t U[t, u] = -\partial_t \tilde{Q}[t, u] = -J \frac{|\mathbf{v}|^2}{4} \tilde{Q} + J e^{J \Theta \cdot \diamond} \mathbf{v}_a \diamond_a Q_w - \partial_t w_i \frac{\partial}{\partial w_i} \tilde{Q},$$

where  $\tilde{Q} = e^{J \Theta \cdot \diamond} Q_w$ , see (2.32). By (4.1) for  $\mathcal{E} = \mathbf{E}$  and by  $\dot{w}_i = \frac{d}{dt} w_i = \partial_t w_i + \{w_i, \mathbf{E}\}$ , we get

$$\begin{aligned} \frac{d}{dt} F(U[t, u]) &= d_U F(U) \partial_t U[t, u] + \{F(U[t, u]), \mathbf{E}\} = d_U F(U[t, u]) J \nabla_u \mathbf{E}(u) \\ &\quad - \dot{w}_i d_U F(U) \partial_{w_i} \tilde{Q} - d_U F(U) J \left( \frac{|\mathbf{v}|^2}{4} \tilde{Q} - J e^{J \Theta \cdot \diamond} \mathbf{v}_a \diamond_a Q_w \right). \end{aligned} \quad (4.6)$$

We have  $\nabla_u \mathbf{E} = -\Delta u + V(\cdot + \mathbf{v}t + y_0)u + \beta(|u|^2)u$  and  $\beta(|u|^2)u = \beta(|\tilde{Q}|^2)\tilde{Q} + \beta(|U|^2)U + \mathbf{f}(U, \tilde{Q})$ . We expand

$$\nabla \mathbf{E}(u) = \nabla \mathbf{E}(U) + \nabla \mathbf{E}(\tilde{Q}) + \mathbf{f}(U, \tilde{Q}). \quad (4.7)$$

We have

$$\nabla \mathbf{E}(\tilde{Q}) = e^{J\frac{\mathbf{v} \cdot \mathbf{x}}{2} + J\frac{|\mathbf{v}|^2}{4}t} \left( \nabla \mathbf{E}(Q_w(\cdot + \mathbf{v}t + y_0)) - \mathbf{v} \cdot \diamond Q_w(\cdot + \mathbf{v}t + y_0) + \frac{\mathbf{v}^2}{4} Q_w(\cdot + \mathbf{v}t + y_0) \right). \quad (4.8)$$

By (1.4) we have  $\nabla \mathbf{E}(Q_w(\cdot + \mathbf{v}t + y_0)) = E_w Q_w(\cdot + \mathbf{v}t + y_0)$ . So various terms cancel and we get

$$\frac{d}{dt} F(U[t, u]) = d_U F(U) J[\nabla_U \mathbf{E}(U) + \mathbf{f}(U, \tilde{Q})] - \dot{w}_i d_U F(U) \partial_{w_i} \tilde{Q} + E_w d_U F(U) J \tilde{Q}.$$

We finally obtain (4.3) because by (1.5) we have  $J\tilde{Q} = w_2 \partial_{w_1} \tilde{Q} - w_1 \partial_{w_2} \tilde{Q}$ .  $\square$

Using the notation of Lemma 4.2 and of Lemma 2.5 we get the following elementary lemma.

**Lemma 4.3.** *We have, in the notation of Lemma 4.2 and of Lemma 2.5,*

$$\tilde{\beta}(u) = \tilde{\beta}(e^{J\tau \cdot \diamond} \Phi_p) + \tilde{\beta}(\tilde{Q} + e^{J\tau \cdot \diamond} P(p)P(\pi)r') + \mathbf{f}(e^{J\tau \cdot \diamond} \Phi_p, \tilde{Q} + e^{J\tau \cdot \diamond} P(p)P(\pi)r'). \quad (4.9)$$

$\square$

#### 4.1 Set up for the discrete mode associated to the potential $V$

We start stating following elementary and standard fact.

**Lemma 4.4.** *Consider the function  $Q_w$  of Prop. 1.1. Consider the operator*

$$\begin{aligned} \tilde{\mathfrak{h}} := & -\Delta + \mathbf{v} \cdot \diamond + 4^{-1}|\mathbf{v}|^2 + V(\cdot + \mathbf{v}t + y_0) - E_w \\ & + \begin{pmatrix} \beta(|\tilde{Q}|^2) + 2\beta'(|\tilde{Q}|^2) \operatorname{Re} \tilde{Q} & 2\beta'(|\tilde{Q}|^2) \operatorname{Re} \tilde{Q} \\ 2\beta'(|\tilde{Q}|^2) \operatorname{Im} \tilde{Q} & \beta(|\tilde{Q}|^2) + 2\beta'(|\tilde{Q}|^2) \operatorname{Im} \tilde{Q} \end{pmatrix}. \end{aligned} \quad (4.10)$$

Then we have the following equality:

$$\tilde{\mathfrak{h}} \frac{\partial}{\partial w_i} \tilde{Q} = \left( \frac{\partial}{\partial w_i} E_w \right) \tilde{Q}. \quad (4.11)$$

$\square$

We write equation (1.12) with a special view at the evolution of the variable  $w$ . Here we assume that for a certain interval of time we have  $(t, u(t)) \in B(\varepsilon_2)$  with  $B(\varepsilon_2)$  as in Proposition 2.1. Substituting (2.32) in (1.12) and using twice an expansion like (4.7) we get for  $\eta := e^{J\tau' \cdot \diamond} P(p')P(\pi)r'$ ,

$$\begin{aligned} & \partial_t (e^{J\tau' \cdot \diamond} \Phi_{p'} + \eta) + \dot{w}_i \partial_{w_i} \tilde{Q} + J4^{-1} \mathbf{v}^2 \tilde{Q} + e^{J\Theta \cdot \diamond} \mathbf{v} \cdot \nabla Q_w \\ & = J\nabla \mathbf{E}(e^{J\tau' \cdot \diamond} \Phi_{p'}) + J\nabla \mathbf{E}(\tilde{Q}) + J\nabla \mathbf{E}(\eta) + J\mathbf{f}(\eta, \tilde{Q}) + J\mathbf{f}(\eta + \tilde{Q}, e^{J\tau' \cdot \diamond} \Phi_{p'}). \end{aligned}$$

We substitute  $\nabla \mathbf{E}(\tilde{Q})$  using (4.8), we use (1.5), that is  $J\tilde{Q} = w_2 \partial_{w_1} \tilde{Q} - w_1 \partial_{w_2} \tilde{Q}$ , and

$$\mathbf{f}(\eta, \tilde{Q}) = \partial_s \tilde{\beta}(s\eta + \tilde{Q})|_{s=0} + \int_{[0,1]^3} \partial_s \partial_{s_1} \partial_{s_2} \tilde{\beta}(ss_2\eta + s_1\tilde{Q}) ds ds_1 ds_2.$$

We then get the following equation:

$$\begin{aligned} & (\dot{w}_1 - E_w w_2) \partial_{w_1} \tilde{Q} + (\dot{w}_2 + E_w w_1) \partial_{w_2} \tilde{Q} - J\tilde{\mathfrak{h}}\eta = -\partial_t (e^{J\tau' \cdot \diamond} \Phi_{p'} + \eta) + J\nabla \mathbf{E}(e^{J\tau' \cdot \diamond} \Phi_{p'}) \\ & + J\nabla \mathbf{E}_P(\eta) + J\mathbf{f}(\eta + \tilde{Q}, e^{J\tau' \cdot \diamond} \Phi_{p'}) + \int_{[0,1]^3} \partial_s \partial_{s_1} \partial_{s_2} \tilde{\beta}(ss_2\eta + s_1\tilde{Q}) ds ds_1 ds_2. \end{aligned} \quad (4.12)$$

Notice now that  $\langle \eta, J\partial_{w_i}\tilde{Q} \rangle = 0$  for  $i = 1, 2$  implies  $\langle \eta, \tilde{Q} \rangle = 0$ . So, see [10],

$$\langle J\tilde{\eta}\eta, J\frac{\partial}{\partial w_i}\tilde{Q} \rangle = \langle \eta, \tilde{\eta}\frac{\partial}{\partial w_i}\tilde{Q} \rangle = \langle \eta, \tilde{Q} \rangle \frac{\partial}{\partial w_i}E_w = 0.$$

Applying  $\langle \cdot, J\partial_{w_i}\tilde{Q} \rangle$  to (4.12) and using the above remarks and (2.26) we get

$$(1 + O(w^2)) \begin{pmatrix} \dot{w}_1 - E_w w_2 \\ -(\dot{w}_2 + E_w w_1) \end{pmatrix} = \begin{pmatrix} \langle \text{rhs}(4.12), J\partial_{w_1}\tilde{Q} \rangle \\ \langle \text{rhs}(4.12), J\partial_{w_2}\tilde{Q} \rangle \end{pmatrix}. \quad (4.13)$$

In the sequel we will use the following lemma.

**Lemma 4.5.** (2.33) implies for  $i = 1, 2$

$$\begin{aligned} \langle e^{J\tau' \cdot \diamond} f', e^{J\frac{1}{2}\mathbf{v} \cdot x} \phi_0(\cdot + \mathbf{v}t + y_0) \rangle &= \sum_i \langle e^{J\tau' \cdot \diamond} \mathbf{S}_{k,m}^{0,1}(i), e^{J\frac{1}{2}\mathbf{v} \cdot x} \partial_{w_i} Q_w(\cdot + \mathbf{v}t + y_0) \rangle \\ &- \cos(4^{-1}t|\mathbf{v}|^2) \langle e^{J\tau' \cdot \diamond} f', e^{J(\frac{1}{2}\mathbf{v} \cdot x + \frac{t}{4}|\mathbf{v}|^2)} J\partial_{w_2} q_w(\cdot + \mathbf{v}t + y_0) \rangle \\ &+ \sin(4^{-1}t|\mathbf{v}|^2) \langle e^{J\tau' \cdot \diamond} f', e^{J(\frac{1}{2}\mathbf{v} \cdot x + \frac{t}{4}|\mathbf{v}|^2)} J\partial_{w_1} q_w(\cdot + \mathbf{v}t + y_0) \rangle \end{aligned} \quad (4.14)$$

where the  $\mathbf{S}_{k,m}^{0,1}(i)$  are  $S_{k,m}^{0,1}(t, \pi, \Pi, \Pi(f'), z', f')$  symbols in the sense of Def. 3.3. We have similarly

$$\begin{aligned} \langle e^{J\tau' \cdot \diamond} f', J e^{J\frac{1}{2}\mathbf{v} \cdot x} \phi_0(\cdot + \mathbf{v}t + y_0) \rangle &= \sum_i \langle e^{J\tau' \cdot \diamond} \mathbf{S}_{k,m}^{0,1}(i), e^{J\frac{1}{2}\mathbf{v} \cdot x} \partial_{w_i} Q_w(\cdot + \mathbf{v}t + y_0) \rangle \\ &+ \sin(4^{-1}t|\mathbf{v}|^2) \langle e^{J\tau' \cdot \diamond} f', e^{J(\frac{1}{2}\mathbf{v} \cdot x + \frac{t}{4}|\mathbf{v}|^2)} J\partial_{w_2} q_w(\cdot + \mathbf{v}t + y_0) \rangle \\ &+ \cos(4^{-1}t|\mathbf{v}|^2) \langle e^{J\tau' \cdot \diamond} f', e^{J(\frac{1}{2}\mathbf{v} \cdot x + \frac{t}{4}|\mathbf{v}|^2)} J\partial_{w_1} q_w(\cdot + \mathbf{v}t + y_0) \rangle. \end{aligned} \quad (4.15)$$

