A Deadbeat Observer for Two and Three-dimensional LTI Systems by a Time/Output-Dependent State Mapping

Gilberto Pin * **Peng Li** ** **Giuseppe Fedele** ***
Thomas Parisini ****

* *Electrolux Professional S.p.A., Italy (e-mail: gilberto.pin@electrolux.it).*
** *Department of Electrical and Electronic Engineering, Imperial College London, UK (e-mail: peng.li13@imperial.ac.uk)*
*** *Department of Informatics, Modeling, Electronics and Systems Engineering, University of Calabria, Italy (e-mail: giuseppe.fedele@unical.it)*
**** *Department of Electrical and Electronic Engineering, Imperial College London, UK and Department of Engineering and Architecture, University of Trieste, Italy (e-mail: t.parisini@gmail.com)*

Abstract: The problem of deadbeat state reconstruction for non-autonomous linear systems has been solved since several decades, but all the architectures formulated since now require either high-gain output injection, which amplifies measurement noises (e.g., in the case of sliding-mode observers), either state augmentation, which yields a non-minimal realization of the deadbeat observer (e.g., in the case of integral methods and delay-based methods). In this context, the present paper presents, for the first time, a finite-time observer for continuous-time linear systems enjoying minimal linear-time-varying dynamics, that is, the observer has the same order of the observed system. The key idea behind the proposed method is the introduction of an almost-always invertible time/output-dependent state mapping which allows to recast the dynamics of the system in a new observer canonical form whose initial conditions are known.

1. INTRODUCTION

State observers are keenly researched in the recent literature due to the fact that state variables play a significant role in the control system theory, while in many practical situations, they are unmeasurable.

The classical observers proposed in Kalman and Bucy (1961) and Luenberger (1964) are the most well-known methodologies to achieve state estimation. However, although these observers can be designed to converge faster, in the context of implementable techniques (finite gain), they can only guarantee asymptotic convergence of the estimation error. However non-asymptotic observers providing finite-time convergence with fast speed are undoubtedly desirable in most of engineering cases.

Several algorithms have been conceived after the first convolutional deadbeat state estimator proposed by Kalman in the '60s [Kalman et al. (1969)], able to reconstruct the initial conditions and the system's state in finite-time. Besides, the moving-window observers presented in Fuksa and Byrski (1984) obtain deadbeat estimates through convolution and optimization techniques. Instead of using the integral operation adopted in the methodologies above, a kind of delay-based deadbeat observer has been proposed in Gilchrist (1966). Nonetheless, all the aforementioned observers are suffering either from the huge memory consumption or from the heavy computational burden.

In the recent two decades, several novel methods have been developed in the deadbeat observer design. Detailed in Fliess and Sira-Ramırez (2003) and Reger (2007), the algebraic state reconstructors achieve the arbitrary finite-time convergence without using the delay or buffering techniques. However, as a consequence of their internal instability, the algebraic state observers need periodic resetting otherwise the estimates will diverge. The effect of resetting has been shown in Reger et al. (2006). Impulsive observers proposed in Raff et al. (2006) and Raff and Allgöwer (2007) provide non-asymptotic state estimation by updating the state estimates on predefined time instant based on the current measurement. Nonetheless, the impulsive observer and the aforementioned delay-based observer as well as the integral methods have the system dimension at least two times larger than that of the observed system. A different class of finite-time observers making use of sliding-mode adaptation is designed in Haskara (1998). However it is an inevitable problem that the finite-time convergence of the sliding-mode methodology usually relies on (saturated) high-gain injection, which makes the observer vulnerable to measurement noise and limits its applicability.

A new deadbeat state observation technique without discontinuous high-gain injection is introduced in Pin et al. (2013), eliminating the effect of the unknown initial conditions by using Volterra integral operators with suitably shaped kernel functions enjoying internally stable implementation. Furthermore, an alternative non-asymptotic observer resorting to a univariate modulating function is proposed in Pin et al. (2015). The modulating function observer has been exploited for harmonics detection in Power electronic systems (see Chen et al. (2016)). However, for the integral modulation and the kernel-based techniques, state augmentation is still necessary.

In this paper, a deadbeat observer based on time/output-dependent transformations with minimal dimension is proposed. More specifically, we show that after the coordinate transformations, the system can be recast in a *"deadbeat observer canonical form"* defined herein whose initial conditions are known and therefore, arbitrarily short time convergence of the error can be achieved from the measurements. Notably, the proposed deadbeat observer has the same dimension as the observed system. Moreover, the state estimator has no high-gain injection and is natively capable to deal with MIMO system. At the same time, the noise immunity can be enhanced by shaping the modulating function. Moreover, the robustness of the proposed observer is analyzed under the presence of disturbance on both input and output. Numerical simulations are carried out to examine the effectiveness of the proposed methodology.