*Proof.* The starting point is (2.33), that is  $\langle e^{J\tau' \cdot \diamond} P(p')P(\pi)r', J e^{J\Theta \cdot \diamond} \partial_{w_i} Q_w \rangle = 0$ . We first have  $P(p')P(\pi)r' = P(\pi)r' + S_{k,m}^{0,1}(\pi, \Pi, \Pi(f'), z', f')$ . We next use (2.44) to get  $P(\pi)r' = P_c(\pi)f' + S_{k,m}^{0,1} = f' + S_{k,m}^{0,1}$ . We therefore get

$$\langle e^{J\tau' \cdot \diamond} f', J e^{J(\frac{1}{2}\mathbf{v} \cdot x + \frac{t}{4}|\mathbf{v}|^2)} \partial_{w_i} Q_w(\cdot + \mathbf{v}t + y_0) \rangle = \langle e^{J\tau' \cdot \diamond} \mathbf{S}_{k,m}^{0,1}, J e^{J\frac{1}{2}\mathbf{v} \cdot x} \partial_{w_i} Q_w(\cdot + \mathbf{v}t + y_0) \rangle.$$

Now recall from Prop. 1.1 that  $\partial_{w_1} Q_w = \phi_0 + \partial_{w_1} q_w$  and  $\partial_{w_2} Q_w = -J\phi_0 + \partial_{w_2} q_w$ . Use also

$$\begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} e^{J\alpha} \phi_0 \\ J e^{J\alpha} \phi_0 \end{pmatrix} = \begin{pmatrix} \phi_0 \\ J\phi_0 \end{pmatrix}.$$

This yields the desired formulas (4.14)–(4.15).  $\square$

## 4.2 Set up for $\Pi$ , $\tau$ , $z$ and $f$

Given a function  $F(\pi, u)$  and if  $\pi = \pi(t)$  has a given evolution in  $t$ , we have

$$\frac{d}{dt} F(\pi, u) = \partial_\pi F(\pi, u) \cdot \dot{\pi} + \{F(\pi, u), \mathbf{E}(u)\}. \quad (4.16)$$

By continuity, by Proposition 2.1 we know that there exists a  $T > 0$  and an interval  $I_T = [0, T]$  s.t.  $(t, u(t)) \in B(\varepsilon_2)$  for  $t \in I_T$  for all  $u_0$  in Theor. 1.4 if  $\varepsilon_0$  small enough. Then the representation (2.32) is true for  $u(t)$  with  $t \in I_T$ . We set  $\pi(t) = \Pi(U[t, u(t)])$ .

The functions  $\Pi_j(U)$  are invariant by the changes of variables in (3.8), and so in particular do not depend on the parameter  $\pi$ . So we have by (4.3)

$$\dot{\Pi}_j = \langle \nabla_U \Pi_j, J \nabla_U \mathbf{E}(U) \rangle + d_U \Pi_j \mathbf{A}.$$

Then we have

$$\begin{aligned} \dot{\Pi}_4 &= d_U \Pi_4 \mathbf{A} \text{ and for } a \leq 3 \\ \dot{\Pi}_a &= -\langle V(\cdot + \mathbf{v}t + y_0) \partial_{x_a} [\Phi_{p'} + P(p')P(\pi)r'], \Phi_{p'} + P(p')P(\pi)r' \rangle + d_U \Pi_a \mathbf{A}. \end{aligned} \quad (4.17)$$

We have  $\tau' = \tau'(U[t, u])$ . In particular, by Lemma 4.2 we have

$$\dot{D}'_a = \langle \nabla_U D'_a, J \nabla_U \mathbf{E}(U) \rangle + d_U D'_a \mathbf{A}.$$

We have  $D' = D + \mathcal{R}_{k,m}^{0,2}$  by Theorem 3.5. Then by Claim 8 in Theorem 3.5, see also Lemma 2.8 [4], we have

$$\begin{aligned} \dot{D}'_a - v'_a &= \partial_{\Pi_a} K_0 + \frac{1}{2} \partial_{\Pi_a} \langle V(\cdot + \mathbf{v}t + y_0) (\Phi_{p'} + P(p')P(\pi)r'), \Phi_{p'} + P(p')P(\pi)r' \rangle \\ &\quad + \{\mathcal{R}_{k,m}^{0,2}, K_0\} + 2^{-1} \{\mathcal{R}_{k,m}^{0,2}, \langle V(\cdot + \mathbf{v}t + y_0)U, U \rangle\} + d_U D'_a \mathbf{A}. \end{aligned} \quad (4.18)$$

We similarly have

$$\begin{aligned} \dot{\vartheta}' - \omega' - 2^{-1}(v')^2 &= \{\mathcal{R}_{k,m}^{0,2}, K_0\} + 2^{-1} \{\mathcal{R}_{k,m}^{0,2}, \langle V(\cdot + \mathbf{v}t + y_0)U, U \rangle\} - \\ &\quad \partial_{\Pi_4} K_0 - 2^{-1} \partial_{\Pi_4} \langle V(\cdot + \mathbf{v}t + y_0) (\Phi_{p'} + P(p')P(\pi)r'), \Phi_{p'} + P(p')P(\pi)r' \rangle + d_U \vartheta' \mathbf{A}. \end{aligned} \quad (4.19)$$

We have

$$\dot{z}_j = -i \partial_{z_j} K_0 + \dot{\Pi} \cdot \partial_{\pi} z_j - i 2^{-1} \partial_{z_j} \langle V(\cdot + \mathbf{v}t + y_0)U, U \rangle + d_U z_j \mathbf{A}. \quad (4.20)$$

We have

$$\begin{aligned} \dot{f} &= \dot{\Pi} \cdot \partial_{\pi} f + (P_c(p_0)P_c(\pi)P_c(p_0))^{-1} J \nabla_f K' + d_U f \mathbf{A}, \\ \nabla_f K' &:= \nabla_f K_0 + 2^{-1} \nabla_f \langle V(\cdot + \mathbf{v}t + y_0)U, U \rangle. \end{aligned} \quad (4.21)$$

We couple equations (4.17), (4.18), (4.19), (4.20) and (4.21) with (4.13).

## 5 Bootstrapping

As in [4], Theorem 1.4 follows from the following Theorem.

**Theorem 5.1.** *Consider the constants  $0 < \epsilon < \epsilon_0$  of Theorem 1.4. Then there is a fixed and we have  $C > 0$  such that we have  $(t, u(t)) \in B(\epsilon_2)$  for all  $t \in I = [0, \infty)$*

$$\|f\|_{L_t^p(I, W_x^{1,q})} \leq C\epsilon \text{ for all admissible pairs } (p, q), \quad (5.1)$$

$$\|z^\mu\|_{L_t^2(I)} \leq C\epsilon \text{ for all multi indices } \mu \text{ with } \mathbf{e} \cdot \mu > \omega_0, \quad (5.2)$$

$$\|z_j\|_{W_t^{1,\infty}(I)} \leq C\epsilon \text{ for all } j \in \{1, \dots, \mathbf{n}\} \quad (5.3)$$

$$\|\omega' - \omega_0\|_{L_t^\infty(I)} \leq C\epsilon, \quad \|v' - v_0\|_{L_t^\infty(I)} \leq C\epsilon \quad (5.4)$$

$$\|(\dot{w}_1 - E_w w_2, \dot{w}_2 + E_w w_1)\|_{L_t^\infty(I) \cap L_t^1(I)} \leq C\epsilon. \quad (5.5)$$

Furthermore, there exist  $\omega_+$  and  $v_+$  such that

$$\lim_{t \rightarrow +\infty} \omega'(t) = \omega_+, \quad \lim_{t \rightarrow +\infty} v'(t) = v_+ \quad (5.6)$$

$$\lim_{t \rightarrow +\infty} \dot{D}'(t) = v_+, \quad \lim_{t \rightarrow +\infty} \dot{\vartheta}'(t) = \omega_+ + 4^{-1}v_+^2 \quad (5.7)$$

$$\lim_{t \rightarrow +\infty} z(t) = 0. \quad (5.8)$$

Theorem 5.1 will be obtained as a consequence of the following Proposition.

**Proposition 5.2.** *Consider the constants  $0 < \epsilon < \epsilon_0$  of Theorem 1.4. There exist a constant  $c_0 > 0$  such that for any  $C_0 > c_0$  there is an  $\epsilon_0 > 0$  such that if  $(t, u(t)) \in B(\epsilon_2)$  for all  $t \in I = [0, T]$  for some  $T > 0$  and the inequalities (5.1)–(5.5) hold for this  $I$  and for  $C = C_0$ , and if furthermore for  $t \in I$*

$$\|\dot{D}' - v'\|_{L^1(0,t)} < C\epsilon\langle t \rangle, \quad (5.9)$$

$$\|p' - p_0\|_{L^\infty(I)} < C\epsilon, \quad (5.10)$$

*then in fact for  $I = [0, T]$  the inequalities (5.1)–(5.5) hold for  $C = C_0/2$  and the inequalities (5.9)–(5.10) hold for  $C = c$  with  $c$  a fixed constant.*

The proof of Theorem 5.1 and of Proposition 5.2 is very similar to the proof of Theorem 6.6 and Proposition 6.7 in [4].

## 5.1 Proof that Proposition 5.2 implies Theorem 5.1

We start with the following lemma from [4].

**Lemma 5.3.** *Assume the hypotheses of Proposition 5.2 and consider a fixed  $S_{2k,0}^{0,0}$  where  $k > 3$  and a fixed  $q \in \mathcal{S}(\mathbb{R}^3)$ . Then for  $\epsilon_0$  small enough there exists a fixed constant  $c$  dependent on  $c_1$ ,  $S_{2k,0}^{0,0}$  and  $q$  s.t.*

$$\|q(\cdot + \mathbf{v}t + D' + y_0)S_{2k,0}^{0,0}\|_{L^1((0,T),L_x^p)} \leq c\epsilon \text{ for all } p \geq 1. \quad (5.11)$$

*Proof.* This is Lemma 7.3 [4] but we reproduce the proof partially. We have by  $k > 3$  and Sobolev embedding,

$$\|q(\cdot + \mathbf{v}t + D' + y_0)S_{2k,0}^{0,0}\|_{L_x^p} \leq C_{q,k}\|S_{2k,0}^{0,0}\|_{\Sigma_{2k}}\langle D'(t) + t\mathbf{v} + y_0 \rangle^{-k}.$$

Then for a fixed  $C = C_{q,k,S}$

$$\begin{aligned} \|q(\cdot + \mathbf{v}t + D' + y_0)S_{2k,0}^{0,0}\|_{L^1((0,T),L_x^p)} &\leq C\|\langle D'(s) + s\mathbf{v} + y_0 \rangle^{-k}\|_{L^1(0,T)}, \\ \|\langle D'(s) + s\mathbf{v} + y_0 \rangle^{-k}\|_{L^1(0,T)} &= \|\langle D'(0) + s\mathbf{v} + I(s) + y_0 \rangle^{-k}\|_{L^1(0,T)} \text{ where} \\ I(s) &:= sv_0 + \int_0^s (\dot{D}'(\tau) - v_0)d\tau, \quad |\dot{I}(s)| \leq 3C_0\epsilon, \end{aligned} \quad (5.12)$$

where  $|\dot{I}(s)| \leq 3C_0\epsilon$  follows by (5.9)–(5.10) and by  $|v_0| \lesssim \epsilon$ . Then (5.11) follows by Lemma 5.5 below.  $\square$

By  $D'(0) = (\tau_1(0, u_0), \tau_2(0, u_0), \tau_3(0, u_0))$  we get  $|D'(0)| < C\epsilon$  for fixed  $C$  by (1.15) and Proposition 2.1, see also the discussion at the end of Sect. 2.1. After Lemma 2.9 [4] the following is proved.

**Lemma 5.4.** For  $\varepsilon_0$  in (1.15) small enough we have

$$\sup_{\text{dist}_{S^2}(\vec{e}, \frac{\mathbf{v}}{|\mathbf{v}|}) \leq \varepsilon_1} \int_0^\infty (1 + |\mathbf{v}| \vec{e} t + D'(0) + y_0|^2)^{-1} dt < 10\epsilon. \quad (5.13)$$

□

We now prove the following lemma.

**Lemma 5.5.** For  $\varepsilon_0 > 0$  in (1.15) sufficiently small, we have for a fixed  $c$

$$\| \langle D'(s) + s\mathbf{v} + y_0 \rangle^{-k} \|_{L^1(0,T)} < c\epsilon. \quad (5.14)$$

*Proof.* Set  $d_0 := D'(0) + y_0$ . If  $|\langle d_0 + s\mathbf{v} \rangle| \geq 6C_0\epsilon s$  for all  $s > 0$ , then since  $|I(s)| \leq 3C_0\epsilon s$  by (5.12) we get  $\langle D'(s) + s\mathbf{v} + y_0 \rangle \sim \langle d_0 + s\mathbf{v} \rangle$  with fixed constants for all  $s > 0$ . Then (5.14) follows from (5.13).

Suppose for an  $s_0 > 0$  that  $|d_0 + s_0\mathbf{v}| < 6C_0\epsilon s_0$ . Squaring this inequality and for  $C_1 = (6C_0)^2|\mathbf{v}|^{-2}$  we get

$$|\mathbf{v}|^2(1 - C_1\epsilon^2)s_0^2 + 2d_0 \cdot \mathbf{v}s_0 + |d_0|^2 < 0.$$

This implies  $(d_0 \cdot \mathbf{v})^2 > |d_0|^2 |\mathbf{v}|^2(1 - C_1\epsilon^2)$  for the discriminant and

$$d_0 \cdot \mathbf{v} < -|d_0| |\mathbf{v}| \sqrt{1 - C_1\epsilon^2}.$$

This implies  $d_0 \neq 0$  and  $\text{dist}_{S^2}(-\frac{d_0}{|d_0|}, \frac{\mathbf{v}}{|\mathbf{v}|}) = O(\epsilon^2)$ . From (5.13) we get

$$|\mathbf{v}|^{-1} \|\langle d_0 - \frac{d_0}{|d_0|} s \rangle^{-k}\|_{L^1(\mathbb{R}_+)} = |\mathbf{v}|^{-1} \|\langle |d_0| - s \rangle^{-k}\|_{L^1(\mathbb{R}_+)} < 10\epsilon.$$

For  $\epsilon_0 > 0$  in (1.15) small, we get  $|\mathbf{v}|^{-1} < \kappa\epsilon$  for  $\kappa = 20/\|\langle t \rangle^{-k}\|_{L^1(\mathbb{R})}$ . We have

$$\|\langle D'(s) + s\mathbf{v} + y_0 \rangle^{-k}\|_{L^1(0,T)} \leq |\mathbf{v}|^{-1} \|\langle d_0 + s + I_1(s/|\mathbf{v}|) \rangle^{-k}\|_{L^1(0,|\mathbf{v}|T)},$$

where  $\frac{d}{ds}[I_1(s/|\mathbf{v}|)] \leq 3C_0\epsilon/|\mathbf{v}|$ . We complete the proof of (5.14) by

$$\begin{aligned} \|\langle D'(s) + s\mathbf{v} + y_0 \rangle^{-k}\|_{L^1(0,T)} &\leq |\mathbf{v}|^{-1} \|\langle d_0/|\mathbf{v}| + s + I_1(s/|\mathbf{v}|) \rangle^{-k}\|_{L^1(0,|\mathbf{v}|T)} \\ &\leq 2|\mathbf{v}|^{-1} \|\langle t \rangle^{-k}\|_{L^1(\mathbb{R})} < 40\epsilon \text{ for } 3C_0\epsilon_0/|\mathbf{v}| < 1/2. \end{aligned} \quad \square$$

□

**Lemma 5.6.** Let  $0 < \varepsilon_4 < \varepsilon_2$  and let  $B(\varepsilon_4)$  an open neighborhood of  $\phi_{\omega_1}$  in  $H^1(\mathbb{R}^3, \mathbb{R}^2)$  defined like (2.14) but with  $\varepsilon_4$  instead of  $\varepsilon_2$ . Then under the hypotheses of Prop. 5.2 for the  $\varepsilon_0 > 0$  in (1.15) sufficiently small we have  $\tau'(t) \in \mathcal{T}(t, \delta)$  (where  $\delta > 0$  is given in Lemma 2.4) and  $(t, u(t)) \in B(\varepsilon_4)$  for  $t \in [0, T]$ .

*Proof.* By Lemma 5.4 and by the argument in Lemma 2.3 for any preassigned  $M > 0$  if  $\varepsilon_0 > 0$  in (1.15) is sufficiently small we either have  $|\mathbf{v}| \geq \epsilon^{-\frac{1}{2}}$  or  $|D'(0) + \mathbf{v}t + y_0| \geq M$ . Furthermore, the argument in Lemma 5.5 shows that either  $\langle D'(s) + t\mathbf{v} + y_0 \rangle \sim \langle D'(0) + \mathbf{v}t + y_0 \rangle$  in  $[0, T]$  for fixed constants or  $|\mathbf{v}| \geq c_o\epsilon^{-1}$  for a fixed  $c_o > 0$ . In any case, we conclude that for any fixed  $\delta_1 > 0$  for  $\varepsilon_0 > 0$  sufficiently small we have  $\tau'(t) \in \mathcal{T}(t, \delta_1)$  for  $t \in [0, T]$ .

Since (5.1)–(5.5) and (5.9)–(5.10) imply for  $\varepsilon_0 > 0$  sufficiently small that  $u(t) \in e^{J\tau'(t) \cdot \diamond} B_{H^1}(\varepsilon_4)$  for all  $t \in [0, T]$  we conclude  $(t, u(t)) \in B(\varepsilon_4)$  for  $t \in [0, T]$ .

□



**Lemma 5.7.** *Under the hypotheses of Proposition 5.2 and for  $\varepsilon_0$  small enough we have  $\|\dot{\Pi}_j\|_{L^1(I)} \leq c\varepsilon$  for a fixed  $c$  for all  $j$ .*

*Proof.* We have  $\|\langle V(\cdot + \mathbf{v}t + y_0)\partial_{x_a}(\Phi_{p'} + P(p')P(\pi)r'), \Phi_{p'} + P(p')P(\pi)r' \rangle\|_{L^1(I)} \leq c\varepsilon$  by an argument in [4]. We focus now on the additional terms not already present in [4]. We have

$$|d_U \Pi_j \mathbf{A}| \leq |\langle U, \mathbf{f}(U, \tilde{Q}) \rangle| + |\dot{w}_1 - E_w w_2| |\langle U, \diamond_j \partial_{w_1} \tilde{Q} \rangle| + |\dot{w}_2 + E_w w_1| |\langle U, \diamond_j \partial_{w_2} \tilde{Q} \rangle|.$$

By (3.4)

$$U = \mathbf{S}_{n',m'}^{0,0}(\pi, \Pi, \varrho', z', f') + e^{J\tau' \cdot \diamond} P(p') P_c(\pi) f'.$$

with  $\varrho' = \Pi(f')$ . Composing with the map in (3.8) we obtain

$$U = \mathbf{S}_{n,m}^{0,0}(\pi, \Pi, \varrho, z, f) + e^{J\tau \cdot \diamond} e^{J\mathcal{R}_{n,m}^{0,2} \cdot \diamond} P(p) P_c(\pi) f. \quad (5.15)$$

with  $\varrho = \Pi(f)$  for any preassigned pair  $(n, m)$ . This is obtained by taking both  $n'$  and  $m'$  sufficiently large, using the fact that the pullback of symbols  $\mathbf{S}_{n',m'}^{i,j}$  and  $\mathcal{R}_{n',m'}^{i,j}$  are symbols  $\mathbf{S}_{n,m}^{i,j}$  and  $\mathcal{R}_{n,m}^{i,j}$  for any  $n \leq n' - CN_1$  and  $m \leq m' - CN_1$  for a fixed  $C$ . Furthermore we have  $p' = p + \mathcal{R}_{n,m}^{0,2}$ . For all this, see [1]. We now have

$$\|\langle U, \mathbf{f}(U, \tilde{Q}) \rangle\|_{L_t^1} \leq \int_{[0,1]^2} \|\langle \mathbf{S}_{n,m}^{0,0} + e^{J\tau \cdot \diamond} e^{J\mathcal{R}_{n,m}^{1,2} \cdot \diamond} P(p) P_c(\pi) f, \partial_t \partial_s [\tilde{\beta}(\iota U + s\tilde{Q})] \rangle\|_{L_t^1} d\iota ds.$$

We have

$$\|\langle \mathbf{S}_{n,m}^{0,0}, \partial_t \partial_s [\tilde{\beta}(\iota U + s\tilde{Q})] \rangle\|_{L_t^1} \leq \|\mathbf{S}_{n,m}^{0,0} \tilde{Q}\|_{L_t^1 L_x^2} \|\tilde{\beta}''(\iota U + s\tilde{Q}) U\|_{L_t^\infty L_x^2}.$$

We have  $\|\tilde{\beta}''(\iota U + s\tilde{Q}) U\|_{L_t^\infty L_x^2} \leq c_1$  for a fixed  $c_1$  by (H3) and by (5.1)–(5.3) and (5.10). These imply also  $\|\mathbf{S}_{n,m}^{0,0} \tilde{Q}\|_{L_t^1 L_x^2} \leq c_2 \varepsilon$  for a fixed  $c_2$  by Lemma 5.3.