2. PROBLEM STATEMENT

Consider the MISO linear system [1]

$$\begin{array}{l} x^{(1)}(t) = Ax(t) + Bu(t) \\ y(t) = c\,x(t) \end{array} \qquad (1)$$

with $x \in \mathbb{R}^n$, $y \in \mathbb{R}$, $u \in \mathbb{R}^m$. Assume that the system is fully controllable. Then there exists a vector $l \in \mathbb{R}^n$ such that $A - lc$ is a nilpotent matrix. Then, the observed system (1) can be written in the following form:

$$x^{(1)}(t) = (A - lc)x(t) + ly(t) + Bu(t). \qquad (2)$$

Letting G and T be respectively a Schur (upper-triangular) and a unitary-orthogonal matrices such that $A - lc = TGT^\top$, then the matrix $G = T^\top(A - lc)T$ has null-diagonal elements ($g_{11} = 0, g_{22} = 0, ..., g_{nn} = 0$). By the change of coordinates $z(t) \triangleq T^\top x(t)$, we can recast the above system in strict-upper-triangular form

$$\begin{array}{rcl} z^{(1)}(t) &=& Gz(t) + v(u(t), y(t)) \\ y(t) &=& hz(t), \end{array} \qquad (3)$$

where $v(u(t), y(t)) \triangleq T^\top ly(t) + T^\top Bu(t)$ and $h \triangleq cT$.

The nilpotency of the matrix G in the above system and its strict-upper triangular structure are instrumental to simplify the algebra in the forthcoming analysis.

A further change of coordinates is operated now to recast the system in a convenient form, named the *"deadbeat observer canonical form"*. The proposed observer relies on a time/output-dependent coordinate transformation $\Psi : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^n$ given by:

$$\xi = \Psi(z \,|\, t, y) = Y(t)z + \zeta(t, y) \qquad (4)$$

where the separator ` | ´ is introduced with the aim of separating the state vector z, which is the main argument of the transformation, from the instrumental arguments t and y. In (4) the term $\zeta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^n$ is a specified function of the time and of the measurable output, such that $\zeta(0, y) = 0, \forall y \in \mathbb{R}$, while $Y(t) : \mathbb{R} \to \mathbb{R}^n$ is a known time-dependent matrix, such that $Y(0) = 0$ and whose inverse is well-defined (nonsingular) for any $t > 0$ (strictly).

Due to the structure of Y and ζ, that will be described in the following sections for the two distinct cases $n = 2$ and $n = 3$, the transformed coordinate ξ has a known initial condition whatever be the value of $z_0 = z(0)$:

$$\xi_0 = \xi(0) = \Psi(z_0|0, y(0)) = Y(0)z_0 + \zeta\,(0, hz_0) \equiv 0,$$

for all $z_0 \in \mathbb{R}^n$.

Moreover, the dynamics of ξ are fully characterized and observable, so that a conventional observer can be deployed to obtain an estimate $\hat{\xi}$.

Since the inverse of the mapping Ψ with respect to the first argument, Ψ^{-1}, exists for any $t > 0$ (strictly) for the specific state transformation provided in the following Sections, then it can be used to retrieve an estimate $\hat{z}$ of the state z by:

$$\hat{z} = \Psi^{-1}(\hat{\xi} \,|\, t, y) = Y(t)^{-1}\big(\hat{\xi} - \zeta(t, y)\big), \quad \forall t > 0 \qquad (5)$$

from which, finally, an estimate for the original state-vector $\hat{x}$ can be trivially obtained by linear projection:

$$\hat{x} = T^{-\top}z.$$

3. TWO DIMENSIONAL SYSTEM

In this case, we will consider a system having dimension $n = 2$. Letting $\xi = [\xi_1 \;\; \xi_2]^\top$, the transformation $\Psi(z|t, y)$ takes the form

$$\begin{array}{l} \xi_1(t) = k(t)hz(t) \\ \xi_2(t) = k(t)hGz(t) - k^{(1)}(t)y(t) \end{array} \qquad (6)$$

with $k(\cdot)$ a user-defined function of time with scalar output such that $k(0) = 0, k^{(1)}(0) = 0$ and $k(t) > 0, \forall t > 0$. Moreover, let $\Gamma = [1 \; 0]$. By this choice, the output of the ξ system can be obtained by

$$\gamma(t) = \Gamma\xi(t) = \xi_1(t) = k(t)y(t).$$

Note that, by choosing k according to the prescriptions above, then the initial value of ξ is known ($\xi(0) = 0$).