We have

$$\begin{aligned} & \|\langle e^{J\tau \cdot \diamond} e^{J\mathcal{R}_{n,m}^{1,2} \cdot \diamond} P(p) P_c(\pi) f, \partial_t \partial_s [\tilde{\beta}(\iota U + s\tilde{Q})] \rangle\|_{L_t^1} \\ & \leq \|f\|_{L_t^\infty L_x^2} \|\tilde{\beta}''(\iota U + s\tilde{Q}) \tilde{Q} \mathbf{S}_{n,m}^{0,0}\|_{L_t^1 L_x^2} \quad (\text{this is } O(\varepsilon^2)) \\ & + \|f\|_{L_t^2 L_x^6} \|\tilde{\beta}''(\iota U + s\tilde{Q}) \tilde{Q} e^{J\tau \cdot \diamond} e^{J\mathcal{R}_{n,m}^{1,2} \cdot \diamond} P(p) P_c(\pi) f\|_{L_t^2 L_x^{\frac{6}{5}}} \quad (\text{this is } O(\|f\|_{L_t^2 L_x^6}^2) = O(\varepsilon^2)). \end{aligned}$$

Then we conclude  $\|\langle U, \mathbf{f}(U, \tilde{Q}) \rangle\|_{L_t^1} \leq c\varepsilon$  for a fixed  $c$ .

We consider

$$\begin{aligned} & \|\dot{w}_1 - E_w w_2\|_{L_t^2} \|\langle U, \diamond_j \partial_{w_1} \tilde{Q} \rangle\|_{L_t^2} \\ & \leq C\varepsilon \left( \|\langle \mathbf{S}_{n,m}^{0,0}, \diamond_j \partial_{w_1} \tilde{Q} \rangle\|_{L_t^2} + \|\langle e^{J\tau \cdot \diamond} e^{J\mathcal{R}_{n,m}^{1,2} \cdot \diamond} P(p) P_c(\pi) f, \diamond_j \partial_{w_1} \tilde{Q} \rangle\|_{L_t^2} \right). \end{aligned}$$

This is  $O(\varepsilon^{\frac{3}{2}})$  because for a fixed  $C$  and using Lemma 5.3

$$\begin{aligned} & \|\langle e^{J\tau \cdot \diamond} e^{J\mathcal{R}_{n,m}^{1,2} \cdot \diamond} P(p) P_c(\pi) f, \diamond_j \partial_{w_1} \tilde{Q} \rangle\|_{L_t^2} \leq C \|f\|_{L_t^2 L_x^6} \leq CC_0 \varepsilon \\ & \|\langle \mathbf{S}_{n,m}^{0,0}, \diamond_j \partial_{w_1} \tilde{Q} \rangle\|_{L_t^2}^2 \leq \|\langle \mathbf{S}_{n,m}^{0,0}, \diamond_j \partial_{w_1} \tilde{Q} \rangle\|_{L_t^1} \|\langle \mathbf{S}_{n,m}^{0,0}, \diamond_j \partial_{w_1} \tilde{Q} \rangle\|_{L_t^\infty} \leq C\varepsilon. \end{aligned}$$

□

**Lemma 5.8.** *Under the hypotheses of Prop. 5.2 for  $\varepsilon_0 > 0$  sufficiently small, for any preassigned  $c > 0$  we have in  $[0, T]$*

$$\|\dot{w}_1 - E_w w_2\|_{L^1 \cap L^2} + \|\dot{w}_2 + E_w w_1\|_{L^1 \cap L^2} \leq c\varepsilon. \quad (5.16)$$

*Proof.* We will bound only the first term in the left. We use (4.13). Furthermore we will only bound

$$\|\langle \text{rhs}(4.12), J\partial_{w_1} \tilde{Q} \rangle\|_{L^1 \cap L^\infty} \leq c\varepsilon. \quad (5.17)$$

All the other terms can be bounded similarly. By Lemma 5.3 we have, see (4.4) for  $\tilde{\beta}$ ,

$$\|\langle J\nabla \mathbf{E}(e^{J\tau' \cdot \diamond} \Phi_{p'}), J\partial_{w_1} \tilde{Q} \rangle\|_{L^1} \leq c\varepsilon.$$

Schematically, omitting factors irrelevant in the computation, we have

$$\begin{aligned} \langle J\nabla \mathbf{E}_P(e^{J\tau' \cdot \diamond} P(p')P(\pi)r'), J\partial_{w_1} \tilde{Q} \rangle &\sim \langle \tilde{\beta}(P(p')P(\pi)r'), \phi_0(\cdot + D' + y_0) \rangle \\ &= \langle \tilde{\beta}(\mathbf{S}_{k,m}^{0,1} + e^{J\mathcal{R}_{k,m}^{0,2} \cdot \diamond} f), \phi_0(\cdot + D' + y_0) \rangle = \langle \tilde{\beta}(\mathbf{S}_{k,m}^{0,1}), \phi_0(\cdot + D' + y_0) \rangle \\ &\quad + \langle \tilde{\beta}(e^{J\mathcal{R}_{k,m}^{0,2} \cdot \diamond} f), \phi_0(\cdot + D' + y_0) \rangle + \langle \mathbf{f}(e^{J\mathcal{R}_{k,m}^{0,2} \cdot \diamond} f, \mathbf{S}_{k,m}^{0,1}), \phi_0(\cdot + D' + y_0) \rangle. \end{aligned}$$

Then bounding one by one the terms in the r.h.s. by routine arguments and using Lemma 5.3, we get

$$\|\langle J\nabla \mathbf{E}_P(e^{J\tau' \cdot \diamond} P(p')P(\pi)r'), J\partial_{w_1} \tilde{Q} \rangle\|_{L^1 \cap L^\infty} \leq c\varepsilon.$$

Similarly

$$\int_{[0,1]^3} ds ds_1 ds_2 \|\langle \partial_s \partial_{s_1} \partial_{s_2} \tilde{\beta}(ss_2 e^{J\tau' \cdot \diamond} P(p')P(\pi)r' + s_1 \tilde{Q}), J\partial_{w_1} \tilde{Q} \rangle\|_{L^1 \cap L^\infty} \leq c\varepsilon$$

and

$$\begin{aligned} &\|\langle J\mathbf{f}(e^{J\tau' \cdot \diamond} P(p')P(\pi)r' + \tilde{Q}, e^{J\tau' \cdot \diamond} \Phi_{p'}), J\partial_{w_1} \tilde{Q} \rangle\|_{L^1 \cap L^\infty} \leq \\ &\int_{[0,1]^2} du ds \|\langle \tilde{\beta}''(\iota(e^{J\tau' \cdot \diamond} P(p')P(\pi)r' + \tilde{Q}) + se^{J\tau' \cdot \diamond} \Phi_{p'})e^{J\tau' \cdot \diamond} P(p')P(\pi)r' \tilde{Q}, \partial_{w_1} \tilde{Q} \rangle\|_{L^1 \cap L^\infty} \leq c\varepsilon. \end{aligned}$$

Schematically we have

$$\begin{aligned} \langle \partial_t(e^{J\tau' \cdot \diamond} P(p')P(\pi)r'), J\partial_{w_1} \tilde{Q} \rangle &\sim \langle \dot{\tau}' P(p')P(\pi)r', \diamond \phi_0(\cdot + D' + y_0) \rangle \\ &\quad + \langle (\partial_t P(p')P(\pi))r' + P(p')P(\pi)\dot{r}', \phi_0(\cdot + D' + y_0) \rangle. \end{aligned} \quad (5.18)$$

We have  $p' = p + \mathcal{R}_{k,m}^{0,2}$ , see [1]. We also have  $\tau' = \tau + \mathcal{R}_{k,m}^{0,2}$  by (3.8). For the time derivatives we use also the equations in Sect. 4. In particular we have  $(\partial_t P(p')P(\pi))r' = S_{k,m}^{0,1}$  and one by one the terms in the r.h.s. of (5.18) satisfy the desired bounds. Similarly it is elementary to see also

$$\|\langle \partial_t e^{J\tau' \cdot \diamond} \Phi_{p'}, \partial_{w_1} \tilde{Q} \rangle\|_{L^1 \cap L^\infty} \leq c\varepsilon.$$

□

**Lemma 5.9.** *Under the hypotheses of we can extend  $u(t)$  for all  $t \geq 0$  with  $(t, u(t)) \in B(\varepsilon_2)$ . Furthermore (5.1)–(5.5) hold for a fixed  $C$  in  $[0, \infty)$  and we have  $\lim_{t \nearrow \infty} z(t) = 0$ .*

*Proof.* We can apply a standard continuity argument, Prop. 5.2 and Lemma 5.6 to conclude that  $(t, u(t)) \in B(\varepsilon_2)$  for all  $t \geq 0$  and that (5.1)–(5.5) hold on  $[0, \infty)$ . The fact that  $\lim_{t \nearrow \infty} z(t) = 0$  follows by Lemma 7.1 [4].  $\square$

**Lemma 5.10.** *There is a fixed  $C$  and  $f_+ \in H^1$  and a function  $\varsigma : [0, \infty) \rightarrow \mathbb{R}^4$  such that for the variable  $f$  in (5.1) we have*

$$\lim_{t \nearrow \infty} \|f(t) - e^{J\varsigma(t) \cdot \diamond} e^{-Jt\Delta} f_+\|_{H^1} = 0. \quad (5.19)$$

*Proof.* The proof of Lemma 5.10 is the same of Sect. 11 in [4] and is a standard consequence of the estimates (5.1)–(5.3), of (6.9) and (6.3) below in  $I = [0, T) = [0, \infty)$  applied to (6.13) below, where  $h = M^{-1}e^{\frac{1}{2}Jv_0 \cdot x}f$ .  $\square$

We can now apply [4] which proves the following facts, that yields Theor. 5.1 assuming Prop. 5.2.