Omitting, for the sake of brevity, the explicit dependence on time of $k, k^{(1)}, \xi, u, y$ and z, and defining the functions

$$Y = Y(t) \triangleq \begin{bmatrix} hk \\ hGk \end{bmatrix}, \zeta = \zeta(t, y) \triangleq \begin{bmatrix} 0 \\ -k^{(1)}y \end{bmatrix}, \qquad (7)$$

then (6) can be written in the form

$$\xi = Yz + \zeta. \qquad (8)$$

Now, taking the time-derivative of ξ_1 and ξ_2, the dynamics of the system in the transformed coordinates write

$$\begin{array}{rcl} \xi_1^{(1)} &=& hGkz + k^{(1)}y + khv \\ \xi_2^{(1)} &=& hG^2kz - k^{(2)}y + khGv - k^{(1)}hv. \end{array} \qquad (9)$$

Moreover, defining

[1] The requirement for the system to be single-output is instrumental to simplify the derivation of the deadbeat observer, but it does not prevent the application of the developed method to MIMO systems. Indeed, any observable MIMO system can be reduced to a MISO one by resorting to the method described in Lemma 9.4.4 of Willems and Polderman (1997) . Formally, the method is described to achieve input reduction (MIMO to SIMO), but can be trivially modified for the task of output reduction exploting the control/observer duality in the framework of linear systems.

$$M = M(t) \triangleq \begin{bmatrix} hGk \\ hG^2k \end{bmatrix} \tag{10}$$

and

$$\eta = \eta(t,u,y) \triangleq \begin{bmatrix} k^{(1)}y + khv(u,y) \\ -k^{(2)}y + k(t)hGv(u,y) - k^{(1)}hv(u,y) \end{bmatrix}$$

then in view of (8), (3) can be rewritten as follows

$$\xi^{(1)} = X\xi - X\zeta + \eta \tag{11}$$

where we have posed $X \triangleq MY^{-1}$. Since G is upper-triangular and nilpotent, then in view of (9) and (10) the matrix X becomes

$$X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \tag{12}$$

By the structure of X and Γ it is readily seen that the ξ-system is fully observable; therefore, an output-error injection matrix K can be designed to build an observer in the form of (13) with stable error dynamics.

Since both the initial conditions and the dynamics of the ξ-system (11) are known and observable, then a classical linear error-feedback observer can be arranged as follows:

$$\begin{aligned} \hat{\xi}^{(1)}(t) &= X\hat{\xi}(t) - X\zeta(t, y(t)) + \eta(t, u(t), y(t)) \\ &\quad + K(\gamma(t) - \Gamma\hat{\xi}(t)), \end{aligned} \tag{13}$$

initialized with the known initial conditions $\hat{\xi}(0) = 0$. In (13), $K \in \mathbb{R}^n$ is a conventional (Luenberger) observer gain which is aimed at stabilizing the dynamics of the observer error. Moreover, ζ and η are known functions of the measured input $u(t)$ and output $y(t)$.

4. THREE DIMENSIONAL SYSTEM

Let us introduce a transformation $\Psi(z|t, y)$. To this end, let $k(\cdot)$ be a user-defined time-function such that $k(0) = 0$, $k^{(1)}(0) = 0$, $k^{(2)}(0) = 0$ and $k(t) > 0$, $\forall t > 0$. Let us define

$$Y = Y(t) \triangleq \begin{bmatrix} hk \\ hGk \\ hG^2k - hGk^{(1)} \end{bmatrix}, \tag{14}$$

and

$$\zeta = \zeta(t, y) \triangleq \begin{bmatrix} 0 \\ -2k^{(1)}y \\ k^{(2)}y \end{bmatrix}. \tag{15}$$

Similarly with the two-dimensional case, the dynamics of the system in the transformed coordinates write

$$\begin{aligned} \xi_1^{(1)} &= hGkz + k^{(1)}y + khv \\ \xi_2^{(1)} &= hG^2kz - 2k^{(2)}y - k^{(1)}hGz + khGv - 2k^{(1)}hv \\ \xi_3^{(1)} &= hG^3kz + k^{(3)}y + khG^2v - k^{(1)}hGv + k^{(2)}hv. \end{aligned}$$

Moreover, defining

$$M = M(t) \triangleq \begin{bmatrix} hGk \\ hG^2k - k^{(1)}hG \\ hG^3k \end{bmatrix} \tag{16}$$

omitting the dependence of v on u and y, introducing the function

$$\eta = \eta(t,u,y) \triangleq \begin{bmatrix} k^{(1)}y + khv \\ -2k^{(2)}y + khGv - 2k^{(1)}hv \\ k^{(3)}y + khG^2v - k^{(1)}hGv + k^{(2)}hv \end{bmatrix}$$

and defining $X \triangleq MY^{-1}$, then the dynamics of ξ can be written as

$$\xi^{(1)} = X(\xi - \zeta) + \eta$$

where the matrix X takes the form

$$X = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}. \tag{17}$$

Notably, with $\Gamma = [1\ 0\ 0]$, the system is fully observable. The observer (13) can be deployed also in this case.

5. ROBUSTNESS ANALYSIS

Consider the circumstance that the measurements are contaminated by bounded noise:

$$u_d(t) = u(t) + d_u(t), \quad y_d(t) = y(t) + d_y(t),$$

where $d_u(t) \leq \bar{d}_u \in \mathbb{R}^m$ and $d_y(t) \leq \bar{d}_y \in \mathbb{R}$. Correspondingly, the state observer takes the form:

$$\begin{aligned} \hat{\xi}_d^{(1)}(t) &= X\hat{\xi}_d(t) - X\zeta_d(t, y_d(t)) + \eta_d(t, u_d(t), y_d(t)) \\ &\quad + K(\gamma_d(t) - \Gamma\hat{\xi}_d(t)), \end{aligned}$$

where all the variables with subscript "d" except $\gamma_d(t)$ denote the noisy counterparts of the variables in (13) and can be obtained by replacing $y(t)$ and $u(t)$ by the noisy measurements $y_d(t)$ and $u_d(t)$ in (15) and (4).

Defining the estimation error $\tilde{\xi}_d(t) = \hat{\xi}_d(t) - \xi(t)$ and considering (11), the error dynamics can be obtained as:

$$\begin{aligned} \tilde{\xi}_d^{(1)}(t) &= (X - K\Gamma)\tilde{\xi}_d(t) - X(\zeta_d - \zeta) + (\eta_d - \eta) \\ &\quad + Kk(t)d_y(t). \end{aligned} \tag{18}$$

Remarkably, in (18), the terms $\zeta_d - \zeta$, $\eta_d - \eta$ and Kkd_y are functions of the disturbance d_u and d_y modulated by the function $k(t)$ and its derivatives. Therefore, the stability of the error dynamics can be guaranteed by choosing the observer gain K such that all the eigenvalues of the matrix $X - K\Gamma$ are on the left-half plane. Being $X - K\Gamma$ Hurwitz, there exists a positive definite matrix P satisfying the algebraic Lyapunov equation: $P(X - K\Gamma) + (X - K\Gamma)^\top P = -I$. Let us choose the quadratic form $V(x) \triangleq \tilde{\xi}_d^\top P\tilde{\xi}_d$ as the Lyapunov candidate. Therefore, denoting

$$r(t, k, d_u, d_y) = -X(\zeta_d - \zeta) + (\eta_d - \eta) + Kk(t)d_y(t),$$

one can obtain the derivative of $V(t)$ along the system's state trajectory:

$$\begin{aligned} \dot{V}(t) &= \frac{\partial V}{\partial \tilde{\xi}_d}\tilde{\xi}_d^{(1)} \\ &\leq -\|\tilde{\xi}_d(t)\|^2 + 2\|P\|\|r(t, k(t), d_u(t), d_y(t))\|\|\tilde{\xi}_d(t)\|. \end{aligned}$$

It is obvious that $\dot{V}(t) \leq 0$ as long as the following condition is verified

$$\|\tilde{\xi}_d(t)\| \geq 2\|P\|\|r(t, k(t), d_u(t), d_y(t))\|.$$

We can conclude that function $r(t, k(t), d_u(t), d_y(t))$ should be designed as small as possible by tuning the function $k(t)$ and the observer gain K.

A further source of errors is represented by the output-dependent inverse mapping used to retrieve $\hat{x}$ from $\hat{\xi}$. Indeed the additive noise perturbation affects the estimate through the noisy term $\zeta_d(t, y_d)$:

$$\hat{x}_d(t) = T^{-\top}Y(t)^{-1}(\hat{\xi}_d(t) - \zeta_d(t, y_d)). \tag{19}$$

In turn, the estimation error in the noisy scenario is

$$\tilde{x}_d(t) = T^{-\top}Y^{-1}(t)(\tilde{\xi}_d(t) - \tilde{\zeta}_d(t,d_y))$$
$$\Rightarrow \|\tilde{x}_d(t)\| \leq \|T^{-\top}\|\|Y^{-1}(t)\|(\|\tilde{\xi}_d(t)\| + \|\tilde{\zeta}_d(t,d_y)\|)$$

where $\tilde{\zeta}(t,d_y) \triangleq \zeta_d(t,y_d) - \zeta(t,y)$ is a function of the disturbance d_y and the derivatives of the function $k(t)$.

The results above indicate that by suitably choosing the observer gain K and the modulating function $k(t)$, the proposed deadbeat observer is ISS with respect to the bounded measurement noise d_u and d_y. Moreover, the time-behavior of the state estimation error is related to $k(t)$ as well as the values of the disturbances. In the next few lines, we are going to calculate an upper bound of the estimation error.