- For  $\varepsilon_0$  small enough, (5.10) holds for  $C = c < C_0/2$  with  $c$  a fixed constant. Furthermore, (5.4) holds for  $C = c < C_0/2$  with  $c$  a fixed constant.

- We have

$$|D'(t) + t\mathbf{v} + y_0| \geq t2^{-1}|\mathbf{v}| - |D'(0) + y_0| \quad (5.20)$$

- We have

$$\lim_{t \rightarrow +\infty} (\dot{D}' - v') = 0, \quad \lim_{t \rightarrow +\infty} (\dot{v}' - \omega' - 4^{-1}(v')^2) = 0. \quad (5.21)$$

- There exist  $\omega_+$  and  $v_+$  such that the limits (5.6) are true.

## 6 Proof of Proposition 5.2

**Lemma 6.1.** *Assume the hypotheses of Prop. 5.2. Then there is a fixed  $c$  such that for all admissible pairs  $(p, q)$*

$$\|f\|_{L_t^p([0, T], W_x^{1, q})} \leq c\varepsilon + c \sum_{\mathbf{e} \cdot \mu > \omega_0} |z^\mu|^2_{L_t^2(0, T)} \quad (6.1)$$

where we sum only on multiindices such that  $\mathbf{e} \cdot \mu - \mathbf{e}_j < \omega_0$  for any  $j$  such that for the  $j$ -th component of  $\mu$  we have  $\mu_j \neq 0$ .

*Proof.* Compared to [4], the one additional term in (4.21) here is the term  $d_U f \mathbf{A}$ , which we now analyze. By the fact that the inverse of (3.8) has the same structure (the flows which yield (3.8) when reversed yields the inverse of (3.8), see Lemma 3.4 [4]) we have

$$f = e^{J\mathcal{R}_{k,m}^{0,2}(\pi, \Pi, \Pi(f'), z', f') \cdot \diamond} f' + \mathbf{S}_{k,m}^{1,1}(\pi, \Pi, \Pi(f'), z', f').$$

Hence

$$d_U f = e^{J\mathcal{R}_{k,m}^{0,2} \cdot \diamond} d_U f' + J d_U \mathcal{R}_{k,m}^{0,2} \cdot \diamond (f - \mathbf{S}_{k,m}^{1,1}) + d_U \mathbf{S}_{k,m}^{1,1}.$$

Notice that we have  $d_U \mathcal{R}_{k,m}^{0,2} \in B(\Sigma_{-k'}, \mathbb{R}^4)$  and  $d_U \mathbf{S}_{k,m}^{1,1} \in B(\Sigma_{-k'}, \Sigma_{k'})$  with norms

$$\begin{aligned} \|d_U \mathcal{R}_{k,m}^{0,2}\|_{B(\Sigma_{-k'}, \mathbb{R}^4)} &\leq C \|r'\|_{\Sigma_{-k'}} \\ \|d_U \mathbf{S}_{k,m}^{1,1}\|_{B(\Sigma_{-k'}, \Sigma_{k'})} &\leq C(|\pi - \Pi| + |\Pi(f')| + \|r'\|_{\Sigma_{-k'}}). \end{aligned}$$

Then

$$\begin{aligned} \|d_U \mathbf{S}_{k,m}^{1,1} \mathbf{f}(U, \tilde{Q})\|_{L_t^1 H^1 + L_t^2 H^{1,s}} &\lesssim C_0 \epsilon \int_{[0,1]^2} d\iota d\kappa \|\tilde{\beta}''(\iota U + \kappa \tilde{Q}) \tilde{Q} U\|_{L_t^1 L^{2,-s} + L_t^2 L^{2,-s}} \\ &\lesssim C_0 \epsilon \int_{[0,1]^2} d\iota d\kappa (\|\tilde{Q} \mathbf{S}_{n,m}^{0,0}\|_{L_t^1 L_x^2} + \|\tilde{Q} e^{J\tau \cdot \diamond} e^{J\mathcal{R}_{n,m}^{0,2} \cdot \diamond} P(p) P_c(\pi) f\|_{L_t^2 L_x^2}) = O(\epsilon^2) \end{aligned}$$

and similarly

$$\|d_U \mathcal{R}_{k,m}^{0,2} \mathbf{f}(U, \tilde{Q})\|_{L_t^\infty + L_t^1} \lesssim C_0 \epsilon \int_{[0,1]^2} d\iota d\kappa \|\tilde{\beta}''(\iota U + \kappa \tilde{Q}) \tilde{Q} U\|_{L_t^\infty L^{2,-s} + L_t^1 L^{2,-s}} = O(\epsilon^2).$$

So we conclude

$$J d_U \mathcal{R}_{k,m}^{0,2} \mathbf{f}(U, \tilde{Q}) \cdot \diamond f - d_U \mathcal{R}_{k,m}^{0,2} \mathbf{f}(U, \tilde{Q}) \cdot \diamond \mathbf{S}_{k,m}^{1,1} + d_U \mathbf{S}_{k,m}^{1,1} \mathbf{f}(U, \tilde{Q}) = \mathcal{A} \cdot \diamond f + R_1 + R_2 \quad (6.2)$$

with for any preassigned  $c$

$$\|\mathcal{A}\|_{L^\infty(0,T) \cap L^1(0,T)} + \|R_1\|_{L^1([0,T], H^1)} + \|R_2\|_{L^2([0,T], H^{1,s})} \leq c\epsilon. \quad (6.3)$$

We have

$$\begin{aligned} d_U f' &= (P_c(\pi) P_c(p_0))^{-1} P_c(\pi) P(p_0) d_U r', \\ d_U r' &= (P(p') P(\pi) P(p_0))^{-1} P(p') [e^{-J\tau' \cdot \diamond} - J \diamond_j P(p') r' d_U \tau_j' - \partial_{p_j'} P(p') r' d_U p_j']. \end{aligned}$$

Proceeding like above we conclude that

$$e^{J\mathcal{R}_{k,m}^{0,2} \cdot \diamond} d_U f' \mathbf{f}(U, \tilde{Q}) = e^{J(\mathcal{R}_{k,m}^{0,2} - \tau') \cdot \diamond} P_c(p_0) \mathbf{f}(U, \tilde{Q}) + \mathcal{A} \cdot \diamond f + R_1 + R_2,$$

where the last three terms are like those in the r.h.s. of (6.2). We have

$$\begin{aligned} \|\mathbf{f}(U, \tilde{Q})\|_{L_t^1 H^1 + L_t^2 H^{1,s}} &\leq \int_{[0,1]^2} d\iota d\kappa \|\tilde{\beta}''(\iota U + \kappa \tilde{Q}) \tilde{Q} U\|_{L_t^1 H^1 + L_t^2 H^{1,s}} \\ &\leq C \|\tilde{Q} \mathbf{S}_{n,m}^{0,0}\|_{L_t^1 L_x^2} (1 + \|f\|_{L_t^1 H_x^1}) + \|\tilde{Q} (e^{J\tau \cdot \diamond} e^{J\mathcal{R}_{n,m}^{0,2} \cdot \diamond} P(p) P_c(\pi) f)^2\|_{L_t^2 H_x^1} \\ &\leq c\epsilon + \|\tilde{Q}\|_{L_t^\infty W_x^{1,3}} \|f\|_{L_t^2 W_x^{1,6}} \leq c\epsilon + C(C_0) \epsilon^2. \end{aligned}$$

Therefore  $\mathbf{f}(U, \tilde{Q})$  is of the form  $R_1 + R_2$  with the estimate in (6.3).

Summing up, for  $h = M^{-1} e^{\frac{1}{2} J v_0 \cdot x} f$  with  $M$  defined in (2.40), we have

$$\begin{aligned} i\dot{h} &= \mathcal{K}_{\omega_0} h + \sigma_3 P_c(\mathcal{K}_{\omega_0}) V(\cdot + \mathbf{v}t + y_0 + D' + \mathcal{R}_{k,m}^{0,2}) h + \sigma_3 \mathcal{A}_4(t) P_c(\mathcal{K}_{\omega_0}) h \\ &\quad - \sum_{a=1}^3 i \mathcal{A}_a(t) P_c(\mathcal{K}_{\omega_0}) \partial_{x_a} h + \sum_{|\mathbf{e} \cdot (\mu - \nu)| > \omega_0} z^\mu \bar{z}^\nu \mathbf{G}_{\mu\nu}(t, \Pi(f)) + R_1 + R_2, \end{aligned} \quad (6.4)$$

where:

$$\mathbf{G}_{\mu\nu}(t, \Pi(f)) := M^{-1} e^{J \frac{v_0 \cdot x}{2}} G_{\mu\nu}(t, \Pi(f)), \quad (6.5)$$

with  $G_{\mu\nu}(t, \Pi(f))$  the coefficients of  $Z_1$ , see (3.12) and where (6.3) are satisfied.

Notice that in (6.4) we can drop  $\mathcal{R}_{k,m}^{0,2}$  from the argument of  $V$ , absorbing the difference inside  $R_1 + R_2$ , so that  $\sigma_3 P_c(\mathcal{K}_{\omega_0}) V(\cdot + \mathbf{v}t + y_0 + D') h$  becomes the second term in the r.h.s of (6.4).

Set  $\mathbf{D} := \mathbf{v}t + y_0 + D'$ . Set

$$\tilde{g}(t)u(t, x) := e^{i\sigma_3(-\frac{t}{4}\mathbf{v}^2 - \frac{\mathbf{v}\cdot x}{2})}u(t, x + \mathbf{D}(t)). \quad (6.6)$$

Recall that

$$\begin{aligned} \tilde{g}(t)^{-1}u &= e^{i\sigma_3(\frac{t}{4}\mathbf{v}^2 + \frac{\mathbf{v}\cdot(x-\mathbf{D}(t))}{2})}u(t, x - \mathbf{D}(t)), \\ [\tilde{g}(t)^{-1}, i\partial_t - \mathcal{K}_0]u &= i(\dot{\mathbf{D}} - \mathbf{v}) \cdot \nabla_x(\tilde{g}^{-1}(t)u), \\ [\tilde{g}(t)^{-1}, \partial_{x_j}]u &= -i\sigma_3 \frac{\mathbf{v}_j}{2} \tilde{g}^{-1}(t)u. \end{aligned} \quad (6.7)$$

Set now  $g(t) = \tilde{g}(t)e^{i\sigma_3 \int_0^t \hat{\varphi}(s)ds}$  for a  $\hat{\varphi}$  which will be introduced later. Then, irrespective of the  $\hat{\varphi}$ , we have  $V(\cdot + \mathbf{D}) = \tilde{g}V\tilde{g}^{-1}$ . We now set

$$\mathbf{P}_D := g\mathbf{P}g^{-1} \text{ where } \mathbf{P} := \phi_0\langle \cdot, \phi_0 \rangle + \sigma_1\phi_0\langle \cdot, \sigma_1\phi_0 \rangle. \quad (6.8)$$

Then, for a fixed  $\delta > 0$ , we add to (6.4) the term  $i\delta\mathbf{P}_D h - i\delta\mathbf{P}_D h = 0$ . We will think of  $-i\delta\mathbf{P}_D h$  as a damping term in (6.4) and  $i\delta\mathbf{P}_D h$  as a reminder term, since it can be absorbed inside the reminder  $R_1 + R_2$ , as we show now.

**Lemma 6.2.** *Under the hypotheses of Prop. 5.2, for  $\varepsilon_0$  small enough we have for any preassigned  $c > 0$  and irrespective of the  $\hat{\varphi}$ ,*

$$\|\mathbf{P}_D h\|_{L^1([0,T], H^1) + L^2([0,T], W^{1,6})} \leq c\varepsilon. \quad (6.9)$$

*Proof.* Obviously it is enough to prove

$$\|\langle e^{-i\sigma_3 \int_0^t \hat{\varphi}(s)ds} \tilde{g}^{-1}h, \psi \rangle\|_{L^1(0,T) + L^2(0,T)} \leq c\varepsilon \text{ for } \psi = \phi_0, \sigma_1\phi_0. \quad (6.10)$$

We will consider the case  $\psi = \phi_0$ . The other case is similar. We have from  $h = M^{-1}e^{\frac{1}{2}Jv_0 \cdot x}f$

$$\begin{aligned} \langle e^{-i\sigma_3 \int_0^t \hat{\varphi}(s)ds} \tilde{g}^{-1}h, \phi_0 \rangle &= \langle M^{-1}e^{J\frac{v_0 \cdot x}{2}}f, e^{i\sigma_3(\frac{t}{4}\mathbf{v}^2 + \frac{\mathbf{v}\cdot x}{2} - \int_0^t \hat{\varphi}(s)ds)}\phi_0(\cdot + \mathbf{D}) \rangle \\ &= \langle e^{J\frac{v_0 \cdot x}{2}}f, (M^{-1})^T e^{i\sigma_3(\frac{t}{4}\mathbf{v}^2 + \frac{\mathbf{v}\cdot x}{2} - \int_0^t \hat{\varphi}(s)ds)} M^T (M^{-1})^T \phi_0(\cdot + \mathbf{D}) \rangle \\ &= \langle e^{J\frac{v_0 \cdot (x+\mathbf{D})}{2}}f, e^{J(\frac{t}{4}\mathbf{v}^2 + \frac{\mathbf{v}\cdot x}{2} + \frac{v_0 \cdot \mathbf{D}}{2} - \int_0^t \hat{\varphi}(s)ds)} (M^{-1})^T \phi_0(\cdot + \mathbf{D}) \rangle \\ &= \langle f, e^{J(\frac{t}{4}\mathbf{v}^2 + \frac{v_0 \cdot \mathbf{D}}{2} + \frac{\mathbf{v}\cdot x}{2} - \int_0^t \hat{\varphi}(s)ds)} (M^{-1})^T \phi_0(\cdot + \mathbf{D}) \rangle + O(\varepsilon\|f\|_{L_x^6}). \end{aligned} \quad (6.11)$$

We used  $(M^{-1})^T i\sigma_3 M^T = \overline{M} i\sigma_3 \overline{M}^{-1} = J$ . We have  $\|O(\varepsilon\|f\|_{L_x^6})\|_{L^2(0,T)} \leq C(C_0)\varepsilon^2$ . Ignoring the  $O(\varepsilon\|f\|_{L_x^6})$  term, we can write the last line in (6.11) in the form

$$\begin{aligned} &\langle f, e^{J\frac{\mathbf{v}\cdot x}{2}} e^{J\lambda(t)} \phi_0(\cdot + \mathbf{D}) \rangle + i \langle f, e^{J\frac{\mathbf{v}\cdot x}{2}} e^{J\lambda(t)} J \phi_0(\cdot + \mathbf{D}) \rangle \\ &= e^{-i\lambda(t)} \langle f, e^{J\frac{\mathbf{v}\cdot x}{2}} \phi_0(\cdot + \mathbf{D}) \rangle + (\sin \lambda(t) + i \cos \lambda(t)) \langle f, e^{J\frac{\mathbf{v}\cdot x}{2}} J \phi_0(\cdot + \mathbf{D}) \rangle, \end{aligned} \quad (6.12)$$

for some real valued function  $\lambda(t)$ .

By the fact that  $\phi_0$  is a Schwartz function and by Lemmas 4.5 and 5.3, we conclude that the  $L^1(0, T) + L^2(0, T)$  norm of (6.12) is bounded by  $C(C_0)\varepsilon^2$ , independently of  $\lambda(t)$ . This yields (6.10) for  $\psi = \phi_0$ . The case  $\psi = \sigma_1\phi_0$  is similar.  $\square$

We can rewrite (6.4)

$$\begin{aligned} i\dot{h} &= \mathcal{K}_{\omega_0} h + \sigma_3 P_c(\mathcal{K}_{\omega_0}) V(\cdot + \mathbf{v}t + y_0 + D') h - i\delta \mathbf{P}_D h + \sigma_3 \mathcal{A}_4(t) P_c(\mathcal{K}_{\omega_0}) h \\ &\quad - \sum_{a=1}^3 i\mathcal{A}_a(t) P_c(\mathcal{K}_{\omega_0}) \partial_{x_a} h + \sum_{|\mathbf{e} \cdot (\mu - \nu)| > \omega_0} z^\mu \bar{z}^\nu \mathbf{G}_{\mu\nu}(t, \Pi(f)) + R_1 + R_2, \end{aligned} \quad (6.13)$$

Then the proof of Lemma 6.1 is exactly the same as in [4] using Theorem 7.1 below.

We set now

$$g = h + Y, \quad Y := \sum_{|\mathbf{e} \cdot (\mu - \nu)| > \omega_0} z^\mu \bar{z}^\nu R_{\mathcal{K}_{\omega_0}}^+(\mathbf{e} \cdot (\mu - \nu)) \mathbf{G}_{\mu\nu}(t, 0). \quad (6.14)$$

**Lemma 6.3.** *Assume the hypotheses of Prop. 5.2 and let  $T > \varepsilon_0^{-1}$ . Then for fixed  $s > 1$  there exist a fixed  $c$  such that if  $\varepsilon_0$  is sufficiently small, for any preassigned and large  $L > 1$  we have  $\|g\|_{L^2((0,T), L_x^{2-s})} \leq (c + C_0 L^{-1})\epsilon$ .*

*Proof.* The proof is exactly the same of Lemma 8.5 in [4].  $\square$

**Lemma 6.4.** *There is a set of variables  $\zeta = z + O(z^2)$  such that for a fixed  $C$  we have*

$$\|\zeta - z\|_{L_t^2} \leq CC_0 \epsilon^2, \quad \|\zeta - z\|_{L_t^\infty} \leq C\epsilon^3 \quad (6.15)$$

$$\partial_t \sum_{j=1}^n e_j |\zeta_j|^2 = -\Gamma(\zeta) + \mathfrak{r} \quad (6.16)$$

and s.t., for a fixed constant  $c_0$  and a preassigned but arbitrarily large constant  $L$ , we have

$$\begin{aligned} \Gamma(\zeta) &:= 4 \sum_{\Lambda > \omega_0} \Lambda \operatorname{Im} \left\langle R_{\mathcal{H}_{\omega_0}}^+(\Lambda) \sum_{\mathbf{e} \cdot \alpha = \Lambda} \zeta^\alpha \mathbf{G}_{\alpha 0}(t, 0), \sigma_3 \sum_{\mathbf{e} \cdot \alpha = \Lambda} \bar{\zeta}^\alpha \overline{\mathbf{G}_{\alpha 0}}(t, 0) \right\rangle, \\ \|\mathfrak{r}\|_{L^1[0,T]} &\leq (1 + C_0)(c_0 + C_0 L^{-1})\epsilon^2. \end{aligned} \quad (6.17)$$

For the proof see [4, 2]. By [2] Lemma 10.5 we have  $\Gamma(\zeta) \geq 0$ . We make now the following hypothesis:

(H11) there exists a fixed constant  $\Gamma > 0$  s.t. for all  $\zeta \in \mathbb{C}^{\mathbf{n}}$  we have:

$$\Gamma(\zeta) \geq \Gamma \sum_{\substack{\mathbf{e} \cdot \alpha > \omega_0 \\ \mathbf{e} \cdot \alpha - \mathbf{e}_k < \omega_0 \\ \forall k \text{ s.t. } \alpha_k \neq 0}} |\zeta^\alpha|^2. \quad (6.18)$$

Then integrating and exploiting (6.15) we get for  $t \in [0, T]$  and fixed  $c$

$$\sum_j \mathbf{e}_j |z_j(t)|^2 + 4\Gamma \sum_{\alpha \text{ as in (H11)}} \|z^\alpha\|_{L^2(0,t)}^2 \leq c(1 + C_0 + C_0^2 L^{-1})\epsilon^2.$$

From the last inequality and from Lemma 6.1 we conclude that for  $\varepsilon_0 > 0$  sufficiently small and any  $T > 0$ , (5.1)–(5.3) in  $I = [0, T]$  and with  $C = C_0$  imply (5.1)–(5.3) in  $I = [0, T]$  with  $C = c(1 + \sqrt{C_0} + C_0 L^{-\frac{1}{2}})$  for fixed  $c$ .