According to the error dynamics (18), the estimation error has the expression:

$$\tilde{\xi}_d(t) = \int_0^t e^{(X-K\Gamma)(t-\tau)} r\big(\tau, k(\tau), d_y(\tau), d_u(\tau)\big)d\tau. \quad (20)$$

Define a vector $\rho \triangleq \zeta/y \in \mathbb{R}^3$, one can immediately obtain $\zeta_d - \zeta = \rho d_y(t)$ and $\eta = \rho^{(1)}y + \phi y + (Y + \rho h)v$, where $\phi \triangleq \left[\, k^{(1)}\ 0 \ldots 0 \,\right]^\top \in \mathbb{R}^n$. Therefore, one can obtain

$$\eta_d - \eta = \big(\rho^{(1)} + \phi + (Y+\rho h)T^\top l\big)d_y + (Y+\rho h)T^\top B d_u.$$

All in all,

$$r(t,k,d_u,d_y) = (-X\rho + \rho^{(1)} + \phi + (Y+\rho h)T^\top l + Kk)d_y + (Y+\rho h)T^\top B d_u.$$

Hence, the estimation error $\tilde{\xi}_d(t)$ verifies the following bound:

$$\|\tilde{\xi}_d(t)\| \leq \|r_1(t)\||\bar{d}_y| + \|r_2(t)\|\|\bar{d}_u\| \triangleq \bar{\tilde{\xi}}_d \quad (21)$$

where

$$r_1(t) = \int_0^t e^{(X-K\Gamma)(t-\tau)}\big[-X\rho(\tau) + \rho^{(1)}(\tau) + \phi(\tau)\big]d\tau + \int_0^t e^{(X-K\Gamma)(t-\tau)}\big[(Y(\tau) + \rho(\tau)h)T^\top l + Kk(\tau)\big]d\tau,$$
$$r_2(t) = \int_0^t e^{(X-K\Gamma)(t-\tau)}(Y(\tau) + \rho(\tau)h)T^\top B d\tau.$$

Furthermore,

$$\|\tilde{x}_d(t)\| \leq \|T^{-\top}\|\|Y^{-1}\|\big(\bar{\tilde{\xi}}_d + \|\rho(t)\||d_y|\big) = \|T^{-\top}\|\|Y^{-1}\|\big[\big(\|r_1(t)\| + \|\rho(t)\|\big)|\bar{d}_y| + \|r_2(t)\|\|\bar{d}_u\|\big].$$

6. NUMERICAL EXAMPLES

The behavior of the proposed method is compared with a recent delay-based finite-time observer proposed in Engel and Kreisselmeier (2002). The simulations are carried out on the Matlab/Simulation environment, with *ode4 (Runge Kutta)* solver and sampling time $T_s = 10^{-3}s$.

6.1 Two dimensional system

Consider a two dimensional system:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = cx(t), \quad t \in \mathbb{R}_{\geq 0} \end{cases} \quad (22)$$

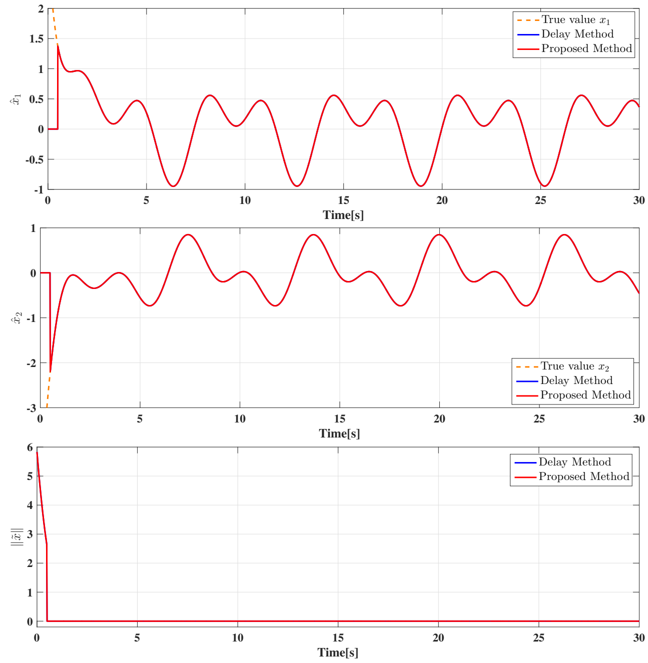

Fig. 1. Time behavior of the second order observer in noise-free scenario.

where

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}, c = [\,1\ \ 2\,],$$

with $x(t) \in \mathbb{R}^2$ and initial condition $x_0 = [3\ \ -5]^\top$. The system is driven by the input $u(t) = [\sin(t)\ \ 0.5\sin(2t)]^\top$. According to (22) we can find $l = [0\ \ -1]^\top$ such that $A - lc$ is nilpotent and can be decomposed as $A - lc = TGT^\top$ with

$$G = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

The gain vector of the observer (13) is chosen as $K = [3\ 2]^\top$ placing the poles of the observer at -1 and -2. The function $k(t)$ is designed as $(1 - e^{-t})^3$ to ensure that $\xi(0) = 0$. The algorithm is activated at $t_a = 0.5s$.