We bound the r.h.s. of (4.13). By Lemma 5.3 we have for a fixed  $c$

$$\|\langle |\partial_t e^{J\tau' \cdot \diamond} \Phi_{p'}| + |\nabla \mathbf{E}(e^{J\tau' \cdot \diamond} \Phi_{p'})| + J|\mathbf{f}(\eta + \tilde{Q}, e^{J\tau' \cdot \diamond} \Phi_{p'})|, J|\partial_{w_i} \tilde{Q}| \rangle\|_{L_t^1} \leq c\epsilon.$$

We have

$$\|\langle \nabla \mathbf{E}_P(\eta), \partial_{w_i} \tilde{Q} \rangle\|_{L_t^1} = \|\langle \beta(|e^{J\tau' \cdot \diamond} P(p') P(\pi) r'|^2) e^{J\tau' \cdot \diamond} P(p') P(\pi) r', \partial_{w_i} \tilde{Q} \rangle\|_{L_t^1}.$$

Next,  $r' = S_{k,m}^{0,1} + e^{J\mathcal{R}_{k,m}^{0,2} \cdot \diamond} f$ . Then the above can be bounded by

$$\begin{aligned} & \|\langle \nabla \mathbf{E}_P(e^{J\tau' \cdot \diamond} S_{k,m}^{0,1}), \partial_{w_i} \tilde{Q} \rangle\|_{L_t^1} + \|\langle \nabla \mathbf{E}_P(e^{J(\mathcal{R}_{k,m}^{0,2} + \tau') \cdot \diamond} f), \partial_{w_i} \tilde{Q} \rangle\|_{L_t^1} \\ & + \|\langle \mathbf{f}(e^{J\tau' \cdot \diamond} S_{k,m}^{0,1}, e^{J(\mathcal{R}_{k,m}^{0,2} + \tau') \cdot \diamond} f), \partial_{w_i} \tilde{Q} \rangle\|_{L_t^1} \leq c\epsilon. \end{aligned}$$

This completes the proof of Proposition 5.2.

## 7 Linear dispersion

Set  $\mathcal{K}_0 = \sigma_3(-\Delta + \omega_0)$ ,  $\mathcal{K}_1 = \mathcal{K}_{\omega_0} = \mathcal{K}_0 + \mathcal{V}_1$ ,  $\mathcal{K}_2 = \mathcal{H}_0 + \mathcal{V}_2$  where  $\mathcal{V}_2 = \sigma_3 V$ . Set  $\mathcal{V}_2^D(t, x) := \mathcal{V}_2(x + \mathbf{D}(t))$   $P_c := P_c(\mathcal{K}_1)$ ,  $\mathcal{K}(t) = \mathcal{K}_0 + \mathcal{V}_1 + \mathcal{V}_2^D(t)$  We have the following result.

**Theorem 7.1.** *Consider for  $P_c F(t) = F(t)$  and  $P_c u(0) = u_0$  the equation*

$$i\dot{u} - P_c \mathcal{K}(t) P_c u - i P_c v(t) \cdot \nabla_x u + \varphi(t) P_c \sigma_3 u = F - i\delta \mathbf{P}_D u \quad (7.1)$$

for  $(v(t), \varphi(t)) \in C^1([0, T], \mathbb{R}^3 \times \mathbb{R})$ . Fix  $\delta_0 > |e_0 + \omega_0|$ . For  $\mathbf{v}$  the vector in Theor.1.4, set

$$c(T) := \|(\varphi(t), v(t))\|_{L_t^\infty[0, T] + L_t^1[0, T]} + \|\mathbf{v} - \dot{\mathbf{D}}(t)\|_{L_t^\infty[0, T]}. \quad (7.2)$$

Then for any  $\sigma_0 > 3/2$  there exist a  $c_0 > 0$  and a  $C > 0$  such that, if  $c(T) < c_0$ ,  $\sigma > \sigma_0$  and  $\delta > \delta_0$ , then for any admissible pair  $(p, q)$ , see (1.7), we have for  $i = 0, 1$

$$\|u\|_{L_t^p([0, T], W_x^{i, q})} \leq C(\|u_0\|_{H^i} + \|F\|_{L_t^2([0, T], H_x^{i, \sigma}) + L_t^1([0, T], H_x^i)}). \quad (7.3)$$

*Proof.* Consider the problem

$$i\dot{u} - \mathcal{K}_0 u - i v(t) \cdot \nabla_x u + \varphi(t) \sigma_3 u = \mathcal{V}_2^D u + G u - i\delta P_d u - i\delta \mathbf{P}_D u, \quad u(t_0) = u_0, \quad (7.4)$$

where  $P_d = 1 - P_c$  and

$$G(t) := \mathcal{V}_1 - P_d \mathcal{K}(t) P_c - \mathcal{K}(t) P_d.$$

By the proof of Theorem 9.1 in [4], Theorem 7.1 is a consequence of Proposition 7.2 below.  $\square$

**Proposition 7.2.** *Let  $U(t, t_0)$  be the group associated to (7.4). Then for  $\sigma > 3/2$  there exists a fixed  $C > 0$  such that for all  $0 \leq t_0 < t \leq T$*

$$\|\langle x - x_0 \rangle^{-\sigma} U(t, t_0) \langle x - x_1 \rangle^{-\sigma}\|_{L^2 \rightarrow L^2} \leq C \langle t - t_0 \rangle^{-\frac{3}{2}} \quad \forall (x_0, x_1) \in \mathbb{R}^6. \quad (7.5)$$

and

$$\int_0^T \|\langle x - x(t) \rangle^{-\sigma} U(t, t_0) u_0\|_{L_x^2}^2 dt \leq C \|u_0\|_{L_x^2}^2 \quad \forall x(t) \in C^0([0, T], \mathbb{R}^3). \quad (7.6)$$

The proof is the same of Proposition 9.2 in [4] with a small difference. Notice that in [4] the operator  $\sigma_3(-\Delta + \omega_0 + V)$  does not have eigenvalues, while here it does have the eigenvalues  $\pm(e_0 + \omega_0)$ , with projection on the vector space generated by the eigenspaces given by the operator  $\mathbf{P}$  introduced in (6.8).

Now, the proof is exactly the same of Proposition 9.2 in [4] except for the following modification. The analogue of (9.43) [4] is now

$$\begin{aligned} & (\mathrm{i}\partial_t - \mathcal{K}_0)g^{-1}u - \mathrm{i}\hat{v}(t) \cdot \nabla_x g^{-1}u + \mathrm{i}\delta \mathbf{P}g^{-1}u = \sigma_3 V g^{-1}u \\ & + g^{-1}\mathcal{V}_1 - \mathrm{i}\delta P_d - \mathcal{K}_1 P_d + P_d \sigma_3 V(\cdot + \mathbf{D})P_c - \sigma_3 V(\cdot + \mathbf{D})P_d]u \end{aligned} \quad (7.7)$$

where  $g(t) = \tilde{g}(t)e^{\mathrm{i}\sigma_3 \int_0^t \hat{\varphi}(s)ds}$  like after (6.7), where we choose the same  $\hat{\varphi}$  of [4] and where  $\sigma_3 V(x) = g^{-1}(t)\sigma_3 V(x + \mathbf{D})g(t)$  and by (6.8) we have  $\mathbf{P} = g^{-1}(t)\mathbf{P}_{\mathbf{D}}g(t)$ .

The operator of formula (9.46) in [4] has to be changed into

$$T_1 f(s) := W_2 \int_{t_0}^s e^{-\mathrm{i}(s-\tau)\sigma_3(-\Delta+\omega_0+V) - (s-\tau)\delta \mathbf{P}} W_1 f(\tau) d\tau,$$

where  $W_1 W_2 = \sigma_3 V - \mathrm{i}\delta \mathbf{P}$ . Then, for

$$T_0 f(s) := W_2 \int_{t_0}^s e^{-\mathrm{i}(s-\tau)(-\Delta+\omega_0)} W_1 f(\tau) d\tau$$

we have  $(1 - \mathrm{i}T_1)(1 + \mathrm{i}T_0) = 1$ . Furthermore, we have for a fixed  $C_\sigma$  for any  $\sigma > 5/2$

$$\|\langle x - x_0 \rangle^{-\sigma} e^{-\mathrm{i}t\sigma_3(-\Delta+\omega_0+V) - t\delta \mathbf{P}} \langle x - x_1 \rangle^{-\sigma}\|_{L^2 \rightarrow L^2} \leq C_\sigma \langle t \rangle^{-\frac{3}{2}} \quad \forall t \geq 0 \text{ and } (x_0, x_1) \in \mathbb{R}^6$$

which follows by the condition  $\delta \geq \delta_0 > |e_0 + \omega_0|$ . Then the proof in [4] yields Proposition 7.2.

## 8 Dropping the hypothesis $u_0 \in \Sigma_2$

Up to now we have assumed  $u_0 \in \Sigma_2$ , that is (1.18), to guarantee that as we remark at the end of Sect. 3 the coordinates of  $U[t, u(t)]$  belong to the image of the map (3.8) in the sense of (3.9)–(3.11). For the same reason in the series [2, 3, 4] it is assumed that  $u_0 \in \Sigma_\ell$  for fixed  $\ell \gg 1$  with depends on the  $N = N_1$  in Hypothesis (H7). This is used only in order to make sense of the pullback by means of (3.8) of the form  $\Omega$  discussed in claim (8) of Theorem 3.5. However everywhere in [2, 3, 4] and here the distance of  $u(t)$  and of  $u_0$  from ground states is measured only with the metric of  $H^1(\mathbb{R}^3)$ .

Now we discuss briefly the fact that we can drop (1.18) and assume only  $u_0 \in H^1$ . Let  $u_0 \in H^1$  with  $u_0 \notin \Sigma_2$  and let  $\{u_n(0)\}_{n \geq 1}$  be a sequence with  $u_n(0) \rightarrow u_0$  in  $H^1$  and with  $u_n(0) \in \Sigma_2$  for any  $n \geq 1$ . We can apply our result to each solution  $u_n(t)$ . By the well posedness of (1.3) and by the continuity of the maps defined in Proposition 2.1, in (2.36) and at the beginning of Sect. 3 we have for the coordinates of  $u_n(t)$  and of  $u(t)$

$$(\tau'_n(t), p'_n(t), z'_n(t), f'_n(t), w_n(t)) \xrightarrow{n \rightarrow \infty} (\tau'(t), p'(t), z'(t), f'(t), w(t)) \quad (8.1)$$

in  $\mathbb{R}^8 \times \mathbb{C}^{\mathbf{n}} \times H^1 \times \mathbb{C}$ . Furthermore, since (3.8) is a local homeomorphism of  $\mathbb{R}^4 \times \mathbb{C}^{\mathbf{n}} \times (H^1 \cap L_c^2(p_0))$ , see (3.10) and the comments immediately below (3.10), we also have a limit

$$(\tau_n(t), p_n(t), z_n(t), f_n(t), w_n(t)) \xrightarrow{n \rightarrow \infty} (\tau(t), p(t), z(t), f(t), w(t)) \quad (8.2)$$

with on the left the final coordinates of  $u_n(t)$ . Notice that on the right of (8.2) we have the final coordinates of  $u(t)$  since map (3.8) makes them correspond to the initial coordinates of  $u(t)$ .



No use in Sections 5–7 is made of the hypothesis that  $u_0 \in \Sigma_2$ . Hence Theorem 5.1 holds also for the coordinates on the right in (8.2). From this and Lemma 2.5 we conclude that

$$\begin{aligned} u(t) &= e^{J\tau'(t) \cdot \diamond} (\Phi_{p'(t)} + P(p'(t))P(\Pi(t))r'(t)) + e^{J(\frac{1}{2}\mathbf{v} \cdot x + \frac{t}{4}|\mathbf{v}|^2)} Q_w(\cdot + \mathbf{v}t + y_0), \\ p'(t) &= \Pi(t) + \mathcal{R}_{k,m}^{0,2}(\Pi(t), t, z(t), f(t)), \\ r'(t) &= e^{J\mathcal{R}_{k,m}^{0,2}(\Pi(t), z(t), f(t)) \cdot \diamond} (f + \mathbf{S}_{k,m}^{0,1}(\Pi(t), z(t), f(t))), \end{aligned} \quad (8.3)$$

where we are making use of (3.8) and of claims (6)–(7) of Theorem 3.5.

Finally, the proof that (8.3) yields (1.16) is in [4], especially in Sect. 12. Notice that the proof in [4] of the facts we list now makes only use of  $u_0 \in H^1$ .

The facts needed to obtain (1.16) are Lemma 5.7,  $\lim_{t \nearrow \infty} \mathbf{S}_{k,m}^{0,1} = 0$  in  $H^1$ ,  $\lim_{t \nearrow \infty} \mathcal{R}_{k,m}^{0,2} = 0$  in  $\mathbb{R}^4$  and  $\lim_{t \nearrow \infty} (\tau'(t) + \varsigma(t)) = \zeta_0$  for  $\varsigma$  the function in Lemma 5.10 and for some  $\zeta_0 \in \mathbb{R}^4$ . This is proved in [4].

## A Implicit function theorem

**Theorem A.1.** *Let  $F \in C^\infty(B_X(0, \delta_0) \times B_Y(0, \delta_0); Y)$  with  $F(0, 0) = 0$ . Further, assume there exists  $\delta_1, \delta_2 > 0$  s.t.*

$$\sup_{(x,y) \in B_X(0, \delta_1) \times B_Y(0, \delta_2)} \|D_y F(x, y)^{-1}\| \leq 2. \quad (\text{A.1})$$

Now, set  $\delta_3 \in (0, \delta_1)$  s.t.

$$\sup_{x \in B_X(0, \delta_3)} \|F(x, 0)\| \leq \frac{1}{8} \delta_4, \quad (\text{A.2})$$

where

$$\delta_4 := \min \left( \delta_2, \frac{1}{8} \left( \sup_{x \in B_X(0, \delta_1), y \in B_Y(0, \delta_2)} \|D_{yy} F(x, y)\| \right)^{-1} \right). \quad (\text{A.3})$$

Then there exists a function  $y(\cdot) \in C^\infty(B_X(0, \delta_3); B_Y(0, \delta_4))$  s.t. for any  $x \in B_X(0, \delta_3)$  and for  $y \in B_Y(0, \delta_4)$  we have  $F(x, y) = 0$  if and only if  $y = y(x)$ .

*Proof.* First, for  $(x, y) \in B_X(0, \delta_1) \times B_Y(0, \delta_2)$ , we have

$$F(x, y) = 0 \quad \Leftrightarrow \quad y = y - (D_y F(x, 0))^{-1} F(x, y).$$

So, we set

$$\Phi(x; y) := y - (D_y F(x, 0))^{-1} F(x, y)$$

and seek for the fixed point of  $\Phi$ .

Now, set

$$y_0 = 0, \quad y_{n+1} = \Phi(x; y_{n-1}) \text{ for } n \in \mathbb{N}.$$

We show

- $\forall n \in \mathbb{N}, y_n \in B_Y(0, \delta_4)$
- $y_n$  converges.

Indeed, by the continuity of  $F$  w.r.t.  $y$ ,  $\lim y_n$  is the fixed point of  $\Phi(x; \cdot)$ .

Now, let  $y, y' \in B_Y(0, \delta_4)$ , we have

$$\begin{aligned}\Phi(x; y) - \Phi(x; y') &= (D_y F(x, 0))^{-1} \int_0^1 (D_y F(x, 0) - D_y F(x, y' + t(y - y')))(y - y') dt \\ &= - (D_y F(x, 0))^{-1} \int_0^1 \int_0^1 (D_{yy} F(x, s(y' + t(y - y'))))(y' + t(y - y'))(y - y') ds dt.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\|\Phi(x; y) - \Phi(x; y')\| &\leq \| (D_y F(x, 0))^{-1} \| \\ &\times \int_0^1 \int_0^1 \| (D_{yy} F(x, s(y' + t(y - y'))))(y' + t(y - y')) \| ds dt \|y - y'\| \\ &2 \left( \sup_{x \in B_X(0, \delta_1), y \in B_Y(0, \delta_2)} \|D_{yy} F(x, y)\| \right) \delta_4 \|y - y'\| \\ &\leq \frac{1}{4} \|y - y'\|.\end{aligned}$$

On the other hand,

$$\|y_1\| = \|\Phi(x; 0)\| = \|D_y F(x, 0)F(x, 0)\| \leq 2 \sup_{x \in B_X(0, \delta_3)} \|F(x, 0)\| \leq \frac{1}{4} \delta_3.$$

Therefore, we have

$$\|y_n\| \leq \sum_{k=1}^n \|y_k - y_{k-1}\| \leq \sum_{k=1}^n 4^{-k} \|y_1\| \leq 2 \|y_1\| \leq \frac{1}{2} \delta_3.$$

Therefore, for all  $n \in \mathbb{N}$ ,  $y_n \in B_Y(0, \delta_3)$ . Further, we by

$$\|y_n - y_m\| \leq \sum_{k=m+1}^n \|y_k - y_{k-1}\| \leq \sum_{k=m+1}^n 4^{-k} \|y_1\|.$$

$\{y_n\}$  is a Cauchy sequence so it has a limit.

Finally, if there exist two  $y, y' \in B_Y(0, \delta_3)$  s.t.  $F(x, y) = F(x, y') = 0$  we have

$$\|y - y'\| = \|\Phi(x; y) - \Phi(x; y')\| \leq \frac{1}{4} \|y - y'\|$$

So, we have  $y = y'$ . This gives the uniqueness.  $\square$

**B**  $\omega \mapsto \phi_\omega$  in  $C^1(\mathcal{O}, H^2)$  implies  $\omega \mapsto \phi_\omega$  in  $C^\infty(\mathcal{O}, \Sigma_n)$  for any  $n \in \mathbb{N}$

**Proposition B.1.** Assume (H1)–(H3), (H6) and

(H4)' *There exists an open interval  $\mathcal{O} \subset \mathbb{R}_+$  such that equation (1.11) admits a positive radial solution  $\phi_\omega \in H^2$  for  $\omega \in \mathcal{O}$ . Further, assume  $\omega \mapsto \phi_\omega$  is in  $C^1(\mathcal{O}, H^2)$ .*

*Then, the map  $\omega \mapsto \phi_\omega$  is in  $C^\infty(\mathcal{O}, \Sigma_n)$  for arbitrary  $n \in \mathbb{N}$ .*

*Proof.* (Sketch). By a standard bootstrapping argument one can show  $\phi_\omega \in H^n$  for arbitrary  $n$ . Further, by maximum principle, one can show  $\phi_\omega$  decays exponentially. Therefore,  $\phi_\omega \in \Sigma_n$  for arbitrary  $n$ . Further,  $\omega \mapsto \phi_\omega$  is in  $C^0(\mathcal{O}, \Sigma_n)$ .

Next, fix  $\omega_0 \in \mathcal{O}$ . Differentiating

$$0 = -\Delta\phi_\omega + \omega\phi_\omega + \beta(\phi_\omega^2)\phi_\omega,$$

with respect to  $\omega$ , we have

$$-\phi_\omega = (-\Delta + \omega + \beta(\phi_\omega^2) + 2\beta'(\phi_\omega^2)\phi_\omega^2) \partial_\omega \phi_\omega. \quad (\text{B.1})$$

Now, set

$$\begin{aligned} A &:= -\Delta + \omega_0 + \beta(\phi_{\omega_0}^2) + 2\beta'(\phi_{\omega_0}^2)\phi_{\omega_0}^2 \\ B_\varepsilon &:= \varepsilon + \beta(\phi_{\omega_0+\varepsilon}^2) + 2\beta'(\phi_{\omega_0+\varepsilon}^2)\phi_{\omega_0+\varepsilon}^2 - \beta(\phi_{\omega_0}^2) - 2\beta'(\phi_{\omega_0}^2)\phi_{\omega_0}^2 \end{aligned}$$

Then,  $A$  is invertible as an operator on  $A : L_{\text{rad}}^2(\mathbb{R}^3) \rightarrow L_{\text{rad}}^2(\mathbb{R}^3)$ . Since (B.1) can be written as

$$-\phi_{\omega_0+\varepsilon} = (A + B_\varepsilon) \partial_\omega \phi_{\omega_0+\varepsilon}.$$

Therefore, since  $B_0 = 0$ , for sufficiently small  $\varepsilon$ , we have

$$\partial_\omega \phi_{\omega_0+\varepsilon} = - \left( \sum_{k=0}^{\infty} (-1)^k (A^{-1} B_\varepsilon)^k \right) A^{-1} \phi_{\omega_0+\varepsilon}. \quad (\text{B.2})$$

Now, we can show that if  $\omega \mapsto \phi_\omega$  is in  $C^m(\mathcal{O}, \Sigma_n)$ , then  $\varepsilon \rightarrow (A + B_\varepsilon)^{-1}$  is  $C^m$  with values in  $B(\Sigma^n, \Sigma^n)$ . By induction, one can show  $\omega \mapsto \phi_\omega$  is in  $C^\infty(\mathcal{O}, \Sigma_n)$ .  $\square$

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