The parameters of the delay-based observer are chosen as $L_1 = [-1\ 1]^\top$ and $L_2 = [-17\ 11]\top$, with pole vectors $p_1 = [-1\ \ -2]$ and $p_2 = [-3\ \ -4]$ and delay $D = 0.5s$.

In the ideal case, the time behaviors of the two deadbeat observers are shown in Fig. 1. From the simulation results, one can conclude that both methods performs exactly the same providing deadbeat state estimation within finite time.

In order to examine their immunity against perturbation, we simulate the disturbances adding on the output $y(t)$ and the input $u(t)$ uniformly distributed random sequences ranging within the interval $[-0.5, 0.5]$ (shown in Fig. 2 and Fig. 3). With the same parameter settings, we obtain the state estimates in Fig. 4, where the estimation error bound calculated in Section 5 is also plotted.

It is shown that under the perturbation of measurement noise, both algorithms are stable and able to converge in a neighborhood of the true states. Although there are oscillations under the error bound in the initial phase (in the first $3s$), as the derivatives of the function $k(t)$ decays

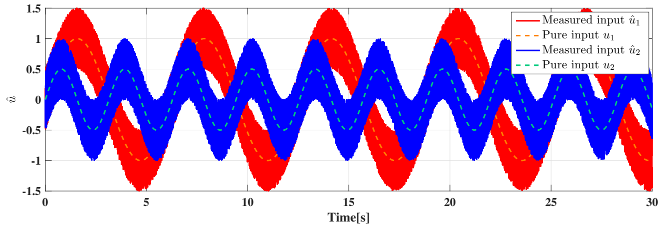

Fig. 2. Noisy measurement and the pure signal of $u(t)$.

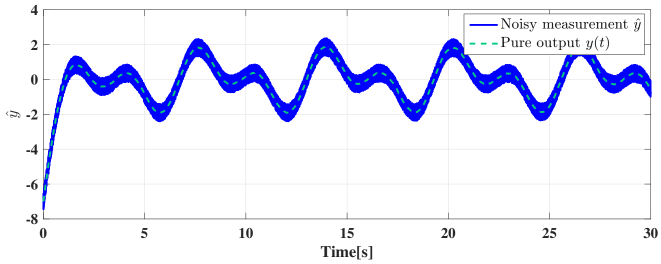

Fig. 3. Noisy measurement and the pure signal of $y(t)$.

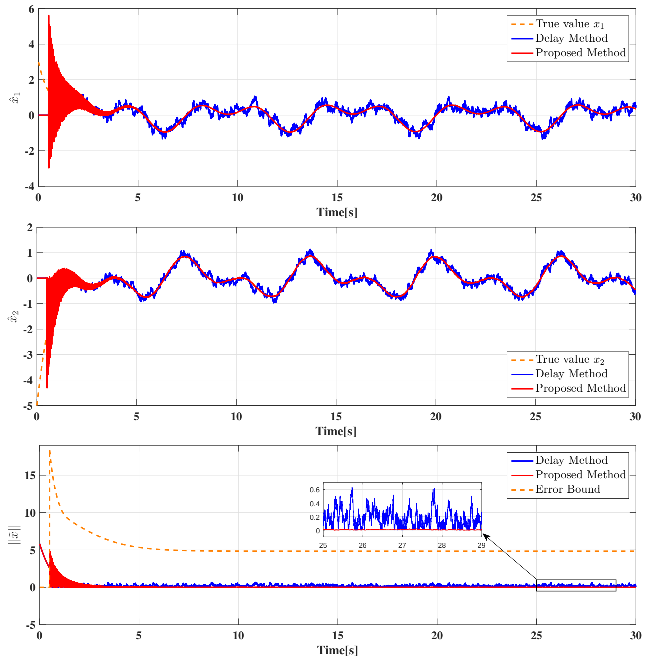

Fig. 4. Time behavior of the second order observer in noisy scenario.

to zero, the proposed observer gives the state estimates with higher accuracy.

6.2 Three dimensional system

Consider the following three dimensional system:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = cx(t), \quad t \in \mathbb{R}_{\geq 0} \end{cases} \tag{23}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 0 & 2 \\ 1 & 0 \end{bmatrix}, \ c = [\, 1 \ 2 \ 1 \,],$$

with $x(t) \in \mathbb{R}^3$ and initial condition $x_0 = [3 \ \ -5 \ \ -7]^\top$.The input is chosen as $u(t) = [\sin(t) \ \ 0.5]^\top$. We can find $l = [-2 \ 4 \ \ -9]^\top$ making $A - lc$ a nilpotent matrix and

$$\mathbf{G} = \begin{bmatrix} 0 & 9.3917 & 19.7131 \\ 0 & 0 & 1.0911 \\ 0 & 0 & 0 \end{bmatrix}.$$

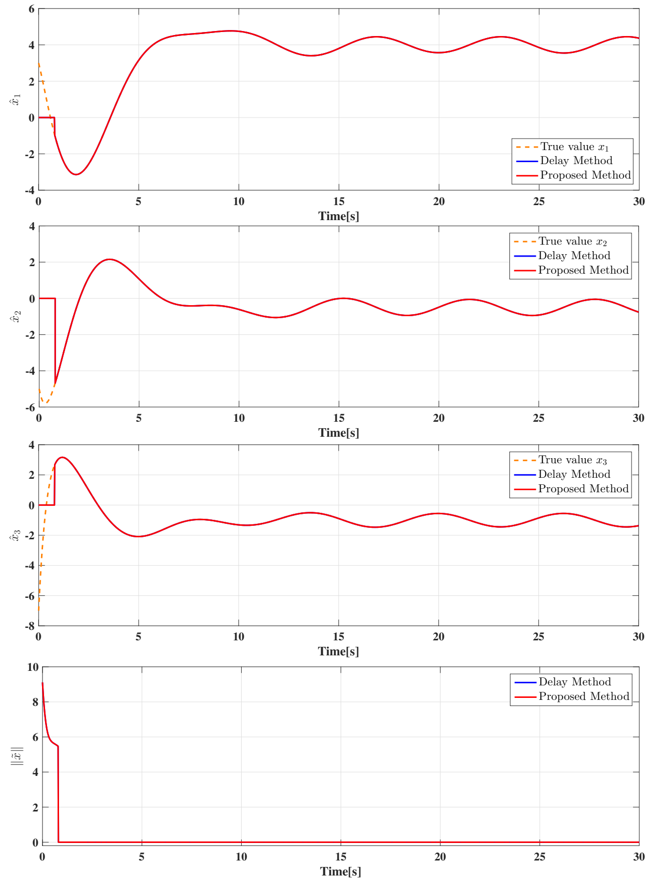

Fig. 5. Time behavior of the third order observer in noise-free scenario.

Parameters of observer (16) are designed as $K = [6 \ 11 \ 6]^\top$, therefore the roots of the observer's characteristic polynomial are -1, -2 and -3. The function $k(t)$ is still designed as $k(t) = (1 - e^{-t})^3$ guaranteeing that $\xi(0) = 0$. We set the activation time of the observer $t_a = 0.8s$.

The delay-based method is parameterized by $L_1 = [-2 \ 3 \ -1]^\top$, $L_2 = [-107 \ 48 \ 23]^\top$ with poles at $p_1 = [-1 \ -2 \ -3]$ and $p_2 = [-4 \ -5 \ -6]$. The length of the delay $D = 0.8s$. The results of the two observers in absence of noise are shown as Fig. 5.

Let us consider also the noisy scenario by adding the distributed random sequences ranging within $[-0.5, 0.5]$ on the output $y(t)$ and $u(t)$ (see Fig. 6 and Fig. 7). With all the parameters unchanged, the time behavior of the two observers with the noisy measurements are depicted in the Fig. 8, in which the estimation error of the proposed method is compared with the error bound.

Similarly with the two dimensional examples, both observers converge in finite-time in the noise-free scenario. With noisy measurements, according to the results, the proposed observer gives the state estimates entering a smaller neighborhood of the true states in the steady state, albeit there are some oscillations in the beginning.

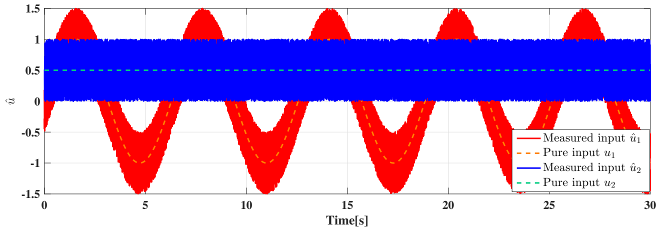

Fig. 6. Noisy measurement and the pure signal of $u(t)$.

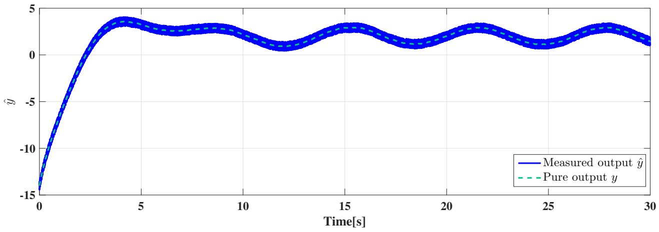

Fig. 7. Noisy measurement and the pure signal of $y(t)$.

7. CONCLUDING REMARKS

In this work, a new finite-time observer with minimal order realization is proposed for continuous-time LTI systems having order two and three. The algorithm consists in recasting the original system in a novel deadbeat observer canonical form through a linear-time-varying mapping which eliminates the effect of the initial conditions. Future efforts will be devoted to extend the present formulation for systems of higher dimensions.

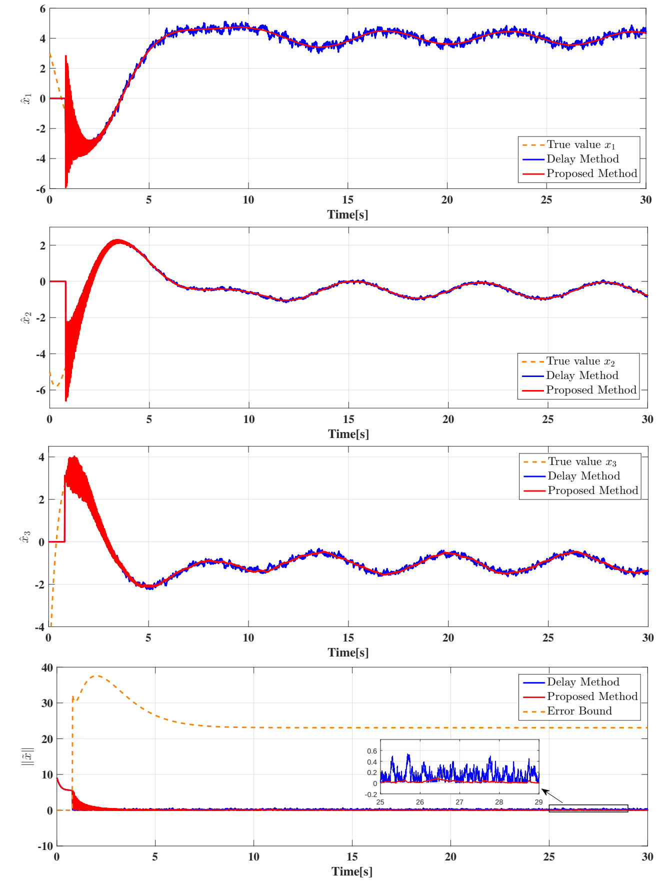

Fig. 8. Time behavior of the third order observer in noisy scenario.

REFERENCES

Chen, B., Pin, G., Ng, W., Parisini, T., and Hui, S. (2016). A fast-convergent modulation integral observer for online detection of the fundamental and harmonics in grid-connected power electronics systems. *IEEE Transactions on Power Electronics*, PP(99), 1–1. doi: 10.1109/TPEL.2016.2570821.

Engel, R. and Kreisselmeier, G. (2002). A continuous-time observer which converges in finite time. *IEEE Transactions on Automatic Control*, 47(7), 1202–1204.

Fliess, M. and Sira-Ramırez, H. (2003). State reconstructors: a possible alternative to asymptotic observers and kalman filters. In *Proceedings of CESA*.

Fuksa, S. and Byrski, W. (1984). General approach to linear optimal estimator of finite number of parameters. *Automatic Control, IEEE Transactions on*, 29(5), 470–472.

Gilchrist, J.D. (1966). n-observability for linear systems. In *Joint Automatic Control Conference*, 4, 703–713.

Haskara, I. (1998). On sliding mode observers via equivalent control approach. *International Journal of control*, 71(6), 1051–1067.

Kalman, R.E., Falb, P.L., and Arbib, M.A. (1969). *Topics in mathematical system theory*, volume 33. McGraw-Hill New York.

Kalman, R.E. and Bucy, R.S. (1961). New results in linear filtering and prediction theory. *Journal of basic engineering*, 83(1), 95–108.

Luenberger, D.G. (1964). Observing the state of a linear system. *Military Electronics, IEEE Transactions on*, 8(2), 74–80.

Pin, G., Chen, B., and Parisini, T. (2015). The modulation integral observer for linear continuous-time systems. In *Control Conference (ECC), 2015 European*, 2932–2939. doi:10.1109/ECC.2015.7330983.

Pin, G., Lovera, M., Assalone, A., and Parisini, T. (2013). Kernel-based non-asymptotic state estimation for linear continuous-time systems. In *2013 American Control Conference*, 3123–3128. IEEE.

Raff, T. and Allgöwer, F. (2007). An impulsive observer that estimates the exact state of a linear continuous-time system in predetermined finite time. In *Control & Automation, 2007. MED'07. Mediterranean Conference on*, 1–3. IEEE.

Raff, T., Lachner, F., and Allgöwer, F. (2006). A finite time unknown input observer for linear systems. In *Control and Automation, 2006. MED'06. 14th Mediterranean Conference on*, 1–5. IEEE.

Reger, J., Mai, P., and Sira-Ramirez, H. (2006). Robust algebraic state estimation of chaotic systems. In *2006 IEEE International Conference on Control Applications*, 326–331. doi:10.1109/CCA.2006.285913.

Reger, J. (2007). On algebraic time-derivative estimation and deadbeat state reconstruction. *arXiv preprint arXiv:0710.0010.*

Willems, J. and Polderman, J. (1997). *Introduction to Mathematical Systems Theory*. Springer.